数理方程.

~~$\frac{\partial^2 u}{\partial x \partial y}=0$.~~

1.1 一些基本偏微分方程的概念.

$\frac{\partial^2 u}{\partial x \partial y}=0$. $f(x)+g(y)$.

$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$

$\xi = x+at$. $\eta = x-at$.

$\frac{\partial^2 u}{\partial \xi \partial \eta}=0$

$u(x,t)=f(x-at)+g(x+at)$.

三个最基本物理方程.

1. 弦振动方程.

$$\begin{cases} u_{tt}=a^2u_{xx} \ (\text{泛定方程}). \\ u(0,x)=\varphi(x).\ u_t(0,x)=\psi(x) \end{cases}$$

若有外力, $\begin{cases} u_{tt}=a^2u_{xx}+f(t,x) \\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x) \end{cases}$

$f(t,x)=\frac{g(t,x)}{\rho} \to$ 外力;

2. 热传导方程

~~$u_t=a^2u_{xx}$ (泛定方程)~~

$\{ u_t=a^2u_{xx}+f(t,x) \leftarrow =\frac{g(t,x)}{c\rho}$ 外界热源

3. 泊松方程

$\Delta u=g(t,x)$

1.3 定解条件与定解问题.

(1) 初始条件与初始问题

$t=0$时, $u(t,x)$与各阶偏导取值.

$\begin{cases} u_{tt}=a^2u_{xx} \\ u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x). \end{cases}$ 二阶导,两个初始条件

$\begin{cases} u_t=a^2u_{xx}+f(t,x) \\ u(0,x)=\varphi(x). \end{cases} \leftarrow$ 一阶方程.

只需一个初始条件.

(2) 边界条件与边值问题

$(\alpha\frac{\partial u}{\partial n}+\beta u)|_S=\varphi(x,y,z)$

$\alpha=0$, 第一类边界条件

$\beta=0$ 第二类边界条件

$\alpha^2+\beta^2\neq 0$. 第三类边界条件.

(3) 混合问题.

同时含有初始条件与边界条件.

对于热方程来说:

i) 第一类边界条件: 已知S上物体温度.

ii) 第二类边界条件:

$\frac{\partial u}{\partial n}|_S=-\frac{q(t,x,y,z)}{k}$

从S上流出的热流密度为q

iii) 第三类边界条件: 与外部介质有热交换

$(k\frac{\partial u}{\partial n}+hu)|_S=h\theta$.

$\frac{\partial u}{\partial n}|_{x=0}=-\frac{\partial u}{\partial x}|_{x=0}$

$\frac{\partial u}{\partial n}|_{x=l} = \frac{\partial u}{\partial x}|_{x=l}$

对于弦振动边界条件:

$$\begin{cases} u(t,0)=\mu(t) \\ u(t,l)=\nu(t). \end{cases}$$

若弦受外力

$$\begin{cases} u_x(t,0)=\frac{\mu(t)}{T} \\ u_x(t,l)=\frac{\nu(t)}{T}. \end{cases}$$

若固定在弹簧自由端上

$$\begin{cases} T u_x(t,0)-k u(t,0)=\mu(t) \\ T u_x(t,l)+k u(t,l)=\nu(t) \end{cases}$$

1.5 叠加原理与齐次化原理

1. $L[\sum c_i u_i]=\sum c_i L[u_i]=\sum c_i f_i$

2. 级数齐次化

$u=\sum c_i u_i$ 收敛

则 Lu 收敛于 $\sum c_i f_i$

3. $Lu=f$.

$U=\int u(M)dM$ 积分齐次化

$LU=\int f\,dM$

二. 齐次化原理.

齐次化原理1.

$$\begin{cases} u_{tt}=Lu+f(t,M) \\ u|_{t=0}=0,\ u_t|_{t=0}=0 \end{cases}$$

的解可由 $\begin{cases} \frac{\partial^2 w}{\partial t^2}=Lw \\ w|_{t=\tau}=0,\ w_t(\tau,0)=f(\tau,M) \end{cases}$

$w(t,0)$

$\int_0^t w(t,M,\tau)d\tau$ 得出

齐次化原理2

$$\begin{cases} \frac{\partial w}{\partial t}=Lw \\ w|_{t=\tau}=f(\tau,M) \end{cases}$$

是 $\begin{cases} u_t=Lu+f(t,M) \\ u(t,0)=0 \end{cases}$

$\int_0^t w(t,M,\tau)d\tau$

的解.

~~第一章课后题~~.

Tip. 特征线法:

$u_x \to \frac{1}{dx}$

$u_y \to -\frac{1}{dy}$

$u_{xy} \to \frac{-1}{dx\,dy}$

$u_{xx} \Rightarrow \frac{1}{(dx)^2}$ $u_{yy} \Rightarrow \left(\frac{1}{dy}\right)^2$

课本,定解

7.(1) $\begin{cases} u_t=a^2u_{xx} \\ u(t,0)=0\ \ u(t,l)=u_0 \end{cases}$

(2) $\begin{cases} u_t=a^2u_{xx} \\ u_x(t,0)=-\frac{q_1}{k},\ u_x(t,l)=\frac{q_2}{k} \end{cases}$

(3) $\begin{cases} u_t=a^2u_{xx} \\ u(t,0)=\mu(t) \end{cases}$ $k u_x(t,l)+h u=h\theta(t)$

8. $\begin{cases} u_{tt}=a^2u_{xx} \\ u_x(t,l)=u_x(t,x)=0 \end{cases}$

$u(0,x)=\{$

想清楚,初始条件 边界条件. 都不要漏

半弦问题 $u(t,0)=0$. 奇延拓

$u_x(t,0)=0$ 偶延拓

☆

9.(2) $\begin{cases} u_{tt} = a^2 \Delta_3 u \\ u(0,r) = \varphi(r) \\ u_t(0,r) = \psi(r) \end{cases}$

$u_{tt} = a^2(u_{rr} + \frac{2}{r} u_r)$

$r u_{tt} = a^2 r u_{rr} + 2a^2 u_r$

令 $v = ru$. $v_{rr} = r u_{rr} + 2u_r$

则 $\begin{cases} v_{tt} = a^2 v_{rr} \\ v(0,r) = r\varphi(r) = \varphi_1(r) \\ v_t(0,r) = r\psi(r) = \psi_1(r) \end{cases}$

$v(t,r) = \frac{\varphi_1(r+at) + \varphi_1(r-at)}{2} + \frac{1}{2a}\int_{r-at}^{r+at} \psi_1(r)\,dr$

$u(t,r) = \frac{(r+at)\varphi(r+at) + (r-at)\varphi(r-at)}{2r} + \frac{1}{2ar}\int_{r-at}^{r+at} r\psi(r)\,dr$

(4) $\begin{cases} u_{tt} = u_{xx} \\ u|_{t+x=0} = \varphi(x),\ \varphi(0) = \psi(0) \\ u|_{t-x=0} = \psi(x) \end{cases}$

令 $x+t = \xi$, $x-t = \eta$.

$u_t = u_\xi \xi_t = u_\xi - u_\eta$

$u_{tt} = u_{\xi\xi}\xi_t - u_{\xi\eta}\xi_t + u_{\xi\eta}\eta_t - u_{\eta\eta}\eta_t = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$.

$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$.

$\therefore u_{\xi\eta} = 0$.

$\begin{cases} u_{\xi\eta} = 0 \\ u(0,\eta) = \varphi(-\frac{\eta}{2}) \\ u(\xi,0) = \psi(\frac{\xi}{2}) \end{cases}$

$u(\xi,\eta) = f(\xi) + g(\eta)$.

$u(\xi,0) = f(\xi) + g(0) = \psi(\frac{\xi}{2})$

$u(0,\eta) = f(0) + g(\eta) = \varphi(\frac{\eta}{2})$

$\therefore u(\xi,\eta) = \varphi(\frac{\eta}{2}) + \psi(\frac{\xi}{2}) - f(0) - g(0)$

$u(x,t) = \psi(\frac{x+at}{2}) + \varphi(\frac{x-at}{2}) - \varphi(0)$

10. 利用叠加原理和齐次化原理求解.

$\begin{cases} u_t + a u_x = f(t,x) \\ u(0,x) = \varphi(x) \end{cases}$

分为两组解.

$\begin{cases} v_t + a v_x = 0 \\ v(0,x) = \varphi(x) \end{cases}$

令 $\xi = x - at$, $\eta = t$

$\begin{cases} v_\eta = 0. \end{cases}$

$v(x,t) = \varphi(x-at)$.

$\begin{cases} w_t + a w_x = f(t,x) \\ w(0,x) = 0. \end{cases}$

$\begin{cases} w'_t = -a w'_x \\ w'(\tau,x) = f(\tau,x) \end{cases}$ $w'(t,x) = g(x-at)$

$\varphi(x-at) + \int_0^t f(x-a(t-\tau), \tau)d\tau$

第二章 分离变量法

2.1 有界弦的自由振动

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x) \end{cases}$$

~~设~~ $u(t,x) = X(x)T(t)$

$X(x)T''(t) = a^2 X''(x)T(t)$

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)}$$

$T'' + \lambda a^2 T = 0$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{cases}$$

对于 $\lambda \le 0$ 均无解。

$\lambda > 0$: $X(x) = A\cos kx + B\sin kx$.

$X(0) = 0 \quad A = 0 \cdot X(l) = 0 \quad k = \frac{n\pi}{l}$

$\lambda = (\frac{n\pi}{l})^2$. $X(x) = B_n \sin\frac{n\pi x}{l}$.

$T(t) = C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l}$.

代入 $u(0,x) = \varphi(x)$

$$\varphi(x) = \sum_{n=1}^{+\infty} C_n \sin\frac{n\pi x}{l}$$

$$C_n = \frac{\langle X_n, \varphi(x)\rangle}{\|X_n\|^2}$$

$$\|X_n\|^2 = \int_0^l \sin^2\frac{n\pi x}{l}$$

$$\langle \varphi(x), X_n\rangle = \int_0^l \varphi(x)\cdot\sin\frac{n\pi x}{l}dx.$$

代入 $u_t(0,x) = \psi(x)$.

$$\psi(x) = \sum_{n=1}^{+\infty} \frac{n\pi a}{l} D_n \sin\frac{n\pi x}{l}$$

$$D_n = \frac{\langle \psi(x), X_n\rangle}{\|X_n\|^2} \cdot \frac{l}{n\pi a}.$$

2.2 极坐标下 $\Delta_2 u = 0$ 的边值问题

$$\Delta_2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

令 $u(r,\theta) = R(r)\Theta(\theta)$

$[R''(r) + \frac{R'(r)}{r}]\Theta(\theta) + \frac{1}{r^2}\Theta''(\theta) = 0$

$[r^2R''(r) + rR'(r)]\Theta(\theta) = -R(r)\Theta''(\theta)$

$$-\frac{r^2R''(r) + rR'(r)}{R(r)} = \frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda$$

$r^2R''(r) + rR'(r) ~~+~~ - \lambda R(r) = 0$

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(\theta) = \Theta(\theta + 2\pi) \end{cases}$$

$\therefore \lambda = k^2,\ k = 1, 2, \ldots$

令 $r = e^t$

$\frac{dR}{dt} = r\frac{dR}{dr} \qquad \frac{d^2R}{dt^2} = r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}$

$\therefore$ 原方程 $R''(t) - \lambda R(t) = 0$

$R(t) = Ae^{kt} + Be^{-kt} = A_k r^k + B_k r^{-k}$

$\therefore u(r,\theta) = A_0 + B_0\ln r$

$$+ \sum_{n=1}^{+\infty}(A_n r^n + B_n r^{-n})(C_n\cos n\theta + D_n\sin n\theta)$$

例: $\begin{cases} \Delta_2 u = 0 \\ u|_{x^2+y^2=R^2} = F(x,y) \end{cases}$

变换极坐标，由于为圆域

$$\begin{cases} \Delta_2 u = 0 \\ u|_{r=R} = F(R\cos\theta, R\sin\theta) = f(\theta) \end{cases}$$

$u(r,\theta) = A_0 + B_0\ln r + \sum_{n=1}^{+\infty}(A_n \cdots)(\cdots)$

由于圆域内，$u(0,\theta)$ 有界.

$\therefore$ ~~$B_0 = 0$~~ $B_n = 0$, $(n = 1, 2, \ldots)$

代入即可.

2.3. 施图姆-刘维尔理论

施刘标准型.

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) - q(x)y + \lambda p(x)y = 0$$

(施刘定理)

(1) 可数性: 存在固有值

$(\lambda_0=0) < \lambda_1 < \lambda_2 < \cdots < \lambda_n$.

(2) 非负性 $\lambda_0 = 0$ 的充要条件为,

$q(x) \equiv 0$ 且 不取一三类边界条件.

(3) 正交性. (4) 完备性.

2.4 非齐次情形

$$\begin{cases} L_t u + L_x u = f(t,x) \\ \alpha_1 u_x(t,x_1) - \beta_1 u(t,x_1) = 0 \\ \alpha_2 u_x(t,x_2) + \beta_2 u(t,x_2) = 0 \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x) \end{cases}$$

一般两种方法. 固有函数法或齐次化原理法.

例:

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(t,x) \\ u(t,0) = 0,\ u(t,l) = 0 \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x) \end{cases}$$

$$\begin{cases} v_{tt} = a^2 v_{xx} \\ v(t,0) = v(t,l) = 0 \\ v(0,x) = \varphi(x),\ v_t(0,x) = \psi(x) \end{cases}$$

达朗贝尔公式

$$\frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(x)\,dx.$$

$$\begin{cases} w_{tt} = a^2 w_{xx} + f(t,x) \\ w(t,0) = 0,\ w(t,l) = 0 \\ w(0,x) = w_t(0,x) = 0 \end{cases}$$

求法1: $\begin{cases} w_{tt} = a^2 w_{xx} \\ w(\tau,x) = 0,\ w_t(\tau,x) = f(\tau,x) \end{cases}$

$w(t,x) = f(x-at) + g(x+at)$

$w(\tau,x) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,x)\,dx$

法2. $u(t,x) = \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\,d\xi\,d\tau$

$T''(t)X(x) = a^2 T(t) X''(x)$

$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad X(x) = \sin\frac{n\pi x}{l}$

$f(t,x) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{l} \quad f_n(t) = \frac{\int_0^l f(t,x)\sin\frac{n\pi x}{l}\,dx}{\|X_n\|^2}$

代入, $\begin{cases} T''(t) + \lambda_n a^2 T(t) = f_n(t) \\ T_n(0) = T_n'(0) = 0 \end{cases}$

$\bar{T}_n p^2 + \left(\frac{n\pi a}{l}\right)^2 \bar{T} = \bar{f}_n$.

得 $\bar{T}_n = \frac{\bar{f}_n}{\lambda a^2 + p^2}$

$T_n(t) = f_n(t) * \frac{l}{n\pi a}\sin\frac{n\pi a t}{l}$.

Tip: 若非齐次项 $f(t,x)$ 不依赖于时间t 可先求一特解 $v(x)$, 然后化为齐次

(3) 泊松方程边界问题

根据边界条件

和解的一般形式

去凑

第一、二章例题.

eg1: $u_{xx}+2u_{xy}-3u_{yy}=0$. (特征线法).

$$(dx)^2+\frac{2}{dxdy}-(\frac{3}{dy^2})=0$$

$$(\frac{1}{dx}-\frac{3}{dy})(\frac{1}{dx}+\frac{1}{dy})=0$$

令 $\xi=3x-y$. $\eta=x+y$

$u_{\xi\eta}=0$.

$$u(x,y)=f(\frac{3x-y}{4})+g(\frac{x+y}{4})$$

eg2: (定解问题书写).

有一厚圆筒，初始温度u_0，内表面温度增加与t成线性关系，外表面与温度u_1的介质有热交换. 写出定解问题.

$$\begin{cases} u_t=D\Delta_2 u=D(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}) \\ u(t,r)=u_0 \\ \frac{\partial u}{\partial n}|_{r=r_1}+hu=hu_1 \end{cases}$$

$u(t,r_2)=at+b$.

$$\frac{\partial u}{\partial n}|_{r=r_1}=\frac{\partial u}{\partial r}|_{r=r_1}\cdot\vec{e_r}$$

$(u+hu_r)|_{r=r_1}=u_1$.

行波法. 即为达朗贝尔公式

eg3: (奇延拓).

$$\begin{cases} u_{tt}=a^2u_{xx}\ (0<x<+\infty,\ t>0) \\ u(0,x)=g_1(x).\ u_t(0,x)=g_2(x) \\ u(t,0)=0 \end{cases}$$

作奇延拓 $\hat{g}_1(x)=\begin{cases} g_1(x) & x>0 \\ -g_1(-x) & x<0 \end{cases}$

$\hat{g}_2(x)=\begin{cases} g_2(x) \\ -g_2(-x) \end{cases}$

$$u(t,x)=\frac{\hat{g}_1(x+at)+\hat{g}_1(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\hat{g}_2(x)dx$$

$x>at$.

$\hat{g}_1(x)=g_1(x)$. $\hat{g}_2(x)=g_2(x)$

$x<at$

$$\frac{g_1(x+at)-g_1(at-x)}{2}+\frac{1}{2a}\int_0^{x+at}g_2(x)dx+\frac{1}{2a}\int_0^{at-x}-g_2(x)dx.$$

eg8. (偶延拓).

$$\begin{cases} u_{tt}=a^2u_{xx} \\ u(0,x)=g_1(x)\ \ u_t(0,x)=g_2(x) \\ u_x(t,0)=0 \end{cases}$$

$x-at>0$. 原式

$x-at<0$

$$\frac{g_1(x-at)+g_1(at-x)}{2}+\frac{1}{2a}\int_0^{x+at}g_2(x)dx+\frac{1}{2a}\int_0^{at-x}g_2(x)dx$$

eg.9. 通过函数变换转为一维无界区域波动方程问题.

$$\begin{cases} \frac{\partial}{\partial x}[(1-\frac{x}{h})^2\frac{\partial u}{\partial x}]=\frac{1}{a^2}(1-\frac{x}{h})^2\frac{\partial^2 u}{\partial t^2} \\ u(x,0)=\varphi(x).\ \ u_t(x,0)=\psi(x). \end{cases}$$

注意方程为 $u_{xx}(1-\frac{x}{h})^2-\frac{2}{h}(1-\frac{x}{h})u_x-\frac{1}{a^2}(1-\frac{x}{h})u_{tt}=0$.

$u(x,t)=w(x)v(x,t)$.

$u_x=w_xv+wv_x$.

$u_{xx}=wv_{xx}+2w_xv_x+w_{xx}v$

$u_{tt}=wv_{tt}$

$$(1-\frac{x}{h})^2wv_{tt}=a^2(1-\frac{x}{h})^2wv_{xx}+[2a^2(1-\frac{x}{h})w_x-\frac{2a^2}{h}w]v_x+[a^2(1-\frac{x}{h})w_{xx}-\frac{2a^2}{h}w_x]v$$

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$2a^2(1-\frac{x}{h})w'' - \frac{2a^2}{h}w_x = 0$

$\frac{w_x}{w} = \frac{1}{h-x}$. $\frac{dw}{dw} = \frac{dx}{h-x}$

$w = \frac{1}{h-x}$ $w_x = \frac{1}{(h-x)^2}$. $w_{xx} = \frac{2}{(h-x)^3}$

$V_{xx} = a^2 V_{tt}$.

$u(x,t) = w(x) \cdot V(t,x)$

$= [f_1(x+at) + f_2(x-at)]/(h-x)$

~~第二章~~ eg.10.(通解法求解.)

$$\begin{cases} u_{xy} + u_x = 0 \\ u(0,y) = \varphi(y).\ u(x,0) = \psi(x). \end{cases}$$

$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y} + 1) = 0$

$u_y + u = g(y)$.

$\frac{du}{dy} + u = g(y)$

$\frac{\partial}{\partial y}(e^y u) = g(y) e^y$.

$e^y u = \int e^y g(y) dy + h(x)$

$u = \int \cancel{e^y g(y) dy}\ f(y) + h(x) e^{-y}$

$u(0,y) = h(0) e^{-y} + f(y) = \varphi(y)$.

$u(x,0) = h(x) + f(0) = \psi(x)$

$h(x) = \psi(x) - f(0)$

$f(y) = \varphi(y) - h(0) e^{-y}$

$u(x,y) = \varphi(y) + e^{-y}[\psi(x) - \psi(0)]$

第二章例题.

eg1.(高阶方程分离变量问题的处理.)

$$\begin{cases} u_{tt} + a^2 u_{xxxx} = 0. \\ u(0,t) = u_{xx}(0,t) = 0. \\ u(l,t) = u_{xx}(l,t) = 0 \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$$

令 $u(t,x) = T(t) X(x)$.

$X''''(x) T(t) a^2 + T''(t) + X(x) = 0$.

$-\frac{T''(t)}{a^2 T(t)} = \frac{X''''(x)}{X(x)} = -\lambda^2$

$X(x) = C_1 \operatorname{sh} x + C_2 \operatorname{ch} x + C_3 \sin x + C_4 \cos x$

$X(0) = X''(0) = 0$. $C_2 = C_4 = 0$.

$X(l) = X''(l) = 0$.

$C_1 = 0$. $\lambda = (\frac{n\pi}{l})^2$

继续求解即可.

eg2.(根据自然语言描述物理问题书写定解问题并求解.)

长为 l 杆,侧面,$x=0$ 处绝热 $x=l$ 处与外界($u=0$)交换.

$u(0,x) = u_0$

$$\begin{cases} u_t - a^2 u_{xx} = 0 \\ u_x(t,0) = 0.\ (u + \frac{1}{h} u_x)|_{x=l} = 0. \\ u(0,x) = u_0. \end{cases}$$

$X(l)+\alpha h X'(l)=0$

$\cot\lambda_n=\lambda_n h/l$

求解即可

第三章 特殊函数

3.1 贝塞尔函数

在柱坐标下分离变量时，

有可能出现 ν 阶贝塞尔函数：

$x^2y''+xy'+(x^2-\nu^2)y=0$

$$\begin{cases}\Delta_3 u=0 & x^2+y^2<R^2 \quad 0<z<H\\ u|_{r=\rho}=0 \\ u(x,y,0)=g_1,\ u(x,y,H)=g_2\end{cases}$$

$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+u_{zz}=0$

$\Theta(\theta)=\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix}$

$R(r)=J(\sqrt{\mu_n}r)\quad J_a'(\sqrt{\mu_n}a)=0$

$Z''-\mu_n Z=0.$

$Z=A_1e^{\sqrt{\mu_n}z}+A_2e^{-\sqrt{\mu_n}z}$

$$\triangle\begin{cases}J_\nu(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\\ J_{-\nu}(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\,\Gamma(k-\nu+1)}\left(\frac{x}{2}\right)^{2k-\nu}\end{cases}$$

当 $\nu=n$ 为整数时，$J_\nu(x)$ 与 $J_{-\nu}(x)$

线性相关。

$N_\nu(x)=\frac{J_\nu(x)\cos\nu\pi-J_{-\nu}(x)}{\sin\nu\pi}$

$\lim_{\nu\to n}=\frac{\frac{\partial J_\nu}{\partial\nu}-(-1)^n\frac{\partial J_{-\nu}}{\partial\nu}}{\pi}$

3.2 贝塞尔函数的性质

(1). 贝塞尔函数的母函数

$\exp\frac{x}{2}(\xi-\xi^{-1})=\sum_{-\infty}^{+\infty}J_n(x)\xi^n$

$J_n(x+y)=\sum_{-\infty}^{+\infty}J_k(x)J_{n-k}(y).$

微分性质：

☆ ① $(x^\nu J_\nu)'=x^\nu J_{\nu-1}=x^\nu J_\nu'+\nu x^{\nu-1}J_\nu$

② $\left(\frac{J_\nu}{x^\nu}\right)'=-\frac{J_{\nu+1}}{x^\nu}=\frac{J_\nu'}{x^\nu}-\frac{\nu J_\nu}{x^{\nu+1}}$

☆ $J_\nu'=\frac{1}{2}(J_{\nu-1}-J_{\nu+1})\quad \frac{2\nu}{x}J_\nu(x)=(J_{\nu-1}+J_{\nu+1})$

固有值为0当且仅当边界条件为第二类，固有函数为1.

模长：由题/试卷会给。

eg1. 设 $J_0(w_nx)$ 对应 $w_n: 0<w_1<w_2<\cdots<w_n$

分别将 1. x^2 分解为 $J_0(w_nx)$ 的级数。

$$\begin{cases}x^2y''+xy'+w_n x^2 y=0\\ |y(0)|<\infty,\ y(1)=0.\end{cases}$$

$N_0^2=\frac{1}{2}J_1^2(w_i).$

分子为：$\int_0^1 J_0(w_ix)\cdot x\,dx$

$=\int_0^{w_i}\frac{J_0(s)\cdot s\,ds}{w_i^2}$

$=\frac{1}{w_i^2}\int_0^{w_i}J_0(s)s\,ds$

$=\frac{1}{w_i^2}(J_1(s)s)\Big|_0^{w_i}$

$=\frac{J_1(w_i)}{w_i}$

$1=\sum_{n=1}^{+\infty}\frac{\langle J_n,1\rangle}{\langle J_n,J_n\rangle}=\sum_{n=1}^{+\infty}\frac{J_1(w_n)}{\frac{1}{2}J_1^2(w_n)\,w_n}J_0(w_nx)$

$=\sum_{n=1}^{+\infty}\frac{2J_0(w_nx)}{w_nJ_1(w_n)}$

$\langle x^2,J_0(w_nx)\rangle=\int_0^1 x^2\cdot xJ_0(w_nx)\,dx$

$=\frac{\int_0^{w_n}s^3J_0(s)\,ds}{w_n^4}$

$=\frac{(s^3J_1(s))\Big|_0^{w_n}-2\int_0^{w_n}s^2J_1(s)\,ds}{w_n^4}$

$=\frac{J_1(w_n)}{w_n}-2\frac{J_2(w_n)}{w_n^2}$

$=\frac{w_nJ_1(w_n)-2J_2(w_n)}{w_n^2}$

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$x^2 = \sum_{n=1}^{+\infty} \frac{\langle J_0, x^2\rangle}{\|J_n(w_n)\|^2} = \sum_{n=1}^{+\infty} \frac{2(w_n J_1(w_n) - 2J_2(w_n))}{w_n J_1 w_n^2}$

$w_n J_1 0 (w_n) = \frac{1}{2} J_2(w_n)$

$2J_2(w_n) = \frac{4 J_1(w_n)}{w_n}$

eg2. 金属圆柱，半径为1.

上下温度为 $1-r^2$，无热源，

侧面温度为0

$\begin{cases} \Delta_3 u = 0 \\ u|_{z=M} = 1-r^2 \quad u|_{r=1} = 0 \\ u|_{z=1} = 0 \end{cases}$

u与θ无关

$rR''Z + R'Z + rRZ'' = 0$

$\frac{rR''+R'}{rR} = \frac{-Z''}{Z} = -\mu$.

$\begin{cases} rR''+R'+(\mu r)R=0 \\ R(1)=0. \quad |R(0)|<\infty \end{cases}$

为 $J_0(w_n r)$　w_n 为 $J_0(w_n)=0$

$Z = C_1 e^{-w_n x} + C_2 e^{w_n x}$

$u = \sum_{n=1}^{+\infty} J_0(w_n x)(C_1 e^{w_n x} + C_2 e^{-w_n x})$

$u(r,z)$　$u = \sum_{n=1}^{+\infty} J_0(w_n x)(C_n \operatorname{sh} w_n x$

$u(r,0) = 0$.

展开计算即可.

$J_0(w_n x)$

eg3. $\begin{cases} u_t = \Delta_3 u \\ u|_{r=1} = 0. \quad u(0,r) = 1-r^2 \end{cases}$

$RT' = R''T + R'T$

$\frac{T'}{T} = \frac{rR''+R'}{rR} = -\mu$

$\begin{cases} rR''+rR'+\mu r^2 R = 0 \\ R(1)=0. \end{cases}$

3.4 勒让德方程的固有值.

在球坐标中分离变量时，会产生勒让德方程：$\begin{cases} (1-x^2)y'' - 2xy' + \lambda y = 0 \\ |y(\pm 1)| < \infty. \end{cases}$

$\lambda_n = n(n+1)$

$R_n = A_n e^{nt} + B_n e^{-(n+1)t}$,

$u(r,\theta) = \sum_{n=0}^{+\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos\theta)$.

确定系数 $\begin{cases} ①\text{实心球}\ B_n = 0 \\ ②\text{球壳}. \\ ③\text{球外}.\ A_n = 0 \end{cases}$

的系图：$\langle f\cos\theta, P_n\rangle$

(一般好像都是直接展开，不用算内积

eg: $\begin{cases} \Delta_3 u = 0 \quad r<a \\ u|_{r=a} = \cos^2\theta. \end{cases}$

$u = \sum_{n=1}^{+\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos\theta)$

$|r(0)| < \infty$

$B_n = 0$.

$P_0 = 1.\ P_1 = x.\ P_2 = \frac{1}{2}(3x^2-1)$

$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0$

$\therefore \frac{2}{3}\cancel{a} = A_2 a^2 \cdot \frac{1}{3} = A_0$

其余为0.

2. 半球问题.

$$\begin{cases} \Delta u = 0 \quad r<a,\ \theta\in[0,\frac{\pi}{2}]\ \varphi\in[0,2\pi] \\ u|_{r=a} = \cos^2\theta \cdots \cancel{u|_{\theta=\frac{\pi}{2}}=0\ (I)} \end{cases}$$

$\cancel{u|_{\theta=0} = 0\ (II)}$

(1) 底部绝热. 温度上下对称

(2) 底部恒温为0. 上下反对称

3.5 勒让德递推公式及其母函数.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{+\infty} P_n(x) t^n.$$

定理. 递推公式.

① $(n+1)P_{n+1}(x) - x(2n+1)P_n(x) + nP_{n-1}(x) = 0$

② $nP_n - xP_n' + P_{n-1}' = 0$

③ $nP_{n-1} - P_n' + xP_{n-1}' = 0$.

④ $P_{n+1}' - P_{n-1}' = (2n+1)P_n$.

$\int_0^1 P_n(x) x^m dx$.

① $m=0$. $\int_0^1 P_n(x) dx = \cancel{\ }$

$\frac{1}{2n+1}(P_{n-1} - P_{n+1}) = \frac{1}{2n+1}[P_{n-1}(0) - P_{n+1}(0)]$

$P_n(1)$ 都相等.

② $\cancel{\ }$ $n=0$, $\int_0^1 x^m \cancel{\ } = 0$.

③ $mn \neq 0$.

$I_{m,n} = I_{m-1,n-1} \cdot \frac{m}{m+n+1}$.

3.6 函数的傅立叶-勒让德展开.

$$C_n = \frac{\int f(x) P_n(x) dx}{\|P_n\|^2}$$

$$= \frac{2n+1}{2}\int f(x) P_n(x) dx$$

题目:

3.12. 设 $J_0(2w_n)=0$ 把函数

$$\begin{cases} 1 & (0<x<1) \\ \frac{1}{2} & x=1 \\ 0 & (1<x<2) \end{cases}$$

展开为 $J_0(w_n x)$ 的级数.

$$\begin{cases} x^2y'' + xy' + (w_n x)^2 y = 0 \\ |y(0)| < \infty \quad y(2) = 0. \end{cases}$$

$\|J_0(w_n)\|^2 = \cancel{\ } \frac{2^2}{2} \cdot J_1^2(w_n 2)$.

$\langle f, J_0(w_n)\rangle = \int_0^1 r J_0(w_n r) dr$

$$= \frac{\int_0^{w_n} s J_0(s) ds}{(w_n)^2}$$

$$= \frac{J_1(w_n)}{w_n}$$

$$C_n = \frac{J_1(w_n)}{2J_1^2(2w_n) w_n}$$

$$f = \sum_{n=1}^{+\infty} \frac{J_1(w_n)}{2w_n J_1^2(2w_n)} \cdot J_0(w_n x)$$

3.16. 半径为 R 的无限长圆柱体侧表面保持一定温度 u_0, 柱内初始温度为0. 求柱体温度分布.

$$\begin{cases} u_t = \Delta u. \\ u|_{t=0} = 0. \quad u|_{r=R} = u_0 \end{cases}$$

$T'R = (R'' + \frac{1}{r}R')T$

$\frac{T'}{T} = -\lambda = \frac{r^2R'' + rR'}{r^2 R}$

$T(t) = \cancel{C_1 \operatorname{sh} w_n t + C_2 \operatorname{ch} w_n t}$

$C_1 e^{w_n t} + C_2 e^{-w_n t}$

$C_2 = 0$

$R(r) = J_0(w_n r)$

$t \to \infty$ 有界

齐次化 $\begin{cases} u_t = a^2\Delta u \\ u_{t=0} = -u_0 \quad u|_{r=R} = 0 \end{cases}$

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$u(t,r) = \sum_{n=1}^{+\infty} C_n J_0(\omega_n r) e^{-\omega_n t}$

$u(0,r) = \sum_{n=1}^{+\infty} C_n J_0(\omega_n r) = -u_0$

$\langle f, J \rangle = \int_0^R r J_0(\omega_n r) dr$

$= \int_0^{\omega_n R} \frac{s J_0(s) ds}{\omega_n^2} = \frac{R J_1(\omega_n R)}{\omega_n}$

$\|N\| = \frac{R^2}{2} J_1^2(\omega_n R)$

$C_n = -u_0 \cdot \frac{2}{R\omega_n J_1(\omega_n R)}$

$\therefore u(t,r) = u_0 - 2u_0 \sum_{n=1}^{+\infty} \frac{2J_0(\omega_n r)}{R\omega_n J_1(\omega_n R)} e^{-\omega_n t}$

3.18. 解定解问题.

$\begin{cases} u_{rr} + \frac{1}{r}u_r + u_{zz} = 0 \\ u(a,z) = 0 \\ u(r,0) = 0 \quad u(r,l) = T_0 \end{cases}$

令 $u(r,z) = R(r)Z(z)$

$\begin{cases} r^2R'' + rR' + \lambda r^2 R = 0 \\ |R(0)| < \infty. \quad R(a) = 0. \end{cases}$

$R(r) = J_0(\omega_n r)$

$Z = \operatorname{sh}\omega_n z$

$u(r,z) = \sum_{n=1}^{+\infty} C_n J_0(\omega_n r) \operatorname{sh}\omega_n z$

$u(r,l) = \sum_{n=1}^{+\infty} C_n J_0(\omega_n r) \cdot \operatorname{sh}\omega_n l = T_0$

$C_n = \frac{\langle J, \frac{T_0}{\operatorname{sh}\omega_n l} \rangle}{\|J\|^2}$

$\langle J, f \rangle = \int_0^a J_0(\omega_n r) r dr$

$= \int_0^{\omega_n a} \frac{J_0(s) s ds}{\omega_n^2} = \frac{a J_1(\omega_n a)}{\omega_n}$

$\|N\| = \frac{a^2}{2} J_1^2(\omega_n a)$

$C_n = \frac{2aJ_1(\omega_n a)}{a^2 J_1^2(\omega_n a)} = \frac{2}{\omega_n a J_1(\omega_n a)} \frac{T_0}{\operatorname{sh}\omega_n l}$

$\therefore u(r,z) = \sum_{n=1}^{+\infty} \frac{T_0}{\operatorname{sh}\omega_n l} \frac{2}{\omega_n a J_1(\omega_n a)} J_0(\omega_n r) \operatorname{sh}\omega_n z$

第四章: 积分变换解题

4.1 傅立叶变换

$F(\lambda) = \int_{-\infty}^{+\infty} f(x) e^{i\lambda x} dx$

$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\lambda) e^{-i\lambda x} d\lambda.$

微分性质: $F[f^{(k)}(x)] = (-i\lambda)^k F(\lambda)$

卷积性质: $F[f * g] = F[f] * F[g]$

高维傅立叶变换与反变换:

$F(\lambda,\mu,\nu) = \iiint_{-\infty}^{+\infty} f(x,y,z) e^{i(\lambda x + \mu y + \nu z)} dx dy dz$

$f(x,y,z) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} F(\lambda,\mu,\nu) e^{i(\lambda x + \mu y + \nu z)} d\lambda d\mu d\nu$

eg1. 解初值问题.

$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \\ u(0,x) = \varphi(x) \end{cases}$

令 $\bar{u}(t,\lambda) = \int_{-\infty}^{+\infty} u(t,x) e^{i\lambda x} dx$

$\frac{\partial \bar{u}}{\partial t} = -a^2\lambda^2 \bar{u} \qquad \bar{u}(t,\lambda) = Ae^{-\lambda^2 a^2 t}$

$\bar{u}(0,\lambda) = \Phi(\lambda). \qquad \bar{u}(t,\lambda) = \Phi(\lambda) e^{-\lambda^2 a^2 t}.$

$u(t,x) = \varphi(x) * F^{-1}[e^{-\lambda^2 a^2 t}]$

$F^{-1}[e^{-\lambda^2 a^2 t}] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\lambda^2 a^2 t} e^{-i\lambda x} d\lambda$.

$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-a^2 t(\lambda - \frac{ix}{2a^2t})^2} \cdot e^{-\frac{x^2}{4a^2t}} d\lambda$

$= \frac{1}{2\pi\sqrt{a^2t}} \cdot \sqrt{\pi}\, e^{-\frac{x^2}{4a^2t}}$

$= \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}}$

$\therefore u(t,x) = \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4a^2t}} \cdot \varphi(\xi)\, d\xi$

eg2. 解定解问题.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = 0 \\ u(0,x) = \varphi(x),\ u_t(0,x) = 0 \end{cases}$$

$\bar{u}(t,\lambda) = \int \cdots$

$$\begin{cases} \frac{\partial^2 \bar{u}}{\partial t^2} + a^2\lambda^4 \bar{u} = 0 \\ \bar{u}(0,\lambda) = \bar{\varphi}(\lambda). \quad \bar{u}_t(0,\lambda) = 0 \end{cases}$$

$\bar{u}(t,\lambda) = C_1 \sin a\lambda^2 t + C_2 \cos\lambda^2 a t$.

$\bar{u}_t(0,\lambda) = 0.\quad C_1 = 0.$

$C_2 = \bar{\varphi}(\lambda)$

$F^{-1}[\cos\lambda^2 a t] + iF^{-1}[\sin\lambda^2 a t]$

$= F^{-1}[e^{i\lambda^2 a t}] =$

$= \frac{1}{2\pi} \exp(-\frac{ix^2}{4at}) \int_{-\infty}^{+\infty} \exp[ia t(\lambda - \frac{x}{2at})^2] d\lambda$

$= \frac{1}{2\pi} \exp(-\frac{ix^2}{4at}) \int_{-\infty}^{+\infty} \exp\, iay^2 dy$.

$= \frac{1}{2\pi} \exp(-\frac{ix^2}{4at}) \int_{-\infty}^{+\infty} \cos ay^2 + i \sin ay^2 dy$.

$\int_{-\infty}^{+\infty} \cos x^2 = \int_{-\infty}^{+\infty} \sin x^2 = \sqrt{\frac{\pi}{2}}$

$\therefore F^{-1}[\cos\lambda^2 a t]$

$= \frac{1}{2\sqrt{\pi a t}} (\cos\frac{x^2}{4at} + \sin\frac{x^2}{4at})$

对于半直线上的定解问题.
可使用正弦变换或余弦变换.

$\bar{f}_s(\lambda) = \int_0^{+\infty} f(x) \sin\lambda x\, dx$

$\bar{f}_c(\lambda) = \int_0^{+\infty} f(x) \cos\lambda x\, dx$

且有反演公式. $f(x) = \frac{2}{\pi}\int_0^{+\infty} \bar{f}_s(\lambda) \sin\lambda x\, d\lambda$

$= \frac{2}{\pi}\int_0^{+\infty} \bar{f}_c(\lambda) \cos\lambda x\, d\lambda$.

eg3. 用余弦变换解以下定解问题.

$$\begin{cases} u_t = a^2 u_{xx} \\ u(0,x) = 0,\ u_x(t,0) = Q. \end{cases}$$

$u(t,+\infty) = u_x(t,+\infty) = 0.$

~~$\bar{u}_t = -a^2\lambda^2\bar{u}$~~

~~$\bar{u}(0,\lambda) = 0$~~

注意微分性质.

① $(\bar{f}_s)' = -\lambda \bar{f}_c$ ② $(\bar{f}_c)' = \lambda \bar{f}_s - f(0)$

$\bar{u}_t = -\lambda^2 a^2 \bar{u} - a^2 Q$.

待解 $\bar{u} = \bar{v} - \frac{Q}{\lambda^2}$

$\bar{v}_t = -\lambda^2 a^2 \bar{v}$

$\bar{v}(t,\lambda) = e^{-\lambda^2 a^2 t} \cdot A$

$\bar{u}(0,\lambda) = A - \frac{Q}{\lambda^2} = 0.\ A = \frac{Q}{\lambda^2}$

$\bar{u} = \frac{Q}{\lambda^2}(e^{-\lambda^2 a^2 t} - 1)$

$= -a^2 Q \int_0^t e^{-\lambda^2 a^2 \tau} d\tau$

$F^{-1} e^{-\lambda^2 a^2 \tau} \longrightarrow \frac{1}{a\sqrt{\pi \tau}} e^{-\frac{x^2}{4a^2\tau}}$

$\therefore u(t,x) = \frac{-aQ}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{\tau}} e^{-\frac{x^2}{4a^2\tau}} d\tau$

eg 4. $\begin{cases} u_{tt}+a^2u_{xxxx}=0 \\ u(0,x)=\varphi(x),\ u_t(0,x)=0. \end{cases}$

解 $\frac{d^2\bar{u}}{dt^2}+\lambda^4a^2\bar{u}=0$

$\begin{cases} \bar{u}(0,\lambda)=\bar{\varphi}(\lambda) \quad \bar{u}_t=0 \end{cases}$

$\bar{u}=C_1\cos\lambda^2a^2t+C_2\sin\lambda^2at$

$\bar{u}=C\cos\lambda^2at$

$C=\bar{\varphi}(\lambda)$

$u=\varphi(x)*\frac{1}{2\sqrt{\pi at}}\left(\sin\frac{x^2}{4at}+\cos\frac{x^2}{4at}\right)$

4.2 用拉普拉斯变换解题.

$F(p)=\int_0^{+\infty}f(t)e^{-pt}dt.$

$L[f^{(n)}(t)]=p^nF(p)-p^{n-1}f(0)-\cdots-pf^{(n)}(0).$

eg1.

$\begin{cases} \frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}+f(t) \quad (0<x<\infty,\ t>0) \\ u|_{t=0}=0 \quad u_t|_{t=0}=0 \\ u|_{x=0}=0 \end{cases}$

$\bar{u}(p,x)=\int_0^{+\infty}u(t,x)e^{-pt}dt$

$p^2F(p)-pf(0)-pf'(0)=\cancel{pF(p)}$

$\Rightarrow p^2\bar{u}(p,x)=a^2\frac{\partial^2\bar{u}}{\partial x^2}+\bar{f}(p)$

$\begin{cases} \bar{u}(p,0)=0 \end{cases}$

特解 $v=\frac{\bar{f}(p)}{p^2}.$

$p^2\bar{w}(p,x)=a^2\frac{\partial^2\bar{w}}{\partial x^2}.$

$\bar{w}(p,x)=C_1e^{\frac{p}{a}x}+C_2e^{-\frac{p}{a}x}$

$\bar{w}(p,0)=0.\ \bar{w}(p,x)=0$

$L(p)$ 有界. $C_1(p)=0$

$\bar{u}(p,x)=C_2e^{-\frac{p}{a}x}+\frac{\bar{f}(p)}{p^2}$

$\bar{u}(p,0)=0.$

$\therefore \bar{u}(p,x)=\frac{\bar{f}(p)}{p^2}(-e^{-\frac{px}{a}}+1)$

$L[\frac{1}{p^2}]=t \quad L^{-1}[\bar{f}(p)]=f(t)$

$\therefore L^{-1}[\frac{\bar{f}(p)}{p^2}]=f*t.$

$L^{-1}[\frac{\bar{f}(p)}{p^2}e^{-\frac{p}{a}x}]$

$=h(t-\frac{x}{a})g(t-\frac{x}{a})$

eg2. 一根无限长杆, 端点温度变化已知, 杆的初始温度为0, 求温度变化规律.

$\begin{cases} \frac{\partial u}{\partial t}=a^2\frac{\partial^2u}{\partial x^2} \\ u(0,x)=0. \\ u(t,0)=f(t). \end{cases}$

解 $pF(p)-f(0)$

$\Rightarrow p\bar{u}=a^2\frac{\partial^2\bar{u}}{\partial x^2}.$

$\bar{u}(p,0)=\bar{f}(p)$

$\bar{u}(p,x)=C_1e^{\frac{\sqrt{p}}{a}x}+C_2e^{-\frac{\sqrt{p}}{a}x}$

$\bar{u}$ 有界. $C_1=0$

$\bar{u}(p,x) = C(p) e^{-\frac{p}{a}x}$.

$\bar{u}(p,0) = C(p) = \bar{f}(p)$

$\therefore \bar{u}(p,x) = \bar{f}(p) \cdot e^{-\frac{p}{a}x}$

$L^{-1}[e^{-\frac{p}{a}x}] = \frac{x}{2a\sqrt{\pi t^3}} e^{-\frac{x^2}{4a^2 t}}$

$f * u(t)$

eg3. $\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \\ u(t,0) = 0 \quad u_x(t,l) = A\sin wt \\ u(0,x) = 0. \quad u_t(0,x) = 0 \end{cases}$

$p^2 F(p) - p f(0) - f'(0)$.

$= p^2 \bar{u} = a^2 \frac{\partial^2 \bar{u}}{\partial x^2}$

$\bar{u}(p,x) = C_1 sh \frac{p}{a} x + C_2 ch \frac{p}{a} x$ $\cancel{C(p) e^{-\frac{p}{a}x}}$.

$\frac{\partial \bar{u}}{\partial x}\Big|_l = \frac{Aw}{p^2+w^2}$.

$\therefore C_2 = 0$

$C \frac{p}{a} \cdot ch \frac{p}{a} l = \frac{Aw}{p^2+w^2}$

$\therefore C = \frac{Aw}{(p^2+w^2)(\frac{p}{a} ch \frac{p}{a} l)}$

$\therefore \bar{u}(p,x) = \frac{Aaw}{(p^2+w^2)(p\, ch \frac{p}{a} l)} sh \frac{p}{a} x$

$u(t,x) = \sum_k Res[\bar{u}(p,x) p_k]$

$p_k = 0. \ \pm wi \quad ch \frac{pl}{a} = 0$

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第五章 基本解和解的积分表达式.

5.1 δ函数.

(1) $\delta(x) = \begin{cases} 0 & else \\ +\infty & x=0 \end{cases}$

(2) $\int_{-\infty}^{+\infty} \delta(x) dx = 1$

$\int_a^b \delta(x) dx = \begin{cases} 1 & 0 \in [a,b] \\ 0 & 0 \notin [a,b] \end{cases}$

$\delta(x) f(x) = f(0) \delta(x)$.

$\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0)$.

$\int_{-\infty}^{+\infty} \delta(x-\xi) f(\xi) d\xi = f(x)$

性质: (1) δ函数为偶函数

(2) δ函数导数

$\int_{-\infty}^{+\infty} \delta^{(n)}(x) f(x) dx = (-1)^n f^{(n)}(0)$.

(3) δ的傅立叶变换.

$F[\delta(x)] = 1$. $F^{-1}[1] = \delta(x)$

$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda x} d\lambda$.

$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos\lambda x d\lambda$

三维δ函数. $\delta(x,y,z)$

$= \delta(x)\delta(y)\delta(z)$.

eg1. 计算 $I = \frac{2a}{\pi} \int_0^{+\infty} d\lambda \int_0^t f(\tau) \sin\lambda x \sin a\lambda(t-\tau) d\tau$

$\frac{1}{\pi} \int_0^{+\infty} \sin\lambda x \sin a\lambda(t-\tau) d\lambda$.

$= \frac{1}{\pi} \int_0^{+\infty} \cos\lambda[x-a(t-\tau)] - \cos\lambda[x+a(t-\tau)] d\lambda$

$= \delta[x-a(t-\tau)] - \delta[x+a(t-\tau)]$

$I = a \int_0^t f(\tau)[\delta(x-a(t-\tau))] d\tau$

$= \int_{x-at}^{x} \delta(s) f(\frac{1}{a}(s-x+at)) ds$

$= f(t-\frac{x}{a})$

5.2 场势方程的边值问题.

5.2.1 $Lu=0$ 的方程基本解:

方程 $Lu=\delta(M)$ 称为 $Lu=f(M)$ 的基本解.

~~也称基~~ 定理: U为基本解.
则. $U*f$ 为 $Lu=f$ 的解.

三维场势方程基本解为

$u=-\frac{1}{4\pi r}$

5.2.2 格林函数及其物理意义☆

求解静电场边值问题

$$I_1:\begin{cases}\Delta u=-f(x,y,z)\\ u|_S=\varphi(x,y,z)\end{cases}$$

考虑基本解 I_2: $\begin{cases}\Delta G=-\delta(x-\xi,y-\eta,z-\zeta)\\ G|_S=0\end{cases}$

$G=u_1+u_2$. u_1为 $\frac{1}{4\pi r(M,M_0)}$

$$\begin{cases}\Delta u_2=0\\ u_2|_S=(G-u_1)|_S=-\frac{1}{4\pi r(M,M_0)}\Big|_{M\in S}\end{cases}$$

定理: $f(M)$, $\varphi(M)$ 连续. I_1 解为

$$u(M)=-\iint\varphi(M)\frac{\partial G}{\partial n}\,dS+\iiint G f(M_0)\,dM_0$$

5.2.3 镜像法求格林函数.

(1) 半空间格林函数.

$$\begin{cases}G=-\delta(x-\xi,y-\eta,\cancel{z}\,z-\zeta)\\ G|_{z=0}=0.\end{cases}$$

为满足边界条件. 不妨在 $(\xi,\eta,-\zeta)$ 处放置一负电荷 ε_0. $G=\frac{1}{4\pi}\left[\frac{1}{r_0}-\frac{1}{r_1}\right]$

即满足方程.

此时 $\frac{\partial G}{\partial n}=-\frac{\partial G}{\partial z}\Big|_{z=0}$

$=\frac{1}{4\pi}\left[\frac{z-\zeta}{r_0^3}-\frac{z+\zeta}{r_1^3}\right]\Big|_{z=0}$

$=-\frac{\zeta}{2\pi}\left[(x-\xi)^2+(y-\eta)^2+|\zeta|^2\right]^{-\frac{3}{2}}$

$$u(\xi,\eta,\zeta)=-\frac{1}{2\pi}\iint\frac{\varphi(x,y)\,\zeta}{[(x-\xi)^2+(y-\eta)^2+\zeta^2]^{\frac{3}{2}}}\,dx\,dy.$$

(2) 球域上的格林函数.

思路大致一样

$G(M)=\frac{1}{4\pi}\left[\frac{1}{r_0}-\frac{R}{\rho_0 r_1}\right]$

$\frac{\partial G}{\partial n}\Big|_S=\frac{\partial G}{\partial r}\Big|_{r=R}$

$=-\frac{R^2-\rho_0^2}{4\pi R(R^2+\rho_0^2-2R\rho_0\cos\psi)^{\frac{3}{2}}}$

5.2.4 二维情形.

$\Delta_2 u=\delta(x,y)$

$u=\frac{1}{2\pi}\ln r=-\frac{1}{2\pi}\ln\frac{1}{r}$

记 (x,y) 为 M. 点 (ξ,η) 为 M_0. D为某个平面区域. L为其边界. 则边值问题

$$\begin{cases}\Delta G=-\delta(M-M_0)\\ G|_L=0\end{cases}$$

$$u(\xi,\eta)=-\oint_L\varphi(M)\frac{\partial G(M,M_0)}{\partial n}\,dl+\iint_D G(M,M_0)f(M)\,dA$$

电像法对格林函数仍然适用.

如: $\begin{cases}\Delta G=-\delta(M-M_0)\\ G|_{y=0}=0\end{cases}$

$G(M, M_0) = \frac{1}{2\pi}\left(\ln\frac{1}{r_0} - \ln\frac{1}{r_1}\right)$

$M_0(\xi,\eta)$, $M_1(-\xi,\eta)$

对圆内格林函数

$G = \frac{1}{2\pi}\left[\ln\frac{1}{r_0} - \ln\frac{R}{\rho_0}\cdot\frac{1}{r_1}\right]$

5.3 $u_t = Lu$ 型方程柯西问题基本解

I_1: $\begin{cases} \frac{\partial u}{\partial t} = Lu + f(t,x,y,z) \\ u(0,x,y,z) = \varphi(x,y,z) \end{cases}$ 的基本解为 $\begin{cases} \frac{\partial U}{\partial t} = LU \\ U(0,x,y,z) = \delta(x,y,z) \end{cases}$

且原方程解为 ~~$U*\varphi + \int_0^t U(t-\tau,M)*f(\tau,M)d\tau$~~

$U*\varphi + \int_0^t U(t-\tau,M)*f(\tau,M)d\tau$

tip: 热方程 $\begin{cases} u_t = a^2\Delta u \\ u(0,x,y,z) = \delta(x,y,z) \end{cases}$

基本解为 $\left(\frac{1}{2a\sqrt{\pi t}}\right)^3 \exp\left(-\frac{x^2+y^2+z^2}{4a^2t}\right)$

特别地，一维的有：

$\frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2t}\right)$

5.4 $u_{tt} = Lu$ 方程柯西问题的基本解

$\begin{cases} \frac{\partial^2 U}{\partial t^2} = LU \\ U(0,M) = 0,\ U_t(0,M) = \delta(M) \end{cases}$

为 $\begin{cases} \frac{\partial^2 u}{\partial t^2} = Lu + f(t,M) \\ u(0,M) = \varphi(M),\ u_t(0,M) = \psi(M) \end{cases}$

的基本解

$u(t,M) = \frac{\partial}{\partial t}[U*\varphi(M)] + U(t,M)*\psi(M) + \int_0^t U(t-\tau,M)*f(\tau,M)d\tau$

三维波动方程基本解为 $\frac{1}{4\pi a r}\delta(r-at)$

一维为 $\frac{h}{2a}(a^2t^2-x^2)$

5.4.2 降维法

为求解二维波动方程的柯西问题

$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ u(0,x,y) = \varphi(x,y) \\ u_t(0,x,y) = \psi(x,y) \end{cases}$

我们可考虑

$\begin{cases} u_{tt} = a^2\Delta u & u(0,x,y,z) = \varphi(x,y) \\ u_t(0,x,y,z) = \psi(x,y) \end{cases}$

$$u(t,x,y) = \frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\iint \frac{\varphi(\xi,\eta)}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}d\xi d\eta + \iint \frac{\psi(\xi,\eta)}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}d\xi d\eta\right]$$

三维自由波动方程解为

$M_{at}(\varphi) = \frac{1}{4\pi(at)^2}\iint \varphi(\xi,\eta,\zeta)dS$

$u = \frac{\partial}{\partial t}[tM_{at}(\varphi)] + tM_{at}(\psi)$