量子力学 B
2021 秋季学期
作业 9 *(截止期: 12 月 8 号周三课上)*

1. 设$\left|\ell m\right\rangle$为力学量完全集$\{\hat{\vec{\ell}}^2, \hat{\ell}_z\}$的共同本征态，其中$\hat{\vec{\ell}}$, $\hat{\ell}_z$分别为轨道角动量及其在z方向的分量，ℓ, m为相应量子数。如果只考虑$\ell = 1$的子空间，请解答下述问题:

a. 请写出$\hat{\ell}_y^2$在此子空间内的矩阵表示，并注明矩阵表示所用的基。

b. 如果在$\left|\ell = 1, m = 1\right\rangle$态上测$\hat{\ell}_y^2$，可能的测值和相应的几率是多少?

c. 求$\left\langle \ell = 1, m = 1\left|\hat{\ell}_y^2\right|\ell = 1, m = 1\right\rangle$。

2. 证明:
a. $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$
b. $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$
c. 设$\sigma_0 = I$，则任意 2x2 矩阵 M 均可以写做$M = \sum_{i=0,x,y,z} M_i\,\sigma_i$，证明$M_i = \frac{1}{2}Tr(M\sigma_i)$。

3. 请把$e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_\alpha e^{-i\lambda\hat{\sigma}_z}$ ($\alpha = x, y, z$)表示成单位阵和 Pauli 矩阵的线性组合。

4. 假设某粒子自旋初态为 z 方向自旋的本征态$\left|\downarrow\right\rangle$。从 0 时刻到$t$时刻，体系的 Hamiltonian 为$\hat{H}_I = A\hbar\hat{\sigma}_x$；从$t$时刻到$t + \tau$时刻，体系的 Hamiltonian 为$\hat{H}_{II} = \frac{1}{2}\hbar\omega(\hat{\sigma}_z + 1)$；从$t + \tau$时刻到$2t + \tau$时刻，体系的 Hamiltonian 为$\hat{H}_I = A\hbar\hat{\sigma}_x$。试求在$2t + \tau$时刻测量粒子处在$\left|\downarrow\right\rangle$的几率。

1. 设$\left|\ell m\right\rangle$为力学量完全集$\{\hat{\vec{\ell}}^2, \hat{\ell}_z\}$的共同本征态，其中$\hat{\vec{\ell}}, \hat{\ell}_z$分别为轨道角动量及其在$z$方向的分量，$\ell, m$为相应量子数。如果只考虑$\ell = 1$的子空间，请解答下述问题：

a. 请写出$\hat{\ell}_y^2$在此子空间内的矩阵表示，并注明矩阵表示所用的基。

b. 如果在$\left|\ell = 1, m = 1\right\rangle$态上测$\hat{\ell}_y^2$，可能的测值和相应的几率是多少？

c. 求$\left\langle \ell = 1, m = 1\middle|\hat{\ell}_y^2\middle|\ell = 1, m = 1\right\rangle$。

解：以$\{\hat{\vec{\ell}}^2, \hat{\ell}_z\}$的共同本征态作为基

由上次作业结论，有

$\hat{l}_+$、$\hat{l}_-$对应的矩阵为 $\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\hbar$、$\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}\hbar$

由于 $\hat{l}_y = \frac{1}{2i}(\hat{l}_+ - \hat{l}_-)$

则 $\hat{l}_y$ 对应的矩阵为

$$\frac{\sqrt{2}}{2i}\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

则 $\hat{l}_y^2$ 对应的矩阵为

$$\frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

所用的基依次为 $|1,-1\rangle$
$|1,0\rangle$
$|1,-1\rangle$

b. 令 $\det \left| \begin{pmatrix} \lambda-1 & 0 & 1 \\ 0 & \lambda-2 & 0 \\ 1 & 0 & \lambda-1 \end{pmatrix} \right| = 0$

得到 $\lambda_1 = 0 \quad \lambda_2 = \lambda_2 = 2$.

相应的归一化本征态为 $|\alpha\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ $|\beta\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $|\gamma\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

则可能的测值为 $0, \hbar^2$.

测得为 0 的概率 $P(l_y^2=0) = \left|\frac{\sqrt{2}}{2}(-1,0,-1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{2}$

测得为 $\hbar^2$ 的概率 $P(l_y^2=\hbar^2) = 1 - P(l_y^2=0) = \frac{1}{2}$

C. $\langle l=1, m=1|\hat{l}_y^2|l=1, m=1\rangle$

$= (0,0,1)\frac{\hbar^2}{2}\begin{pmatrix}1 & 0 & -1\\0 & 2 & 0\\-1 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{\hbar^2}{2}$

2. 证明:

a. $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

b. $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$

c. 设$\sigma_0 = I$，则任意 2x2 矩阵 M 均可以写做$M = \sum_{i=0,x,y,z} M_i\,\sigma_i$，证明$M_i = \frac{1}{2}Tr(M\sigma_i)$。

证明:

a. $\{\sigma_x, \sigma_x\} = 2\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = 2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$\{\sigma_y, \sigma_y\} = 2\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = 2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$\{\sigma_z, \sigma_z\} = 2\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = 2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$\{\sigma_x, \sigma_y\} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

$\{\sigma_x, \sigma_z\} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

$\{\sigma_y, \sigma_z\} = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

b. $\sigma_x\sigma_x = \hat{I} = \sigma_y\sigma_y = \sigma_z\sigma_z$ 所以 $\sigma_j\sigma_j = \hat{I}$

$$\sigma_x\sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_z \qquad \sigma_y\sigma_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_x$$

$$\sigma_z\sigma_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y \qquad \sigma_y\sigma_x = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_z$$

$$\sigma_z\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\sigma_x \qquad \sigma_x\sigma_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y$$

所以 $\sigma_i\sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\sigma_k$

c. $$M = M_0\sigma_0 + M_1\sigma_x + M_2\sigma_y + M_3\sigma_z = \begin{pmatrix} M_0+M_3 & M_1-iM_2 \\ M_1+iM_2 & M_0-M_3 \end{pmatrix}$$

$$\frac{1}{2}\mathrm{Tr}(M\sigma_0) = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} M_0+M_3 & M_1-iM_2 \\ M_1+iM_2 & M_0-M_3 \end{pmatrix} = M_0$$

$$\frac{1}{2}\mathrm{Tr}(M\sigma_x) = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} M_1-iM_2 & M_0+M_3 \\ M_0-M_3 & M_1+iM_2 \end{pmatrix} = M_1$$

$$\frac{1}{2}\mathrm{Tr}(M\sigma_y) = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} M_2+iM_1 & -i(M_0+M_3) \\ i(M_0-M_3) & M_2-iM_1 \end{pmatrix} = M_2$$

$$\frac{1}{2}\mathrm{Tr}(M\sigma_z) = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} M_0+M_3 & -M_3+iM_2 \\ M_1+iM_2 & M_3-M_0 \end{pmatrix} = M_3$$

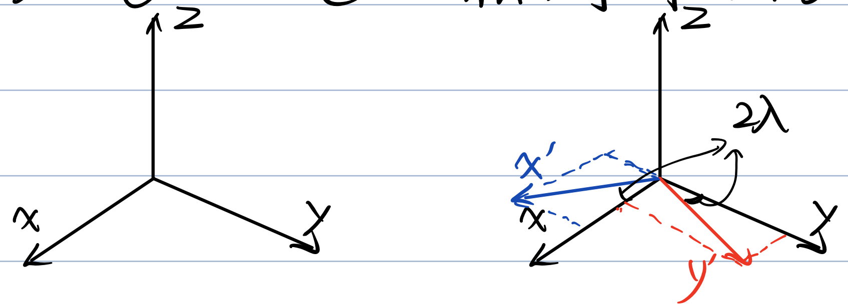
3. 请把 $e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_\alpha e^{-i\lambda\hat{\sigma}_z}$ ($\alpha = x, y, z$)表示成单位阵和 Pauli 矩阵的线性组合。

解: $$e^{-i\lambda\hat{\sigma}_z} = \cos\lambda\,\hat{I} - i\,\hat{\sigma}_z\sin\varphi.$$
$$e^{i\lambda\hat{\sigma}_z} = \cos\lambda\,\hat{I} + i\,\hat{\sigma}_z\sin\varphi$$

则 $$e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_x e^{-i\lambda\hat{\sigma}_z} = (\cos\lambda\,\hat{I} + i\hat{\sigma}_z\sin\lambda)\,\hat{\sigma}_x\,(\cos\lambda\,\hat{I} - i\hat{\sigma}_z\sin\lambda)$$
$$= \cos 2\lambda\,\hat{\sigma}_x - \sin 2\lambda\,\hat{\sigma}_y$$

$$e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_y e^{-i\lambda\hat{\sigma}_z} = (\cos\lambda\hat{I} + i\hat{\sigma}_z\sin\lambda)\hat{\sigma}_y(\cos\lambda\hat{I} - i\hat{\sigma}_z\sin\lambda)$$
$$= \cos 2\lambda\,\hat{\sigma}_y + \sin 2\lambda\,\hat{\sigma}_x$$
$$e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_z e^{-i\lambda\hat{\sigma}_z} = \hat{\sigma}_z$$

注：$e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_\alpha e^{-i\lambda\hat{\sigma}_z}$ 相当于将 $\hat{\sigma}_\alpha$ 绕 $\hat{\sigma}_z$ 顺时针转 2λ.



4. 假设某粒子自旋初态为 z 方向自旋的本征态 $|\downarrow\rangle$。从 0 时刻到 t 时刻，体系的 Hamiltonian 为 $\hat{H}_I = A\hbar\hat{\sigma}_x$；从 t 时刻到 $t+\tau$ 时刻，体系的 Hamiltonian 为 $\hat{H}_{II} = \frac{1}{2}\hbar\omega(\hat{\sigma}_z + 1)$；从 $t+\tau$ 时刻到 $2t+\tau$ 时刻，体系的 Hamiltonian 为 $\hat{H}_I = A\hbar\hat{\sigma}_x$。试求在 $2t+\tau$ 时刻测量粒子处在 $|\downarrow\rangle$ 的几率。

解：由课上的结论：
$$\begin{cases} |\uparrow\rangle_z = \frac{\sqrt{2}}{2}(|\uparrow\rangle_x + |\downarrow\rangle_x) \\ |\downarrow\rangle_z = \frac{\sqrt{2}}{2}(|\uparrow\rangle_x - |\downarrow\rangle_x) \end{cases}$$

则 $e^{\frac{-i\hat{H}_I t}{\hbar}}|\downarrow\rangle_z$

$$= e^{-iAt\hat{\sigma}_x}\frac{\sqrt{2}}{2}(|\uparrow\rangle_x - |\downarrow\rangle_x)$$
$$= \frac{\sqrt{2}}{2}(e^{-iAt}|\uparrow\rangle_x - e^{iAt}|\downarrow\rangle_x) = |\psi(t)\rangle$$

而
$$\begin{cases} |\uparrow\rangle_x = \frac{\sqrt{2}}{2}(|\uparrow\rangle_z + |\downarrow\rangle_z) \\ |\downarrow\rangle_x = \frac{\sqrt{2}}{2}(|\uparrow\rangle_z - |\downarrow\rangle_z) \end{cases}$$

则 $|\psi(t)\rangle = e^{-iAt}\frac{1}{2}(|\uparrow\rangle_z+|\downarrow\rangle_z) - e^{iAt}\frac{1}{2}(|\uparrow\rangle_z-|\downarrow\rangle_z)$

$= \frac{1}{2}(e^{-iAt}-e^{iAt})|\uparrow\rangle_z + \frac{1}{2}(e^{-iAt}+e^{iAt})|\downarrow\rangle_z$

则 $|\psi(t+\tau)\rangle = e^{\frac{-i\hat{H}_{II}\tau}{\hbar}}|\psi(t)\rangle$

$= e^{-i\frac{\omega}{2}(\hat{\sigma}_z+1)\tau}|\psi(t)\rangle$

$= \frac{1}{2}(e^{-iAt}-e^{iAt})e^{-i\omega\tau}|\uparrow\rangle_z + \frac{1}{2}(e^{-iAt}+e^{iAt})|\downarrow\rangle_z$

$= -i\sin At\, e^{-i\omega\tau}|\uparrow\rangle_z + \cos At\,|\downarrow\rangle_z$

$= -\frac{\sqrt{2}}{2}i\sin At\, e^{-i\omega\tau}(|\uparrow\rangle_x+|\downarrow\rangle_x) + \frac{\sqrt{2}}{2}\cos At(|\uparrow\rangle_x-|\downarrow\rangle_x)$

$= \frac{\sqrt{2}}{2}(\cos At - i\sin At\, e^{-i\omega\tau})|\uparrow\rangle_x - \frac{\sqrt{2}}{2}(i\sin At\, e^{-i\omega\tau} + \cos At)|\downarrow\rangle_x$

则 $|\psi(2t+\tau)\rangle = e^{\frac{-i\hat{H}_I t}{\hbar}}|\psi(t+\tau)\rangle$

$= e^{-iAt\hat{\sigma}_x}|\psi(t+\tau)\rangle$

$= \frac{\sqrt{2}}{2}(\cos At - i\sin At\, e^{-i\omega\tau})e^{-iAt}|\uparrow\rangle_x - \frac{\sqrt{2}}{2}(i\sin At\, e^{-i\omega\tau}+\cos At)e^{iAt}|\downarrow\rangle_x$

$= \frac{1}{2}(\cos At - i\sin At\, e^{-i\omega\tau})e^{-iAt}(|\uparrow\rangle_z+|\downarrow\rangle_z)$
$- \frac{1}{2}(i\sin At\, e^{-i\omega\tau}+\cos At)e^{iAt}(|\uparrow\rangle_z-|\downarrow\rangle_z)$

$|\downarrow\rangle_z$ 前的系数 $C_1 = \frac{1}{2}(\cos At - i\sin At\, e^{-i\omega\tau})e^{-iAt}$
$+ \frac{1}{2}(\cos At + i\sin At\, e^{-i\omega\tau})e^{iAt}$

$= \cos^2 At + i\sin At\, e^{-i\omega\tau}(i\sin At)$

$= \cos^2 At - \sin^2 At\, e^{-i\omega\tau}$

则几率为 $|C_1|^2 = |\cos^2 At - \sin^2 At\, e^{-i\omega\tau}|^2$

$= [\cos^2 At - \sin^2 At\cos(\omega\tau)]^2 +$

$\sin^2 At \sin^2 \omega\tau$

$= \cos^4 At - 2\sin^2 At \cos^2 At \cos^2 \omega\tau + \sin^4 At$

$= 1 - 2\cos^2 At \sin^2 At - 2\sin^2 At \cos^2 At \cos^2 \omega\tau$

$= 1 - 2\cos^2 At \sin^2 At (1 + \cos^2 \omega\tau)$

量子力学第九次作业答案

By 鸽子

1.

1. 解: a. 只考虑 $l=1$ 的子空间. m 可能的取值为 ± 1 和 0.

记 $|1\,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\quad |1\,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad |1\,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$

由 $\begin{cases}\hat{L}_+ = \hat{L}_x + i\hat{L}_y \\ \hat{L}_- = \hat{L}_x - i\hat{L}_y\end{cases}$ $\quad \hat{L}_\pm |l,m\rangle = \sqrt{(l\mp m)(l\pm m+1)}\,\hbar\,|l,m\pm 1\rangle$.

可得 $\hat{L}_y$ 的矩阵表示为 $\dfrac{\hbar}{\sqrt{2}\,i}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}$

$\therefore \hat{L}_y^2$ 的矩阵表示为 $\dfrac{\hbar}{\sqrt{2}(-i)}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}\dfrac{\hbar}{\sqrt{2}\,i}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}$

$$= \frac{\hbar^2}{2}\begin{array}{c}\begin{array}{ccc}|1,1\rangle & |1,0\rangle & |1,-1\rangle\end{array}\\ \begin{pmatrix}1&0&-1\\0&2&0\\-1&0&1\end{pmatrix}\end{array}$$

b. 设 $\hat{L}_y^2$ 的本征值为 λ

令 $$|\hat{L}_y^2-\lambda I| = \begin{vmatrix}\frac{\hbar^2}{2}-\lambda & 0 & -\frac{\hbar^2}{2}\\ 0 & \hbar^2-\lambda & 0\\ -\frac{\hbar^2}{2} & 0 & \frac{\hbar^2}{2}-\lambda\end{vmatrix} = \left(\frac{\hbar^2}{2}-\lambda\right)^2(\hbar^2-\lambda)-\left(\frac{\hbar^2}{2}\right)^2(\hbar^2-\lambda)=0$$

$\therefore \lambda = \hbar^2$ 或 0

① $\lambda=\hbar^2$ $\begin{pmatrix} \frac{\hbar^2}{2} & 0 & -\frac{\hbar^2}{2} \\ 0 & \hbar^2 & 0 \\ -\frac{\hbar^2}{2} & 0 & \frac{\hbar^2}{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar^2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ -a \end{pmatrix}$

本征值简并，在该子空间内取两个互相正交的基.

$|A_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}$ $|A_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ -1 \end{pmatrix}$

② $\lambda=0$ $\begin{pmatrix} \frac{\hbar^2}{2} & 0 & -\frac{\hbar^2}{2} \\ 0 & \hbar^2 & 0 \\ -\frac{\hbar^2}{2} & 0 & \frac{\hbar^2}{2} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$ $\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = |A_3\rangle$

$|l=1, m=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{(|A_1\rangle + |A_2\rangle) + \sqrt{2}|A_3\rangle}{2}$

$\therefore$ 在该态上测量 $\hat{L}_y^2$，可能的测值为 $\hbar^2$ 或 ~~±~~ 0

其中测值为 $\hbar^2$ 的几率为 $P_1 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

测值为 $-\hbar^2$ 的几率为 $P_2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$

c. $\langle l=1, m=1 | \hat{L}_y^2 | l=1, m=1 \rangle = \frac{\hbar^2}{2}$

2.

2. 证明:

a. $\{\sigma_x,\sigma_x\} = 2\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = 2\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\{\sigma_y,\sigma_y\} = 2\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = 2\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\{\sigma_z,\sigma_z\} = 2\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = 2\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\{\sigma_x,\sigma_y\} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\{\sigma_x,\sigma_z\} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\{\sigma_y,\sigma_z\} = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

b. $\sigma_x\sigma_x = \sigma_y\sigma_y = \sigma_z\sigma_z = I$，即 $\sigma_i\sigma_i = I$

$\begin{cases}\sigma_x\sigma_y = \begin{pmatrix}i&0\\0&-i\end{pmatrix} = i\sigma_z \\ \sigma_y\sigma_x = \begin{pmatrix}-i&0\\0&i\end{pmatrix} = -i\sigma_z\end{cases}$ $\begin{cases}\sigma_y\sigma_z = \begin{pmatrix}0&i\\i&0\end{pmatrix} = -i\sigma_x \\ \sigma_z\sigma_y = \begin{pmatrix}0&-i\\-i&0\end{pmatrix} = -i\sigma_x\end{cases}$

$\begin{cases}\sigma_x\sigma_z = \begin{pmatrix}0&-1\\1&0\end{pmatrix} = -i\sigma_y \\ \sigma_z\sigma_x = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = i\sigma_y\end{cases}$

$\Rightarrow \sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$

c. $M = M_0\sigma_0 + M_1\sigma_x + M_2\sigma_y + M_3\sigma_z = \begin{pmatrix}M_0+M_3 & M_1-iM_2\\ M_1+iM_2 & M_0-M_3\end{pmatrix}$

$\frac{1}{2}Tr(M\sigma_0) = \frac{1}{2}Tr\begin{pmatrix}M_0+M_3 & M_1-iM_2\\ M_1+iM_2 & M_0-M_3\end{pmatrix} = M_0$

$\frac{1}{2}Tr(M\sigma_x) = \frac{1}{2}Tr\begin{pmatrix}M_1-iM_2 & M_0+M_3\\ M_0-M_3 & M_1+iM_2\end{pmatrix} = M_1$

$\frac{1}{2}Tr(M\sigma_y) = \frac{1}{2}Tr\begin{pmatrix}M_2+iM_1 & -i(M_0+M_3)\\ i(M_0-M_3) & M_2-iM_1\end{pmatrix} = M_2$

$\frac{1}{2}Tr(M\sigma_z) = \frac{1}{2}Tr\begin{pmatrix}M_0+M_3 & -M_3+iM_2\\ M_1+iM_2 & M_3-M_0\end{pmatrix} = M_3$

3.

3. 解: $e^{-i\lambda\hat{\sigma}_z} = \cos\lambda - i\hat{\sigma}_z \sin\varphi$

$e^{i\lambda\hat{\sigma}_z} = \cos\lambda + i\hat{\sigma}_z \sin\varphi$

$e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_x e^{-i\lambda\hat{\sigma}_z} = (\cos\lambda + i\hat{\sigma}_z\sin\lambda)\hat{\sigma}_z(\cos\lambda - i\hat{\sigma}_z\sin\lambda)$

$= \cos 2\lambda\,\hat{\sigma}_x - \sin 2\lambda\,\hat{\sigma}_y$

$e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_y e^{-i\lambda\hat{\sigma}_z} = (\cos\lambda + i\hat{\sigma}_z\sin\lambda)\hat{\sigma}_y(\cos\lambda\hat{I} - i\sigma_z\sin\lambda)$

$= \cos 2\lambda\,\hat{\sigma}_y + \sin 2\lambda\,\hat{\sigma}_x$

$e^{i\lambda\hat{\sigma}_z}\hat{\sigma}_z e^{-i\lambda\hat{\sigma}_z} = \hat{\sigma}_z$.

4.

解: 时间演化算符 $\hat{u} = e^{-i\frac{\hat{H}}{\hbar}t}$

$$\hat{u}_I = e^{i\frac{\hat{H}}{\hbar}t} = e^{-iA\hat{\sigma}_x t} = \begin{pmatrix} \frac{e^{-iAt}+e^{iAt}}{2} & \frac{e^{-iAt}-e^{iAt}}{2} \\ \frac{e^{-iAt}-e^{iAt}}{2} & \frac{e^{-iAt}+e^{iAt}}{2} \end{pmatrix}$$

注: 此处利用 $e^{\hat{A}} = \sum_n e^{A_n}|\psi_n\rangle\langle\psi_n|$

$$\hat{u}_{II} = e^{-i\frac{\hat{\sigma}_z+\hat{I}}{2}\omega\tau} = e^{-i\begin{pmatrix}1&0\\0&0\end{pmatrix}\omega\tau} = \begin{pmatrix} e^{-i\omega\tau} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\omega\tau} & 0 \\ 0 & 1 \end{pmatrix}$$

$\hat{u}_{III} = e^{-i\frac{\hat{H}_I}{\hbar}t} = \hat{u}_I$ 记 $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$\therefore$ 终态 $|\psi(2t+\tau)\rangle = \hat{u}_{III}\hat{u}_{II}\hat{u}_I|\psi(0)\rangle$

$$= \begin{pmatrix} \frac{e^{-iAt}+e^{iAt}}{2} & \frac{e^{-iAt}-e^{iAt}}{2} \\ \frac{e^{-iAt}-e^{iAt}}{2} & \frac{e^{-iAt}+e^{iAt}}{2} \end{pmatrix}\begin{pmatrix} e^{-i\omega\tau} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{e^{-iAt}+e^{iAt}}{2} & \frac{e^{-iAt}-e^{iAt}}{2} \\ \frac{e^{-iAt}-e^{iAt}}{2} & \frac{e^{-iAt}+e^{iAt}}{2} \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix} \cos At & -i\sin At \\ -i\sin At & \cos At \end{pmatrix}\begin{pmatrix} -i\sin At\cdot e^{-i\omega\tau} \\ \cos At \end{pmatrix}$$

$$= \begin{pmatrix} -i\sin At\cos At\,(1+e^{-i\omega\tau}) \\ \cos^2 At - \sin^2 At\, e^{-i\omega\tau} \end{pmatrix}$$

$= -i\sin At\cos At\,(1+e^{-i\omega\tau})|\uparrow\rangle + (\cos^2 At - \sin^2 At\, e^{-i\omega\tau})|\downarrow\rangle$

∴ $2t+\tau$ 时刻测量粒子处在 $|\downarrow\rangle$ 的几率为

$$P = \left|\cos^2 At - \sin^2 At\, e^{-i\omega\tau}\right|^2 = \cos^4 At - 2\cos^2 At \sin^2 At \cos\omega\tau + \sin^4 At$$