P69 弦截法收敛阶证明

证明. 根据以下弦截法迭代格式：

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}, \qquad k = 1, 2, \dots$$

可以计算第 $k+1$ 步误差为：

$$\begin{aligned}
e_{k+1} &= x_{k+1} - x^* \\
&= x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} - x^* \\
&= \frac{x_k f(x_k) - x_k f(x_{k-1}) - f(x_k)(x_k - x_{k-1}) - x^* f(x_k) + x^* f(x_{k-1})}{f(x_k) - f(x_{k-1})} \\
&= \frac{f(x_k)(x_{k-1} - x^*) - f(x_{k-1})(x_k - x^*)}{f(x_k) - f(x_{k-1})} \\
&= \frac{f(x_k)e_{k-1} - f(x_{k-1})e_k}{f(x_k) - f(x_{k-1})} \\
&= \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \frac{\frac{f(x_k)}{e_k} - \frac{f(x_{k-1})}{e_{k-1}}}{x_k - x_{k-1}} e_k e_{k-1}
\end{aligned} \tag{1}$$

将 $f(x_k) = f(x^* + e_k)$ 泰勒展开可得：

$$f(x_k) = f(x^* + e_k) \approx f(x^*) + e_k f'(x^*) + \frac{1}{2!} e_k^2 f''(x^*) + o(e_k^3)$$

由于 x^* 是方程 $f(x) = 0$ 的解，那么有：

$$f(x^*) = 0 \Longrightarrow \frac{f(x_k)}{e_k} = f'(x^*) + \frac{1}{2!} e_k f''(x^*) + o(e_k^2) \tag{2}$$

类似的将 $f(x_{k-1}) = f(x^* + e_{k-1})$ 泰勒展开可得：

$$\frac{f(x_{k-1})}{e_{k-1}} = f'(x^*) + \frac{1}{2!} e_{k-1} f''(x^*) + o(e_{k-1}^2) \tag{3}$$

将 (2) 和 (3) 做差可得：

$$\frac{f(x_k)}{e_k} - \frac{f(x_{k-1})}{e_{k-1}} = \frac{1}{2!}(e_k - e_{k-1}) f''(x^*) + o(e_{k-1}^2)$$

由于 $x_k - x_{k-1} = e_k - e_{k-1}$，那么有：

$$\frac{\frac{f(x_k)}{e_k} - \frac{f(x_{k-1})}{e_{k-1}}}{x_k - x_{k-1}} \approx \frac{1}{2!} f''(x^*)$$

根据一阶差分：

$$\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \approx \frac{1}{f'(x^*)}$$

最终带入回式 (1) 可以得到：

$$e_{k+1} \approx \frac{1}{2} \frac{f''(x^*)}{f'(x^*)} e_k e_{k-1} = c e_k e_{k-1} \tag{4}$$

如果先假设弦截法的收敛阶为 α，那么由收敛阶的定义，存在 $A>0$，

$$\lim_{k\to\infty}\frac{|e_{k+1}|}{|e_k|^{\alpha}}=A$$

那么有：

$$\begin{aligned}
&|e_{k+1}|\sim A|e_k|^{\alpha}\\
&|e_k|\sim A|e_{k-1}|^{\alpha}\\
&|e_{k-1}|\sim (A^{-1}|e_k|)^{\frac{1}{\alpha}}
\end{aligned}$$

带入到式 (4) 可得：

$$A|e_k|^{\alpha}\sim|c||e_k|A^{-\frac{1}{\alpha}}|e_k|^{\frac{1}{\alpha}}\Longrightarrow A^{1+\frac{1}{\alpha}}|c|^{-1}\sim|e_k|^{1+\frac{1}{\alpha}-\alpha}$$

左边为常数，那么右边也为常数，因此可得:

$$1+\frac{1}{\alpha}-\alpha=0\Longrightarrow\alpha=\frac{1+\sqrt{5}}{2}=1.618$$

□