1. 假设力学量算符 $\hat{A}$ 的本征值集合为$\{A_n\}$，其中某本征值 A_m 对应的本征态有 s 重简并。请分别写出 A_m 对应的简并子空间中任意量子态及与此简并子空间正交的量子态的一般形式。

解：首先我们可以通过施密特正交化手段得到 A_m 对应的简并子空间的 S 个正交归一的基矢 $\{|A_{m\alpha}\rangle\}$，$\alpha = 1, 2, \dots S$.

则简并子空间中任意量子态 $|\psi\rangle$ 可以表示为：

$$|\psi\rangle = \sum_\alpha C_\alpha |A_{m\alpha}\rangle, \quad \alpha = 1, 2, \dots S.$$

设 $\hat{A}$ 对应的本征值为 $A_1, A_2 \dots A_m \dots A_n$，对应的本征态为 $|A_{11}\rangle, |A_{12}\rangle \dots |A_{1l}\rangle, |A_{21}\rangle, |A_{22}\rangle \dots |A_{2k}\rangle \dots |A_{nj}\rangle \dots$

则与 A_m 对应的简并子空间中任意的量子态可以表示为

$$|\varphi\rangle = \sum_{ij} C_{ij} |A_{ij}\rangle.$$

其中 $i = 1, 2, \dots m-1, m+1, \dots n$.

$j = 1, 2, \dots g$.

g 为 A_i 对应简并子空间的维数.

2. 利用产生/湮灭算符和坐标/动量算符的关系，证明一维简谐振子的哈密顿量

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

可以表示为

$$\hat{H} = (\hat{a}^\dagger \hat{a} + \frac{1}{2})\hbar\omega$$

$$\begin{cases} \hat{a}^+ = \dfrac{1}{\sqrt{2\hbar m\omega}} (m\omega\hat{x} - i\hat{p}) \\ \hat{a} = \dfrac{1}{\sqrt{2\hbar m\omega}} (m\omega\hat{x} + i\hat{p}) \end{cases}$$

记住.

证明：
$$\begin{cases} \hat{x} = \sqrt{\hbar/2m\omega}\, (\hat{a} + \hat{a}^+) \\ \hat{p} = -i\sqrt{m\omega\hbar/2}\, (\hat{a} - \hat{a}^+) \end{cases}$$

$$\hat{x}^2 = \hbar/2m\omega\, [\hat{a}^2 + (\hat{a}^+)^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}]$$

$$\hat{p}^2 = -m\omega\hbar/2\, [\hat{a}^2 + (\hat{a}^+)^2 - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a}]$$

代入 $\hat{H}$ 的表示式中

$$\hat{H}=-\frac{\omega\hbar}{4}[\hat{a}^2+(\hat{a}^+)^2-\hat{a}\hat{a}^+-\hat{a}^+\hat{a}]+\frac{\omega\hbar}{4}[\hat{a}^2+(\hat{a}^+)^2+\hat{a}\hat{a}^++\hat{a}^+\hat{a}]$$
$$=\frac{\omega\hbar}{2}[\hat{a}\hat{a}^++\hat{a}^+\hat{a}]$$

考察 $[\hat{a},\hat{a}^+]$

$$=\hat{a}\hat{a}^+-\hat{a}^+\hat{a}$$
$$=\frac{1}{2}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x}+\frac{i}{\sqrt{m\omega\hbar}}\hat{p}\right)\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x}-\frac{i}{\sqrt{m\omega\hbar}}\hat{p}\right)$$
$$-\frac{1}{2}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x}-\frac{i}{\sqrt{m\omega\hbar}}\hat{p}\right)\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x}+\frac{i}{\sqrt{m\omega\hbar}}\hat{p}\right)$$
$$=-\frac{i}{\hbar}\hat{x}\hat{p}+\frac{i}{\hbar}\hat{p}\hat{x}$$
$$=-\frac{i}{\hbar}[\hat{x},\hat{p}]=1$$

即 $[\hat{a},\hat{a}^+]=\hat{a}\hat{a}^+-\hat{a}^+\hat{a}=1$

代入 $\hat{H}$ 的表达示

$$\hat{H}=\frac{\hbar\omega}{2}[\hat{a}\hat{a}^++\hat{a}^+\hat{a}]$$
$$=\frac{\hbar\omega}{2}[2\hat{a}^+\hat{a}+1]=\left(\hat{a}^+\hat{a}+\frac{1}{2}\right)\hbar\omega$$

3. 请利用产生/湮灭算符的对易关系,证明 Fock 态的正交归一性，即

$$\langle n|m\rangle=\delta_{nm},\quad \text{where}\quad |n\rangle=\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$[\hat{a},\hat{a}^+]=1$

$\hat{N}=\hat{a}^+\hat{a}$

$[\hat{x},\hat{p}]=i\hbar$

证明: $\langle n|m\rangle=\frac{1}{\sqrt{m!n!}}\langle 0|\hat{a}^n(\hat{a}^+)^m|0\rangle$

① $n=m$. $\langle n|n\rangle=\frac{1}{n!}\langle 0|\hat{a}^n\hat{a}^{+n}|0\rangle$ $\quad |n\rangle=\frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$

考察 $\hat{a}^n(\hat{a}^+)^n = \hat{a}^{n-1}(\hat{a}\hat{a}^+)(\hat{a}^+)^{n-1} = \hat{a}^{n-1}(\hat{N}+1)(\hat{a}^+)^{n-1}$

而 $[\hat{N}+1, \hat{a}^+] = [\hat{a}^+\hat{a}+1, \hat{a}^+] = [\hat{a}^+\hat{a}, \hat{a}^+] = \hat{a}^+$

$\Rightarrow (\hat{N}+1)\hat{a}^+ - \hat{a}^+(\hat{N}+1) = \hat{a}^+$

$\Rightarrow (\hat{N}+1)\hat{a}^+ = \hat{a}^+(\hat{N}+2)$

于是 $\hat{a}^{n-1}(\hat{N}+1)(\hat{a}^+)^{n-1} = \hat{a}^{n-1}\hat{a}^+(\hat{N}+2)(\hat{a}^+)^{n-2}$

$\vdots$

$= \hat{a}^{n-1}(\hat{a}^+)^{n-1}(\hat{N}+n)$

$= \hat{a}^{n-2}(\hat{N}+1)(\hat{a}^+)^{n-2}(\hat{N}+n)$

$\vdots$

$= (\hat{N}+1)(\hat{N}+2)\dots(\hat{N}+n)$

则 $\langle n|n\rangle = \frac{1}{n!}\langle 0|(\hat{N}+1)(\hat{N}+2)\dots(\hat{N}+n)|0\rangle$

$= 1$

② $n \neq m$ 时，不妨设 $n > m$.

$\langle n|m\rangle = \frac{1}{\sqrt{n!m!}}\langle 0|\hat{a}^n(\hat{a}^+)^m|0\rangle$

与①同样. $\hat{a}^n(\hat{a}^+)^m = \hat{a}^{n-m}\hat{a}^m(\hat{a}^+)^m = \hat{a}^{n-m}(\hat{N}+1)\cdots(\hat{N}+m)$

则 $\langle 0|\hat{a}^{n-m}(\hat{N}+1)\dots(\hat{N}+m)|0\rangle$

$= (\langle 0|\hat{a})\cdot\hat{a}^{n-m-1}(\hat{N}+1)\dots(\hat{N}+m)|0\rangle$

$= 0$

由① ②可得 $\langle n|m\rangle = \delta_{nm}$.

4. 请验证 Fock 态下的不确定关系

$a.\ \langle n|\hat{x}|n\rangle = 0,\ \langle n|\hat{p}|n\rangle = 0$

$b.\ \langle n|\hat{x}^2|n\rangle = \frac{\hbar}{m\omega}(n+\frac{1}{2}),\ \langle n|\hat{p}^2|n\rangle = m\hbar\omega(n+\frac{1}{2})$

$c.\ \Delta x\Delta p = (n+\frac{1}{2})\hbar$

证明:

a. $\langle n|\hat{x}|n\rangle$

$= \sqrt{\hbar/2m\omega}\,\langle n|\hat{a}+\hat{a}^+|n\rangle$

$= \sqrt{\hbar/2m\omega}\,(\langle n|\hat{a}|n\rangle + \langle \hat{n}|\hat{a}^+|n\rangle)$

$= \sqrt{\hbar/2m\omega}\,(\sqrt{n}\langle n|n-1\rangle + \sqrt{n+1}\,\langle n|n+1\rangle) = 0$

$\langle n|\hat{p}|n\rangle$

$= -i\sqrt{m\omega\hbar/2}\,\langle n|\hat{a}-\hat{a}^+|n\rangle$

$= -i\sqrt{m\omega\hbar/2}\,(\sqrt{n}\langle n|n-1\rangle - \sqrt{n+1}\,\langle n|n+1\rangle)$

$=0$

b. $\hat{x}^2 = \hbar/2m\omega\,[\hat{a}^2 + (\hat{a}^+)^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a}]$

$\hat{p}^2 = -m\omega\hbar/2\,[\hat{a}^2 + (\hat{a}^+)^2 - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a}]$

$\langle n|\hat{x}^2|n\rangle = \hbar/2m\omega\,(\langle n|\hat{a}^2+(\hat{a}^+)^2|n\rangle + \langle n|\hat{a}\hat{a}^+|n\rangle + \langle n|\hat{a}^+\hat{a}|n\rangle)$

$= \hbar/2m\omega\,\langle n|2\hat{a}^+\hat{a}+1|n\rangle$

$= \hbar/2m\omega\,\langle n|2\hat{N}+1|n\rangle$

$= \hbar/2m\omega\,(2n+1)$

$= \frac{\hbar}{m\omega}(n+\frac{1}{2})$

$\langle n|\hat{p}^2|n\rangle = \frac{m\omega\hbar}{2}\langle n|\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}|n\rangle$

$= \frac{m\omega\hbar}{2}\langle n|2\hat{N}+1|n\rangle$

$= m\omega\hbar\,(n+\frac{1}{2})$

c. $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{m\omega}(n+\frac{1}{2})}$

$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{m\omega\hbar(n+\frac{1}{2})}$

$\Delta x \Delta p = \hbar(n+\frac{1}{2})$

5. 证明如下关系 (提示：利用之前作业的结论)

a. $e^{\alpha\hat{a}^\dagger}\hat{a}e^{-\alpha\hat{a}^\dagger} = \hat{a} - \alpha$

b. $e^{\lambda\hat{a}}|0\rangle = |0\rangle$

$f(\alpha) = e^{\alpha\hat{a}^+}\hat{a}e^{-\alpha\hat{a}^+}$

$f'(\alpha) = -1 \quad f(0) = \hat{a} \Rightarrow f(\alpha) = \hat{a} - \alpha$

证明:

a. $[\hat{a}, e^{-\alpha\hat{a}^+}] = \sum_n [\hat{a}, \frac{1}{n!}(-\alpha\hat{a}^+)^n]$

$= \sum_n \frac{(-\alpha)^n}{n!}[\hat{a}, (\hat{a}^+)^n]$

$= -\alpha[\hat{a}, \hat{a}^+]\sum_n \frac{(-\alpha)^n}{n!}(\hat{a}^+)^n$

$= -\alpha e^{-\alpha\hat{a}^+}$

$= \hat{a}e^{-\alpha\hat{a}^+} - e^{-\alpha\hat{a}^+}\hat{a}$

$\Rightarrow \hat{a}e^{-\alpha\hat{a}^+} = e^{-\alpha\hat{a}^+}\hat{a} - \alpha e^{-\alpha\hat{a}^+}$

于是 $e^{\alpha\hat{a}^+}\hat{a}e^{-\alpha\hat{a}^+}$

$= e^{\alpha\hat{a}^+}e^{-\alpha\hat{a}^+}(\hat{a}-\alpha)$

而 $e^{\alpha(\hat{a}^+-\hat{a}^+)} = e^{\alpha\hat{a}^+}e^{-\alpha\hat{a}^+}e^{-\frac{1}{2}\alpha^2[\hat{a}^+,\hat{a}^+]} = e^{\alpha\hat{a}^+}e^{-\alpha\hat{a}^+}$

代入上式可得

$e^{\alpha\hat{a}^+}\hat{a}e^{-\alpha\hat{a}^+} = e^{\alpha\hat{a}^+}e^{-\alpha\hat{a}^+}(\hat{a}-\alpha) = e^{\alpha(\hat{a}^+-\hat{a}^+)}(\hat{a}-\alpha) = \hat{a}-\alpha$

b. $e^{\lambda\hat{a}}|0\rangle$

$$= \sum_n \frac{1}{n!}(\lambda\hat{a})^n|0\rangle$$

$$= \left[1+\lambda\hat{a}+\frac{1}{2!}(\lambda a)^2+\dots\right]|0\rangle$$

$$= |0\rangle. \quad (\hat{a}|0\rangle = 0)$$

$$[\hat{a}, f(\hat{a},\hat{a}^+)] = \frac{\partial f}{\partial \hat{a}^+}$$

$$[\hat{a}^+, f(\hat{a},\hat{a}^+)] = -\partial f/\partial \hat{a}$$

$$e^{x\hat{a}}\hat{a}^+ e^{-x\hat{a}} = \hat{a}^+ + x$$ (x 与 $\hat{a}, \hat{a}^+$ 对易)

$$e^{-x\hat{a}^+}\hat{a}\, e^{x\hat{a}^+} = \hat{a} + x$$

量子力学第三次作业答案

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2021.10.21

1 题目

1.1 第一题

1.2 第二题

利用产生/湮灭算符和坐标/动量算符的关系, 证明一维简谐振子的哈密顿量

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}mw^2\,\hat{x}^2$$

可以表示为

$$\hat{H}=\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\hbar w$$

1.3 第三题

请利用产生/湮灭算符的对易关系,证明 Fock 态的正交归一性，即

$$\langle n|m\rangle=\delta_{nm},\mathrm{where}|n\rangle=\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

1.4 第四题

请验证 Fock 态下的不确定关系:

1.$\langle n|\hat{x}|n\rangle=0,\langle n|\hat{p}|n\rangle=0$

2.$\langle n|\hat{x}^2|n\rangle=\frac{\hbar}{mw}\left(n+\frac{1}{2}\right),\langle n|\hat{p}^2|n\rangle=m\hbar w\left(n+\frac{1}{2}\right)$

3.$\Delta x\Delta p = \left(n + \frac{1}{2}\right)\hbar$

1.5 第五题

证明：

1. $\exp(\alpha\hat{a}^\dagger)\hat{a}\exp(-\alpha\hat{a}^\dagger) = \hat{a} - \alpha$

2.$\exp(\lambda\hat{a})|0\rangle = |0\rangle$

2 答案

2.1 第一题

2.2 第二题

对于谐振子，我们有重要公式：

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m w}}(m w\,\hat{x} - i\,\hat{p}), \hat{a} = \frac{1}{\sqrt{2\hbar m w}}(m w\,\hat{x} + i\,\hat{p})$$

$$\hat{N} = \hat{a}^\dagger\hat{a} = \frac{1}{2\hbar m w}(m w\,\hat{x} - i\,\hat{p})(m w\,\hat{x} + i\,\hat{p}) = \frac{m w}{2\hbar}\hat{x}^2 + \frac{\hat{p}^2}{2\hbar m w} - \frac{1}{2}$$

则

$$\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\hbar w=\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}mw^2\,\hat{x}^2$$

2.3 第三题

对于谐振子，我们有重要公式：

$$[\hat{a},\hat{a}^\dagger]=1;[\hat{x},\hat{p}]=i\hbar$$

已知条件：

$$<0|0>=1,\hat{N}=\hat{N}^\dagger=\hat{a}^\dagger\hat{a}$$

由

$$n\langle n|m\rangle=(\hat{N}|n\rangle\,)^\dagger|m\rangle=\langle n|\hat{N}|m\rangle=m\langle n|m\rangle$$

当$n\neq m$时,必有$\langle n|m\rangle=0$;

当$n=m$时，考虑$\langle n|\hat{N}|n\rangle$：

首先有 $\langle n|\hat{N}|n\rangle=n\langle n|n\rangle$;

当$n\geqslant 1$时,$\langle n|\hat{N}|n\rangle=\langle n|\hat{a}^\dagger\hat{a}|n\rangle=(\hat{a}|n\rangle)^\dagger\hat{a}|n\rangle$

由$|n\rangle=\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$得：

$$\hat{a}|n\rangle=\sqrt{n}|n-1\rangle$$

代入上式有

$$\langle n|\hat{N}|n\rangle=\langle n|\hat{a}^\dagger\hat{a}|n\rangle=(\hat{a}|n\rangle)^\dagger\hat{a}|n\rangle=n\langle n-1|n-1\rangle$$

即有

$$\langle n|n\rangle=\langle n-1|n-1\rangle$$

重复上述操作可得到：

$$\langle n|n\rangle=<0|0>=1$$

综上，

$$\langle n|m\rangle=\delta_{nm},\mathrm{where}|n\rangle=\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

用归纳法证明也可

2.4 第四题

由第二题可知：

$$\hat{x}=\sqrt{\frac{\hbar}{2mw}}(\hat{a}+\hat{a}^{\dagger}),\hat{p}=-i\sqrt{\frac{1}{2}mw\hbar}(\hat{a}-\hat{a}^{\dagger})$$

1.

$$\langle n|\hat{x}|n\rangle=\left\langle n\middle|\sqrt{\frac{\hbar}{2mw}}(\hat{a}+\hat{a}^{\dagger})\middle|n\right\rangle=0$$

$$\langle n|\hat{p}|n\rangle=\left\langle n\middle|-i\sqrt{\frac{1}{2}mw\hbar}(\hat{a}-\hat{a}^{\dagger})\middle|n\right\rangle=0$$

2.

由$[\hat{a},\hat{a}^{\dagger}]=1$,$\hat{a}^{\dagger}\hat{a}=\hat{N}$

$$\hat{x}^2=\frac{\hbar}{2mw}(\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^{\dagger}\hat{a}+1),\hat{p}^2=-\frac{1}{2}mw\hbar(\hat{a}^2+\hat{a}^{\dagger 2}-2\hat{a}^{\dagger}\hat{a}-1)$$

则

$\langle n|\hat{x}^2|n\rangle=\langle n|\frac{\hbar}{2mw}(\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^{\dagger}\hat{a}+1)|n\rangle=\frac{\hbar}{mw}(n+\frac{1}{2})$

$\langle n|\hat{p}^2|n\rangle=\langle n|-\frac{1}{2}mw\hbar(\hat{a}^2+\hat{a}^{\dagger 2}-2\hat{a}^{\dagger}\hat{a}-1)|n\rangle=m\hbar w(n+\frac{1}{2})$

3.

代入2.中的结果得：

$\Delta x\Delta p=\sqrt{(<x^2>-<x>^2)(<p^2>-<p>^2)}=(n+\frac{1}{2})\hbar$

2.5 第五题

证：1.法一：

$$令f(\alpha)=\exp(\alpha\hat{a}^{\dagger})\hat{a}\exp(-\alpha\hat{a}^{\dagger})$$

$$f'(\alpha)=\exp(\alpha\hat{a}^{\dagger})[\hat{a}^{\dagger},\hat{a}]\exp(-\alpha\hat{a}^{\dagger})=-1$$

令$\alpha=0$, 得到 $f(0)=\hat{a}$, 即

$$f(\alpha)=\exp(\alpha\hat{a}^{\dagger})\hat{a}\exp(-\alpha\hat{a}^{\dagger})=\hat{a}-\alpha$$

法二：利用第二次作业的公式即 ：

已知$[\hat{a},\hat{a}^{\dagger}]=1,[\hat{a},\exp(\alpha\hat{a}^{\dagger})]=\alpha[\hat{a},\hat{a}^{\dagger}]\exp(\alpha\hat{a}^{\dagger})=\alpha\exp(\alpha\hat{a}^{\dagger})$

$\exp(\alpha\hat{a}^{\dagger})\hat{a}\exp(-\alpha\hat{a}^{\dagger})=(\hat{a}\exp(\alpha\hat{a}^{\dagger})-\alpha\exp(\alpha\hat{a}^{\dagger}))\exp(-\alpha\hat{a}^{\dagger})=\hat{a}-\alpha$

2.

已知 $\hat{a}|0>=0$

$$\exp(\alpha\hat{a})|0>=\sum_{m=0}\frac{\hat{a}^m}{m!}|0>=|0>$$

3 简单总结

从这几次作业我们能得到非常重要的关系：

1.基的正交完备归一性。

I为Hilbert空间的单位算符。

对于离散情况：

$$\text{完备性：}\sum_n|\psi_n><\psi_n|=I$$

$$\text{正交归一性：}<\psi_n|\psi_m>=\delta_{nm}$$

对于连续情况(1d)：

$$\text{完备性：}\int dx|x><x|=\int dp|p><p|=I$$

$$\text{正交归一性：}<x'|x>=\delta\left(x-x'\right)$$

2.x表象和E表象的变换：

我们需要根据问题决定在哪个表象 $i.e.$取哪组基去解决问题。

对于谐振子：

$$\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)\hbar w=\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}mw^2\hat{x}^2$$

$$\hat{a}^{\dagger}=\frac{1}{\sqrt{2\hbar mw}}(mw\hat{x}-i\hat{p}),\hat{a}=\frac{1}{\sqrt{2\hbar mw}}(mw\hat{x}+i\hat{p})$$

$$\hat{x}=\sqrt{\frac{\hbar}{2mw}}(\hat{a}+\hat{a}^{\dagger}),\hat{p}=-i\sqrt{\frac{1}{2}mw\hbar}(\hat{a}-\hat{a}^{\dagger})$$

薛定谔方程：

$$A_S\text{一般不含时}$$

$$\frac{\partial}{\partial t}|\psi\rangle_S(t) = \hat{H}|\psi\rangle_S(t)$$

海森堡方程：

$$\frac{d}{dt}A_H = [A_H, H] + \frac{\partial}{\partial t}A_S$$

$$|\psi\rangle_H = |\psi\rangle_S(0), \text{始终为初态}$$