一、位势方程：

$\Delta u = f(x)$ in $\Omega \subset \mathbb{R}^n$. Poisson方程.

一类边值 $u|_{\partial\Omega} = g(x)$.

二 - - - - - $\frac{\partial u}{\partial \vec{n}}\Big|_{\partial\Omega} = g(x)$.

三类边值 $\left(hu + \frac{\partial u}{\partial \vec{n}}\right)\Big|_{\partial\Omega} = g(x)$

特别地：$\Delta u = 0$ 调和方程.

二、基本解与Green公式

$\Delta u = \delta$，其中 δ 支集为一个孤立点，且 $\int_{\mathbb{R}^n} \delta \, dx = 1$.

不妨设为零点.

设 $u(x) = u(|x|) = u(r)$ 径向解.

则 $\left(\partial_r^2 + \frac{n-1}{r}\partial_r\right) u + \underbrace{\frac{1}{r^2}\Delta_{S^{n-1}} u}_{\text{径向}\to\text{与角方向无关} = 0} = 0$

$\therefore \partial_r^2 u + \frac{n-1}{r}\partial_r u = 0$

令 $v = \partial_r u$ $\Rightarrow \begin{cases} v = c \cdot r^{-(n-1)} \\ u = c_1 \cdot r^{-(n-2)} + C_2 & n \geq 3 \\ u = c_1 \cdot \log r & n = 2 \end{cases}$

Δu 是线性的，不妨令 $C_2 = 0$.

定义：$k(x) = \begin{cases} \frac{1}{2\pi}\ln|x| & n=2 \\ \frac{1}{n(2-n)\omega_n}|x|^{-(n-2)} & n \geq 3. \end{cases}$

$n \cdot \omega_n \to n$ 维 Euclide 单位球面表面积.

特别地 $n=3$ $k(x) = -\frac{1}{4\pi}|x|^{-1}$

① 第一 Green 公式：

若 $u,v\in C^2(\Omega)\cap C^1(\bar{\Omega})$

则 $\int_\Omega u\Delta v\,dx=\int_\Omega \nabla(u\nabla v)-\nabla u\cdot\nabla v\,dx$

$=\int_{\partial\Omega} u\nabla v\cdot\vec{n}\,ds-\int_\Omega \nabla u\cdot\nabla v\,dx$

$=\int_{\partial\Omega} u\frac{\partial v}{\partial\vec{n}}\,ds-\int_\Omega \nabla u\cdot\nabla v\,dx$.

② 第二 Green 公式： $u,v\in C^2(\Omega)\cap C^1(\bar{\Omega})$

$\int_\Omega v\Delta u\,dx$

$=\int_{\partial\Omega} v\frac{\partial u}{\partial\vec{n}}\,ds-\int_\Omega \nabla u\cdot\nabla v\,dx$

$\Rightarrow\int_\Omega u\Delta v-v\Delta u\,dx$

$=\int_{\partial\Omega} u\frac{\partial v}{\partial\vec{n}}-v\frac{\partial u}{\partial\vec{n}}\,dS$

☆ $\begin{cases}\Delta u=0\\ u|_{\partial\Omega}=\varphi(x)\end{cases}$

claim: $n=3$ $u\in C^2(\Omega)\cap C^1(\bar{\Omega})$ 满足 $\Delta u=0$ in Ω.

则 $\forall x_0\in\Omega$ $u(x_0)=\frac{1}{4\pi}\int_{\partial\Omega}-u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{|x-x_0|}\frac{\partial u}{\partial\vec{n}}\,dS$ (*)

proof: 若 $u(x)$ 是 $\Delta u=0$ 的解，则 $u(x+x_0)$ 也是 $\Delta u=0$ 的解.

(平移不变)

不妨设 $x_0=0$ 否则设(*)对 $\overset{=0}{x_0}$ 成立

则对 $u(y+x_0)$ 应用(*).

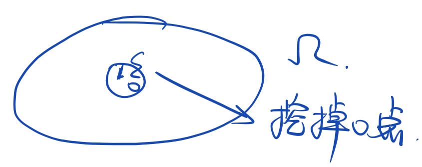
$$\frac{1}{4\pi}\int_{\partial(\Omega-x_0)}-u(y+x_0)\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|y|}\right)+\frac{1}{|y|}\frac{\partial u}{\partial\vec{n}}(y+x_0)\,ds$$

$$\underline{\underline{y+x_0=x}}\ \frac{1}{4\pi}\int_{\partial\Omega}-u(x)\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{|x-x_0|}\frac{\partial u}{\partial\vec{n}}(x)\,dS=u(x_0)$$

要证：$0\in\Omega$. ⇒

$$u(0)=\frac{1}{4\pi}\int_{\partial\Omega}-u(x)\cdot\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\frac{\partial u}{\partial\vec{n}}(x)\,dS$$

令 $v=\frac{-1}{4\pi}\cdot\frac{1}{|x|}$ （在0处不可微）



v 在 $\Omega \setminus \overset{=\Omega_\varepsilon}{B_\varepsilon(0)}$ 上应用第二 Green 公式

$\Delta u=0$ $\underset{\sim}{\Delta v=0}$ (极坐标)

$$\overset{\downarrow}{0} = \frac{1}{4\pi}\int_{\partial\Omega_\varepsilon} -u\cdot\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|x|}\right) + \frac{1}{|x|}\frac{\partial u}{\partial\vec{n}}(x)\,dS$$

$$= \frac{1}{4\pi}\int_{\partial\Omega} -u\cdot\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|x|}\right) + \frac{1}{|x|}\frac{\partial u}{\partial\vec{n}}(x)\,dS$$

$$+ \frac{1}{4\pi}\int_{\partial B_\varepsilon} -u\cdot\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|x|}\right) + \frac{1}{|x|}\frac{\partial u}{\partial\vec{n}}(x)\,dS.$$

$$\frac{\partial f}{\partial\vec{n}} = \vec{n}\cdot\nabla f = -\frac{x}{r}\nabla f = -\partial_r f.$$

$$\frac{1}{4\pi}\int_{\partial B_\varepsilon} -u\cdot\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|x|}\right)dS = \frac{1}{4\pi}\int_{\partial B_\varepsilon} -u\left(-\frac{\partial}{\partial r}\cdot\frac{1}{r}\right)dS$$

$$= \frac{-1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon} u\,dS \longrightarrow -u(0).$$

claim: $\displaystyle \frac{1}{\sigma(\partial B_\varepsilon)}\int_{\partial B_\varepsilon} u\,dx = u(0).\quad (\varepsilon\to 0)$

proof: 构造 $\displaystyle \frac{1}{\sigma(\partial B_\varepsilon)}\int_{\partial B_\varepsilon}(u(x)-u(0))\,dx$

$$\frac{1}{4\pi}\int_{\partial B_\varepsilon}\frac{1}{|x|}\frac{\partial u}{\partial\vec{n}}(x)\,dS = \frac{1}{4\pi\varepsilon}\int_{B_\varepsilon}\Delta u\,dx = 0 \qquad \forall\varepsilon>0.$$

$$\therefore\ u(0) = \frac{1}{4\pi}\int_{\partial\Omega} -u\frac{\partial}{\partial\vec{n}}\left(\frac{1}{|x|}\right) + \frac{1}{|x|}\frac{\partial u}{\partial\vec{n}}(x)\,dS.$$

$\because$ 0 满足上式 $\therefore$ $\forall x_0\in\Omega$ 仍满足上式

若存在 g 在 Ω 上调和（$\Delta g=0$），且 $\displaystyle g\Big|_{\partial\Omega} = \frac{1}{4\pi|x-x_0|}$

由第二 Green 公式 $\Rightarrow$ $\displaystyle 0 = \int_{\partial\Omega} u\frac{\partial g}{\partial\vec{n}} - g\frac{\partial u}{\partial\vec{n}}\,dS$

与(*)相加: $u(x_0) = \int_{\partial\Omega} u\frac{\partial}{\partial \vec{n}}\underbrace{\left(-\frac{1}{4\pi|x-x_0|}+g\right)}_{=G(x)} + \underbrace{\left(\frac{1}{4\pi|x-x_0|}-g\right)}_{=0}\frac{\partial u}{\partial \vec{n}}\, ds$

$$= \int_{\partial\Omega} u\cdot \frac{\partial}{\partial\vec{n}} G\, ds$$

定义(Green函数): 满足:

① $G(x)\in C^2(\Omega)$ 且 $\Delta G=0$ $\quad \forall x\neq x_0$

② $G(x)=0$ $\quad \forall x\in\partial\Omega$

③ $G+\frac{1}{4\pi|x-x_0|}$ 在 x_0 处有限. 处处二阶连续可微. 且 x_0 处调和.

性质: $G(x,x_0)=G(x_0,x)$ $\quad \forall x\neq x_0$ 对易原理.

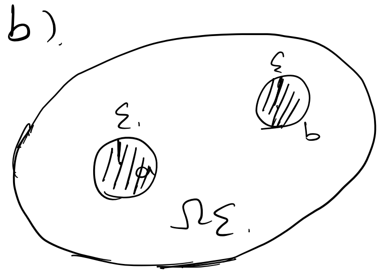
(本质: $\Delta G=\delta_{x_0}$ 等效).

proof: 欲证明 $G(a,b)=G(b,a)$, $\quad \forall a\neq b$

令 $u(x)=G(x,a)$ $\quad v(x)=G(x,b)$

对 u, v 在 上用第二Green公式

$\Omega\setminus B_\varepsilon(a)\cup B_\varepsilon(b)$



$$0=\int_{\partial\Omega}+\int_{\partial B_\varepsilon(a)}+\int_{\partial B_\varepsilon(b)} u\cdot\frac{\partial v}{\partial\vec{n}}-v\frac{\partial u}{\partial \vec{n}}\,dS$$

∵ u, v 均为 $\underbrace{\text{Green 函数}}_{\text{性质②}}$ $\quad \int_{\partial\Omega}=0$

$$\int_{\partial B_\varepsilon(a)} + \int_{\partial B_\varepsilon(b)} = 0$$

第二 Green公式

0.

$$\int_{\partial B_\varepsilon(a)} \underbrace{\left[G(x,a) + \frac{1}{4\pi|x-a|}\right]\frac{\partial v}{\partial \vec{n}} - v\frac{\partial}{\partial \vec{n}}\left[G(x,a) + \frac{1}{4\pi|x-a|}\right]}_{0} - \frac{1}{4\pi|x-a|}\frac{\partial v}{\partial \vec{n}} + \frac{\partial}{\partial \vec{n}}\left(\frac{1}{4\pi|x-a|}\right)v \, dS$$

$$= \frac{-1}{4\pi}\int_{\partial B_\varepsilon(a)} \frac{1}{|x-a|}\frac{\partial v}{\partial \vec{n}} - \frac{\partial}{\partial \vec{n}}\left(\frac{1}{|x-a|}\right)v \, dS$$

$$= \underbrace{\frac{-1}{4\pi\varepsilon}\int_{\partial B_\varepsilon(a)} \frac{\partial v}{\partial \vec{n}}}_{①} + \underbrace{\frac{1}{4\pi}\int_{\partial B_\varepsilon(a)} v\left(-\frac{\partial}{\partial r}\frac{1}{|x-a|}\right) dS.}_{②}$$

? 负号

$$\Rightarrow ② = \frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon(a)} v\,dx \longrightarrow v(a).$$

$$① \leq \frac{-4\pi\varepsilon^2}{4\pi\varepsilon}\cdot \max\frac{\partial v}{\partial \vec{n}} \longrightarrow 0 \quad (\varepsilon \to 0)$$

$$\therefore \int_{\partial B_\varepsilon(a)} \longrightarrow v(a) \quad \varepsilon \to 0.$$

$$= G(a,b)$$

$$\int_{\partial B_\varepsilon(b)} \longrightarrow -u(b) \quad \varepsilon \to 0.$$

$$= -G(b,a).$$

∴ 综上，

$$G(a,b) = G(b,a).$$

$$\Delta u = 0 \quad \text{in } \Omega.$$

$$u|_{\partial\Omega} = \varphi(x).$$

则 $u(x) = \int_{\partial\Omega} \varphi(y) \frac{\partial G}{\partial \vec{n}}(y,x)\, dy$

特别地，

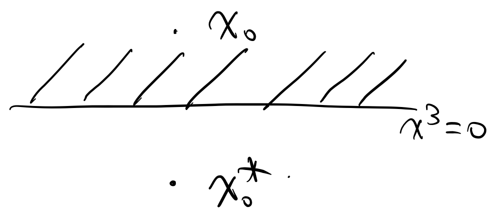
$$\begin{cases} \Delta u = f & \text{in } \Omega. \\ u|_{\partial\Omega} = \varphi(x) \end{cases}$$

非齐次方程.

$$\Rightarrow u(x) = \int_{\Omega} G f \, dy + \int_{\partial\Omega} \varphi \frac{\partial G}{\partial \vec{n}}(y,x)\, dy$$

☆ Green 函数求法:

1、n=3 半空间 电像法



令 $G(x) = -\frac{1}{4\pi|x-x_0|} + \frac{1}{4\pi|x-x_0^*|}$ $\quad x_0^* = (x_0^1, x_0^2, -x_0^3)$

坐标

验证 ① $G(x)$ 在 $\{x^3>0\} \setminus \{x_0\}$ 二阶连续可导. 调和 ✓.

② $G(x) = 0 \quad \forall x \in (x^1, x^2, 0)$.

③ trivial.

例: $\begin{cases} \Delta u = 0 & (x_3 > 0) \\ u|_{x_3=0} = \varphi(x) \end{cases}$

$$\Rightarrow u = \int_{\partial\Omega} \varphi(x) \frac{\partial G}{\partial \vec{n}} \, dx_1 dx_2$$

$$= \int_{\partial\Omega} \varphi(x) \left(-\frac{\partial G}{\partial x_3}\right) dx_1 dx_2$$

$$\frac{\partial G}{\partial x_3} = \frac{1}{4\pi |x-x_0|^2} \cdot \frac{(x_3 - x_0^3)}{|x-x_0|}$$

$x_3 = 0$

$$u(x_0) = \int_{\mathbb{R}^2} \varphi(x) \frac{-(x_3 - x_0^3)}{4\pi |x-x_0|^3} dx_1 dx_2$$

$$= \int_{\mathbb{R}^2} \varphi(x) \cdot \frac{2x_0^3}{4\pi |x-x_0|^3} \, dx_1 dx_2$$

或

$$u(x) = \int_{\mathbb{R}^2} \varphi(y) \frac{2x_0^3}{4\pi |y-x|^3} \, dy$$

2. 球面上的 Green 函数. $\Omega = B_R(0) \subset \mathbb{R}^3$.

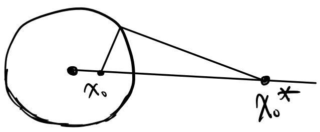
选取 x_0^* 在 $0 \sim x_0$ 的延长线上的一点

令 $G(x) = \dfrac{1}{4\pi |x-x_0|} + \dfrac{C}{4\pi |x-x_0^*|}$

令 $\rho = |x-x_0| \quad \rho^* = |x-x_0^*|$

欲: $\forall |x| = R \quad -\dfrac{1}{\rho} + \dfrac{C}{\rho^*} = 0$

Green 函数 ② 条件.



阿波罗尼斯圆 $\Longrightarrow$ 只需 $\dfrac{\rho^*}{\rho} = \dfrac{R}{|x_0|} \triangleq C$

$$\Nearrow \quad C = \frac{\frac{R^2}{|x_0|} - R}{R - |x_0|} = \frac{R}{|x_0|}$$

令 $G(x) = -\dfrac{1}{4\pi|x-x_0|} + \dfrac{1}{4\pi}\cdot\dfrac{R}{|x_0|}\cdot\dfrac{1}{|x-x_0^*|}$

而 x_0^* 距原点距离为 $\dfrac{R^2}{|x_0|}$ $\qquad x_0^* = \dfrac{R^2}{|x_0|}\cdot\dfrac{x_0}{|x_0|}$.

$$\frac{\partial G}{\partial \vec{n}} = \nabla G\cdot\vec{n} = \frac{x-x_0}{4\pi|x-x_0|^3} - \frac{1}{4\pi}\frac{R}{|x_0|}\cdot\frac{x-x_0^*}{|x-x_0^*|^3}$$

当$|x|=R$时 $\rho^* = |x-x_0^*| = \dfrac{R}{|x_0|}\cdot|x-x_0|$.

换: $\dfrac{x-x_0}{4\pi|x-x_0|^3} - \dfrac{1}{4\pi}\dfrac{R}{|x_0|}\cdot\dfrac{x-x_0^*}{(\frac{R}{|x_0|})^3\cdot|x-x_0|^3}$

$$= \frac{x}{4\pi|x-x_0|^3}\left(1-\frac{|x_0|^2}{R^2}\right)$$

$$\therefore \nabla G\cdot\vec{n} = \frac{x}{R}\cdot\frac{x}{4\pi|x-x_0|^3}\cdot\left(1-\frac{|x_0|^2}{R^2}\right)$$

$$= \frac{R^2 \quad (R^2-|x_0|^2)}{R \quad 4\pi|x-x_0|^3\cdot R^2}$$

$$= \frac{R^2-|x_0|^2}{4\pi R}\int_{|x|=R}\frac{\varphi(x)}{|x-x_0|^3}\,dx.$$

调和函数的性质:

$\Delta u=0$ in Ω 称 u 在 Ω 上是调和的.

* 极坐标公式: $\int_{\mathbb{R}} f(x)\,dx = \int_0^{+\infty}\int_{B_r(x)} f(y)\,dS(y)\,dr$.

$$\frac{d}{dr}\int_{B_r(x_0)} f(y)\,dy = \frac{d}{dr}\int_0^r d\rho \int_{\partial B_\rho(x_0)} f(y)\,dS(y)$$

$$= \int_{\partial B_r(x_0)} f(y)\,dS(y).$$

定义: 若 $u\in C(\Omega)$.

① 称 u 满足平均值性质: 如果 $\forall B_r(x)\subset\Omega$

$$u(x)=\frac{3}{4\pi r^3}\int_{B_r(x)} u(y)\,dy. \qquad (三维)$$

② 称 u 满足第二平均值性质: 如果 $\forall B_r(x)\subset\Omega$.

$$u(x)=\frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(y)\,dS(y).$$

命题: ① $\Longleftrightarrow$ ②

proof: "$\Leftarrow$" $\displaystyle\int_{B_r(x)} u(y)\,dy=\int_0^r d\rho\int_{\partial B_\rho(x)} u(y)\,dS(y)$

$$=\int_0^r 4\pi\rho^2\cdot u(x)\,d\rho$$

$$=\frac{4}{3}\pi r^3 u(x)$$

$$\Rightarrow u(x)=\frac{3}{4\pi r^3}\int_{B_r(x)} u(y)\,dy. \qquad \square$$

"$\Rightarrow$" $\frac{4}{3}\pi r^3 \cdot u(x) = \int_{B_r(x)} u(y)\, dy$

两边对 r 求导: $4\pi r^2 \cdot u(x) = \int_{\partial B_r(x)} u(y)\, dy$

$$\Rightarrow u(x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(y)\, dy \qquad \square$$

定理. 若 $u \in C^2(\Omega)$ 在 Ω 上调和，则 u 满足平均值性质.

proof: $\forall \rho > 0 \quad B_\rho(x) \subset \Omega$.

$$0 = \int_{B_\rho(x)} \Delta u \, dy \overset{\text{散度定理}}{=\!=\!=} \int_{\partial B_\rho(x)} \nabla u \cdot \vec{n}\, ds$$

$\vec{n}$ 为球面外法向

$\vec{n} = \frac{y-x}{\rho}$

$$= \int_{\partial B_\rho(x)} \frac{y-x}{\rho} \cdot \nabla u \, dS(y)$$

$$\overset{w=\frac{y-x}{\rho}}{=\!=\!=} \int_{|w|=1} w \cdot \nabla u(\rho w + x)\, \rho^2 \, dS(w)$$

$$= \rho^2 \int_{|w|=1} \frac{\partial u}{\partial \rho}\, dS(w) = \rho^2 \cdot \frac{\partial}{\partial \rho}\int_{|w|=1} u(\rho w + x)\, dS(w)$$

$= 0$

$\therefore \int_{|w|=1} u(\rho w + x)\, dS(w)$ 关于 ρ 是常数.

$$\Rightarrow \int_{|w|=1} u(x)\, dS(w) = \int_{|w|=1} u(rw + x)\, dS(w)$$

$$\Rightarrow 4\pi \cdot u(x) = \frac{1}{r^2}\int_{|w|=r} u(w+x)\, dS(w)$$

$$\Rightarrow u(x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(y)\, dS(y) \qquad \square$$

定理：若 $u\in C(\overline{\Omega})$ 在 Ω 内满足平均值性质，则 u 在 Ω 上光滑且调和

思路：若 $u=u*\varphi$，φ 光滑，则 u 光滑

proof：令 $\varphi\in C_0^\infty(B_1(0))$ 表示 $\varphi\equiv 0$ on $\overline{B_1(0)}^c$

或 $\varphi=0$，$\forall |x|>1$

且 $\underbrace{\int\varphi=1}$ 且 $\varphi\geq 0$ radial. $(\varphi(x)=\varphi(|x|))$ 镜像.

则 $\int\varphi\,dx=\int_0^\infty dr\int_{\partial B_r(x)}\varphi(r\omega)\,dS$

$=\int_0^\infty dr\int_{\partial B_r(x)}\varphi(r)\,dS=\int_0^\infty 4\pi r^2\cdot\varphi(r)\,dr.$

$=1$

即 $\int\varphi=1\Rightarrow\int_0^\infty 4\pi r^2\varphi(r)\,dr=1.$

令 $\varphi_\varepsilon(z)=\frac{1}{\varepsilon^n}\varphi(\frac{z}{\varepsilon})$　则 φ_ε 的 supp 是 $|x|<\varepsilon$.

即：$\underbrace{\varphi_\varepsilon\in C_0^\infty(B_\varepsilon(0))}$.

且 $\int\frac{1}{\varepsilon^n}\varphi(\frac{z}{\varepsilon})\,dz\overset{y=\frac{z}{\varepsilon}}{=}\int\varphi(y)\,dy=1.$

$\forall x\in\Omega$　$(u*\varphi_\varepsilon)(x)$　　φ_ε：mollfier 光滑子.

$=\int_{\mathbb{R}^3}u(y)\cdot\varphi_\varepsilon(x-y)\,dy.$

$=\int_{|x-y|<\varepsilon}u(y)\cdot\varphi_\varepsilon(x-y)\,dy.$

取 $0<\varepsilon<$ distance $(x,\partial\Omega)$.

$\overset{\varphi\text{镜像}}{=}\int_{|x-y|<\varepsilon}u(y)\cdot\varphi_\varepsilon(y-x)\,dy$

$\overset{z=y-x}{=}\int_{|z|<\varepsilon}u(z+x)\cdot\varphi_\varepsilon(z)\,dz.$

$$\overset{y=\frac{z}{\varepsilon}}{=\!=\!=} \int_{|y|<1} u(\varepsilon y+x)\cdot\varphi(y)\,dy \qquad (\varepsilon\text{ 可以充分小}).$$

$$\overset{\text{极坐标}}{=\!=\!=} \int_0^1 dr \underline{\int_{|w|=1} u(\varepsilon rw+x)\cdot\varphi(r)\ dS(rw)}.$$

↙ 平均值定理

$$=\int_0^1 \varphi(r)\ dr\cdot \frac{4\pi(\varepsilon r)^2\cdot u(x)}{\varepsilon^2} \Leftarrow \int_{|w|=1} u(\varepsilon rw+x)\varphi(r)\cdot\frac{1}{\varepsilon^2}dS(\varepsilon rw)$$

$$= u(x)\int_0^1 \varphi(r)\cdot 4\pi r^2\,dr. = u(x). \qquad \Box.$$

考虑 Δu $\because u\in C^\infty(\Omega)$. $\forall x\in\Omega$ 取 $B_r(x)\subset\Omega$.

见前面证明

$$\int_{B_r(x)}\Delta u\,dy = r^2\frac{d}{dr}\int_{|w|=1} u(rw+x)\,dS(w).$$

$$\overset{\text{平均值}}{=\!=\!=} r^2\cdot\frac{d}{dr}(4\pi u(x)) = 0$$

u 与 r 无关.
则 u 满足平均值性质.

claim: $\Delta u=0$ in Ω.

否则 $\exists x_0$ s.t. $\Delta u(x_0)>0$ 则 $\exists r_0>0$ s.t. $\Delta u(x)>0$ $\forall x\in B_r(x_0)$.

$\therefore$ 矛盾. $\Box$.

命题: 若 $u\in C(\bar\Omega)$ 满足平均值性质, 则 u 只在边界上达到最大值或最小值, 或 $u\equiv$ const.

proof: 令 $\Sigma=\{x\in\Omega \mid u(x)=\max\limits_{\bar\Omega} u \triangleq M\}$.

只需证明 $\Sigma=\emptyset$ $\Rightarrow$ 只在边界取到

或 $\Sigma=\Omega$ $\Rightarrow$ const.

下面证明: Σ 既开又闭. (相对于 Ω 是既开又闭的).

① 设 $\{x_n\} \subset \Sigma$ $\quad x_n \to \bar{x}$ $\quad$ 则 $\lim\limits_{n\to\infty} x_n = \bar{x}$.

$\therefore \lim\limits_{n\to\infty} u(x_n) = u(\bar{x}) = M$ $\quad \therefore \bar{x} \in \Sigma$.

② $\forall x_0 \in \Sigma$ 由平均值性质

$$M = u(x_0) = \frac{3}{4\pi r^3} \int_{B_r(x_0)} u(y)\, dy$$

而 $u(y) \leqslant M$

$$\therefore \int_{B_r(x_0)} u(y) - M \, dy \equiv 0 \Rightarrow u(y) \equiv M \quad \forall y \in B_r(x_0)$$

$\square$.

命题: 若 $u \in C(\overline{B_R})$ 是调和的, $B_R = B_R(x_0)$

则 $|Du(x_0)| \leqslant \frac{n}{R} \max\limits_{B_R} |u|$. $\quad$ 光滑.

proof: 不妨设 $u \in C^1(\overline{B_R})$. $\quad$ (为简单起见) $\quad$ 令 $x_0 = 0$.

$\nabla u = 0 \Rightarrow \nabla \cdot \partial_{x_i} u = 0 \quad \therefore \partial_{x_i} u$ 满足平均值性质.

$$\partial_{x_i} u = \frac{1}{\alpha(B_R(0))} \int_{B_R(0)} \partial_{x_i} u(y) d(y).$$

散度定理 $\quad \frac{1}{\alpha(B_R(0))} \int_{\partial B_R(0)} \overset{\text{内积}}{u(y) \cdot \nu_i} \, dS$

$$\leqslant \frac{1}{\alpha(B_R(0))} \int_{\partial B_R(0)} |u| \, dS$$

$$\leqslant \frac{\alpha(\partial B_R(0))}{\alpha(B_R(0))} \cdot \max_{B_R} |u|. \qquad \alpha(\partial B_R(0)) = \frac{n}{R} \alpha(B_R(0)).$$

$$\alpha(B_R(0)) = \int_0^R d\rho \int_{\partial B_\rho(0)} dS.$$

$$= \frac{n}{R} \max_{B_R} |u|$$

命题：设 $u \in C(\overline{B_R})$ 是 $B_R = B_R(x_0)$ 上的非负调和函数，

则成立 $|Du(x_0)| \le \frac{n}{R} u(x_0)$.

非负

proof: $|\partial_{x_i} u(x_0)| \le \frac{1}{\gamma(B_R(x_0))} \int_{\partial B_R(x_0)} u(y) \, dS(y)$

$\underline{\underline{\text{平均值}}} \quad \frac{\gamma(\partial B_R(x_0))}{\gamma(B_R(x_0))} \cdot u(x_0) = \frac{n}{R} u(x_0)$

推论：(Liouville)

$\mathbb{R}^n$ 上的上有界或下有界的调和函数是常数.

proof: 设 $u \le M$ 且 $\Delta u = 0$ 　令 $v = M - u$.

则 v 为 $\mathbb{R}^n$ 上非负调和函数.

由梯度估计 $\forall x_0 \quad \forall R > 0$

$$|Dv(x)| \le \frac{n}{R} v(x_0) \quad \forall x \in B_R(x_0)$$

令 $R \to \infty$ 　$Dv(x) \equiv 0$ 　$\square$.

定理 (解析性) 调和函数是解析的. (不作要求)

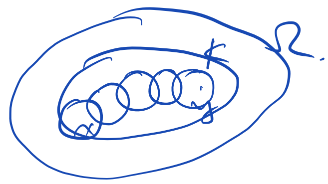
Harnack不等式

定理. 设 u 在 Ω 上调和. 则对 $\forall K$ 紧集 $\subset\Omega$, 存在正常数 $C=C(\Omega,K)$ 使得若 $u>0$ in Ω, 则 $\forall x,y\in K$

$$\frac{1}{C}u(y)\le u(x)\le Cu(y)$$

K中点相互控制

proof: step 1: 先证一个小球成立. 由有限覆盖定理, x, y 可以用有限个小球相连.



取 $R=\frac{1}{4}\text{distance}(K,\partial\Omega)$ 则 $\forall x\in K$ $B_{4R}(x)\subset\Omega$.

$\forall y\in B_R(x)$ 则 $B_R(y)\subset B_{2R}(x)$

由平均值性质: $u(x)=\frac{3}{4\pi(2R)^3}\int_{B_{2R}(x)}u(z)dz$

$$\ge\frac{3}{4\pi(2R)^3}\int_{B_R(y)}u(z)dz$$

$$=\frac{1}{8}u(y)$$

由对称性 $u(y)\ge\frac{1}{8}u(x)$ $\quad\therefore\frac{1}{8}u(y)\le u(x)\le 8u(y)$ □.

step 2: 由K是紧集 $\exists$有限个半径为R的开球构成K的开覆盖

$\therefore\forall x,y$ 用有限个球连接起来

$$\frac{1}{2^{3N}}u(y)\le u(x)\le 2^{3N}u(y)$$

Harnack 得证. □

命题：设 u 在 $B_R(x_0)$ 内调和，且 $u\geq 0$，则

$$\frac{R(R-r)}{(R+r)^2}u(x_0)\leq u(x)\leq \frac{R(R+r)}{(R-r)^2}u(x_0) \quad \forall r=|x-x_0|<R$$

proof：不妨设 $x_0=0$

由 Poisson 公式 $u(x)=\dfrac{R^2-|x|^2}{4\pi R}\displaystyle\int_{|y|=R}\frac{u(y)}{|x-y|^3}dS(y).$

$\because |y|<R$ 则 $R-|x|<|x-y|<R+|x|.$

$$\Rightarrow \frac{R^2-|x|^2}{4\pi R}\cdot\frac{1}{(R+|x|)^3}\int_{|y|=R}u(y)dS(y)\leq u(x)\leq \frac{R^2-|x|^2}{4\pi R}\cdot\frac{1}{(R-|x|)^3}\int_{|y|=R}u(y)dS(y)$$

平均值性质 $\Rightarrow$ $\dfrac{R^2-|x|^2}{(R+|x|)^3}\cdot R\cdot u(0)\leq u(x)\leq \dfrac{R^2-|x|^2}{(R-|x|)^3}\cdot R\cdot u(0)$

$$=\frac{(R-|x|)R}{(R+|x|)^2} \qquad\qquad =\frac{(R+|x|)R}{(R-|x|)^2}$$

□

定理.（Liouville）

若 u 是 $\mathbb{R}^3$ 上的上有界或下有界调和函数，则 u 是常数.

proof：不妨设 u 上有界 记 $M=\sup\limits_{\mathbb{R}^3} u.$

则 $v=M-u$ 为调和函数.

取 $R>|x|$ 在 $B_R(0)$ 上用 Harnack：

$$\frac{R(R-|x|)}{(R+|x|)^2}(M-u(0)) \leq M-u(x) \leq \frac{R(R+|x|)}{(R-|x|)^2}(M-u(0))$$

令 $R\to\infty \Rightarrow u(x)\equiv u(0) \quad \forall x\in\mathbb{R}^3$

$\therefore u(x)$ 在 $\mathbb{R}^3$ 上是常数 □.

奇点可去定理.

设 u 在 $B_R(0)\setminus\{0\}$ 是调和的, 且满足 $u(x)=\begin{cases} o(\ln|x|) & n=2 \\ o(|x|^{-1}) & n=3 \end{cases}$

则可重新定义 u 在 0 点的值, 使得 u 在 $B_R(0)$ 上 C^2 且调和.

proof: 只证 $n=3$.

令 v 满足: $\begin{cases} \Delta v=0 & \text{in } B_R(0). \\ v\equiv u & \text{on } \partial B_R(0). \end{cases}$

由 Poisson 公式: v 一定存在 $\quad v\in C^2(\Omega)\cap C(\bar\Omega)$.

要证: $u\equiv v$ on $B_R(0)\setminus\{0\}$.

令 $w(x)=u(x)-v(x)$.

则 $\forall\, 0<r<R \quad \begin{cases} \Delta w=0 & \text{in } B_r(0). \\ w=0 & \text{on } \partial B_r(0). \end{cases}$

由极大值原理: w 的最大值、最小值均在边界取到 一层一层敲

$\max\limits_{A}|w|=\max\limits_{\partial A}|w|. \quad A=\{0<|x|<R\}$

$w(x)=0 \quad \forall |x|=R$

设 $M_r=\max\limits_{|x|=r}|w(x)|$. 则 $\boxed{|w(x)|\leq\frac{r}{|x|}M_r} \quad \forall |x|=r$.

令 $W(x)=\frac{r}{|x|}M_r-w(x)$ 则 $\Delta W(x)=0$ (给定 r).

且 $|x|=r$ 时 $W(x)\geq 0$ $|x|=R$ 时 $W(x)>0$

对 $W(x)$ 在 A 区域 由极大值原理. $A=\{x \mid r<|x|<R\}$.

证明:

$W(x)\geq 0$ in A

故: $w(x)\leq \frac{r}{|x|}\cdot M_r$ $r<|x|<R$.

下面估计. $M_r=\max\limits_{|x|=r}|w(x)|\leq \max\limits_{|x|=r}|v(x)|+\max\limits_{|x|=r}|u(x)|$.

而 $\max\limits_{|x|=r}|v(x)|\leq \max\limits_{\partial B_R}|v(x)|=\max\limits_{\partial B_R}|u|\triangleq M$.

$$|w(x)|\leq \underline{\frac{r}{|x|}M+\frac{r}{|x|}\max_{|x|=r}|u(x)|} \quad o\left(\frac{1}{x}\right) \qquad \forall\ \underline{r<|x|<R}.$$

由条件. 令 $r\to 0$.

$\longrightarrow 0$

$\therefore |w(x)|\equiv 0$ $\forall\ \underline{0<|x|<R}$ □.

极大值原理　第二边值解的唯一性

定理：设 $u\in C^2(B_1)\cap C(\overline{B_1})$ 是 B_1 上的次调和函数，即 $\Delta u\geq 0$，

则有 $\sup\limits_{B_1} u \leq \sup\limits_{\partial B_1} u$.

考虑 $|x|^2$ 正定

且 $\Delta|x|^2$ 为常数

proof：令 $V_\varepsilon(x) = u(x)+\varepsilon\cdot|x|^2$

则 $\Delta V_\varepsilon(x) = \Delta u + 2n\varepsilon > 0$

设 $V_\varepsilon(x)$ 在内部一点 (A) 达到最大值，则在 (A) 点 $\Delta V_\varepsilon(x)\leq 0$

$\therefore\ \sup\limits_{\overline{B_1}} V_\varepsilon = \sup\limits_{\partial B_1} V_\varepsilon = \sup\limits_{\partial B_1} u+\varepsilon$.

放缩使左边与 ε 无关，

再令 $\varepsilon\to 0$.

且 $\sup\limits_{\overline{B_1}} u \leq \sup\limits_{\overline{B_1}} V$

令 $\varepsilon\to 0$　$\sup\limits_{\overline{B_1}} u \leq \sup\limits_{\partial B_1} u$.

定理(内部梯度估计)

设 u 在 B_1 内调和，则有 $\sup\limits_{B_{\frac{1}{2}}} |Du| \leq C\cdot \sup\limits_{\partial B_1} |u|$.

其中 $C=C(n)$ 为正常数.　$(Du=\nabla u)$.

proof：$|Du|=|\nabla u| = \sqrt{(\frac{\partial u}{\partial x_1})^2+\cdots+(\frac{\partial u}{\partial x_n})^2}$.

令 $w=|Du|^2$

则 $\Delta v = \Delta|Du|^2 = \Delta\left((\frac{\partial u}{\partial x_1})^2+\cdots+(\frac{\partial u}{\partial x_n})^2\right)$

$= \Delta\left(\sum\limits_{i=1}^{n} \partial_i^2 u\right)$

$$\sum_{i=1}^{n} 2\partial_i u \, \partial_{ij} u$$

$\downarrow$ 对 j

$$\sum_{j=1}^{n}\sum_{i=1}^{n} 2\partial_{ij} u \, \partial_{ij} u + 2\partial_i u \, \underbrace{\partial_{ijj} u}_{\Delta u = 0}$$

$$= 2 \sum_{i=1}^{n}\sum_{j=1}^{n} |\partial_{ij} u|^2 \geq 0$$

$\therefore \Delta V \geq 0$

令 $\eta \in C_0^\infty(B_1)$ 且 $\operatorname{supp}\eta \subseteq B_1$. $\eta \equiv 1$ on $B_{\frac{1}{2}}$.

则 $$\max_{B_{\frac{1}{2}}} |Du|^2 = \max_{B_{\frac{1}{2}}} \eta |Du|^2 \leq \max_{B_1} \eta |Du|^2.$$

$$\Delta(\eta^2 |Du|^2) = \Delta\eta^2 \cdot |Du|^2 + \eta^2 \Delta |Du|^2 + \nabla\eta^2 \cdot \nabla |Du|^2$$

$$= \Delta\eta^2 \cdot |Du|^2 + 2\eta^2 \sum_{i=1}^{n}\sum_{j=1}^{n} |\partial_{ij} u|^2 + 8\sum_{i,j} \eta \partial_i \eta \, \partial_j u \, \partial_{ij} u.$$

$$\boxed{ab \leq \varepsilon a^2 + \frac{c}{\varepsilon} b^2 \qquad \sqrt{2\varepsilon}\, a \frac{1}{\sqrt{2\varepsilon}} b \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2}$$

$$\eta \partial_i \eta \, \partial_j u \, \partial_{ij} u \leq \varepsilon \eta^2 |\partial_{ij} u|^2 + \frac{8|\partial_i \eta \, \partial_j u|^2}{4\varepsilon}$$

$$\geq -C \cdot |Du|^2 - \underbrace{\sum_{i,j} \varepsilon \eta^2 |\partial_{ij} u|^2}_{\varepsilon = 1} + \frac{8|\partial_i \eta|^2}{4\varepsilon} |\partial_j u|^2 + \underline{2\eta^2 \sum_{i,j} |\partial_{ij} u|^2}$$

$$\geq -C \cdot |Du|^2$$

吸收进 C 中

$$\underline{\Delta(u^2)} = \sum_{i=1}^{n} \partial_i^2 (u^2) = \sum_{i=1}^{n} \partial_i \left(2u \, \partial_i u \right)$$

$$= \sum_{i=1}^{n} 2(\partial_i u)^2 + 2u\cdot\partial_{ii} u. \quad = 2|Du|^2 + 2u\cdot\Delta u.$$

$$(\Delta u = 0) \qquad = 2|Du|^2.$$

用 Δu^2 处理 $|Du|^2$.

$$\therefore \Delta\left(\eta^2|Du|^2 + \frac{C}{2}u^2\right) \geq -C|Du|^2 + \frac{C}{2}\Delta(u^2) = 0.$$

由极大值原理.

$$\max_{\overline{B_{\frac12}}} |Du|^2 \leq \max_{\overline{B_1}} \eta^2|Du|^2 + \frac{C}{2}u^2 = \max_{\partial B_1} \underbrace{\eta^2}_{\eta \equiv 0\ (|x|=1)}|Du|^2 + \frac{C}{2}u^2.$$

$$= \frac{C}{2}\max_{\partial B_1} u^2.$$

$$\therefore \max_{\overline{B_{\frac12}}} |Du| \leq \sqrt{\frac{C}{2}} \max_{\partial B_1} |u|. \qquad \square.$$

引理. 设 u 是 B_1 上的非负调和函数. 则 $\exists C = C(n)$.

s.t. $$\sup_{\overline{B_{\frac12}}} |D\ln u| \leq C.$$

推论. (Harnack).

设 u 是 B_1 上的非负调和函数. 则 $\exists$ 常数 $C = C(n)$.

s.t. $$u(x_1) \leq C\,u(x_2) \qquad \forall\, x_1, x_2 \in B_{\frac12}.$$

proof: 若 $u(x^*)=0$ 则由极值原理 $x^* \in \partial B_1$.

不妨设 $u>0$ in B_1.

则: $\ln u(x_1) - \ln u(x_2) = \int_0^1 \frac{d}{dt}\ln u(tx_1+(1-t)x_2)\,dt.$

$$= \left|\int_0^1 D\ln u\,(tx_1+(1-t)x_2)\,dt\right| |x_1-x_2|.$$

$$\overset{引理}{\leq} C$$

$\frac{d}{dt}u(t\vec{x})$

$=[\nabla u(t\vec{x})]\cdot\vec{x}.$

$\therefore \quad \frac{u(x_1)}{u(x_2)} \leq e^C.$

命题: (Hopf 引理) 设 $u\in C(\overline{B_1})$ 是 $B_1=B_1(0)$ 上的调和函数

若存在 $x_0\in\partial B_1$ s.t. $u(x)<u(x_0)$ $\forall x\in B_1$.

则 $\exists\, C=C(n)$ s.t. $\frac{\partial u}{\partial \vec{n}}(x_0) \geq C\left(u(x_0)-u(0)\right) \underset{\triangle}{>} 0.$

△ 严格大.

proof: step1: 令 $v(x)=e^{-\alpha|x|^2}-e^{-\alpha}.$

则 $\forall |x|<1$ $v(x)>0$

$$\Delta v = \sum_{i=1}^n \partial_i\left(\partial_i e^{-\alpha|x|^2}\right) = \sum_{i=1}^n \partial_i\left(e^{-\alpha|x|^2}\cdot 2\alpha x_i\right).$$

$$= \sum_{i=1}^n 2\alpha e^{-\alpha|x|^2} + 4\alpha^2 x_i^2\cdot e^{-\alpha|x|^2}$$

$$= 2n\alpha e^{-\alpha|x|^2} + 4\alpha^2|x|^2\cdot e^{-\alpha|x|^2}.$$

当 $|x|>\frac{1}{2}$

$$\Delta v \geq (\alpha^2-2\alpha n)e^{-\alpha|x|^2} = 0$$

取 $\alpha=\frac{1}{2n}$

step 2:

令 $h_\varepsilon(x) = u(x) - u(x_0) + \varepsilon v(x)$.

$\Delta h_\varepsilon(x) \geq 0 \qquad \forall \frac{1}{2}<|x|<1$.

由极大值原理. $h_\varepsilon(x)$ 在 $\{x=\frac{1}{2}\}\cup\{x=1\}$ 取到最大值.

$|x|=\frac{1}{2}$: $h_\varepsilon(x) = u(x) - u(x_0) + \varepsilon\left(e^{-\frac{\alpha}{4}} - e^{-\alpha}\right)$

$$\leq \max_{|x|=\frac{1}{2}} u(x) - u(x_0) + \varepsilon\left(e^{-\frac{\alpha}{4}} - e^{-\alpha}\right).$$

使 ε 充分小. $h_\varepsilon(x)<0$.

$|x|=1$: $h_\varepsilon(x) = u(x) - u(x_0)$.

$$\leq 0$$

$\therefore$ $h_\varepsilon(x)$ 在$\underset{\frac{1}{2}<|x|<1}{\underline{内部}}$一定严格小于0

且 $h_\varepsilon(x_0)=0$ $\quad\therefore$ x_0点处 $h_\varepsilon(x)$ 达到最大值.

$\therefore \dfrac{\partial h_\varepsilon(x_0)}{\partial \vec{n}} \geq 0$. $\qquad \boxed{\dfrac{\partial v}{\partial \vec{n}} = \dfrac{x}{|x|}\cdot \nabla v}$

$\therefore \dfrac{\partial u}{\partial \vec{n}}(x_0) + \varepsilon \dfrac{\partial v}{\partial \vec{n}}(x_0) = \dfrac{\partial u}{\partial \vec{n}}(x_0) + \varepsilon\left(-2\alpha e^{-\alpha}\right)$

$\therefore \dfrac{\partial u}{\partial \vec{n}}(x_0) > \varepsilon 2\alpha e^{-\alpha} > 0$

step3: $\quad w(x) = u(x_0) - u(x) > 0 \qquad$ 则 $\Delta w = 0$ in B_1

由 Harnack 不等式

$$w(0) \leq C \inf_{B_{\frac{1}{2}}} w(x)$$

$$\Longleftrightarrow \quad u(x_0) - u(0) \leq C \inf_{B_{\frac{1}{2}}} \left(u(x_0) - u(x) \right)$$

$$= C\left(u(x_0) - \sup_{B_{\frac{1}{2}}} u(x) \right).$$

$$\Rightarrow \sup_{B_{\frac{1}{2}}} u(x) \leq u(x_0) - \frac{1}{C}\left(u(x_0) - u(0) \right)$$

$$\Rightarrow \sup_{B_{\frac{1}{2}}} u(x) - u(x_0) \leq -\frac{1}{C}\left(u(x_0) - u(0) \right) < 0$$

取 ε 充分小：$\varepsilon = \frac{\delta}{C}\left(u(x_0) - u(0) \right)$ $\quad \delta \to 0.$

$\therefore$ 由 step2 结束 $\quad \therefore$ 证完

称Ω满足内切球条件.

若$\forall x_0\in\partial\Omega$ 当Γ, 存在开球$B\subset\Omega$ s.t $x_0\in\partial B$

且球在x_0处与Ω相切. 相同法线

定理: 设Ω具有内切球性质, $u\in C(\bar{\Omega})$, u不恒为常数且$\Delta u=0$

它在$x_0\in\partial\Omega$取到$\bar{\Omega}$上最大值, 则$\frac{\partial u}{\partial n}(x_0)>0$.

严格大.

proof: $\because$ u不为常数 由极大值原理, u只在边界取最大值.

$\because\Omega$满足内切球性质 $\exists B\subset\Omega$ B在x_0处与Ω相切.

$\therefore u(x)<u(x_0)\quad\forall x\in B$

由Hopf引理. $\frac{\partial u}{\partial n}(x_0)>0$

定理. (第二边值解的唯一性).

若Ω满足内切球性质, 则第二边值问题的解在$u\in C(\bar{\Omega})\cap C^2(\Omega)$

(相差一个常数意义下) 唯一.

proof: $\begin{cases}\Delta u=0 & \text{in}\ \Omega.\\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega}=\varphi(x)\end{cases}$

只须证明: $\begin{cases}\Delta u=0 & \text{in}\ \Omega\\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega}=0\end{cases}$ 只有常数解.

若u不为常数 由极大值原理.

u的最大值只能在边界上取到

设 u在 $x_0 \in \partial\Omega$ 达到最大值.

由前定理: $\frac{\partial u}{\partial n}(x_0) > 0$ 与边界条件矛盾. □

定理.(最大模估计).

设 $u \in C^2(\Omega) \cap C(\bar{\Omega})$ 是 $\begin{cases} \Delta u = f & \text{in } \Omega. \\ u = \varphi(x) & \text{on } \partial\Omega. \end{cases}$ 的解,

则 $\exists C = C(n, \Omega)$ 使得 $\max\limits_{\bar{\Omega}} |u| \le \Phi + CF$

其中: $F = \max\limits_{\Omega} |f|$ $\Phi = \max\limits_{\partial\Omega} |\varphi(x)|$ ($\Omega \subset \mathbb{R}^n$ 有界区域)

proof: 不妨设 Ω 在区域 $0 < x_1 < d$ 上.

作辅助函数 (KEY: $\Delta u \ge 0$ $u|_{\partial\Omega} \le 0$)

$$v(x) = \Phi + (e^{ad} - e^{ax_1}) F, \quad a > 1$$

$$\text{则 } \Delta(u - v) = \Delta u + a^2 F \cdot e^{ax_1}$$
$$= f + a^2 F \cdot e^{ax_1}$$
$$\ge -F + a^2 e^{ax_1} \cdot F = (\underbrace{a^2 e^{ax_1} - 1}) F \ge 0$$

$x_1 > 0$.

$$(u - v)|_{\partial\Omega} = \underbrace{\varphi - \Phi}_{\le 0} - \underbrace{(e^{ad} - e^{ax_1}) F}_{\ge 0} \le 0 \quad \text{on } \partial\Omega.$$

由极大值原理: $u - v \le 0$ in Ω.

$$\Rightarrow u \le \Phi + (e^{ad} - e^{ax_1}) F.$$

$$\le \Phi + \underbrace{(e^{ad}-1)}_{\shortparallel\, C} F.$$

对 $-u$ 同样操作 $\Rightarrow$ $|u| \le \Phi + CF$. $\square$

注：利用最大模估计，可以得到：解的稳定性与唯一性.

能量法：（唯一性）

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\int_\Omega u\Delta u\, dx = 0 \Rightarrow \int_\Omega \nabla(u\nabla u) - |\nabla u|^2\, dx = 0$$

$$\Rightarrow \int_{\partial\Omega} u\cdot\nabla u\cdot\vec{n}\, dx = \int_\Omega |\nabla u|^2\, dx$$

$$\Rightarrow \nabla u = 0 \quad \text{in } \Omega \Rightarrow u \text{ 为常数}.$$

Summary

PDE 的求解方法与理论分析.

一、波动方程：1、求解方法：①分离变量法（有界区域）

②行波法（无界区域：直线，半直线，$\mathbb{R}^2$、$\mathbb{R}^3$）

2. 理论：①能量估计（唯一性、稳定性：

$\times \partial_t u$ ☆有限传播速度）

②有限传播速度与 Huygens 原理

二、热方程：1、求解方法：①分离变量法（有界区域）

② Fourier 变换（$\mathbb{R}^n$）

2. 理论：①极值原理（辅助函数构造）

②能量估计

$\times u$

三、位势方程：1、求解方法：①分离变量法（有界区域）

② Green 法

2. 理论：①调和函数的性质：

（平均值性质，Harnack 不等式）

②极值原理

（Hopf 引理，唯一性、稳定性）

局限性：1、求解方法依赖于特殊区域.

2. 正则性（可微、可积）太高，

└→ 随机过程，"点电荷".

3. 常系数标准方程.（变系数、低阶）.

4. 双曲型方程未提及.

实分析 → 泛函分析 → 偏微分方程

偏微分方程 ⇓ 非线性方程

泛函分析 → 非线性方程

高等实分析 → 调和分析 → 非线性方程

拓扑 → 非线性方程

变分 → 非线性方程

动力系统 → 非线性方程