量子物理作业

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第二章

1

B 的原子序数为 5，B^{4+} 为类氢离子，$Z = 5$

$$E_i = \lim_{n\to\infty} k(\frac{1}{1^2} - \frac{1}{n^2}) = k$$

$$E = Z^2 k(\frac{1}{1^2} - \frac{1}{2^2}) = \frac{3}{4} Z^2 E_i = 255eV$$

$$\lambda = \frac{hc}{E} = 4.862nm$$

2

设该类氢离子原子序数为 Z，三条谱线波长对应的原能级数分别为 n_1, n_2, n_3

$$\frac{1}{\lambda_1} - \frac{1}{\lambda_2} = Z^2 R(\frac{1}{n_1^2} - \frac{1}{n_2^2})$$
$$\frac{1}{\lambda_1} - \frac{1}{\lambda_3} = Z^2 R(\frac{1}{n_1^2} - \frac{1}{n_3^2})$$

$$\frac{\frac{1}{n_1^2}-\frac{1}{n_3^2}}{\frac{1}{\lambda_1}-\frac{1}{\lambda_3}}=\frac{\frac{1}{n_1^2}-\frac{1}{n_2^2}}{\frac{1}{\lambda_1}-\frac{1}{\lambda_2}}$$

$$\frac{1-\frac{n_1^2}{n_3^2}}{1-\frac{\lambda_1}{\lambda_3}}=\frac{1-\frac{n_1^2}{n_2^2}}{1-\frac{\lambda_1}{\lambda_2}}$$

$$\frac{1-\frac{n_1^2}{n_3^2}}{1-\frac{\lambda_1}{\lambda_3}}\div\frac{1-\frac{n_1^2}{n_2^2}}{1-\frac{\lambda_1}{\lambda_2}}-1=A-1=0$$

因为此式量纲为 1，适合进行验证

编写 python 程序在适当的范围内寻找 $minBias = min\{|A-1|\}$,解得 minBias=0.00333820467648982，其对应解为 n1,n2,n3=[7, 5, 4]

```
c=(1-99.2/121.5)/(1-99.2/108.5)
bias=2
for k in range(1,10):
    for j in range(k+1,10):
        for i in range(j+1,10):
            temp=abs((1-float(i)**2/float(k)**2)/c/((1-float(i)**2/float(j)**2))-1)
            if(temp<bias):
                bias=temp
                list=[i,j,k]

print("minBias=",end ='')
print(bias)
print("n1,n2,n3=",end ='')
print(list)
```

```
$ python3 2.py
minBias=0.00333820467648982
n1,n2,n3=[7, 5, 4]
```

故这三条谱线原本的能级分别是 7,5,4，可以继续预测该线系的其他谱线波长

$$\lambda = \left(\frac{1}{\lambda_1} - \frac{\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\left(\frac{1}{n_1^2} - \frac{1}{n^2}\right)}{\frac{1}{n_1^2} - \frac{1}{n_2^2}} \right)^{-1}$$

取值不同的 n 得到 |n|λ/nm| |–|–| |6|102.50| |8|97.17| |9|95.82| |10|94.88| |11|94.20| |12|93.68|

3

已知氢原子电离能为 13.6eV，故解方程

$$13.6(1 - \frac{1}{x^2}) = 12.2$$

解得 $x = 3.117$，故氢原子能级最多升为 3 级

则可能有三种谱线：

- $3 \to 2$，$\lambda = (R_H(\frac{1}{2^2} - \frac{1}{3^2}))^{-1} = 656.5nm$
- $3 \to 1$，$\lambda = (R_H(\frac{1}{1^2} - \frac{1}{3^2}))^{-1} = 102.57nm$
- $2 \to 1$，$\lambda = (R_H(\frac{1}{1^2} - \frac{1}{2^2}))^{-1} = 121.57nm$

4

波长最长即能量最小，即从第二级跃迁到第一级，能量为

$$E = -3.4 + 13.6 = 10.2eV$$

5

假设质子不动

$$L = n\hbar = mvr$$

$$F = \frac{e^2}{4\pi\varepsilon_0 r^2} = m\frac{v^2}{r}$$

$$\Rightarrow r = \frac{4\pi\varepsilon_0 n^2\hbar^2}{e^2 m}$$

替换 m 为 $\mu = \frac{m_p m_{\mu^-}}{m_p + m_{\mu^-}}$

$$r_n = \frac{4\pi\varepsilon_0 n^2\hbar^2(m_p + m_{\mu^-})}{e^2 m_p m_{\mu^-}} = 2.845n^2 \times 10^{-4} nm$$

$$E_n = -\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 r} = -\frac{1}{2}\frac{e^4 m_p m_{\mu^-}}{(4\pi\varepsilon_0)^2 n^2\hbar^2(m_p + m_{\mu^-})} = -\frac{e^4 m_p m_{\mu^-}}{8\varepsilon_0^2 n^2 h^2(m_p + m_{\mu^-})} = -\frac{2.531 \times 10^3}{n^2} eV$$

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考虑前后的四维矢量，原来静止时原子的四维矢量为 P^μ，跃迁后光子和原子的四维矢量分别为 P_ν^μ，P_0^μ，设能级四级和一级的能量差为 ΔE

$$P^\mu = (\frac{E_0 + \Delta E}{c}, 0, 0, 0)$$

$$P_\nu^\mu = (\frac{h\nu}{c}, \frac{h\nu}{c}, 0, 0)$$

$$P_0^\mu = (\frac{\sqrt{E_0^2 + p^2c^2}}{c}, -p, 0, 0)$$

则可得

$$P^\mu = P_\nu^\mu + P_0^\mu$$

$$P^\mu - P_\nu^\mu = P_0^\mu$$

两边取模，有

$$(P^{\mu}-P_{\nu}^{\mu})(P_{\mu}-(P_{\nu})_{\mu})=P_{0}^{\mu}(P_{0})_{\mu}$$

$$(\frac{E_0+\Delta E}{c})^2+0-2\frac{E_0+\Delta E}{c}\frac{h\nu}{c}=(\frac{E_0}{c})^2$$

解得 $\nu=\dfrac{\Delta E(2E_0+\Delta E)}{2h(E_0+\Delta E)}$

$\Delta E=13.6(1-\dfrac{1}{4^2})=12.75eV$
$E_0=m_pc^2+m_ec^2+E_c$，其中 $E_c=-13.6eV$。因为 $m_pc^2>>m_ec^2,|E_c|$，故可近似为 $E_0=m_pc^2$

解得 $\nu=3.08\times10^{15}Hz$

原子核反冲的动量 $p=\frac{h\nu}{c}$，假设反冲速度 $v<<c$，有 $v=\frac{h\nu}{m_pc}=4.074m/s$，假设成立

$$\lambda=\frac{c}{\nu}=97.33nm$$

若假设原子静止，$\nu_0=\dfrac{\Delta E}{h}$

$$\frac{\lambda_0}{\lambda}=\frac{\nu}{\nu_0}=\frac{2E_0+\Delta E}{2(E_0+\Delta E)}=1-\frac{\Delta E}{2(E_0+\Delta E)}$$

则

$$\frac{\Delta\lambda}{\lambda}=\frac{\Delta E}{2(E_0+\Delta E)}=6.7944\times10^{-9}$$

第三章

1

$$p=\frac{h}{\lambda}$$

$$E_k = E - E_0 = \sqrt{p^2c^2 + E_0^2} - E_0 = \sqrt{\left(\frac{h}{\lambda}\right)^2 c^2 + m_e^2c^4} - m_ec^2$$

简单估算有 $\frac{h}{\lambda}c << m_ec^2$，故原式可化简为

$$\sqrt{\left(\frac{h}{\lambda}\right)^2 c^2 + m_e^2c^4} - m_ec^2 \approx m_ec^2\left(1 + \frac{1}{2}\frac{\left(\frac{h}{\lambda}\right)^2 c^2}{m_e^2c^4}\right) - m_ec^2 = \frac{h^2}{2m_e\lambda^2} = 4.9723 \times 10^{-6}eV$$

2

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{\frac{(E+m_ec^2)^2}{c^2} - m_e^2c^2}} = \frac{hc}{\sqrt{E^2 + 2Em_ec^2}} = 5.355 \times 10^{-12}m$$

3

$$\rho = \psi\psi^* = (x+iy)e^{-(x^2+y^2)}(x-iy)e^{-(x^2+y^2)} = (x^2+y^2)e^{-2(x^2+y^2)}$$

归一化

$$\rho' = \frac{\rho}{\iint_{R^2}\rho dS} = \frac{\rho}{\pi/4}$$

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(1)

$$\int_{-\infty}^{+\infty}|\psi(x)|^2dx = 2\int_0^{+\infty}e^{-2x}dx = 1$$

故原函数已经归一化

(2)

$$\varphi(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\psi(x)e^{-\frac{i}{\hbar}px}dx=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}e^{-|x|}e^{-\frac{i}{\hbar}px}dx=\frac{1}{\sqrt{2\pi\hbar}}\left(\int_{0}^{+\infty}e^{-x}e^{-\frac{i}{\hbar}px}dx+\int_{-\infty}^{0}e^{x}e^{-\frac{i}{\hbar}px}\right.$$

$$=\frac{1}{\sqrt{2\pi\hbar}}\left(\int_{0}^{+\infty}e^{-x}e^{-\frac{i}{\hbar}px}dx+\int_{0}^{+\infty}e^{-x}e^{\frac{i}{\hbar}px}dx\right)=\frac{1}{\sqrt{2\pi\hbar}}\int_{0}^{+\infty}e^{-x}(e^{\frac{i}{\hbar}px}+e^{-\frac{i}{\hbar}px})dx$$

$$=\frac{2}{\sqrt{2\pi\hbar}}\int_{0}^{+\infty}cos\left(\frac{px}{\hbar}\right)e^{-x}dx=\sqrt{\frac{2}{\pi\hbar}}\frac{1}{1+\left(\frac{p}{\hbar}\right)^{2}}$$

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归一化，有

$$\int_{-\infty}^{+\infty}C^{2}e^{-\frac{2x^{2}}{\sigma^{2}}}dx=1\Rightarrow\frac{\sigma C^{2}}{\sqrt{2}}\sqrt{\pi}=1\Rightarrow C=\frac{1}{\sqrt{\sigma}}(\frac{2}{\pi})^{\frac{1}{4}}$$

$$\psi(x)=\frac{1}{\sqrt{\sigma}}(\frac{2}{\pi})^{\frac{1}{4}}e^{-\frac{x^{2}}{\sigma^{2}}}$$

由对称性

$$\langle x\rangle=\int_{-\infty}^{+\infty}\psi^{*}(x)x\psi(x)dx=0$$

$$\langle p\rangle=\int_{-\infty}^{+\infty}\psi^{*}(x)(-i\hbar\partial_{x})\psi(x)dx=-i\hbar\int_{-\infty}^{+\infty}\psi^{*}(x)(-\frac{2x}{\sigma^{2}})\psi(x)dx=0$$

$$\langle T\rangle=\frac{1}{2m}\int_{-\infty}^{+\infty}\psi^{*}(x)(-\hbar^{2}\partial_{xx}^{2})\psi(x)dx=-\frac{\hbar^{2}}{2m}\int_{-\infty}^{+\infty}\psi^{*}(x)\partial_{x}(-\frac{2x}{\sigma^{2}}\psi(x))dx$$

$$=\frac{\hbar^{2}}{m\sigma^{2}}\int_{-\infty}^{+\infty}\psi^{*}(x)(\psi(x)+x(-\frac{2x}{\sigma^{2}})\psi(x))dx$$

$$= \frac{\hbar^2}{m\sigma^2}(1 - \frac{2C^2}{\sigma^2}\int_{-\infty}^{+\infty} x^2 e^{-\frac{2x^2}{\sigma^2}} dx) = \frac{\hbar^2}{2m\sigma^2}$$

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归一化，有波函数为

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$\partial_x \psi(x) = \frac{1}{\sqrt{2}}\delta(x+1) - \frac{1}{\sqrt{2}}\delta(x-1)$$

$$\Delta x = \sqrt{\int_{-1}^{1} x^2 \frac{1}{2} dx - 0} = \frac{1}{\sqrt{3}}$$

$$\Delta p = \sqrt{\int_{-\infty}^{+\infty} \psi(-\hbar^2)\partial_x(\frac{1}{\sqrt{2}}\delta(x+1) - \frac{1}{\sqrt{2}}\delta(x-1))dx - 0}$$
$$= \sqrt{-\hbar^2 \int_{-\infty}^{+\infty} \psi d(\frac{1}{\sqrt{2}}\delta(x+1) - \frac{1}{\sqrt{2}}\delta(x-1))}$$
$$= \sqrt{-\hbar^2(\psi(\frac{1}{\sqrt{2}}\delta(x+1) - \frac{1}{\sqrt{2}}\delta(x-1))\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}(\frac{1}{\sqrt{2}}\delta(x+1) - \frac{1}{\sqrt{2}}\delta(x-1))d\psi)}$$

$$= \sqrt{\hbar^2 \int_{-\infty}^{+\infty}(\frac{1}{\sqrt{2}}\delta(x+1) - \frac{1}{\sqrt{2}}\delta(x-1))^2 dx}$$

$$= \frac{\hbar}{\sqrt{2}}\sqrt{\int_{-\infty}^{+\infty}(\delta(x+1) - \delta(x-1))^2 dx} = \sqrt{2}\hbar\delta(0) = +\infty$$

不确定性关系显然成立

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$$\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{3}{2m}\langle p_x^2\rangle$$

$$\langle p_x^2\rangle \geq \langle p_x\rangle^2 + (\frac{\hbar}{2\Delta x})^2$$

要使 $\langle p_x^2\rangle$ 最小，有等号成立且 $\langle p_x\rangle = 0$，Δx 最大

显然当粒子只在 $x = \pm L$ 出现时，$\Delta x = L$ 最大

$$\langle p_x^2\rangle_{min} = (\frac{\hbar}{2L})^2$$

$$\langle T\rangle_{min} = \frac{3}{2m}(\frac{\hbar}{2L})^2$$

(1)

$$\langle T\rangle_{min} = \frac{3}{2m}(\frac{\hbar}{2L})^2 = 4.578\times 10^{-19}J = 2.857eV$$

(2)

$$\langle T\rangle_{min} = \frac{3}{2m}(\frac{\hbar}{2L})^2 = 2.490\times 10^{-14}J = 1.554\times 10^{5}eV$$

(3)

$$\langle T\rangle_{min} = \frac{3}{2m}(\frac{\hbar}{2L})^2 = 4.170\times 10^{-49}J = 2.603\times 10^{-30}eV$$

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不妨考虑一维的情况

$$P=\int_{-\infty}^{+\infty}\psi(x)\psi^*(x)dx$$

$$\frac{dP}{dt}=\frac{d}{dt}\int_{-\infty}^{+\infty}\psi(x)\psi^*(x)dx=\int_{-\infty}^{+\infty}\left(\frac{\partial\psi(x)}{\partial t}\psi^*(x)+\psi(x)\frac{\partial\psi^*(x)}{\partial t}\right)dx$$

代入薛定谔方程

$$=\int_{-\infty}^{+\infty}-\frac{i}{\hbar}\left(\psi^*\hat{H}\psi-\psi\hat{H}\psi^*\right)dx=-\frac{i}{\hbar}\left((\psi,\hat{H}\psi)-(\psi^*,\hat{H}\psi^*)\right)=-\frac{i}{\hbar}\left(\langle H\rangle-\langle\bar{H}\rangle\right)$$

要使 $\frac{dP}{dt}=0$，则

$$\langle H\rangle=\overline{\langle H\rangle}$$

对任意本征态成立，故其本征值为实数

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由题，有

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x)+V(x)\psi(x)=E\psi(x)$$

取复共轭，有

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi^*(x)+V(x)\psi^*(x)=E\psi^*(x)$$

故

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(\psi(x)+\psi^*(x)\right)+V(x)\left(\psi(x)+\psi^*(x)\right)=E\left(\psi(x)+\psi^*(x)\right)$$

即 $\psi(x)+\psi^*(x)$ 也是该方程的解，且为实数解

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假设某一本征态本征值 E_n 满足 $E_n < V_{min} \Rightarrow T(x)=E_n-V(x)<0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x)+V(x)\psi(x)=E_n\psi(x)$$

$$\frac{\partial^2}{\partial x^2}\psi(x)=-\frac{2m}{\hbar^2}T(x)\psi(x)$$

由第九题结论，不妨取其实数解，可知实数解 $\psi(x)$ 在 R 上为严格单调增函数，故

$$\int_{-\infty}^{+\infty}\psi(x)\psi^*(x)dx=\int_{-\infty}^{+\infty}\psi^2(x)dx=+\infty$$

矛盾，故 $E_n \geq V_{min}$

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假设其定态能量小于等于 0

- $E<0$：

由第十题证得这种情况不成立

- $E=0$：

$$\frac{\partial^2}{\partial x^2}\psi(x)=0$$

$$\psi(x)=kx$$

又因为

$$\lim_{x\to\frac{D}{2}^-}\psi(x)=0$$

则

$$\psi(x)=0$$

显然不成立

综上，定态能量 E 一定大于 0

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n 为奇数时

$$\phi_n(x)=Acos(k_nx),A=\sqrt{\frac{2}{D}},k_n=\frac{n\pi}{D}$$

$$\langle x\rangle=\int_{-\frac{D}{2}}^{\frac{D}{2}}xA^2cos^2(k_nx)dx=0$$

$$\langle x^2\rangle=\int_{-\frac{D}{2}}^{\frac{D}{2}}x^2A^2cos^2(k_nx)dx=A^2D^3(\frac{1}{24}-\frac{1}{4n^2\pi^2})=D^2(\frac{1}{12}-\frac{1}{2n^2\pi^2})$$

$$\langle p\rangle=\int_{-\frac{D}{2}}^{\frac{D}{2}}Acos(k_nx)(-i\hbar\partial_x)(Acos(k_nx))dx=\int_{-\frac{D}{2}}^{\frac{D}{2}}i\hbar k_nA^2cos(k_nx)sin(k_nx)dx=0$$

$$\langle p^2\rangle=\int_{-\frac{D}{2}}^{\frac{D}{2}}Acos(k_nx)(-\hbar^2\partial_{xx}^2)(Acos(k_nx))dx=\int_{-\frac{D}{2}}^{\frac{D}{2}}\hbar^2A^2k_n^2cos^2(k_nx)dx=\hbar^2A^2k_n^2\frac{D}{2}=\frac{\pi^2\hbar^2}{D^2}n^2$$

n 为偶数时

$$\phi_n(x)=Asin(k_nx),A=\sqrt{\frac{2}{D}},k_n=\frac{n\pi}{D}$$

$$\langle x\rangle=\int_{-\frac{D}{2}}^{\frac{D}{2}}xA^2sin^2(k_nx)dx=0$$

$$\langle x^2\rangle = \int_{-\frac{D}{2}}^{\frac{D}{2}} x^2A^2sin^2(k_nx)dx = D^2(\frac{1}{12}-\frac{1}{2n^2\pi^2})$$

$$\langle p\rangle = \int_{-\frac{D}{2}}^{\frac{D}{2}} Asin(k_nx)(-i\hbar\partial_x)(Asin(k_nx))dx = 0$$

$$\langle p^2\rangle = \int_{-\frac{D}{2}}^{\frac{D}{2}} Asin(k_nx)(-\hbar^2\partial_{xx}^2)(Asin(k_nx))dx = \frac{\pi^2\hbar^2}{D^2}n^2$$

由上，得

$$\Delta x = D\sqrt{\frac{1}{12}-\frac{1}{2n^2\pi^2}}$$

$$\Delta p = \frac{\pi\hbar}{D}n$$

$$\Delta x\Delta p = \frac{\hbar}{2}\sqrt{\frac{n^2\pi^2}{3}-2} \geq \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3}-2} \approx 1.13\frac{\hbar}{2}$$

故基态 $n=1$ 时最接近不等式极限

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a.

$$\int_{-\infty}^{+\infty}(\phi_1(x)+i\phi_2(x))(\phi_1(x)-i\phi_2(x))dx = \int_{-\infty}^{+\infty}\phi_1^2(x)+\phi_2^2(x)dx = 2$$

故 $\psi(x,0)=\frac{1}{\sqrt{2}}(\phi_1(x)+i\phi_2(x))$

b.

$$\psi(x,t) = \frac{1}{\sqrt{2}}(\phi_1(x)e^{-\frac{iE_1t}{\hbar}} + i\phi_2(x)e^{-\frac{iE_2t}{\hbar}})$$

$$|\psi(x,t)|^2 = \psi(x,t)\psi^*(x,t) = \frac{1}{2}(\phi_1(x)e^{-\frac{iE_1t}{\hbar}} + i\phi_2(x)e^{-\frac{iE_2t}{\hbar}})(\phi_1(x)e^{\frac{iE_1t}{\hbar}} - i\phi_2(x)e^{\frac{iE_2t}{\hbar}})$$

$$= \frac{1}{2}(\phi_1^2(x)+\phi_2^2(x)+i\phi_1(x)\phi_2(x)(e^{\frac{i(E_1-E_2)t}{\hbar}} - e^{-\frac{i(E_1-E_2)t}{\hbar}})) = \frac{1}{2}(\phi_1^2(x)+\phi_2^2(x)-2\phi_1(x)\phi_2(x)sin\frac{(E_1-E_2)t}{\hbar})$$

$$= \frac{1}{D}(cos^2\frac{\pi x}{D} + sin^2\frac{2\pi x}{D} - 2cos\frac{\pi x}{D}sin\frac{2\pi x}{D}sin\frac{(E_1-E_2)t}{\hbar})$$

c.

$$\langle x\rangle = \int_{-\frac{D}{2}}^{\frac{D}{2}} x|\psi(x,t)|^2dx = -\frac{2}{D}sin\frac{(E_1-E_2)t}{\hbar}\int_{-\frac{D}{2}}^{\frac{D}{2}} xcos\frac{\pi x}{D}sin\frac{2\pi x}{D}dx = -\frac{16D}{9\pi^2}sin\frac{(E_1-E_2)t}{\hbar} = \frac{16D}{9\pi^2}sin\frac{3}{2}$$

$$\langle x^2\rangle = \int_{-\frac{D}{2}}^{\frac{D}{2}} x^2|\psi(x,t)|^2dx = \frac{1}{2}D^2(\frac{1}{12} - \frac{1}{2\pi^2} + \frac{1}{12} - \frac{1}{2\times 2^2\pi^2}) = D^2(\frac{1}{12} - \frac{5}{16\pi^2})$$

$$\langle p\rangle = \int_{-\frac{D}{2}}^{\frac{D}{2}} \left(\phi_1(x)e^{\frac{iE_1t}{\hbar}} - i\phi_2(x)e^{\frac{iE_2t}{\hbar}}\right)(-i\hbar\partial_x)\left(\phi_1(x)e^{-\frac{iE_1t}{\hbar}} + i\phi_2(x)e^{-\frac{iE_2t}{\hbar}}\right)dx = \frac{8\hbar}{3D}cos\frac{3\pi^2\hbar t}{2mD^2}$$

$$\langle p^2\rangle = \frac{1}{2}(\frac{\pi^2\hbar^2}{D^2} + \frac{\pi^2\hbar^2}{D^2}2^2) = \frac{5}{2}\frac{\pi^2\hbar^2}{D^2}$$

d.

$$\langle H\rangle = \int_{-\frac{D}{2}}^{\frac{D}{2}} \psi^*(x,t)\hat{H}\psi(x,t)dx = \int_{-\frac{D}{2}}^{\frac{D}{2}} \frac{1}{\sqrt{2}}(\phi_1^*(x)e^{\frac{iE_1t}{\hbar}} - i\phi_2^*(x)e^{\frac{iE_2t}{\hbar}})\hat{H}\frac{1}{\sqrt{2}}(\phi_1(x)e^{-\frac{iE_1t}{\hbar}} + i\phi_2(x)e^{-\frac{iE_2t}{\hbar}})dx$$

$$\int_{-\frac{D}{2}}^{\frac{D}{2}} \frac{1}{2}(\phi_1^*(x)e^{\frac{iE_1t}{\hbar}} - i\phi_2^*(x)e^{\frac{iE_2t}{\hbar}})(E_1\phi_1(x)e^{-\frac{iE_1t}{\hbar}} + iE_2\phi_2(x)e^{-\frac{iE_2t}{\hbar}})dx$$

$$=\frac{1}{2}(E_1+E_2+\int_{-\frac{D}{2}}^{\frac{D}{2}} iE_2\phi_1^*(x)\phi_2(x)e^{\frac{i(E_1-E_2)t}{\hbar}}-iE_1\phi_1(x)\phi_2^*(x)e^{-\frac{i(E_1-E_2)t}{\hbar}}dx)$$

$$=\frac{1}{2}(E_1+E_2)$$

得到能量的结果为 E_1 或 E_2，概率均为 $\frac{1}{2}$

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$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x)=E\phi(x)$$

得到波函数方程形式

$$\phi_n(x)=Asink_nx+Bcosk_nx,\ k_n=\sqrt{\frac{2mE}{\hbar^2}}$$

代入 $x=0,a,\phi(x)=0\Rightarrow B=0,E_n=\frac{n^2\pi^2\hbar^2}{2ma^2},k_n=\frac{n\pi}{a}$

由归一化

$$\phi_n(x)=\sqrt{\frac{2}{a}}sin\frac{n\pi x}{a}$$

$$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}$$

由三倍角公式

$$\psi(x,0)=\frac{A}{4}\sqrt{\frac{a}{2}}(3\sqrt{\frac{2}{a}}sin\frac{\pi x}{a}-\sqrt{\frac{2}{a}}sin\frac{3\pi x}{a})=\frac{A}{4}\sqrt{\frac{a}{2}}(3\phi_1-\phi_3)$$

由正交性易得

$$\int_{-\infty}^{+\infty}\psi(x,0)^*\psi(x,0)dx=\frac{A^2a}{32}(9+1)=\frac{5A^2a}{16}=1$$

$$A=\frac{4}{\sqrt{5a}}$$

$$\psi(x,t)=\frac{1}{\sqrt{10}}(3\phi_1 e^{-\frac{iE_1t}{\hbar}}-\phi_3 e^{-\frac{iE_3t}{\hbar}})$$

$$|\psi(x,t)|^2=\frac{1}{10}(9\phi_1^2+\phi_3^2-6\phi_1\phi_3 cos\frac{(E_3-E_1)t}{\hbar})$$

$$\langle x\rangle=\int_0^a x|\psi(x,t)|^2dx=\frac{a}{2}-\frac{3}{5}cos\frac{8\pi^2\hbar t}{2ma^2}\int_0^a x\phi_1\phi_3dx=\frac{a}{2}$$

同理计算得

$$\langle p\rangle=0$$

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体系初始

$$\phi(x)=\sqrt{\frac{2}{a}}sin\frac{\pi x}{a}$$

变化后，体系的本征波函数和能量如下

$$\phi_n(x)=\sqrt{\frac{1}{a}}sin\frac{n\pi x}{2a}$$

$$E_n=\frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

在 $[0,a]$，有

$$\psi(x,0)=\phi(x)=\sqrt{\frac{2}{a}}sin\frac{\pi x}{a}$$

$$\psi(x,0)=\sum_{n=1}^{+\infty}c_n\phi_n(x)=\sum_{i=1}^{+\infty}c_n\sqrt{\frac{1}{a}}sin\frac{n\pi x}{2a}$$

上式 $\psi(x,0)$ 其系数满足

$$c_n=\int_0^{2a}\phi_n^*(x)\psi(x,0)=\int_0^a\sqrt{\frac{1}{a}}sin\frac{n\pi x}{2a}\sqrt{\frac{2}{a}}sin\frac{\pi x}{a}dx=\frac{\sqrt{2}}{a}\int_0^a sin\frac{n\pi x}{2a}sin\frac{\pi x}{a}dx$$

$$=\begin{cases}\frac{\sqrt{2}}{2},\ n=2\\ 0,\ n=4,6,8,\cdots\\ (-1)^{\frac{n+1}{2}}\frac{4\sqrt{2}}{(n^2-4)\pi},\ n=1,3,5,7,\cdots\end{cases}$$

故每个本征态的概率为

$$P=\begin{cases}\frac{1}{2},\ n=2\\ 0,\ n=4,6,8,\cdots\\ \frac{32}{(n^2-4)^2\pi^2},\ n=1,3,5,7,\cdots\end{cases}$$

故最有可能观测到的是 $n=2$ 的能量，其概率为 $\frac{1}{2}$，能量为

$$E_2=\frac{4\pi^2\hbar^2}{2m(2a)^2}=\frac{\pi^2\hbar^2}{2ma^2}$$

因为能量不随时间变化，则求初态的能量平均值即可

$$\int_0^{2a}\psi^*(x,0)\hat{H}\psi(x,0)dx=\int_0^a\phi^*(x)\hat{H}\phi(x)dx=\frac{\pi^2\hbar^2}{2ma^2}$$

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$x<0$ 时，$\phi(x)=0$

$x>0$ 时，波函数的解为 $\phi_n(x)=N_n e^{-\frac{1}{2}(\frac{x}{l_T})^2}H_n(\frac{x}{l_T})$

由连续性

$$\lim_{x\to 0^+}\phi_n(x)=0$$

$$\Rightarrow n=2k-1,k\geq 1$$

故 n 只能取为奇数，则其波函数为

$$\phi_n=\begin{cases}0 & x<0\\ N_n e^{-\frac{1}{2}(\frac{x}{l_T})^2}H_n(\frac{x}{l_T}) & x>0\end{cases}$$

对应的能量为

$$E_n=(n+\frac{1}{2})\hbar\omega$$

上面的 n 均为奇数

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若波函数为偶函数

$$\phi(x)=\begin{cases}A\cos kx-B\sin kx, & -a<x<0\\ A\cos kx+B\sin kx, & 0<x<a\end{cases}$$

连续性自然满足，令 $\phi(a)=\phi(-a)=0$，解得

$$A\cos ka=B\sin ka$$

结合 $x=0$ 的波函数跃变

$$\phi'(0^+) - \phi'(0^-) = -\frac{2m}{\hbar^2}\gamma\phi(0)$$

$$Bk - (-Bk) = -\frac{2m}{\hbar^2}\gamma A \Rightarrow B = -\frac{m\gamma}{k\hbar^2}A$$

则

$$A(\cos ka + \frac{m\gamma}{k\hbar^2}\sin ka) = 0$$

$$\Rightarrow \tan ka = -\frac{k\hbar^2}{m\gamma}$$

该超越方程有多解，每个解为一个能级

$$E_n = \frac{\hbar^2 k_n^2}{2m}$$

若波函数为奇函数

$$\phi(x) = \begin{cases} -A\cos kx + B\sin kx, & -a < x < 0 \\ A\cos kx + B\sin kx, & 0 < x < a \end{cases}$$

$$\phi(0) = 0$$

$$\Rightarrow A = 0$$

$$\phi(x) = B\sin kx, -a < x < a$$

结合 $x = 0$ 的波函数跃变

$$\phi'(0^+) - \phi'(0^-) = -\frac{2m}{\hbar^2}\gamma\phi(0)$$

上式显然成立，则根据无限方势阱的能级，其为

$$E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

因为奇函数在奇点处 $\phi(x) = 0$，故该点没有导函数的突变，因此奇函数不受 $\delta-$ 函数势垒影响

18

反证法，假设有两个束缚态本征波函数 ϕ_1, ϕ_2 具有相同的本征能量 E

设 $\omega^2(x) = \dfrac{2m}{\hbar^2}(V(x) - E) > 0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

$$\frac{\partial^2}{\partial x^2}\phi(x) = \omega^2(x)\phi(x)$$

$$\Rightarrow \frac{\phi_1''}{\phi_1} = \frac{\phi_2''}{\phi_2}$$

$$\Rightarrow \phi_1''\phi_2 = \phi_2''\phi_1$$

$$\Rightarrow (\phi_1'\phi_2 - \phi_2'\phi_1)' = 0$$

$$\Rightarrow \phi_1'\phi_2 - \phi_2'\phi_1 = C$$

无穷远处波函数为 0

$$C = \lim_{x\to+\infty}(\phi_1'\phi_2 - \phi_2'\phi_1) = 0$$

$$\Rightarrow \frac{\phi_1'}{\phi_1} = \frac{\phi_2'}{\phi_2}$$

$$\Rightarrow \phi_1 = C'\phi_2$$

故 ϕ_1, ϕ_2 并不简并

第四章

2

设能级为 1 时能量为 E_0，且 $n=1, l=0, m=0$，则

$$T = E_0 - V = E_0 + \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} < 0$$

$$r > -\frac{e^2}{4\pi\varepsilon_0 E_0} = r_0$$

$$P = \int_{V_0} \phi^*_{nlm}\phi_{nlm} d^3\vec{r} = \int_0^{2\pi} \Phi^*_m \Phi_m d\varphi \int_0^{\pi} \Theta^*_{lm}\Theta_{lm} \sin\theta d\theta \int_{r_0}^{+\infty} R^*_{nl} R_{nl} r^2 dr$$

$$= \int_{r_0}^{+\infty} R^*_{nl} R_{nl} r^2 dr = \int_{-\frac{e^2}{4\pi\varepsilon_0 E_0}}^{+\infty} R^*_{nl} R_{nl} r^2 dr = \int_{-\frac{e^2}{4\pi\varepsilon_0 E_0}}^{+\infty} r^2 \frac{4}{a_0^3} e^{-\frac{2r}{a_0}} dr = \frac{1}{2}\int_{-\frac{e^2}{2\pi\varepsilon_0 E_0 a_0}}^{+\infty} x^2 e^{-x} dx$$

$$= \frac{1}{2}\int_4^{+\infty} x^2 e^{-x} dx = \frac{1}{2}(-e^{-x}(x^2+2x+2))|_4^{+\infty} = 13e^{-4}$$

4

$$\psi = \frac{\alpha^{\frac{5}{2}}}{\sqrt{\pi}} z e^{-\alpha(x^2+y^2+z^2)} = \frac{\alpha^{\frac{5}{2}}}{\sqrt{\pi}} r\cos\theta e^{-\alpha r^2} = f(r)\cos\theta = g(r) Y_{1,0}(\theta,\phi)$$

则由波函数形式可知该粒子处在角动量本征态上，且 $l=1, m=0$

对应 L^2 和 L_z 的本征值为 $2\hbar^2$ 和 0

6

在 L_z 的本征态下，有

$$\hat{L}_z\psi = m\hbar\psi$$

由对易关系有

$$[\hat{L}_y, \hat{L}_z] = \hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y = i\hbar\hat{L}_x$$
$$[\hat{L}_z, \hat{L}_x] = \hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z = i\hbar\hat{L}_y$$

则

$$i\hbar\langle L_x\rangle = \int_V \psi^*(\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y)\psi d^3\vec{r} = m\hbar\langle L_y\rangle - \int_V \psi^*\hat{L}_z\hat{L}_y\psi d^3\vec{r}$$

由厄米算符性质，有

$$\int_V \psi^*\hat{L}_z(\hat{L}_y\psi)d^3\vec{r} = \int_V (\hat{L}_y\psi)(\hat{L}_z\psi)^* d^3\vec{r} = m\hbar\langle L_y\rangle$$

故

$$\langle L_x\rangle = 0$$

同理

$$\langle L_y\rangle = 0$$

设沿与 z 方向成 θ 角方向上的方向矢量为 $\vec{n}$ 分量，该方向角动量算符为

$$\hat{L}_n = (\vec{n}\cdot\vec{i})\hat{L}_x + (\vec{n}\cdot\vec{j})\hat{L}_y + (\vec{n}\cdot\vec{k})\hat{L}_z$$

$$\langle L_n\rangle = (\vec{n}\cdot\vec{k})\langle L_z\rangle = m\hbar\cos\theta$$

9

自寻查询**选择定则**

Ch5

1

$|\alpha\rangle = |0\rangle - 2|1\rangle + 2i|2\rangle$

$|\beta\rangle = i|0\rangle - 3|2\rangle$

a

$\langle\alpha| = \langle 0| - 2\langle 1| - 2i\langle 2|$

$\langle\beta| = -i\langle 0| - 3\langle 2|$

b

$$\langle\alpha|\beta\rangle = (\langle 0| - 2\langle 1| - 2i\langle 2|)(i|0\rangle - 3|2\rangle) = i + 6i = 7i$$

$$\langle\beta|\alpha\rangle = (-i\langle 0| - 3\langle 2|)(|0\rangle - 2|1\rangle + 2i|2\rangle) = -i - 6i = -7i$$

$$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$$

c

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} 1 \\ -2 \\ 2i \end{pmatrix} \begin{pmatrix} -i & 0 & -3 \end{pmatrix} = \begin{pmatrix} -i & 0 & -3 \\ 2i & 0 & 6 \\ 2 & 0 & -6i \end{pmatrix}$$

$$|\beta\rangle\langle\alpha| = \begin{pmatrix} i \\ 0 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -2i \end{pmatrix} = \begin{pmatrix} i & -2i & 2 \\ 0 & 0 & 0 \\ -3 & 6 & 6i \end{pmatrix}$$

$$(|\alpha\rangle\langle\beta|)^\dagger = (|\alpha\rangle\langle\beta|)^H = \begin{pmatrix} i & -2i & 2 \\ 0 & 0 & 0 \\ -3 & 6 & 6i \end{pmatrix} = |\beta\rangle\langle\alpha|$$

2

a

$$H = E \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

b

$$H\psi = E'\psi$$

设 $E' = \lambda E$，有

$$\det \begin{pmatrix} \lambda - 1 & -i \\ i & \lambda + 1 \end{pmatrix} = 0 \Rightarrow \lambda^2 = 2 \Rightarrow \lambda = \pm\sqrt{2}$$

特征向量分别为

$$|\psi_1\rangle = \begin{pmatrix} i(\sqrt{2}+1) \\ 1 \end{pmatrix}, E_1 = \sqrt{2}E$$

$$|\psi_2\rangle = \begin{pmatrix} i(\sqrt{2}-1) \\ -1 \end{pmatrix}, E_2 = -\sqrt{2}E$$

c

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{i}{2\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$$

$$|\psi(t)\rangle = -\frac{i}{2\sqrt{2}}(|\psi_1\rangle e^{-\frac{iE_1 t}{\hbar}} + |\psi_2\rangle e^{-\frac{iE_2 t}{\hbar}}) = -\frac{i}{2\sqrt{2}}\left(\begin{pmatrix} i(\sqrt{2}+1) \\ 1 \end{pmatrix} e^{-\frac{i\sqrt{2}Et}{\hbar}} + \begin{pmatrix} i(\sqrt{2}-1) \\ -1 \end{pmatrix} e^{\frac{i\sqrt{2}Et}{\hbar}}\right)$$

$$= -\frac{i}{2\sqrt{2}}\left(\begin{pmatrix} i\sqrt{2} \\ 0 \end{pmatrix}(e^{\frac{i\sqrt{2}Et}{\hbar}} + e^{-\frac{i\sqrt{2}Et}{\hbar}}) - \begin{pmatrix} i \\ 1 \end{pmatrix}(e^{\frac{i\sqrt{2}Et}{\hbar}} - e^{-\frac{i\sqrt{2}Et}{\hbar}})\right)$$

$$= \begin{pmatrix} \cos\frac{\sqrt{2}Et}{\hbar} - \frac{i}{\sqrt{2}}\sin\frac{\sqrt{2}Et}{\hbar} \\ -\frac{1}{\sqrt{2}}\sin\frac{\sqrt{2}Et}{\hbar} \end{pmatrix}$$

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$AB = (AB)^\dagger = B^\dagger A^\dagger = BA$

即 $[A,B]=0$ 时，AB 为厄米算符

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$$[V,x] = Vx - xV = 0$$

$$[H,x] = \frac{1}{2\mu}[p^2,x] = \frac{1}{2\mu}(p[p,x] + [p,x]p) = -\frac{i\hbar}{\mu}p$$

$$[[H,x],x] = [-\frac{i\hbar}{\mu}p, x] = -\frac{\hbar^2}{\mu}$$

同时

$$[[H,x],x] = [Hx - xH, x] = Hx^2 + x^2H - 2xHx$$

$$\langle m|[[H,x],x]|m\rangle = \langle m|Hx^2|m\rangle + \langle m|x^2H|m\rangle - 2\langle m|xHx|m\rangle = 2E_m\langle m|x^2|m\rangle - 2\langle m|xHx|m\rangle$$

$$= \sum_n 2E_m\langle m|x|n\rangle\langle n|x|m\rangle - \sum_n 2\langle m|xH|n\rangle\langle n|x|m\rangle = \sum_n 2(E_m - E_n)\langle m|x|n\rangle\langle n|x|m\rangle$$

$$= \sum_n 2(E_m - E_n)\langle n|x|m\rangle^*\langle n|x|m\rangle = -2\sum_n (E_n - E_m)|\langle n|x|m\rangle|^2$$

且

$$\langle m|[[H,x],x]|m\rangle = -\frac{\hbar^2}{\mu}\langle m|m\rangle = -\frac{\hbar^2}{\mu}$$

$$\Rightarrow \sum_n (E_n - E_m)|\langle n|x|m\rangle|^2 = \frac{\hbar^2}{2\mu}$$

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a

$$e^{iA} = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}$$

$$(e^{iA})^\dagger = (\sum_{n=0}^{\infty} \frac{(iA)^n}{n!})^\dagger = \sum_{n=0}^{\infty} \frac{(-iA^\dagger)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-iA)^n}{n!} = e^{-iA}$$

$$e^{iA}(e^{iA})^\dagger = e^{iA}e^{-iA} = I$$

b

采用爱因斯坦求和约定，令 $V = \delta_{ij}|u_i\rangle\langle v_j|$

则有

$$V|v_n\rangle = \delta_{ij}|u_i\rangle\langle v_j|v_n\rangle = \delta_{ij}|u_i\rangle\delta_{jn} = |u_n\rangle$$

下证其为幺正算符

$$V^\dagger = \delta_{ij}(|u_i\rangle\langle v_j|)^\dagger = \delta_{ij}|v_j\rangle\langle u_i|$$

$$VV^\dagger = \delta_{ij}|u_i\rangle\langle v_j|\delta_{mn}|v_n\rangle\langle u_m| = \delta_{ij}\delta_{mn}\delta_{jn}|u_i\rangle\langle u_m| = |u_m\rangle\langle u_m| = I$$

$V^\dagger V$ 同理，故 $V = \delta_{ij}|u_i\rangle\langle v_j|$ 符合要求

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归纳法，当 $n = 1$ 时，A^1 为厄米算符

假设当 $n = k - 1$ 时，A^{k-1} 为厄米算符，且有

$$[A^{k-1}, A] = 0$$

则 $A^k = A^{k-1}A$ 也为厄米算符，故 $\forall n \in N, A^n$ 均为厄米算符

则 $\sum_n c_n A^n$ 也是厄米的

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$$A + iB = \frac{1}{2}(U^\dagger + U) + i\frac{i}{2}(U^\dagger - U) = U$$

且 $U^\dagger$ 存在，则 A, B 均存在，分解成立

$$A^2+B^2=(\frac{1}{2}(U^\dagger+U))^2+(\frac{i}{2}(U^\dagger-U))^2=\frac{1}{4}(U^{\dagger 2}+U^2+2I-(U^{\dagger 2}+U^2-2I))=I$$

$$[A,B]=AB-BA=(\frac{1}{2}(U^\dagger+U))(\frac{i}{2}(U^\dagger-U))-(\frac{i}{2}(U^\dagger-U))(\frac{1}{2}(U^\dagger+U))$$
$$=\frac{i}{4}(U^{\dagger 2}-U^2-(U^{\dagger 2}-U^2))=0$$

17

先证一个引理，若对任意的算符 A 和 $\alpha,\beta,[A_\alpha,A_\beta]=0$，则 $\epsilon_{\alpha\beta\gamma}A_\beta A_\gamma=0$

$$\epsilon_{\alpha\beta\gamma}A_\beta A_\gamma=\frac{1}{2}(\epsilon_{\alpha\beta\gamma}A_\beta A_\gamma+\epsilon_{\alpha\beta\gamma}A_\beta A_\gamma)=\frac{1}{2}(\epsilon_{\alpha\beta\gamma}A_\beta A_\gamma+\epsilon_{\alpha\beta\gamma}A_\gamma A_\beta)$$

$$=\frac{1}{2}(\epsilon_{\alpha\beta\gamma}A_\beta A_\gamma-\epsilon_{\alpha\gamma\beta}A_\gamma A_\beta)=0$$

首先角动量有

$$L_\alpha=\epsilon_{\alpha\beta\gamma}r_\beta p_\gamma$$

则

$$\vec{r}\cdot\vec{L}=\delta_{\alpha\beta}r_\alpha L_\beta=\delta_{\alpha\beta}\epsilon_{\beta\gamma\lambda}r_\alpha r_\gamma p_\lambda=\epsilon_{\alpha\gamma\lambda}r_\alpha r_\gamma p_\lambda=0$$

$$\vec{L}\cdot\vec{r}=\delta_{\alpha\beta}L_\beta r_\alpha=\delta_{\alpha\beta}\epsilon_{\beta\gamma\lambda}r_\gamma p_\lambda r_\alpha=\epsilon_{\alpha\gamma\lambda}r_\gamma p_\lambda r_\alpha=0$$

$$\vec{p}\cdot\vec{L}=\delta_{\alpha\lambda}p_\lambda\epsilon_{\alpha\beta\gamma}r_\beta p_\gamma=\epsilon_{\alpha\beta\gamma}p_\alpha r_\beta p_\gamma=\epsilon_{\alpha\beta\gamma}p_\alpha(p_\gamma r_\beta+i\hbar\delta_{\gamma\beta})=\epsilon_{\alpha\beta\gamma}p_\alpha p_\gamma r_\beta+i\hbar\epsilon_{\alpha\beta\gamma}\delta_{\gamma\beta}p_\alpha$$

$$= \epsilon_{\alpha\beta\gamma}p_\alpha p_\gamma r_\beta + i\hbar\epsilon_{\alpha\beta\gamma}\delta_{\gamma\beta}p_\alpha = 0 + i\hbar\epsilon_{\alpha\beta\beta}p_\alpha = 0$$

$$\vec{L}\cdot\vec{p} = \delta_{\alpha\lambda}\epsilon_{\alpha\beta\gamma}r_\beta p_\gamma p_\lambda = r_\beta(\epsilon_{\alpha\beta\gamma}p_\gamma p_\alpha) = 0$$

$$(\vec{L}\times\vec{p})_\alpha = \epsilon_{\alpha\beta\gamma}L_\beta p_\gamma = \epsilon_{\gamma\alpha\beta}\epsilon_{\beta lm}r_l p_m p_\gamma = (\delta_{\gamma l}\delta_{\alpha m} - \delta_{\gamma m}\delta_{\alpha l})r_l p_m p_\gamma$$

$$(\vec{p}\times\vec{L})_\alpha = \epsilon_{\alpha\beta\gamma}p_\beta L_\gamma = \epsilon_{\alpha\beta\gamma}\epsilon_{\gamma lm}p_\beta r_l p_m = (\delta_{\alpha l}\delta_{\beta m} - \delta_{\alpha m}\delta_{\beta l})p_\beta r_l p_m$$

则有

$$(\vec{L}\times\vec{p})\cdot\vec{p} = \delta_{\alpha\beta}(\vec{L}\times\vec{p})_\alpha p_\beta = \delta_{\alpha\beta}(\delta_{\gamma l}\delta_{\alpha m} - \delta_{\gamma m}\delta_{\alpha l})r_l p_m p_\gamma p_\beta = (\delta_{\gamma l}\delta_{\alpha m} - \delta_{\gamma m}\delta_{\alpha l})r_l p_m p_\gamma p_\alpha$$

$$= r_l p_m p_l p_m - r_l p_m p_m p_l = r_l p_m(p_l p_m - p_m p_l) = r_l p_m[p_l, p_m] = 0$$

$$\vec{p}\cdot(\vec{p}\times\vec{L}) = \delta_{\alpha\beta}p_\beta(\vec{L}\times\vec{p})_\alpha = \delta_{\alpha\beta}p_\beta(\delta_{\alpha l}\delta_{\beta m} - \delta_{\alpha m}\delta_{\beta l})p_\beta r_l p_m = (\delta_{\alpha l}\delta_{\beta m} - \delta_{\alpha m}\delta_{\beta l})p_\alpha p_\beta r_l p_m$$

$$= p_l p_m r_l p_m - p_m p_l r_l p_m = [p_l, p_m]r_l p_m = 0$$

$$(\vec{p}\times\vec{L})\cdot\vec{p} = \delta_{\alpha n}(\vec{p}\times\vec{L})_\alpha p_n = \delta_{\alpha n}\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma lm}p_\beta r_l p_m p_n = \epsilon_{\alpha\beta\gamma}\epsilon_{\gamma lm}p_\beta r_l p_m p_\alpha = \epsilon_{\alpha\beta\gamma}\epsilon_{\gamma lm}(r_l p_\beta - i\hbar\delta_{l\beta})p_m p_\alpha$$

$$= \epsilon_{lm\gamma}\epsilon_{\gamma\alpha\beta}r_l p_m p_\alpha p_\beta - i\hbar\delta_{l\beta}\epsilon_{\alpha\beta\gamma}\epsilon_{\gamma lm}p_m p_\alpha = \delta_{n\gamma}\epsilon_{lmn}r_l p_m(\epsilon_{\gamma\alpha\beta}p_\alpha p_\beta) + i\hbar\epsilon_{\alpha l\gamma}\epsilon_{ml\gamma}p_m p_\alpha$$

$$= 0 + 2i\hbar\delta_{\alpha m} p_m p_\alpha = 2i\hbar\vec{p}^2$$

$$\vec{p} \cdot (\vec{L} \times \vec{p}) = \delta_{n\alpha}\epsilon_{\gamma\alpha\beta}\epsilon_{\beta lm} p_n r_l p_m p_\gamma = \epsilon_{\gamma\alpha\beta}\epsilon_{\beta lm} p_\alpha r_l p_m p_\gamma$$

进行 (α, β, γ) 的轮换，有

$$\vec{p} \cdot (\vec{L} \times \vec{p}) = \epsilon_{\alpha\beta\gamma}\epsilon_{\gamma lm} p_\beta r_l p_m p_\alpha = (\vec{p} \times \vec{L}) \cdot \vec{p} = 2i\hbar\vec{p}^2$$

$$(\vec{L} \times \vec{p})_\alpha + (\vec{p} \times \vec{L})_\alpha = \epsilon_{\gamma\alpha\beta}\epsilon_{\beta lm} r_l p_m p_\gamma - \epsilon_{\gamma\alpha\beta}\epsilon_{\beta lm} p_\gamma r_l p_m = \epsilon_{\gamma\alpha\beta}\epsilon_{\beta lm}(r_l p_m p_\gamma - p_\gamma r_l p_m)$$

$$= \epsilon_{\gamma\alpha\beta}\epsilon_{\beta lm}(r_l p_m p_\gamma - (r_l p_\gamma - i\hbar\delta_{l\gamma}) p_m) = i\hbar\delta_{l\gamma}\epsilon_{\gamma\alpha\beta}\epsilon_{\beta lm} p_m = i\hbar\epsilon_{\gamma\alpha\beta}\epsilon_{\beta\gamma m} p_m$$

$$= 2i\hbar\delta_{\alpha m} p_m = 2i\hbar p_\alpha$$

$$\Rightarrow \vec{L} \times \vec{p} + \vec{p} \times \vec{L} = 2i\hbar\vec{p}$$

$$[L^2, p_\alpha] = [L_i L_i, p_\alpha] = L_i[\epsilon_{i\beta\gamma} r_\beta p_\gamma, p_\alpha] + [\epsilon_{i\beta\gamma} r_\beta p_\gamma, p_\alpha] L_i$$

$$= \epsilon_{i\beta\gamma} L_i [r_\beta, p_\alpha] p_\gamma + \epsilon_{i\beta\gamma}[r_\beta, p_\alpha] p_\gamma L_i = i\hbar\delta_{\alpha\beta}\epsilon_{i\beta\gamma}(L_i p_\gamma + p_\gamma L_i)$$

$$= i\hbar\epsilon_{i\alpha\gamma}(L_i p_\gamma + p_\gamma L_i) = i\hbar(\vec{p} \times \vec{L} - \vec{L} \times \vec{p})_\alpha$$

$$\Rightarrow [L^2, \vec{p}] = i\hbar(\vec{p} \times \vec{L} - \vec{L} \times \vec{p})$$