波动方程：

$$\begin{cases}\partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = g(x) \qquad \partial_t u\big|_{t=0} = h(x)\end{cases}$$

给定两个初值.

$$(\partial_t + \partial_x)(\partial_t - \partial_x)\, u = \partial_t^2 u - \partial_x^2 u.$$

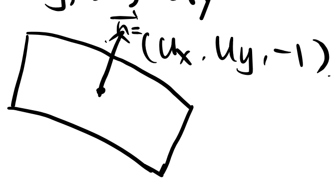
考虑齐次项 $f=0$: $(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0$

一、一阶拟线性方程：

令 $v = (\partial_t - \partial_x)u \Rightarrow$ 解 $(\partial_t + \partial_x)v = 0$

1、考虑 $a(x,y,u)\cdot u_x + b(x,y,u)\cdot u_y - c(x,y,u) = 0$

$z = u(x,y)$.



$$\Rightarrow (a, b, c)\cdot\begin{pmatrix}u_x\\u_y\\-1\end{pmatrix} = 0$$

找曲面使法向与(a,b,c)处处垂直

设 $\gamma(s) = \big(x(s), y(s), z(s)\big)$ 是曲面的一条参数曲线

$$\frac{dx}{ds} = a(x,y,z) \qquad \frac{dy}{dt} = b(x,y,z) \qquad \frac{dz}{dt} = c(x,y,z)$$

令 $U(s) = u(x(s), y(s)) - z$

$$\frac{dU}{ds} = \frac{\partial u}{\partial x}\cdot\dot{x} + \frac{\partial u}{\partial y}\cdot\dot{y} - \dot{z}$$

$$= a\cdot\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} - c = 0$$

$\therefore U(s) = U(s_0)$

即曲线若有一点在曲线上，则一直在曲线上.

若 a, b, c 关于 x, y, z C^1 的，则由 Picard 存在唯一性：

$[s_0-\delta, s_0+\delta)$ 存在且唯一.

例：$\begin{cases} u_t + a u_x = 0 & x\in\mathbb{R}\ \ t>0. \\ u(x,0) = g(x) \end{cases}$

solution: $\dfrac{dt(s)}{ds} = 1 \qquad \dfrac{dx(s)}{ds} = a \qquad \dfrac{dz(s)}{ds} = 0$

由初值 $\gamma(s)$ 经过 $(0, s_0, g(s_0))$

$$\Rightarrow \begin{array}{l} t(s) = s - s_0 \qquad x(s) = a(s-s_0) + s_0 \\ z(s) \equiv g(s_0) \end{array}$$

$$\therefore g(s_0) = u\Big(\underbrace{a(s-s_0)+s_0}_{\substack{\parallel \\ x}}, \underbrace{s-s_0}_{\substack{\parallel \\ t}}\Big).$$

$$\Rightarrow \boxed{u(x,t) = g(x-at)}$$

例：$\begin{cases} u_t + a u_x = f(x,t). \\ u(x,0) = 0 \end{cases}$

solution: $\dfrac{dt(s)}{ds} = 1 \qquad \dfrac{dx(s)}{ds} = a \qquad \dfrac{dz(s)}{ds} = f(x,t)$

$t(s_0) = 0 \qquad x(s_0) = s_0, \qquad z(s_0) = 0$

$\therefore t(s) = s - s_0 \qquad x(s) = a(s-s_0) + s$

$$z(s) = z(s_0) + \int_{s_0}^{s} f(x(s), t(s))\,ds.$$

$$\Downarrow$$

$$z(s)=\int_{s_0}^{s} f(x(\sigma),t(\sigma))\,ds$$

$$=\int_{s_0}^{s} f(a(\tilde{\sigma}-s_0)+\tilde{\sigma},\ \tilde{\sigma}-s_0)\,d\tilde{\sigma}.$$

$$\overset{\sigma=\tilde{\sigma}-s_0}{=\!=\!=}\int_0^t f(a\sigma+x-at,\ \sigma)\,d\sigma.$$

$\boxed{\text{传输方程}}$

$$\therefore u(x,t)=\int_0^t f(x-a(t-\sigma),\sigma)\,d\sigma$$

二. 一维波动方程初值问题.

$$\begin{cases}\partial_t^2 u-\partial_x^2 u=f(x,t).\\ u(x,0)=\varphi(x) \quad \partial_t u(x,0)=\psi(x).\end{cases} \qquad (WE)$$

$$\begin{cases}\partial_t^2 u_1-\partial_x^2 u_1=0\\ u_1(x,0)=\varphi(x) \quad \partial_t u_1(x,0)=\psi(x)\end{cases} (*) \qquad \begin{cases}\partial_t^2 u_2-\partial_x^2 u_2=f(x,t)\\ u_2(x,0)=0 \quad \partial_t u_2(x,0)=0\end{cases} (**)$$

叠加原理.

claim $U=U_1+U_2$ is solution of WE

proof: $\partial_t^2 U-\partial_x^2 U=\partial_t^2 U_1-\partial_x^2 U_1-\partial_t^2 U_2+\partial_x^2 U_2.$
$=f(x,t).$

$$U(x,0)=U_1(x,0)+U_2(x,0)=\varphi(x).$$

$$\partial_t U(x,0)=\partial_t U_1(x,0)+\partial_t U_2(x,0)=\psi(x).$$

Solution: 令 $V(x,t) = (\partial_t - \partial_x)U$

则 $\partial_t V + \partial_x V = 0$

$$V(x,0) = \partial_t U(x,0) - \partial_x U(x,0) = \psi - \varphi'$$

代入传输方程公式

$$\Rightarrow V = (\psi - \varphi')(x-t)$$

而 $\begin{cases} \partial_t U - \partial_x U = V \\ U(x,0) = \varphi(x) \end{cases} \xrightarrow{\text{代入}} U = \varphi(x+t) + \int_0^t V(x+t-\sigma, \sigma)\, d\sigma$

$$= \varphi(x+t) + \int_0^t \psi(\underbrace{x+t-\sigma-\sigma}_{=y}) - \varphi'(\underbrace{x+t-\sigma-\sigma}_{=y})\, d\sigma$$

lemma: 传输方程也可以用叠加原理.

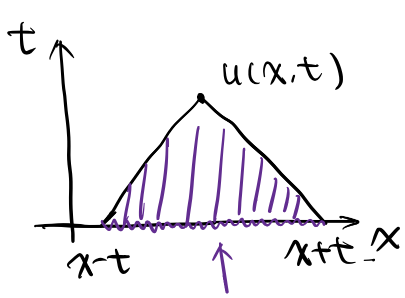
$$\begin{cases} \partial_t u + a\,\partial_x u = f(x,t) \\ u(x,0) = g(x) \end{cases}$$

$$u = g(x-at) + \int_0^t f(x - a(t-\sigma), \sigma)\, d\sigma$$

$$= \varphi(x+t) + \frac{1}{2}\int_{x-t}^{x+t} \psi(y) - \varphi'(y)\, dy$$

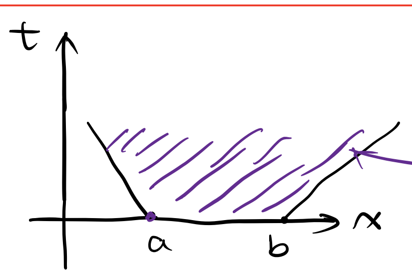
$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[\varphi(x+t) + \varphi(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\, dy}$$

D'Alembert 公式



U只与[x-t, x+t]有关

称为依赖区间



影响区域

若 $[a,b] = \operatorname{supp} f$.

则随t演化: $\operatorname{supp} f$ 逐渐变大.

波解具有有限传播速度.

例：
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \boxed{x \geq 0} \quad t > 0 \\ u(x,0) = g(x) & \partial_t u(x,0) = h(x). \\ u(0,t) = 0 \end{cases}$$

对 x 的区间作了限制

solution：（相容性条件）① $u(0,0) = g(0) = 0$

② $\partial_t u(0,t) = h'(t) \Rightarrow h'(0) = 0$

③ $\partial_t^2 u \big|_{(0,t)} = 0 \qquad \partial_x^2 u \big|_{(x,0)} = g''(x)$

$\Rightarrow g''(0) = 0$

作奇延拓：
$$\begin{cases} \tilde{u}(x,t) = \begin{cases} u(x,t) & x \geq 0 \\ -u(-x,t) & x < 0 \end{cases} \\ \tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases} \\ \tilde{h}(x) = \begin{cases} h(x) & x \geq 0 \\ -h(-x) & x < 0 \end{cases} \end{cases}$$

← 初值，

$u(0,t) \equiv 0$

当 $x<0$ 时 $\quad \partial_t^2 \tilde{u} = -\partial_t^2 u(-x,t)$

$\qquad \partial_x^2 \tilde{u} = -\partial_x^2 u(-x,t)$

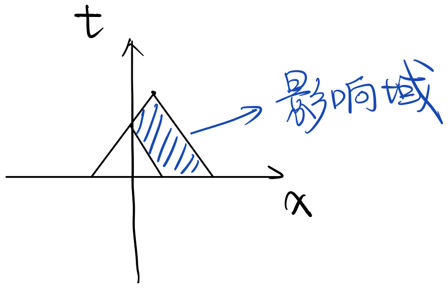
$\therefore \forall x \in \mathbb{R}$，$\tilde{u}$ 满足 $\begin{cases} \partial_t^2 \tilde{u} - \partial_t^2 \tilde{u} = 0 \\ \tilde{u}(x,t) = \tilde{g} \quad \partial_t \tilde{u} = \tilde{h}. \end{cases}$ 仍满足原方程

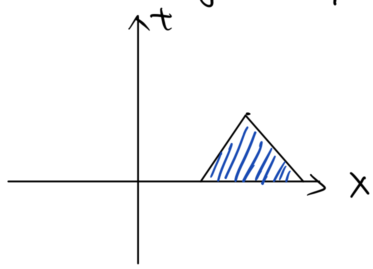
由D'Alembert 公式：

$$\tilde{u} = \frac{1}{2}\left(\tilde{g}(x+t) + \tilde{g}(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(s)\, ds.$$

若 $x>t$ 则 $u=\frac{1}{2}\left(g(x+t)+g(x-t)\right)+\frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds$.

若 $x<t$ 则 $u=\frac{1}{2}\left(g(x+t)-g(t-x)\right)+\frac{1}{2}\int_{t-x}^{x+t} h(s)\,ds$.





三. 特征线法推导二阶波方程.

① $\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 \\ u(x,0)=g(x) \qquad \partial_t u(x,0)=h(x). \end{cases}$

solution.

令 $\xi = x+t \qquad \eta = x-t \qquad \therefore x=\frac{\xi+\eta}{2} \qquad t=\frac{\xi-\eta}{2}$

$$\tilde{u}(\xi,\eta)=u(x,t)=u\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right).$$

$$\partial_t u = \partial_t \tilde{u} = \partial_\xi \tilde{u} - \partial_\eta \tilde{u}.$$

$$\partial_t^2 u = \partial_{\xi\xi}\tilde{u} - 2\partial_{\xi\eta}\tilde{u} + \partial_{\eta\eta}\tilde{u}$$

$$\partial_x^2 u = \partial_{\xi\xi}\tilde{u} + 2\partial_{\xi\eta}\tilde{u} + \partial_{\eta\eta}\tilde{u}.$$

$$\partial_{tt}u - \partial_{xx}u = 0 \Rightarrow \partial_{\xi\eta}\tilde{u} = 0$$

$$\therefore \tilde{u} = F(\xi) + G(\eta).$$

初值 $\xi=\eta$ 时 $u(x,0)=u(\xi,\eta)=g(\xi)=g(\eta)$.

$$\partial_t u(\xi,\eta)=\partial_\xi \tilde{u}-\partial_\eta \tilde{u}=h(\xi)=h(\eta).$$

即 $\begin{cases} F(\xi)+G(\xi)=g(\xi). \\ F'(\xi)-G'(\xi)=h(\xi). \end{cases}$ $\Rightarrow F(\xi)-G(\xi)=\int_0^\xi h(s)\,ds+\underset{\triangle}{C}$

$$F(\xi)=\frac{1}{2}g(\xi)+\frac{1}{2}\int_0^\xi h(s)\,ds+\frac{C}{2}.$$

$$G(\xi)=\frac{1}{2}g(\xi)-\frac{1}{2}\int_0^\xi h(s)\,ds-\frac{C}{2}.$$

$$\Rightarrow \tilde{u}=F(\xi)+G(\eta)=\frac{1}{2}\left(g(\xi)+g(\eta)\right)+\frac{1}{2}\int_\eta^\xi h(s)\,ds$$

$$\Rightarrow u=\frac{1}{2}\left(g(x+t)+g(x-t)\right)+\frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds.$$

---

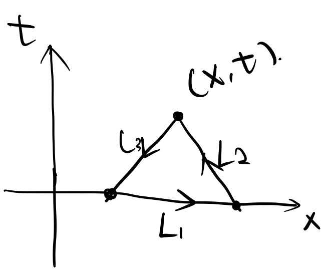
② $\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) \\ u(x,0) = 0 \qquad \partial_t u(x,0) = 0 \end{cases}$

claim: $\forall (x,t)$ $\quad u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dy\,ds$

Green公式: $\iint_\Delta (P_x - Q_t)\,dx\,dt = \int_{\partial\Delta} P\,dt + Q\,dx$

proof:

$$\iint_\Delta f\,dx\,dt = \iint_\Delta (\partial_t^2 u - \partial_x^2 u)\,dx\,dt$$



$$= \iint \underline{\left((\partial_t u)_t - (\partial_x u)_x\right)}\,dx\,dt$$

dxdt

$= (-\partial_x u)_x - (-\partial_t u)_t$

⇑

严格按照

Green公式

$$= \int_{\partial\Delta} (-\partial_x u)\,dt - \partial_t u\,dx$$

L_1 上: 原式 $= 0$

L_2 上: 原式 $= \int_{L_2} \partial_x u\,dx + \partial_t u\,dt$

$\overset{dt=-dx}{}$

$$= u(x,t) - u(x+t,0) = u(x,t)$$

L_3 上 原式 $= -\int_{L_3} \partial_x u\,dx + \partial_t u\,dt$

$\overset{dx=dt}{}$

$$= -\left(u(x-t,0) - u(x,t)\right) = u(x,t),$$

$$\therefore u(x,t) = \frac{1}{2}\iint_\Delta f(x,t)\,dx\,dt.$$

四：一维波动方程的分离变量法

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & 0 < x < L \\ u(0,t) = 0 & u(L,t) = 0 \\ u(x,0) = f(x) & \partial_t u(x,0) = g(x) \end{cases}$$

两端固定的弦.

设 $u(x,t) = T(t)X(x)$

由方程： $T''(t)X(x) - T(t)X''(x) = 0$

$$\Rightarrow \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} \triangleq -\lambda \quad \text{（一定是一个常数）}$$

$$\Rightarrow \begin{cases} T''(t) + \lambda T(t) = 0 \\ X''(x) + \lambda X(x) = 0 \end{cases}$$

考虑边值条件： $\begin{cases} T(t)X(0) = 0 \\ T(t)X(L) = 0 \end{cases} \Rightarrow \begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$

解： $\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \quad X(L) = 0 \end{cases}$ （不考虑 $X(x) \equiv 0$）.

设 $\lambda < 0$ $\quad X(t) = c_1 \cdot e^{-\sqrt{-\lambda}\, t} + c_2 e^{\sqrt{-\lambda}\, t}$.

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 \cdot e^{-\sqrt{-\lambda} L} + c_2 \cdot e^{-\sqrt{-\lambda} L} = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$$ 没有意义

设 $\lambda = 0$ $\quad X(x) = ax + b$ 有两个零点 $X(x) \equiv 0$

设 $\lambda > 0$ $\quad X(x) = C_1 \sin(\sqrt{\lambda}\, x) + C_2 \cdot \cos(\sqrt{\lambda}\, t)$

$$\therefore \begin{cases} X(0) = C_2 = 0 \\ X(L) = C_1 \cdot \sin\sqrt{\lambda} L = 0 \end{cases}$$

选取: $\sqrt{\lambda} = \frac{k\pi}{L}$ $\quad k = 1, 2 \cdots$ $\quad \lambda = \left(\frac{k\pi}{L}\right)^2$

$$T''(t) + \left(\frac{k\pi}{L}\right)^2 T(t) = 0$$

$$T_k(t) = D_1 \cdot \sin\left(\frac{k\pi}{L} t\right) + D_2 \cdot \cos\left(\frac{k\pi}{L} t\right).$$

$$\therefore u(x,t) = \sum_{k=1}^{\infty} \left(D_{1k} \sin\left(\frac{k\pi}{L} t\right) + D_{2k} \cdot \cos\left(\frac{k\pi}{L} t\right) \right) \cdot \sin\left(\frac{k\pi}{L} x\right).$$

$$u(x,0) = \sum_{k=1}^{\infty} D_{2k} \cdot \sin\left(\frac{k\pi}{L} x\right) = f(x).$$

$$\partial_t u(x,0) = \sum_{k=1}^{\infty} D_{1k} \cdot \frac{k\pi}{L} \sin\left(\frac{k\pi}{L} x\right) = g(x).$$

$$\xrightarrow{\text{Fourier}} \begin{cases} D_{2k} = \frac{2}{l} \int_0^l f(x) \cdot \sin\left(\frac{k\pi}{L} x\right) dx \\ D_{1k} = \frac{2}{k\pi} \int_0^l g(x) \cdot \sin\left(\frac{k\pi}{L} x\right) dx. \end{cases}$$

作正弦展开.
将 $f(x)$ 奇延拓.

Sturm-Liouville 问题:

考虑方程:

$$\begin{cases} A(t)\cdot U_{tt} + C(x)\cdot U_{xx} + D(t)\cdot U_t + E(x)\cdot U_x \\ \quad + [F_1(t)+F_2(x)]\cdot U = 0 \quad a\le x\le b \\ U(x,0)=\varphi(x) \qquad \partial_t U(x,0)=\psi(x). \\ \alpha_1 U(a,t)+\alpha_2 U_x(a,t)=0, \quad \beta_1 U(b,t)+\beta_2 U_x(b,t)=0 \\ \alpha_1^2+\alpha_2^2\ne 0 \end{cases}$$

solution: 令: $U(x,t)=T(t)X(x)$

$$A(t)T''(t)X(x)+C(x)T(t)X''(x)+D(t)T'(t)X(x)+E(x)\cdot T(t)\cdot X'(x)+(F_1(t)+F_2(x))(T(t)X(x))=0$$

$$\underbrace{\frac{A(t)T''+D(t)\cdot T'+F_1(t)\cdot T}{T}}_{\text{只与 } t \text{ 有关}} + \underbrace{\frac{C(x)\cdot X''+EX'+F_2X}{X}}_{\text{只与 } x \text{ 有关}} = 0$$

$$\therefore \begin{cases} CX''+EX'+F_2X-\lambda X=0 \\ AT''+DT'+F_1T+\lambda T=0 \end{cases}$$

边值:
$$\begin{cases} T(t)\cdot \underbrace{(\alpha_1 X(a)+\alpha_2 X'(a))}_{=0}=0 & \forall t>0. \\ T(t)\cdot \underbrace{(\beta_1 X(b)+\beta_2 X'(b))}_{=0}=0 & \forall t>0 \end{cases}$$

考虑:
$$\begin{cases} CX''+EX'+F_2X-\lambda X=0 \\ \alpha_1 X(a)+\alpha_2 X'(a)=0 \\ \beta_1 X(b)+\beta_2 X'(b)=0 \end{cases}$$

记 $L = C\cdot\frac{d}{dx^2} + E\frac{d}{dx} + F_2$　　　$LX=\lambda X$

即λ一定是L的特征值

$$SCX'' + SEx' + SF_2X = \lambda SX$$

$$(SCX')' - (SC)'X' + ESX' + F_2SX = \lambda SX$$

令S s.t $(SC)' = ES$ $\Rightarrow$ $S = \frac{1}{c}e^{\int_0^x \frac{E}{C}(s)ds}$

则 $(SCX')' + F_2SX = \lambda SX$

令 $p = SC > 0$　$q = F_2S$

$$\Rightarrow \begin{cases} (px')' + qX = \lambda SX \\ \alpha_1 X(a) + \alpha_2 X'(a) = 0 \\ \beta_1 X(b) + \beta_2 X'(b) = 0 \end{cases} \qquad (**)$$

定理：设 X_1, X_2 是同一特征值的特征函数

则 $\exists C$ s.t $X_1 = CX_2$　　（C为const）

proof: $\begin{cases} \alpha_1 X_1(a) + \alpha_2 X_1'(a) = 0 \\ \alpha_1 X_2(a) + \alpha_2 X_2'(a) = 0 \end{cases}$　　$\begin{pmatrix} X_1(a) & X_1'(a) \\ X_2(a) & X_2'(a) \end{pmatrix}\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} = 0$

有非零解

$\Rightarrow \det\begin{bmatrix} X_1(a) & X_1'(a) \\ X_2(a) & X_2'(a) \end{bmatrix} = 0$　　即 $\det W(a) = 0$.

$\therefore$ wronsky行列式为0　$\Rightarrow$ 解线性相关.

定理：若 x_i, x_j 分别是对应 $\lambda_i \neq \lambda_j$ 的特征函数。

则 x_i, x_j 在 $[a,b]$ 上正交，即 $\int_a^b x_i x_j S(x)\,dx = 0$

proof:
$$\begin{cases} (Px_i')' + qx_i - \lambda_i S x_i = 0 \quad ① \longleftarrow \text{乘 } x_j \\ (Px_j')' + qx_j - \lambda_j S x_j = 0 \quad \longleftarrow \text{乘 } x_i \end{cases}$$

$$x_j(Px_i')' = (Px_i'x_j)' - Px_i'x_j'$$

$$① \Rightarrow (Px_i'x_j)' - Px_i'x_j' + qx_ix_j - \lambda_i S x_i x_j = 0$$

$$② \Rightarrow (Px_j'x_i)' - Px_i'x_j' + qx_ix_j - \lambda_j S x_i x_j = 0$$

$$\Rightarrow \underbrace{Px_i'x_j\Big|_a^b - Px_ix_j'\Big|_a^b}_{\parallel} - (\lambda_i - \lambda_j)\int_a^b S x_i x_j\,dx = 0$$

$$P(b)x_i'(b)x_j(b) - P(a)x_i'(a)x_j(a) - P(b)x_i(b)x_j'(b) - P(a)x_i(a)x_j'(b)$$

$$\overset{\text{边值}}{=} P(b)\left[\frac{\beta_1}{\beta_2}x_i(b)x_j(b) - \frac{\beta_1}{\beta_2}x_i(b)x_j(b)\right] + \underbrace{\overset{0}{\cdots}}$$

$$= 0$$

$$\Rightarrow (\lambda_i - \lambda_j)\underbrace{\int_a^b S x_i x_j\,dx} = 0 \qquad \therefore \text{正交性得证}$$

定理：Sturm-Liouville 有可数个特征值

且 $\{\lambda_i\}$ 满足 ①：$\lambda_i > 0 \quad \forall i$ ② $\lambda_i \to +\infty$

记 λ_i 对应的特征函数 φ_i

$\{\varphi_i\}$ 是 $L^2[a,b]$ 中的完备正交基.

注：$L^2[a,b]$ 是使 $\int_a^b |f(x)|^2 dx$ 有限 的 f 全体.

例：(热传导方程边值问题)

$$\begin{cases} U_t = a^2 U_{xx} & 0<x<l \quad t>0. \\ U(x,0) = \varphi(x,0). \\ U(0,t) = 0 & U_x(l,t) + h\,u(l,t) = 0 \end{cases}$$

注：区间上的PDE ⇒ 考虑分离变量法

令 $u(x,t) = T(t)X(x)$

$$\Rightarrow \quad T'(t)X(x) = a^2 T(t) X''(x)$$

$$\therefore \frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} \triangleq -\lambda.$$

$$\therefore \quad T'(t) + \lambda a^2 T(t) = 0$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \quad X'(l) + h X(l) = 0 \end{cases} \qquad ①$$

由Sturm－Liouville定理：①一定有可数特征值 λ ……

① $\Rightarrow \quad X''(x)\cdot X(x)+\lambda X^2(x)=0$

注：能量法，乘以函数本身或它的导数。

$\Rightarrow (X\cdot X')'-(X')^2+\lambda X^2(x)=0$

$\int_0^L$

$X(L)X'(L)-X(0)X'(0)-\int_0^l (X')^2dx+\int_0^l \lambda X^2(x)\,dx=0$

$\therefore \lambda\int_0^l X^2(x)dx=\int_0^l (X')^2dx+h\cdot X^2(l) \Rightarrow \boxed{\lambda\geq 0}$

1' 若 $\lambda=0$ $\quad X=C_1x+C_2$ （显然不满足第二个边值条件）

2' 若 $\lambda>0$ $\quad$ 则 $X(x)=C_1\cdot\cos(\sqrt{\lambda}t)+C_2\sin(\sqrt{\lambda}t)$

代入边值条件：$X(0)=0 \Rightarrow C_1=0$

$X'(l)+hX(l)=0 \Rightarrow \sqrt{\lambda}C_2\cdot\cos(\sqrt{\lambda}l)+h\cdot C_2\sin(\sqrt{\lambda}l)=0$

$\Rightarrow \tan(\sqrt{\lambda}l)=-\frac{\sqrt{\lambda}}{h}$

记 $\sqrt{\lambda}l=\mu$ $\quad$ 则 $\tan\mu=-\frac{\mu}{lh}$

令 μ_n 是方程的解。 $\quad \lambda_n=\left(\frac{\mu_n}{l}\right)^2$

从一个角度验证了Sturm－Liouville定理

$\therefore X_n(x)=\sin\left(\frac{\mu_n}{l}t\right)$

由 $T' + a^2 \lambda_n T = 0 \Rightarrow T_n = e^{-a^2 \lambda_n t}$.

$$= e^{-a^2 \left(\frac{\mu_n}{l}\right)^2 t}, \quad C_n.$$

解为：$y = \sum_{n=1}^{\infty} C_n e^{-a^2 \left(\frac{\mu_n}{l}\right)^2 t} \cdot \sin\left(\frac{\mu_n}{l} x\right)$

初值：$y(x,0) = \sum_{n=1}^{\infty} C_n \cdot \sin\left(\frac{\mu_n}{l} x\right) = \varphi(x)$.

C_n 为 $\varphi(x)$ 的正弦展开系数.

注：$\int_0^l \underbrace{\sum_{n=1}^{\infty} C_n \cdot \sin\left(\frac{\mu_n}{l} x\right) \cdot \sin\left(\frac{\mu_n}{l} x\right)} = \int_0^l \varphi(x) \cdot \sin\left(\frac{\mu_n}{l} x\right)$

$\Rightarrow \mu_n C_n = \int_0^l \varphi(x) \cdot \sin\left(\frac{\mu_n}{l} x\right)$.

$\Rightarrow \quad C_n = \frac{1}{\mu_n} \int_0^l \varphi(x) \cdot \sin\left(\frac{\mu_n}{l} x\right) dx$

例：圆形区域热传导方程。 $\Omega=\{(x,y)\mid x^2+y^2<1\}$

$$\begin{cases}\dfrac{\partial u}{\partial t}=\Delta u & x\in\Omega \quad t>0\\ u(x,0)=\varphi(x) & \\ u(x,t)\big|_{\partial\Omega}=0 & \end{cases}$$

solution：令 $u=T(t)X(x)$

$$\Rightarrow T'(t)X(x)=T(t)\cdot\Delta X(x)$$

$$\Rightarrow \frac{T'(t)}{T(t)}=\frac{\Delta X(x)}{X(x)}\triangleq-\lambda$$

$$\begin{cases}\Delta X(x)+\lambda X(x)=0\\ X\big|_{\partial\Omega}=0\end{cases}$$

记 $x=r\cos\theta \quad y=r\sin\theta$

$$\Delta=\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2$$

令 $X(r,\theta)=R(r)\Theta(\theta)$.

$$\Rightarrow\begin{cases}R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)+\lambda R(r)\Theta(\theta)=0\\ R(1)\Theta(\theta)=0 \quad\Rightarrow \underline{R(1)=0}\end{cases}$$

且 $\boxed{\Theta(\theta+2\pi)=\Theta(\theta)}$

$$\Rightarrow \underbrace{r^2\frac{R''(r)+\frac{1}{r}R'(r)+\lambda R(r)}{R(r)}}_{\triangleq+\mu.}+\underbrace{\frac{\Theta''(\theta)}{\Theta(\theta)}}_{\triangleq\mu}=0$$

μ 为 const

$$\Rightarrow \begin{cases} \Theta''(\theta)+\mu\Theta(\theta)=0 \\ \Theta(\theta+2\pi)=\Theta(\theta). \end{cases}$$

若$\mu\leq 0$ 不满足边值条件⊗

若$\mu>0$

$\Theta(\theta)=C_1\sin(\sqrt{\mu}\theta)+C_2\cos(\sqrt{\mu}\theta)$

由边值 $\sqrt{\mu}\in\mathbb{Z}$ $\Rightarrow \mu=k^2$ $k=1,2\cdots$

$$\begin{cases} r^2R''(r)+rR'(r)+(\lambda r^2-k^2)R=0 \\ R(1)=0 \end{cases}$$

令: $\boxed{S(\rho)=R(r) \quad \rho=cr}$

$R'=c\cdot S'(\rho)$

$R''=c^2S''(\rho)$

$\Rightarrow c^2r^2S''(\rho)+crS'(\rho)$

$+(\frac{\lambda}{c^2}\rho^2-k^2)S(\rho)=0$

取$c^2=\lambda$ $\Rightarrow$ $\rho^2S''(\rho)+\rho S'(\rho)+(\rho^2-k^2)S(\rho)=0$

幂级数: 解为k阶Bessel函数.

记解为 $S=J_k(\rho)$ $\Rightarrow$ $R(r)=J_k(\sqrt{\lambda}r)$

由初值: $R(1)=0$ $\Rightarrow$ $J_k(\sqrt{\lambda})=0$

设J_k的零点为 $\mu_1^{(k)}$ ---- $\mu_n^{(k)}$ ----

$\Rightarrow$ $\lambda=(\mu_i^{(k)})^2$ $i=1,2\cdots$

$$X(r,\theta)=\sum_{k=1}^{\infty}[C_1\sin(k\theta)+C_2\cos(k\theta)]\cdot J_k(\mu_m^{(k)}r)$$

考虑 $T'(t)+\lambda T(t)=0$ $\lambda=(\mu_i^{(k)})^2$.

$\therefore T(t)=C\cdot e^{-(\mu_m^{(k)})^2t}$

$$u(x,t)=\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}e^{-(\mu_m^{(k)})t}J_k(\mu_m^{(k)}r)[A_m^k\cos(k\theta)+B_m^k\sin(k\theta)]$$

$$
\text{边值 } u(x,0)=\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\underbrace{J_k(\mu_m^{(k)}r)}_{\text{关于 } r \text{ 是 } L^2 \text{ 完备正交基}}\Big[\underbrace{A_m^k\cos(k\theta)+B_m^k\sin(k\theta)}_{\text{关于 } \theta \text{ 是 } L^2 \text{ 完备正交基}}\Big]
$$

$$
=\varphi(x)
$$

A_m^k、B_m^k 是 $\varphi(x)$ 在 r、θ 展开的系数.

非齐次方程：

1、先考虑齐次方程 ⇒ 特征值与特征函数（构成完备正交基）.

2. 将初值和非齐次项按上述基展开

例：
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) & 0\le x\le l \quad t>0 \\ u(0,t)=0 \quad \partial_t u(x,0)=0 & 0\le x\le l \\ u(l,t)=0 \quad u(x,0)=0 & \end{cases}$$

solution: $X_n = \sin\left(\frac{n\pi}{l}x\right)$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\cdot\sin\left(\frac{n\pi}{l}x\right)$$

其中：$f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\left(\frac{n\pi}{l}x\right)dx$

令 $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\cdot\sin\left(\frac{n\pi}{l}x\right)$.

则 $\partial_t^2 u = \sum_{n=1}^{\infty} T''_n(t)\sin\left(\frac{n\pi}{l}x\right)$

$$\partial_x^2 u = -\sum_{n=1}^{\infty} T_n(t)\left(\frac{n\pi}{l}\right)^2\sin\left(\frac{n\pi}{l}x\right)$$

$$\sum_{n=1}^{\infty}\left[T_n''(t) + \left(\frac{n\pi}{l}\right)^2 T_n(t)\right]\sin\left(\frac{n\pi}{l}x\right) = \sum_{n=1}^{\infty} f_n(t)\sin\left(\frac{n\pi}{l}x\right).$$

$$\Longleftrightarrow T_n''(t) + \left(\frac{n\pi}{l}\right)^2 T_n(t) = f_n(t).$$

初值：$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin\left(\frac{n\pi}{l}x\right) = 0 \quad \therefore T_n(0)=0$

? $\partial_t u(x,0) = \sum_{n=1}^{\infty} T_n'(0)\cdot \sin\left(\frac{n\pi}{l}x\right) = 0$ $\quad T_n'(0)=0$

$$T_n(t) = \frac{l}{n\pi}\int_0^t f_n(\tau)\sin\left(\frac{n\pi}{l}(t-\tau)\right)d\tau$$

非齐次边界：

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f & 0 \le x \le l \\ U_x(0,t) = \mu(t) & U(l,t) = \nu(t) \\ U(x,0) = \varphi(x) & \partial_t U(x,0) = \psi(x) \end{cases}$$

solution: 令: $V = u(x,t) - \mu(t)(x-l) - \nu(t)$

$$\underline{V_x'(0,t) = 0 \qquad V(l,t) = 0}$$

$u = V + \mu(t)\cdot(x-l) + \nu(t)$

$\partial_t^2 u - \partial_x^2 u = \underline{\partial_t^2 V - \partial_x^2 V + \mu''(t)\cdot(x-l) + \nu''(t) = f}$

边值: $V_x(0,t) = 0 \qquad V(l,t) = 0$

$V(x,0) = \varphi(x) - \mu(0)\cdot(x-l) - \nu(0)$

$\partial_t V(x,0) = \psi(x) - \mu'(0)(x-l) - \nu'(0)$

高维波动方程初值问题:

$$\begin{cases} \partial_t u - \Delta u = f(x,t) \\ u(x,0)=\varphi(x) \quad \partial_t u(x,0) = \psi(x) \\ \text{边界条件} \end{cases}$$

$x \in \Omega \subseteq \mathbb{R}^n$. $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.

$\leftarrow$ 初值.

↳ ① 第一类边界条件: $u(x,t)\big|_{\partial\Omega} = \mu(x,t)$ → Diriclet 边值.

② 第二类边界条件: $\frac{\partial u}{\partial \vec{v}}\big|_{\partial\Omega} = \mu(x,t)$ → Neuman 边值

$\vec{v}$ 外法向量

③ …三…………: $\left(\sigma u + \frac{\partial u}{\partial \vec{v}}\right)\Big|_{\partial\Omega} = \mu(x,t)$ → Robin 边值.

考虑三维情形(自由波方程).

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0)=\varphi(x) \quad \partial_t u(x,0) = \psi(x) \end{cases} \quad (W).$$

极坐标: $\Delta_{\mathbb{R}^N} = \partial_r^2 + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}$

$$\partial_t^2 u - \left(\partial_r^2 u + \frac{N-1}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u\right) = 0$$

若 $u(t,r,\omega)$ 与 ω 无关 则 $u = u(t,r)$.

$$\Rightarrow \partial_t^2 u - \left(\partial_r^2 u + \frac{2}{r}\partial_r u\right) = 0$$

变换形式 ☆

$u(r) = v(\rho)$.

$u \mapsto fu$.

令 $v=r^k u$ 则 $u=\dfrac{v}{r^k}$

代入后:

$$\frac{\partial_t^2 v}{r^k}-\left(\frac{\partial_r^2 v}{r^k}-2k\frac{\partial_r v}{r^{k+1}}+k(k+1)\cdot\frac{v}{r^{k+2}}+\frac{N-1}{r}\left(\frac{\partial_r v}{r^k}-k\frac{v}{r^{k+1}}\right)\right)=0$$

欲使其变换为一维波方程

$$\begin{cases}-2k+N-1=0\\(k+1)-(N-1)=0\end{cases}\Rightarrow\begin{cases}N=3\\k=1\end{cases}\Rightarrow \text{三维空间.}$$

令 $v=u\cdot r \Rightarrow \partial_t^2 v-\partial_r^2 v=0$

其中: $v(x,0)=r\varphi(x)$ $\partial_t v(x,0)=r\psi(x)$. $r\geq 0$.

边值 $v(0,t)=0$

半直线上的波方程 $\Rightarrow$ 奇延拓

解为 $v=\begin{cases}\dfrac{(r+t)\varphi(r+t)+(r-t)\varphi(r-t)}{2}+\dfrac{1}{2}\displaystyle\int_{r-t}^{r+t}s\psi(s)\,ds & r\geq t.\\[2ex]\dfrac{(r+t)\varphi(r+t)+(t-r)\varphi(t-r)}{2}+\dfrac{1}{2}\displaystyle\int_{t-r}^{r+t}s\psi(s)\,ds & r<t.\end{cases}$

为球面对称解 \ 径向解. (v 与 w 无关).

考虑非球面对称解：

$$\partial_t^2 u-\left(\partial_r^2 u+\frac{2}{r}\partial_r u+\Delta_{S^2}u\right)=0 \quad ①$$

注：$\Delta_{S^2}u=\mathrm{div}\cdot(\nabla\cdot u)$

散度定理：$\int_\Omega \mathrm{div}\cdot\vec{F}=\int_{\partial\Omega}\vec{F}\cdot\vec{n}\,dS$

而 S^2（球面） $\partial S^2=\emptyset$

$\therefore \int_{S^2}\Delta_{S^2}f\,dS=0$ （边界为空集）

对①式两边积分：

$$\int_{S^2}\left(\partial_t^2 u-\left(\partial_r^2 u+\frac{2}{r}\partial_r u+\Delta_{S^2}u\right)\right)d\sigma=0$$

$$\Rightarrow \partial_t^2\left(\int_{S^2}u\,d\sigma\right)=\partial_r^2\left(\int_{S^2}u\,d\sigma\right)+\frac{2}{r}\partial_r\left(\int_{S^2}u\,d\sigma\right)$$

令 $\bar{u}(r,t)=\frac{1}{4\pi}\int_{S^2}u(r,w)\,d\sigma(w)$.

则 $\partial_t^2\bar{u}=\partial_r^2\bar{u}+\frac{2}{r}\bar{u} \quad r>0$

令 $v=r\bar{u} \quad \Rightarrow \partial_t^2 v-\partial_r^2 v=0$

初值 $v(r,0)=r\bar{\varphi}(r) \quad \partial_t v=r\bar{\psi}(r)$

将 $\bar{u}$ 作偶延拓：

$$\Rightarrow v=\frac{(r+t)\bar{\varphi}(r+t)+(r-t)\bar{\varphi}(r-t)}{2}+\frac{1}{2}\int_{r-t}^{r+t}s\bar{\psi}(s)\,ds.$$

$\bar{u}=\frac{1}{r}v$ 　而 $\bar{u}=\frac{1}{4\pi}\int_{S^2}u(r,w,t)\,d\sigma(w)$.

已知 $\bar{u}$ 解 u.　　　　$r=0$ 处与 ω 无关

step 1: $\bar{u}(0,t) = \frac{1}{4\pi}\int_{S^2} u(0,\underset{\triangle}{\omega},t)\, d\sigma(\omega)$.

$$= u(0,t) = \partial_r(r\bar{u})\Big|_{r=0}$$

$\partial_r(r\bar{u})$

$= \frac{1}{2}(\bar{\varphi}(t) + t\bar{\varphi}'(t) + \bar{\varphi}(-t) - t\bar{\varphi}'(-t))$

$+\frac{1}{2}(t\bar{\psi}(t) - (-t)\bar{\psi}(-t))$　　∵ u 偶延拓 ψ, φ 是偶函数

$= \bar{\varphi}(t) + t\bar{\varphi}'(t) + t\bar{\psi}(t)$.

$= \frac{d(t\varphi(t))}{dt} + t\bar{\psi}(t)$.

$= \frac{d}{dt}\left(t\cdot\frac{1}{4\pi}\int_{S^2}\varphi(t\omega)\,d\sigma(\omega)\right) + \frac{t}{4\pi}\int_{S^2}\psi(t\omega)\,d\sigma(\omega)$

三维自由波方程

step2: 若 $u(x,t)$ 是 (W) 的解，则 $\forall x_0 \in \mathbb{R}^3$

$u(x+x_0,t)$ 也是 (W) 的解 初值为 $\varphi(x+x_0)$.

$\Downarrow$　　　　　　　　　　　　　　　　　　$\psi(x+x_0)$.

平移不变性

由 step 1: 令 $x=0$

$$u(x_0,t) = \frac{d}{dt}\left(t\cdot\frac{1}{4\pi}\int_{S^2}\varphi(x_0+t\omega)\,d\sigma(\omega)\right)$$

$$+\frac{t}{4\pi}\int_{S^2}\psi(x_0+t\omega)\,d\sigma$$

$$\therefore u(x,t)=\frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(x+t\omega)\,d\sigma(\omega)\right)$$

$$+\frac{t}{4\pi}\int_{S^2}\psi(x+t\omega)\,d\sigma(\omega)$$

$\boxed{\text{Kirchhoff公式}}$

更进一步地:

$$u(x,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|x-y|=t}\varphi(y)\,dS(y)\right)$$

$$+\frac{1}{4\pi t}\int_{|x-y|=t}\psi(y)\,dS(y)$$

重点: ① 球坐标 Laplace;

② 平移不变性.

紧接着三维情形 $\rightarrow$ 二维波方程

$$\begin{cases} \partial_t^2 u - (\partial_{x_1}^2 + \partial_{x_2}^2 u) = 0 \\ u(x,0) = \varphi(x_1,x_2) \quad \partial_t u(x,0) = \psi(x_1,x_2) \end{cases}$$

转换为三维情形.

令 $\tilde{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$.

$\tilde{\varphi}(x_1,x_2,x_3) = \varphi(x_1,x_2)$.

$\tilde{\psi}(x_1,x_2,x_3) = \psi(x_1,x_2)$

$$\begin{cases} \partial_t^2 \tilde{u} - (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2)\tilde{u} = 0 \\ \tilde{u}\big|_{t=0} = \tilde{\varphi} \qquad \partial_t \tilde{u}\big|_{t=0} = \tilde{\psi} \end{cases}$$

$$\tilde{u} = \underbrace{\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|x-y|=t} \tilde{\varphi}(y)\,dS(y)\right) + \frac{1}{4\pi t}\int_{|x-y|=t} \tilde{\psi}(y)\,dS(y)}$$

不妨设 $x_3 \equiv 0$

$x_3 \equiv 0$

$u(x_1,x_2,t) =$

计算:

$$u(0,0,t) = \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y|=t} \varphi(y_1,y_2)\,dS(y)\right)$$

$$+ \frac{1}{4\pi t}\int_{|y|=t} \psi(y_1,y_2)\,dS(y)$$

$$\int_{|y|=t} \varphi(y_1,y_2)\, dS(y) = 2\int_{y_3=\sqrt{t^2-y_1^2-y_2^2}} \varphi(y_1,y_2)\, dS$$

$$= 2\int_{y_1^2+y_2^2\le t^2} \varphi(y_1,y_2)\sqrt{1+\left(\frac{\partial y_3}{\partial y_1}\right)^2+\left(\frac{\partial y_3}{\partial y_2}\right)^2}\, dy_1 dy_2$$

$$= 2\int_{y_1^2+y_2^2\le t^2} \varphi(y_1,y_2)\cdot\sqrt{\frac{t^2}{t^2-y_1^2-y_2^2}}\, dy_1 dy_2 .$$

故 $u(0,t) = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{y_1^2+y_2^2\le t^2} \frac{\varphi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\, dy_1 dy_2\right)$

$$+\frac{1}{2\pi}\int_{y_1^2+y_2^2\le t^2} \frac{\psi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\, dy_1 dy_2$$

利用平移不变性: $\forall x_0\in\mathbb{R}^2$.

$$u(x_0,t) = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{y_1^2+y_2^2\le t^2} \frac{\varphi(y+x_0)}{\sqrt{t^2-y^2}}\, dy_1 dy_2\right)$$

$$+\frac{1}{2\pi}\int_{y^2\le t^2} \frac{\psi(y+x_0)}{\sqrt{t^2-y^2}}\, dy_2 dy_2$$

$$= \frac{d}{dt}\left(\frac{1}{2\pi}\int_{|x-y|\le t} \frac{\varphi(y)}{\sqrt{t^2-(x-y)^2}}\, dy_1 dy_2\right)$$

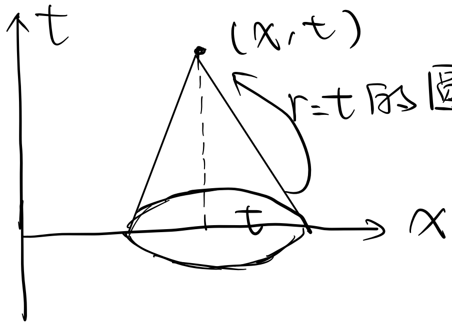
$$+\frac{1}{2\pi}\int_{|x-y|\le t} \frac{\psi(y)}{\sqrt{t^2-(x-y)^2}}\, dy_1 dy_2 .$$

$\boxed{\text{Poisson公式}}$

Poisson 公式的影响域（依赖域）

惠更斯原理？

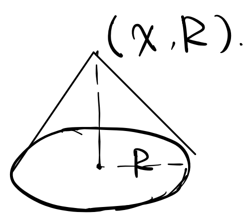
1、



r=t的圆盘影响 $u(x,t)$ 的值.

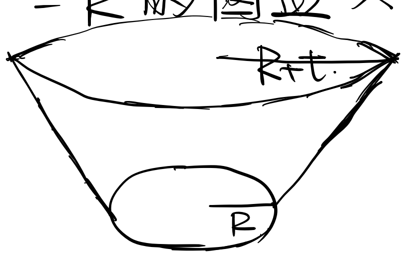
称 r=t 的圆盘为 $\boxed{\text{决定区域}}$

2. r=R的圆盘只能决定如下的锥形



光锥？

r=R的圆盘只能影响如下的圆盘.



即 Supp $u(x,t)$ 的增长速率有限.

有限传播速度.

Kirchhoff公式的不同

$$u(x,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{\underline{|x-y|=t}}\varphi(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|x-y|=t}\psi(t)\,dS(y)$$

注意到只与球面有关

Huygens原理:

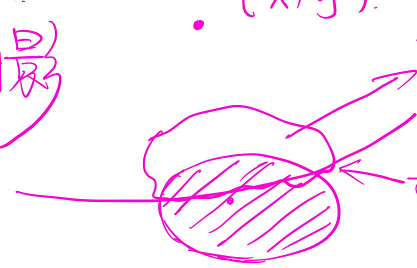
奇数维波 只与 球面有关

偶数维波 只与 球体有关.

设: 空间中有点(x,y).

初值的支集为阴影

(x,y).



偶数维与球体有关

奇数维只与该球面有关

Duhamel 原理

$$\begin{cases} \partial_t^2 u - \Delta u = f \\ u(x,0)=0 \qquad \partial_t u(x,0)=0 \end{cases} \quad (NW)$$

以τ作为初始时刻

$$\begin{cases} \partial_t^2 w - \Delta w = 0 \\ w(x,\tau,\tau)=0 \quad \partial_t w(x,\tau,\tau) = f(x,\tau) \end{cases}$$

w是关于 x,t 的，但初始时刻选在τ

则 $u = \int_0^t w(x,t;\tau)\,d\tau$.

验证：u是 NW 的解

$$\frac{\partial u}{\partial t} = \underbrace{w(x,t,t)}_{=0} + \int_0^t \frac{\partial w}{\partial t}\,d\tau = \int_0^t \partial_t w(x,t;\tau)\,d\tau$$

$$\frac{\partial^2 u}{\partial t^2} = \partial_t w(x,t;t) + \int_0^t \partial_{tt} w(x,t,\tau)\,d\tau$$

$$= f(x,t) + \int_0^t \partial_{tt} w(x,t,\tau)\,d\tau.$$

$$\Delta u = \int_0^t \Delta w(x,t;\tau)\,d\tau.$$

$$\because \partial_{tt} u - \Delta u = f(x,t) + \int_0^t \partial_{tt} w - \Delta w \,d\tau = f(x,t).$$

初值易证明. $\square$

能量法:

对于 $\partial_t^2 u-\Delta u=0$

平移不变性 $\begin{cases} u(x,t)\longmapsto u(x+x_0,t) & 空间 \\ u(x,t)\longmapsto u(x,t+t_0) & 时间 \\ u(x,t)\longmapsto u(\lambda x,\lambda t) & 伸缩. \end{cases}$

考虑 $\underline{\partial_t u(\partial_t^2 u-\Delta u)=0}$ ①

而 $\sum_{i=1}^{n}\partial_t\partial_{x_i}^2 u=\partial_{x_i}(\partial_t u\,\partial_{x_i}u)-\partial_{x_i}\partial_t u\,\partial_{x_i}u.$

$$
\begin{aligned}
① &= \frac{1}{2}\partial_t(\partial_t u)^2-\sum_{i=1}^{n}\left(\partial_{x_i}(\partial_t u\,\partial_{x_i}u)-\frac{1}{2}\partial_t(\partial_{x_i}u)^2\right) \\
&\text{验证} \\
&= \frac{1}{2}\partial_t(\partial_t u)^2-\mathrm{div}\left(\partial_t u\cdot\nabla u\right)+\frac{1}{2}\partial_t|\nabla u|^2 \\
&= \partial_t\underbrace{\left(\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2\right)}_{\text{称为能量密度. 记作 } e(u).}-\mathrm{div}(\partial_t u\cdot\nabla u)=0
\end{aligned}
$$

$\Rightarrow\ \partial_t[e(u)]-\mathrm{div}(\partial_t u\cdot\nabla u)=0$　能量等式微分形式

若乘以 $\partial_{x_i}u$，则得到动量等式　Neothur定理.

考虑: $\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times [0,+\infty) \\ \left(\frac{\partial u}{\partial \nu} + \alpha u\right)\Big|_{\partial\Omega} = 0 & \alpha > 0 \end{cases}$

$\because \partial_t (e(u)) - \operatorname{div}(\partial_t u \cdot \nabla u) = 0$

$\Rightarrow \partial_t \int_\Omega e(u)\, dx - \int_\Omega \operatorname{div}(\partial_t u \cdot \nabla u)\, dx = 0$

由散度定理:

$\partial_t \int_\Omega e(u)\, dx - \int_{\partial\Omega} \vec{F} \cdot \vec{n}\, dS = 0$

$\Rightarrow \partial_t \int_\Omega e(u)\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \cdot \partial_t u\, dS = 0$

$\Rightarrow \partial_t \int_\Omega e(u)\, dx + \alpha \int_{\partial\Omega} \partial_t u \cdot (\quad u)\, dS = 0$

$\Rightarrow \partial_t \int_\Omega e(u)\, dx + \frac{1}{2}\alpha\, \partial_t \int_{\partial\Omega} u^2\, dS = 0$

$\Rightarrow \partial_t \Big(\underbrace{\int_\Omega e(u)\, dx + \frac{1}{2}\alpha \int_{\partial\Omega} u^2\, dS}_{= E(u)} \Big) = 0$

$\Rightarrow E(u) = \text{const}.$ 能量守恒.

定理. $\begin{cases} \partial_t^2 u - \Delta u = f & \text{in } \Omega \times [0,+\infty) \\ u(x,0) = \varphi(x) \quad \partial_t u(x,0) = \psi(x) & \text{in } \Omega \\ \left(\frac{\partial u}{\partial \vec{n}} + \alpha u\right)\Big|_{\partial\Omega} = h. & t \geq 0 \end{cases}$ (**)

最多有一个解

proof: $v = u_1 - u_2$

$$\therefore \begin{cases} \partial_t^2 v - \Delta v = 0 \\ v(x,0) = 0 \qquad \partial_t v(x,0) = 0 \\ \left(\frac{\partial v}{\partial n} + \sigma v\right)\Big|_{\partial\Omega} = 0 \end{cases}$$

由能量估计

$$\int_\Omega \frac{1}{2}\left((\partial_t v)^2 + (\nabla v)^2\right) dx + \frac{\sigma}{2}\int_{\partial\Omega} v^2 dS = E(v).$$

初始时刻 $\begin{cases} v(x,0) = 0 \quad \Rightarrow \quad \nabla \cdot v = 0 \\ \partial_t(x,0) = 0 \end{cases}$

$\therefore E(v) = 0$ 而 $E(v)$ 能量守恒 $\therefore E(v) \equiv 0$.

而各项非负 $\begin{cases} v(x,t)\big|_{\partial\Omega} = 0 \\ \partial_t v \equiv 0 \\ \nabla v \equiv 0 \end{cases} \Rightarrow v(x,t) = 0$

稳定性定理: 对于(**)

$\forall \varepsilon > 0 \quad \exists \eta = \eta(\varepsilon, T) > 0$ s.t 若 $\|\varphi_1 - \varphi_2\|_{L^2(\Omega)} \leq \eta \quad \|\nabla\varphi_1 - \nabla\varphi_2\|_{L^2(\Omega)} \leq \eta$

$\|\varphi_1 - \varphi_2\|_{L^2(\partial\Omega)} \leq \eta$, $\|f_1 - f_2\|_{L^2[0,1]\times\Omega} \leq \eta$

则 u_1 与 u_2 在 $0 \leq t \leq T$ 上满足

$$\|u_1 - u_2\|_{L^2(\Omega)} + \|\nabla u_1 - \nabla u_2\|_{L^2(\Omega)} \leq \varepsilon, \quad \|\partial_t u_1 - \partial_t u_2\|_{L^2(\Omega)} \leq \varepsilon$$

注: $\|\varphi_1 - \varphi_2\|_{L^2(\Omega)} = \left[\int_\Omega (\varphi_1 - \varphi_2)^2 dx\right]^{1/2}$.

proof: 令 $u = u_1 - u_2$ $\varphi = \varphi_1 - \varphi_2$ $\psi = \psi_1 - \psi_2$ $f = f_1 - f_2$.

$$\therefore \begin{cases} \partial_{tt} u - \Delta u = f \\ u(x,0) = \varphi(x) \quad \partial_t u(x,0) = \psi(x) \\ \left(\frac{\partial u}{\partial \vec{n}} + \sigma u\right)\Big|_{\partial\Omega} = 0 \end{cases}$$

$$\partial_t u(\partial_{tt} u - \Delta u) = f \partial_t u.$$

$$\partial_t\left(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2\right) - \operatorname{div}(\partial_t u \cdot \nabla u) =$$

$$\partial_t \underbrace{\int_\Omega \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 \, dx + \frac{\sigma}{2}\int_{\partial\Omega} u^2 ds}_{E(t)} = \int_\Omega f \partial_t u \, dx$$

$$\left|\int_\Omega f \partial_t u \, dx\right| \le \frac{\int_\Omega f^2 dx}{2} + \frac{\int_\Omega (\partial_t u)^2 dx}{2}$$ Cauchy 不等式

$$\therefore \partial_t E(t) \le \frac{1}{2}\int_\Omega f^2 dx + \frac{1}{2}\int_\Omega (\partial_t u)^2 dx.$$

$$\frac{dE(t)}{dt} \le \frac{1}{2}\int_\Omega f^2 dx + E(t).$$

由 Gronwall 不等式：

$$\frac{d}{dt}\left(E(t)e^{-t}\right) \le e^{-t} \frac{1}{2}\int_\Omega f^2 dx. \le \frac{1}{2}\int_\Omega f^2 dx.$$

$$E(t) \cdot e^{-t} - E(0) \le \frac{1}{2}\int_0^t \int_\Omega |f|^2 dx dt.$$

$$\therefore E(t) \le e^t E(0) + \frac{1}{2} e^t \int_0^t \int_\Omega |f|^2 dx dt.$$

$$\le e^T\left(E(0) + \int_0^t \int_\Omega |f|^2 dx dt\right).$$

$$E(0)= \frac{1}{2}\int_\Omega \psi^2 dx+\frac{1}{2}\int_\Omega (\nabla\varphi)^2 dx+\frac{\sigma}{2}\int_{\partial\Omega}|\varphi|^2 dx$$

而 $\int_0^t\int_\Omega |f^2|\,dx\,dt \le \eta^2$

$$\because E(t)\le e^T\left(\frac{1}{2}\eta^2+\frac{1}{2}\eta^2+\frac{\sigma}{2}\eta^2\right)\longrightarrow 0$$

而 $\underline{\frac{d}{dt}\int_\Omega |u^2|\,dx} = \int_\Omega 2u\cdot\partial_t u\,dx$

$$\le \int_\Omega u^2+(\partial_t u)^2\,dx \le \underline{\int_\Omega u^2 dx} + 2E(t).$$

由Gronwall不等式:

$$\int_\Omega |u^2|\,dx \le C\left(\int_\Omega |\varphi|^2 dx + 2\int_0^t E(t)\,dt\right)\longrightarrow 0$$

重点: ① 有限传播速度

② 分离变量法

③ 能量估计.