$\partial_t u - \Delta u = 0 \quad \text{in } \Omega \times [0, +\infty)$

$u(x,0) = u_0(x)$

边值：① Diriclet 边值 $\quad u(x,t)\big|_{\Omega} = g(x,t) \quad t>0$

② Neuman 边值 $\quad \frac{\partial u}{\partial \vec{n}}(x,t)\Big|_{\partial\Omega} = g(x,t) \quad t>0$

③ Robin 边值 $\quad \left(u + \sigma \frac{\partial u}{\partial \vec{n}}\right)\Big|_{\partial\Omega} = g(x,t) \quad t>0$

解法：① 分离变量法（Ω 是区间，$\mathbb{R}$，圆盘）

② Fourier 变换

(Fourier)：

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times [0,+\infty] \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases}$$

记 $\mathcal{S}(\mathbb{R}^n) = \left\{ u \in C^\infty(\mathbb{R}^n) \,\middle|\, \sup_{x \in \mathbb{R}^n} x^\beta |\partial^\alpha u| < +\infty \right\}$

即 $|\partial^\alpha u| < \frac{M}{|x|^\beta}$ 称 $\mathcal{S}(\mathbb{R}^n)$ 为速降函数空间

e.g. $e^{-|x|^2} \in \mathcal{S}(\mathbb{R})$

定义：f 的 Fourier 变换

$\forall f \in \mathcal{S}(\mathbb{R}^n) \quad \hat{f}(\xi) = \int_{\mathbb{R}^n} f \cdot e^{-ix\cdot\xi}\, dx$

$$\boxed{\begin{array}{c} \text{物理空间 } f \\ \downarrow \\ \text{频率空间 } \hat{f} \end{array}}$$

性质: 若 $f \in \mathcal{S}(\mathbb{R}^n)$

1、 $\tau_{x_0} f(x) = f(x - x_0)$ 则 $\widehat{\tau_{x_0} f}(x) = e^{-i x_0 \cdot \xi} \cdot \hat{f}(\xi)$.

proof: $\widehat{\tau_{x_0} f}(x) = \int_{\mathbb{R}^n} e^{-i x \cdot \xi} \cdot f(x - x_0)\, dx$

$$= e^{-i x_0 \xi}\, \hat{f}(\xi).$$

相旋转.

2. $S_\lambda f(x) = f(\lambda x)$ 则 $\widehat{S_\lambda f}(x) = \lambda^{-n} \hat{f}(\lambda^{-1} \xi)$.

proof: $\widehat{S_\lambda f}(x) = \int_{\mathbb{R}} e^{-i x \cdot \xi} f(\lambda x)\, dx$

$$= \int_{\mathbb{R}} e^{-i y \cdot \lambda^{-1} \xi} f(y) \cdot \lambda^{-n}\, dy$$

$$= \lambda^{-1} f(\lambda^{-1} \xi).$$

3. $\alpha = (\alpha_1 \cdots \alpha_n)$ $|\alpha| = \alpha_1 + \cdots + \alpha_n$ $x^\alpha = x_1^{\alpha_1} + \cdots + x_n^{\alpha_n}$.

$\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ 则: $\widehat{\partial^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$

proof: $\widehat{\partial^\alpha f}(\xi) = \int \partial^\alpha f(x) \cdot e^{-i x \cdot \xi}\, dx$

考虑: $\int \partial_{x_1} f(x) e^{-i x \xi}\, dx = -\int f(x) \cdot \partial_{x_1}(e^{-i x \cdot \xi})\, dx + \underbrace{f(x) \cdot e^{-i x \xi} \Big|^{x=+\infty}}_{\to 0}$

$$= \underline{i \xi_1 \int f(x) \cdot e^{-i x \xi}\, dx}$$

4. $\widehat{(-ix)^\alpha f}(\xi) = \partial_\xi^\alpha \hat{f}(\xi)$.

proof: $\widehat{(-ix)^\alpha f}(\xi) = \int_{\mathbb{R}^n} (-ix)^\alpha f(x) \cdot e^{-ix\cdot\xi}\, dx$

$$= \int_{\mathbb{R}^n} f(x)\ \partial_\xi^\alpha e^{-ix\cdot\xi}\, dx$$

$$= \partial_\xi^\alpha \int_{\mathbb{R}^n} f(x)\cdot e^{-ix\xi}\, dx = \partial_\xi^\alpha \hat{f}(\xi).$$

5. (卷积) $f*g = \int f(x-t)g(t)\,dt = \int f(t)g(x-t)\,dt$.

则 $\widehat{f*g} = \hat{f}\,\hat{g}$ $\qquad \widehat{fg} = \hat{f} * \hat{g}$

例:

$f(x) = e^{-x^2}$ $x\in\mathbb{R}$ $\qquad \hat{f}(\xi) = \sqrt{\pi}\, e^{-\frac{\xi^2}{4}}$ $\xi\in\mathbb{R}$.

令 $F(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-x^2} e^{-ix\cdot\xi}\, dx$

则 $\frac{dF}{d\xi} = \int_{\mathbb{R}} (-ix)\ e^{-x^2} e^{-ix\xi}\, dx$

$$= i\int_{\mathbb{R}} (-x)\, e^{-x^2} e^{-ix\xi}\, dx$$

$$= \frac{i}{2} \int_{\mathbb{R}} (e^{-x^2})'\, e^{-ix\xi}\, dx$$

$$= \frac{i}{2}\, \widehat{f'}(\xi) = \frac{i}{2}\cdot i\xi \cdot \hat{f}(\xi)$$

$$= -\frac{\xi}{2}\hat{f}(\xi).$$

故 $\frac{dF}{d\xi}+\frac{\xi}{2}F(\xi)=0$ $\quad F(0)=\sqrt{\pi}$. $\quad$ ODE.

$$F(\xi)=\sqrt{\pi}e^{-\frac{\xi^2}{4}}$$

若 $f\in\mathcal{S}(\mathbb{R}^n)$，定义 Fourier 逆变换

$$\check{f}(x)=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n}f(\xi)\,e^{ix\xi}\,d\xi$$

命题 若 $f\in\mathcal{S}(\mathbb{R}^n)$ 则 $\hat{f}\in\mathcal{S}(\mathbb{R}^n)$ $\quad (\hat{f})^{\vee}=f$

例：$(e^{-\xi^2 t})^{\vee}(x)=\frac{1}{(4\pi t)^{\frac{1}{2}}}e^{-\frac{x^2}{4t}}$

$n\geq 2$：$\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n}e^{-\xi_1 t}e^{-\xi_2 t}\cdots\cdots e^{-\xi_n t}e^{ix_1\xi_1}\cdots e^{ix_n\xi_n}\,d\xi_1\cdots d\xi_n.$

$$=\prod_{m=1}^{n}\underbrace{\left(\frac{1}{2\pi}\right)\int_{\xi}e^{-\xi_m t}e^{ix_m\xi_m}\,d\xi_m}_{\Downarrow}.$$

一维的结果：$G(x,t)=\frac{1}{(4\pi t)^{\frac{1}{2}}}e^{-\frac{x^2}{4t}}$.

$$\therefore\ (e^{-\xi^2 t})^{\vee}(x)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

对 $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u(x,0) = u_0(x). & \text{in } \mathbb{R}^n. \end{cases}$

性质 3.

$$\Rightarrow \begin{cases} \partial_t \hat{u}(\xi,t) - \left(-\xi_1^2 \hat{u}(\xi,t) - \cdots - \xi_n^2 \hat{u}(\xi,t)\right) = 0 \\ \hat{u}(\xi,0) = \widehat{U_0}(\xi). \end{cases}$$

ODE

$$\Rightarrow \hat{u}(\xi,t) = \widehat{u_0}(\xi)\, e^{-|\xi|^2 t}$$

$$\therefore u(x,t) = \left(\widehat{u_0}(\xi) \cdot e^{-|\xi|^2 t}\right)^{\wedge}$$

$$= \int_{\mathbb{R}^n} \widehat{u_0}(x-y)\, e^{-|y|^2 t}\, dy$$

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy = K_t * U_0(x).$$

令 $K(x) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}}$ $\quad K_t(x) = t^{-\frac{n}{2}} K(t^{-\frac{1}{2}} x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$

则 $K_t(x)$ 是一组单位逼近 (t 充分小)

命题: 若 u 连续有界 则 $\lim_{t\to 0^+} u(x,t) = u_0(x)$.

proof: $u(x,t) - u_0(x)$

$$= \int K_t(y)\, U_0(x-y)\, dy - U_0(x) \int K_t(y)\, dy$$

$$= \int K_t(y) \cdot [U_0(x-y) - U_0(x)]\, dy$$

$$= \int_{|y|\le\delta} K_t(y)\cdot\underbrace{[U_0(x-y)-U_0(x)]}_{\text{连续}}\,dy + \int_{|y|\ge\delta} K_t(y)\cdot \underset{\text{有界}}{M}$$

$\longrightarrow 0$

性质：由卷积的微分性质：

① $u(x,t)\in C^\infty \quad \forall t>0$ 温度分布瞬间光滑

② $\sup\limits_{x\in\mathbb{R}^n} |u(x,t)| \le \sup\limits_{x\in\mathbb{R}^n} |U_0(x)|$. 温度最大值单减

③ $u(x,-t)$ 不满足原方程 不能时间反向演化.

(宏观).

④ 记 $U_0(x)$ 的 support 为 Ω。 无限传播速度

而 $U(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int u_0(y)\, e^{-\frac{|y-x|^2}{4t}}\,dy$

$u(x,t)\geqslant 0$

⑤ $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}$

非齐次方程.

$$\begin{cases} \partial_t u - \Delta u = f & x \in \mathbb{R}^n \quad t > 0. \\ u(x,0) = 0. \end{cases}$$

$\xrightarrow{\text{Fourier}}$ $\partial_t \hat{u} + |\xi|^2 \hat{u} = \hat{f} \qquad \hat{u}(\xi,0) = 0$

$$\hat{u}(\xi,t) = \int_0^t \hat{f} e^{-|\xi|^2(t-s)} ds$$

$\xrightarrow[\text{逆变换}]{\text{Fourier}}$ $\displaystyle\int_0^t \left(e^{-|\xi|^2(t-s)}\right)^\vee * f \, ds$

$$= \int_0^t \frac{1}{4\pi(t-s)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s) \, dy ds.$$

注: 波动方程

$\partial_t^2 u - \Delta u = 0 \qquad$ 初值 $u(x,0) = \varphi \qquad \partial_t u(x,0) = \psi$

$\Rightarrow \quad \partial_t^2 \hat{u} + |\xi|^2 \hat{u} = 0 \qquad \lambda^2 + |\xi|^2 = 0 \quad \lambda = \pm i|\xi|$

$\therefore \hat{u} = C_1 \cos(|\xi|t) + C_2 \sin(|\xi|t).$

代入初值. $\hat{u} = \hat{\varphi}(\xi) \cos(|\xi|t) + \dfrac{\hat{\psi}(\xi)}{|\xi|} \sin(|\xi|t).$

记 $\widehat{\cos(t\sqrt{-\Delta})f} = \cos(t|\xi|) \hat{f}(\xi).$

$\widehat{\dfrac{\sin(t\sqrt{-\Delta})f}{\sqrt{-\Delta}}} = \dfrac{\sin(t|\xi|)}{|\xi|} \hat{f}(\xi).$

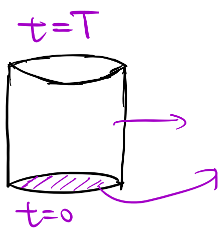
考虑 $\widehat{\sqrt{-\Delta}} = \sqrt{-(-|\xi|^2)}$
$= |\xi|.$

$$\xrightarrow[\text{逆变换}]{\text{Fourier}} u(x,t) = \cos(t\sqrt{-\Delta})\varphi + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\psi$$

极大值原理：

令 $Q_T = \Omega \times (0,T]$

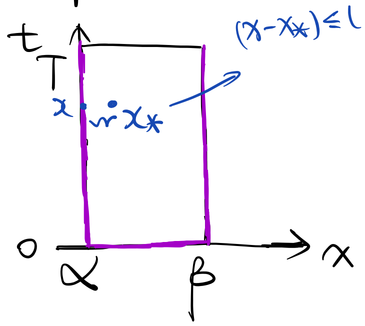
抛物边界 $\Gamma_T = \overline{Q_T} \setminus Q_T$



底面+侧面 ⇒ 抛物边界.

定理.（极值原理）

令 $u(x,t)$ 是矩形 $R_T = \{\alpha < x < \beta,\ 0 < t \le T\}$ 上连续，且满足方程 $\partial_t u - \partial_x^2 u = 0$。则它在抛物边界上达到最大（小）值.



$(x-x_*) \le (\beta-\alpha)$.

proof.（最大值）

令 $M = \max\limits_{R_T} u$ $\qquad$ $m = \max\limits_{\Gamma_T} u$.

要证明：$M = m$.

否则 $m < M$ $\quad$ 设 $u(x_*, t_*) = M$ $\quad$ $(x_*, t_*) \in R_T$

$$\begin{cases} u_t(x_*, t_*) \ge 0 \\ u_x(x_*, t_*) = 0 \\ u_{xx}(u_*, t_*) \le 0 \end{cases}$$

令 $v(x,t) = u(x,t) + h(x)$.

$v_t = u_t$

$v_{xx} = u_{xx} + h_{xx}$

$\because u_t = u_{xx}$

$\Rightarrow v_t - v_{xx} = -h_{xx}$

not im-portant

$$V(x,t) = u(x,t) + \frac{M-m}{4L^2}(x-x_*)^2 \qquad (L=\beta-\alpha).$$

则 $\max\limits_{P_T} V(x,t) \le \underbrace{\max\limits_{P_T} u(x,t)}_{m} + \frac{M-m}{4L^2}\cdot L^2 < M$

且 $\max\limits_{R_T} V(x,t) \ge M.$

$\therefore$ V 在内部达到最大值.

设 V 在 $(\tilde{x},\tilde{t})$ 达到最大值　则 $V_t(\tilde{x},\tilde{t}) \ge 0$ ↗ $\tilde{x},\tilde{t}$ 内部 $=0$　上边界 ≥ 0

$V_{xx}(\tilde{x},\tilde{t}) \le 0$

$$\begin{cases} U_t = V_t \\ U_{xx} = V_{xx} - \frac{M-m}{2L^2} \end{cases}$$

$$\Rightarrow U_t - U_{xx} = V_t - V_{xx} + \frac{M-m}{2L^2} \ge \frac{M-m}{2L^2} > 0$$

$\therefore$ $U_t - U_{xx} > 0$　与 U 是解矛盾.

证明过程相同

注: $U_t - U_{xx} = f \le 0$　in $\Omega \times (0,T]$ ↗ 在边界取到.

↳ 依然满足极大值原理.

$U_t - U_{xx} = f \ge 0$　in $\Omega \times [0,T]$

↳ 极小值原理.

定理（唯一性和稳定性）

热传导方程：$\begin{cases} \partial_t u - \Delta u = f & \text{in } R_T \\ u(x,0) = \varphi \\ u(\alpha,t) = \mu_1(t) & u(\beta,t) = \mu_2(t). \end{cases}$

在 R_T 上解是唯一的，且连续依赖于边界 Γ_T 上给定的初始条件和边界条件.

proof: $u = U_1 - U_2$

则：$\begin{cases} \partial_t u - \Delta u = 0 \\ u(x,0) = 0 \\ u(\alpha,t) = 0 \quad u(\beta,t) = 0 \end{cases}$

由极值原理：

$$\max_{R_T} u = \min_{R_T} u = 0$$

$\therefore u \equiv 0$.

设 $U_1 \Rightarrow \begin{cases} \partial_t u - \partial_{xx} u = f \\ u(x,0) = \varphi_1(x) \\ u(\alpha,t) = \mu_{11}(t) \\ u(\beta,t) = \mu_{12}(t) \end{cases}$

$U_2 \Rightarrow \begin{cases} \partial_t u - \partial_{xx} u = f \\ u(x,0) = \varphi_2(x) \\ u(\alpha,t) = \mu_{21}(t) \\ u(\beta,t) = \mu_{22}(t) \end{cases}$

$u = U_1 - U_2 \Rightarrow \begin{cases} \partial_t u - \partial_{xx} u = 0 \\ u(x,0) = \varphi_1(x) - \varphi_2(x). \\ u(\alpha,t) = \mu_{11}(t) - \mu_{21}(t). \\ u(\beta,t) = \mu_{12}(t) - \mu_{22}(t). \end{cases}$

证明 $|u| \leq \max\left\{ \max_{\alpha\leq x\leq\beta} |\varphi_1(x) - \varphi_2(x)|,\ \max_{0\leq t\leq T} |\mu_{11}(t) - \mu_{21}(t)|,\right.$

$\left. \max_{0\leq t\leq T} |\mu_{12}(t) - \mu_{22}(t)| \right\}$

① u 有正的最大值

由极大值原理.

$$\max_{\overline{R_T}} u = \max_{\Gamma_T} u = \max\{\max \varphi_1-\varphi_2,\ \max \mu_{11}-\mu_{21},\ \max \mu_{12}-\mu_{22}\}$$

$$\leq \max\{\max|\varphi_1-\varphi_2|,\ \max|\mu_{11}-\mu_{21}|,\ \max|\mu_{12}-\mu_{22}|\}$$

② u 有负的最小值.

$$\min_{\overline{R_T}} u = \max_{\Gamma_T} u = \min\{\min \varphi_1-\varphi_2,\ \min \mu_{11}-\mu_{21},\ \min. \mu_{12}-\mu_{22}\}$$

$\Rightarrow$

$$-\min_{\overline{R_T}} u = \max_{\Gamma_T} u = -\min\{\min \varphi_1-\varphi_2,\ \min \mu_{11}-\mu_{21},\ \min. \mu_{12}-\mu_{22}\}$$

$$\Rightarrow \left|\min_{R_T} u\right| \leq \max\{\max|\varphi_1-\varphi_2|,\ \max|\mu_{11}-\mu_{21}|,\ \max|\mu_{12}-\mu_{22}|\}$$

第三类边值问题:

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & 0<x<l \\ u(0,t) = \mu_1(t) & (u_x + hu)(l,t) = \mu_2(t) \quad h>0 \\ u(x,0) = \varphi(x) \end{cases}$$

的唯一性:

若解非零
则解有正的最大值
或负的最小值

proof: $\begin{cases} \partial_t u - \partial_x^2 u = 0 \\ u(0,t)=0 \quad (u_x+hu)(l,t) = 0 \\ u(x,0)=0 \end{cases}$

一定在此处取到

① u 在 Γ_T 上取得正的最大值.

$\therefore \exists \bar{t}$ s.t $\max\limits_{R_T} u = \max\limits_{\Gamma_T} u = u(l,\bar{t}) > 0$

最大值: $u_x \geq 0$

$\because u_x(l,\bar{t}) + h\cdot u(l,\bar{t}) = 0 \quad \therefore u_x(l,\bar{t}) < 0$

与 u 在 $(l,\bar{t})$ 取极大值矛盾.

$\therefore u$ 的最大值非正

② 同理: u 的最小值非负 $\quad \therefore u \equiv 0$

第二类边值问题:

$$\begin{cases}\partial_t u - \partial_x^2 u = f & 0<x<l \quad 0<t\leq T\\ u(0,t)=\mu_1(t) & U_x(l,t)=\mu_2(t).\\ u(x,0)=\varphi(x). & \end{cases}$$

的唯一性.

proof:

$$\begin{cases}\partial_t u - \partial_x^2 u = 0 & \\ u(0,t)=0 & U_x(l,t)=0\\ U(x,0)=0 & \end{cases}$$

← 构造: 使其变成第三类边值.

令 $V(x,t) = W(x)\cdot U(x,t)$ $\qquad U = \dfrac{V(x,t)}{W(x)}$.

则: $\partial_t u = \dfrac{\partial_t V}{W}$ $\qquad \partial_x U = \dfrac{\partial_x V}{W} - \dfrac{V}{W^2}\cdot \partial_x W$

$$\partial_{xx} U = \frac{\partial_{xx} V}{W} - 2\frac{\partial_x V\,\partial_x W}{W^2} + \frac{2V\cdot(\partial_x W)^2}{W^3} - \frac{V}{W^2}\partial_{xx}W$$

$-\frac{V}{W^2}\partial_{xx}W$

代入原式 $\Rightarrow \dfrac{\partial_t V}{W} - \left(\dfrac{\partial_{xx}V}{W} - 2\dfrac{\partial_x V\partial_x W}{W^2} + \dfrac{2V(\partial_x W)^2}{W^3}\right) = 0$

$$\Rightarrow \partial_t V - \left(\partial_{xx}V - 2\frac{\partial_x V\partial_x W}{W} + \frac{2V(\partial_x W)^2}{W^2} - \frac{V}{W}\partial_{xx}W\right) = 0$$

$V(0,t) = 0$ $\qquad \left(-V\dfrac{\partial_x W}{W^2} + \dfrac{1}{W}\partial_x V\right)\Big|_{x=l} = 0$

$V(x,0)=0$

换元动机: $\begin{cases} w(l,t)\equiv 1.\\ \partial_x W(l,t) = -1\end{cases} \Rightarrow W = -x+l+1$

令 $W(x) = -x+l+1$

原方程: $\partial_t V - \left(\partial_{xx}V + 2\dfrac{\partial_x V}{-x+l+1} + 2\dfrac{1}{(-x+l+1)^2}V\right) = 0$

$V(0,t)=0$ $\qquad \left(\dfrac{V}{(-x+l+1)^2} + \dfrac{\partial_x V}{-x+l+1}\right)\Big|_{x=l} = (V+V_x)\Big|_{x=l} = 0$

$V(x,0)=0$

动机: 此时 $\partial_t V = 0$ $\quad \partial_{xx} \leq 0.$ $\quad \partial_x V = 0$

但 V 控制不住.

令: $\tilde{v} = e^{-\lambda t} v(x,t)$ $\Rightarrow$ $v = e^{\lambda t} \cdot \tilde{v}$

$$\Rightarrow \lambda e^{\lambda t}\cdot \tilde{v} + e^{\lambda t}\partial_t \tilde{v} - e^{\lambda t}\left(\partial_x^2 \tilde{v} + \frac{2}{-x+l+1}\partial_x \tilde{v} + 2\frac{1}{-x+l+1}\tilde{v}\right) = 0$$

$$\Rightarrow \begin{cases} \partial_t \tilde{v} - \partial_x^2 \tilde{v} - \frac{2}{-x+l+1}\partial_x \tilde{v} + \left(\lambda - 2\frac{1}{(-x+l+1)^2}\right)\tilde{v} = 0 \\ \tilde{v}(0,t) = 0 \qquad (\tilde{v} + \tilde{v}_x)\big|_{x=l} = 0 \\ \tilde{v}(x,0) = 0. \end{cases}$$

Consider: $0 < x < l$ $\Rightarrow$ $\frac{2}{-x+l+1} \leq 2$ 取 $\lambda > 2$

则: claim. $\tilde{v}$ 满足极大值原理.

否则 $\tilde{v}$ 在 (x^*, t^*) $0 < x^* < l$ $0 < t^* \leq T$,

上取到正的最大值

$$\partial_t \tilde{v}(x^*, t^*) \geq 0 \qquad \partial_x^2 \tilde{v}(x^*, t^*) \leq 0$$

$$\partial_x \tilde{v}(x^*, t^*) = 0$$

$$\left(\lambda - \frac{2}{(-x+l+1)^2}\right) v(x^*, t^*) > 0$$ 矛盾.

$\therefore$ $\tilde{v}$ 在抛物边界达到正最大值.

下面证明: $\tilde{v} \equiv 0$ 否则 $\tilde{v}$ 在抛物边界达到正最大值或负最小值

则 $(\tilde{v} + \tilde{v}_x)\big|_{x=l} = 0$ 依第三类导出矛盾. $\square$

$$\Rightarrow V \equiv 0 \overset{w(x)\neq 0.}{\Rightarrow} U \equiv 0$$

定理. $\begin{cases} \partial_t u - \Delta u = 0 \\ u(x,0) = \varphi(x). \end{cases}$ $\quad \mathbb{R}^n \times (0, +\infty]$

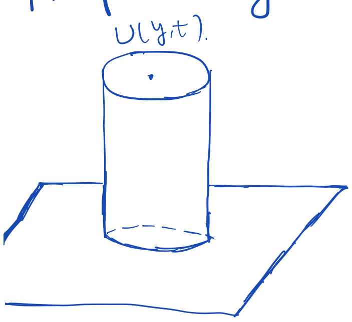
无热源
温度递减

若 $|u(x,t)| \le A e^{a|x|^2} \quad \forall x \in \mathbb{R}^n, \ 0 < t \le T$

$A, a > 0$

则有 $\sup\limits_{(\mathbb{R}^n, (0,+\infty])} u(x,t) \le \max\limits_{x \in \mathbb{R}^n} \varphi(x).$

proof: $u(y,t) \le \max \varphi(x). \quad \forall y \in \mathbb{R}^n, t > 0$



Step 1. 使T充分小

令 $|x-y| = r \quad 0 < t \le T$

令 $v(x,t) = u(x,t) - \underbrace{\dfrac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}}_{}$

是$\partial_t u - \Delta u = 0$ 的解.

$\therefore \partial_t v - \Delta v = 0$

而 ① $v(x,0) = u(x,0) - \underbrace{\dfrac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}}}_{\text{正数}} \le \max \varphi(x).$

② $|x-y| = r \quad v(x,t) \le A e^{a|x|^2} - \dfrac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}}$

$|x| \le |y| + r.$

$\le A e^{a(|y|+r)^2} - \dfrac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}}$

T充分小 $\left(\dfrac{1}{4(T+\varepsilon)} = a + \nu \quad \nu > 0 \right) \Rightarrow$ $(T+\varepsilon) = \dfrac{1}{4(a+\nu)} < \dfrac{1}{4a}$

只需 $T < \frac{1}{4a}$

就能找到 ε, ν.

$$\leq A e^{a(|y|+r^2)} - \mu\cdot(a+\varepsilon)^{n/2}\cdot e^{(a+\gamma)r^2}$$

$$\to -\infty \quad (r\to+\infty).$$

$\therefore$ 当 r 充分大时 $\quad V(x,t) \leq \max\limits_{x\in\mathbb{R}^n} \varphi(x).$

由极大值原理: $\quad V(x,t) \leq \max\limits_{x\in\mathbb{R}^n} \varphi(x).$

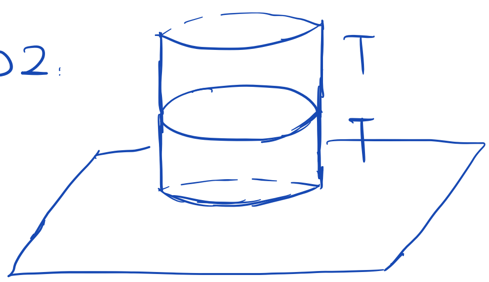
$$\Rightarrow \quad u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}} \leq \max \varphi(x).$$

$$\Rightarrow u(x,t) \leq \max\varphi(x) + \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{\varepsilon}}$$ 令 $\mu\to 0$

$$\therefore u(x,t) \leq \max_{x\in\mathbb{R}^n} \varphi(x).$$

T 不必无限小
(a 是给定的)

Step 2:



一直往上叠,而 T 只比 $\frac{1}{4a}$ 小一点

$\therefore$ nT 是发散的

能量估计

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,T] \\ u(x,0) = \varphi(x) & \text{in } \Omega. \\ u(x,t)\big|_{\partial\Omega} = g(x,t). \end{cases}$$

有唯一解.

proof: $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times [0,T] \\ u(x,0) = 0 \\ u(x,t)\big|_{\partial\Omega} = 0 \end{cases}$ 只有零解.

波方程乘 $\partial_t u$.

热方程乘 u.

$$0 = \int_\Omega u(u_t - \Delta u) \, ds = \frac{1}{2}\partial_t \int_\Omega u^2 dx - \int_\Omega \nabla\cdot(u\nabla u) - |\nabla u|^2 dx$$

$$= \frac{1}{2}\partial_t \int_\Omega u^2 dx - \underbrace{\int_{\partial\Omega} u \cdot \nabla u \cdot \vec{n} \, dS}_{=0\ (\text{边值条件})} + \int_\Omega |\nabla u|^2 dx$$

$$\Rightarrow \frac{1}{2}\partial_t \int_\Omega u^2 dx = -\int_\Omega |\nabla u|^2 dx \le 0$$

$$\therefore \left(\int_\Omega u^2 dx\right)\Big|_{t=t} \le \left(\int_\Omega u^2 dx\right)\Big|_{t=0} = 0$$

$$\therefore u \equiv 0$$

更一般地:

① $$\frac{1}{2}\partial_t \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx = \int_\Omega u \cdot f \, dx$$

$$\le \int_\Omega \frac{1}{2} u^2 dx + \int_\Omega \frac{1}{2} f^2 dx$$

若 $u(x,0) \neq 0$

则 ② $\frac{1}{2}\int_\Omega u^2(x,t)\,dx + \int_0^t\int_\Omega |\nabla u|^2\,dx\,dt \leq \frac{1}{2}\int_\Omega u^2(x,0)\,dx$