1. 2011-2012第一学期 第一次测试

1. $\forall \varepsilon>0$, $\exists\, n_0=\left[\frac{1}{4\varepsilon}+\frac{1}{2}\right]+1$, $\forall n>n_0$，有

$$\left|\frac{n}{2n+\sin n}-\frac{1}{2}\right|=\frac{\frac{1}{2}|\sin n|}{2n+\sin n}=\frac{1}{2\left(\frac{2n}{|\sin n|}+1\right)}\le\frac{1}{2(2n-1)}<\varepsilon$$

2. ~~$\exists n_0$~~ $\forall\varepsilon>0$, $\exists X_0>0$，$\forall X_1,X_2>X_0$，有 $|f(x_1)-f(x_2)|<\varepsilon$

3. (1) $=\lim\limits_{n\to\infty} n^{\frac{3}{2}}\cdot\dfrac{\sin\frac{1}{n}}{\sqrt{n+\sin\frac{1}{n}}+\sqrt{n}}=\lim\limits_{n\to\infty} n^{\frac{3}{2}}\dfrac{\frac{1}{n}}{2\sqrt{n}}=\dfrac{1}{2}$

(2) $=\lim\limits_{x\to1}\dfrac{x^2(x-1)+2(x-1)}{(x+1)(x-1)}=\lim\limits_{x\to1}\dfrac{x^2+2}{x+1}=\dfrac{3}{2}$

(3) $\lim\limits_{n\to\infty}\left[2^n\cdot\left(1+\left(\frac{3}{2}\right)^n\right)\right]^{\frac{1}{n}}=2\lim\limits_{n\to\infty}\left[1+\left(\frac{3}{2}\right)^n\right]^{\frac{1}{n}}=3$

Δ: $x>1$时 $\lim\limits_{n\to\infty}(1+x^n)^{\frac{1}{n}}=x$：

(省略lim) $(1+x^n)^{\frac{1}{n}}=\exp\dfrac{\ln(1+x^n)}{n}=\exp\dfrac{\ln x^n+\ln(x^{-n}+1)}{n}=\exp\dfrac{\ln x^n}{n}=x$

$x<1$时 $\lim\limits_{n\to\infty}(1+x^n)^{\frac{1}{n}}=1$：

$1=1^{\frac{1}{n}}<(1+x^n)^{\frac{1}{n}}<2^{\frac{1}{n}}\to1$

(4) $=\lim\limits_{x\to0}\left(1+\frac{1}{2}x^2\right)^{\frac{1}{x^2}}=\sqrt{\lim\limits_{x\to0}\left(1+\frac{x^2}{2}\right)^{\frac{2}{x^2}}}=\sqrt{e}$

4. $a_n\le b^{n-1}a_1$

~~$\forall\varepsilon>0$, $\exists n_0=$ ，$\forall n_1,n_2>n_0$，有 (设$n_1<n_2$)~~

~~$S_{n_1}-S_{n_2}=\sum_{k=n_1+1}^{n_2}a_k\le$~~

$S_n\nearrow$ (单调)

$S_n\le\sum\limits_{k=1}^{n}b^{k-1}a_1=a_1\cdot\dfrac{1-b^n}{1-b}<\dfrac{a_1}{1-b}$ (有界) $\Big\}\Rightarrow$ 收敛

~~先用数归证 $a_n\in(0,1]$~~

5. $a_{n+1}<\dfrac{a_n+a_n}{2}=a_n$

$\therefore a_n\downarrow$ (单调)
$a_n>0$ (有界) $\Big\}\Rightarrow$ 收敛

设 $\lim\limits_{n\to\infty}a_n=A$

$\therefore A=\dfrac{A+\sin A}{2}\Rightarrow A=\sin A\Rightarrow A=0$

6. $x_n=\dfrac{x_{n-1}+a}{2}>\dfrac{x_{n-1}+x_{n-1}}{2}=x_{n-1}$ (单调)
$<\dfrac{a+a}{2}=a$ (有界)

$\lim\limits_{n\to\infty}x_n\triangleq x_0\Rightarrow x_0=\dfrac{x_0+a}{2}\Rightarrow x_0=a$

7. (1) 令 $y=0$，$|xf(x)|\le M|x|\Rightarrow|f(x)|\le M\Rightarrow$ 有界

$\therefore\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=0$

(2) 令$a=0$即可

(题目有误，应把条件改为 $|yf(x)-xf(y)|\le M|x|+M|y|$)：

(1) $\left|\dfrac{f(x)}{x}-\dfrac{f(y)}{y}\right|\le M\left|\dfrac{1}{y}\right|+M\left|\dfrac{1}{x}\right|$

$\forall\varepsilon>0$, $\exists X_0=\dfrac{2M}{\varepsilon}$，$\forall|x|,|y|>X_0$，有

$\left|\dfrac{f(x)}{x}-\dfrac{f(y)}{y}\right|\le M\left(\left|\dfrac{1}{x}\right|+\left|\dfrac{1}{y}\right|\right)<\dfrac{2M}{X_0}=\varepsilon$

$\xRightarrow{\text{柯西}}$ 收敛

(2) ~~设 $a=\lim\limits_{x\to\infty}\frac{f(x)}{x}$，则 $\forall\varepsilon>0$，$\left|\frac{f(x)}{x}-a\right|<\varepsilon$~~

$\left|f(x)-\dfrac{x}{y}f(y)\right|\le M\dfrac{|x|}{|y|}+M$

令 $y\to\infty$，设 $a=\lim\limits_{y\to\infty}\dfrac{f(y)}{y}$，则 $|f(x)-ax|\le M$ (保号性)

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2011-2012 第一学期 第二次测试

1.(1) $f'(x)=e^{-x^2}-2x^2e^{-x^2}=e^{-x^2}(1-2x^2)$

令 $f'(x)=0 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$

$f''(x)=-2xe^{-x^2}(1-2x^2)+e^{-x^2}\cdot(-4x)$

$=2xe^{-x^2}(2x^2-3)$

令 $f''(x)=0 \Rightarrow x=0,\pm\sqrt{\frac{3}{2}}$

$f(\frac{1}{\sqrt{2}})=\frac{1}{\sqrt{2e}}$, $f(-\frac{1}{\sqrt{2}})=-\frac{1}{\sqrt{2e}}$

$\lim\limits_{x\to+\infty}f(x)=0$, $\lim\limits_{x\to-\infty}f(x)=0$

$\therefore f(x)_{max}=\frac{1}{\sqrt{2e}}$, $f(x)_{min}=-\frac{1}{\sqrt{2e}}$.

凹: $(-\infty,-\sqrt{\frac{3}{2}})$, $(0,\sqrt{\frac{3}{2}})$

凸: $(-\sqrt{\frac{3}{2}},0)$, $(\sqrt{\frac{3}{2}},+\infty)$

(2) $=\lim\limits_{x\to+\infty}\left[(1+\frac{1}{x})^x\right]^x e^{-x}=1$

(3) $=\lim\limits_{x\to0}\frac{2(1-\frac{x^2}{2})-(1+3x-\frac{9}{2}x^2)-(1-3x+\frac{9x^2}{2})}{-x^2}=\lim\limits_{x\to0}\frac{-x^2}{-x^2}=1$

2. $f'(x)=\prod\limits_{i=1}^{k}(x-x_i)^{n_i-1}\cdot\sum\limits_{i=1}^{k}n_i\prod\limits_{\substack{j=1\\(j\neq i)}}^{k}(x-x_j)$

~~$\therefore \sum\limits_{i=1}^{n}n_i\prod\limits_{\substack{j=1\\(j\neq i)}}^{k}(x-x_j)=\sum\limits_{i=1}^{k}n_i\prod\limits_{j=1}^{k-1}(x-\xi_j)$~~

$f(x_i)=0 \Rightarrow \exists\xi_i, f'(\xi_i)=0, \xi_i\in(x_i,x_{i+1})$

又 $g(\xi_i)=0\cdots$

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3. $\because y=\ln x$ 是凹函数

$\therefore \ln(\frac{x_1^p}{p}+\frac{x_2^q}{q})\geq\frac{1}{p}\ln(x_1^p)+\frac{1}{q}\ln(x_2^q)=\ln(x_1x_2)$

$\therefore x_1x_2\leq\frac{x_1^p}{p}+\frac{x_2^q}{q}$

4. $\frac{df'}{dx}=-f$

$\Rightarrow f'\frac{df'}{dx}=-f\frac{df}{dx}$

$\Rightarrow f'df'=-fdf$

$\Rightarrow \frac{1}{2}f'^2=-\frac{1}{2}f^2+C$

$\Rightarrow f^2+f'^2=2C$

$G(x)=f^2+f'^2$

$G'(x)=0$

$\Rightarrow G(x)=c^2$

$f'=\pm\sqrt{c^2-f^2}$

设 $f=A\sin\theta$，则 $f'=A\cos\theta\cdot\frac{d\theta}{dx}=A\cos\theta$

$\therefore \frac{d\theta}{dx}=1$

$\therefore f=A\sin(x+\varphi)$ ， $f'=A\cos(x+\varphi)=A\cos x\cos\varphi-A\sin x\sin\varphi$

$=A\sin x\cos\varphi+A\cos x\sin\varphi$

而 $f(0)=A\sin\varphi$, $f'(0)=A\cos\varphi$

$\therefore f(x)=f(0)\cos x+f'(0)\sin x$

5.(1) 不妨设 $f'(a)>0$，则 $f'(b)>0$

$\therefore \exists\delta>0$，使 $f(a+\delta)=\underbrace{f(a)}_{=0}+\underbrace{\delta}_{>0}\underbrace{f'(a+\varepsilon)}_{>0\text{(保号性)}}>0$

$f(b-\delta)=\underbrace{f(b)}_{=0}-\underbrace{\delta}_{>0}\underbrace{f'(b-\varepsilon)}_{>0\text{(保号性)}}<0$

$\therefore$ 由零点定理，$\exists\xi\in(a+\delta,b-\delta)$, $f(\xi)=0$

(2) $g(x)\triangleq f(x)e^{-x}$, $g'(x)=[f'(x)-f(x)]e^{-x}$

$\because g(a)=g(b)=g(\xi)=0$

$\therefore$ ~~$\exists\xi\in(a,b)$, $g'(\xi)=0$ 即 $f'(\xi)=f(\xi)$~~

$\exists\xi_1\in(a,\xi)$, $\xi_2\in(\xi,b)$, $g'(\xi_1)=g'(\xi_2)=0$

(3) $h(x)\triangleq[f(x)+f'(x)]e^{+x}$, $h'(x)=[f(x)-f''(x)]e^{x}$

$\because h(\xi_1)=h(\xi_2)=0$

$\therefore \exists\eta\in(\xi_1,\xi_2)$, $h'(\eta)=0$

$\boxed{3}$ 2012-2013 第一学期 第一次测试

1. (1) × $a_n=(-1)^n$, $a=1$

(2) √ ~~$a_n-a_N=0$~~ $\forall \varepsilon>0$, $\exists N>0$, $\forall n_1>N$, $n_2>N$, $|a_{n_1}-a_N|<\frac{\varepsilon}{2}$,

$|a_{n_2}-a_N|<\frac{\varepsilon}{2} \Rightarrow |a_{n_1}-a_{n_2}|<\varepsilon \xrightarrow[\text{收敛}]{\text{柯西}}$ 收敛

(3) √ $g(x)=f(x)-x$

$g(-2)\geq -1-(-2)=1>0$, $g(2)\leq 1-2=-1<0$

$\therefore \exists x_0\in[-2,2]$, $f(x_0)=x_0$

(4) × $f(x)=x$

2. (1) $=1$

(2) $=\lim\limits_{n\to\infty}\left(n-\frac{(n-1)^{3/2}}{\sqrt{n}}\right)=\lim\limits_{n\to\infty} n\left[1-\left(1-\frac{1}{n}\right)^{3/2}\right]=\lim\limits_{n\to\infty} n\left[1-\left(1-\frac{3}{2n}\right)\right]=\frac{3}{2}$

(3) $=\lim\limits_{x\to+\infty}\left[\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}}\right]^4\Big/\left(1+\frac{2}{x+1}\right)^2=e^4$

(4) $=\lim\limits_{x\to0}\frac{\tan x\cdot\frac{1}{2}\sin x}{\frac{1}{2}\sin^2x}=\lim\limits_{x\to0}\frac{1}{\cos x}=1$

3. 当 $x>0$ 时 $f(x)=\lim\limits_{nx\to\infty}\frac{1+x^2e^{nx}}{x+e^{nx}}=x^2$

当 $x=0$ 时 $f(x)=1$

当 $x<0$ 时 $f(x)=\lim\limits_{nx\to-\infty}\frac{1+x^2e^{nx}}{x+e^{nx}}=\frac{1}{x}$

$\therefore f(x)=\begin{cases}x^2, & x>0\\ 1, & x=0\\ \frac{1}{x}, & x<0\end{cases}$

而 $\lim\limits_{x\to0^+}f(x)=0$, $\lim\limits_{x\to0^-}f(x)=-\infty$

$\therefore f(x)$ 在 $x=0$ 处不连续，其他地方连续

4. $f(x)=f(x^{\frac{1}{2}})=f(x^{\frac{1}{4}})=f(x^{\frac{1}{2^n}})$

$\because f(x)$ 在 $x=1$ 处连续

$\therefore \lim\limits_{n\to\infty}f(x^{\frac{1}{2^n}})=f(\lim\limits_{n\to\infty}x^{\frac{1}{2^n}})=f(1)$

$\therefore f(x)\equiv f(1)$ 为常数

5. $x_{n+1}=\frac{\alpha+\alpha x_n}{\alpha+x_n}>\frac{\alpha+x_n}{\alpha+x_n}=1$ (用数归证 $\forall n\geq2, x_n>1$)

$x_{n+1}=\frac{\alpha+\alpha x_n}{\alpha+x_n}<\frac{\alpha^2+\alpha x_n}{\alpha+x_n}=\alpha$ (用数归证 $\forall n\geq 2$, $x_n<\alpha$) (有界)

$x_{n+1}-x_n=\frac{\alpha+\alpha x_n-\alpha x_n-x_n^2}{\alpha+x_n}=\alpha-x_n>0$

$\therefore \{x_n\}\nearrow$ (单调)

$\therefore$ 有极限

$\therefore$ 设 $\lim\limits_{n\to\infty}x_n=x_0$, $x_0=\frac{\alpha(1+x_0)}{\alpha+x_0}$

$\therefore x_0=\sqrt{\alpha}$

6. ~~$\forall\varepsilon>0$ $\exists N>\frac{1}{\varepsilon}$, $\forall n_1>N$, $n_2>N$, 有 (不妨设 $n_2>n_1$)~~

~~$|a_{n_1}-a_{n_2}|=\left|\sum\limits_{k=n_1+1}^{n_2}\frac{(-1)^{k-1}}{k}\right|$~~

~~$=\begin{cases}\text{若 } n_2-n_1 \text{ 为奇：} \left|\sum\limits_{k=n_1+1}^{n_2}\frac{(-1)^{k-n_1-1}}{k}\right|<\frac{n_2-n_1}{n_1}+\frac{1}{n_2}\\ \text{若 } n_2-n_1 \text{ 为偶：}\end{cases}$~~

S_{2n} 单调有界
课本P259

交错级数审敛法(莱布尼兹定理) ⇒ 收敛

7. $\because \lim\limits_{n\to\infty}\frac{a_n}{n}=0$ ~~$\forall k>N$~~ $\left|\frac{a_k}{k}\right|$ $\therefore \forall\varepsilon>0$, $\exists N>0$, $\forall k>N$, 有 $|a_k|<\varepsilon|k|\leq\varepsilon n$

~~$\therefore \forall\varepsilon>0, \exists N>0, \left|\frac{a_k}{k}\right|<0$~~

$\forall\varepsilon>0$, $\exists N_0\in N_+$, $\forall n>N_0$, $|a_n|<n\varepsilon$

$M=\max\limits_{i=1,\cdots,N_0}\{a_i\}$, $\forall n>\max\{N_0,[\frac{M}{\varepsilon}+1]\}$, $\forall i\leq n$, $\frac{|a_i|}{n}\leq$

~~$\therefore \left|\frac{a_k}{n}\right|<0$~~

而 $\forall k\leq N$, $|a_k|<\varepsilon n$ 显然成立

$\therefore \frac{1}{n}\max\limits_{1\leq k\leq n}\{a_k\}<\frac{1}{n}\cdot\varepsilon n=\varepsilon$ $\therefore \lim\limits_{n\to\infty}\frac{1}{n}\max\limits_{1\leq k\leq n}\{a_k\}=0$

$\boxed{4}$ 2012-2013 第一学期 第二次测试

1.(1) $(x^2e^x)^{(n)} = x^2e^x + 2nxe^x + \frac{n(n-1)}{2}\cdot 2e^x = e^x(x^2+2nx+n^2-n)$

(2) $\cos(xy)\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 1$

$$(\cos(xy)\cdot x + 2y)\frac{dy}{dx} = 1-\cos(xy)\cdot y$$

$$\frac{dy}{dx} = \frac{1-y\cdot\cos(xy)}{x\cdot\cos(xy)+2y}$$

(3) $= \lim_{x\to0}\frac{\sin x^2}{x^2\sin x^2} = \infty$

(4) $= \cancel{\lim_{x\to\infty} x a^{\frac1x}\left(1-\left(\frac ba\right)^{\frac1x}\right)}\ \lim_{t\to0}\frac{a^t-b^t}{t} = \lim_{t\to0} a^t\ln a - b^t\ln b$

$= \ln a - \ln b$

(5) $= e^{\lim_{x\to0}\frac1x\ln\frac{(1+x)^{\frac1x}}{e}} = \exp\left[\lim_{x\to0}\frac1x\left(\frac1x\ln(1+x)-1\right)\right]$

$= \exp\left[\lim_{x\to0}\frac1x\left(\frac1x\cdot\left(x-\frac{x^2}{2}\right)-1\right)\right] = \exp\left(-\frac12\right) = \frac{1}{\sqrt e}$

(6) $= \lim_{x\to0}\frac{\sin x\cos2x\cos3x + 2\cos x\sin2x\cos3x + 3\cos x\cos2x\sin3x}{\sin x}$

$= \lim_{x\to0}\left(\cos2x\cos3x + 4\cos^2x\cos3x + 3\cos x\cos2x(3-4\sin^2x)\right)$

$= 1+4+9=14$

(7) $\rho = \frac{(1+y'^2)^{3/2}}{|y''|} = \frac{\left(1+\left(\frac1x\right)^2\right)^{3/2}}{\left|-\frac{1}{x^2}\right|} = x^2\left[1+\frac{1}{x^2}\right]^{3/2}$

$\frac1\rho = \frac{1}{x^2\left(1+\frac1{x^2}\right)^{3/2}}$

2. 令 $f(x)=x^{\frac1x}$, $x\in[1,+\infty)$

$f'(x) = x^{\frac1x}\cdot\left(-\frac1{x^2}\ln x+\frac1{x^2}\right) = x^{\frac1x-2}(1-\ln x)$

$\therefore (1,e)\nearrow$, $(e,+\infty)\searrow$

而 $f(e)=e^{\frac1e}$ $\therefore \sqrt[n]{n}\le e^{\frac1e}$, $\forall n\in N^+$

又 $\because \sqrt{2} < \sqrt[3]{3}$

$\therefore A=\sqrt[3]{3}$, $n_0=3$

3. ~~x>0时, f'(x)=~~

只需 $\lim_{x\to0^+}f(x)=\lim_{x\to0^-}f(x)$ 且 $f'_+(x)=f'_-(x)$

$\lim_{x\to0^+}f(x)=0$, $\lim_{x\to0^-}f(x)=b \Rightarrow b=0$

$f'_+(x)=\lim_{x\to0^+}\frac{\tan^2(ax)}{x^2}=a^2$, $f'_-(x)=\lim_{x\to0^-}\frac{(2a-1)x+b}{x}=2a-1 \Rightarrow a=1$

4. 充分性: $\forall x_1,x_2\in I$, (设 $x_1<x_2$) $\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|}=|f'(\xi)|<M$, $\xi\in(x_1,x_2)$

$\therefore \forall\varepsilon>0, \exists\delta=\frac{\varepsilon}{M}, \forall |x_1-x_2|<\delta$, 有 $|f(x_1)-f(x_2)|<\delta M=\varepsilon$

必要性不成立: eg. $f(x)=\sqrt x$, $I=$ ~~[0,1]~~ $(0,1)$

5. ~~假设无界, 即∀M>0, ∃x₀, 使|f(x₀)|>M~~

~~不妨设M≥f(a), M≥f(b).~~ 令 $M=\max\{|f(a)|,|f(b)|,|f(\frac{a+b}{2})|\}$

~~令x=a, λx+(1-λ)y=x₀, 调整λ, 可使y取遍(x₀,b)~~

则 $f(\lambda a+(1-\lambda)b)\le\lambda f(a)+(1-\lambda)f(b)\le M$

$\therefore$ $\cancel{f(x)<-M}$ $f(x)\le M$, $\forall x\in[a,b]$

又取 $x=a$, $\lambda x+(1-\lambda)y=\frac{a+b}{2}$, 调整 $\lambda \in(0,\frac12)$, 可使 y 取遍 $(\frac{a+b}{2},b)$

则 $f(\frac{a+b}{2})\le\lambda f(a)+(1-\lambda)f(y)$

$-M\le\lambda M+(1-\lambda)f(y)$

$f(y)\ge-\frac{1+\lambda}{1-\lambda}M\ge-3M$, $\forall y\in(\frac{a+b}{2},b)$

同理 $f(y)\ge-3M$, $\forall y\in(a,\frac{a+b}{2})$ 闭区间凸函数⇒闭区间连续⇒有界

$\therefore \forall x\in[a,b]$, 有 $|f(x)|\le3M$

6. $\cancel{\frac{f(x)-f(0)}{x-0}=\lim_{\Delta x\to0}\frac{f(\theta x+\Delta x)-f(\theta x)}{\Delta x}}$

(导) $f'(x)=\theta f''(\theta x)x+f'(\theta x)$

(除) $\frac{f'(x)}{x}=\theta f''(\theta x)+\frac{f'(\theta x)}{x}$

(洛) $f''(x)=\theta f''(\theta x)+\theta f''(\theta x)$ (谢惠民P197有另解)

$\therefore \theta=\frac12$

5 2013-2014 第一学期 第一次测试

1. $\forall \varepsilon>0$, $\exists N=[\frac{\sqrt{2}}{2\varepsilon}]+1$, $\forall n>N$，有

$$\left|\frac{n}{2n+(-1)^n\sqrt{2}}-\frac{1}{2}\right|=\left|\frac{(-1)^n\sqrt{2}}{2(2n+(-1)^n\sqrt{2})}\right|\le\frac{\sqrt{2}}{2n}<\varepsilon$$

2. (1) × ~~$\exists\varepsilon=\frac{1}{2},\forall N>0,\exists x$~~ 取 $a_n=2n-1$, $b_n=2n$.

有 $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=+\infty$. 但 $\lim_{n\to\infty}f(a_n)=-1$, $\lim_{n\to\infty}f(b_n)=1$.

∴不收敛

(2) √ $u_k\triangleq\frac{\ln k}{k^3}$, $u_{k+1}<u_k$

交错级数审敛法 ⇒ 收敛

3. (1) $\sum_{k=1}^{n}\frac{1}{n+(-1)^k\sqrt{k}}>\sum_{k=1}^{n}\frac{1}{n+\sqrt{k}}>\sum_{k=1}^{n}\frac{1}{n+\sqrt{n}}=\frac{n}{n+\sqrt{n}}$

而 $\lim_{n\to\infty}\frac{n}{n+\sqrt{n}}=1$ ⇒ 原式 ≥ 1

同理，原式 ≤ 1

∴原式 $=1$

(2) $=\lim_{x\to0}(1+2x)^{\frac{1}{x}}=\sqrt{\lim_{x\to0}(1+2x)^{\frac{1}{2x}}}=\sqrt{e}$

(3) $\lim_{x\to0}\frac{\frac{1}{n}x+1-1}{x}=\frac{1}{n}$

(4) $=\lim_{n\to\infty}n^\alpha\left[\left(1+\frac{\ln n}{n}\right)^\alpha-1\right]=\lim_{n\to\infty}n^\alpha\cdot\frac{\alpha\ln n}{n}=\lim_{n\to\infty}\frac{\alpha\ln n}{n^{1-\alpha}}$

$=\lim_{x\to+\infty}\frac{\alpha\ln x}{x^{1-\alpha}}=\lim_{x\to+\infty}\frac{\alpha/x}{(1-\alpha)x^{-\alpha}}=0$

4. $A=\{f(x)\mid a<x<x_0\}$ 有上确界 β

$\forall\varepsilon>0$, $\exists x_1\in(a,x_0)$，有 $\beta-\varepsilon<f(x_1)\le\beta$. 取 $\delta=x_0-x_1>0$

$\forall\ 0<x_0-x<\delta$，有 $x_1<x<x_0$， $\beta-\varepsilon<f(x_1)<f(x)\le\beta$ 即 $|f(x)-\beta|<\varepsilon$

∴ $f(x_0^-)=\beta$ 存在，同理 $f(x_0^+)$ 也存在 (以上f改为g)

单侧连续的复合 ⇒ $f(x_0^-)$, $f(x_0^+)$ 存在

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5. $\forall\varepsilon>0$, $\exists N>0$, $\forall n>N$, 有 $|a_n-a|<\varepsilon$

$|a_{n_1}-a|<\varepsilon$, $|a_{n_2}-a|<\varepsilon\Rightarrow|a_{n_1}-a_{n_2}|\le|a_{n_1}-a|+|a_{n_2}-a|<2\varepsilon$

取 $\varepsilon=\frac{1}{2}$，则 $a_{n_1}=a_{n_2}$，即 $\forall n>N$，有 $a_n=C\Rightarrow|a_n-C|<\varepsilon$

∴ $\lim_{n\to\infty}a_n=C\Rightarrow a=C$

∴ $\exists N>0$, $\forall n>N$, 有 $a_n=a$

6. (1) ① $a_1\ge1$

② 假设 $a_k\ge1$，则 $a_{k+1}=1+\frac{2}{a_k}\ge1$

∴ $a_n\ge1$, $\forall n\Rightarrow a_n=1+\frac{2}{a_{n-1}}\le3$

~~$a_{n+1}-a_n=1+\frac{2}{a_n}-a_n=\frac{-a_n^2+a_n+2}{a_n}=-\frac{(a_n-2)(a_n+1)}{a_n}$~~

~~$a_{n+1}-a_n=\frac{2}{a_n}-\frac{2}{a_{n-1}}=\frac{2(a_{n-1}-a_n)}{a_na_{n-1}}$~~ (保留) ⇒ $a_{n+1}-a_n$ 与 $a_{n-1}-a_{n-2}$ 同号 ⇓

$a_{2k-1}<a_{2k}$, $a_{2k}>a_{2k+1}$

$a_n-a_{n-2}=\frac{2}{a_{n-1}}-\frac{2}{a_{n-3}}=\frac{2(a_{n-3}-a_{n-1})}{a_{n-1}a_{n-3}}$

∴ a_n-a_{n-2} 与 $a_{n-3}-a_{n-1}$ 同号

∴ a_n-a_{n-2} 与 $a_{n-2}-a_{n-4}$ 同号

而 $a_1=1$, $a_2=3$, $a_3=\frac{5}{3}$, $a_4=\frac{11}{5}\Rightarrow a_1<a_3$, $a_2>a_4$

∴ $a_{2k-1}<a_{2k+1}$, $a_{2k}>a_{2k+2}$

∴ $\{a_{2k-1}\}\nearrow$, $\{a_{2k}\}\downarrow$ 单调有界 ⇒ 收敛

∵ $a_{2k-1}<a_{2k}$

∴(保号性) $\lim_{k\to\infty}a_{2k-1}\le\lim_{k\to\infty}a_{2k}$

∵ $a_{2k}>a_{2k+1}$

∴(保号性) $\lim_{k\to\infty}a_{2k}\ge\lim_{k\to\infty}a_{2k+1}=\lim_{k\to\infty}a_{2k-1}$

→ $\lim_{k\to\infty}a_{2k-1}=\lim_{k\to\infty}a_{2k}=\lim_{n\to\infty}a_n$

(2) $a=1+\frac{2}{a}\Rightarrow\lim_{n\to\infty}a_n=a=2$

另解: $a_{2n+1}=3-\frac{4}{a_{2n-1}+2}$

归纳得 $a_{2k+1}<a_{2k+3}<2$ 单调有界

[6] 2013-2014 第一学期 第二次测试

1. $|x|>1$时，$f(x)=\cancel{0}\ \frac{1}{x}$

$$f(1)=\frac{1+a+b}{2}=\lim_{x\to1^+}f(x)=1\Rightarrow a+b=1$$
$$f(-1)=\frac{-1+a-b}{2}=\lim_{x\to1^-}f(x)=-1\Rightarrow a-b=-1$$
$$\Rightarrow a=0,\ b=1$$

验证：$|x|<1$时，$f(x)=x$，成立

~~$a=1,b=0$~~ $\therefore a=0,b=1$

2. $f(0)=\lim_{x\to0}f(x)=\lim_{x\to0}\frac{f(x)}{x}\cdot\lim_{x\to0}x=0$

$$f'(0)=\cancel{\lim_{x\to0}f'(x)=}\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=0$$

$$\lim_{x\to0}\frac{f(x)}{x^2}=\lim_{x\to0}\frac{f'(x)}{2x}=\lim_{x\to0}\frac{f''(x)}{2}=\frac{f''(0)}{2}$$

3. $$\lim_{x\to x_0}\frac{(x-x_0)f'(x_0)-[f(x)-f(x_0)]}{[f(x)-f(x_0)](x-x_0)f'(x_0)}=\lim_{x\to x_0}\frac{-\frac{1}{2}(x-x_0)^2f''(\cancel{x_0})}{[f(x)-f(x_0)](x-x_0)f'(x_0)}$$

$$=\lim_{x\to x_0}\frac{-\frac{1}{2}f''(x_0)}{f'(x_0)f'(\xi)}=-\frac{f''(x_0)}{2[f'(x_0)]^2}$$

4. $$\frac{dy}{dx}=\frac{y'}{x'}=\frac{e^t(\cos t-\sin t)}{e^t(\sin t+\cos t)}=\frac{1-\tan t}{\tan t+1}$$

$$\frac{d^2y}{dx^2}=\frac{-\sec^2x(\tan t+1)-(1-\tan t)\sec^2x}{(\tan t+1)^2e^t(\sin t+\cos t)}=-\frac{2\sec^2x}{(\tan t+1)^2e^t(\sin t+\cos t)}$$

5. $g(x)\triangleq|f(x)|$

~~$g(x)=$~~ 若在$x=0$的小邻域内$f(x)$同号，则 $\begin{cases}f(0)\geq0: & g'(0)=f'(0)\\ f(0)<0: & g'(0)=-f'(0)\end{cases}$

否则$f(0)=0$. 不妨设 $x\in(-\delta,0)$时 $f(x)<0$；$x\in(0,\delta)$时，$f(x)>0$.

则 $g'_+(0)=f'_+(0)=f'(0)$，$g'_-(0)=-f'_-(0)=-f'(0)$

要使 $g'(0)=g'_+(0)=g'_-(0)$，则必有 $f'(0)=0$

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6. ~~令$x=y$，$f(x^2)=2f(x)$~~

令$x=y=1$ $\begin{cases}f(1)=2f(1)\Rightarrow f(1)=0\\ \cancel{f(x)=2f(x^{\frac{1}{2}})=2^nf(x^{\frac{1}{2^n}})=\lim_{n\to\infty}2^nf(x^{\frac{1}{2^n}})}\end{cases}$

$$\lim_{y\to1}f(x_0y)=f(x_0)+\lim_{y\to1}f(y)=f(x_0)$$

$$\lim_{x\to x_0}f(x)\rightarrow 连续$$

7. $\Leftarrow\sqrt{n+1}\ln\sqrt{n}>\sqrt{n}\ln\sqrt{n+1}$

$\Leftarrow\frac{2\ln n}{\sqrt{n}}>\frac{2\ln(n+1)}{\sqrt{n+1}}$

$\Leftarrow\frac{\ln n}{\sqrt{n}}>\frac{\ln(n+1)}{\sqrt{n+1}}$

$f(x)\triangleq\frac{\ln x}{\sqrt{x}}$，$f'(x)=\frac{\frac{\sqrt{x}}{x}-\frac{\ln x}{2\sqrt{x}}}{x}=\frac{2-\ln x}{2x\sqrt{x}}<0$ ($x\geq10$时)

$\therefore f(x)$在$[10,+\infty)\downarrow$

$\therefore f(n)>f(n+1)$

8. ~~$\int_0^{10}v(t)dt=100$~~ 设$x(t)$表示第$t\sim t+1$秒跑过的路程.

若$\forall t\in[0,9]$，$x(t)>10$，则 $\sum_{t=0}^{9}x(t)>100$. ×

若$\forall t\in[0,9]$，$x(t)<10$，则 $\sum_{t=0}^{9}x(t)<100$. ×

若$\exists t_0\in[0,9]$，$x(t_0)=10$，则已成立

若$\exists t_1,t_2\in[0,9]$，$x(t_1)<10$，$x(t_2)>10$，则由介值定理，

$\exists t_0\in(t_1,t_2)$或$t_0\in(t_2,t_1)$，使$x(t_0)=10$.

9. 若$\exists f(t)\neq0$

$\Rightarrow\exists$~~最~~极大值$f(x_0)>0$ (~~最~~极小值同理)

$f'(x_0)=0$，$f''(x_0)\leq0$ 与 $f''(x_0)=e^{x_0}f(x_0)>0$矛盾

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7 2015-2016 第一学期 第一次测试

1. $=\lim\limits_{x\to1}\frac{x-1}{(x-1)(x+1)^2}=\lim\limits_{x\to1}\frac{1}{(x+1)^2}=\frac{1}{4}$ (不想用定义)

2. $=\frac{\sqrt{n}(-1)^n}{\sqrt{n+(-1)^n}+\sqrt{n}}$

$\lim\limits_{n\to\infty}[\sqrt{n}(\sqrt{n+(-1)^n}-\sqrt{n})]=\lim\limits_{n\to\infty}\frac{\sqrt{n}(-1)^n}{2\sqrt{n}}=\lim\limits_{n\to\infty}\frac{(-1)^n}{2}$ 不收敛

(2个子列分别收敛于 $\pm\frac{1}{2}$)

3. (1) $=0$

(2) $=\exp[\lim\limits_{n\to\infty}n\ln(1+2\sin\frac{1}{n})]=\exp[\lim\limits_{n\to\infty}n\ln(1+\frac{2}{n}+o(\frac{1}{n}))]=\exp[\lim\limits_{n\to\infty}n(\frac{2}{n}+o(\frac{1}{n}))]$ $=e^2$

4. (1) $=\lim\limits_{x\to1}\frac{(x+3)(x-1)}{2(x-1)(x+5)}=\lim\limits_{x\to1}\frac{x+3}{2(x+5)}=\frac{1}{3}$

(2) $=\lim\limits_{x\to0}\frac{1-\frac{1}{2}x^2-1+\frac{1}{2}(2x)^2}{x^2}=\frac{3}{2}$

5. 无界. $\forall M>0$, $\exists x_0=\frac{\pi}{2}M$, $x_0\sin x_0=\frac{\pi}{2}M>M$.

不是无穷大量. $\exists M>0$, $\forall x_0>0$, $\exists x=\pi x_0$, $|x\sin x|=0<M$

6. $\overline{\lim\limits_{n\to\infty}}(\ )=\frac{1}{2}$, $\underline{\lim\limits_{x\to\infty}}(\ )=-\frac{1}{2}$

7. $\lim\limits_{n\to\infty}\frac{a_n^2}{n}=\lim\limits_{n\to\infty}a_n^2-a_{n-1}^2=\lim\limits_{n\to\infty}\frac{a_n+a_{n-1}}{a_{n-1}}=\lim\limits_{n\to\infty}\frac{2a_{n-1}+\frac{1}{a_{n-1}}}{a_{n-1}}=2$

$\therefore\lim\limits_{n\to\infty}\frac{a_n}{\sqrt{n}}=\sqrt{2}$ $\quad(\lim\limits_{n\to\infty}\frac{1}{a_n}=0)$

8. $a_{n+1}-a_n=\frac{1}{2}(1-\frac{1}{n})\overset{>0}{(a_n-a_{n-1})}<\frac{1}{2}(a_n-a_{n-1})<\cdots<\frac{1}{2^n}$

$\therefore a_n<1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}<3$

单调有界 $\Rightarrow$ 收敛

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8 2015-2016 第一学期 第二次测试

1. (1) $(cx+d)f(x)=ax+b$

$(cx+d)C_n^0f^{(n)}(x)+C_n^1cf^{(n-1)}(x)=0$, $n\geq2$时

$\Rightarrow f^{(n)}(x)=-\frac{nc}{(cx+d)}f^{(n-1)}(x)=(-1)^{n-1}\frac{c^{n-1}n!}{(cx+d)^{n-1}}\left(\frac{ax+b}{cx+d}\right)'=(-1)^{n-1}n!c^{n-1}\cdot\frac{ad-bc}{(cx+d)^{n+1}}$

(2) $x'=1-\cos t$, $y'=\sin t$

$\frac{dy}{dx}=\frac{\sin t}{1-\cos t}$

$\frac{d^2y}{dx^2}=\frac{\cos t(1-\cos t)-(\sin t)^2}{(1-\cos t)^3}=-\frac{1}{(1-\cos t)^2}$

2. (1) $=\lim\limits_{x\to0}\frac{1-x\cot x}{x^2}=\lim\limits_{x\to0}\frac{-\cot x+x/\sin^2x}{2x}=\lim\limits_{x\to0}\frac{-\cos x+\frac{x}{\sin x}}{2x\sin x}$

$=\lim\limits_{x\to0}\frac{-\frac{1}{2}\sin2x+x}{2x^3}=\lim\limits_{x\to0}\frac{-\cos2x+1}{6x^2}=\lim\limits_{x\to0}\frac{2\sin2x}{12x}=\frac{1}{3}$

(2) $=\lim\limits_{x\to+\infty}[\frac{e}{2}x+x^2(e^{x\ln(1+\frac{1}{x})}-e)]$

$=\lim\limits_{x\to+\infty}[\frac{e}{2}x+x^2(e^{x(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3})}-e)]$

$=\lim\limits_{x\to+\infty}[\frac{e}{2}x+x^2e(e^{-\frac{1}{2x}+\frac{1}{3x^2}}-1)]$

$=\lim\limits_{x\to+\infty}[\frac{e}{2}x+x^2e(-\frac{1}{2x}+\frac{1}{3x^2}+\frac{1}{8x^2})]$

$=\frac{11e}{24}$

3. (1) ~~$f(t)=t^{\frac{1}{n}}$, $f'(t)=\frac{1}{n}t^{\frac{1}{n}-1}>0$, $f''(t)=(\frac{1}{n}-1)\frac{1}{n}t^{\frac{1}{n}-2}<0$~~

$f(x)\triangleq x^{\frac{1}{n}}-y^{\frac{1}{n}}-(x-y)^{\frac{1}{n}}$, $x\geq y$

$f'(x)=\frac{1}{n}x^{\frac{1}{n}-1}-\frac{1}{n}(x-y)^{\frac{1}{n}-1}=\frac{1}{n}\left(\frac{1}{x^{1-\frac{1}{n}}}-\frac{1}{(x-y)^{1-\frac{1}{n}}}\right)\leq0$

$\therefore f(x)\downarrow \Rightarrow f(x)\leq f(y)=0$

(2) $f(x)=\cos x-1+\frac{x^2}{2}$

$f'(x)=\cancel{\frac{1}{\sin^2 x}}\cancel{+x}-\sin x+x \quad (-\infty,0)\downarrow,\ (0,+\infty)\nearrow \Rightarrow f(x)>f(0)=0$

$f''(x)=1-\cos x\geqslant 0$

4. $f'(x)=-e^{-x}\sum\limits_{k=0}^{n}\frac{x^k}{k!}+e^{-x}\sum\limits_{k=1}^{n}\frac{x^{k-1}}{(k-1)!}$

$=-e^{-x}\cdot\frac{x^n}{n!}$

令 $f'(x)=0\Rightarrow x=0$

若n为奇，$f_{极大值}(x)=f(0)=1$，无极小值

若n为偶，$f'(x)\leqslant 0$，$f(x)$无极大、小值

5. $\because f(x)$一致连续

$\therefore \forall \varepsilon>0,\exists\delta>0,\ \forall |x_1-x_2|<\delta$, s.t. $|f(x_1)-f(x_2)|<\varepsilon$

令 $h(y_1)=y_1^{\alpha}-y_2^{\alpha}-(y_1-y_2)^{\alpha},\ y_1\geqslant y_2$

$h'(y_1)=\alpha y_1^{\alpha-1}-\alpha(y_1-y_2)^{\alpha-1}\leqslant 0$

$\therefore h(y_1)\downarrow\Rightarrow h(y_1)\leqslant h(y_2)=0$

$\therefore y_1^{\alpha}-y_2^{\alpha}\leqslant(y_1-y_2)^{\alpha}$

$\therefore \forall\varepsilon>0,\exists\delta>0,\ \forall|x_1-x_2|<\delta$, s.t. $|f(x_1)-f(x_2)|<\varepsilon^{\frac{1}{\alpha}}$

$\Rightarrow |g(x_1)-g(x_2)|\leqslant|f(x_1)-f(x_2)|^{\alpha}<\varepsilon$

6. $\cancel{P'(x)=\sum\limits_{k=1}^{n}x^{k-1}=\sum\limits_{k=0}^{n-1}x^k=\begin{cases}\frac{1-x^n}{1-x}, & x\neq 1\\ n, & x=1\end{cases}}$

~~$x\neq1$时，若n是偶数，$(-\infty,-1)\downarrow$，$(-1,1)\nearrow$，$(1,+\infty)\nearrow\Rightarrow P(x)\geqslant P(-1)$~~

$P(x)$的零点就是$f(x)=e^{-x}\sum\limits_{k=0}^{n}\frac{x^k}{k!}$的零点.

由4知，n为奇时，$f(x)$在$(-\infty,0)\nearrow$，$(0,+\infty)\downarrow$，在$(-\infty,0)$有一零点

n为偶时，$f(x)$在$R\downarrow$，$\lim\limits_{x\to-\infty}f(x)=+\infty$，$\lim\limits_{x\to+\infty}f(x)=0$，无零点.

$\boxed{15}$ 2019−2020 第一学期 期中考试

1.(1) $0<a_n<M$

~~$\forall\varepsilon>0,\exists n_0=\quad,\ \forall n>n_0$，有~~

$\cancel{\frac{a_n}{a_1+a_2+\cdots+a_n}}$ $\forall\varepsilon>0$,

①若$\exists\delta>0$，有无穷多项$a_k>\delta$，则$\exists n_0$，使$a_i(i=1,\cdots,n_0)$中有$\left[\frac{M}{\varepsilon\delta}\right]+1$项$>\delta$，则$\underbrace{\frac{a_n}{a_1+a_2+\cdots+a_n}}_{\forall n>n_0}<\frac{M}{\left[\frac{M}{\varepsilon\delta}+1\right]\cdot\delta}\leqslant\varepsilon$

$\therefore \lim\limits_{n\to\infty}\frac{a_n}{a_1+a_2+\cdots+a_n}=0$

②若$\forall\delta>0$，仅有有限项$a_k>\delta$，则$\exists n_0$，使$\forall n>n_0$，有$a_n\leqslant\delta$

取$\delta=\varepsilon a_1$，则$\frac{a_n}{a_1+a_2+\cdots+a_n}<\frac{\varepsilon a_1}{a_1}=\varepsilon$

$\therefore \lim\limits_{n\to\infty}\frac{a_n}{a_1+a_2+\cdots+a_n}=0$

(2) $=\lim\limits_{x\to+\infty}\left(x\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}+ax+b\right)$

$=\lim\limits_{x\to+\infty}\left(x\left(1+\frac{3}{2x}\right)+ax+b\right)$

$=\lim\limits_{x\to+\infty}\left[(1+a)x+\left(\frac{3}{2}+b\right)\right]=0\Rightarrow a=-1,\ b=-\frac{3}{2}$

(3) $=\lim\limits_{x\to x_0}\frac{(x-x_0)f'(x_0)-[f(x)-f(x_0)]}{[f(x)-f(x_0)](x-x_0)f'(x_0)}$

$=\lim\limits_{x\to x_0}\frac{\cancel{\frac{1}{2}f''(x_0)}[f'(x_0)\cdot x-f(x)]-[f'(x_0)\cdot x_0-f(x_0)]}{[f(x)-f(x_0)](x-x_0)f'(x_0)}$

同$\boxed{6}$T3

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(4) $\frac{dy}{dx}=\frac{\frac{2t}{1+t^2}}{\frac{1}{1+t^2}}=2t$

$\frac{d^2y}{dx^2}=2(1+t^2)$

(5) $f^{(n)}(0)=\cancel{x^2[\ln(1-x^2)]^{(n)}}\quad 2[\ln(1-x^2)]^{(n-2)}_{x=0}$

$=2\left[\frac{-2x}{1-x^2}\right]^{(n-3)}_{x=0}$

设 $g(x)=\frac{2x}{x^2-1}$，则 $(x^2-1)g(x)=2x$

$(x^2-1)g^{(k)}(x)+k\cdot 2x g^{(k-1)}(x)+k(k-1)g^{(k-2)}(x)=0$，$k\geqslant 2$时

令$x=0$，得 $g^{(k)}(0)=k(k-1)g^{(k-2)}(0)$

又$\because g(0)=0$，$g'(0)=-2$

$\therefore g^{(k)}(0)=\begin{cases}0, & k\text{为偶}\\ -2\cdot k!, & k\text{为奇}\end{cases}$

$\therefore f^{(n)}(0)=\begin{cases}0, & n\text{为奇}\\ -4\cdot(n-3)!, & n\text{为偶}\end{cases}$

(6) $=\lim\limits_{x\to 0}\frac{(1-\frac{1}{2}\sin^2x+\frac{1}{24}\sin^4x)-(1-\frac{1}{2}x^2+\frac{1}{24}x^4)}{x^4}$

$=\lim\limits_{x\to 0}\frac{1}{x^4}\left[-\frac{1}{2}(x-\frac{1}{6}x^3)^2+\frac{1}{24}x^4+\frac{1}{2}x^2-\frac{1}{24}x^4\right]$

$=\lim\limits_{x\to 0}\frac{1}{x^4}\cdot(\frac{1}{6}x^4)=\frac{1}{6}$

~~2. $g(x)=f^{-1}(x)$~~

~~$[f(x^2)]''=f''(x^2)\cdot 2x\Rightarrow [f(x^2)]''_{x=0}=0$~~

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3\. $f'(0)=\lim\limits_{x\to 0}\frac{f(x)}{x}=x^{\alpha-1}\sin\frac{1}{x}$

(1) $\alpha\leq 1$

(2) $x\neq 0$时，$f'(x)=\alpha x^{\alpha-1}\sin\frac{1}{x}+x^{\alpha}\cos\frac{1}{x}\cdot(-\frac{1}{x^2})$

$=x^{\alpha-1}(\alpha\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x})$

$\lim\limits_{x\to 0}f'(x)\neq f'(0)=0\Rightarrow 1<\alpha\leq 2$

(3) $\alpha>2$

4\. ~~$x_0=\lim\limits_{n\to\infty}x_n$~~ 若$\exists x_1,x_2\in[a,b]$，~~$f(x_1)f(x_2)<0$~~ $(f(x_1)-A)(f(x_2)-A)<0$，则$\exists x_0$介于x_1,x_2之间，$f(x_0)=A$

否则，不妨设 $f(x)\geqslant A$.

$\because \forall\varepsilon>0$，$\exists n_0>0$，$\forall n>n_0$，$|f(x_n)-A|<\varepsilon\Rightarrow A\leq f(x_n)<A+\varepsilon$

$\therefore \inf\{f(x_n)\}=A\Rightarrow$ ✓

(否则，设 $\inf\{f(x_n)\}=m>A$，取$\varepsilon=m-A$，则$\forall n$，$f(x_n)\geqslant A+\varepsilon$，矛盾)

收敛子列 $\{x_{n_k}\}$，$f(x_0=\lim\limits_{k\to\infty}x_{n_k})=\lim\limits_{k\to\infty}f(x_{n_k})=\lim\limits_{n\to\infty}f(x_n)=A$

5\. (1) $\forall\varepsilon>0$，$\exists\delta=\frac{\varepsilon}{\max\{|f'(x)|\}}$，$\forall x_1,x_2\geqslant a$且$|x_1-x_2|<\delta$，有

$|f(x_1)-f(x_2)|=|f'(\xi)(x_1-x_2)|=|f'(\xi)|\cdot|x_1-x_2|<\varepsilon$

(2) $\frac{|f(x)|}{x}\leq\frac{|f'(\xi)|(x-a)+|f(a)|}{x}<|f'(\xi)|+\frac{|f(a)|}{a}<K$ (有界)

$\left|\frac{f(x_1)}{x_1}-\frac{f(x_2)}{x_2}\right|=\frac{|x_2f(x_1)-x_1f(x_2)|}{x_1x_2}=\frac{|(x_2-x_1)(f(x_1)-f(x_2))+(x_2f(x_2)-x_1f(x_1))|}{x_1x_2}$

$\leq\frac{(x_1-x_2)^2|f'(\xi_1)|}{x_1x_2}+\frac{|f(\xi_2)+\xi_2f'(\xi_2)|\cdot|x_1-x_2|}{x_1x_2}$

$\leq|x_1-x_2|\cdot\left(\frac{|x_1-x_2|\cdot M}{a^2}+\frac{|f(\xi_2)|+\xi_2|f'(\xi_2)|}{a\xi_2}\right)$

$<\frac{\delta^2M}{a^2}+\cancel{\frac{2\delta M}{a}}\ \frac{\delta K}{a}+\frac{\delta M}{a}<\varepsilon$ $(\exists\delta)$

6\. (1) $g(x)=e^{-x}f(x)$

$g'(x)=-e^{-x}(f(x)-f'(x))<0$

$g(x)\downarrow$，$g(x)<g(0)=1\Rightarrow f(x)<e^x$

(2) $g(x)=e^{-x}(f(x)+f'(x))$

$g'(x)=-e^{-x}(f(x)-f''(x))<0$

$g(x)<g(0)\leq 2\Rightarrow f(x)+f'(x)\leq 2e^x$

~~$h(x)=e^x(f(x)-f'(x))>h(0)\geq 0$~~

$h(x)=e^x(f(x)-e^x)<h(0)=0$

$\therefore f(x)<e^x$

16 2020-2021 第一学期 期中考试

1.(1) $\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}} < \sqrt[n]{1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\cdots+\frac{1}{2^{[\log_2 n]}}} < \sqrt[n]{[\log_2 n]+2}$

$< \sqrt[n]{\log_2 n+3} < 1+\varepsilon$

$\sqrt[n]{n} = 1+\lambda_n \to 0$

$n = 1+n\lambda_n+\frac{n(n-1)}{2}\lambda_n^2$

$> 1+\frac{n(n-1)}{2}\lambda_n^2$

$\lambda_n < \sqrt{\frac{2}{n}}$

$\Leftarrow \log_2 n+3 < (1+\varepsilon)^n$

$\Leftarrow \log_2 n+2 < n\varepsilon$

而 $\lim_{n\to\infty}\frac{\log_2 n+2}{n}=0$

∴成立，即 $\lim_{n\to\infty}\sqrt[n]{1+\frac{1}{2}+\cdots+\frac{1}{n}}=1$

(2) $=\lim_{x\to\infty}\left(2e^{\frac{1}{x}}+\frac{\cos x}{x}\right)=2+0=2$

(3) $=\lim_{x\to\infty}\frac{\frac{1}{2}x-\frac{1}{6}x}{\frac{1}{3}x}=1$

(4) $f'(x)=(\sin x)^{\cos x}(\cos x\cdot\ln\sin x)'$

$=(\sin x)^{\cos x}\left(-\sin x\cdot\ln\sin x+\cos x\cdot\frac{1}{\sin x}\cdot\cos x\right)$

$=(\sin x)^{\cos x}\left(-\sin x\cdot\ln\sin x+\frac{\cos^2 x}{\sin x}\right)$

(5) $f(x)=\left(2-1+\frac{1}{2}x^2-\frac{1}{24}x^4\right)^{\frac{1}{3}}+o(x^5)$

$=\left(1+\frac{1}{2}x^2-\frac{1}{24}x^4\right)^{\frac{1}{3}}+o(x^5)$

$=1+\frac{1}{3}\left(\frac{1}{2}x^2-\frac{1}{24}x^4\right)+\frac{\frac{1}{3}\cdot(-\frac{2}{3})}{2}\cdot\left(\frac{1}{2}x^2-\frac{1}{24}x^4\right)^2+o(x^5)$

$=1+\frac{1}{6}x^2-\frac{1}{72}x^4-\frac{1}{9}\left(\frac{1}{4}x^4\right)+o(x^5)$

$=1+\frac{1}{6}x^2-\frac{1}{24}x^4+o(x^5)$

2. $a_{n+1}=-(a_n-1)^2+1\le 1$ (有界)

归纳可知 $a_n>0$

$a_{n+1}-a_n=a_n(1-a_n)\ge 0$ (单调)

∴$\{a_n\}$收敛

设 $\lim_{n\to+\infty}a_n=a_0$，则 $a_0=a_0(2-a_0)\Rightarrow a_0=1$

3. $\lim_{x\to0^+}f(x)=0\Rightarrow a=2$

$\lim_{x\to0^-}f(x)=0\Rightarrow b=1$

此时，$f'_+(x)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{x^2\sin\frac{1}{x}}{x}=\lim_{x\to0^+}x\sin\frac{1}{x}=\lim_{x\to0^+}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=1$

$f'_-(x)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{x\cos x}{x}=\lim_{x\to0^+}\cos x=1$

∴可导

4. $f(x)=x^2-x\sin x-\cos x+\frac{1}{2}$

$f'(x)=2x-x\cos x-\sin x+\sin x=(2-\cos x)x$

∴$f(x)$在$(-\infty,0)\downarrow$, $(0,+\infty)\uparrow$

而 $\lim_{x\to-\infty}f(x)=+\infty$, $\lim_{x\to+\infty}f(x)=+\infty$, $f(0)=-\frac{1}{2}<0$

∴恰只 $\exists x_1\in(-\infty,0)$, $x_2\in(0,+\infty)$, 使 $f(x_{1,2})=0$

5. $\forall\varepsilon>0$, $\exists\delta>0$, $\forall|x_1-x_2|<\delta$,

① 若 $x_1,x_2\in(a,b]$ 或 $x_1,x_2\in[b,c)$，则已成立

② ~~若x~~ 否则，不妨设 $x_1\in(a,b]$, $x_2\in[b,c)$. 取 $\delta=\min\{\frac{b-a}{2},\frac{c-b}{2}\}$

∵在$(a,b]$和$[b,c)$连续

∴在(a,c)连续

∴在$[b-\delta,b+\delta]$连续

∴在$[b-\delta,b+\delta]$一致连续

∵$x_1,x_2\in[b-\delta,b+\delta]$

∴$|f(x_1)-f(x_2)|<\varepsilon$

∴$f(x)$在(a,c)上一致连续

$\le|f(x_1)-f(b)|+|f(b)-f(x_2)|$

6. ~~$\frac{xdx}{a^2}+\frac{ydy}{b^2}=0$~~

在$(x_0,y_0)(x_0>0,y_0>0)$处的切线为 $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$

令$x=0$，得 $y=\frac{b^2}{y_0}$

令$y=0$，得 $x=\frac{a^2}{x_0}$

$\therefore S=\frac{1}{2}\cdot\frac{b^2}{y_0}\cdot\frac{a^2}{x_0}=\frac{ab}{2\sin\theta\cos\theta}=\frac{ab}{\sin 2\theta}$

当且仅当 $\theta=\frac{\pi}{4}$ 时有 $S_{min}=ab$

此时，$x_0=a\sin\theta=\frac{\sqrt{2}}{2}a$，$y_0=\frac{\sqrt{2}}{2}b$

切线：$\frac{\sqrt{2}x}{2a}+\frac{\sqrt{2}y}{2b}=1$

7. 原式⇔ $\frac{\frac{f(b)}{b}-\frac{f(a)}{a}}{\frac{1}{b}-\frac{1}{a}}=f(\xi)-\xi f'(\xi)$

设 $g(x)=\frac{f(x)}{x}$，$h(x)=\frac{1}{x}$

由柯西中值定理，$\exists\xi\in(a,b)$ $\frac{g(b)-g(a)}{h(b)-h(a)}=\frac{g'(\xi)}{h'(\xi)}=\frac{\frac{f'(\xi)\xi-f(\xi)}{\xi^2}}{-\frac{1}{\xi^2}}=f(\xi)-\xi f'(\xi)$

8. $\lim_{x\to0}\frac{f(x)-f(ax)}{x-ax}=\frac{A}{1-a}$

$\lim_{x\to0}\frac{f(x)-f(ax)}{x}=\lim_{x\to0}\left[\frac{f(x)-f(0)}{x-0}-\frac{f(ax)-f(0)}{x-0}\right]$

$=\lim_{x\to0}\left[\frac{f(x)-f(0)}{x-0}-a\cdot\frac{f(ax)-f(0)}{ax-0}\right]$

$=f'(0)-af'(0)$

$A'\triangleq\frac{A}{1-a}$ $\quad A'-\frac{\varepsilon}{2}<\frac{f(a^{n-1}x)-f(a^nx)}{a^{n-1}x-a^nx}<A'+\frac{\varepsilon}{2}$

$A'-\frac{\varepsilon}{2}<\frac{f(x)-f(a^nx)}{x-a^nx}<A'+\frac{\varepsilon}{2}$

令$n\to\infty$ $\quad A'-\frac{\varepsilon}{2}\le\frac{f(x)-f(0)}{x-0}\le A'+\frac{\varepsilon}{2}$

$\boxed{13}$ 2017-2018第一学期期中考试

2. $f(x)=\sin\frac{1}{x}$

取$\varepsilon=1$，$\forall\delta>0$，$\exists x_1=\frac{1}{2n\pi-\frac{\pi}{2}}$，$x_2=\frac{1}{2n\pi+\frac{\pi}{2}}$ $(n=[\max\{\frac{1}{2\delta},1\}]+1)$

$|x_1-x_2|=\frac{\pi}{(2n\pi)^2-\frac{\pi^2}{4}}<\frac{\pi}{2n\pi}=\frac{1}{2n}<\delta$

而$|f(x_1)-f(x_2)|=2>\varepsilon$ ⇒ 不一致连续

3. (1) $=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1}\cdot\frac{1}{1-\frac{1}{n+1}}$

$=\frac{1}{e}$

(2) $=\lim_{x\to0}\frac{x\cdot\frac{x}{2}\cdot x^2}{[(x-\frac{x^3}{6})-x(1-\frac{x^2}{2})]x}=\frac{3}{4}$

(3) $=\lim_{x\to1^-}\frac{\ln(1-x)}{\frac{1}{\ln x}}=\lim_{x\to1^-}\frac{\frac{1}{x-1}}{-\frac{1}{\ln^2x}\cdot\frac{1}{x}}=\lim_{x\to1^-}\frac{\ln^2x}{1-x}$

$=\lim_{x\to1^-}\frac{2\ln x\cdot\frac{1}{x}}{-1}=0$

(4) $=\lim_{x\to\infty}\left(x-x^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)\right)=\frac{1}{2}$

4. $\sum_{k=1}^{n}\frac{k}{n^2+k}>\sum_{k=1}^{n}\frac{k}{n^2+n}=\frac{(1+n)n/2}{n^2+n}\to\frac{1}{2}$，$n\to\infty$时

$\sum_{k=1}^{n}\frac{k}{n^2+k}<\sum_{k=1}^{n}\frac{k}{n^2}=\frac{(1+n)n/2}{n^2}\to\frac{1}{2}$，$n\to\infty$时

$\therefore\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2+k}=\frac{1}{2}$

5. 同$\boxed{4}$T3.

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6. $y'+2^y\ln2\cdot y'-1-\cos x=0$

$x=0$时 $y+2^y=1 \Rightarrow y=0$

代入上式,得 $y'+\ln2\cdot y'-2=0 \Rightarrow y'(0)=\frac{2}{\ln2+1}$

7. $g(x)=e^{-\frac{x}{b-a}}(f(x)-f(a))$, $g(a)=0$

$g'(x)=-e^{-\frac{x}{b-a}}\cancel{\left(\frac{f(x)}{b-a}-f'(x)\right)}\cdot\left(\frac{f(x)-f(a)}{b-a}-f'(x)\right)$

若$\exists c\in(a,b]$, $f(c)=f(a)\Rightarrow g(c)=0$, 则$\exists\xi\in(a,c)$, $g'(\xi)=0\Rightarrow\checkmark$

否则,不妨设 $\forall x\in(a,b]$, $f(x)>f(a)$, 则

$g'(x_0)=-e^{-\frac{x_0}{b-a}}\cdot\left(\frac{f(x_0)-f(a)}{b-a}-0\right)<0$

~~$\lim_{x\to a^+} f'_+(a)>0\Rightarrow f(a+\delta)>0\ (\exists\delta>0)$~~

~~$\therefore g'(a+\delta$~~

$g(x_0)>0\Rightarrow g'(\eta)=\frac{g(x_0)-g(a)}{x_0-a}>0\quad(\exists\eta\in(a,x_0))$

$\therefore\exists\xi\in(\eta,x_0)$, s.t. $g'(\xi)=0$

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$\boxed{14}$ 2018-2019 第一学期 期中考试

2.(1) $=\lim_{n\to\infty}[\ln(n+1)+\ln n]\cdot[\ln(n+1)-\ln n]$

$=\lim_{n\to\infty}\ln[n(n+1)]\cdot\ln\frac{n+1}{n}$

$=\lim_{n\to\infty}[\ln(n+1)+\ln n]\cdot\ln(1+\frac{1}{n})$

$=\lim_{n\to\infty}\frac{\ln(n+1)+\ln n}{n}$

$=\lim_{n\to\infty}[\ln(n+1)-\ln(n-1)]=0$

(2) $=\lim_{n\to\infty}n^{\frac{2}{3}}\cdot n^{\frac{1}{3}}[(1+\frac{1}{n})^{\frac{1}{3}}-1]$

$=\lim_{n\to\infty}n\cdot\frac{1}{3n}=\frac{1}{3}$

(3) $=\lim_{x\to+\infty}\left(1-\frac{1}{x+2}\right)^{x+2}\cdot\left(1-\frac{1}{x+2}\right)^{-2}=\frac{1}{e}$

(4) $=\cancel{\lim_{x\to+\infty}}-\frac{e}{2}$

3.(1) $=\frac{1}{\tan\frac{x}{2}}\cdot\sec^2\frac{x}{2}\cdot\frac{1}{2}=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{1}{\sin x}$

(2) $=\frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\cdot\left(-\frac{2}{(1+x^2)^2}\right)2x=-\frac{2}{1+x^2}$

(3) $=\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)=-\frac{x}{\sqrt{1-x^2}}$

(4) $=xe^x+ne^x$

4. $a_n\in[1,2]$

$a_{n+1}-a_n=\frac{1}{a_n}-\frac{1}{a_{n-1}}=\frac{a_{n-1}-a_n}{a_na_{n-1}}$

$\{a_{2k-1}\}\nearrow$, $\{a_{2k}\}\searrow$ 同$\boxed{5}$T6

6. $g(x)=xf(x)+x-1$

$g(0)=-1<0$, $g(1)=f(1)\geq 0$

$\therefore \exists x_0\in(0,1]$, 使 $g(x_0)=0 \Rightarrow f(x_0)=\frac{1-x_0}{x_0}$

设 $x_1\neq x_2$ 满足 $f(x_1)=\frac{1-x_1}{x_1}$, $f(x_2)=\frac{1-x_2}{x_2}$

则 $f(x_1)-f(x_2)=\frac{x_2-x_1}{x_1x_2}$

$|f(x_1)-f(x_2)|=\frac{1}{x_1x_2}|x_2-x_1|>|x_2-x_1|$, 不符

$\therefore x_0$唯一

7. $(f''(x))^2\leq(f'(x))^2$

$[f'(x)-f''(x)]\cdot[f'(x)+f''(x)]\geq 0$

~~$e^x[f'(x)-f''(x)]\cdot e^{+x}[f'(x)+f''(x)]\geq 0$~~

~~$[e^{-x}f'(x)]'\cdot[e^x f'(x)]'\leq 0$~~

若不严格单调, 则 $\exists x_0$, s.t. $f'(x_0)=0$, ~~$f''(x_0)\neq 0$~~

代入上式, 得 $-[f''(x_0)]^2\geq 0$, 矛盾

(反证法)若不(严格单调), 则 $\exists x_0$, s.t. $f'(x_0)=0$

设 $M=\max\limits_{x\in[x_0-\frac12,x_0+\frac12]}|f'(x)|=|f'(x_m)|$

(微分中值定理) $\frac{M}{\frac12}\leq\left|\frac{f'(x_m)-f'(x_0)}{x_m-x_0}\right|=|f''(\xi)|\leq|f'(\xi)|\leq M\Rightarrow M=0$

$\therefore f'(x)\equiv 0$, $\forall x\in[x_0-\frac12,x_0+\frac12]$

一直往外拓, 得 $f'(x)\equiv 0$, $\forall x\in(-\infty,+\infty)$, 与"非常值"矛盾

$\boxed{12}$ 2007-2008第一学期期中考试

2. (1) $\sum_{k=1}^{n}\frac{1}{(n^3+n-k)^{\frac13}}<\sum_{k=1}^{n}\frac{1}{n^{3\cdot\frac13}}=1$

$\sum_{k=1}^{n}\frac{1}{(n^3+n-k)^{\frac13}}>\sum_{k=1}^{n}\frac{1}{(n^3+n)^{\frac13}}=\frac{1}{(1+\frac{1}{n^2})^{\frac13}}\to 1$

$\therefore$原式$=1$

(2) $=\lim\limits_{x\to0}\frac{x^3\sin^2\frac1x}{\frac12x^2}=\lim\limits_{x\to0}2x\sin^2\frac1x=0$

(3) $=\lim\limits_{x\to+\infty}\left(\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}}\right)^2\cdot\left(1+\frac{2}{x-1}\right)=e^2$

(4) $=\lim\limits_{x\to0}\frac{\frac16x^3}{x^3}=\frac16$

3. (1) $=2^x\ln2\cdot\sin x+2^x\cos x+\frac{2x}{x^2+1}$

4. ~~$f'(x)=$~~ $f(x)=\frac12\sin 2x(\sin x+\cos x)$

$=\frac{\sqrt2}{2}\sin 2x\cdot\sin(x+\frac{\pi}{4})$

$=\frac{\sqrt2}{4}(\cos(3x+\frac{\pi}{4})+\cos(x-\frac{\pi}{4}))$

$f'(x)=\frac{\sqrt2}{4}(+3\sin(3x+\frac{\pi}{4})-\sin(x-\frac{\pi}{4}))$

$x\in[0,\frac{\pi}{4})$时, $f'(x)\geq0$; $x=\frac{\pi}{4}$时, $f'(x)=0$; $x\in(\frac{\pi}{4},\frac{\pi}{2}]$时, $f'(x)\leq0$

$\therefore f(x)_{max}=$ ~~$f(\frac{\pi}{4})-\max\{f(0),f(\frac{\pi}{2})\}=0$~~ $f(\frac{\pi}{4})=\frac{\sqrt2}{2}$

5. $g(x)=e^{x^2}f(x)$, $g(a)=g(b)=0$

$g'(x)=e^{x^2}(2xf(x)-f'(x))\leq 0$

若$\exists c\in(a,b)$, s.t. $f(c)>0$ (<0同理) $\Rightarrow g(c)>0$

则$\exists\xi\in(a,c)$, s.t. $g'(\xi)=\frac{g(c)-g(a)}{c-a}>0$, 矛盾

$\therefore f(x)\equiv 0$, $\forall x\in[a,b]$

6. $g(x)=A[f(x)]^2+B[f'(x)]^2$

$g'(x)=A\cdot 2f(x)\cdot f'(x)+B\cdot 2f'(x)\cdot f''(x)$

$=2f'(x)[Af(x)+Bf''(x)]$

令$A=1, B=\frac{1}{2}$，有 $g'(x)=-x[f'(x)]^2$

∴ $x<0$时 $g'(x)>0$, $g(x)\nearrow$; $x>0$时 $g'(x)<0$, $g(x)\searrow$

$\Rightarrow \underset{max}{g(x)}=g(0)$

∴ $f(x)$和$f'(x)$都有界

7.(1) $f(x)=x+x^n$ 在$(0,+\infty)\nearrow$

$f(0)=0<1$, $f(1)=2>1$

∴恰有一正根

(2) $x_n+x_n^n=1$

$x_{n+1}+x_{n+1}^{n+1}=1$

若 $x_{n+1}\geqslant x_n$，则 $x_{n+1}^{n+1} > x_n^n$，$x_{n+1}+x_{n+1}^{n+1}>x_n+x_n^n=1$，矛盾

∴ $x_{n+1}<x_n$

而 $x_n\in(0,2)$

∴ $\{x_n\}$收敛

若 $\lim\limits_{n\to\infty} x_n=x_0<1$，则 $\lim\limits_{n\to\infty}(x_n+x_n^n)=x_0+\lim\limits_{n\to\infty}x_0^n<1+0=1$，矛盾

若 $x_0>1$ 同理

∴ $\lim\limits_{n\to\infty} x_n=x_0=1$

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18 2012-2013 第一学期 第三次测试

1.(1) $=\int x(x^7-7x^6+21x^5-35x^4+35x^3-21x^2+7x-1)dx$

$=\int(x^8-7x^7+21x^6-35x^5+35x^4-21x^3+7x^2-x)dx$

$=\frac{1}{9}x^9-\frac{7}{8}x^8+3x^7-\frac{35}{6}x^6+7x^5-\frac{21}{4}x^4+\frac{7}{3}x^3-\frac{1}{2}x^2+C$

(2) $=\int 2\sin x\cos^3x\,dx$

$=\int -2\cos^3x\,d\cos x$

$=-\frac{1}{2}\cos^4x+C$

(3) 令 $t^2=x$, $t=\sqrt{x}$

原式 $=2\int t\sin t\,dt=-2\int t\,d\cos t$

$=-2(t\cos t-\int\cos t\,dt)$

$=-2t\cos t+2\sin t+C$

$=-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C$

(4) 令 $x=\sinh t$

原式 $=\int\ln(\sinh t+\cosh t)\,d\sinh t$

$\cancel{=\int t\cosh t\,dt}=\int t\,d\sinh t$

$=t\sinh t-\int\sinh t\,dt$

$=t\sinh t-\cosh t+C$

2.(1) $=-\int_0^1 x\,d\arcsin x+x\arcsin x\Big|_0^1$

$=-\int_0^{\frac{\pi}{2}}\sin t\,dt+\frac{\pi}{2}=-\cos t\Big|_0^{\frac{\pi}{2}}+\frac{\pi}{2}=1+\frac{\pi}{2}$

(2) 令 $x=\cancel{\sin t}\tan t$

原式 $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{t}{\tan^2t}\cdot\sec^2t\cdot dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{t}{\sin^2t}dt=-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}t\,d\cot t$

$=-t\cot t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot t\,dt=\frac{\pi}{4}+\ln\sin t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{\pi}{4}+\ln\sqrt{2}$

(3) $=\int_0^1\left(-\frac{x}{x^2+1}+\frac{1}{x+1}\right)dx$

$=-\int_0^1\frac{dx^2}{2(x^2+1)}+\int_0^1\frac{dx}{x+1}$

$=-\frac{1}{2}\ln(x^2+1)\Big|_0^1+\ln(x+1)\Big|_0^1=\frac{1}{2}\ln 2$

(4) ~~设 $t=\sqrt{1-\cos x}$~~ $\because f(x)$关于$x=\pi$是奇函数

$\therefore$原式$=0$

3. (1) $\lim\limits_{x\to\infty}\frac{1}{4x^3}\cdot\frac{x^8}{1+x^6}=0$

(2) ~~$=\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}\frac{1}{1+\frac{k}{n^2}}\cdot\frac{1}{n}$~~

$\sum\limits_{k=1}^{n}\frac{n}{n^2+k}<\sum\limits_{k=1}^{n}\frac{n}{n^2}=1$

$\sum\limits_{k=1}^{n}\frac{n}{n^2+k}>\sum\limits_{k=1}^{n}\frac{n}{n^2+n}=\frac{n}{n+1}\to 1\ (n\to\infty)$

$\therefore$原式$=1$

4. $s=\int_0^{+\infty}\sqrt{r^2+r'^2}\,d\theta$

$=\int_0^{+\infty}\sqrt{2e^{-2\theta}}\,d\theta=\int_0^{+\infty}\sqrt{2}\cdot e^{-\theta}d\theta=-\sqrt{2}e^{-\theta}\Big|_0^{+\infty}=\sqrt{2}$

5. $I^2=\left(\int_0^1 x f(x)\,dx\right)^2\leq\left(\int_0^1 x^2dx\right)\left(\int_0^1 f^2(x)dx\right)=\frac{1}{3}\cdot\frac{1}{3}(6I-1)$

$9I^2-6I+1\leq 0\Rightarrow$ "="

$\therefore\frac{f(x)}{x}=C$ 代入得$C=1$

$\Rightarrow f(x)=x$

6. $g(x,y)=\int_0^x f(t+y)\,dt-\int_0^x f(t)\,dt$

$=\int_y^{x+y}f(t)\,dt-\int_0^x f(t)\,dt$

$=\left(\int_y^{x+y}f(t)dt+\int_x^y f(t)\,dt\right)-\left(\int_0^x f(t)\,dt+\int_x^y f(t)\,dt\right)$

$=\int_x^{x+y}f(t)\,dt-\int_0^y f(t)\,dt=g(y,x)$

$\boxed{20}$ 2013-2014第一学期第三次测试

1. (1) $=\int\frac{(x-3)+8}{(x-3)^2+4}\,d(x-3)$

设 $x-3=2\tan t$

原式$=\int(\tan t+4)\cos^2 t\,d\tan t$

$=\int(\tan t+4)\,dt$

$=-\ln\cos t+4t+C$

(2) 设 $\sqrt{1-x}=t$, $x=1-t^2$

原式$=\int_1^0(1-t^2)t\,d(1-t^2)$

$=\int_1^0 2t^2(t^2-1)\,dt=\int_1^0(2t^4-2t^2)dt$

$=\left(\frac{2}{5}t^5-\frac{2}{3}t^3\right)\Big|_1^0=\frac{4}{15}$

(3) $=-\int_1^{+\infty}\ln x\,d\frac{1}{1+x}=-\frac{\ln x}{1+x}\Big|_1^{+\infty}+\int_1^{+\infty}\frac{1}{(1+x)x}dx$

$=\int_1^{+\infty}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx=\left(\ln x-\ln(x+1)\right)\Big|_1^{+\infty}=\ln\frac{x}{x+1}\Big|_1^{+\infty}=\ln 2$

(4) $=\begin{cases}\frac{1}{3}x^3-\frac{2}{3}+C, & x\leq -1\\ x+C, & -1<x<1\\ \frac{1}{4}x^4+\frac{3}{4}+C, & x\geq 1\end{cases}$

2. (1) $=\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n\sqrt{1+\frac{3k}{n}}}=\int_0^1\frac{1}{\sqrt{1+3x}}dx$

$=\frac{2}{3}\sqrt{1+3x}\Big|_0^1=\frac{2}{3}$

(2) $=\lim\limits_{\xi\to+\infty}a\frac{(\ln\xi)^n}{\xi+2}=\begin{cases}0, & n=0,1\\ \lim\limits_{\xi\to+\infty}an\frac{(\ln\xi)^{n-1}}{\xi}=0, & n\geq 2\end{cases}=0$, $n\in N$

3. $\begin{cases}x=2y^2\\ x=1+y^2\end{cases}\Rightarrow(x,y)=(2,\pm1)$

$S=\int_{-1}^1(1+y^2-2y^2)\,dy=\int_{-1}^1(1-y^2)\,dy=\left(y-\frac{1}{3}y^3\right)\Big|_{-1}^1=\frac{4}{3}$

4. (1) $V_1=\int_a^2 \pi y^2 dx = 4\pi\int_a^2 x^4 dx = 4\pi\cdot\frac{1}{5}x^5\Big|_a^2=\frac{4\pi}{5}(32-a^5)$

$V_2=\int_0^{2a^2}\pi(a^2-x^2)dy=\int_0^{2a^2}\pi(a^2-\frac{y}{2})dy$

$=(\pi a^2 y-\frac{\pi y^2}{4})\Big|_0^{2a^2}=2\pi a^4-\pi a^4=\pi a^4$

(2) $V_1+V_2=\frac{4\pi}{5}(32-a^5)+\pi a^4 \cancel{=a^4} \triangleq f(a)$

$f'(a)=-4\pi a^4+\cancel{(8\pi-4)}\,4\pi\, a^3 = 4\pi a^3(-a+1)$

令 $f'(a)=0\Rightarrow a=1$

$f(a)_{max}=f(1)=\frac{4\pi}{5}\times 31+\pi=\frac{129\pi}{5}$

5. √

√6. $\int_a^b xf(x)dx=\int_a^b x\,dF(x)= xF(x)\Big|_a^b-\int_a^b F(x)dx$ $\quad (F(x)\triangleq\int_a^x f(t)dt)$

$=b\int_a^b f(t)dt-\int_a^b F(x)dx$

同理 $\int_a^b xg(x)dx=b\int_a^b g(t)dt-\int_a^b G(x)dx$

$\therefore \int_a^b xf(x)dx-\int_a^b xg(x)dx=\int_a^b[G(x)-F(x)]dx\le 0$

7. $F(x)\triangleq\int_a^x f(x)dx\Rightarrow F(a)=F(b)=0$

~~$\exists x_0\in(a,b)$, s.t. $f(x_0)=F'(x_0)=0$~~

$\mathcal{F}(x)\triangleq\int_a^x F(x)dx$

$0=\int_a^b xf(x)dx=b\int_a^b f(t)dt-\int_a^b F(x)dx=-\mathcal{F}(b)$

$\mathcal{F}(a)=\mathcal{F}(b)=0\Rightarrow \exists x_0\in(a,b)$, s.t. $F(x_0)=\mathcal{F}(x_0)=0$

$\therefore \exists x_1\in(a,x_0),\ x_2\in(x_0,b)$, s.t. $f(x_1)=f(x_2)=0$

[2] 2015-2016 第一学期 第三次测试

1. (1) $=\int(t+1)t^5dt=\int(t^6+t^5)dt$

$=\frac{1}{7}t^7+\frac{1}{6}t^6=\frac{1}{7}(x-1)^7+\frac{1}{6}(x-1)^6+C$

(2) 设 $x=\sin^2 t$

原式 $=\int\frac{2\sin t\cos t}{\sin t\cos t}dt=2t+C=2\arcsin\sqrt{x}+C$

(3) $=\int\frac{1}{2\sin x(1+\cos x)}dx$ ~~$=\int\frac{1}{8}(t^2+1)(t+\frac{1}{t})\,d(\arctan 2t)$ $(t=\tan\frac{x}{2})$~~

~~$=\int\frac{2(t^2+1)^2}{8(4t^2+1)t}dt$~~

$=\int\frac{\sin x\,dx}{2(1-\cos^2x)(1+\cos x)}=\int-\frac{du}{2(1-u^2)(1+u)}$ $\quad(u=\cos x)$

$=\int-\frac{1}{8}(\frac{u+3}{(u+1)^2}-\frac{1}{u-1})du=-\int\frac{1}{8}(\frac{1}{u+1}+\frac{2}{(u+1)^2}-\frac{1}{u-1})du$

$=-\frac{1}{8}(\ln(u+1)-\frac{2}{u+1}-\ln(u+1))+C$

$=-\frac{1}{8}(\ln\frac{\cos x+1}{-\cos x+1}-\frac{2}{\cos x+1})+C$

(4) $=\int\ln x\,d(x^2-x)=-\int(x^2-x)\,d\ln x+\ln x\cdot(x^2-x)$

$=-\int(x-1)dx+\ln x\cdot(x^2-x)=-\frac{1}{2}x^2+x+\ln x\cdot(x^2-x)+C$

2. (1) $=-\int_0^1x^2de^{-x}=-x^2e^{-x}\Big|_0^1+\int_0^1 2e^{-x}x\,dx$

$=-\frac{1}{e}+2\int_0^1 -x\,de^{-x}=-\frac{1}{e}-2xe^{-x}\Big|_0^1+2\int_0^1e^{-x}dx$

$=-\frac{3}{e}+(-2e^{-x})\Big|_0^1=2-\frac{5}{e}$

(2) $=\int_0^1(\frac{x+2}{x^2+1}-\frac{1}{x+1})dx=\int_0^{\frac{\pi}{4}}(\tan t+2)dt-\int_0^1\frac{dx}{x+1}$ $\quad(x=\tan t)$

$=[-\ln(\cos t)+2t]\Big|_0^{\frac{\pi}{4}}-\ln(x+1)\Big|_0^1=-\frac{1}{2}\ln 2+\frac{\pi}{2}$

(3) $=\int_{-3}^{-\sqrt{2}} x^2\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} 2\,dx + \int_{\sqrt{2}}^{2} x^2\,dx$

$= \frac{1}{3}x^3\Big|_{-3}^{-\sqrt{2}} + 4\sqrt{2} + \frac{1}{3}x^3\Big|_{\sqrt{2}}^{2} = \frac{35+8\sqrt{2}}{3}$

(4) $=\int_{-1}^{1} \frac{x^2e^x}{e^x+1}\,dx$ $\cancel{= \int_{-1}^{1} x^2 (e^x+1)\, d\frac{1}{e^x+1}}$

$\cancel{= x^2\Big|_{-1}^{1} + \int_{-1}^{1} \frac{1}{e^x+1}\, d[x^2(e^x+1)] = +\int_{-1}^{1}\left(2x + \frac{x^2e^x}{e^x+1}\right)dx}$

$\cancel{= + x^2\Big|_{-1}^{1} + I = +I}$

$\cancel{\therefore I = 0}$

$= \int_{-1}^{0} \frac{x^2}{e^{-x}+1}\,dx + \int_{0}^{1} \frac{x^2e^x}{e^x+1}\,dx$

$= \int_{0}^{1} \frac{x^2}{e^x+1}\,dx + \int_{0}^{1} \frac{x^2e^x}{e^x+1}\,dx = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$

3.(1) $= \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)\right)^{\frac{1}{n}} = \exp\left[\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)\right]$

$= \exp\left[\int_0^1 \ln(1+x)\,dx\right] = \exp\left[(1+x)(\ln(1+x)-1)\Big|_0^1\right]$

$= e^{2\ln 2-1} = \frac{4}{e}$

(2) $= \lim_{x\to 0}\frac{x\int_0^x f(t)\,dt - \int_0^x t f(t)\,dt}{x\int_0^x f(t)\,dt}$

$= 1 - \lim_{x\to0}\frac{\int_0^x t f(t)\,dt}{x\int_0^x f(t)\,dt} = 1 - \lim_{x\to 0}\frac{x f(x)}{\int_0^x f(t)\,dt + x f(x)}$

$= 1 - \frac{1}{\lim_{x\to0}\frac{\int_0^x f(t)\,dt}{x f(x)} + 1} = 1 - \frac{1}{\lim_{x\to0}\cancel{\frac{f(x)}{f(x)+}}\ \frac{\int_0^x f(t)\,dt}{x}\cdot\frac{1}{f(x)} + 1}$

$= 1 - \frac{1}{\frac{f(0)}{f(0)} + 1} = \frac{1}{2}$

4. 设 $g(x) = f(x) + f(-x)$

$0 = \int_{-1}^{1} f(x)g(x)\,dx = \int_{1}^{-1} f(-x)g(-x)\,d(-x) = \int_{-1}^{1} f(-x)g(x)\,dx$

$\therefore \int_{-1}^{1}[f(x)+f(-x)]\,g(x)\,dx = 0$

$\Rightarrow \int_{-1}^{1}[f(x)+f(-x)]^2\,dx = 0$

$\Rightarrow f(x)+f(-x) \equiv 0 \Rightarrow f(x)$奇

5. $\forall x\in[a,b]$, RHS $= |f(\xi)| + \int_a^b |f'(x)|\,dx \quad (\exists \xi\in[a,b])$

$\geq |f(\xi)| + \int_\xi^x |f'(t)|\,dt$

$\geq \left|f(\xi) + \int_\xi^x f'(t)\,dt\right|$

$= |f(\xi) + f(x) - f(\xi)| = |f(x)|$

22 2016-2017第一学期第三次测试

1. $\forall \varepsilon>0$，$\exists \delta>0$，当划分$\|T\|<\delta$时，任取$x_i\in[a_i, a_{i+1}]$，均有$\left|\sum_{i=1}^{n}(a_{i+1}-a_i)f(x_i)-I\right|<\varepsilon$，则$f(x)$在$[a_1, a_{n+1}]$上Riemann可积.

2. (1) $t\triangleq 1+\sqrt{x}$，$x=(t-1)^2$

$$原式=\int t^{100}d(t-1)^2=\int(2t^{101}-2t^{100})dt=\frac{1}{51}t^{102}-\frac{2}{101}t^{101}+C$$

$$=\frac{1}{51}(1+\sqrt{x})^{102}-\frac{2}{101}(1+\sqrt{x})^{101}+C$$

(2) $x\triangleq \sin t$

$$原式=\int\frac{t}{\cos t}\cdot\frac{1+\sin^2 t}{\sin^2 t}\cdot\cos t\,dt=\int\frac{t(1+\sin^2 t)}{\sin^2 t}dt$$

$$=\int(t\csc^2 t+t)\,dt=-\int t\,d\cot t+\int t\,dt$$

$$=-t\cot t+\int\cot t\,dt+\frac{1}{2}t^2$$

$$=-t\cot t+\ln(\sin t)+\frac{1}{2}t^2+C$$

$$=-\arcsin x\cdot\frac{\sqrt{1-x^2}}{x}+\ln x+\frac{1}{2}\arcsin^2 x+C$$

3. (1) $=\frac{1}{2}\left(\int_0^{n\pi}x|\sin x|dx+\int_0^{n\pi}(n\pi-x)|\sin x|dx\right)=\frac{1}{2}\int_0^{n\pi}n\pi|\sin x|dx$

$$=\frac{1}{2}n^2\pi\int_0^{\pi}\sin x\,dx=-\frac{1}{2}n^2\pi\cos x\Big|_0^{\pi}=n^2\pi$$

(2) $\cancel{=\frac{1}{2}\left(\int_0^1\frac{x}{e^x+e^{2-x}}dx+\int_0^1\frac{1-x}{e^{1-x}+e^{1+x}}dx\right)}$

$$=\int_0^1\frac{2}{e^x+e^{2-x}}dx=\int_1^e\frac{2}{y+e^2/y}d\ln y=\int_1^e\frac{2}{y^2+e^2}dy=\int_{\frac{1}{e}}^1\frac{2}{e(t^2+1)}dt=\frac{2}{e}\arctan t\Big|_{\frac{1}{e}}^1$$

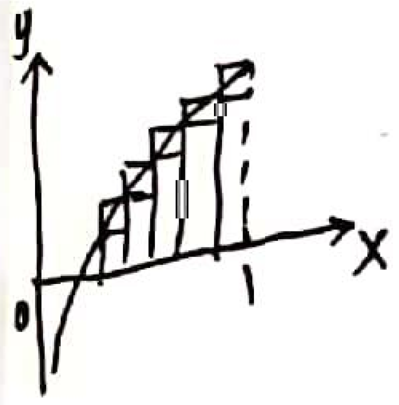
$$=\frac{2}{e}\left(\frac{\pi}{4}-\arctan\frac{1}{e}\right)$$

(3) $=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos t\,dt}{(2-\sin^2 t)\cos t}=2\int_0^{\frac{\pi}{2}}\frac{\sin^2 t+\cos^2 t}{\sin^2 t+2\cos^2 t}dt=2\int_0^{1}\frac{p^2+1}{p^2+2}\cdot\frac{dp}{p^2+1}$

$(x=\cancel{\cos}\sin t$，$p=\tan t)$ $=2\int_0^1\frac{1}{p^2+2}dp=\sqrt{2}\arctan\frac{p}{\sqrt{2}}\Big|_0^1$

$$=\sqrt{2}\arctan\frac{\sqrt{2}}{2}$$

4. 不妨设单调递增. 当$\frac{k}{n}\le x\le\frac{k+1}{n}$时有$f(\frac{k}{n})\le f(x)\le f(\frac{k+1}{n})$ 累加得…



由图知 $\sum_{k=1}^{n-1}f(\frac{k}{n})\cdot\frac{1}{n}\le\int_0^1 f(x)dx\le\sum_{k=1}^{n}f(\frac{k}{n})\cdot\frac{1}{n}$

令$n\to\infty$，有 $\lim_{n\to\infty}\sum_{k=1}^{n}f(\frac{k}{n})\cdot\frac{1}{n}=\lim_{n\to\infty}\sum_{k=1}^{n-1}f(\frac{k}{n})\cdot\frac{1}{n}$

$\le\int_0^1 f(x)dx\le\lim_{n\to\infty}\sum_{k=1}^{n}f(\frac{k}{n})\cdot\frac{1}{n}$ ∴取等号

5. $F'(0)=\lim_{t\to 0}t\sin\frac{1}{t}=0$

$G'(0)=\lim_{x\to 0}\frac{\int_0^x\cos\frac{1}{t}dt}{x}$ $\cancel{=\lim_{x\to 0}\frac{\int_{\infty}^{\frac{1}{x}}-\frac{1}{p^2}\cos p\,dp}{x}}$

凑

$=\lim_{x\to 0}\frac{-\int_0^x t^2 d\sin\frac{1}{t}}{x}=\lim_{x\to 0}\frac{-x^2\sin\frac{1}{x}+2\int_0^x t\sin\frac{1}{t}dt}{x}=\lim_{x\to 0}\frac{2\int_0^x t^3 d\cos\frac{1}{t}}{x}=\lim_{x\to 0}\frac{2x^3\cos\frac{1}{x}-6\int_0^x t^2\cos\frac{1}{t}dt}{x}$

$=\lim_{x\to 0}\frac{-6x^2\cos\frac{1}{x}}{x}=0$

6. (1) 不一定. $f(x)=\begin{cases}0, & a\le x\le\frac{a+b}{2}\\ 1, & \frac{a+b}{2}<x\le b\end{cases}$

(2) 不一定. $f(x)=x\sin\frac{1}{x}$，$x\in[0,1]$

7. $I_n=\int_0^{\pi}\cos^n x\cdot\cos nx\,dx$ $\cancel{=\int_0^{\pi}\cos^{n-1}x\cos nx\,d\sin x}$

$\cancel{=-\int_0^{\pi}\sin x\,d(\cos^{n-1}x\cdot\cos nx)=-\int_0^{\pi}-\sin^2 x\cdot(n-1)\cdot\cos^{n-2}x\cos nx}$

$=\int_0^{\pi}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^n\cdot\frac{e^{inx}+e^{-inx}}{2}dx=\int_0^{\pi}\frac{1}{2^{n+1}}\sum_{k=0}^{2n}C_{2n}^k e^{i(k-n)x}\cdot(e^{inx}+e^{-inx})dx$

$=\int_0^{\pi}\frac{1}{2^{n+1}}\sum_{k=0}^{2n}C_{2n}^k(e^{ikx}+e^{-ikx})dx=\frac{1}{2^{n+1}}\left[\sum_{k=1}^{2n}C_{2n}^k\left(\frac{1}{ik}e^{ikx}-\frac{1}{ik}e^{-ikx}\right)+2x\right]\Big|_0^{\pi}=\frac{1}{2^n}\left(\sum_{k=0}^{2n}C_{2n}^k\sin kx+x\right)\Big|_0^{\pi}=\frac{\pi}{2^n}$

8. $LHS=\int_0^1(e^{-x}f(x))'e^x dx$ $\cancel{\int_0^1(e^{-x}f(x))'de^x}$

$=\int_0^1 e^x d(e^{-x}f(x))=f(x)\Big|_0^1-\int_0^1 e^{-x}f(x)\,de^x$

$=f(1)-f(0)-\int_0^1 f(x)dx$ ✗

(注: $F(x)=e^{-x}f(x)$

$F(0)=0$，$F(1)=\frac{1}{e}$

$\exists\xi$，$F'(\xi)>0$

不变号$\Rightarrow F'(x)>0$)

$LHS>\int_0^1(e^{-x}f(x))'dx=e^{-x}f(x)\Big|_0^1=\frac{1}{e}$

缩放

Memo NO. ____

Date / /

X-Talk

19 2012－2013 第一学期第四次测试

1.(1) $\frac{dy}{dx}=\frac{x+2}{y-2}$

$(y-2)dy=(x+2)dx$

$\frac{1}{2}y^2-2y=\frac{1}{2}x^2+2x+C$

(2) $\begin{cases}x+y+2=0\\x-y+4=0\end{cases}\Rightarrow\begin{cases}x_0=-3\\y_0=1\end{cases}$

令 $\begin{cases}u=x+3\\v=y-1\end{cases}\Rightarrow\frac{dv}{du}=\frac{u+v}{u-v}$

令 $k=\frac{v}{u}$，$v=ku$，$\frac{dv}{du}=k+\frac{dk}{du}\Rightarrow k+\frac{dk}{du}=\frac{1+k}{1-k}$

$\frac{1-k}{1+k^2}dk=du$

$\therefore u=\arctan k-\frac{1}{2}\ln(1+k^2)+C$

$=\arctan\frac{v}{u}-\frac{1}{2}\ln(1+\frac{v^2}{u^2})+C$

即 $x=\arctan\frac{y-1}{x+3}-\frac{1}{2}\ln\left(1+\frac{(y-1)^2}{(x+3)^2}\right)+C'$

2. 令 $z=y'$，$z'=y''\Rightarrow z'-xz-x^3z^3=0$

$z^{-3}z'-xz^{-2}=x^3$

令 $u=z^{-2}\Rightarrow\frac{du}{dx}+2xu=-2x^3$ (*)

$\frac{du}{dx}=-2xu\Rightarrow\frac{du}{u}=-2xdu\Rightarrow\ln u=-x^2+C$

$y'(0)=1\Rightarrow u=e^{-x^2}$

将 $u=ve^{-x^2}$ 代入(*)，得 $\frac{dv}{dx}e^{-x^2}=-2x^3$

$dv=-x^2e^{x^2}dx^2=-e^t t\,dt$ $(t=x^2)$

$v=-\int t\,de^t=-te^t+\int e^t dt=(1-t)e^t+C$

$\therefore u=1-x^2+Ce^{-x^2}$

$y'(0)=1\Rightarrow u=1-x^2\Rightarrow z=\pm\sqrt{1-x^2}$

Memo NO. ____

Date / /

X-Talk

$\frac{dy}{dx}=\pm\frac{1}{\sqrt{1-x^2}}$

$y=\pm\int\frac{1}{\sqrt{1-x^2}}dx=\pm\int\frac{1}{\sqrt{1-\sin^2 t}}d\sin t=\pm\int dt=\pm\arcsin x+C$

$y(0)=0\Rightarrow y=\pm\arcsin x$

3.(1) $\sim\sum_{n=1}^{\infty}\frac{\sqrt{2n\pi}}{e^n}$ 收敛

(2) $\int_2^{+\infty}\frac{1}{x(\ln x)^2}dx=-\frac{1}{\ln x}\Big|_2^{+\infty}=\frac{1}{\ln 2}$ 收敛

(3) $<\sum_{n=1}^{\infty}\frac{n}{2^n}$ 收敛

(4) $=\sum_{n=1}^{\infty}\frac{(-1)^n}{[\ln(2n+1)]^3}$ 条件收敛

(5) $\frac{1}{\sqrt{1+x}}=1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}x^n$ $a_n>\frac{(2n-2)!!}{(2n)!!}=\frac{1}{2n}$

令 $x=1$ 得 $=\frac{\sqrt{2}}{2}-1$ ~~$\frac{a_{n+1}}{a_n}=\frac{2n+1}{2n+2}<1$ 而~~ $\therefore$条件收敛

4.(1) $\lim_{n\to\infty}f_n(0)=0$

当 $0<x\le1$，$\lim_{n\to\infty}f_n(x)=0$

$\therefore$收敛 $\alpha\in R$

(2) $f_n'(x)=n^\alpha(2xe^{-nx}-nx^2e^{-nx})=n^\alpha xe^{-nx}(2-nx)$ 用求导找最值

$\lim_{n\to\infty}\sup_{x\in[0,1]}f_n(x)=\lim_{n\to\infty}f_n(\frac{2}{n})=\lim_{n\to\infty}n^{\alpha-2}\cdot\frac{4}{e^2}=0$

$\therefore\alpha<2$

(3) $RHS=0$ $\int_0^1 f_n(x)dx<\varepsilon\Leftrightarrow\int_0^1|f_n(x)|dx<\varepsilon$

$LHS=0\Leftrightarrow\forall\varepsilon>0,\exists N>0,\forall n>N,$ ~~$x\in[0,1],|f_n(x)-0|<\varepsilon$~~ $\Leftrightarrow|f_n(x)|<\varepsilon,\forall x$

$\Leftrightarrow\alpha<2$

直接积

5. $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{n+2}{n+3}=1$

$\therefore R=1$

当$x=1$时不收敛，当$x=-1$时条件收敛

$\therefore -1\le x<1$

$\because \frac{1}{1-x}=\sum\limits_{n=0}^{\infty}x^n \Rightarrow \frac{x^2}{1-x}=\sum\limits_{n=0}^{\infty}x^{n+2}$

$\therefore$ 求导，得 $\frac{2x-x^2}{(1-x)^2}=\sum\limits_{n=0}^{\infty}\frac{x^{n+1}}{n+2} \Rightarrow \frac{2-x}{(1-x)^2}=\sum\limits_{n=0}^{\infty}\frac{x^n}{n+2}$

6. $y=\arctan\frac{3x-1}{3x+1}$

$\tan y=\frac{3x-1}{3x+1} \Rightarrow y=\arctan 3x-\frac{\pi}{4}$

由 $\arctan x=\sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ $(-1\le x\le 1)$ (*)

得 $y=\sum\limits_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n+1}}{2n+1}-\frac{\pi}{4}$ $\left(-\frac{1}{3}<x\le\frac{1}{3}\right)$

$\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\arctan 1=\frac{\pi}{4}$

4.(3) $\int_0^1 n^\alpha x^2 e^{-nx}dx=-\frac{n^\alpha}{n}\int_0^1 x^2 de^{-nx}$

$=-n^{\alpha-1}\left(x^2e^{-nx}\Big|_0^1-\int_0^1 e^{-nx}dx^2\right)$

$=-n^{\alpha-1}\left(e^{-n}-2\int_0^1 xe^{-nx}dx\right)$

$=-n^{\alpha-1}\left(e^{-n}+\frac{2}{n}\int_0^1 x de^{-nx}\right)$

$=-n^{\alpha-1}\left(e^{-n}+\frac{2}{n}xe^{-nx}\Big|_0^1-\frac{2}{n}\int_0^1 e^{-nx}dx\right)$

$=-n^{\alpha-1}\left[\left(\frac{2}{n}+1\right)e^{-n}+\frac{2}{n^2}e^{-nx}\Big|_0^1\right]$

$=-n^{\alpha-1}\left[\left(\frac{2}{n}+1\right)e^{-n}+\frac{2}{n^2}(e^{-n}-1)\right]\to\frac{2}{n^{3-\alpha}}$

$\therefore \alpha<3$

Memo NO. ____ X-Talk
Date / /

$\boxed{29}$ 2011-2012 第一学期期末考试

1. (1) $=\lim\limits_{x\to 0}\frac{\frac{1}{2}x^2}{x^2}=\frac{1}{2}$

(2) $=\lim\limits_{x\to+\infty}\left(\left(1+\frac{3}{x-1}\right)^{\frac{x-1}{3}}\right)^3\left(1+\frac{3}{x-1}\right)=e^3$

(3) $=\lim\limits_{n\to\infty}\frac{\sin^2\xi}{\xi}$ $(\xi\in(n,n+1))$

$=\lim\limits_{\xi\to+\infty}\frac{\sin^2\xi}{\xi}=0$

(4) $=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}=\int_0^1\frac{1}{1+x^2}dx=\arctan x\Big|_0^1=\frac{\pi}{4}$

2. (1) $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{\sqrt{\sin^2t(1-\sin^2t)}}d\sin^2t=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}dt=\frac{\pi}{2}$

(2) $=\int_0^{+\infty}\frac{1}{2}\left(\frac{1}{1+x}+\frac{1-x}{1+x^2}\right)dx$

$=\frac{1}{2}\left(\int_0^{+\infty}\frac{d(1+x)}{1+x}+\int_0^{+\infty}\frac{dx}{1+x^2}-\frac{1}{2}\int_0^{+\infty}\frac{d(1+x^2)}{1+x^2}\right)$

$=\frac{1}{2}\left[\ln(1+x)+\arctan x-\frac{1}{2}\ln(1+x^2)\right]\Big|_0^{+\infty}$

$=\frac{1}{2}\arctan x\Big|_0^{+\infty}=\frac{\pi}{4}$

3. (1) $a_n=\frac{n+1}{3n-2}$

$\lim\limits_{n\to\infty}a_n=\frac{1}{3}\ne 0$

$\therefore$ 不收敛（即发散）

(2) $a_n=\frac{n^{n-1}}{n!e^n}=\frac{n^{n-1}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^n e^{\theta_n/12n}}=\frac{1}{n\sqrt{2\pi n}\,e^{\theta_n/12n}}$

$\lim\limits_{n\to\infty}\frac{a_n}{n^{3/2}}=\frac{1}{\sqrt{2\pi}}$ 而 $\sum\limits_{n=1}^{\infty}\frac{1}{n^{3/2}}$ 收敛

$\therefore$ 收敛

单调有界 ⇒ 收敛

4. 若 $\lim\limits_{x\to-\infty} f'(x)=a>0$

则 $\forall \varepsilon>0, \exists x_0<0, \forall x<x_0, |f'(x)-a|<\varepsilon$

即 $\varepsilon=a$，则 $f'(x)>a$，$\forall x<x_0$

由 $f''(x)>0$ 知 $f'(x)>a$，$\forall x\in R$

$f(x)=f(0)+\int_0^x f'(t)\,dt < f(0)-\int_x^0 a\,dt$ （当 $x<0$ 时）

$=f(0)+ax$

取 $x_1=-\frac{f(0)}{a}$，有 $f(x_1)<f(0)+ax_1=0$，矛盾

$\therefore \lim\limits_{x\to-\infty} f'(x)=0$

5. 设 $z=y'$，$z'=y''$，则 ~~$z'+z$~~ $z'-z=(2x+2)e^x$ $(*)$

$z'-z=0 \Rightarrow z=Ce^x$

将 $z=ue^x$ 代入 $(*)$ 得 $u'(x)e^x=(2x+2)e^x \Rightarrow u(x)=x^2+2x$

$\therefore y'=(x^2+2x)e^x$

$y=\int(x^2+2x)e^x\,dx=\int(x^2+2x)\,de^x$

$=(x^2+2x)e^x-\int e^x(2x+2)\,dx$

$=(x^2+2x)e^x-(2x+2)e^x+\int 2e^x\,dx$

$=x^2e^x+C$

6. (1) $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}e^{(2n+1)x}<1 \Rightarrow x>0$

$x=0$ 时发散

$\therefore x>0$

~~(2) 否，取 $x=\frac{1}{2n+1}$，则 $\lim\limits_{n\to\infty} a_n=\frac{1}{e}\neq0$~~

~~(3)~~ (2) 是. $\{\frac{1}{n}\}$ 单调递减，一致趋于0.

$\sum\limits_{k=1}^{n}\frac{1}{e^{k^2x}}<\sum\limits_{k=1}^{n}\frac{1}{e^{kx}}=\frac{1}{e^x}\sum\limits_{k=1}^{n}\left(\frac{1}{e}\right)^k<\frac{1}{e^x}\cdot\frac{1}{e}\cdot\frac{1}{1-\frac{1}{e}}<\frac{1}{e-1}$

$\therefore \sum\limits_{k=1}^{n}\frac{1}{e^{k^2x}}$ 一致有界

由 Dirichlet 判别法知其一致收敛.

(3) 由一致收敛级数的性质可知.

7. 由 Cauchy 准则，有 $\forall\varepsilon>0, \exists N>0, \forall n>N, p>0$，有

$\sum\limits_{i=n+1}^{n+p}\left(\frac{a_{i+1}}{a_i}-1\right)=\sum\limits_{i=n+1}^{n+p}\frac{a_{i+1}-a_i}{a_i}<\varepsilon$

$LHS\geqslant\frac{1}{a_{n+p+1}}\sum\limits_{i=n+1}^{n+p}(a_{i+1}-a_i)=\frac{a_{n+p+1}-a_{n+1}}{a_{n+p+1}}=1-\frac{a_{n+1}}{a_{n+p+1}}$

取 $\varepsilon=\frac{1}{2}$ 得 $a_{n+p+1}\leqslant 2a_{n+1}$

选定 n，变动 p，可知 $\{a_n\}$ 有界.

8. $LHS=\int_0^1\left|(e^xf(x))'e^{-x}\right|dx\geqslant\frac{1}{e}\int_0^1 d(e^xf(x))$

$=\frac{1}{e}(e^xf(x))\Big|_0^1=1$

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30 2012-2013 第一学期期末考试

1. $\cancel{\exists a\in R, \forall \varepsilon>0, \exists X_0>0, \forall X>X_0, 有|f(x)-a|<\varepsilon}$

$\forall \varepsilon>0, \exists X_0>0, \forall X_1, X_2>X_0$, 有 $|f(X_1)-f(X_2)|<\varepsilon$

2.(1) 式 $\lim\limits_{x\to+\infty}\frac{1}{4x^3}\cdot\frac{x^8+|\cos x^2|}{1+x^6}\cdot 2x=\lim\limits_{x\to+\infty}\frac{x^8}{2x^2(1+x^6)}=\frac{1}{2}$

(2) ✓

3.(1) $=\int_0^{\frac{\pi}{2}}\frac{\sin 2x\,dx}{\frac{3+\cos 2x}{2}}=\int_0^{\frac{\pi}{2}}\frac{d\cos 2x}{3+\cos 2x}=\ln(3+\cos 2x)\Big|_0^{\frac{\pi}{2}}=-\ln 2$

(2) $I_n=\int_0^{+\infty}x^{2n+1}e^{-x^2}dx=-\frac{1}{2}\int_0^{+\infty}x^{2n}d\,e^{-x^2}$

$=-\frac{1}{2}x^{2n}e^{-x^2}\Big|_0^{+\infty}+\frac{1}{2}\int_0^{+\infty}e^{-x^2}dx^{2n}$

$=n\int_0^{+\infty}x^{2n-1}e^{-x^2}dx=nI_{n-1}=n!\,I_0$

$=n!\int_0^{+\infty}(-\frac{1}{2})e^{-x^2}d(-x^2)=-\frac{1}{2}n!\,e^{-x^2}\Big|_0^{+\infty}=\frac{1}{2}n!$

4.(1) $\frac{dy}{dx}+1+\frac{y}{x}=0$

令 $u=\frac{y}{x}$, $\frac{dy}{dx}=\frac{du}{dx}x+u\Rightarrow\frac{du}{dx}x+2u+1=0$

$\therefore\frac{du}{2u+1}=-\frac{dx}{x}$

积得 $\frac{1}{2}\ln(2u+1)=-\ln x+\ln C'$

即 $(\frac{2y}{x}+1)^2x=C'$

$y=\frac{1}{2}(\pm\sqrt{Cx}-x)$

?✓(2) $z\triangleq y'\Rightarrow z'+z^2=z$ $y''=y'\frac{dy'}{dy}$ $y'e^y=e^y+C$

$\frac{dz}{z-z^2}=dx$ $y'\frac{dy'}{dy}+(y')^2=y'$ $y=x+1$

$x=\int\frac{dz}{z(1-z)}=\int\frac{1}{z}dz-\int\frac{dz}{z-1}=\ln|z|-\ln|z-1|+C$

$\frac{z}{z-1}=C'e^x$ 即 $1-\frac{1}{z}=C''e^{-x}$ $z(0)=1\Rightarrow C''=0$

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5.(1) $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1\Rightarrow R=1$

$x=1$ 时 收敛；$x=-1$ 时 发散

$\therefore$ 收敛域 $(-1,1]$

(2) 设该级数为 $f(x)$, 则 $f(x)=x(x+1)\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n-1}}{n}=x(x+1)g(x)$

$\because-\sum\limits_{n=1}^{\infty}(-x)^n=x\cdot\frac{1}{1-(-x)}=\frac{x}{1+x}$

$\therefore g(x)=(\frac{x}{1+x})'=\frac{1}{(1+x)^2}$

$\therefore f(x)=\frac{x}{1+x}$

(3) $\cancel{|u_n|=\left|\frac{(x+1)x^n}{n}\right|\le\frac{2}{n}}$

$\{(x+1)x^n\}$ 单调 ↓，且 $|(x+1)x^n|\le 2$ (一致有界)

又 $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ 一致收敛

$\therefore$ 级数一致收敛

6. $g(x)=e^{cx}f(x)$

$g(a)=g(b)=0$

微分中值定理 $\Rightarrow g'(\xi)=e^{c\xi}(f'(\xi)+cf(\xi))=0\Rightarrow$ ✓

7.(1) $\Leftarrow(\beta-\alpha)\int_\alpha^\beta f(x)dx+(\beta-\alpha)\int_\beta^1 f(x)dx\geqslant(1-\alpha)\int_\alpha^\beta f(x)dx$

$\Leftarrow\frac{1}{1-\beta}\int_\beta^1 f(x)dx\geqslant\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx$

$\Leftarrow f(\xi_2)\geqslant f(\xi_1)$ $(\xi_1\in(\alpha,\beta),\ \xi_2\in(\beta,1))$

$\Leftarrow\xi_2\geqslant\xi_1$

? (2) 令 $\alpha\to0$, $\beta\to1$, 则 $\int_\alpha^\beta f(x)dx\to\int_0^1 f(x)dx$, $\frac{1-\alpha}{\beta-\alpha}\to1$

不能更大

31 2013-2014第一学期期末考试

1.(1) $=\frac{1}{1+e^{x\sin x}}\cdot e^{x\sin x}\cdot(\sin x+x\cos x)$

(2) $=(x^2+1)\sin^{(n)}x+n\cdot 2x\cdot\sin^{(n-1)}x+n(n-1)x\cdot\sin^{(n-2)}x=\{$ （按mod4分类讨论）

2.(1) $\lim_{n\to\infty} n\cdot\frac{\arctan\xi}{\xi}\quad(\xi\in(n^2,n^2+n))$

$=\lim_{n\to\infty} n\cdot\frac{\pi/2}{n^2}=0$

(2) $=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{1-(\frac{k}{n})^2}}\cdot\frac{1}{n}=\int_0^1\frac{dx}{\sqrt{1-x^2}}$

$=\int_0^1\frac{d\sin t}{\cos t}=\int_0^{\frac{\pi}{2}}dt=\frac{\pi}{2}$

3.(1) $=\int\ln x\,dxe^x=xe^x\ln x-\int xe^x\,d\ln x$

$=xe^x\ln x-\int e^x dx=xe^x\ln x-e^x+C$

(2) $=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2x\cos x\,d\sin x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin x\cos x)^2dx$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\frac{1}{2}\sin 2x)^2dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{8}(1-\cos 4x)dx$

$=(\frac{1}{8}x-\frac{1}{32}\sin 4x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{\pi}{8}$

4. $\frac{y\cos x-y'\sin x}{y^2}=(1-\sin x)\cos x$

$u'=\left(\frac{\sin x}{y}\right)'=(1-\sin x)\cos x$

$u=\int du=\int(1-\sin x)\,d\sin x=\sin x-\frac{1}{2}\sin^2x+C$

$y=\frac{\sin x}{\sin x-\frac{1}{2}\sin^2x+C}$

5. $\frac{xe^{-nx}}{\sqrt{n}}\le\frac{x}{\sqrt{n}}\cdot\frac{1}{nx}=\frac{1}{n^{3/2}}$ 而 $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ 收敛

∴一致收敛

✓6. $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}\quad(-1<x<1)$

$\sum_{n=1}^{\infty}x^{n-1}=\frac{x}{x(1-x)}=\frac{1}{1-x}\quad(x\neq 0)$

（积）$\sum_{n=1}^{\infty}\frac{x^n}{n}=-\ln(1-x)$

（积）$\sum_{n=1}^{\infty}\frac{x^{n+1}}{n(n+1)}=\cancel{x(\ln|x|-1)+}(1-x)(\ln(1-x)-1)+1$

∴ $\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}=\frac{(1-x)(\ln(1-x)-1)+1}{x}\quad(-1<x<1$ 且 $x\neq 0)$

$x=0$时，级数$=0$；$x=1$时，级数$=\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1})=1$；$x=-1$时，上式仍成立.

收敛区域：$[-1,1]$

7. $\cancel{f(x)=e^x+e^y+xy-(e^{xy}+1)}$

$\cancel{f'(x)=e^x+y-ye^{xy}}$

$\cancel{f''(x)=e^x-y^2e^{xy}}$

$(e^x-1)(e^y-1)>(1+x-1)(1+y-1)=xy$

8. (1) ✓

(2) $\Leftarrow e^{dx}f'(x)\le ce^{dx}f(x)$

$\Leftarrow(e^{dx}f'(x))'\le c(e^{dx}f(x))'$

$\Leftarrow df'(x)+f''(x)\le cdf(x)+cf'(x)$

$\Leftarrow f''(x)\le cdf(x)+(c-d)f'(x)$

取 $cd=a$，$c-d=b$ 即可

32 2015-2016第一学期期末考试

1.(1) $=\lim\limits_{n\to\infty}\sqrt{n}\cos n\xi\sin\xi^n=\lim\limits_{n\to\infty}\sqrt{n}\,\xi^n\cos n\xi=0\quad(\xi\in(0,1))$

(2) $=\lim\limits_{x\to0}\dfrac{\sin x^2\cdot 2x}{\frac{4x^3}{1+x^4}}=\lim\limits_{x\to0}\dfrac{2x^3}{4x^3}=\dfrac{1}{2}$

2.(1) $=\frac{1}{3}\int\arctan x\,dx^3=\frac{1}{3}x^3\arctan x-\frac{1}{3}\int x^3\,d\arctan x$

$=\frac{1}{3}x^3\arctan x-\frac{1}{6}\int\frac{x^2}{1+x^2}dx^2$

$=\frac{1}{3}x^3\arctan x-\frac{1}{6}\int dx^2+\frac{1}{6}\int\frac{1}{1+x^2}d(1+x^2)$

$=\frac{1}{3}x^3\arctan x-\frac{1}{6}x^2+\frac{1}{6}\ln(1+x^2)+C$

(2) $=\int\frac{1}{\frac{1}{t}(1+\frac{1}{t^4})}d\frac{1}{t}=\int-\frac{t^3}{t^4+1}dt=-\frac{1}{4}\int\frac{d(t^4+1)}{t^4+1}=-\frac{1}{4}\ln(t^4+1)+C$

$=-\frac{1}{4}\ln(\frac{1}{x^4}+1)+C$

(3) $I=\int_0^1\frac{(1-x)^2e^x}{(1+x^2)^2}dx$, $J=\int_0^1\frac{(1+x)^2e^x}{(1+x^2)^2}dx$

$I+J=\int_0^1\frac{2e^x}{1+x^2}dx$, $J-I=\int_0^1\frac{4xe^x}{(1+x^2)^2}dx=-2\int_0^1e^x d\frac{1}{1+x^2}=-\frac{2e^x}{1+x^2}\Big|_0^1+2\int_0^1\frac{e^x}{1+x^2}dx=2-e+I+J$

$\therefore I=\frac{1}{2}e-1$

(4) $=\frac{1}{3}\int_0^{+\infty}\left(\frac{1}{x+1}+\frac{-x+2}{x^2-x+1}\right)dx=\frac{1}{3}\left[\ln(x+1)\Big|_0^{+\infty}-\int_0^{+\infty}\frac{(x-\frac{1}{2})-\frac{3}{2}}{(x-\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}dx\right]$

设 $x-\frac{1}{2}=\frac{\sqrt3}{2}y$，则后一积分 $=\int_0^{+\infty}\frac{y-\sqrt3}{y^2+1}d\left(\frac{\sqrt3}{2}y+\frac{1}{2}\right)$

$=\frac{\sqrt3}{2}\int_{-\frac{1}{\sqrt3}}^{+\infty}\frac{y-\sqrt3}{y^2+1}dy=\frac{\sqrt3}{4}\int_{-\frac{1}{\sqrt3}}^{+\infty}\frac{dy^2}{y^2+1}-\frac{3}{2}\int_{-\frac{1}{\sqrt3}}^{+\infty}\frac{dy}{y^2+1}$

$=\left[\frac{\sqrt3}{4}\ln(y^2+1)-\frac{3}{2}\arctan y\right]\Big|_{-\frac{1}{\sqrt3}}^{+\infty}=\frac{\sqrt3}{4}\ln(y^2+1)\Big|_{-\frac{1}{\sqrt3}}^{+\infty}-\frac{3}{2}\left(\frac{\pi}{2}+\frac{\pi}{6}\right)$

$=\int\left(\frac{1}{1+x}+\frac{2-x}{x^2-x+1}\right)dx=\ln|1+x|-\frac{1}{2}\int\frac{d(x^2-x+1)}{x^2-x+1}+\frac{3}{2}\int\frac{dx}{x^2-x+1}$

3. $=x[\ln(1+x^2)]^{(n)}+n[\ln(1+x^2)]^{(n-1)}=n[\ln(1+x^2)]^{(n-1)}$

设 $g(x)=\ln(1+x^2)$

$g'(x)=\frac{2x}{1+x^2}$ 即 $(1+x^2)g'(x)=2x$

$\therefore(1+x^2)g^{(k+1)}(x)+2kxg^{(k)}(x)+k(k-1)g^{(k-1)}(x)=0$

则 $g^{(k+1)}(0)+k(k-1)g^{(k-1)}(0)=0\quad(k\geq2)$

而 $g(0)=g'(0)=0$, $g''(0)=2$

$\therefore g^{(k)}(0)=\begin{cases}0, & k=0\text{ 或}k\text{为奇}\\ 2\cdot(k-1)!, & k\text{为偶}\end{cases}$

$\therefore f^{(n)}(0)=\begin{cases}0, & n=1\text{或}n\text{为偶}\\ 2n\cdot(n-2)!, & n\text{为奇}\end{cases}$

4. $\int_0^t\sqrt{1+f'(x)}\,dx=\int_0^t f(x)\,dx,\ \forall t>0$

$\therefore\sqrt{1+f'(x)}\equiv f(x)$

$1+\frac{df}{dx}=f^2$

$\frac{df}{(f-1)(f+1)}=dx$

$x=\frac{1}{2}\int\frac{df}{f-1}-\frac{1}{2}\int\frac{df}{f+1}=\frac{1}{2}[\ln(f-1)-\ln(f+1)]+C$

$\therefore f(x)=\frac{2}{1-e^{2x-2C}}-1$

由 $f(0)=1$，得 $C=+\infty$，则 $f(x)\equiv1$

$\therefore V=\pi t$

5.(1) 设 $z=y'\cancel{\cos x}$，则 $\sin x\cdot z'-\cos x\cdot z=\sin^2x+1$

$\sin x\cdot z'-\cos x\cdot z=0\Rightarrow z=C_1\sin x$

$dy=C_1\sin x\,dx\Rightarrow y=-C_1\cos x+C_2$

(2) 设 $z=y'-y$，则 $z'-2z=2x\Rightarrow z=-x-\frac{1}{2}+Ce^{2x}$

$\cancel{(ye^{-x})'=-e^{-x}(x+\frac{1}{2})}\Rightarrow y=x+\frac{3}{2}+C_1e^{2x}+C_2e^x$

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6. $LHS=\int_0^a xf(x)dx+\int_a^{2a} xf(x)dx$

$=\int_0^a xf(x)dx+\int_0^a (x+a)f(x+a)dx$

$=\int_0^a xf(x)dx+\int_0^a -(x+a)f(x)dx$

$=-a\int_0^a f(x)dx$

7. $\Leftarrow 2\int_0^x xf(x)dx \geq \left(\int_0^x f(x)dx\right)^2, \forall x\in[0,1]$

$\Leftarrow$ (求导) $2xf(x)\geq 2f(x)\int_0^x f(x)dx$

$\Leftarrow$ ~~若~~ $x \geq \int_0^x f(x)dx$

$\Leftarrow \int_0^x (1-f(x))dx \geq 0$

"=" eg. $f(x)\equiv 0$ 或 $f(x)\equiv 1$

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$\boxed{33}$ 2016-2017第一学期期末考试

1.(1) $=\lim\limits_{x\to 0^+}\dfrac{\sin x\cdot 2x}{\frac{3x^2}{1+x^3}}=\lim\limits_{x\to 0^+}\dfrac{2x^2}{3x^2}=\dfrac{2}{3}$

(2) $=\lim\limits_{n\to\infty} n(1-\xi)\cos(n\xi)\sin\xi^n \quad (\xi\in(0,1))$

$=\lim\limits_{n\to\infty} n(1-\xi)\cos(n\xi)\xi^n=0$

2.(1) $=\frac{1}{2}\int \arctan x\, dx^2=\frac{1}{2}x^2\arctan x-\frac{1}{2}\int x^2 d\arctan x$

$=\frac{1}{2}x^2\arctan x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$

$=\frac{1}{2}x^2\arctan x-\frac{1}{2}x+\frac{1}{2}\int\frac{1}{1+x^2}dx$

$=\frac{1}{2}x^2\arctan x-\frac{1}{2}x+\frac{1}{2}\arctan x+C$

(2) $=\int_1^{+\infty}\left(\frac{1}{x}-\frac{1}{1+x}\right)dx=\ln\frac{x}{1+x}\Big|_1^{+\infty}=\ln 2$

(3) $=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\cos x\, d\sin x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin x\cos x)^2dx=\checkmark$

3. 在$[\delta,+\infty)$上一致收敛

$|u_n|\leq\frac{1}{n^2\delta}$

在$(0,+\infty)$上不一致收敛

取$x=\frac{1}{n^2}$, $\lim\limits_{n\to\infty}\sup\limits_{x\in(0,+\infty)} u_n(x)\geq\lim\limits_{n\to\infty}u_n\left(\frac{1}{n^2}\right)=\lim\limits_{n\to\infty}\dfrac{\cos\frac{1}{n}}{2}=\frac{1}{2}>0$

4. $\lim\limits_{n\to\infty}\dfrac{a_n}{a_{n+1}}=2\Rightarrow R=$ ~~#~~ 2

设 $y=\frac{x}{2}$, 则 级数 $=\sum\limits_{n=1}^{\infty}\frac{y^n}{n}\triangleq f(y)$

$\frac{f(y)}{y}=\sum\limits_{n=1}^{\infty}\frac{y^{n-1}}{n}=\left(\sum\limits_{n=1}^{\infty}y^n\right)'=\left(\frac{1}{1-y}\right)'=\frac{1}{(1-y)^2}$

$\therefore$ 级数的和函数$=\frac{y}{(1-y)^2}=\frac{2x}{(2-x)^2}$

5.(1) $y^{-2}y'+y^{-1}=x$

设 $z=y^{-1}$，则 $-z'+z=x$

将 $z=ue^{x}$ 代入，得 $-u'e^{x}=x \Rightarrow u=-\int\frac{x}{e^{x}}dx=e^{-x}(1+x)+C$

$\therefore y=\frac{1}{z}=\frac{1}{1+x+Ce^{x}}$

显然 $y\equiv0$ 也成立

(2) 设 $z=y'$，则 $xz'+2z=2$

$$\frac{dz}{2(1-z)}=\frac{dx}{x}$$

$$-\frac{1}{2}\ln(1-z)=\ln x+\ln C_1 \Rightarrow z=1-\frac{C}{x^2}$$

$\cancel{1-z=1/Cx^2}$ $\quad y=\int(1-\frac{C}{x^2})dx=x+\frac{C}{x}+C'$

$\cancel{y'=1-Cx^2\ \frac{1}{1-Cx^2}}$

$\cancel{\therefore y=x-\frac{1}{3}Cx^3+C'=x-C_0x^3+C'}$

6. 设 $z=y'$，则 $z'-2z=e^{4x}$

将 $z=ue^{2x}$ 代入，得 $u'e^{2x}=e^{4x}\Rightarrow u=\int e^{2x}dx=\frac{1}{2}e^{2x}+C$

$\therefore y'=z=\frac{1}{2}e^{4x}+Ce^{2x}$ $\quad y'(0)=0\Rightarrow C=-\frac{1}{2}$

$y=\frac{1}{8}e^{4x}+C'e^{2x}+C''$ $\quad y(0)=0\Rightarrow C''=\frac{1}{8}$

$\therefore y=\frac{1}{8}e^{4x}-\frac{1}{4}e^{2x}+\frac{1}{8}$

7.(1) 由数学归纳法.

$$\frac{f_{n+1}(x)}{f_n(x)}=\frac{2}{1+\frac{f_{n-1}(x)}{f_n(x)}}\le\frac{2}{1+\frac{1}{C_n}}=\frac{2C_n}{C_n+1}=C_{n+1}$$

(2) $\cancel{g_n\triangleq|f_n(x)-f_{n-1}(x)|=}$

若 $f_1\ge f_0$ ($f_1<f_0$ 同理可证)，则 $f_{n+1}\ge f_n$

而 $f_n\le C_nf_{n-1}\le 2f_{n-1}$，故 $(f_n-2f_{n-1})(f_n-f_{n-1})\le0$

$6f_n^2-5f_n(f_n+f_{n-1})+2f_{n-1}(f_n+f_{n-1})\le0$

$3f_{n+1}-5f_n+2f_{n-1}\le0$

$f_{n+1}-f_n\le\frac{2}{3}(f_n-f_{n-1})$

$\therefore|f_{n+1}(x)-f_n(x)|\le(\frac{2}{3})^n|f_1(x)-f_0(x)|$

(3) 由柯西收敛准则. 设 $M=\max|f_1(x)-f_0(x)|$

$\forall\varepsilon>0,\exists N=[\log_{\frac{2}{3}}\frac{\varepsilon}{3M}]$，$\forall n>N, p>0$，有

$$|f_n(x)-f_{n+p}(x)|\le|f_n(x)-f_{n+1}(x)|+\cdots+|f_{n+p-1}(x)-f_{n+p}(x)|$$

$$\le\left[(\frac{2}{3})^{n+1}+\cdots+(\frac{2}{3})^{n+p}\right]|f_1(x)-f_0(x)|$$

$$\le(\frac{2}{3})^{n+1}\cdot\frac{1}{1-\frac{2}{3}}\cdot M=3M(\frac{2}{3})^{n+1}<\varepsilon$$

8. 充分性：$\{a_n\}$ 单调有界 $\Rightarrow$ 收敛 $\Rightarrow\sum_{n=1}^{\infty}(a_{n+1}-a_n)$ 收敛

$\Downarrow$ $\quad\Downarrow$

$a_n>1$ 且递增 $\Rightarrow\frac{1}{a_n\ln a_{n+1}}$ 单调有界 $\Rightarrow\sum_{n=1}^{\infty}\frac{a_{n+1}-a_n}{a_n\ln a_{n+1}}$ 收敛

(或：$\sum_{n=1}^{\infty}\frac{a_{n+1}-a_n}{a_n\ln a_{n+1}}\le\sum_{n=1}^{\infty}\frac{a_{n+1}-a_n}{a_1\ln a_2}=\frac{a_\infty-a_1}{a_1\ln a_2}$)

必要性：(若题目改为 $\sum_{n=1}^{\infty}\frac{a_{n+1}-a_n}{a_n\ln a_n}$ 则：

反证法：若 $\{a_n\}$ 无界，则 $\frac{\ln\ln a_{n+1}-\ln\ln a_n}{a_{n+1}-a_n}=\frac{1}{a_{n+\xi}\ln a_{n+\xi}}<\frac{1}{a_n\ln a_n}$

级数 $>\sum_{n=1}^{\infty}(\ln\ln a_{n+1}-\ln\ln a_n)$ 无界 $\Rightarrow$ 发散)

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34 2017-2018 第一学期期末考试

1. $P_n(x)=\sum_{k=0}^{n} a_k(x-x_0)^k$

2.(1) $=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}}dx=\int\frac{d\tan\frac{x}{2}}{\tan\frac{x}{2}}=\ln\left|\tan\frac{x}{2}\right|+C$

(2) $=\int e^t\,dt^3=3\int t^2de^t=3\left(t^2e^t-\int e^t dt^2\right)$

$=3\left(t^2e^t-2\int t\,de^t\right)=3\left(t^2e^t-2te^t+2\int e^t dt\right)$

$=3\left(t^2e^t-2te^t+2e^t\right)+C=3\left(x^{\frac{2}{3}}e^{\sqrt[3]{x}}-2\sqrt[3]{x}e^{\sqrt[3]{x}}+2e^{\sqrt[3]{x}}\right)$

(3) $=\int_0^1\arctan x\,dx-\int_0^1\arctan x\,d\arctan x$

$=x\arctan x\Big|_0^1-\int_0^1\frac{x}{1+x^2}dx-\frac{1}{2}(\arctan x)^2\Big|_0^1$

$=\frac{\pi}{4}-\frac{1}{2}\ln(1+x^2)\Big|_0^1-\frac{1}{2}\left(\frac{\pi}{4}\right)^2=\frac{\pi}{4}-\frac{1}{2}\ln 2-\frac{\pi^2}{32}$

(4) $=\int_{-\frac{\pi}{2}}^{0}\frac{\sin^2x}{1+e^x}dx+\int_0^{\frac{\pi}{2}}\frac{\sin^2x}{1+e^x}dx=\int_0^{\frac{\pi}{2}}\sin^2x\left(\frac{1}{1+e^{-x}}+\frac{1}{1+e^x}\right)dx$

$=\int_0^{\frac{\pi}{2}}\sin^2x\,dx=\int_0^{\frac{\pi}{2}}\frac{1-\cos2x}{2}dx=\left(\frac{x}{2}-\frac{\sin2x}{4}\right)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{4}$

3.(1) $\lambda^3+\lambda^2+\lambda+1=0\Rightarrow\lambda=-1,\pm i$

$\therefore y=C_0e^{-x}+C_1\cos x+C_2\sin x$

(2) $y'+2xy=0\Rightarrow y=Ce^{-x^2}$

将 $y=ue^{-x^2}$ 代入,得 $u'e^{-x^2}=4x\Rightarrow u=\cancel{\int\frac{2dx^2}{e^{-x^2}}}\int 2e^{x^2}dx^2=2e^{x^2}+C$

$\therefore y=2+Ce^{-x^2}$

由 $y(0)=0\Rightarrow C=-2\Rightarrow y=2-2e^{-x^2}$

√4. $\Leftarrow x\int_0^x f(t)\,d\sin^2\frac{t}{2}\geqslant\sin^2\frac{x}{2}\int_0^x f(t)dt$

$\Leftarrow$(分部积分) $xf(x)\sin^2\frac{x}{2}-x\int_0^x\sin^2\frac{t}{2}df(t)\geqslant xf(x)\sin^2\frac{x}{2}-\sin^2\frac{x}{2}\int_0^x t\,df(t)$

$\Leftarrow\int_0^x\frac{t}{x}df(t)\geqslant\int_0^x\frac{\sin^2\frac{t}{2}}{\sin^2\frac{x}{2}}df(t)\Leftarrow\int_0^x\left(\frac{t}{x}-\frac{\sin^2\frac{t}{2}}{\sin^2\frac{x}{2}}\right)f'(t)\,dt\geqslant0$ (若 $f(x)$ 可导即证)

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(不可导的证法)

$\Leftarrow x\int_0^x f(t)\,d\sin^2\frac{t}{2}\geqslant\sin^2\frac{x}{2}\int_0^x f(t)\,dt$

$\Leftarrow\int_0^x f(t)\,d\frac{\sin^2\frac{t}{2}}{\sin^2\frac{x}{2}}\geqslant\int_0^x f(t)\,d\frac{t}{x}$

(设 $g(t)=\sin^2\frac{t}{2}/\sin^2\frac{x}{2}$, $h(t)=\frac{t}{x}$)

$\Leftarrow\int_0^1 f(g^{-1}(x))dx\geqslant\int_0^1 f(h^{-1}(x))dx$

$\Leftarrow g^{-1}(x)\geqslant h^{-1}(x),\ \forall x\in[0,1]$

$\Leftarrow g(t)\leqslant h(t),\ \forall t\in[0,x]$

$\Leftarrow g(t)$ 为下凸函数

5.(1) $\cancel{\sum_{n=1}^{\infty}a_n}$ $a_n=(-1)^n$ 部分和有界

$b_n=xe^{-nx}$ 单调↓

$x>0$ 时, $b_n\leqslant\frac{x}{nx}=\frac{1}{n}\to0$; $x=0$ 时, $b_n=0$

$\therefore$ 一致收敛

(2) $|u_n(x)|=\frac{x}{(e^x)^n}$

$x>0$ 时 $\because e^x>1$

$\therefore$ 收敛

$x=0$ 时, 级数$=0$　$\therefore$ 收敛

(3) $\cancel{\lim_{n\to\infty}\sup_{x\in[0,1]}|u_n(x)|\geqslant\lim_{n\to\infty}\left|u_n\left(\frac{1}{n^2}\right)\right|}$

若其一致收敛, 则 $\sum_{n=1}^{\infty}|u_n(x)|=\frac{x}{e^x}\cdot\frac{1}{1-\frac{1}{e^x}}=\frac{x}{e^x-1}\triangleq f(x),x\in(0,1]$

由连续性知 $\lim_{x\to0}f(x)\overset{L'H}{=}\lim_{x\to0}\frac{1}{e^x}=1$

而 $\sum_{n=1}^{\infty}|u_n(0)|=0$, 矛盾

$\therefore$ 不一致收敛

6. 设 $x-2=3y$, 则 $f(x)=\ln(3y+3)=\ln3+\ln(1+y)$

$=\ln3+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{y^n}{n}$　$(-1<y\leqslant1)$

$=\ln3+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n\cdot3^n}$　$(-1<x\leqslant5)$

√7. $g(x)$ 在 x_0 可导: $g'(x_0)=\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{1}{x-x_0}\cdot\left[\frac{f(x)-f(x_0)}{x-x_0}-f'(x_0)\right]$

见反面

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8. 设 $|f'(x_1)|=M=\max|f'(x)|$

$|f'(x_2)|=m=\min|f'(x)|$

则 $\int_0^1|f'(x)|dx\le M$

$\int_0^1|f''(x)|dx \ge \int_{x_2}^{x_1}|f''(x)|dx \ge \left|\int_{x_2}^{x_1}f''(x)dx\right| = |f'(x_1)-f'(x_2)| = M-m$

$f(0)f(1)\ge0 \Rightarrow$ 不妨设 $f(0),f(1)\ge0$ ∴只需证 $\int_0^1|f(x)|dx\ge\frac{m}{2}$

若∃$x_0\in[0,1]$, s.t. $f'(x_0)=0$, 则 $m=0$，显然成立

若不存在，不妨设 $\forall x\in[0,1], f'(x)>0 \Rightarrow m=\min f'(x)$

则 $f(x)\ge f(0)+\int_0^x f'(x)dx \ge f(0)+mx$

$\int_0^1|f(x)|dx \ge \int_0^1 mx\,dx = m\cdot\frac12x^2\Big|_0^1=\frac m2$ ∴得证

7. $g'(x_0)=\lim\limits_{h\to0}\frac{g(x_0+h)-g(x_0)}{h}=\lim\limits_{h\to0}\frac{\frac{f(x_0+h)-f(x_0)}{h}-f'(x_0)}{h}=\lim\limits_{h\to0}\frac{f(x_0+h)-f(x_0)-f'(x_0)h}{h^2}$

$\overset{L'H}{=}\lim\limits_{h\to0}\frac{f'(x_0+h)-f'(x_0)}{2h}$

$\lim\limits_{x\to x_0}g'(x)=$ ~~$\lim\limits_{h\to0}g'(x_0+h)=\lim\limits_{h\to0}\frac{f(x}{h}$~~ $\frac{f'(x)(x-x_0)-f(x)+f(x_0)}{(x-x_0)^2}=\lim\limits_{h\to0}\frac{f'(x_0+h)h-f(x_0+h)+f(x_0)}{h^2}$

∴ $g'(x_0)-\lim\limits_{x\to x_0}g'(x)=\lim\limits_{h\to0}\frac{2(f(x_0+h)-f(x_0))-(f'(x_0)+f'(x_0+h))h}{h^2}$

(还有小量)

$=\lim\limits_{h\to0}\frac{\cancel{2f'(x_0)h}-(f'(x_0)+f'(x_0+h))h}{h^2}=\lim\limits_{h\to0}\frac{f'(x_0)-f'(x_0+h)}{h}=-f''(x_0)$

~~$=\lim\limits_{h\to0}\frac{f'(x_0+h)h-f'(x_0)h}{h^2}$~~

∴ $g'(x_0)-\lim\limits_{x\to x_0}g'(x)=\lim\limits_{h\to0}\frac{f'(x_0+h)h-f'(x_0)h-2f'(x_0+h)h+2f(x_0+h)-2f(x_0)}{2h^2}$

$=\lim\limits_{h\to0}\frac{-(f'(x_0+h)-f'(x_0))h-2f'(x_0)h+2f(x_0+h)-2f(x_0)}{2h^2}=-\frac{f''(x_0)}{2}+\lim\limits_{h\to0}\frac{-f'(x_0)h+f(x_0+h)-f(x_0)}{h^2}$

$\overset{L'H}{=}-\frac{f''(x_0)}{2}+\lim\limits_{h\to0}\frac{-f'(x_0)+f'(x_0+h)}{2h}=-\frac{f''(x_0)}{2}+\frac{f''(x_0)}{2}=0$

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35 2018-2019 第一学期期末考试

1. $(1-x)f(x)=e^x$, $f(0)=1$

~~$(1-x)f^{(n)}(x)+n(1-x)f$~~

两边求n次导, $(1-x)f^{(n)}(x)-nf^{(n-1)}(x)=e^x$

$f^{(n)}(0)-nf^{(n-1)}(0)=1$

$\therefore f^{(n)}(0)=nf^{(n-1)}(0)+1=1+n(1+(n-1)f^{(n-2)}(0))$

$=\cdots=1+n+n(n-1)+\cdots+n(n-1)\cdots3\cdot2+n!f^{(0)}(0)$

$=\sum_{k=0}^{n}\frac{n!}{k!}$

2.(1) $=\int dx-\int\frac{e^{2x}}{1+e^{2x}}dx=x-\frac12\int d\ln(1+e^{2x})=x-\frac12\ln(1+e^{2x})+C$

(2) $=\int\frac{dx}{(x-1)(x+1)^2}=\frac14\int\left(\frac{1}{x-1}-\frac{x+3}{(x+1)^2}\right)dx$

$=\frac14\int\left(\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{(x+1)^2}\right)dx$

$=\frac14\left[\ln(x-1)-\ln(x+1)+\frac{2}{x+1}\right]+C$

(3) $=\frac12\int_0^1\arctan x\,dx^2=\frac12x^2\arctan x\Big|_0^1-\frac12\int_0^1\frac{x^2}{1+x^2}dx$

$=\frac\pi8-\frac12\left(\int_0^1dx-\int_0^1\frac{1}{1+x^2}dx\right)$

$=\frac\pi8-\frac12(x-\arctan x)\Big|_0^1=\frac\pi4-\frac12$

(4) 令 $t=\sqrt{x-1}$, $x=t^2+1$

$=\int_0^{+\infty}\frac{2t\,dt}{(t^2+1)t}=2\int_0^{+\infty}\frac{dt}{t^2+1}=2\arctan t\Big|_0^{+\infty}=\pi$

3.(1) $((1+x^2)y')'=x$

$(1+x^2)y'=\frac12x^2+\frac C2$

$y'=\frac12\cdot\frac{x^2+C}{x^2+1}=\frac12\left(1+\frac{C'}{x^2+1}\right)$

$y=\frac12x+C_1\arctan x+C_2$

(2) $z \triangleq y'-y$，则 $z'-2z=2x-3$ ✓

4. $LHS=\int_0^{\frac{\pi}{2}} xf(|\cos x|)dx+\int_{\frac{\pi}{2}}^{\pi} xf(|\cos x|)dx$

$$=\int_0^{\frac{\pi}{2}} xf(|\cos x|)dx+\int_0^{\frac{\pi}{2}}(\pi-x)f(|\cos x|)dx=RHS$$

5. 一致收敛. 令 $x=1+\alpha$.

由微分中值定理 $\frac{1}{(n-1)^\alpha}-\frac{1}{n^\alpha}=\frac{\alpha}{(n-\theta)^{\alpha+1}}>\frac{\alpha}{n^x}$

$$\cancel{\sum_{n=1}^{\infty}\frac{(x-1)^2}{n^x}<(x-1)^2+\sum_{n=2}^{\infty}\left[\frac{1}{(n-1)^\alpha}-\frac{1}{n^\alpha}\right]=(x-1)^2+1}$$

$\forall\varepsilon>0,\exists N=[e^{\frac{1}{\varepsilon}}]+1,\forall n>N,p>0,x>1$，有

$$\frac{(x-1)^2}{(n+1)^x}+\frac{(x-1)^2}{(n+2)^x}+\cdots+\frac{(x-1)^2}{(n+p)^x}<\alpha\left(\frac{1}{n^\alpha}-\frac{1}{(n+p)^\alpha}\right)<\frac{\alpha}{n^\alpha}<\frac{\frac{1}{\ln n}}{n^{\frac{1}{\ln n}}}<\frac{1}{\ln n}<\varepsilon$$

6. ✓

7. (1) $\frac{a_n}{a_{n+1}}\leq\frac{n\ln n+\ln n+1}{n\ln n}+\frac{1}{n^2}<\frac{(n+1)\ln(n+1)}{n\ln n}+\frac{(n+1)\ln(n+1)}{n\ln n}\cdot\frac{1}{n^2}$

(2) $\ln\frac{a_n}{a_{n+1}}\leq\ln\left(1+\frac{1}{n}+\frac{1}{n\ln n}+\frac{1}{n^2}\right)\leq\frac{1}{n}+\frac{1}{n\ln n}+\frac{1}{n^2}$

累加 $\Rightarrow \ln a_3-\ln a_n\leq\int_2^n\frac{1}{x}dx+\int_2^n\frac{1}{x\ln x}dx+C=\ln(x\ln x)\Big|_{x=2}^{n}+C$

$$=\ln(n\ln n)+C'$$

$\Rightarrow a_n\geqslant\frac{C''}{n\ln n}>0$　而 $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ 发散，故 $\sum_{n=2}^{\infty}a_n$ 发散

8. 对于任意给定的 x：

若 $f'(x)=0$，则已成立

若 $f'(x)\neq0$，不妨设 $f'(x)>0$（<0 同理）

取 $h=(f'(x))^{\frac{1}{\alpha}}$，则 $\int_{x-h}^{x}(f'(x)-f'(t))dt=-f(x)+f(x-h)+f'(x)h$

$$\int_{x-h}^{x}(x-t)^\alpha dt\geqslant \qquad\qquad \geqslant -f(x)+(f'(x))^{\frac{\alpha+1}{\alpha}}$$

$$\|$$

$$\int_0^h x^\alpha dx=\frac{x^{\alpha+1}}{\alpha+1}\Big|_0^h=\frac{(f'(x))^{\frac{\alpha+1}{\alpha}}}{\alpha+1}$$

此题取 $\alpha=\frac{1}{2}$ 即可

$\boxed{36}$ 2019-2020 第一学期期末考试

1. (1) $=\frac{1}{2}\int\left(-\frac{1}{x^2-1}+\frac{1}{x^2+1}\right)dx=-\frac{1}{4}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)+\frac{1}{2}\int\frac{1}{x^2+1}dx$

$$=-\frac{1}{4}[\ln(x-1)-\ln(x+1)]+\frac{1}{2}\arctan x+C$$

(2) $t\triangleq\sqrt{x}$，$x=t^2$

$$=\int_0^2\frac{2t}{1+t}dt=\int_0^2\left(2-\frac{2}{1+t}\right)dt=[2t-2\ln(1+t)]\Big|_0^2=4-2\ln3$$

(3) $I=-\int_0^{+\infty}\cos x\,de^{-x}=-\cos x\cdot e^{-x}\Big|_0^{+\infty}+\int_0^{+\infty}e^{-x}d\cos x$

$$=1+\int_0^{+\infty}\sin x\,de^{-x}=1+\sin x\cdot e^{-x}\Big|_0^{+\infty}-\int_0^{+\infty}e^{-x}\cos x\,dx=1-I$$

$\therefore I=\frac{1}{2}$

(4) 当 $x\geqslant1$ 时 $=\int\ln x\,dx=x(\ln x-1)+C_1$

当 $x<1$ 时 $=-\int\ln x\,dx=-x(\ln x-1)+C_2=-x(\ln x-1)+C_1-2$

(5) $=\lim\limits_{n\to\infty}e^{(1^p+2^p+\cdots+n^p)\cdot\frac{1}{n^{p+1}}}=\exp\left(\lim\limits_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p\right)$

$$=\exp\left(\int_0^1x^pdx\right)=\exp\left(\frac{x^{p+1}}{p+1}\Big|_0^1\right)=e^{\frac{1}{p+1}}$$

2. $y'(0)=2$，$y(0)=0$

设 $z=y'-3y$，则 $z(0)=2$

$z'-3z=e^{3x}$　(*)

$z'-3z=0\Rightarrow z=Ce^{3x}$

将 $z=ue^{3x}$ 代入 (*) 得 $u'e^{3x}=e^{3x}\Rightarrow u=x+C$

$\therefore z=(x+C)e^{3x}$，$z(0)=C=2$

$\therefore y'-3y=(x+2)e^{3x}$

$y=e^{3x}\left(\int(x+2)dx+C\right)=e^{3x}\left(\frac{1}{2}x^2+2x+C\right)$

$\therefore y=x\left(\frac{1}{2}x+2\right)e^{3x}$

3. $x,y>0$

当$x,y>1$时 $\ln x+\ln y=1 \Rightarrow xy=e$

$$S_1=\int_1^e\left(\frac{e}{x}-1\right)dx=(e\ln x-x)\Big|_1^e=1$$

当$x>1,y<1$时 $\ln x-\ln y=1 \Rightarrow y=\frac{x}{e}$

$$S_2=\frac{1}{2}\left(1-\frac{1}{e}\right)(e-1)=\frac{(e-1)^2}{2e}$$

当$x<1,y>1$时 $S_3=S_2$

当$x,y<1$时 $-\ln x-\ln y=1 \Rightarrow xy=\frac{1}{e}$

$$S_4=\int_{\frac{1}{e}}^1\left(1-\frac{1}{ex}\right)dx=\left(x-\frac{1}{e}\ln x\right)\Big|_{\frac{1}{e}}^1=1-\frac{2}{e}$$

$$\therefore S=1+\frac{(e-1)^2}{e}+1-\frac{2}{e}=e-\frac{1}{e}$$

4. $\alpha>0$或$(\beta<0$且$\alpha>\beta)$时: $f(0^+)=0$ 连续 $\Rightarrow$ 可积

$\alpha=0$时 或$(\beta<0$且$\alpha=\beta)$: 有界+至多有限个间断点 $\Rightarrow$ 可积

否则: 无界 $\Rightarrow$ 不可积

5.(1) $\frac{1}{(k-1)^{\alpha-1}}-\frac{1}{k^{\alpha-1}}=\frac{\alpha-1}{(k-\theta)^\alpha}>\frac{\alpha-1}{k^\alpha}$ 见$\boxed{35}$T5

$$\therefore \sum_{k=1}^n\frac{1}{k^\alpha}<\left(1+1-\frac{1}{n^{\alpha-1}}\right)\cdot\frac{1}{\alpha-1}=\frac{1}{\alpha-1}\left(2-\frac{1}{n^{\alpha-1}}\right)\quad(n\geqslant 2)$$

$$\sum_{n=1}^\infty\frac{1}{n^2}\cdot\frac{1}{\alpha-1}\left(2-\frac{1}{n^{\alpha-1}}\right)=\sum_{n=1}^\infty\frac{1}{\alpha-1}\left(\frac{2}{n^2}-\frac{1}{n^{\alpha+1}}\right)$$ 收敛 ∴收敛

(2) ~~$n\geq A$且~~ $n\to\infty$时, $\frac{x}{n}-\sin\frac{x}{n}\ \cancel{=\frac{x}{n}} \sim \frac{x}{n}-\frac{x}{n}+o\left(\left(\frac{x}{n}\right)^3\right)=o\left(\left(\frac{x}{n}\right)^3\right)$

而 $\sum_{n=1}^\infty\frac{x^3}{n^3}$ 一致收敛 ∴一致收敛

6. 设 $x=f(t)$, $t=f^{-1}(x)$

则 $\int f^{-1}(x)dx=\int t\,dx=tx-\int x\,dt=tx-\int f(t)dt$

$=tx-F(t)+C=xf^{-1}(x)-F(f^{-1}(x))+C$

7. 必要性: $\int_0^T f(x)dx=F(T)-F(0)=0$

充分性: $0=\int_0^{x_0}f(x)dx+\int_{x_0}^T f(x)dx$

$=\int_0^{x_0}f(x+T)dx+\int_{x_0}^T f(x)dx$

$=\int_{x_0}^{T+x_0}f(x)dx=F(x_0+T)-F(x_0)$, $\forall x_0$

~~$f(T+x_0$~~ ∴以T为周期

8. $x=1$时发散.

由Abel定理知$|x|>1$时也发散.

$|x|<1$时: 设$|a_n|\leq M$, $\forall n$

对于给定的$x\in(-1,1)$, $\forall\varepsilon>0$, $\exists N=\left[\log_x\frac{\varepsilon(1-x)}{M}\right]$ $\forall n>N, p>0$, 有

$|a_{n+1}x^{n+1}+a_{n+2}x^{n+2}+\cdots+a_{n+p}x^{n+p}|$

$\leq|Mx^{n+1}+Mx^{n+2}+\cdots+Mx^{n+p}|$

$=Mx^{n+1}\cdot\frac{1-x^p}{1-x}<\frac{Mx^{n+1}}{1-x}<\varepsilon$

由Cauchy收敛准则知其收敛

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1.(1) $=\int \frac{\sin x(1-\cos^2 x)}{\cos x}dx = \int(\tan x - \frac{1}{2}\sin 2x)dx = -\ln|\cos x| + \frac{1}{4}\cos 2x + C$

(2) $= x(\ln x - 1)\Big|_0^1 = -1$

(3) $= \lim_{x\to 0}\frac{\int_0^x t^2 dt}{x^3} = \lim_{x\to 0}\frac{x^2}{3x^2} = \frac{1}{3}$

(4) $y' - \frac{2}{x}y = 0 \Rightarrow \frac{dy}{y} = \frac{2dx}{x} \Rightarrow \ln y = 2\ln x + C \Rightarrow y = Cx^2$

将 $y=ux^2$ 代入，得 $u'x^2 = 2x^2 \Rightarrow u = 2x + C$

$\therefore y = 2x^3 + Cx^2$

(5) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $(-1<x<1)$

求导，得 $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(x-1)^2}$ 即 $\sum_{n=1}^{\infty} nx^n = \frac{x}{(x-1)^2}$

$x=\pm 1$ 时不收敛

$\therefore (-1,1)$ $S(x) = \frac{x}{(x-1)^2}$

2. $r = 1+\cos\theta$

$S = 2\pi\int_0^{\pi} r\sin\theta\sqrt{r^2+r'^2}\,d\theta = 2\pi\int_0^{\pi}(1+\cos\theta)\sin\theta\sqrt{2+2\cos\theta}\,d\theta$

$= 2\pi\int_0^{\pi} 2\cos^2\frac{\theta}{2}\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot 2\cos\frac{\theta}{2}\,d\theta$

$= 32\pi\int_{\pi}^{0}\cos^4\frac{\theta}{2}\,d\cos\frac{\theta}{2} = 32\pi\cdot\frac{1}{5}\cos^5\frac{\theta}{2}\Big|_{\pi}^{0} = \frac{32}{5}\pi$

3. 设 $z = y' - 2iy$，则 $z' + 2iz = 9x\sin x$

$(e^{2ix}z)' = 9x\sin x\cdot e^{2ix}$

题中方程为 $y''+4y = 9xe^{ix}$ 的虚部

$y = z\cdot e^{ix} \Rightarrow z'' + 2iz' + 3z = 9x$

待定系数法得特解 $z = 3x - 2i \Rightarrow y_0 = (3x-2i)e^{ix}$

分离出虚部，得 $y_0 = 3x\sin x - 2\cos x$

对应齐次方程 $y''+4y=0$ 的基本解组为 $\{\cos 2x, \sin 2x\}$

$\therefore$ 通解 $y = C_1\cos 2x + C_2\sin 2x + 3x\sin x - 2\cos x$

$\cancel{\int_{\frac{\pi}{2}}^{\pi}\frac{(\pi-x)\sin x}{\sin x+\cos x}dx}$ $-\int_{\frac{\pi}{2}}^{0}\frac{(\pi-t)\sin(\pi-t)}{\sin(\pi-t)-\cos(\pi-t)}dt$

4. $I = \int_0^{\frac{\pi}{2}}\frac{x\sin x}{\sin x+\cos x}dx + \int_{\frac{\pi}{2}}^{\pi}\frac{x\sin x}{\sin x-\cos x}dx$

$= \int_0^{\frac{\pi}{2}}\frac{x\sin x}{\sin x+\cos x}dx + \int_0^{\frac{\pi}{2}}\frac{(\pi-x)\sin x}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{2}}\frac{\pi\sin x}{\sin x+\cos x}dx$

又 $I = \int_0^{\frac{\pi}{2}}\frac{\pi\sin(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x)}dx = \int_0^{\frac{\pi}{2}}\frac{\pi\cos x}{\sin x+\cos x}dx$

$\therefore 2I = \int_0^{\frac{\pi}{2}}\pi\,dx = \frac{\pi^2}{2} \Rightarrow I = \frac{\pi^2}{4}$

5. $f(x) = \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+4)-3} - \frac{1}{(x+4)-2}$

$= \frac{1}{3}\cdot\frac{1}{\frac{x+4}{3}-1} - \frac{1}{2}\cdot\frac{1}{\frac{x+4}{2}-1}$

$= -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x+4}{3}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x+4}{2}\right)^n$

$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right)(x+4)^n$ $(-6,-2)$

6. $\cancel{e^{nx} > nx+1 > nx \Rightarrow 2^{nx} >}$

~~求导可证~~ 逐点收敛显然

不一致收敛：法I：$x=0$ 处不连续

法II：$f(x) \triangleq \frac{x}{2^{nx}}$，$f'(x) = \frac{2^{nx} - x\cdot 2^{nx}\cdot n\ln 2}{(2^{nx})^2}$

令 $f'(x)=0 \Rightarrow x = \frac{1}{n\ln 2}$

$\therefore \lim_{n\to\infty}\sup_{x\in J}|u_n| = \lim_{n\to\infty}u_n\left(\frac{1}{n\ln 2}\right) = 0$ × (不可行)

7. 单调↓→0 ⇒ 收敛

$\sum_{k=1}^{n}a_n = \sum_{k=1}^{n}\int_0^1\frac{1}{(1+t^3)^{2k}}dt = \int_0^1\sum_{k=1}^{n}\frac{1}{(1+t^3)^{2k}}dt \le \int_0^1\frac{1}{(1+t^3)^2}\cdot\frac{1}{1-\frac{1}{(1+t^3)^2}}dt$

$\cancel{= \int_0^1\frac{1}{(1+t^3)^2-1}dt = \int_0^1\frac{1}{t^3(t^3+1)}dt}$ ×