1、试推导向后Euler方法和中心Euler方法的误差.

(1) 向后Euler法: $\frac{u_{n+1}-u_n}{h}=f(x_{n+1},u_{n+1})$. ($\frac{du}{dx}=f(x,u)$

将 $u(x)$ 在 $x=x_{n+1}$ 处展开, 令 $x=x_n$ 可得.

$u(x_n)=u(x_{n+1})-hu'(x_{n+1})+\frac{h^2}{2}u''(\xi)$, $\xi\in(x_n,x_{n+1})$

于是局部截断误差为 $u(x_{n+1})-u_{n+1}=-\frac{h^2}{2}u''(\xi)$. □

(2) 中心Euler法: $\frac{u_{n+1}-u_{n-1}}{2h}=f(x_n,u_n)$

将 $u(x)$ 在 $x=x_n$ 处展开 并令 $x=x_{n-1}$, x_{n+1}, 得.

$u(x_{n-1})=u(x_n)-hu'(x_n)+\frac{h^2}{2}u''(x_n)-\frac{h^3}{6}u'''(\xi_1)$, $\xi_1\in(x_{n-1},x_n$

$u(x_{n+1})=u(x_n)+hu'(x_n)+\frac{h^2}{2}u''(x_n)+\frac{h^3}{6}u'''(\xi_2)$, $\xi_2\in(x_n,$

于是局部截断误差为:

$u(x_n)+hu'(x_n)+\frac{h^2}{2}u''(x_n)+\frac{h^3}{6}u'''(\xi_2)-[2hf(x_n,u_n)+u(x_n)-hu'(x_n)+\frac{h^2}{2}$

$-\frac{h^3}{6}u'''$

$=\frac{h^3}{6}(u'''(\xi_1)+u'''(\xi_2))=\frac{h^3}{3}u'''(\xi)$, $\xi\in(x_{n-1},x_{n+1})$

2. 正交多项式定理:

如下归纳定义的多项式序列是正交的.

$p_n(x)=(x-a_n)p_{n-1}(x)-b_np_{n-2}(x)$ $(n\geq 2)$

$p_0(x)=1$, $p_1(x)=x-a_1$, 这里

$a_n=\langle xp_{n-1},p_{n-1}\rangle/\langle p_{n-1},p_{n-1}\rangle$

$b_n=\langle xp_{n-1},p_{n-2}\rangle/\langle p_{n-2},p_{n-2}\rangle$

若 $f(x)$ 充分可微，设有数值公式

$$\int_{-2}^{2} \omega(x) f(x)\,dx = A f(x_0) + B f(x_1)$$

其中权函数 $\omega(x) = \frac{1}{\sqrt{4-x^2}}$

(1) 确定常数 A, B, x_0, x_1，使上式具有最高阶精度 3

(2) 求此公式的截断误差.

解：(1) 由上述结论，求出 $P_0(x)=1$.

$a_1 = \langle xP_0, P_0\rangle / \langle P_0, P_0\rangle = 0$ (奇函数) $\Rightarrow$ $P_1(x)=x$.

$a_2 = \langle xP_1, P_1\rangle / \langle P_1, P_1\rangle = 0$ (奇).

$b_2 = \langle xP_1, P_0\rangle / \langle P_0, P_0\rangle$

$\langle x^2, 1\rangle = \int_{-2}^{2} \frac{x^2}{\sqrt{4-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\sin^2 t}{2\cos t} 2\cos t\,dt = 2\pi$.

$\langle 1, 1\rangle = \int_{-2}^{2} \frac{1}{\sqrt{4-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos t}{2\cos t} dt = \pi$

$\Rightarrow b_2 = 2$.

$\Rightarrow P_2(x) = x^2 - 2 \Rightarrow x_0, x_1 = \pm\sqrt{2}$.

令 $f(x) \equiv 1 \Rightarrow \int_{-2}^{2} \omega(x)\,dx = \pi = A + B$

$f(x) = x \Rightarrow -\frac{\sqrt{2}}{2}A + \frac{\sqrt{2}}{2}B = 0 \Rightarrow A = B = \frac{\pi}{2}$

(2) 见 PDF.

3、设二元函数 $f(x,y)$ 在 $a\le x\le b$，$-\infty<y<\infty$ 上连续，且对 y 满足 Lipschitz 条件. $|f(x,y_1)-f(x,y_2)|\le L|y_1-y_2|$

设有 ODE 初值问题

$$\begin{cases} \frac{dy}{dx}=f(x,y) & a\le x\le b. \\ y(a)=y_0 \end{cases}$$

若精确解 $y(x)\in C^2[a,b]$. 且 $|y''(x)|\le M$，$\forall x\in[a,b]$.

证明 Euler 向前公式 $|y(x_n)-y_n|\le \frac{hM}{2L}(e^{L(b-a)}-1)$.

$y_{n+1}=y_n+hf(x_n,y_n)$.

证：记 $e_n=y(x_n)-y_n$.

$$e_{n+1}=y(x_{n+1})-y_{n+1}=y(x_n)+hf(x_n,y(x_n))+\frac{h^2}{2}y''(\xi_n)-y_{n+1}$$

$$=y(x_n)-y_n+hf(x_n,y(x_n))-hf(x_n,y_n)+\frac{h^2}{2}y''(\xi_n)$$

$$\Rightarrow |e_{n+1}|\le|y(x_n)-y_n|+h|f(x_n,y(x_n))-f(x_n,y_n)|+\frac{h^2}{2}M$$

$$\le(1+hL)|e_n|+\frac{h^2}{2}M.$$

$$\Rightarrow |e_n|\le(1+hL)|e_{n-1}|+\frac{h^2}{2}M\le(1+hL)\left[(1+hL)|e_{n-2}|+\frac{h^2}{2}M\right]+\frac{h^2}{2}M$$

$$\le\cdots\le(1+hL)^n|e_0|+\frac{h^2}{2}M\left[1+(1+hL)+\cdots+(1+hL)^{n-1}\right]$$

$$=\frac{h^2M}{2}\cdot\frac{1-(1+hL)^n}{1-(1+hL)}=\frac{M}{2L}\left[(1+hL)^n-1\right]$$

$$\le\frac{M}{2L}\left[(1+hL)^{\frac{nhL}{hL}}-1\right]\le\frac{M}{2L}\left[e^{nhL}-1\right]$$

求常数 a,b,c，使得 ODE $y'(x)=f(x,y)$ 的数值公式 $y_{n+1}=y_n+chK$，$K=f(x_n+ah,\ y_n+bhK)$ 局部截断误差阶数尽可能高.

解：$K=f(x_n,y_n)+ahf_x(x_n,y_n)+bhKf_y(x_n,y_n)+O(h^2)$

$$\Rightarrow K=\frac{f(x_n,y_n)+ahf_x(x_n,y_n)+O(h^2)}{1-bhf_y(x_n,y_n)}.$$

$$=f(x_n,y_n)+ahf_x(x_n,y_n)+bhf_y(x_n,y_n)f(x_n,y_n)$$

$$\Rightarrow y_{n+1}-y(x_{n+1})=cKh-y'(x_n)h-\frac{1}{2}y''(x_n)h^2+O(h^3).$$

$$=cfh+caf_xh^2+cbf_yfh^2-fh-\frac{1}{2}f_xh^2-\frac{1}{2}f_yfh^2+O($$

$$=(c-1)fh+(ca-\frac{1}{2})f_xh^2+(cb-\frac{1}{2})f_yfh^2+O(h^3$$

$$\Rightarrow c=1,\ a=b=\frac{1}{2}.$$

A-n阶对称实方阵 ~~随机产生向量 $x_0 \in R^n$~~ $\lambda_1 > \cdots > \lambda_n$.

取实数 s，令 $B = A - sI$，随机产生非零向量 $x_0 \in R^n$

构造 $x_{k+1} = Bx_k / \|Bx_k\|_2$，$k \in N$.

(1) 求 s 范围使得 x_k 收敛到 λ_1 对应的特征向量.

(2) 求 s 使得收敛速度最快.

解.(1) $A = P\Lambda P^T$，$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

$B = P\,\mathrm{diag}(\lambda_1 - s, \cdots, \lambda_n - s)P^T$，$P$ 正交.

~~$x_k = \frac{P\Lambda^k P^T}{\|\Lambda^k\|} \to P(\ \ )P^T$，它是 λ_n 特征向量，因为 $P\Lambda P^T \cdot P(\ \ )P^T = P\Lambda(\ \ )P^T$~~

$$x_k = \frac{P(\Lambda - s)^k P^T}{\|\ \ \|} x_0 \to P \cdot e_k.$$

$$\Rightarrow |\lambda_1 - s| > \max_{2 \le i \le n} |\lambda_i - s| \Leftrightarrow s \in \left(-\infty, \frac{\lambda_1 + [illegible]}{2}\right.$$

(2) 收敛速度依赖于 $f(s) = \dfrac{\max\limits_{2 \le i \le n} |\lambda_i - s|}{|\lambda_1 - s|}$，

$s = \frac{\lambda_2 + \lambda_n}{2}$ 时，$f(s)$ 最小.