数理方程B第一章参考答案

1. 解：

(1) 首先极坐标形式下的拉普方程为

$$\Delta_2 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial u}{\partial \theta}=0$$

由于求形如$u=u(r),(r=\sqrt{x^2+y^2}\neq 0)$的解, 因此$u$的自变量只有$r$, 则有

$$\Delta_2 u=\frac{d^2 u}{dr^2}+\frac{1}{r}\frac{du}{dr}=0,(r\neq 0)$$

$$\text{亦即}r^2\frac{d^2 u}{dr^2}+r\frac{du}{dr}=0$$

该方程为一欧拉方程.

做变量替换$r=e^t$, 则有

$$\frac{du}{dr}=\frac{du}{dt}\frac{dt}{dr}=\frac{1}{r}\frac{du}{dt}\Rightarrow r\frac{du}{dr}=\frac{du}{dt}$$

$$\frac{d^2u}{dr^2}=\frac{d\left(\frac{1}{r}\frac{du}{dt}\right)}{dr}=-\frac{1}{r^2}\frac{du}{dt}+\frac{1}{r}\frac{d\left(\frac{du}{dt}\right)}{dr}==-\frac{1}{r^2}\frac{du}{dt}+\frac{1}{r}\frac{d\left(\frac{du}{dt}\right)}{dt}\frac{dt}{dr}$$

$$=-\frac{1}{r^2}\frac{du}{dt}+\frac{1}{r^2}\frac{d^2u}{dt^2}\Rightarrow r^2\frac{d^2u}{dr^2}=-\frac{du}{dt}+\frac{d^2u}{dt^2}$$

所以有

$$\frac{d^2u}{dt^2}=0$$

故其通解为

$$u(t)=C_1+C_2t\Rightarrow u(r)=C_1+C_2lnr,C_1,C_2\text{为常数},r>0$$

(2) 依题，首先将方程化成如下形式：

$$\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}+k^2u=0,(k\text{为正常数})$$

两边同乘以r，得到

$$r\frac{d^2u}{dr^2}+\frac{2du}{dr}+k^2ur=0$$

设

$$f(r)=ur$$

则

$$\frac{df}{dr}=r\frac{du}{dr}+u$$

$$\frac{d^2f}{dr^2}=\frac{du}{dr}+r\frac{d^2u}{dr^2}+\frac{du}{dr}=r\frac{d^2u}{dr^2}+\frac{2du}{dr}$$

所以

$$\frac{d^2f}{dr^2}+k^2ur=0$$

其特征方程为

$$\lambda^2+k^2=0$$

k为正常数，故其有两个纯虚的根

$$\lambda_1=-ki,\lambda_2=ki$$

方程通解为

$f=C_1coskr+C_2sinkr \Rightarrow ur=C_1coskr+C_2sinkr, C_1, C_2$为常数

故原方程的通解为

$$u=\frac{1}{r}(C_1coskr+C_2sinkr), C_1, C_2\text{为常数}, r>0$$

2. 解：

$$u_x=F'+G'$$
$$u_y=\lambda_1F'+\lambda_2g'$$
$$u_{xx}=F''+G''$$
$$u_{yy}={\lambda_1}^2F''+{\lambda_2}^2G''$$
$$u_{xy}=\lambda_1F''+\lambda_2G''$$

所以，带入题目中方程左边，有

$${\lambda_1}^2F''+{\lambda_2}^2G''-(\lambda_1+\lambda_2)(\lambda_1F''+\lambda_2G'')+\lambda_1\lambda_2(F''+G'')$$

展开得到上式为 0, 满足方程.

3. 解：

$$u_t=-\frac{1}{2}t^{-\frac{3}{2}}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}+t^{-\frac{1}{2}}\frac{(x-\xi)^2}{4a^2t^2}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}$$

化简得

$$u_t=-\frac{1}{2}t^{-\frac{3}{2}}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}+t^{-\frac{5}{2}}\frac{(x-\xi)^2}{4a^2}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}$$

$$u_x=t^{-\frac{1}{2}}\left(-\frac{x-\xi}{2a^2t}\right)\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}=-\frac{1}{2}t^{-\frac{3}{2}}\frac{x-\xi}{a^2}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}$$

$$u_{xx}=-\frac{1}{2a^2}t^{-\frac{3}{2}}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}+\left(-\frac{1}{2}t^{-\frac{3}{2}}\frac{x-\xi}{a^2}\right)\left(-\frac{x-\xi}{2a^2t}\right)\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}$$

化简得

$$u_{xx}=-\frac{1}{2a^2}t^{-\frac{3}{2}}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}+t^{-\frac{5}{2}}\frac{(x-\xi)^2}{4a^4}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}$$

将其带入，可知其满足方程.

$$\lim_{t\to0}u(t,x)=\lim_{t\to0}\frac{1}{\sqrt{t}}\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\},(x\neq\xi)$$

该极限只与t有关，故其他变量可视为常量，则设

$$\frac{(x-\xi)^2}{4a^2}=k>0, k\text{与}t\text{无关}$$

极限可化为

$$\lim_{t\to 0}\frac{1}{e^{\frac{k}{t}}\sqrt{t}} \xRightarrow{\frac{1}{t}=h,t>0} \lim_{h\to+\infty}\frac{\sqrt{h}}{e^{kh}}=0$$

4. 解：

$$u=axe^{2x+y}$$

则

$$u_x=ae^{2x+y}+2axe^{2x+y}$$
$$u_{xx}=2ae^{2x+y}+2ae^{2x+y}+4axe^{2x+y}=4ae^{2x+y}(1+x)$$
$$u_y=axe^{2x+y}$$

$$u_{yy}=axe^{2x+y}$$

所以

$$u_{xx}-4u_{yy}=4ae^{2x+y}=e^{2x+y}\Rightarrow a=\frac{1}{4}$$

故所求特解为

$$u=\frac{1}{4}xe^{2x+y}$$

5. 解：

$$u_x=yf'$$
$$u_y=xf'$$

带入方程即可知满足

6. 解：

(1) 先考虑一阶线性微分方程通解问题, 即

$$\frac{dy}{dx}+P(x)y=Q(x)$$

的通解为

$$y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+C\right), C\text{为常数}$$

则此题可看成u关于y的一阶线性微分方程, 故

$$u=C(x)e^{-\int a(x,y)dy}$$

注：此题为二变量方程和题设中u有x,y,z三个变量有些不符，故“积分常数”含x或x,z，写哪一种都行。

(2) 设

$$u_y=h(x,y)$$

则

$$u_{xy}=u_{yx}=h_x$$

注：本门课程中涉及到的混合偏导在没有特殊说明的情况下都是可交换的

所以方程化为

$$h_x + h = 0$$

解得

$$h = C_1(y)e^{-x} \Rightarrow u_y = C_1(y)e^{-x} \Rightarrow u = e^{-x}C(y) + D(x)$$

(3) 此题用到了叠加原理

书中提示让我们先求一个形如$v(x)$的特解，设其通解$u(x,t) = v(x) + w(x,t)$则将方程分解为

$$0 = v_{tt} = a^2 v_{xx} + 3x^2$$
$$w_{tt} = a^2 w_{xx}$$

其中第一个方程通解为

$$v(x) = -\frac{1}{4a^2}x^4 + Cx$$

其中第二个方程通解

$$w(x,t) = f(x+at) + g(x-at)$$

故原方程通解为

$$u(x,t) = -\frac{1}{4a^2}x^4 + Cx + f(x+at) + g(x-at)$$

注：所求特解的形式不唯一，选取一种即可，本题中亦可直接写$v(x) = -\frac{1}{4a^2}x^4$.

7. 解：

先确定总的方程，与热有关的有热传导和场势方程，此题中有热的变化故选取热传导方程，物体内部没有热源所以选择其次形式。题干也给出了一个边界条件. 再接着“翻译”物理过程，特别要注意傅里叶定律当中方向导数的求法。

(1)
$$\begin{cases} u_t = a^2 u_{xx}, (t \geq 0, 0 \leq x \leq l) \\ u(0,x) = \varphi(x) \\ u_x(t,0) = 0 \\ u(t,l) = u_0 \end{cases}$$

(2) 不妨考虑左端流进q_1，右端流进q_2，则所得方程及边界条件为：

傅里叶定律：

$$-k\frac{\partial u}{\partial \vec{n}}(\text{流出}) + q(\text{流入}) = 0$$

$$\begin{cases} u_t = a^2 u_{xx}, (t \geq 0, 0 \leq x \leq l) \\ u(0,x) = \varphi(x) \\ -k\frac{\partial u}{\partial \vec{n}}|_{x=0} = -k[-u_x(t,0)] = ku_x(t,0) \Rightarrow ku_x(t,0) + q_1 = 0 \Rightarrow -ku_x(t,0) = q_1 \\ -k\frac{\partial u}{\partial \vec{n}}|_{x=l} = -ku_x(t,l) \Rightarrow -ku_x(t,l) + q_2 = 0 \Rightarrow ku_x(t,l) = q_2 \end{cases}$$

(3) 由牛顿冷却定律，得到方程及边界条件为：

$$\begin{cases} u_t = a^2 u_{xx}, (t \geq 0, 0 \leq x \leq l) \\ u(0, x) = \varphi(x) \\ u(t, 0) = \mu(t) \\ -k\frac{\partial u}{\partial \vec{n}}|_{x=l} = -ku_x(t, l) = h(l)[u(t, l) - \theta(t)] \Rightarrow (ku_x + hu)|_{x=l} = h(l)\theta(t) \end{cases}$$

8. 解：

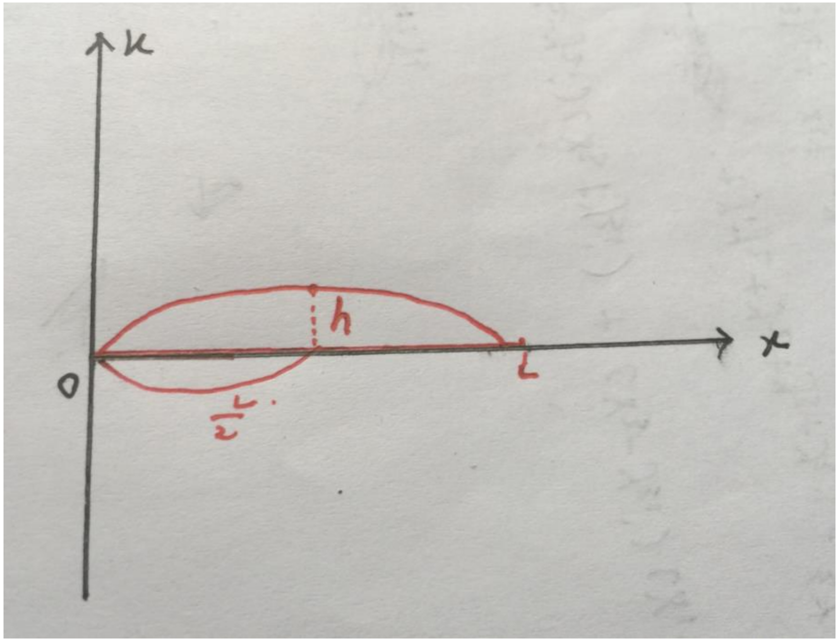
先考虑无条件的一维弦振动，列出方程

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (t \geq 0, 0 \leq l \leq x)$$

考虑第一类边界条件，得

$$u(t, 0) = 0$$
$$u(t, l) = 0$$

如图所示



由于弦做微小横振动，故可将左右两个曲边三角形看成三角形，由三角形相似，得

$$u(0, x) = \frac{2h}{l}x \qquad 0 \leq x \leq \frac{l}{2}$$
$$u(0, x) = \frac{2h}{l}(l - x) \qquad \frac{l}{2} < x \leq l$$

考虑第二类边界条件，得

$$u_t(0, x) = 0$$

所以所得定解问题为：

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} & (t > 0, 0 \le l \le x) \\ u(0,x) = \dfrac{2h}{l} x & 0 \le x \le \dfrac{l}{2} \\ u(0,x) = \dfrac{2h}{l}(l-x) & \dfrac{l}{2} < x \le l \\ u(t,0) = 0 \\ u(t,l) = 0 \\ u_t(0,x) = 0 \end{cases}$$

9. 解：

(1) 由

$$u_t = x^2 \Rightarrow u = x^2 t + C(x) \xRightarrow{u(0,x)=x^2} C(x) = x^2$$

所以

$$u = x^2(t+1)$$

(2) 由球对称可知，u的空间分布只与半径r有关，故可得方程为

$$\begin{cases} u_{tt} = a^2 \left(u_{rr} + \dfrac{2}{r} u_r \right) \\ u(0,r) = \varphi(r) \\ u_t(0,r) = \psi(r) \end{cases}$$

根据提升，设

$$v(t,r) = ru$$

则方程可化为

$$\begin{cases} v_{tt} = a^2 v_{rr} \\ v(0,r) = r\varphi(r) \\ v_t(0,r) = r\psi(r) \end{cases}$$

由达朗贝尔公式，得

$$v(t,r) = \frac{(r+at)\varphi(r+at) + (r-at)\varphi(r-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \xi\psi(\xi)d\xi$$

所以

$$u(t,r) = \frac{(r+at)\varphi(r+at) + (r-at)\varphi(r-at)}{2r} + \frac{1}{2ar}\int_{x-at}^{x+at} \xi\psi(\xi)d\xi$$

(3) 根据提示, 直接研究边界条件

$$u = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

$$\xRightarrow{x^2+y^2+z^2=1} u = \frac{1}{\sqrt{1 - 2xx_0 - 2yy_0 - 2zz_0 + {x_0}^2 + {y_0}^2 + {z_0}^2}}$$

$$= \frac{1}{\sqrt{5+4y}}$$

通过对比系数,得到

$$\begin{cases} x_0 = 0 \\ y_0 = -2 \\ z_0 = 0 \end{cases}$$

满足

$$x_0{}^2 + y_0{}^2 + z_0{}^2 > 1$$

故

$$u = \frac{1}{\sqrt{x^2 + (y+2)^2 + z^2}}$$

(4) 设

$$\begin{cases} \xi = x + t \\ \eta = x - t \end{cases}$$

$$\begin{cases} u_t = u_\xi - u_\eta \\ u_{tt} = u_{\xi\xi} + u_{\eta\eta} \end{cases}$$

$$\begin{cases} u_x = u_\xi + u_\eta \\ u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \end{cases}$$

则原方程化为:

$$\begin{cases} u_{\xi\eta} = 0 \\ u(0,\eta) = \varphi(x) \\ u(\xi,0) = \psi(x) \\ \varphi(0) = \psi(0) \end{cases}$$

由

$$u_{\xi\eta} = 0 \Rightarrow u = f(\xi) + g(\eta)$$

$$u(0,\eta) = f(0) + g(\eta) \xRightarrow{\xi=x+t=0} f(0) + g(2x) = \varphi(2x) \Rightarrow g(x) = \varphi\left(\frac{x}{2}\right) - f(0)$$

$$u(\xi,0) = f(\xi) + g(0) \xRightarrow{\eta=x-t=0} f(\xi) + g(0) = \psi(2x) \Rightarrow f(x) = \psi\left(\frac{x}{2}\right) - g(0)$$

另由

$$\varphi(0) = \psi(0)$$

可得

$$\varphi(0) = \psi(0) = f(0) + g(0)$$

所以

$$u = f(\xi) + g(\eta) = \psi\left(\frac{x+t}{2}\right) + \varphi\left(\frac{x-t}{2}\right) - [f(0) + g(0)]$$
$$= \psi\left(\frac{x+t}{2}\right) + \varphi\left(\frac{x-t}{2}\right) - \varphi(0)$$

10. 解：提示有误，应该是设

$$\begin{cases}\xi = x - at \\ \eta = t\end{cases}$$

依题,方程可化为

$$\begin{cases}\dfrac{\partial u}{\partial t} = -a\dfrac{\partial u}{\partial x} + f(t,x)(t > 0, -\infty < x < +\infty) \\ u(0,x) = \varphi(x)(a \neq 0, a\text{为常数})\end{cases}$$

设

$$u(t,x) = v(t,x) + w(t,x)$$

$v(t,x), w(t,x)$分别满足

$$\begin{cases}\dfrac{\partial v}{\partial t} = -a\dfrac{\partial v}{\partial x} \\ v(0,x) = \varphi(x)\end{cases}$$

$$\begin{cases}\dfrac{\partial w}{\partial t} = -a\dfrac{\partial w}{\partial x} + f(t,x) \\ w(0,x) = 0\end{cases}$$

关于$v(t,x)$为齐次方程,根据变量替换有

$$\frac{\partial v}{\partial t} = -a\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}$$

则方程可化为

$$\begin{cases}\dfrac{\partial v}{\partial \eta} = 0 \\ v(0,x) = \varphi(x)\end{cases}$$

则有

$$\begin{cases}v = f(\xi) = f(x - at) \\ v(x) = \varphi(x)\end{cases}$$

所以

$$v(t,x) = \varphi(x - at)$$

关于$w(t,x)$为齐次边界条件方程方程,则利用齐次化原理求解

则设有函数$g(t,x,\tau)$满足

$$\begin{cases}\dfrac{\partial g}{\partial t} = -a\dfrac{\partial g}{\partial x} \\ g(t,x,\tau)|_{t=\tau} = f(\tau,x)\end{cases}$$

根据提示的变量代换,设

$$\begin{cases}\xi = x - at \\ \eta = t \\ \zeta = \tau\end{cases}$$

则得

$$\begin{cases}\dfrac{\partial g}{\partial \eta} = 0 \\ g(t,x,\tau)|_{t=\tau} = f(\tau,x)\end{cases}$$

则有

$$\begin{cases} g = p(\xi, \zeta) = p(x - at, \tau) \\ p(x - at, \tau)|_{t=\tau} = f(\tau, x) \end{cases}$$

$$p(x - at, \tau)|_{t=\tau} = p(x - a\tau, \tau) = f(\tau, x) \xRightarrow{x = x - a(t-\tau)} g(t, x, \tau) = p(x - at, \tau)$$
$$= f[\tau, x - a(t - \tau)]$$

根据齐次化原理

$$w(t, x) = \int_0^t g(t, x, \tau) d\tau = \int_0^t f[\tau, x - a(t - \tau)] d\tau$$

所以原方程通解为

$$u(t, x) = \varphi(x - at) + \int_0^t f[\tau, x - a(t - \tau)] d\tau$$

11. 解:根据达朗贝尔公式

$$u(t, x) = \frac{\varphi(x + at) + \varphi(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi$$

(1) 依题

$$u(t, 0) = \frac{\varphi(at) + \varphi(-at)}{2} + \frac{1}{2a} \int_{-at}^{at} \psi(\xi) d\xi = 0$$

(2) 依题

$$u_x(t, x) = \frac{\varphi'(x + at) + \varphi'(x - at)}{2} + \frac{1}{2a} [\psi(x + at) - \psi(x - at)]$$

$$\varphi'(x) \text{为奇函数}$$

$$u_x(t, 0) = \frac{\varphi'(at) + \varphi'(-at)}{2} + \frac{1}{2a} [\psi(at) - \psi(-at)] = 0$$

12. 解:

作奇延拓,则有

$$\begin{cases} \tilde{u}_{tt} = a^2 \tilde{u}_{xx} (-\infty < x < +\infty, t > 0) \\ \tilde{u}(0, x) = sinx, \tilde{u}_t(0, x) = kx \\ \tilde{u}(t, 0) = 0 \end{cases}$$

则由达朗贝尔公式

$$\tilde{u}(t, x) = \frac{\sin(x + at) + \sin(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} k\xi d\xi = sinxcosat + ktx$$

所以

$$u(t, x) = sinxcosat + ktx (x > 0, t > 0)$$

第二章作业参考解答

刘炜昊

2021 年 4 月 6 日

目录

1 课本 P_{252} T_1

求方程 $y''+\lambda y=0(0<x<l)$ 在下列边界条件下的固有值和固有函数:
(1) $y'(0)=0, y(l)=0$;
(2) $y'(0)=0, y'(l)+hy(l)=0$;
(3) $y'(0)-ky(0)=0, y'(l)+hy(l)=0(k,h>0)$.
Sol:
由 S-L 定理可知：$\lambda\geq 0$，$\lambda=0$ 当且仅当两端均为第 II 类边界条件.（或分类讨论得出 $\lambda\geq 0$ 且仅第 II 类边界条件时 $\lambda=0$，否则为零解）

(1) 对于问题

$$\begin{cases} y''(x)+\lambda y(x)=0 \\ y'(0)=0,\ y(l)=0 \end{cases}$$

可令 $\lambda\overset{\text{def}}{=}k^2>0$，则得到 $y(x)=A\cos kx+B\sin kx, 0\leq x\leq l$.
由 $y'(0)=kB=0$ 得到 $B=0$，则 $y(x)=A\cos kx, 0\leq x\leq l$.
由 $y(l)=A\cos kl=0$，欲求非零解则 $A\neq 0$，故 $kl=\frac{\pi}{2}+n\pi, n=0,1,2,\cdots$
因此 $k_n=\frac{2n+1}{2l}\pi$，则固有值为 $\lambda_n=\left(\frac{2n+1}{2l}\pi\right)^2$，固有函数为 $y_n(x)=\cos\frac{(2n+1)\pi x}{2l}, n\in\mathbb{N}$.

(2) 对于问题

$$\begin{cases} y''(x)+\lambda y(x)=0 \\ y'(0)=0,\ y'(l)+hy(l)=0 \end{cases}$$

可令 $\lambda\overset{\text{def}}{=}k^2>0$，则得到 $y(x)=A\cos kx+B\sin kx, 0\leq x\leq l$.
由 $y'(0)=kB=0$ 得到 $B=0$，则 $y(x)=A\cos kx, 0\leq x\leq l$.
由 $y'(l)+hy(l)=-kA\sin kl+A\cos kl=0\Rightarrow -A(k\sin kl-h\cos kl)=-A\sqrt{k^2+h^2}\sin(kl-\varphi)=0$，其中 $\tan\varphi=\frac{h}{k}$，$\varphi=\arctan\frac{h}{k}$
欲求非零解则 $A\sqrt{k^2+h^2}\neq 0$，故 $k_nl-\arctan\frac{h}{k_n}=n\pi$，固有值为 $\lambda_n=k_n^2$，固有函数为 $y_n(x)=\cos k_nx, n=0,1,2,\cdots$.

(3) 对于问题

$$\begin{cases} y''(x)+\lambda y(x)=0 \\ y'(0)-ky(0)=0,\ y'(l)+hy(l)=0\ (k,h>0) \end{cases}$$

可令 $\lambda\overset{\text{def}}{=}\mu^2>0$，则得到 $y(x)=A\cos\mu x+B\sin\mu x, 0\leq x\leq l$.

根据题目条件可得

$$\begin{cases} y'(0) - ky(0) = \mu B - kA = 0 \\ y'(l) + hy(l) = -\mu A \sin \mu l + \mu B \cos \mu l + hA \cos \mu l + hB \sin \mu l = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} k & -\mu \\ h\cos \mu l - \mu \sin \mu l & h \sin \mu l + \mu \cos \mu l \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

欲使 A 和 B 有非零解，应有系数行列式为 0:

$$\begin{vmatrix} k & -\mu \\ h\cos \mu l - \mu \sin \mu l & h \sin \mu l + \mu \cos \mu l \end{vmatrix} = (kh - \mu^2)\sin \mu l + (k\mu + \mu h)\cos \mu l = 0$$

得到 μ_n 满足:

$$(kh - \mu_n^2)\tan \mu_n l + \mu_n(k+h) = 0, n = 1, 2, \cdots$$

因此固有值为 $\lambda_n = \mu_n^2$，固有函数为 $y_n(x) = \cos \mu_n x + \frac{k}{\mu_n} \sin \mu_n x, n = 1, 2, \cdots$.

2 课本 P_{252} T_2

解下列固有值问题:

(1) $\begin{cases} y'' - 2ay' + \lambda y = 0 \ (0 < x < 1, a \text{ 为常数}), \\ y(0) = y(1) = 0; \end{cases}$

(2) $\begin{cases} (r^2R')' + \lambda r^2 R = 0 \ (0 < r < a), \\ |R(0)| < +\infty, R(a) = 0; \end{cases}$ [提示：令 $y = rR$.]

(3) $\begin{cases} y^{(4)} + \lambda y = 0 \ (0 < x < l), \\ y(0) = y(l) = y''(0) = y''(l) = 0 \end{cases}$

Sol:

(1) 解特征方程：$u^2 - 2au + \lambda = 0$，记 $\Delta \overset{\text{def}}{=} a^2 - \lambda$.

若 $\lambda < a^2$，则 $\Delta > 0, u_{1,2} = a \pm \sqrt{\Delta} \in \mathbb{R}, y = Ae^{u_1 x} + Be^{u_2 x}$，故

$$\begin{cases} y(0) = A + B = 0 \\ y(1) = Ae^{u_1} + Be^{u2} = 0 \end{cases} \Rightarrow A = B = 0.$$

若 $\lambda = a^2$，则 $\Delta = 0, u_{1,2} = a \in \mathbb{R}, y = Ae^{ax} + Bxe^{ax}$，故

$$\begin{cases} y(0) = A = 0 \\ y(1) = Ae^a + Be^a = 0 \end{cases} \Rightarrow A = B = 0.$$

故 $\lambda > a^2$，则 $\Delta < 0, u_{1,2} = a \pm i\sqrt{-\Delta} y = e^{ax}(A\cos\sqrt{\lambda - a^2}x + B\sin\sqrt{\lambda - a^2}x)$，故

$$\begin{cases} y(0) = A = 0 \\ y(1) = e^a(A\cos\sqrt{\lambda - a^2} + B\sin\sqrt{\lambda - a^2}) = 0 \end{cases} \Rightarrow A = 0, \lambda_n - a^2 = (n\pi)^2, n = 1, 2, \cdots.$$

故固有值为 $\lambda_n = (n\pi)^2 + a^2$，固有值函数为 $y_n(x) = e^{ax}\sin n\pi x, n \in \mathbb{N}_+$.

(2) 令 $y(r) = rR(r)$，则 $R(r) = \frac{y(r)}{r}$，那么 $R'(r) = \frac{y'(r)}{r} - \frac{y(r)}{r^2}, R''(r) = \frac{y''(r)}{r} - \frac{2y'(r)}{r^2} + \frac{2y(r)}{r^3}$.
代入泛定方程得：$rR''(r) + 2R'(r) + \lambda rR(r) = y''(r) + \lambda y(r) = 0, 0 < r < a$，即

$$\begin{cases} y''(r) + \lambda y(r) = 0, 0 < r < a \\ y(0) = y(a) = 0 \end{cases}$$

由 Strum-Liouville 定理可知 $\lambda > 0$，可令 $\lambda \overset{\text{def}}{=} k^2 > 0$，则 $y(r) = A\cos kr + B\sin kr$.

由 $y(0) = A = 0$ 可得 $y(r) = B\sin kr$；由 $y(a) = B\sin ka = 0$，欲求得非零解，则 $B \neq 0, k_n a = n\pi, n = 1, 2, \cdots$

因此固有值为 $\lambda_n = \left(\frac{n\pi}{a}\right)^2$，固有函数为 $R_n(r) = \frac{1}{r}\sin\frac{n\pi r}{a}, n \in \mathbb{N}_+$.

(3) 若 $\lambda>0$, 令 $\lambda \overset{\text{def}}{=} \omega^4, (\omega>0)$, 这时特征根为两对共轭复根:

$$k_{1,2}=\omega\left(\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i\right), k_{3,4}=\omega\left(\frac{-\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i\right)$$

因此相应的泛定方程解为

$$y(x)=C_1e^{\frac{\sqrt{2}}{2}\omega x}\cos\frac{\sqrt{2}}{2}\omega x+C_2e^{\frac{\sqrt{2}}{2}\omega x}\sin\frac{\sqrt{2}}{2}\omega x+C_3e^{-\frac{\sqrt{2}}{2}\omega x}\cos\frac{\sqrt{2}}{2}\omega x+C_4e^{\frac{-\sqrt{2}}{2}\omega x}\sin\frac{\sqrt{2}}{2}\omega x$$

代入边界条件可得:$C_1=C_2=C_3=C_4=0$，因此当 $\lambda>0$ 时无相应的固有值.
若 $\lambda=0$，则 $y(x)=C_3x^3+C_2x^2+C_1x+C_0$，则

$$\begin{cases} y(0)=C_0=0 \\ y(l)=l^3C_3+l^2C_2+lC_1+C_0=0 \\ y''(0)=2C_2=0 \\ y''(l)=6lC_3+2C_2=0 \end{cases} \Rightarrow C_1=C_2=C_3=C_4=0\text{（得到零解）}$$

故 $\lambda<0$，令 $\lambda \overset{\text{def}}{=} -\omega^4$, 这样特征方程存在 4 个特征根：$k_{1,2}=\pm\omega,\quad k_{3,4}=\pm i\omega$
因此可得到：$y(x)=C_1e^{\omega x}+C_2e^{-\omega x}+C_3\cos\omega x+C_4\sin\omega x.$
代入边界条件 $y(0)=y(l)=y''(0)=y''(l)=0$, 分别得到:

$$\begin{aligned} C_1+C_2+C_3=0,\ C_1+C_2-C_3=0 \\ C_1e^{\omega l}+C_2e^{-\omega l}+C_3\cos\omega l+C_4\sin\omega l=0 \\ C_1e^{\omega l}+C_2e^{-\omega l}-C_3\cos\omega l-C_4\sin\omega l=0 \end{aligned}$$

解得 $C_1=C_2=C_3=0,\ C_4\sin\omega l=0$
所以欲求非零解，只有 $C_4\neq 0$, 因此 $\sin\omega l=0\ \Rightarrow\ \omega l=n\pi\ \Rightarrow\ \omega=\frac{n\pi}{l},\ n=1,2,\cdots$
因此可得到固有值为 $\lambda_n=-\left(\frac{n\pi}{l}\right)^4$，固有函数为 $y_n(x)=\sin\frac{n\pi x}{l}, n\in\mathbb{N}_+$.

3 课本 P_{252} T_3

一条均匀的弦固定于 $x=0$ 及 $x=l$, 在开始的一瞬间，它的形状是一条以 $\left(\frac{l}{2},h\right)$ 为顶点的抛物线, 初速度为零，且没有外力作用, 求弦作横振动的位移函数.

Sol:

设位移函数为 $u(t,x)$，其满足的方程为

$$\begin{cases} u_{tt}=a^2u_{xx}\ (t>0,0<x<l). \\ u(t,0)=u(t,l)=0 \\ u(0,x)=\dfrac{4h}{l^2}x(l-x),u_t(0,x)=0 \end{cases}$$

令 $u(t,x)=T(t)X(x)$，分离变量得：$\dfrac{1}{a^2}\dfrac{T''(t)}{T(t)}=\dfrac{X''(x)}{X(x)}=-\lambda$，得到固有值问题：

$$\begin{cases} x''+\lambda x=0 \\ X(0)=X(l)=0 \end{cases}$$

由 Strum-Liouvilla 定理可知 $\lambda>0$，令 $\lambda\overset{\text{def}}{=}k^2>0$，得到

$$X(x)=A\cos kx+B\sin kx.$$

由 $X(0)=A=0$, 此时 $X(x)=B\sin kx$；由 $X(l)=B\sin kl=0$, 欲求得非零解则 $B\neq 0$，故 $k_nl=n\pi,n=1,2,\cdots$

因此固有值为 $\lambda_n=k_n^2=\left(\dfrac{n\pi}{l}\right)^2$，固有函数为 $X_n(x)=\sin\dfrac{n\pi x}{l},n\in\mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE：$T''(t)+a^2\lambda_nT(t)=0$，得到 $T_n(t)=A_n\cos\dfrac{n\pi at}{l}+B_n\sin\dfrac{n\pi at}{l}$.

叠加得到：$u(t,x)=\displaystyle\sum_{n=1}^{+\infty}\left(A_n\cos\frac{n\pi at}{l}+B_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$，$t\geq 0,0\leq x\leq l$.

代入初值条件：

$$\begin{cases} u(0,x)=\displaystyle\sum_{n=1}^{+\infty}A_n\sin\frac{n\pi x}{l}=\frac{4h}{l^2}x(l-x)\overset{\text{def}}{=}f(x) \\ u_t(0,x)=\displaystyle\sum_{n=1}^{+\infty}\frac{n\pi a}{l}B_n\sin\frac{n\pi x}{l}=0 \end{cases}\Rightarrow B_n=0(n=1,2,\cdots)$$

以下计算 $A_n=\dfrac{[f(x),X_n(x)]}{||X_n(x)||^2}$，$n=1,2,\cdots$，其中

$$||X_n(x)||^2=\int_0^l\sin^2\frac{n\pi x}{l}\mathrm{d}x=\int_0^l\frac{1}{2}\left(1-\cos\frac{2n\pi x}{l}\right)\mathrm{d}x=\frac{l}{2}$$

$$[f(x),X_n(x)]=\int_0^lf(x)X_n(x)\mathrm{d}x=\int_0^l\frac{4h}{l^2}x(l-x)\sin\frac{n\pi x}{l}\mathrm{d}x=4hl\int_0^1\left(t\sin n\pi t-t^2\sin n\pi t\right)\mathrm{d}t$$

其中

$$\int_0^1 t\sin n\pi t\mathrm{d}t = \frac{-1}{n\pi}\int_0^1 t\mathrm{d}(\cos n\pi t) = -\frac{1}{n\pi}t\cos n\pi t\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi t\mathrm{d}t = -\frac{1}{n\pi}(-1)^n$$

$$\begin{aligned}\int_0^1 t^2\sin n\pi t\mathrm{d}t &= \frac{-1}{n\pi}\int_0^1 t^2\mathrm{d}(\cos n\pi t)\\ &= -\frac{1}{n\pi}t^2\cos n\pi t\Big|_0^1 + \frac{1}{n\pi}\int_0^1 2t\cos n\pi t\mathrm{d}t\\ &= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2}\int_0^1 t\mathrm{d}(\sin n\pi t)\\ &= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2}t\sin n\pi t\Big|_0^1 - \frac{2}{(n\pi)^2}\int_0^1 \sin n\pi t\mathrm{d}t\\ &= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}\cos n\pi t\Big|_0^1\\ &= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}\left[(-1)^n - 1\right]\end{aligned}$$

因此得到

$$A_n = \frac{[f(x), X_n(x)]}{||X_n(x)||^2} = \frac{2}{l}\cdot 4hl\cdot\frac{2}{(n\pi)^3}[1-(-1)^n] = \frac{16h}{(n\pi)^3}[1-(-1)^n] = \begin{cases}\dfrac{32h}{(n\pi)^3}, & n = 2k+1\\ 0, & n = 2k\ (k\in\mathbb{N})\end{cases}$$

故 $u(t,x) = \dfrac{32h}{\pi^3}\displaystyle\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^3}\cos\frac{(2k+1)\pi at}{l}\sin\frac{(2k+1)\pi x}{l},\ t>0, 0<x<l.$

4　课本 P_{252} T_4

利用圆内狄氏问题的一般解式, 解边值问题 $\begin{cases} \Delta_2 u = 0 \ (r < a) \\ u|_{r=a} = f \end{cases}$，其中 f 分别为

(1) $f = A$ (常数);

(2) $f = A\cos\theta$;

(3) $f = Axy$;

(4) $f = \cos\theta\sin 2\theta$;

(5) $f = A\sin^2\theta + B\cos^2\theta$.

Sol:

可知在圆内该泛定方程的通解为 $u(r,\theta) = C_0 + \sum\limits_{k=1}^{+\infty} r^k(C_k\cos k\theta + D_k\sin k\theta), r < a$.

(1) 对于 $u|_{r=a} = f = A$，可知 $u(a,\theta) = C_0 + \sum\limits_{k=1}^{+\infty} r^k(C_k\cos k\theta + D_k\sin k\theta) = A$

因而 $C_k = D_k = 0\ (k\in\mathbb{N}_+), C_0 = A$，故 $u(r,\theta) = A, r < a$.

(2) 对于 $u|_{r=a} = f = A\cos\theta$，可知 $u(a,\theta) = C_0 + \sum\limits_{k=1}^{+\infty} r^k(C_k\cos k\theta + D_k\sin k\theta) = A\cos\theta$

因而 $C_1 = \dfrac{A}{a}, C_k = 0\ (k = 0,2,3,4,\cdots), D_k = 0\ (k\in\mathbb{N}_+)$，故 $u(r,\theta) = \dfrac{A}{a}r\cos\theta, r < a$.

(3) 对于 $u|_{r=a} = f = Axy = Ar^2\sin\theta\cos\theta = \dfrac{A}{2}a^2\sin 2\theta$

可知 $u(a,\theta) = C_0 + \sum\limits_{k=1}^{+\infty} r^k(C_k\cos k\theta + D_k\sin k\theta) = \dfrac{A}{2}a^2\sin 2\theta$

因而 $D_2 = \dfrac{A}{2}, D_k = 0\ (k = 0,1,3,4,5,\cdots), C_k = 0\ (k\in\mathbb{N})$，故 $u(r,\theta) = \dfrac{A}{2}r^2\cos\theta, r < a$.

(4) 对于 $u|_{r=a} = f = \cos\theta\sin 2\theta = \dfrac{1}{2}(\sin 3\theta + \sin\theta)$

可知 $u(a,\theta) = C_0 + \sum\limits_{k=1}^{+\infty} r^k(C_k\cos k\theta + D_k\sin k\theta) = \dfrac{1}{2}(\sin 3\theta + \sin\theta)$

因而 $D_1 = \dfrac{1}{2a}, D_3 = \dfrac{1}{2a^3}$，其余为 0，故 $u(r,\theta) = \dfrac{1}{2}\left[\dfrac{r}{a}\sin\theta + \left(\dfrac{r}{a}\right)^3\sin 3\theta\right], r < a$.

(5) 对于 $u|_{r=a} = f = A\sin^2\theta + B\cos^2\theta = \dfrac{A+B}{2} + \dfrac{B-A}{2}\cos 2\theta$

可知 $u(a,\theta) = C_0 + \sum\limits_{k=1}^{+\infty} r^k(C_k\cos k\theta + D_k\sin k\theta) = \dfrac{A+B}{2} + \dfrac{B-A}{2}\cos 2\theta$

因而 $C_0 = \dfrac{A+B}{2}, C_2 = \dfrac{B-A}{2a^2}$，其余为 0，故 $u(r,\theta) = \dfrac{A+B}{2} + \dfrac{B-A}{2}\dfrac{r^2}{a^2}\cos 2\theta, r < a$.

5　课本 P_{252} T_5

解下列定解问题：

(1) $\begin{cases} u_{tt}=a^2u_{xx}(0<x<l,t>0), \\ u(t,0)=u_x(t,l)=0, \\ u(0,x)=0,u_t(0,x)=x; \end{cases}$

(2) $\begin{cases} u_t=a^2u_{xx}(0<x<l,t>0), \\ u(t,0)=u(t,l)=0, \\ u(0,x)=x(l-x); \end{cases}$

(3) $\begin{cases} u_{tt}=a^2u_{xx}-2hu_t\ \left(0<x<l,t>0,0<h<\frac{\pi a}{l},h\text{ 为常数}\right), \\ u(t,0)=u(t,l)=0, \\ u(0,x)=\varphi(x),u_t(0,x)=\psi(x); \end{cases}$

(4) $\begin{cases} u_{tt}=a^2u_{xx}(0<x<l,t>0), \\ u_x(t,0)=0,u_x(t,l)+hu(t,l)=0(h>0,h\text{ 为常数}), \\ u(0,x)=\varphi(x),u_t(0,x)=\psi(x); \end{cases}$

(5) $\begin{cases} \Delta_2u=0(r<a), \\ u_r(a,\theta)-hu(a,\theta)=f(\theta)(h>0), \end{cases}$ 特别地, 计算 $f(\theta)=\cos^2\theta$ 时 u 的值;

(6) 环域内的狄氏问题

$$\begin{cases} \Delta_2u=0\ (a<r<b) \\ u(a,\theta)=1,u(b,\theta)=0 \end{cases}$$

(7) 扇形域内的狄氏问题

$$\begin{cases} \Delta_2u=0\ (r<a,0<\theta<\alpha), \\ u(r,0)=u(r,\alpha)=0 \\ u(a,\theta)=f(\theta). \end{cases}$$

Sol:

(1) 令 $u(t,x)=T(t)X(x)$，分离变量得：$\dfrac{1}{a^2}\dfrac{T''(t)}{T(t)}=\dfrac{X''(x)}{X(x)}=-\lambda$，得到固有值问题：

$$\begin{cases} x''+\lambda x=0 \\ X(0)=X'(l)=0 \end{cases}$$

由 Strum-Liouvilla 定理可知 $\lambda>0$，令 $\lambda\overset{\text{def}}{=}k^2>0$，得到 $X(x)=A\cos kx+B\sin kx$.

由 $X(0)=A=0$，此时 $X(x)=B\sin kx$；由 $X'(l)=kB\cos kl=0$，欲求得非零解则 $kB\neq 0$，故 $k_nl=\dfrac{2n+1}{2}\pi,n=0,1,2,\cdots$

因此固有值为 $\lambda_n=k_n^2=\left(\dfrac{2n+1}{2l}\pi\right)^2$，固有函数为 $X_n(x)=\sin\dfrac{2n+1}{2l}\pi x,n\in\mathbb{N}$.

随后解关于 $T(t)$ 的 ODE：$T''(t)+a^2\lambda_nT(t)=0$，得到

$$T_n(t)=A_n\cos\frac{2n+1}{2l}\pi at+B_n\sin\frac{2n+1}{2l}\pi at.$$

叠加得到：$u(t,x)=\sum\limits_{n=0}^{+\infty}\left(A_n\cos\dfrac{2n+1}{2l}\pi at+B_n\sin\dfrac{2n+1}{2l}\pi at\right)\sin\dfrac{2n+1}{2l}\pi x.$

代入初值条件：

$$\begin{cases}u(0,x)=\sum\limits_{n=0}^{+\infty}A_n\sin\dfrac{2n+1}{2l}\pi x=0\\ u_t(0,x)=\sum\limits_{n=0}^{+\infty}\dfrac{(2n+1)\pi a}{2l}B_n\sin\dfrac{2n+1}{2l}\pi x=x\end{cases}\Rightarrow A_n=0(n=0,1,2,\cdots)$$

以下计算 $B_n=\dfrac{2l}{(2n+1)\pi a}\dfrac{[x,X_n(x)]}{||X_n(x)||^2}(n=0,1,2,\cdots)$，其中

$$||X_n(x)||^2=\int_0^l\sin^2\frac{2n+1}{2l}\pi x\mathrm{d}x=\int_0^l\frac{1}{2}\left(1-\cos\frac{2n+1}{l}\pi x\right)\mathrm{d}x=\frac{l}{2}$$

$$\begin{aligned}[x,X_n(x)]&=\int_0^l x\sin\frac{2n+1}{2l}\pi x\mathrm{d}x=\frac{-2l}{(2n+1)\pi}\int_0^l x\mathrm{d}(\cos\frac{2n+1}{2l}\pi x)\\&=\frac{-2l}{(2n+1)\pi}x\cos\frac{2n+1}{2l}\pi x\Big|_0^l+\frac{2l}{(2n+1)\pi}\int_0^l\cos\frac{2n+1}{2l}\pi x\mathrm{d}x\\&=\frac{4l^2}{(2n+1)^2\pi^2}\sin\frac{2n+1}{2l}\pi x\Big|_0^l\\&=\frac{4l^2}{(2n+1)^2\pi^2}\cdot(-1)^n\end{aligned}$$

因此得到

$$B_n=\frac{2l}{(2n+1)\pi a}\frac{[x,X_n(x)]}{||X_n(x)||^2}=\frac{2l}{(2n+1)\pi a}\cdot\frac{2}{l}\cdot\frac{4l^2}{(2n+1)^2\pi^2}\cdot(-1)^n=\frac{16l^2}{(2n+1)^3\pi^3a}(-1)^n$$

故 $u(t,x)=\dfrac{16l^2}{\pi^3a}\sum\limits_{n=0}^{+\infty}\dfrac{(-1)^n}{(2n+1)^3}\cdot\sin\dfrac{(2n+1)}{2l}\pi at\cdot\sin\dfrac{2n+1}{2l}\pi x,\ t>0,0<x<l.$

(2) 令 $u(t,x)=T(t)X(x)$，分离变量得：$\dfrac{1}{a^2}\dfrac{T'(t)}{T(t)}=\dfrac{X''(x)}{X(x)}=-\lambda$，得到固有值问题：

$$\begin{cases}x''+\lambda x=0\\ X(0)=X(l)=0\end{cases}$$

由 Strum-Liouvilla 定理可知 $\lambda>0$，令 $\lambda\overset{\text{def}}{=}k^2>0$，得到 $X(x)=A\cos kx+B\sin kx$.

由 $X(0)=A=0$，此时 $X(x)=B\sin kx$；由 $X(l)=B\sin kl=0$，欲求得非零解则 $B\neq0$，故 $k_nl=n\pi,n=1,2,\cdots$

因此固有值为 $\lambda_n=k_n^2=\left(\dfrac{n\pi}{l}\right)^2$，固有函数为 $X_n(x)=\sin\dfrac{n\pi x}{l},n\in\mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE：$T'(t)+a^2\lambda_nT(t)=0$，得到 $T_n(t)=e^{-a^2\lambda_nt}=e^{-(\frac{n\pi a}{l})^2t}$.

叠加得到：$u(t,x)=\sum\limits_{n=1}^{+\infty}C_ne^{-(\frac{n\pi a}{l})^2t}\sin\dfrac{n\pi x}{l}.$

代入初值条件：$u(0,x)=\sum\limits_{n=1}^{+\infty}C_n\sin\dfrac{n\pi x}{l}=x(l-x)$.

以下计算 $C_n=\dfrac{[x(l-x),X_n(x)]}{||X_n(x)||^2}(n=0,1,2,\cdots)$，其中

$$||X_n(x)||^2=\int_0^l\sin^2\frac{n\pi x}{l}\mathrm{d}x=\int_0^l\frac{1}{2}\left(1-\cos\frac{2n\pi x}{l}\right)\mathrm{d}x=\frac{l}{2}$$

$$[x(l-x),X_n(x)]=\int_0^l x(l-x)\sin\frac{n\pi x}{l}\mathrm{d}x=\int_0^1\left(t\sin n\pi t-t^2\sin n\pi t\right)\mathrm{d}t$$

其中

$$\int_0^1 t\sin n\pi t\mathrm{d}t=\frac{-1}{n\pi}\int_0^1 t\mathrm{d}(\cos n\pi t)=-\frac{1}{n\pi}t\cos n\pi t\Big|_0^1+\frac{1}{n\pi}\int_0^1\cos n\pi t\mathrm{d}t=-\frac{1}{n\pi}(-1)^n$$

$$\begin{aligned}\int_0^1 t^2\sin n\pi t\mathrm{d}t&=\frac{-1}{n\pi}\int_0^1 t^2\mathrm{d}(\cos n\pi t)\\&=-\frac{1}{n\pi}t^2\cos n\pi t\Big|_0^1+\frac{1}{n\pi}\int_0^1 2t\cos n\pi t\mathrm{d}t\\&=-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^2}\int_0^1 t\mathrm{d}(\sin n\pi t)\\&=-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^2}t\sin n\pi t\Big|_0^1-\frac{2}{(n\pi)^2}\int_0^1\sin n\pi t\mathrm{d}t\\&=-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^3}\cos n\pi t\Big|_0^1\\&=-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^3}\left[(-1)^n-1\right]\end{aligned}$$

因此得到

$$C_n=\frac{[x(l-x),X_n(x)]}{||X_n(x)||^2}=\frac{2}{l}\cdot\frac{2l^3}{n^3\pi^3}\cdot[1-(-1)^n]=\frac{4l^2}{n^3\pi^3}\cdot[1-(-1)^n]=\begin{cases}\dfrac{8l^2}{n^3\pi^3}&,n=2k+1\\0&,n=2k\ (k\in\mathbb{N})\end{cases}$$

故 $u(t,x)=\dfrac{8l^2}{\pi^3}\sum\limits_{k=0}^{+\infty}\dfrac{1}{(2k+1)^3}\cdot e^{-(\frac{2k+1}{l}\pi a)^2t}\cdot\sin\dfrac{2k+1}{l}\pi x,\ t>0,0<x<l$.

(3) 令 $u(t,x)=T(t)X(x)$，分离变量得：$\dfrac{1}{a^2}\dfrac{T''(t)}{T(t)}+\dfrac{2h}{a^2}\dfrac{T'(t)}{T(t)}=\dfrac{X''(x)}{X(x)}=-\lambda$，得到固有值问题：

$$\begin{cases}x''+\lambda x=0\\X(0)=X(l)=0\end{cases}$$

由 Strum-Liouvilla 定理可知 $\lambda>0$，令 $\lambda\overset{\text{def}}{=}k^2>0$，得到 $X(x)=A\cos kx+B\sin kx$.

由 $X(0)=A=0$，此时 $X(x)=B\sin kx$；由 $X(l)=B\sin kl=0$，欲求得非零解则 $B\neq 0$，故 $k_nl=n\pi,n=1,2,\cdots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l}\right)^2$，固有函数为 $X_n(x) = \sin\frac{n\pi x}{l}, n \in \mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE：$T''(t) + 2hT'(t) + a^2\lambda_n T(t) = 0$，其中 $0 < h < \frac{\pi a}{l}$，其特征方程为 $k^2 + 2hk + (\frac{n\pi a}{l})^2 = 0$，得到特征根 $k_{1,2} = -h + i\sqrt{(\frac{n\pi a}{l})^2 - h^2}$，可记 $\omega_n = \sqrt{(\frac{n\pi a}{l})^2 - h^2}$，由此得到 $T_n(t) = e^{-ht}(A_n\cos\omega t + B_n\sin\omega t)$.

叠加得到：$u(t,x) = \sum\limits_{n=1}^{+\infty} e^{-ht}(A_n\cos\omega t + B_n\sin\omega t)\sin\frac{n\pi x}{l}$.

代入初值条件：$u(0,x) = \sum\limits_{n=1}^{+\infty} A_n\sin\frac{n\pi x}{l} = \varphi(x)$，得到系数为

$$A_n = \frac{[\varphi(x), X_n(x)]}{||X_n(x)||^2} = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

又 $u_t(0,x) = \sum\limits_{n=1}^{+\infty}(\omega_n B_n - hA_n)\sin\frac{n\pi x}{l} = \psi(x)$，得到

$$\omega_n B_n - hA_n = \frac{[\psi(x), X_n(x)]}{||X_n(x)||^2} = \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

因此 $B_n = \frac{hA_n}{\omega_n} + \frac{2}{\omega_n l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\mathrm{d}x$

故 $u(t,x) = \sum\limits_{n=1}^{+\infty} e^{-ht}(A_n\cos\omega t + B_n\sin\omega t)\sin\frac{n\pi x}{l}$，$t > 0, 0 < x < l$，其中

$$\omega_n = \sqrt{(\frac{n\pi a}{l})^2 - h^2},\ A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\mathrm{d}x,\ B_n = \frac{hA_n}{\omega_n} + \frac{2}{\omega_n l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\mathrm{d}x.$$

(4) 令 $u(t,x) = T(t)X(x)$，分离变量得：$\frac{1}{a^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$，得到固有值问题：

$$\begin{cases} x'' + \lambda x = 0 \\ X'(0) = 0,\ X'(l) + hX(l) = 0 \end{cases}$$

由 Strum-Liouvilla 定理可知 $\lambda > 0$，令 $\lambda \overset{\text{def}}{=} k^2 > 0$，得到 $X(x) = A\cos kx + B\sin kx$.

由 $X'(0) = kB = 0 \Rightarrow B = 0$，此时 $X(x) = A\cos kx$；

又由 $X'(l) + hX(l) = -kA\sin kl + hA\cos kl = 0 \Rightarrow A\sqrt{h^2+k^2}\sin(\arctan\frac{h}{k} - kl)$，欲求得非零解则 $A\sqrt{h^2+k^2} \neq 0$，故 $k_n l - \arctan\frac{h}{k_n} = n\pi \Rightarrow \tan k_n l = \frac{h}{k_n}, n = 1, 2, \cdots$

因此固有值为 $\lambda_n = k_n^2$，k_n 满足 $k_n\tan k_n l = h$，固有函数为 $X_n(x) = \cos k_n x, n \in \mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE：$T''(t) + a^2\lambda_n T(t) = 0$，得到 $T_n(t) = A_n\cos k_n at + B_n\sin k_n at$.

叠加得到：$u(t,x) = \sum\limits_{n=1}^{+\infty}(A_n\cos k_n at + B_n\sin k_n at)\cos k_n x$.

代入初值条件：

$$\begin{cases} u(0,x) = \sum\limits_{n=1}^{+\infty} A_n \cos k_n x = \varphi(x) \\ u_t(0,x) = \sum\limits_{n=1}^{+\infty} k_n a B_n \cos k_n x = \psi(x) \end{cases}$$

那么

$$A_n = \frac{[\varphi(x), X_n(x)]}{||X_n(x)||^2} = \frac{1}{||\cos k_n x||^2}\int_0^l \varphi(x)\cos k_n x \mathrm{d}x, n \in \mathbb{N}_+$$

$$B_n = \frac{1}{k_n a}\frac{[\psi(x), X_n(x)]}{||X_n(x)||^2} = \frac{1}{k_n a||\cos k_n x||^2}\int_0^l \psi(x)\cos k_n x \mathrm{d}x, n \in \mathbb{N}_+$$

其中

$$\begin{aligned} ||X_n(x)||^2 &= \int_0^l \cos^2 k_n x \mathrm{d}x = \int_0^l \frac{1}{2}\left(1+\cos 2k_n x\right)\mathrm{d}x = \frac{l}{2} + \frac{1}{4k_n}\sin 2k_n l \\ &= \frac{l}{2} + \frac{1}{2k_n}\cdot\frac{\sin k_n l \cos k_n l}{\sin^2 k_n l + \cos^2 k_n l} = \frac{l}{2} + \frac{1}{2k_n}\cdot\frac{\tan k_n l}{\tan^2 k_n l + 1} \\ &= \frac{l}{2} + \frac{1}{2k_n}\cdot\frac{hk_n}{h^2+k_n^2} = \frac{l}{2} + \frac{h}{2(h^2+k_n^2)}. \end{aligned}$$

故 $u(t,x) = \sum\limits_{n=1}^{+\infty}(a_n \cos k_n at + b_n \sin k_n at)\cos k_n x, t > 0, 0 < x < l$,其中,$\lambda_n = k_n^2$ $(n = 1,2,\cdots), k_n$ 是 $k_n \tan k_n l = h$ 的正实根，且

$$A_n = \frac{1}{\|X_n\|^2}\int_0^l \varphi(x)\cos k_n x \mathrm{d}x,\ B_n = \frac{1}{k_n a\|X_n\|^2}\int_0^l \psi(x)\cos k_n x \mathrm{d}x,\ \|X_n\|^2 = \frac{l}{2} + \frac{h}{2\left(k_n^2+h^2\right)}.$$

(5) 可知在圆内该泛定方程的通解为 $u(r,\theta) = C_0 + \sum\limits_{k=1}^{+\infty} r^k(C_k \cos k\theta + D_k \sin k\theta), r < a$.

代入边界条件：

$$\begin{aligned} f(\theta) =& u_r(a,\theta) - hu(a,\theta) \\ =& \sum_{k=1}^{+\infty} ka^{k-1}\left(C_k \cos k\theta + D_k \sin k\theta\right) - \left[hC_0 + h\sum_{k=1}^{+\infty} a^k\left(C_k \cos k\theta + D_k \sin k\theta\right)\right] \\ =& -hC_0 + \sum_{k=1}^{+\infty}(k-ha)a^{k-1}\left(C_k \cos k\theta + D_k \sin k\theta\right) \end{aligned}$$

其中

$$||\cos k\theta||^2 = \int_0^{2\pi}\cos^2 k\theta \mathrm{d}\theta = \int_0^{2\pi}\frac{1+\cos 2k\theta}{2}\mathrm{d}\theta = \pi,\ k \in \mathbb{N}_+;$$

$$||\sin k\theta||^2 = \int_0^{2\pi}\sin^2 k\theta \mathrm{d}\theta = \int_0^{2\pi}\frac{1-\cos 2k\theta}{2}\mathrm{d}\theta = \pi,\ k \in \mathbb{N}_+.$$

$$\Rightarrow \begin{cases} C_0 = -\dfrac{1}{2\pi h}\displaystyle\int_0^{2\pi} f(\theta)\mathrm{d}\theta \\ C_k = \dfrac{1}{(ka^{k-1}-ha^k)\,\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos k\theta\mathrm{d}\theta,\ k\in\mathbb{N}_+ \\ D_k = \dfrac{1}{(ka^{k-1}-ha^k)\,\pi}\displaystyle\int_0^{2\pi} f(\theta)\sin k\theta\mathrm{d}\theta,\ k\in\mathbb{N}_+ \end{cases}$$

由此得知通解 $u(r,\theta)=C_0+\sum\limits_{k=1}^{+\infty} r^k(C_k\cos k\theta+D_k\sin k\theta), r<a$，系数如上所示.

特别地，当 $f(\theta)=\cos^2\theta=\dfrac{1}{2}+\dfrac{1}{2}\cos 2\theta$ 时，有

$$\frac{1}{2}+\frac{1}{2}\cos 2\theta=-hC_0+\sum_{k=1}^{\infty}(k-ha)a^{k-1}\left(C_k\cos k\theta+D_k\sin k\theta\right)$$

得到：$C_0=-\dfrac{1}{2h}, C_2=\dfrac{1}{2a\,(2-ha)}$，其余为 0. 因而，$u(r,\theta)=-\dfrac{1}{2h}+\dfrac{r^2\cos 2\theta}{2a\,(2-ha)}$，$r<a$.

(6) 可知在环中该泛定方程的通解为 $u(r,\theta)=C_0+D_0\ln r+\sum\limits_{k=1}^{+\infty}(C_kr^k+D_kr^{-k})(A_k\cos k\theta+B_k\sin k\theta)$，$a<r<b$.

代入边界条件，得：

$$\begin{cases} u(a,\theta)=C_0+D_0\ln a+\displaystyle\sum_{k=1}^{+\infty}\left(C_ka^k+D_ka^{-k}\right)\left(A_k\cos k\theta+B_k\sin k\theta\right)=1 \\ u(b,\theta)=c_0+D_0\ln b+\displaystyle\sum_{k=1}^{\infty}\left(C_kb^k+D_kb^{-k}\right)\left(A_k\cos k\theta+B_k\sin k\theta\right)=0 \end{cases}$$

$$\Rightarrow \begin{cases} C_k=D_k=0, k>0 \\ C_0+D_0\ln a=1 \\ C_0+D_0\ln b=0 \end{cases} \Rightarrow \begin{cases} C_0=\dfrac{\ln b}{\ln b-\ln a} \\ D_0=\dfrac{-1}{\ln b-\ln a} \end{cases}$$

因此得到：$u(r,\theta)=\dfrac{\ln b-\ln r}{\ln b-\ln a}$，$a<r<b$.

(7) 分离变量，令 $u(r,\theta)=R(r)\Theta(\theta)$，得到 $r^2\dfrac{R''}{R}+r\dfrac{R'}{R}=\dfrac{\Theta''}{\Theta}\overset{\text{def}}{=}-\lambda$

由此得到固有值问题：

$$\begin{cases} \Theta^{\theta\theta}(\theta)+\lambda\Theta(\theta)=0 \\ \Theta(0)=\Theta(\alpha)=0 \end{cases}$$

由 Strum-Liouvilla 定理可知 $\lambda>0$，可设 $\lambda\overset{\text{def}}{=}k^2>0$，则 $\Theta(\theta)=A\cos k\theta+B\sin k\theta$，$0<\theta<\alpha$.

由 $\Theta(0)=A=0$, 得到 $\Theta(\theta)=B\sin k\theta$; 由 $\Theta(\alpha)=B\sin k\alpha=0$, 欲求非零解则 $B\neq 0$, 那么 $k_n=\dfrac{n\pi}{\alpha}$, $n=1,2,\cdots$, 得到固有值为 $\lambda_n=\left(\dfrac{n\pi}{\alpha}\right)^2$, 固有值函数为 $\Theta_n(\theta)=\sin\dfrac{n\pi\theta}{\alpha}$, $n\in\mathbb{N}_+$.

另外，关于径向部分得到 Euler 方程：$r^2R''(r)+rR'(r)+\left(\dfrac{n\pi}{\alpha}\right)^2R(r)=0$，解得 $R_n(r)=C_nr^{\frac{n\pi}{\alpha}}+D_nr^{-\frac{n\pi}{\alpha}}$.

由此得到通解为 $u(r,\theta)=\sum\limits_{n=1}^{+\infty}(C_nr^{\frac{n\pi}{\alpha}}+D_nr^{-\frac{n\pi}{\alpha}})\sin\dfrac{n\pi\theta}{\alpha}$.

代入边界条件得：

$$\begin{cases}|u(0,\theta)|<+\infty\ \Rightarrow D_n=0,n\in\mathbb{N}_+\\ u(a,\theta)=\displaystyle\sum_{n=1}^{+\infty}C_na^{\frac{n\pi}{\alpha}}\sin\frac{n\pi\theta}{\alpha}=f(\theta)\end{cases}$$

因此系数 $C_n=\dfrac{1}{a^{\frac{n\pi}{\alpha}}}\cdot\dfrac{[f(\theta),\Theta_n(\theta)]}{||\Theta_n(\theta)||^2}, n\in\mathbb{N}_+$，其中

$$||\Theta_n(\theta)||^2=\int_0^\alpha\sin^2\frac{n\pi\theta}{\alpha}\mathrm{d}\theta=\int_0^\alpha\frac{1-\cos\frac{2n\pi\theta}{\alpha}}{2}\mathrm{d}\theta=\frac{\alpha}{2}$$

$$[f(\theta),\Theta_n(\theta)]=\int_0^\alpha f(\theta)\sin\frac{n\pi\theta}{\alpha}\mathrm{d}\theta$$

因此 $C_n=\dfrac{2}{\alpha a^{\frac{n\pi}{\alpha}}}\int_0^\alpha f(\theta)\sin\dfrac{n\pi\theta}{\alpha}\mathrm{d}\theta, n\in\mathbb{N}_+$，故得到解：

$$u(r,\theta)=\frac{2}{\alpha}\sum_{n=1}^{+\infty}\left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}}\int_0^\alpha f(\xi)\sin\frac{n\pi\xi}{\alpha}\mathrm{d}\xi\sin\frac{n\pi\theta}{\alpha},\ r<a,0<\theta<\alpha.$$

6 课本 P_{253} T_6

长为 $2l$ 的均匀杆，两端与侧面均绝热，若初始温度为

$$\varphi(x)=\begin{cases}\dfrac{1}{2A} & (|x-l|<A<l)\\ 0 & (\text{其余的}x),\end{cases}$$

求 $u(x,t)$ 及 $t\to+\infty$ 时的情况. 又当 $A\to 0$ 时，解的极限如何?

Sol:

列出边界问题如下：

$$\begin{cases}u_t=a^2u_{xx},\ t>0,0<x<2l\\ u_x(t,0)=u_x(t,2l)=0\\ u(0,x)=\varphi(x)=\begin{cases}\frac{1}{2A} & ,|x-l|<A<l\\ 0 & ,\text{其余的}x\end{cases}\end{cases}$$

令 $u(t,x)=T(t)X(x)$，分离变量得：$\dfrac{1}{a^2}\dfrac{T'(t)}{T(t)}=\dfrac{X''(x)}{X(x)}=-\lambda$，得到固有值问题：

$$\begin{cases}x''+\lambda x=0\\ X'(0)=X'(2l)=0\end{cases}$$

由 Strum-Liouvilla 定理可知 $\lambda\geq 0$，当 $\lambda=0$ 时得到固有函数可取 $X_0(x)=1$；当 $\lambda>0$ 时，令 $\lambda\overset{\text{def}}{=}k^2>0$，得到 $X(x)=A\cos kx+B\sin kx$，$k\neq 0$.

由 $X'(0)=kB=0$，此时 $X(x)=A\sin kx$；由 $X'(2l)=-kA\sin 2kl=0$，欲求得非零解则 $kB\neq 0$，故 $2k_nl=n\pi,n=1,2,\cdots$

因此固有值为 $\lambda_n=k_n^2=\left(\dfrac{n\pi}{2l}\right)^2$，固有函数为 $X_n(x)=\cos\dfrac{n\pi x}{2l},n\in\mathbb{N}$.

随后解关于 $T(t)$ 的 ODE：$T'(t)+a^2\lambda_nT(t)=0$，得到 $T_n(t)=C_ne^{-(\frac{n\pi a}{2l})^2t},n\in\mathbb{N}$.

叠加得到：$u(t,x)=\sum\limits_{n=0}^{+\infty}C_ne^{-(\frac{n\pi a}{2l})^2t}\cos\dfrac{n\pi x}{l}$.

代入初值条件：$u(0,x)=\sum\limits_{n=0}^{+\infty}C_n\cos\dfrac{n\pi x}{2l}=\varphi(x)$.

那么

$$\begin{aligned}C_0&=\frac{\int_0^{2l}\varphi(x)\mathrm{d}x}{\int_0^{2l}1\cdot\mathrm{d}x}=\frac{1}{2l}\cdot\frac{2A}{2A}=\frac{1}{2l};\\ C_n&=\frac{\int_0^{2l}\cos\dfrac{n\pi x}{2l}\varphi(x)\mathrm{d}x}{\int_0^{2l}\cos^2\dfrac{n\pi x}{2l}\mathrm{d}x}=\frac{1}{l}\int_{l-A}^{l+A}\frac{1}{2A}\cos\frac{n\pi x}{2l}\mathrm{d}x\\ &=\frac{l}{n\pi A}\left[\sin\frac{n\pi(l+A)}{2l}-\sin\frac{n\pi(l-A)}{2l}\right]=\frac{2}{An\pi}\sin\frac{n\pi A}{2l}\cos\frac{n\pi}{2},n\in\mathbb{N}_+\end{aligned}$$

故

$$
\begin{aligned}
u(t,x) &= \frac{1}{2l}+\frac{1}{\pi A}\sum_{n=1}^{+\infty}\frac{2}{n}\sin\frac{n\pi A}{2l}\cos n\pi 2\cdot e^{-(\frac{n\pi a}{2l})^2 t}\cdot\cos\frac{n\pi x}{2l}\\
&= \frac{1}{2l}+\frac{1}{\pi A}\sum_{k=1}^{+\infty}\frac{1}{k}\sin\frac{k\pi A}{l}\cos k\pi\cdot e^{-(\frac{k\pi a}{l})^2 t}\cdot\cos\frac{k\pi x}{l}\\
&= \frac{1}{2l}+\frac{1}{\pi A}\sum_{k=1}^{+\infty}\frac{(-1)^k}{k}\sin\frac{k\pi A}{l}\cdot e^{-(\frac{k\pi a}{l})^2 t}\cdot\cos\frac{k\pi x}{l},\ t>0, 0<x<2l.
\end{aligned}
$$

对于 $A\to 0$ 及 $t\to+\infty$，由于 $u(t,x)\in C^2$，由 Dirichlet 收敛定理可知其 Fourier 级数一致收敛，那么

$$
\begin{aligned}
\lim_{t\to+\infty}u(t,x) &= \frac{1}{2l}+\frac{1}{\pi A}\sum_{k=1}^{+\infty}\frac{(-1)^k}{k}\sin\frac{k\pi A}{l}\cdot\left[\lim_{t\to+\infty}e^{-(\frac{k\pi a}{l})^2 t}\right]\cdot\cos\frac{k\pi x}{l}\\
&= \frac{1}{2l}\\
\lim_{A\to 0}u(t,x) &= \frac{1}{2l}+\sum_{k=1}^{+\infty}\left[\lim_{A\to 0}\frac{1}{\pi A}\sin\frac{k\pi A}{l}\right]\cdot\frac{(-1)^k}{k}\cdot e^{-(\frac{k\pi a}{l})^2 t}\cdot\cos\frac{k\pi x}{l}\\
&= \frac{1}{2l}+\frac{1}{l}\sum_{k=1}^{+\infty}(-1)^k e^{-(\frac{k\pi a}{l})^2 t}\cdot\cos\frac{k\pi x}{l}.
\end{aligned}
$$

7　课本 P_{253} T_7

解下列定解问题:

(1) $\begin{cases} u_t = a^2\Delta_3 u \\ u|_{r=R} = 0, u(t,0)\text{有限}, \\ u|_{t=0} = f(r); \end{cases}$

[提示：采用球坐标系，由定解条件可知 $u = u(t,r)$.]

(2) $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^4 u}{\partial x^4}(t>0, 0<x<l) \\ u(0,x) = x(l-x), u_t(0,x) = 0, \\ u(t,0) = u(t,l) = 0, \\ u_{xx}(t,0) = u_{xx}(t,l) = 0. \end{cases}$

Sol:

(1) 在球坐标系下沿径向展开：$\dfrac{\partial u}{\partial t} = \dfrac{1}{r^2}\dfrac{\partial}{\partial}\left(r^2\dfrac{\partial u}{\partial r}\right)$

分离变量，令 $u(t,r) = T(t)R(r)$，得到 $\dfrac{T'(t)}{T(t)} = \dfrac{1}{r^2}\dfrac{1}{R}(r^2R')' \overset{\text{def}}{=} -\lambda$.

解以下固有值问题:

$$\begin{cases} (r^2R')' + \lambda r^2 R = 0 \\ |R(0)| < +\infty,\ R(r) = 0 \end{cases}$$

由 Strum-Liouville 定理可知 $\lambda > 0$，令 $R(r) = \dfrac{v(r)}{r}$，则 $R'(r) = \dfrac{v'(r)}{r} - \dfrac{v(r)}{r^2}, r^2R'(r) = rv'(r) - v(r), (r^2R'(r))' = r^2v''(r)$.

因此，泛定方程化为 $v''(r) + \lambda v(r)$，令 $\lambda = k^2 > 0, k \neq 0$，则 $v(r) = A\cos kr + B\sin kr$.

由 $v(0) = rR(r)|_{r=0} = A = 0$，得到 $v(r) = B\sin kr$；由 $v(R) = B\sin kR$，欲求非零解则有 $kR = n\pi$，故 $k = \dfrac{n\pi}{R}, n = 1,2,\cdots$

因此得到固有值为 $\lambda_n = \left(\dfrac{n\pi}{R}\right)^2$，固有函数为 $R_n(r) = \dfrac{1}{r}\sin\dfrac{n\pi r}{R}$，$n \in \mathbb{N}_+$.

此时由关于 $T(t)$ 的 ODE：$T'(t) + \lambda T(t) = 0$ 得到 $T_n(t) = C_n e^{-(\frac{n\pi}{R})^2 t}$.

叠加得到：$u(t,r) = \sum\limits_{n=1}^{+\infty} C_n e^{-(\frac{n\pi}{R})^2 t}\dfrac{1}{r}\sin\dfrac{n\pi r}{R}, t > 0, r \geq 0.$

代入初值条件：$u(0,x) = \sum\limits_{n=0}^{+\infty} C_n \dfrac{1}{r}\sin\dfrac{n\pi r}{R} = f(r).$

由泛定方程可知内积权重为 $\rho(x) = x^2$，则

$$||\frac{1}{r}\sin\frac{n\pi r}{R}||^2 = \int_0^R r^2\left(\frac{1}{r}\sin\frac{n\pi r}{R}\right)^2 \mathrm{d}r = \frac{R}{2};$$

$$\Rightarrow\ C_n = \frac{[f(r), \frac{1}{r}\sin\frac{n\pi r}{R}]}{||\frac{1}{r}\sin\frac{n\pi r}{R}||^2} = \frac{2}{R}\int_0^R tf(t)\sin\frac{n\pi t}{R}\mathrm{d}t$$

因此得：$u(t,r)=\dfrac{2}{Rr}\sum\limits_{n=1}^{+\infty}\left(\int_0^R tf(t)\sin\dfrac{n\pi t}{R}\mathrm{d}t\right)\cdot e^{-(\frac{n\pi}{R})^2t}\cdot\sin\dfrac{n\pi r}{R},t>0,r\geq 0.$

(2) 分离变量，令 $u(t,r)=T(t)X(x)$，得到 $\dfrac{1}{a^2}\dfrac{T''(t)}{T(t)}=\dfrac{X^{(4)}(x)}{X(x)}\overset{\text{def}}{=}-\lambda.$

解以下固有值问题：

$$\begin{cases}X^{(4)}(x)+\lambda X(x)=0\\X(0)=X(l)=X''(0)=X''(l)=0\end{cases}$$

由于 $\int_0^l\lambda X^2\mathrm{d}x=-\int_0^l X^{(4)}X\mathrm{d}x=-\int_0^l X\mathrm{d}X^{(3)}=\int_0^l X^{(3)}X'\mathrm{d}x=\int_0^l X'\mathrm{d}X''=-\int_0^l(X'')^2\mathrm{d}x\leq 0$，即 $\lambda\int_0^l X^2\mathrm{d}x\leq 0$，而 $\int_0^l X^2\mathrm{d}x\geq 0$，因此 $\lambda\leq 0$.

当 $\lambda=0$ 时，$X(x)=Ax+B$，由边界条件可得 $A=B=0$，得到零解，故 $\lambda<0$，令 $\lambda\overset{\text{def}}{=}-\omega^4$, 这样特征方程存在 4 个特征根：$k_{1,2}=\pm\omega$，$k_{3,4}=\pm i\omega$

因此可得到：$y(x)=C_1e^{\omega x}+C_2e^{-\omega x}+C_3\cos\omega x+C_4\sin\omega x.$

代入边界条件 $y(0)=y(l)=y''(0)=y''(l)=0$, 分别得到：

$$\begin{aligned}C_1+C_2+C_3=0,\ C_1+C_2-C_3=0\\C_1e^{\omega l}+C_2e^{-\omega l}+C_3\cos\omega l+C_4\sin\omega l=0\\C_1e^{\omega l}+C_2e^{-\omega l}-C_3\cos\omega l-C_4\sin\omega l=0\end{aligned}$$

解得 $C_1=C_2=C_3=0,\ C_4\sin\omega l=0$

所以欲求非零解，只有 $C_4\neq 0$, 因此 $\sin\omega l=0\ \Rightarrow\ \omega l=n\pi\ \Rightarrow\ \omega=\frac{n\pi}{l},\ n=1,2,\cdots$

因此可得到固有值为 $\lambda_n=-\left(\frac{n\pi}{l}\right)^4$，固有函数为 $y_n(x)=\sin\frac{n\pi x}{l},n\in\mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE：$T''(t)+a^2\lambda_nT(t)=0$，得到 $T_n(t)=A_n\cosh\left[\left(\dfrac{n\pi}{l}\right)at\right]+B_n\sinh\left[\left(\dfrac{n\pi}{l}\right)at\right]$.

叠加得到：$u(t,x)=\sum\limits_{n=1}^{+\infty}\left\{A_n\cosh\left[\left(\dfrac{n\pi}{l}\right)at\right]+B_n\sinh\left[\left(\dfrac{n\pi}{l}\right)at\right]\right\}\sin\dfrac{n\pi x}{l},\ t\geq 0,0\leq x\leq l.$

代入初值条件：

$$\begin{cases}u(0,x)=\sum\limits_{n=1}^{+\infty}A_n\sin\dfrac{n\pi x}{l}=x(l-x)\\u_t(0,x)=\sum\limits_{n=1}^{+\infty}\left(\dfrac{n\pi}{l}\right)^2aB_n\sin\dfrac{n\pi x}{l}=0\end{cases}\Rightarrow B_n=0(n=1,2,\cdots)$$

以下计算 $A_n=\dfrac{[x(l-x),X_n(x)]}{||X_n(x)||^2}$，$n=1,2,\cdots$，其中

$$||X_n(x)||^2=\int_0^l\sin^2\frac{n\pi x}{l}\mathrm{d}x=\int_0^l\frac{1}{2}\left(1-\cos\frac{2n\pi x}{l}\right)\mathrm{d}x=\frac{l}{2}$$

$$[x(l-x),X_n(x)]=\int_0^l x(l-x)\sin\frac{n\pi x}{l}\mathrm{d}x=\int_0^1\left(t\sin n\pi t-t^2\sin n\pi t\right)\mathrm{d}t$$

其中

$$\int_0^1 t\sin n\pi t\mathrm{d}t=\frac{-1}{n\pi}\int_0^1 t\mathrm{d}(\cos n\pi t)=-\frac{1}{n\pi}t\cos n\pi t\Big|_0^1+\frac{1}{n\pi}\int_0^1\cos n\pi t\mathrm{d}t=-\frac{1}{n\pi}(-1)^n$$

$$\begin{aligned}\int_0^1 t^2\sin n\pi t\mathrm{d}t&=\frac{-1}{n\pi}\int_0^1 t^2\mathrm{d}(\cos n\pi t)\\&=-\frac{1}{n\pi}t^2\cos n\pi t\Big|_0^1+\frac{1}{n\pi}\int_0^1 2t\cos n\pi t\mathrm{d}t\\&=-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^2}\int_0^1 t\mathrm{d}(\sin n\pi t)\\&=-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^2}t\sin n\pi t\Big|_0^1-\frac{2}{(n\pi)^2}\int_0^1\sin n\pi t\mathrm{d}t\\&=-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^3}\cos n\pi t\Big|_0^1\\&=-\frac{1}{n\pi}(-1)^n+\frac{2}{(n\pi)^3}\left[(-1)^n-1\right]\end{aligned}$$

因此得到

$$C_n=\frac{[x(l-x),X_n(x)]}{||X_n(x)||^2}=\frac{2}{l}\cdot\frac{2l^3}{n^3\pi^3}\cdot[1-(-1)^n]=\frac{4l^2}{n^3\pi^3}\cdot[1-(-1)^n]=\begin{cases}\dfrac{8l^2}{n^3\pi^3}, & n=2k+1\\ 0, & n=2k\ (k\in\mathbb{N})\end{cases}$$

故 $u(t,x)=\dfrac{8l^2}{\pi^3}\displaystyle\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^3}\cdot\cosh\left[\left(\frac{n\pi}{l}\right)at\right]\cdot\sin\frac{2k+1}{l}\pi x,\ t>0,0<x<l.$

8 课本 P_{254} T_8

一半径为 a 的半圆形平板，其圆周边界上的温度保持 $u(a,\theta)=T\theta(\pi-\theta)$，而直径边界上的温度为零度，板的侧面绝缘，试求板内的稳定温度分布.

Sol:

达到稳态后 $u=u(r,\theta)$，得到定解问题：

$$\begin{cases}\Delta_2 u=0\\ u(a,\theta)=T\theta(\pi-\theta)\\ u(a,0)=u(a,\pi)=0\end{cases}$$

分离变量：令 $u(r,\theta)=R(r)\Theta(\theta)$，得到 $r^2\dfrac{R''}{R}+r\dfrac{R'}{R}=\dfrac{\Theta''}{\Theta}\overset{\text{def}}{=}-\lambda$，解以下固有值问题：

$$\begin{cases}\Theta''(\theta)+\Theta(\theta)=0\\ \Theta(0)=\Theta(\pi)=0\end{cases}$$

由 Strum-Liouville 定理可知 $\lambda>0$, 可令 $\lambda\overset{\text{def}}{=}k^2>0$, 得到 $\Theta(\theta)=A\cos k\theta+B\sin k\theta,\ k\neq 0$.

由 $\Theta'(0)=A=0$, 此时 $\Theta(\theta)=B\sin k\theta$; 由 $\Theta(\pi)=B\sin k\pi=0$, 欲求得非零解则 $B\neq 0$, 故 $k_n=n\pi, n=1,2,\cdots$

因此固有值为 $\lambda_n=k_n^2=n^2$，固有函数为 $\Theta_n(\theta)=\sin n\theta, n\in\mathbb{N}_+$.

随后解关于 $R(r)$ 的 ODE: $r^2R''(r)+rR'(r)+\lambda_nR(r)=0$, 得到 $R_n(r)=C_nr^n+D_nr^{-n}, n\in\mathbb{N}_+$.

叠加得到：$u(t,x)=\sum\limits_{n=1}^{+\infty}(C_nr^n+D_nr^{-n})\sin n\theta$.

代入边界条件得：

$$\begin{cases}|u(0,\theta)|<+\infty\ \Rightarrow D_n=0, n\in\mathbb{N}_+\\ u(a,\theta)=\displaystyle\sum_{n=1}^{+\infty}C_na^n\sin n\theta=T\theta(\pi-\theta)\end{cases}$$

以下计算 $C_n=\dfrac{1}{a^n}\cdot\dfrac{[T\theta(\pi-\theta),\Theta_n(\theta)]}{||\Theta_n(\theta)||^2}$，$n=1,2,\cdots$，其中

$$||\Theta_n(\theta)||^2=\int_0^\pi\sin^2 n\theta\mathrm{d}\theta=\int_0^\pi\frac{1}{2}\left(1-\cos 2n\theta\right)\mathrm{d}\theta=\frac{\pi}{2}$$

$$[T\theta(\pi-\theta),\Theta_n(\theta)]=\int_0^\pi T\theta(\pi-\theta)\sin n\theta\mathrm{d}\theta=\pi^3T\int_0^1\left(t\sin n\pi t-t^2\sin n\pi t\right)\mathrm{d}t$$

其中

$$\int_0^1 t\sin n\pi t\mathrm{d}t=\frac{-1}{n\pi}\int_0^1 t\mathrm{d}(\cos n\pi t)=-\frac{1}{n\pi}t\cos n\pi t\Big|_0^1+\frac{1}{n\pi}\int_0^1\cos n\pi t\mathrm{d}t=-\frac{1}{n\pi}(-1)^n$$

$$
\begin{aligned}
&\int_0^1 t^2 \sin n\pi t \mathrm{d}t = \frac{-1}{n\pi}\int_0^1 t^2 \mathrm{d}(\cos n\pi t)\\
=&-\frac{1}{n\pi}t^2\cos n\pi t\Big|_0^1 + \frac{1}{n\pi}\int_0^1 2t\cos n\pi t\mathrm{d}t\\
=&-\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2}\int_0^1 t\mathrm{d}(\sin n\pi t)\\
=&-\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2}t\sin n\pi t\Big|_0^1 - \frac{2}{(n\pi)^2}\int_0^1 \sin n\pi t\mathrm{d}t\\
=&-\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}\cos n\pi t\Big|_0^1\\
=&-\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}\left[(-1)^n - 1\right]
\end{aligned}
$$

因此得到

$$
\begin{aligned}
C_n &= \frac{1}{a^n}\cdot\frac{[T\theta(\pi-\theta),\Theta_n(\theta)]}{||\Theta_n(\theta)||^2}\\
&= \frac{1}{a^n}\cdot\frac{2}{\pi}\cdot\pi^3 T\cdot\frac{2}{n^3\pi^3}\cdot[1-(-1)^n]\\
&= \frac{4T}{\pi n^3 a^n}\cdot[1-(-1)^n] = \begin{cases}\dfrac{8T}{\pi n^3 a^n}, & n=2k+1\\ 0, & n=2k\ (k\in\mathbb{N})\end{cases}
\end{aligned}
$$

故 $u(r,\theta) = \dfrac{8T}{\pi}\sum\limits_{k=1}^{+\infty}\dfrac{1}{(2k+1)^3}\cdot\left(\dfrac{r}{a}\right)^{2k+1}\cdot\sin(2k+1)\theta,\ t>0, 0<r<a, 0<\theta<\pi.$

9　课本 P_{254} T_9

求方程 $u_{xx} - u_y = 0$ 满足条件

$$\lim_{x\to+\infty} u(x,y) = 0$$

的解 $u = X(x)Y(y)$.

Sol:

由于 $u(x,y) = X(x)Y(y)$，因此 $u_{xx} = X''(x)Y(y) = u_y = X(x)Y'(y)$，得到

$$\frac{X''(x)}{X(x)} = \frac{Y'(y)}{Y(y)} \xlongequal{\text{def}} k$$

故可得到关于 X(x) 和 Y(y) 的两个 ODE:

$$X''(x) + kX(x) = 0,\ Y'(y) + kY(y) = 0$$

因此 $Y(y) = C_0 e^{-ky}$.

若 $k < 0$，则 $X(x) = A\cos\sqrt{-k}x + B\sin\sqrt{-k}x$，显然不满足 $\lim\limits_{x\to+\infty} u(x,y) = 0$；若 $k = 0$，则 $X(x) = Ax+B$，显然也不满足 $\lim\limits_{x\to+\infty} u(x,y) = 0$，故 $k > 0$，得到 $X(x) = C_1e^{-\sqrt{k}x} + C_2e^{\sqrt{k}x}$.

由于 $\lim\limits_{x\to+\infty} u(x,y) = 0$，故 $\lim\limits_{x\to+\infty} X(x) = 0$，因此 $C_2 = 0, X(x) = C_1e^{-\sqrt{k}x}$

所以得到 $u(x,y) = X(x)Y(y) = Ce^{-\sqrt{k}x+ky}$，此处 $k > 0$，C 为任意常数.

10 课本 P_{254} T_{10}

(1) $\begin{cases} u_t = a^2 u_{xx} \\ u(t,0) = u_0, u_x(t,l) = 0 \\ u(0,x) = \varphi(x) \end{cases}$

(2) $\begin{cases} u_t = a^2 u_{xx} \\ u(t,0) = 0, u(t,l) = -\dfrac{q}{k} \\ u(0,x) = u_0 \end{cases}$

并求 $\lim\limits_{t\to+\infty} u(t,x)$;

(3) $\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - a^2 \dfrac{\partial u}{\partial t} + A\mathrm{e}^{-2x} = 0 \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = T_0 \end{cases}$

(4) $\begin{cases} u_{tt} = a^2 u_{xx} + b \sinh x \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = u_t(0,x) = 0 \end{cases}$

(5) $\begin{cases} u_{tt} = u_{xx} + g(g \text{ 为常数}), \\ u(t,0) = 0, u_x(t,l) = E(E \text{ 为常数}), \\ u(0,x) = Ex, u_t(0,x) = 0; \end{cases}$

[提示：先求一个满足泛定方程和边界条件的 $v(x)$, 再令 $u(t,x) = w(t,x) + v(x)$.]

(6) $\begin{cases} \Delta_2 u = a + b\left(x^2 - y^2\right)(a, b \text{ 为常数}, r < R), \\ u(R,\theta) = c(c \text{ 为常数}). \end{cases}$

Sol:

(1) 令 $u(t,x) = v(t,x) + u_0$，则 $v(t,x)$ 满足：

$$\begin{cases} v_t = a^2 v_{xx} \\ v(t,0) = v_x(t,l) = 0 \\ v(0,x) = \varphi(x) - u_0 \end{cases}$$

分离变量：令 $v(t,x) = T(t)X(x)$，得到 $\dfrac{1}{a^2}\dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} \overset{\text{def}}{=} -\lambda$，解以下固有值问题：

$$\begin{cases} X''(x) + X(x) = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

由 Strum-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \overset{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A\cos kx + B\sin kx$, $k \neq 0$.

由 $X(0) = A = 0$，此时 $X(x) = B\sin k\theta$；由 $X'(l) = kB\cos kl = 0$，欲求得非零解则 $kB \neq 0$，故 $k_n = \dfrac{(2n+1)\pi}{2l}$，$n = 0, 1, 2, \cdots$

因此固有值为 $\lambda_n = k_n^2 = \left(\dfrac{2n+1}{2l}\pi\right)^2$，固有函数为 $X_n(x) = \sin\dfrac{2n+1}{2l}\pi x, n \in \mathbb{N}_+$.
设 $v(t,x) = \sum\limits_{n=0}^{+\infty} T_n(t)\sin\dfrac{2n+1}{2l}\pi x$.
随后解关于 $T(t)$ 的 ODE：$T'(t) + a^2\lambda T(t) = 0$，得到 $T_n(t) = e^{-a^2\lambda_n t} = e^{-(\frac{2n+1}{2l}a)^2 t}, n \in \mathbb{N}$.
叠加得到：$v(t,x) = \sum\limits_{n=0}^{+\infty} C_n e^{-(\frac{2n+1}{2l}a)^2 t}\sin\dfrac{2n+1}{2l}\pi x$.
代入初始条件得：$v(0,x) = \sum\limits_{n=0}^{+\infty} C_n \sin\dfrac{2n+1}{2l}\pi x = \varphi(x) - u_0$.
那么系数为：

$$\begin{aligned}
C_n &= \frac{[\varphi(x) - u_0, \sin\dfrac{2n+1}{2l}\pi x]}{||\sin\dfrac{2n+1}{2l}\pi x||^2} \\
&= \frac{2}{l}\int_0^l (\varphi(x) - u_0)\sin\frac{2n+1}{2l}\pi x \mathrm{d}x \\
&= \frac{2}{l}\int_0^l \varphi(x)\sin\frac{2n+1}{2l}\pi x \mathrm{d}x - \frac{4u_0}{(2n+1)\pi}
\end{aligned}$$

故解为：

$$u(t,x) = u_0 + \sum_{n=0}^{+\infty}\left[\frac{2}{l}\int_0^l \varphi(x)\sin\frac{2n+1}{2l}\pi x \mathrm{d}x - \frac{4u_0}{(2n+1)\pi}\right] e^{-(\frac{2n+1}{2l}a)^2 t}\sin\frac{2n+1}{2l}\pi x.$$

(2) 令 $u(t,x) = v(t,x) - \dfrac{q}{k}x$，则 $v(t,x)$ 满足

$$\begin{cases}
v_t = a^2 v_{xx} \\
v(t,0) = v_x(t,l) = 0 \\
v(0,x) = u_0 + \dfrac{q}{k}x
\end{cases}$$

分离变量：令 $v(t,x) = T(t)X(x$，得到 $\dfrac{1}{a^2}\dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} \overset{\text{def}}{=} -\lambda$，解以下固有值问题：

$$\begin{cases}
X''(x) + X(x) = 0 \\
X(0) = X'(l) = 0
\end{cases}$$

由 Strum-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \overset{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A\cos kx + B\sin kx$，$k \neq 0$.

由 $X(0) = A = 0$，此时 $X(x) = B\sin k\theta$；由 $X'(l) = kB\cos kl = 0$，欲求得非零解则 $kB \neq 0$，故 $k_n = \dfrac{2n+1}{2l}\pi$，$n = 0, 1, 2, \cdots$

因此固有值为 $\lambda_n=k_n^2=\left(\dfrac{2n+1}{2l}\pi\right)^2$，固有函数为 $X_n(x)=\sin\dfrac{2n+1}{2l}\pi x, n\in\mathbb{N}$.
随后解关于 $T(t)$ 的 ODE：$T'(t)+a^2\lambda T(t)=0$，得到 $T_n(t)=e^{-a^2\lambda_n t}=e^{-(\frac{2n+1}{2l}a)^2t}, n\in\mathbb{N}$.
叠加得到：$v(t,x)=\sum\limits_{n=0}^{+\infty}C_ne^{-(\frac{2n+1}{2l}a)^2t}\sin\dfrac{2n+1}{2l}\pi x$.
代入初始条件得：$v(0,x)=\sum\limits_{n=0}^{+\infty}C_n\sin\dfrac{2n+1}{2l}\pi x=u_0+\dfrac{q}{k}x$.
那么系数为：

$$\begin{aligned}C_n&=\frac{[u_0+\frac{q}{k}x,\sin\frac{2n+1}{2l}\pi x]}{||\sin\frac{2n+1}{2l}\pi x||^2}\\&=\frac{2}{l}\int_0^l\left(u_0+\frac{q}{k}x\right)\sin\frac{2n+1}{2l}\pi x\mathrm{d}x\end{aligned}$$

其中

$$\begin{aligned}\int_0^l\sin\frac{2n+1}{2l}\pi x\mathrm{d}x&=-\frac{2l}{(2n+1)\pi}\cos\frac{2n+1}{2l}\pi x\Big|_0^l=\frac{2l}{(2n+1)\pi}\\\int_0^l x\sin\frac{2n+1}{2l}\pi x\mathrm{d}x&=-\frac{2l}{(2n+1)\pi}x\cos\frac{2n+1}{2l}\pi x\Big|_0^l+\frac{2l}{(2n+1)\pi}\int_0^l\cos\frac{2n+1}{2l}\pi x\mathrm{d}x\\&=\frac{4l^2}{(2n+1)^2\pi^2}\sin\frac{2n+1}{2l}\pi x\Big|_0^l\\&=\frac{4l^2}{(2n+1)^2\pi^2}(-1)^n\end{aligned}$$

因此系数为:$C_n=\dfrac{2u_0}{l}\cdot\dfrac{2l}{(2n+1)\pi}+\dfrac{2q}{kl}\cdot\dfrac{4l^2}{(2n+1)^2\pi^2}(-1)^n=\dfrac{4u_0}{(2n+1)\pi}+\dfrac{8ql}{k(2n+1)^2\pi^2}(-1)^n$.
故解为: $u(t,x)=-\dfrac{q}{k}x+\sum\limits_{n=0}^{+\infty}\left[\dfrac{4u_0}{(2n+1)\pi}+\dfrac{8ql}{k(2n+1)^2\pi^2}(-1)^n\right]e^{-(\frac{2n+1}{2l}a)^2t}\sin\dfrac{2n+1}{2l}\pi x$, 得到 $\lim\limits_{t\to+\infty}u(t,x)=-\dfrac{q}{k}x$.

(3) 先求解对应的齐次问题：

$$\begin{cases}a^2u_t=u_{xx}\\u(t,0)=u(t,l)=0\end{cases}$$

分离变量：令 $u(t,x)=T(t)X(x$，得到 $a^2\dfrac{T'(t)}{T(t)}=\dfrac{X''(x)}{X(x)}\overset{\text{def}}{=}-\lambda$，解以下固有值问题：

$$\begin{cases}X''(x)+X(x)=0\\X(0)=X(l)=0\end{cases}$$

由 Strum-Liouville 定理可知 $\lambda>0$, 可令 $\lambda\overset{\text{def}}{=}k^2>0$, 得到 $X(x)=A\cos kx+B\sin kx$, $k\neq$

0.

由 $X(0)=A=0$，此时 $X(x)=B\sin k\theta$；由 $X(l)=B\sin kl=0$，欲求得非零解则 $B\neq 0$，故 $k_n=\dfrac{n\pi}{l}$，$n=1,2,\cdots$

因此固有值为 $\lambda_n=k_n^2=\left(\dfrac{n\pi}{l}\right)^2$，固有函数为 $X_n(x)=\sin\dfrac{n\pi x}{l},n\in\mathbb{N}_+$.

设 $u(t,x)=\sum\limits_{n=1}^{+\infty}C_nT_n(t)\sin\dfrac{n\pi x}{l}$.

令 $f(x)=Ae^{-2x}=\sum\limits_{n=1}^{+\infty}f_n\sin\dfrac{n\pi x}{l}$，$T_0=\sum\limits_{n=1}^{+\infty}t_n\sin\dfrac{n\pi x}{l}$，则展开系数为：

$$f_n=\frac{2}{l}\int_0^l Ae^{-2x}\sin\frac{n\pi x}{l}\mathrm{d}x=\frac{2n\pi A}{4l^2+(n\pi)^2}[1-(-1)^ne^{-2l}]$$

$$t_n=\frac{2}{l}\int_0^l T_0\sin\frac{n\pi x}{l}\mathrm{d}x=\frac{2T_0}{n\pi}[1-(-1)^n]$$

代入原问题中得：

$$\begin{cases}a^2\sum\limits_{n=1}^{+\infty}T_n'(t)\sin\dfrac{n\pi x}{l}=-\sum\limits_{n=1}^{+\infty}T_n(t)\left(\dfrac{n\pi}{l}\right)^2\sin\dfrac{n\pi x}{l}+\sum\limits_{n=1}^{+\infty}f_n\sin\dfrac{n\pi x}{l}\\ \sum\limits_{n=1}^{+\infty}T_n(0)\sin\dfrac{n\pi x}{l}=\sum\limits_{n=1}^{+\infty}t_n\sin\dfrac{n\pi x}{l}\end{cases}$$

对比系数得：

$$\begin{cases}a^2T_n'(t)+\left(\dfrac{n\pi}{l}\right)^2T_n(t)=f_n\\ T_n(0)=t_n\end{cases}$$

因此得到：

$$\begin{aligned}&\left[T_ne^{(\frac{n\pi}{la})^2t}\right]'=\frac{f_n}{a^2}e^{(\frac{n\pi}{la})^2t}\\ &\Rightarrow T_n(t)=\left(\frac{l}{n\pi}\right)^2f_n+C_ne^{-(\frac{n\pi}{la})^2t}\\ &\text{又 } T_n(0)=\left(\frac{l}{n\pi}\right)^2f_n+C_n=t_n\\ &\Rightarrow C_n=t_n-\left(\frac{l}{n\pi}\right)^2f_n\\ &\Rightarrow T_n(t)=\left[t_n-\left(\frac{l}{n\pi}\right)^2f_n\right]e^{-(\frac{n\pi}{la})^2t}+\left(\frac{l}{n\pi}\right)^2f_n\\ &\Rightarrow u(t,x)=\sum_{n=1}^{+\infty}T_n(t)\sin\frac{n\pi x}{l}\end{aligned}$$

(4) 先求解对应的齐次问题：

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(t,0) = u(t,l) = 0 \end{cases}$$

分离变量：令 $u(t,x) = T(t)X(x$，得到 $\dfrac{1}{a^2}\dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} \overset{\text{def}}{=} -\lambda$，解以下固有值问题：

$$\begin{cases} X''(x) + X(x) = 0 \\ X(0) = X(l) = 0 \end{cases}$$

由 Strum-Liouville 定理可知 $\lambda > 0$，可令 $\lambda \overset{\text{def}}{=} k^2 > 0$，得到 $X(x) = A\cos kx + B\sin kx$，$k \neq 0$.

由 $X(0) = A = 0$，此时 $X(x) = B\sin k\theta$；由 $X(l) = B\sin kl = 0$，欲求得非零解则 $B \neq 0$，故 $k_n = \dfrac{n\pi}{l}$，$n = 1, 2, \cdots$

因此固有值为 $\lambda_n = k_n^2 = \left(\dfrac{n\pi}{l}\right)^2$，固有函数为 $X_n(x) = \sin\dfrac{n\pi x}{l}, n \in \mathbb{N}_+$.

设 $u(t,x) = \sum\limits_{n=1}^{+\infty} T_n(t)\sin\dfrac{n\pi x}{l}$.

令 $f(x) = b\sinh x = \sum\limits_{n=1}^{+\infty} f_n \sin\dfrac{n\pi x}{l}$，$T_0 = \sum\limits_{n=1}^{+\infty} t_n \sin\dfrac{n\pi x}{l}$，则展开系数为：

$$\begin{aligned} f_n &= \frac{2b}{l}\int_0^l b\sinh x\sin\frac{n\pi x}{l}\mathrm{d}x \\ &= \frac{2b}{l}\left[\cosh x\sin\frac{n\pi x}{l}\Big|_0^l - \int_0^l \cosh x\cdot\frac{n\pi}{l}\cos\frac{n\pi x}{l}\mathrm{d}x\right] \\ &= \frac{2b}{l}\left[-\frac{n\pi}{l}\int_0^l \cos\frac{n\pi x}{l}\mathrm{d}\sinh x\right] \\ &= \frac{2b}{l}\left[-\frac{n\pi}{l}\cos\frac{n\pi x}{l}\sinh x\Big|_0^l - \left(\frac{n\pi}{l}\right)^2\int_0^l \sinh x\sin\frac{n\pi x}{l}\mathrm{d}x\right] \\ &= \frac{2b}{l}\left[\frac{1}{1+\left(\frac{n\pi}{l}\right)^2}\cdot\left(-\frac{n\pi}{l}\right)\cdot(-1)^n\sinh l\right] \\ \Rightarrow f_n &= \frac{2b}{l}\cdot\frac{\left(\frac{n\pi}{l}\right)}{1+\left(\frac{n\pi}{l}\right)^2}\cdot(-1)^{n+1}\sinh l \end{aligned}$$

代入原问题中得：

$$\begin{cases} \sum\limits_{n=1}^{+\infty} T_n''(t)\sin\dfrac{n\pi x}{l} = -\sum\limits_{n=1}^{+\infty} T_n(t)\left(\dfrac{n\pi a}{l}\right)^2\sin\dfrac{n\pi x}{l} + \sum\limits_{n=1}^{+\infty} f_n\sin\dfrac{n\pi x}{l} \\ \sum\limits_{n=1}^{+\infty} T_n(0)\sin\dfrac{n\pi x}{l} = \sum\limits_{n=1}^{+\infty} T_n'(0)\sin\dfrac{n\pi x}{l} = 0 \end{cases}$$

对比系数得：

$$\begin{cases} T_n''(t) + \left(\dfrac{n\pi a}{l}\right)^2 T_n(t) = f_n \\ T_n(0) = T_n'(0) = 0 \end{cases}$$

因此得到：

$$T_n(t) = A_n \sin\frac{n\pi at}{l} + B_n \cos\frac{n\pi at}{l} + \left(\frac{l}{n\pi a}\right)^2 f_n$$

由 $T_n(0) = B_n + \left(\dfrac{l}{n\pi a}\right)^2 f_n = 0$，得到 $B_n = -\left(\dfrac{l}{n\pi a}\right)^2 f_n$；由 $T_n'(0) = \dfrac{n\pi a}{l} \cdot A_n = 0$，得到 $A_n = 0$，所以解为：

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{+\infty} \left(\frac{l}{n\pi a}\right)^2 f_n \left(1 - \cos\frac{n\pi at}{l}\right) \sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{+\infty} \left(\frac{l}{n\pi a}\right)^2 \frac{2b}{l} \cdot \frac{\left(\frac{n\pi}{l}\right)}{1 + \left(\frac{n\pi}{l}\right)^2} \cdot (-1)^{n+1} \sinh l \left(1 - \cos\frac{n\pi at}{l}\right) \sin\frac{n\pi x}{l} \\ &= \frac{2bl^2 \sinh l}{\pi a^2} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n(l^2 + n^2\pi^2)} \left(1 - \cos\frac{n\pi at}{l}\right) \sin\frac{n\pi x}{l} \end{aligned}$$

(5) 令 $u(t,x) = v(t,x) + g$，则 $v(t,x)$ 满足：

$$\begin{cases} v_{tt} = v_{xx} + g \\ v(t,0) = v_x(t,l) = 0 \\ v(0,x) = v_t(0,x) = 0 \end{cases}$$

分离变量：令 $v(t,x) = T(t)X(x)$，得到 $\dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} \overset{\text{def}}{=} -\lambda$，解以下固有值问题：

$$\begin{cases} X''(x) + X(x) = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

由 Strum-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \overset{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A\cos kx + B\sin kx$, $k \neq 0$.

由 $X(0) = A = 0$，此时 $X(x) = B\sin k\theta$；由 $X'(l) = kB\cos kl = 0$，欲求得非零解则 $kB \neq 0$，故 $k_n = \dfrac{(2n+1)\pi}{2l}$，$n = 0, 1, 2, \cdots$

因此固有值为 $\lambda_n = k_n^2 = \left(\dfrac{2n+1}{2l}\pi\right)^2$，固有函数为 $X_n(x) = \sin\dfrac{2n+1}{2l}\pi x, n \in \mathbb{N}_+$.

设 $v(t,x)=\sum\limits_{n=0}^{+\infty}C_nT_n(t)\sin\dfrac{2n+1}{2l}\pi x$.

令 $g=\sum\limits_{n=0}^{+\infty}g_n\sin\dfrac{2n+1}{2l}\pi x$，则展开系数为：

$$g_n=\frac{2}{l}\int_0^l g\sin\frac{2n+1}{2l}\pi x\mathrm{d}x=\frac{4g}{(2n+1)\pi}.$$

代入原问题中得：

$$\begin{cases}\sum\limits_{n=0}^{+\infty}T_n''(t)\sin\dfrac{2n+1}{2l}\pi x+\sum\limits_{n=0}^{+\infty}T_n(t)\left(\dfrac{2n+1}{2l}\pi\right)^2\sin\dfrac{2n+1}{2l}\pi x=\sum\limits_{n=0}^{+\infty}g_n\sin\dfrac{2n+1}{2l}\pi x\\ \sum\limits_{n=0}^{+\infty}T_n'(0)\sin\dfrac{2n+1}{2l}\pi x=\sum\limits_{n=0}^{+\infty}T_n(0)\sin\dfrac{2n+1}{2l}\pi x=0\end{cases}$$

对比系数得：

$$\begin{cases}T_n''(t)+\left(\dfrac{2n+1}{2l}\pi\right)^2T_n(t)=g_n\\ T_n'(0)=T_n(0)=0\end{cases}$$

因此得到：

$$T_n(t)=A_n\cos\frac{2n+1}{2l}\pi t+B_n\sin\frac{2n+1}{2l}\pi t+\frac{4l^2g_n}{(2n+1)^2\pi^2}$$

由 $T_n(0)=A_n+\dfrac{4l^2g_n}{(2n+1)^2\pi^2}=0$，得到 $A_n=-\dfrac{4l^2g_n}{(2n+1)^2\pi^2}$；由 $T_n'(0)=\dfrac{2n+1}{2l}\pi\cdot B_n=0$，得到 $B_n=0$，所以得到：

$$\begin{aligned}v(t,x)&=\sum_{n=0}^{+\infty}\frac{4l^2g_n}{(2n+1)^2\pi^2}\left(1-\cos\frac{2n+1}{2l}\pi t\right)\sin\frac{2n+1}{2l}\pi x\\&=\frac{16gl^2}{\pi^3}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^3}\left(1-\cos\frac{2n+1}{2l}\pi t\right)\sin\frac{2n+1}{2l}\pi x\end{aligned}$$

因此，本题的解为：

$$u(t,x)=v(t,x)+Ex=Ex+\frac{16gl^2}{\pi^3}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^3}\left(1-\cos\frac{2n+1}{2l}\pi t\right)\sin\frac{2n+1}{2l}\pi x.$$

(6) 由于 $\Delta_2u=\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}$，所以可以观察出方程显然有特解（对称）：

$$u_1=\frac{a}{4}\left(x^2+y^2\right)+\frac{b}{12}\left(x^4-y^4\right)=\frac{a}{4}r^2+\frac{b}{12}r^4\cos2\theta$$

作变换：$u=v+u_1=V+\frac{a}{4}r^2+\frac{b}{12}r^4\cos 2\theta$，则有：

$$\begin{cases} \Delta_2 v=0,\ (a,b\text{为常数},r<R) \\ v(R,\theta)=C-\frac{a}{4}R^2-\frac{b}{12}R^4\cos 2\theta \end{cases}$$

由齐次 Laplace 方程在圆内解的一般公式，可设

$$v(r,\theta)=A_0+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n(A_n\cos n\theta+B_n\sin n\theta)$$

依据边界条件有：

$$v|_{r=R}=A_0+\sum_{n=1}^{\infty}(A_n\cos n\theta+B_n\sin n\theta)=C-\frac{a}{4}R^2-\frac{b}{12}R^4\cos 2\theta$$

此式比较系数得到：$A_0=C-\frac{a}{4}R^2$，$A_2=-\frac{b}{12}R^4$，其余的 A_n,B_n 都为 0，这样 $v(r,\theta)=C-\frac{a}{4}R^2-\frac{b}{12}R^2r^2\cos 2\theta$，最后得到：

$$u(r,\theta)=v(r,\theta)+u_1=C+\frac{a}{4}\left(r^2-R^2\right)+\frac{b}{12}r^2\left(r^2-R^2\right)\cos 2\theta$$

11 课本 P_{255} T_{11}

在下列条件下，求环域 $a<r<b$ 内泊松方程 $\Delta_2 u = A$（A 为常数）的解:

$$(1) u(a,\theta)=u_1, u(b,\theta)=u_2 \ (u_1,u_2\text{为常数});$$
$$(2) u(a,\theta)=u_1, \frac{\partial u(b,\theta)}{\partial n}=u_2 \ .$$

Sol:

由于边界条件不含 θ，可知：$u=u(r)$，故得到

$$\begin{aligned}
&\Delta_2 u = u''(r)+\frac{1}{r}u'(r)=A\\
&\Rightarrow (ru'(r))' = Ar\\
&\Rightarrow ru'(r)=\frac{A}{2}r^2+C_1\\
&\Rightarrow u'(r)=\frac{A}{2}r+\frac{C_1}{r}\\
&\Rightarrow u(r)=\frac{A}{4}r^2+C_1\ln r+C_2,\ a<r<b
\end{aligned}$$

(1) 代入边界条件得到:

$$\begin{cases} u|_{r=a}=\dfrac{A}{4}a^2+C_1\ln a+C_2=u_1\\ u|_{r=b}=\dfrac{A}{4}b^2+C_1\ln b+C_2=u_2\end{cases} \Rightarrow \begin{cases} C_1=\dfrac{u_1-u_2+\frac{A}{4}(b^2-a^2)}{\ln a-\ln b}\\ C_2=u_2-\dfrac{A}{4}b^2-\dfrac{u_1-u_2+\frac{A}{4}(b^2-a^2)}{\ln a-\ln b}\ln b\end{cases}$$

因此得到解为：$u(r,\theta)=u_2+\dfrac{A}{4}(r^2-b^2)+\dfrac{u_1-u_2+\frac{A}{4}(b^2-a^2)}{\ln a-\ln b}\cdot(\ln r-\ln b),\ a<r<b.$

(2) 代入边界条件得到:

$$\begin{cases} u|_{r=a}=\dfrac{A}{4}a^2+C_1\ln a+C_2=u_1\\ u'|_{r=b}=\dfrac{A}{2}b+\dfrac{C_1}{b}=u_2\end{cases} \Rightarrow \begin{cases} C_1=u_2b-\dfrac{A}{2}b^2\\ C_2=u_1-\dfrac{A}{4}a^2-\left(u_2b-\dfrac{A}{2}b^2\right)\ln a\end{cases}$$

因此得到解为：$u(r,\theta)=u_1+\dfrac{A}{4}(r^2-a^2)+b\left(u_2-\dfrac{Ab}{2}\right)\ln\dfrac{r}{a},\ a<r<b.$

12 课本 P_{255} T_{12}

解下列矩形区域内的定解问题:

$$(1)\begin{cases}\Delta_2 u=f(x,y)\ (0<x<a,0<y<b)\\ u(0,y)=\varphi_1(y),u(a,y)=\varphi_2(y)\\ u(x,0)=\psi_1(x),u(x,b)=\psi_2(x)\end{cases}$$

[提示：先求一个满足以上边界条件得函数 $Ax+B$，然后用固有函数方法求解.]

$$(2)\begin{cases}u_{tt}=a^2\Delta_2 u\ \ (t>0,0<x<l_1,0<y<l_2)\,,\\ u\big|_{x=0}=u\big|_{x=l_1}=u\big|_{y=0}=u\big|_{y=l_2}=0,\\ u\big|_{t=0}=Axy\,(l_1-x)\,(l_2-y)\,,\\ u_t\big|_{t=0}=0;\end{cases}$$

$$(3)\begin{cases}u_t=a^2\Delta_2 u\ (t>0,0<x<l_1,0<y<l_2)\\ u\big|_{x=0}=u\big|_{x=l_1}=u\big|_{y=0}=u\big|_{y=l_2}=0,\\ u\big|_{t=0}=\varphi(x,y)\end{cases}$$

Sol:

(1) 显然，满足以上关于 x 的边界条件的一个特解为：$u_1=\varphi_1(y)+\dfrac{\varphi_2(y)-\varphi_1(y)}{a}x$

此时，记 $u(x,y)=u_1+v(x,y)$，可知 $v(x,y)$ 满足：

$$\begin{cases}v_{xx}+v_{yy}=f(x,y)-\varphi_1''(y)-\dfrac{\varphi_2''(y)-\varphi_1''(y)}{a}x\\ v(0,y)=v(a,y)=0\\ v(x,0)=\psi_1(x)-\varphi_1(0)-\dfrac{\varphi_2(0)-\varphi_1(0)}{a}x\\ v(x,b)=\psi_2(x)-\varphi_1(b)-\dfrac{\varphi_2(b)-\varphi_1(b)}{a}x\end{cases}$$

先求解对应的齐次问题：

$$\begin{cases}v_{xx}+v_{yy}=0\\ v(0,y)=v(a,y)=0\end{cases}$$

分离变量：令 $v(x,y)=X(x)Y(y)$，得到 $\dfrac{X''(t)}{X(t)}=-\dfrac{Y''(y)}{Y(y)}\overset{\text{def}}{=}-\lambda$，解以下固有值问题：

$$\begin{cases}X''(x)+\lambda X(x)=0\\ X(0)=X(a)=0\end{cases}$$

由 Strum-Liouville 定理可知 $\lambda>0$, 可令 $\lambda\overset{\text{def}}{=}k^2>0$, 得到 $X(x)=A\cos kx+B\sin kx,\ k\neq 0.$

由 $X(0)=A=0$，此时 $X(x)=B\sin kx$；由 $X(a)=B\sin ka=0$，欲求得非零解则 $B\neq 0$，故 $k_n=\dfrac{n\pi}{a}$，$n=1,2,\cdots$

因此固有值为 $\lambda_n = k_n^2 = \left(\dfrac{n\pi}{a}\right)^2$，固有函数为 $X_n(x) = \sin\dfrac{n\pi x}{a}, n \in \mathbb{N}$.

设 $v(x,y) = \sum\limits_{n=1}^{+\infty} f_n(y)\sin\dfrac{n\pi x}{a}$，$f(x,y) - \varphi_1''(y) - \dfrac{\varphi_2''(y)-\varphi_1''(y)}{a}x = \sum\limits_{n=1}^{+\infty} C_n \sin\dfrac{n\pi x}{a}$，其中

$$C_n = \frac{2}{a}\int_0^a \left[f(x,y) - \varphi_1''(y) - \frac{\varphi_2''(y)-\varphi_1''(y)}{a}x\right]\sin\frac{n\pi x}{a}\mathrm{d}x$$

代入原问题中得:

$$\begin{cases} -\sum\limits_{n=1}^{+\infty}\left(\dfrac{n\pi}{a}\right)^2 f_n(y)\sin\dfrac{n\pi x}{a} + \sum\limits_{n=1}^{+\infty} f_n''(y)\sin\dfrac{n\pi x}{a} = \sum\limits_{n=0}^{+\infty} C_n\sin\dfrac{n\pi x}{a} \\ \sum\limits_{n=1}^{+\infty} f_n(0)\sin\dfrac{n\pi x}{a} = \psi_1(x) - \varphi_1(0) - \dfrac{\varphi_2(0)-\varphi_1(0)}{a}x \\ \sum\limits_{n=1}^{+\infty} f_n(b)\sin\dfrac{n\pi x}{a} = \psi_2(x) - \varphi_1(b) - \dfrac{\varphi_2(b)-\varphi_1(b)}{a}x \end{cases}$$

对比系数得:

$$\begin{cases} f_n''(y) - \left(\dfrac{n\pi}{a}\right)^2 f_n(y) = \dfrac{2}{a}\displaystyle\int_0^a \left[f(x,y) - \varphi_1''(y) - \dfrac{\varphi_2''(y)-\varphi_1''(y)}{a}x\right]\sin\dfrac{n\pi x}{a}\mathrm{d}x \\ f_n(0) = \dfrac{2}{a}\displaystyle\int_0^a \left[\psi_1(x) - \varphi_1(0) - \dfrac{\varphi_2(0)-\varphi_1(0)}{a}x\right]\sin\dfrac{n\pi x}{a}\mathrm{d}x \\ f_n(b) = \dfrac{2}{a}\displaystyle\int_0^a \left[\psi_2(x) - \varphi_1(b) - \dfrac{\varphi_2(b)-\varphi_1(b)}{a}x\right]\sin\dfrac{n\pi x}{a}\mathrm{d}x \end{cases} \tag{*}$$

因此得到:

$$u(x,y) = \varphi_1(y) + \frac{\varphi_2(y)-\varphi_1(y)}{a}x + \sum_{n=1}^{+\infty} f_n(y)\sin\frac{n\pi x}{a},$$

其中 $f_n(y)$ 由 (*)ODE 边值问题所确定.

(2) 分离变量：令 $u(t,x,y) = T(t)X(x)Y(y)$，得到

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} - \frac{Y''(y)}{Y(y)} = \frac{X''(t)}{X(t)} \overset{\text{def}}{=} -\lambda$$

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} + \lambda = \frac{Y''(y)}{Y(y)} - \overset{\text{def}}{=} -\mu$$

得到以下两个固有值问题:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l_1) = 0 \end{cases} \quad \begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(0) = Y(l_2) = 0 \end{cases}$$

由 Strum-Liouville 定理可知 $\lambda,\mu > 0$，可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$，得到 $X(x) = \tilde{A}\cos kx + B\sin kx$，$k \neq 0$.

由 $X(0) = \tilde{A} = 0$，此时 $X(x) = B\sin kx$；由 $X(l_1) = B\sin kl_1 = 0$，欲求得非零解则 $B \neq 0$，故 $k_n = \dfrac{n\pi}{l_1}$，$n = 1, 2, \cdots$

因此关于 $X(x)$ 的固有值为 $\lambda_n = k_n^2 = \left(\dfrac{n\pi}{l_1}\right)^2$，固有函数为 $X_n(x) = \sin\dfrac{n\pi x}{l_1}, n \in \mathbb{N}$.

同理可得，关于 $Y(y)$ 的固有值为 $\mu_n = \left(\dfrac{n\pi}{l_2}\right)^2$，固有函数为 $Y_n(y) = \sin\dfrac{n\pi y}{l_2}, n \in \mathbb{N}$.

随后解关于 $T(t)$ 的 ODE：$T''(t) + a^2(\lambda+\mu)T(t) = 0$，得到 $T_{mn}(t) = C_{mn}\cos\omega_{mn}at + D_{mn}\sin\omega_{mn}at, m, n \in \mathbb{N}$，其中 $\omega_{mn} = \sqrt{(\frac{m\pi}{l_1})^2 + (\frac{n\pi}{l_2})^2}$.

叠加得到：$u(t,x,y) = \sum\limits_{m=1}^{+\infty}\sum\limits_{n=1}^{+\infty}(A_{mn}\cos\omega_{mn}at + B_{mn}\sin\omega_{mn}at)\sin\dfrac{m\pi x}{l_1}\sin\dfrac{n\pi y}{l_2}$.

代入初始条件得：

$$\begin{cases} u_t(0,x,y) = \sum\limits_{m=1}^{+\infty}\sum\limits_{n=1}^{+\infty}\omega_{mn}aD_{mn}\sin\dfrac{m\pi x}{l_1}\sin\dfrac{n\pi y}{l_2} = 0 \Rightarrow\ D_{mn} = 0, m, n \in \mathbb{N}_+ \\ u(0,x,y) = \sum\limits_{m=1}^{+\infty}\sum\limits_{n=1}^{+\infty}C_{mn}\sin\dfrac{m\pi x}{l_1}\sin\dfrac{n\pi y}{l_2} = Axy(l_1-x)(l_2-y) \end{cases}$$

可知系数 C_{mn} 为：

$$\begin{aligned} C_{mn} &= \frac{4}{l_1l_2}\int_0^{l_1}\int_0^{l_2}Axy(l_1-x)(l_2-y)\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}\mathrm{d}y\mathrm{d}x \\ &= 4Al_1^2l_2^2\left[\int_0^1 x(1-x)\sin m\pi x\mathrm{d}x\right]\cdot\left[\int_0^1 y(1-y)\sin n\pi y\mathrm{d}y\right] \end{aligned}$$

其中

$$\int_0^1 x\sin n\pi x\mathrm{d}x = \frac{-1}{n\pi}\int_0^1 t\mathrm{d}(\cos n\pi x) = -\frac{1}{n\pi}x\cos n\pi x\Big|_0^1 + \frac{1}{n\pi}\int_0^1\cos n\pi x\mathrm{d}x = -\frac{1}{n\pi}(-1)^n$$

$$\begin{aligned} \int_0^1 x^2\sin n\pi x\mathrm{d}x &= \frac{-1}{n\pi}\int_0^1 x^2\mathrm{d}(\cos n\pi x) \\ &= -\frac{1}{n\pi}x^2\cos n\pi x\Big|_0^1 + \frac{1}{n\pi}\int_0^1 2x\cos n\pi x\mathrm{d}x \\ &= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2}\int_0^1 x\mathrm{d}(\sin n\pi x) \\ &= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2}x\sin n\pi x\Big|_0^1 - \frac{2}{(n\pi)^2}\int_0^1\sin n\pi x\mathrm{d}x \\ &= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}\cos n\pi x\Big|_0^1 \\ &= -\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^3}\left[(-1)^n - 1\right] \end{aligned}$$

因此得到

$$C_{mn} = 4Al_1^2l_2^2 \cdot \frac{2}{(m\pi)^3}[1-(-1)^m] \cdot \frac{2}{(n\pi)^3}[1-(-1)^n] = \frac{16Al_1^2l_2^2}{m^3n^3\pi^6}[1-(-1)^m][1-(-1)^n]$$

那么本题解为: $u(t,x,y) = \sum\limits_{m=1}^{+\infty}\sum\limits_{n=1}^{+\infty} C_{mn}\cos\left[\sqrt{(\frac{m\pi}{l_1})^2+(\frac{n\pi}{l_2})^2}at\right]\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}$, 系数 C_{mn} 如上所述.

(3) 分离变量：令 $u(t,x,y) = T(t)X(x)Y(y)$，得到

$$\frac{1}{a^2}\frac{T'(t)}{T(t)} - \frac{Y''(y)}{Y(y)} = \frac{X''(t)}{X(t)} \stackrel{\text{def}}{=} -\lambda$$

$$\frac{1}{a^2}\frac{T'(t)}{T(t)} + \lambda = \frac{Y''(y)}{Y(y)} - \stackrel{\text{def}}{=} -\mu$$

得到以下两个固有值问题：

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l_1) = 0 \end{cases} \begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(0) = Y(l_2) = 0 \end{cases}$$

由 Strum-Liouville 定理可知 $\lambda,\mu > 0$，可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$，得到 $X(x) = A\cos kx + B\sin kx,\ k \neq 0$.

由 $X(0) = A = 0$，此时 $X(x) = B\sin kx$；由 $X(l_1) = B\sin kl_1 = 0$，欲求得非零解则 $B \neq 0$，故 $k_n = \frac{n\pi}{l_1}$，$n = 1,2,\cdots$

因此关于 $X(x)$ 的固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l_1}\right)^2$，固有函数为 $X_n(x) = \sin\frac{n\pi x}{l_1}, n \in \mathbb{N}$.

同理可得，关于 $Y(y)$ 的固有值为 $\mu_n = \left(\frac{n\pi}{l_2}\right)^2$，固有函数为 $Y_n(y) = \sin\frac{n\pi y}{l_2}, n \in \mathbb{N}$.

随后解关于 $T(t)$ 的 ODE：$T'(t) + a^2(\lambda+\mu)T(t) = 0$，得到 $T_{mn}(t) = e^{-a^2(\lambda_m+\mu_n)t} = e^{-\left[(\frac{m\pi}{l_1})^2+(\frac{n\pi}{l_2})^2\right]a^2t}, m,n \in \mathbb{N}$.

叠加得到：$u(t,x,y) = \sum\limits_{m=1}^{+\infty}\sum\limits_{n=1}^{+\infty} C_{mn}e^{-\left[(\frac{m\pi}{l_1})^2+(\frac{n\pi}{l_2})^2\right]a^2t}\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}$.

代入初始条件得：$u(0,x,y) = \sum\limits_{m=1}^{+\infty}\sum\limits_{n=1}^{+\infty} C_{mn}\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2} = \varphi(x,y)$，可知系数 C_{mn} 为：

$$C_{mn} = \frac{4}{l_1l_2}\int_0^{l_1}\int_0^{l_2}\varphi(x,y)\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}\mathrm{d}y\mathrm{d}x$$

那么本题解为：$u(t,x,y) = \sum\limits_{m=1}^{+\infty}\sum\limits_{n=1}^{+\infty} C_{mn}e^{-\left[(\frac{m\pi}{l_1})^2+(\frac{n\pi}{l_2})^2\right]a^2t}\sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}$，系数 C_{mn} 如上所述.

13 写在最后

本答案仅基于本人对数理方程的粗略理解，为方便同学们复习及纠正自己的答案而编写，很多题目也仅用了一种方法，仅提供参考，具体到对每个同学的意见，在批改作业的过程中也已经写了批注。

此外，考虑到本参考答案可能会存在一定的错误，后期可能会有修正版，可以**点击此处**查看最新版，对这些可能存在的错误，还请同学们海涵。

2020-2021 春季学期数理方程 B 助教

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2021 年 4 月 于合肥

数理方程B第三章参考答案

1. 解：

由

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0$$

设

$$u(r,\theta,z)=R(r)\Theta(\theta)Z(z)$$

则

$$R''(r)\Theta(\theta)Z(z)+\frac{1}{r}\ R'(r)\Theta(\theta)Z(z)+\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)+R(r)\Theta(\theta)Z''(z)=0$$

两边同时除以$R(r)\Theta(\theta)Z(z)$得

$$\frac{R''(r)}{R(r)}+\frac{1}{r}\ \frac{R'(r)}{R(r)}+\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}+\frac{Z''(z)}{Z(z)}=0$$

设

$$\begin{cases}\dfrac{Z''(z)}{Z(z)}=\lambda\\ \dfrac{\Theta''(\theta)}{\Theta(\theta)}=-m^2\end{cases}$$

则

$$\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}=-\lambda+\frac{m^2}{r^2}$$

所以

$$\begin{cases}Z''(z)-\lambda Z(z)=0\\ \Theta''(\theta)+m^2\Theta(\theta)=0\\ r^2R''(r)+rR'(r)+(\lambda r^2-m^2)R(r)=0\end{cases}$$

2. 解：

(1)

$$\frac{d}{dx}J_0(ax)\overset{ax=t}{\Longrightarrow}\frac{d}{dx}J_0(ax)=\frac{d}{dt}J_0(t)\frac{dt}{dx}=a\frac{d}{dt}J_0(t)=aJ_{-1}(t)=-aJ_1(t)=-aJ_i(ax)$$

(2)

$$\frac{d}{dx}[xJ_1(ax)]\overset{ax=t}{\Longrightarrow}\frac{d}{dt}\left[\frac{t}{a}J_1(t)\right]\frac{dt}{dx}=\frac{d}{dt}[tJ_1(t)]=tJ_0(t)=axJ_0(ax)$$

3. 解：

$$\begin{aligned}\int_0^{\frac{\pi}{2}}J_0(xcos\theta)cos\theta d\theta&=\int_0^{\frac{\pi}{2}}\sum_{k=0}^{+\infty}(-1)^k\frac{1}{k!\,\Gamma(k+1)}\left(\frac{xcos\theta}{2}\right)^{2k}cos\theta d\theta\\
&=\int_0^{\frac{\pi}{2}}\sum_{k=0}^{+\infty}(-1)^k\frac{1}{k!^2}\left(\frac{x}{2}\right)^{2k}cos^{2k+1}\theta d\theta\\
&=\sum_{k=0}^{+\infty}(-1)^k\frac{1}{k!^2}\left(\frac{x}{2}\right)^{2k}\int_0^{\frac{\pi}{2}}cos^{2k+1}\theta d\theta=\sum_{k=0}^{+\infty}(-1)^k\frac{1}{k!^2}\left(\frac{x}{2}\right)^{2k}\frac{(2k)!!}{(2k+1)!!}=\sum_{k=0}^{+\infty}(-1)^k\frac{1}{k!^2}\left(\frac{x}{2}\right)^{2k}\frac{2^kk!}{(2k+1)!!}\\
&=\sum_{k=0}^{+\infty}(-1)^k\frac{1}{k!}\frac{x^{2k}}{2^k}\frac{1}{(2k+1)!!}=\sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k}}{(2k)!!}\frac{1}{(2k+1)!!}\\
&=\sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k}}{(2k+1)!}=\frac{1}{x}\sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=\frac{sinx}{x}\end{aligned}$$

4. 解

$$\frac{d}{dx}\sqrt{x}J_{\frac{3}{2}}(x)=\frac{d}{dx}x^{-1}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right]=-x^{-2}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right]+x^{-1}\frac{d}{dx}x^{\frac{3}{2}}J_{\frac{3}{2}}(x)=-x^{-\frac{1}{2}}J_{\frac{3}{2}}(x)+x^{-1}\left[x^{\frac{3}{2}}J_{\frac{1}{2}}(x)\right]=-x^{-\frac{1}{2}}J_{\frac{3}{2}}(x)+x^{\frac{1}{2}}J_{\frac{1}{2}}(x)$$

$$\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x)=\frac{d}{dx}-x^{-\frac{1}{2}}J_{\frac{3}{2}}(x)+\frac{d}{dx}x^{\frac{1}{2}}J_{\frac{1}{2}}(x)=\frac{d}{dx}x^{-2}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right]+x^{\frac{1}{2}}J_{-\frac{1}{2}}(x)$$

$$\frac{d}{dx}-x^{-2}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right]=2x^{-3}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right]+(-x^{-2})\frac{d}{dx}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right]=2x^{-\frac{3}{2}}J_{\frac{3}{2}}(x)-x^{-\frac{1}{2}}J_{\frac{1}{2}}(x)$$

所以

$$\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x)=2x^{-\frac{3}{2}}J_{\frac{3}{2}}(x)-x^{-\frac{1}{2}}J_{\frac{1}{2}}(x)+x^{\frac{1}{2}}J_{-\frac{1}{2}}(x)$$

$$x^2\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x)=2x^{\frac{1}{2}}J_{\frac{3}{2}}(x)-x^{\frac{3}{2}}J_{\frac{1}{2}}(x)+x^{\frac{5}{2}}J_{-\frac{1}{2}}(x)$$

$$(x^2-2)\sqrt{x}J_{\frac{3}{2}}(x)=x^{\frac{5}{2}}J_{\frac{3}{2}}(x)-2x^{\frac{1}{2}}J_{\frac{3}{2}}(x)$$

$$x^2\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x)+(x^2-2)\sqrt{x}J_{\frac{3}{2}}(x)=x^{\frac{5}{2}}J_{\frac{3}{2}}(x)-x^{\frac{3}{2}}J_{\frac{1}{2}}(x)+x^{\frac{5}{2}}J_{-\frac{1}{2}}(x)$$

由递推公式

$$J_{v-1}(x)+J_{v+1}(x)=\frac{2v}{x}J_v(x)$$

取

$$v=\frac{1}{2}$$

则有

$$J_{-\frac{1}{2}}(x)+J_{\frac{3}{2}}(x)=\frac{1}{x}J_{\frac{1}{2}}(x)$$

两边同乘以$x^{\frac{5}{2}}$,得

$$x^{\frac{5}{2}}J_{-\frac{1}{2}}(x)+x^{\frac{5}{2}}J_{\frac{3}{2}}(x)=x^{\frac{3}{2}}J_{\frac{1}{2}}(x)\Rightarrow x^{\frac{5}{2}}J_{-\frac{1}{2}}(x)-x^{\frac{3}{2}}J_{\frac{1}{2}}(x)=-x^{\frac{5}{2}}J_{\frac{3}{2}}(x)$$

故

$$x^2\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x)+(x^2-2)\sqrt{x}J_{\frac{3}{2}}(x)=0$$

5. 解:

由例 1 可得等式

$$e^{ixsin\theta}=J_0(x)+2\sum_{k=1}^{+\infty}J_{2k}(x)cos2k\theta+2i\sum_{k=1}^{+\infty}J_{2k-1}(x)\sin(2k-1)\theta$$

(1) 令$\theta=0$,则

$$1=J_0(x)+2\sum_{k=1}^{+\infty}J_{2k}(x)$$

(2) 令$\theta=\frac{\pi}{2}$,则

$$e^{ix}=J_0(x)+2\sum_{k=1}^{+\infty}(-1)^kJ_{2k}(x)+2i\sum_{k=1}^{+\infty}(-1)^{k-1}J_{2k-1}(x)$$

令$\theta=-\frac{\pi}{2}$,则

$$e^{-ix}=J_0(x)+2\sum_{k=1}^{+\infty}(-1)^kJ_{2k}(x)-2i\sum_{k=1}^{+\infty}(-1)^{k-1}J_{2k-1}(x)$$

所以

$$sinx=\frac{e^{ix}-e^{-ix}}{2i}=2\sum_{k=1}^{+\infty}(-1)^{k-1}J_{2k-1}(x)=2\sum_{k=0}^{+\infty}(-1)^kJ_{2k+1}(x)$$

(3) 由(2)得

$$cosx=\frac{e^{ix}+e^{-ix}}{2}=J_0(x)+2\sum_{k=1}^{+\infty}(-1)^kJ_{2k}(x)$$

6. 解：

递推公式为

$$J_v'(x)=J_{v-1}(x)-\frac{v}{x}J_v(x)$$
$$J_v'(x)=\frac{v}{x}J_v(x)-J_{v+1}(x)$$

$$J_{v-1}(x)+J_{v+1}(x)=\frac{2v}{x}J_v(x)$$

$$J_{v-1}(x)-J_{v+1}(x)=2J_v'(x)$$

(1) 在第二个递推公式中取$v=0$，则

$$J_0'(x)=-J_1(x)$$

$$J_0''(x)=-J_1'(x)\xRightarrow{\text{第二个递推公式}}J_0''(x)=J_2(x)-\frac{1}{x}J_1(x)=J_2(x)+\frac{1}{x}J_0'(x)\Rightarrow J_2(x)=J_0''(x)-\frac{1}{x}J_0'(x)$$

(2) 直接证明比较困难，我们可以考虑执果索因

由要证的

$$J_3(x)+3J_0'(x)+4J_0^{(3)}(x)=0$$

观察这个等式我们发现其各项系数均为常数，对比四个递推公式我们首先从第四个入手，取$v=2$，得

$$J_1(x)-J_3(x)=2J_2'(x)$$

即

$$J_1(x)-J_3(x)=2J_2'(x)$$
$$J_0'(x)=-J_1(x)$$

得

$$J_3(x)+J_0'(x)+2J_2'(x)=0$$

对比其与我们要证的式子，即可知需证

$$J_2'(x)-J_0'(x)=2J_0^{(3)}(x)$$

再从第三个式子入手，取$v=1$，得

$$J_0(x)-J_2(x)=2J_1'(x)=-2J_0''(x)\Rightarrow J_2(x)-J_0(x)=2J_0''(x)$$

两边求导，得

$$J_2'(x)-J_0'(x)=2J_0^{(3)}(x)$$

综上可知原式得证

7. 解：(1)

$$\frac{d}{dx}[J_v^2(x)]=2J_v(x)J_v'(x)\xRightarrow{\text{递推公式4}}J_v(x)[J_{v-1}(x)-J_{v+1}(x)]\xRightarrow{\text{递推公式3}}\frac{x}{2v}[J_{v-1}(x)+J_{v+1}(x)][J_{v-1}(x)-J_{v+1}(x)]$$
$$=\frac{x}{2v}[J_{v-1}^2(x)-J_{v+1}^2(x)]$$

(2)

$$\frac{d}{dx}[xJ_0(x)J_1(x)]=J_0'(x)xJ_1(x)+J_0(x)\frac{d}{dx}[xJ_1(x)]=-xJ_1^2(x)+J_0(x)xJ_0(x)=x[J_0^2(x)-J_1^2(x)]$$

8. 解：

$$\int_0^x x^nJ_0(x)dx=\int_0^x x^{n-1}xJ_0(x)dx=\int_0^x x^{n-1}dxJ_1(x)=x^nJ_1(x)-(n-1)\int_0^x x^{n-1}J_1(x)dx=x^nJ_1(x)+(n-1)\int_0^x x^{n-1}J_0'(x)dx$$

$$=x^nJ_1(x)+(n-1)\int_0^x x^{n-1}dJ_0(x)=x^nJ_1(x)+(n-1)x^{n-1}J_0(x)-(n-1)^2\int_0^x x^{n-2}J_0(x)dx$$

(1)

$$\int_0^x x^3 J_0(x)dx = x^3J_1(x) + 2x^2J_0(x) - 4\int_0^x xJ_0(x)dx = x^3J_1(x) + 2x^2J_0(x) - 4xJ_1(x) = 2x^2J_0(x) + (x^3-4x)J_1(x)$$

(2)

$$\int_0^x x^4 J_1(x)dx = -\int_0^x x^4 J_0'(x)dx = -\int_0^x x^4 dJ_0(x) = 4\int_0^x x^3J_0(x)dx - x^4J_0(x) = (8x^2 - x^4)J_0(x) + 4(x^3-4x)J_1(x)$$

9. 解：由递推公式

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

$$\int J_3(x)dx = \int J_1(x) - 2J_2'(x)dx = \int J_1(x)dx - J_2(x) = \int -J_0'(x)dx - 2J_2(x)$$

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_v(x) \Rightarrow J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

所以

$$\int J_3(x)dx = -J_0(x) - 2\left[\frac{2}{x}J_1(x) - J_0(x)\right] + C = J_0(x) - 4x^{-1}J_1(x) + C$$

10. 解：(1)

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

$$\int_0^x x^2J_2(x)dx = \int_0^x 2xJ_1(x) - x^2J_0(x)dx$$

$$\int_0^x xJ_1(x)dx = -\int_0^x xJ_0'(x)dx = -\int_0^x xdJ_0(x) = \int_0^x J_0(x)dx - xJ_0(x)$$

$$\int_0^x x^2J_0(x)dx = \int_0^x xdxJ_1(x) = x^2J_1(x) - \int_0^x xJ_1(x)dx$$

所以

$$\int_0^x x^2J_2(x)dx = \int_0^x 2xJ_1(x) - x^2J_0(x)dx = -x^2J_1(x) + 3\int_0^x xJ_1(x)dx$$

(2)

$$J_1(x) = -J_0{}'(x)$$

$$\int_0^x xJ_1(x)dx = -\int_0^x xJ_0'(x)dx = -\int_0^x xdJ_0(x) = -xJ_0(x) + \int_0^x J_0(x)dx$$

11. 解：(1)

$$\int J_0(x)sinxdx = xJ_0(x)sinx - \int xdJ_0(x)sinx$$

$$\int xdJ_0(x)sinx = \int x[J_0'(x)sinx + J_0(x)cosx]\,dx$$

$$= \int xJ_0'(x)sinxdx + \int xJ_0(x)cosxdx = -\int xJ_1(x)sinxdx + \int cosxdxJ_1(x) = xJ_1(x)cosx + C$$

$$\int xJ_1(x)sinxdx = -\int xJ_1(x)dcosx = -xJ_1(x)cosx + \int cosxdxJ_1(x)$$

所以

$$\int J_0(x)sinxdx = xJ_0(x)sinx - xJ_1(x)cosx + C$$

(2)

$$\int J_0(x)cosxdx = xJ_0(x)cosx - \int xdJ_0(x)cosx$$

$$\int xdJ_0(x)cosx = \int x[J_0'(x)cosx - J_0(x)sinx]\,dx$$

$$= \int xJ_0'(x)cosxdx - \int xJ_0(x)sinxdx = -\int xJ_1(x)cosxdx - \int sinxdxJ_1(x) = -xJ_1(x)sinx + C$$

$$\int xJ_1(x)cosxdx = \int xJ_1(x)dsinx = xJ_1(x)sinx - \int sinxdxJ_1(x)$$

12. 解：

$$f(\mathrm{x}) = \sum_{n=1}^{+\infty} A_n J_0\,(\omega_n x)$$

其中

$$A_n = \frac{\langle f(x), J_0(\omega_n x)\rangle}{\langle J_0(\omega_n x), J_0(\omega_n x)\rangle} = \frac{\int_0^2 xf(x)J_0(\omega_n x)dx}{{N_{01}}^2} = \frac{\int_0^1 xJ_0(\omega_n x)dx}{\frac{2^2}{2}J_1^2(2\omega_n)}$$

$$\int_0^1 xJ_0(\omega_n x)dx \xRightarrow{\omega_n x = t} \frac{1}{{\omega_n}^2}\int_0^{\omega_n} tJ_0(t)dt = \frac{1}{{\omega_n}^2}\omega_n J_1(\omega_n) = \frac{J_1(\omega_n)}{\omega_n}$$

$$A_n = \frac{J_1(\omega_n)}{2\omega_n J_1^2(2\omega_n)}$$

所以

$$f(x) = \sum_{n=1}^{+\infty} A_n J_0\,(\omega_n x) = \sum_{n=1}^{+\infty} \frac{J_1(\omega_n)}{2\omega_n J_1^2(2\omega_n)}(\omega_n x)$$

13. 解：

$$f(x) = \sum_{n=1}^{+\infty} A_n J_1\,(\omega_n x)$$

其中

$$A_n = \frac{\langle f(x), J_1(\omega_n x)\rangle}{\langle J_1(\omega_n x), J_1(\omega_n x)\rangle} = \frac{\int_0^1 xf(x)J_1(\omega_n x)dx}{{N_{11}}^2} = \frac{\int_0^1 x^2 J_1(\omega_n x)dx}{\frac{1^2}{2}J_2^2(\omega_n)}$$

$$\int_0^1 x^2 J_1(\omega_n x)dx \xRightarrow{\omega_n x = t} \frac{1}{{\omega_n}^3}\int_0^{\omega_n} t^2 J_1(t)dt = \frac{1}{{\omega_n}^3}{\omega_n}^2 J_2(\omega_n) = \frac{J_2(\omega_n)}{\omega_n}$$

$$A_n = \frac{2}{\omega_n J_2(\omega_n)}$$

所以

$$f(x) = \sum_{n=1}^{+\infty} A_n J_1\,(\omega_n x) = f(x) = \sum_{n=1}^{+\infty} \frac{2}{\omega_n J_2(\omega_n)} J_1\,(\omega_n x)$$

14. 解：依题

$$\int_0^1 xf^2(x)dx = \int_0^1 x\left[\sum_{n=1}^{+\infty} A_n J_0(\omega_n x)\right]^2 dx$$

$$= \int_0^1 x\left[\sum_{n=1}^{+\infty} A_n^2 J_0^2(\omega_n x) + 2\sum_{n=2}^{+\infty} A_1(A_2+\cdots A_n)J_0(\omega_1 x)J_0(\omega_n x)\right.$$

$$\left.+2\sum_{n=3}^{+\infty} A_2(A_3+\cdots A_n)J_0(\omega_2 x)J_0(\omega_n x)+\cdots 2A_{n-1}A_n J_0(\omega_{n-1}x)J_0(\omega_n x)\right]dx$$

$$= \sum_{n=1}^{+\infty} A_n^2\int_0^1 xJ_0^2(\omega_n x)dx + 2\sum_{n=2}^{+\infty} A_1(A_2+\cdots A_n)\int_0^1 xJ_0(\omega_1 x)J_0(\omega_n x)dx$$

$$+\cdots 2A_{n-1}A_n\int_0^1 J_0(\omega_{n-1}x)J_0(\omega_n x)dx$$

依题利用内积的性质

$$\langle J_0(\omega_n x),J_0(\omega_n x)\rangle = N_{01}^2 = \frac{1^2}{2}J_1^2(\omega_n) = \frac{1}{2}J_1^2(\omega_n)$$

$$\langle J_0(\omega_n x),J_0(\omega_m x)\rangle = 0, n\neq m$$

故

$$\int_0^1 xf^2(x)dx = \sum_{n=1}^{+\infty} A_n^2\int_0^1 xJ_0^2(\omega_n x)dx = \frac{1}{2}\sum_{n=1}^{+\infty} A_n^2 J_1^2(\omega_n)$$

15. 解：

$$1 = \sum_{n=1}^{+\infty}\frac{2}{\omega_n J_1(\omega_n)}J_0(\omega_n x) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n x)$$

对比系数得

$$A_n = \frac{2}{\omega_n J_1(\omega_n)}, n = 1,2,\cdots$$

令$f(x)=1$

$$\frac{1}{2} = \int_0^1 xdx = \frac{1}{2}\sum_{n=1}^{+\infty}\frac{4}{\omega_n^2 J_1^2(\omega_n)}J_1^2(\omega_n) = 2\sum_{n=1}^{+\infty}\frac{1}{\omega_n^2} \Rightarrow \sum_{n=1}^{+\infty}\frac{1}{\omega_n^2} = \frac{1}{4}$$

16. 解:先说分析思路

依题,设温度函数为$u(t,r)$

所以

$$\begin{cases} u_t = u_{rr} + \dfrac{1}{r}u_r \quad (t>0, 0<r<R) \\ u(t,R) = u_0, u(0,r) = 0 \end{cases}$$

解决此类问题，当然要选择适当的边界条件，因此如何选择适当的边界条件成为至关重要的问题。首先先无脑进行分离变量

$$u(t,r) = T(t)R(r)$$

$$\begin{cases} T'(t)R(r) = T(t)R''(r) + \dfrac{1}{r}T(t)R'(r) \\ T(t)R(R) = u_0, T(0)R(r) = 0 \end{cases}$$

即

$$\begin{cases} \dfrac{T'(t)}{T(T)} = \dfrac{R''(r)}{R(r)} + \dfrac{1}{rR(r)}R'(r) = \dfrac{rR''(r)+R'(r)}{rR(r)} = -\lambda \\ T(t)R(R) = 0, T(0)R(r) = 0 \end{cases}$$

则有

$$\begin{cases} T'(t) + \lambda T(t) = 0 \\ r^2R''(r) + rR(r) + \lambda r^2 = 0 \\ T(t)R(R) = 0, T(0)R(r) = 0 \end{cases}$$

得到了一个 0 阶贝塞尔方程,考虑贝塞尔当方程的边界条件

$$\begin{cases}\{x^2y''+xy'+(\lambda x^2-\nu^2)y=0,(0<x<a,\nu\geq 0)\\ \alpha y(a)+\beta y'(a)=0\\ y(0)\text{有界}\end{cases}$$

考虑到本题中的条件

$$T(t)R(R)=u_0,T(0)R(r)=0$$

则需要“调整”为,此题中$R(0)$有界（物理意义上为初始温度不可能为无穷大

$$R(R)=0$$

设

$$u(t,r)=v(t,r)+w(t,r)$$

$v(t,r)$满足分离变量之后的贝塞尔方程

$$\begin{cases}v_t=v_{rr}+\frac{1}{r}v_r\quad(t>0,0<r<R)\\ v(t,R)=0,u(0,r)=?\end{cases}$$

$$\begin{cases}w_t=w_{rr}+\frac{1}{r}w_r\quad(t>0,0<r<R)\\ w(t,R)=u_0,w(0,r)=?\end{cases}$$

那么如何解决两个？条件的匹配呢，可以从找特解要尽可能往简单找的思路考虑。之前分析的$v(t,r)$满足分离变量之后的贝塞尔方程，故其对应齐次方程。则$w(t,r)$为特解。

$$w(t,R)=u_0$$

则可取$w(t,r)=u_0$，

$$\begin{cases}w_t=0\quad(t>0,0<r<R)\\ w(t,R)=u_0,w(0,r)=u_0\end{cases}$$

那么

$$\begin{cases}v_t=v_{rr}+\frac{1}{r}v_r\quad(t>0,0<r<R)\\ v(t,R)=0,v(0,r)=-u_0\end{cases}$$

$v(t,r)$的解决：

$$v(t,r)=T(t)R(r)$$

$$\begin{cases}T'(t)R(r)=T(t)R''(r)+\frac{1}{r}T(t)R'(r)\\ T(t)R(R)=0,T(0)R(r)=-u_0\end{cases}$$

即

$$\begin{cases}\frac{T'(t)}{T(T)}=\frac{R''(r)}{R(r)}+\frac{1}{rR(r)}R'(r)=\frac{rR''(r)+R'(r)}{rR(r)}=-\lambda\\ T(t)R(R)=0,T(0)R(r)=0\end{cases}$$

则有

$$\begin{cases}T'(t)+\lambda T(t)=0\\ r^2R''(r)+rR(r)+\lambda r^2=0\\ R(R)=0,T(0)R(r)=-u_0\end{cases}$$

下面来解决$R(r)$对应的贝塞尔方程：

$$\begin{cases}r^2R''(r)+rR(r)+\lambda r^2=0\\ R(R)=0,R(0)\text{有界}\end{cases}$$

此为一零阶贝塞尔方程,且边界条件为第一类

则其固有值为

$$\lambda_n={\omega_n}^2,n=1,2,\cdots$$

固有函数为

$$R_n(x)=J_0(\omega_n r),n=1,2,\cdots$$

其中ω_n为$J_0(\omega R)=0$的正实根

故

$$T_n{}'(t)+\lambda_n T_n(t)=0$$

得

$$T_n(t) = e^{-\omega_n{}^2 t}$$

所以

$$v(t,r) = \sum_{n=1}^{+\infty} A_n e^{-\omega_n{}^2 t} J_0(\omega_n r)$$

又

$$v(0,r) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n x) = -u_0$$

所以

$$A_n = \frac{\langle J_0(\omega_n r), -u_0\rangle}{\langle J_0(\omega_n r), J_0(\omega_n r)\rangle} = \frac{-u_0 \int_0^R r J_0(\omega_n r) dx}{N_{01}{}^2} = -u_0 \frac{\frac{1}{\omega_n^2}\int_0^{\omega_n R} t J_0(t) dt}{\frac{R^2}{2} J_1{}^2(\omega_n R)} = -u_0 \frac{2}{R\omega_n J_1(\omega_n R)}$$

所以

$$v(t,r) = -2u_0 \frac{1}{R} \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n R)} e^{-\omega_n{}^2 t} J_0(\omega_n r)$$

故

$$u(t,r) = v(t,r) + w(t,r) = u_0 - 2u_0 \frac{1}{R} \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n R)} e^{-\omega_n{}^2 t} J_0(\omega_n r)$$

参考答案实际是对$v(t,r)$作了一个处理

设

$$x = \frac{r}{R}$$

则$v(t,r) = v(t,x)$

$$\begin{cases} v_t = v_{rr} + \frac{1}{r} v_r \quad (t > 0, 0 < r < R) \\ v(t,R) = 0, u(0,r) = -u_0 \end{cases}$$

变为

$$\begin{cases} v_t = \frac{1}{R^2} v_{xx} + \frac{1}{R^2 x} v_x \quad (t > 0, 0 < x < 1) \\ v(t,1) = 0, v(0,x) = -u_0 \end{cases}$$

$v(t,r)$的解决：

$$v(t,x) = T(t)R(x)$$

$$\begin{cases} T'(t)R(x) = \frac{1}{R^2} T(t)R''(x) + \frac{1}{R^2 x} T(t)R'(x) \\ T(t)R(1) = 0, T(0)R(x) = -u_0 \end{cases}$$

即

$$\begin{cases} R^2 \frac{T'(t)}{T(T)} = \frac{R''(x)}{R(x)} + \frac{1}{xR(x)} R'(x) = \frac{xR''(x) + R'(x)}{xR(x)} = -\lambda \\ T(t)R(1) = 0, T(0)R(x) = 0 \end{cases}$$

则有

$$\begin{cases} T'(t) + \frac{\lambda}{R^2} T(t) = 0 \\ x^2 R''(x) + xR(x) + \lambda x^2 = 0 \\ R(1) = 0, T(0)R(x) = -u_0 \end{cases}$$

下面来解决$R(r)$对应的贝塞尔方程：

$$\begin{cases} x^2 R''(x) + xR(x) + \lambda x^2 = 0 \\ R(1) = 0, R(0)\text{有界} \end{cases}$$

此为一零阶贝塞尔方程，且边界条件为第一类

则其固有值为

$$\lambda_n = \omega_n{}^2, n = 1,2,\cdots$$

固有函数为

$$R_n(x) = J_0(\omega_n x), n = 1,2,\cdots$$

其中ω_n为$J_0(\omega) = 0$的正实根

故

$$T_n{}'(t) + \frac{\lambda_n}{R^2} T_n(t) = 0$$

得

$$T_n(t) = e^{-\frac{\omega_n{}^2}{R^2}t}$$

所以

$$v(t,r) = \sum_{n=1}^{+\infty} A_n e^{-\frac{\omega_n{}^2}{R^2}t} J_0(\omega_n x)$$

又

$$v(0,x) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n x) = -u_0$$

所以

$$A_n = \frac{\langle J_0(\omega_n x), -u_0 \rangle}{\langle J_0(\omega_n), J_0(\omega_n x) \rangle} = \frac{-u_0 \int_0^1 x J_0(\omega_n x) dx}{N_{01}{}^2} = -u_0 \frac{\frac{1}{\omega_n^2} \int_0^{\omega_n} t J_0(t) dt}{\frac{1}{2} J_1{}^2(\omega_n)} = -u_0 \frac{2}{\omega_n J_1(\omega_n)}$$

所以

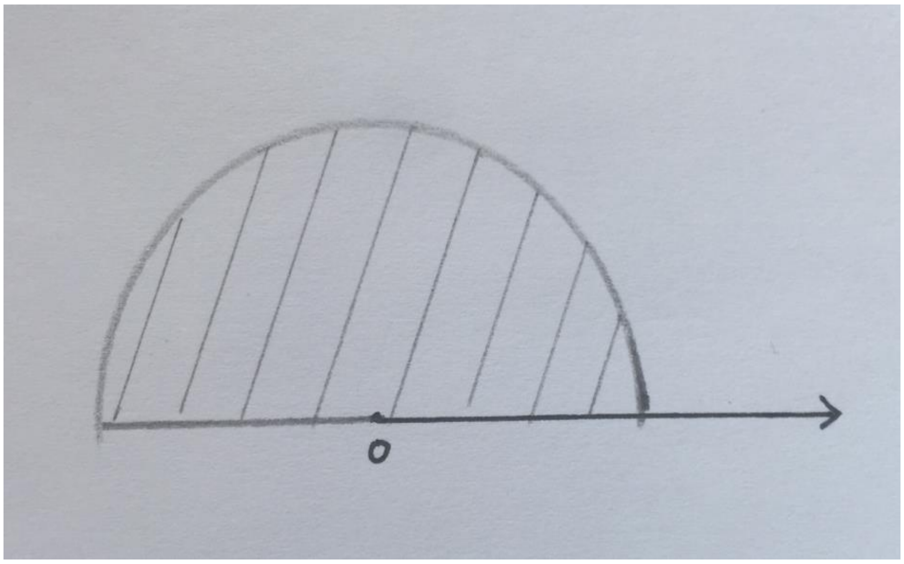
$$v(t,r) = -2u_0 \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n)} e^{-\omega_n{}^2 t} J_0(\omega_n x)$$

故

$$u(t,r) = v(t,r) + w(t,r) = u_0 - 2u_0 \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n)} e^{-\frac{\omega_n{}^2}{R^2}t} J_0(\omega_n x) = u_0 - 2u_0 \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n)} e^{-\frac{\omega_n{}^2}{R^2}t} J_0\left(\omega_n \frac{r}{R}\right)$$

17. 解：边缘固定指的是边缘的曲线部分和直线部分都无横向振动

考虑如图极坐标系



设该薄膜的横向振动函数为$u(t,r,\varphi)$

则根据题意列出方程及边界条件如下：

$$\begin{cases} u_{tt} = a^2 \Delta_2 u = a^2 (u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}) \\ u(t,r,0) = u(t,r,\pi) = 0 \\ u(t,0,\varphi) = u(t,R,\varphi) = 0 \\ (0 < r < R, t > 0, 0 \le \varphi \le \pi) \end{cases}$$

由分离变量法，设$u(t,r,\varphi) = T(t)R(r)\Phi(\varphi)$

则有

$$\begin{cases} T''(t)R(r)\Phi(\varphi) = a^2\Delta_2 u = a^2(T(t)R''(r)\Phi(\varphi) + \frac{1}{r}T(t)R'(r)\Phi(\varphi) + \frac{1}{r^2}T(t)R(r)\Phi''(\varphi)) \\ \Phi(0) = \Phi(\pi) = 0 \\ (0 < r < R, t > 0, 0 \le \varphi \le \pi) \end{cases}$$

即

$$\begin{cases} \frac{T''(t)}{a^2T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Phi''(\varphi)}{\Phi(\varphi)} \\ \Phi(0) = \Phi(\pi) = 0 \\ R(0) = R(b) = 0 \\ (0 < r < R, t > 0, 0 \le \varphi \le \pi) \end{cases}$$

设

$$\frac{\Phi''(\varphi)}{\Phi(\varphi)} = -m^2$$

由

$$\Phi(0) = \Phi(\pi) = 0$$

由第一类边界条件知其特征值大于 0

其特征值为

$$m, m = 1,2,\cdots$$

其特征函数为

$$\Phi_m(\varphi) = sinm\varphi, m = 1,2,\cdots$$

设

$$\frac{T''(t)}{a^2T(t)} = -k^2$$

则有

$$\begin{cases} r^2R''(r) + rR'(r) + (k^2r^2 - m^2)R(r) = 0 \\ R(R) = 0, R(0) = 0(\text{有界}) \end{cases}$$

则该方程的固有值为

$$k_n^2 = \omega_{mn}^2, m = 1,2,\cdots, n = 1,2,\cdots$$

则该方程的固有函数为

$$R_n(r) = J_m(\omega_{mn}r), m = 1,2,\cdots, n = 1,2,\cdots$$

其中ω_{mn}为方程$J_m(\omega b)$的所有正根$m = 1,2,\cdots$， $n = 1,2,\cdots$

所以

$$\frac{{T_n}''(t)}{a^2T_n(t)} = -\omega_{mn}^2$$

$T_n(t) = A_{mn}cos(\omega_{mn}t) + B_{mn}sin(\omega_{mn}t)$

所以

$$u(t, r, \varphi) = \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} J_m(\omega_{mn}r)\, sinm\varphi[A_{mn}cos(\omega_{mn}t) + B_{mn}sin(\omega_{mn}t)]$$

18. 解：

(1) 设$u(r, z) = R(r)Z(z)$

由分离变量法得

$$\begin{cases} R"(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z''(z) = 0 \Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{Z''(z)}{Z(z)} = 0 \\ R(a)Z(z) = 0 \\ R(r)Z(0) = 0, R(r)Z(l) = T_0 \end{cases}$$

设

$$\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}=-\lambda$$

得

$$\begin{cases} r^2R(r)+rR'(r)+\lambda r^2R(r)=0 \\ R(a)=0, R(0)\text{有界} \end{cases}$$

则其固有值为

$$\lambda=\omega_n^2, n=1,2,\cdots$$

其固有函数为

$$R_n(r)=J_0(\omega_n r), n=1,2,\cdots$$

其中ω_n为

$$J_0(\omega a)=0$$

的正根

所以

$$\begin{cases} \dfrac{Z_n''(z)}{Z_n(z)}-\omega_n^2=0 \\ Z(0)=0 \end{cases}$$

则

$$\begin{cases} Z_n(z)=A_ne^{-\omega_n z}+B_ne^{\omega_n z} \\ Z_n(0)=A_n+B_n=0 \Rightarrow A_n=-B_n \end{cases}$$

所以

$$Z_n(z)=C_n sh(\omega_n z)$$

所以

$$u(r,z)=\sum_{n=1}^{+\infty}C_n sh(\omega_n z)J_0(\omega_n r)$$

$$u(r,l)=\sum_{n=1}^{+\infty}C_n sh(\omega_n l)J_0(\omega_n r)=T_0$$

$$C_n sh(\omega_n l)=\frac{\langle T_0, J_0(\omega_n r)\rangle}{\langle J_0(\omega_n r), J_0(\omega_n r)\rangle}=\frac{\int_0^a T_0 rJ_0(\omega_n r)dr}{\dfrac{a^2}{2}{J_1}^2(\omega_n a)}=2T_0\frac{\int_0^{\omega_n a} tJ_0(t)dt}{a^2\omega_n^2{J_1}^2(\omega_n a)}=2T_0\frac{1}{\omega_n aJ_1(\omega_n a)}$$

所以

$$u(r,z)=\sum_{n=1}^{+\infty}C_n sh(\omega_n z)J_0(\omega_n r)=2T_0\sum_{n=1}^{+\infty}\frac{1}{\omega_n aJ_1(\omega_n a)}\frac{sh(\omega_n z)}{sh(\omega_n l)}J_0(\omega_n r)$$

其中ω_n为

$$J_0(\omega a)=0$$

的正根

(2) 设$u(r,t)=R(r)T(t)$

由分离变量法得

$$\begin{cases} T''(t)R(r)+2hT'(t)R(r)=a^2\left(T(t)R''(r)+\dfrac{1}{r}T(t)R'(r)\right) \\ R(0)T(t)\text{有限}, T(t)R'(l)=0 \\ R(r)T(0)=\varphi(r), R(r)T'(0)=0 \end{cases}$$

设

$$\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}=-\lambda$$

得

$$\begin{cases} r^2R(r)+rR'(r)+\lambda r^2R(r)=0 \\ R'(l)=0, R(0)\text{有界} \end{cases}$$

则其固有值为

$$\lambda=\begin{cases}0\\ \omega_n^2,n=1,2,\cdots\end{cases}$$

其固有函数为

$$R_n(r)=\begin{cases}C,C为常数\\ J_0(\omega_n r),n=1,2,\cdots\end{cases}$$

其中ω_n为

$$J'_0(\omega l)=0,即J_1(\omega l)=0$$

的正根，其中

则当$\lambda=0$时

$$\begin{cases}\dfrac{T_0''(t)+2hT_0'(t)}{a^2T_0(t)}=0\\ T_0'(0)=0\end{cases}$$

即

$$\begin{cases}T_0''(t)+2hT_0'(t)=0\\ T_0'(0)=0\end{cases}$$

得该方程对应的特征方程为$\alpha^2+2h\alpha=0$，特征根为$\alpha_1=-2h,\alpha_2=0$.

$$\begin{cases}T_0(t)=C_0+D_0e^{-h2t}\\ T_0'(0)=0\end{cases}$$

则当$\lambda=\omega_n^2,n=1,2,\cdots$时

$$D_0=0$$

$$\begin{cases}\dfrac{T_n''(t)+2hT_n'(t)}{a^2T_n(t)}=-\omega_n^2\\ T_n'(0)=0\end{cases}$$

即

$$\begin{cases}T_n''(t)+2hT_n'(t)+\omega_n^2a^2T_n(t)=0\\ T_n'(0)=0\end{cases}$$

得该方程对应的特征方程为$\alpha^2+2h\alpha+\omega_n^2a^2=0$，判别式为$\Delta=4h^2-4\omega_n^2a^2$.

题干中$h\ll1$，结合一阶贝塞尔函数零点及a的实际物理意义可知判别式小于0.（其实就是根据答案凑的说法）

则该特征方程解为$\alpha_1=-h+i\sqrt{(\omega_n a)^2-h^2},\alpha_2=-h-i\sqrt{(\omega_n a)^2-h^2}$

记$q_n=\sqrt{(\omega_n a)^2-h^2}$

则该方程解为

$$T_n(t)=e^{-ht}(C_n cosq_n t+D_n sinq_n t)$$

$T'_n(0)=0$得

$$-hC_n+q_nD_n=0\Rightarrow D_n=\frac{h}{q_n}$$

故原方程解为

$$u(r,t)=C_0+D_0e^{-h2t}\text{(这里本来要乘 C，但将 C 规划到两个待定系数显得答案比较美观)}$$
$$+\sum_{n=1}^{+\infty}e^{-ht}(C_n cosq_n t+D_n sinq_n t)J_0(\omega_n r)$$

再考虑 $u(r,0)=\varphi(r)$

$$u(r,0)=C_0+\sum_{n=1}^{+\infty}C_nJ_0(\omega_n r)=\sum_{n=0}^{+\infty}C_nJ_0(\omega_n r)=\varphi(r)$$

$$C_n=\frac{\langle\varphi(r),J_0(\omega_n r)\rangle}{\langle J_0(\omega_n r),J_0(\omega_n r)\rangle}=\frac{\int_0^l r\,\varphi(r)J_0(\omega_n r)dr}{\int_0^l rJ_0{}^2(\omega_n r)dr}$$

$$C_0=\frac{\int_0^l r\,\varphi(r)dr}{\int_0^l rdr}=\frac{2}{l^2}\int_0^l r\,\varphi(r)dr$$

$$C_n=\frac{\int_0^l r\,\varphi(r)J_0(\omega_n r)dr}{N_{01}^2}=\frac{\int_0^l r\,\varphi(r)J_0(\omega_n r)dr}{\frac{l^2}{2}J_1^2(\omega_n r)}=\frac{2}{l^2}\frac{\int_0^l r\,\varphi(r)J_0(\omega_n r)dr}{J_1^2(\omega_n r)},n=1,2,\cdots$$

所以

$$u(r,t)=C_0+D_0e^{-h2t}+\sum_{n=1}^{+\infty}e^{-ht}(C_n cosq_n t+D_n sinq_n t)J_0(\omega_n r)$$

其中

$$C_0=\frac{2}{l^2}\int_0^l r\,\varphi(r)dr$$

$$C_n=\frac{2}{l^2}\frac{\int_0^l r\,\varphi(r)J_0(\omega_n r)dr}{J_1^2(\omega_n r)},n=1,2,\cdots$$

$$D_0=0$$

$$D_n=\frac{h}{q_n}$$

19. 解：由分离变量法，设$u(r,z)=R(r)Z(z)$

$$\begin{cases}R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)=0\\R(0)Z(z)=\text{有限},R'(R)Z(z)+kR(R)Z(z)=0\\R(r)Z(0)=0,R(r)Z(h)=f(r)\end{cases}$$

即

$$\begin{cases}\frac{R''(r)}{R(r)}+\frac{R'(r)}{rR(r)}+\frac{Z''(z)}{Z(z)}=0\\R'(R)+kR(R)=0,R(0)\text{有限}\\Z(0)=0,R(r)Z(h)=f(r)\end{cases}$$

设

$$\frac{R''(r)}{R(r)}+\frac{R'(r)}{rR(r)}=-\lambda$$

关于$R(r)$的方程为

$$\begin{cases}r^2R''(r)+rR'(r)+\lambda r^2R(r)=0\\R'(R)+kR(R)=0,R(0)\text{有限}\end{cases}$$

其为 0 阶贝塞尔方程，且具有第三类边界条件 1+

故其固有值为

$$\lambda=\omega_n^2,n=1,2,\cdots$$

固有函数为

$$R_n(r)=J_0(\omega_n r),n=1,2,\cdots$$

其中ω_n为方程$\omega J_0{}'(\omega R)+kJ_0(\omega R)=0[-\omega J_1(\omega R)+kJ_0(\omega R)=0]$的正根.

所以

$$\begin{cases}Z_n''(z)-\omega_n^2Z_n(z)=0\\Z_n(0)=0\end{cases}$$

解得

$$Z_n(z)=A_n sh\omega_n z$$

故

$$u(r,z)=\sum_{n=1}^{+\infty}A_n sh\omega_n zJ_0(\omega_n r)$$

$$u(r,\mathrm{h})=\sum_{n=1}^{+\infty}A_n sh\omega_n hJ_0(\omega_n r)=f(r)$$

$$A_n sh\omega_n h=\frac{\langle f(r),J_0(\omega_n r)\rangle}{\langle J_0(\omega_n r),J_0(\omega_n r)\rangle}=\frac{\int_0^R rf(r)J_0(\omega_n r)dr}{N_{03}^2}=\frac{\int_0^R rf(r)J_0(\omega_n r)dr}{\frac{1}{2}\left[R^2+\left(\frac{R}{\omega_n k}\right)^2\right]J_0^2(\omega R)}=\frac{2}{R^2}\frac{\int_0^R rf(r)J_0(\omega_n r)dr}{\left(1+\frac{1}{\omega_n{}^2k^2}\right)J_0^2(\omega R)}$$

$$u(r,z)=\frac{2}{R^2}\sum_{n=1}^{+\infty}\frac{\int_0^R rf(r)J_0(\omega_n r)dr}{\left(1+\frac{1}{\omega_n{}^2k^2}\right)J_0^2(\omega R)}\frac{sh\omega_n z}{sh\omega_n h}J_0(\omega_n r)$$

20. 解:

由递推公式

$$(n+1)p_{n+1}(x)-x(2n+1)p_n(x)+np_{n-1}(x)=0,n\geq 1$$

及

$$p_0(x)=1$$

$$p_1(x)=x$$

代入$x=0$

$$(n+1)p_{n+1}(0)+np_{n-1}(0)=0,n\geq 1$$

即

$$(n+2)p_{n+2}(0)+(n+1)p_n(0)=0,n\geq 0$$

当$n=2k-2,k=1,2,\cdots$时

$$2kp_{2k}(0)+(2k-1)p_{2k-2}(0)=0$$

即

$$\frac{p_{2k}(0)}{p_{2k-2}(0)}=-\frac{(2k-1)}{2k}$$

$$p_{2k}(0)=\frac{p_{2k}(0)}{p_{2k-2}(0)}\frac{p_{2k-2}(0)}{p_{2k-4}(0)}\cdots\frac{p_2(0)}{p_0(0)}p_0(0)=\left[-\frac{(2k-1)}{2k}\right]\left[-\frac{(2k-3)}{(2k-2)}\right]\cdots\left[-\frac{1}{2}\right]\cdot 1=(-1)^k\frac{(2k-1)!!}{(2k)!!}$$

k由来:观察底下的$2,\cdots,2k$,共k项

当$n=2k-1,k=1,2,\cdots$时

$$(2k+1)p_{2k+1}(0)+2kp_{2k-1}(0)=0$$

$$p_1(x)=x\Rightarrow p_1(0)=0$$

$$k=1,3p_3(0)+2p_1(0)=0\Rightarrow p_3(0)=0$$

$$\cdots$$

所以

$$p_{2k-1}(0)=0$$

综上

$$p_n(0)=\begin{cases}1,n=0\\(-1)^k\frac{(2k-1)!!}{(2k)!!},n=2k\\0,n=2k-1\end{cases},k=1,2,\cdots$$

由递推公式

$$np_{n-1}(x)-p_n'(x)+xp_{n-1}{}'(x)=0$$

代入$x=0$

$$np_{n-1}(0)-p_n'(0)=0$$

$$p_n'(0)=np_{n-1}(0),n\geq 2$$

当$n=2k+1,k=0,1,2,\cdots$时

$$p_{2k+1}'(0)=(2k+1)p_{2k}(0)=(2k+1)(-1)^k\frac{(2k-1)!!}{(2k)!!}=(-1)^k\frac{(2k+1)!!}{(2k)!!}$$

当$n=2k,k=1,2,\cdots$时

$$p'_{2k}(0) = 2kp_{2k-1}(0) = 0$$
$$p_0(0) = 1 \Rightarrow p'_0(0) = 0$$

所以

$$p'_n(0) = \begin{cases} 0, n = 2k \\ (-1)^k \dfrac{(2k+1)!!}{(2k)!!}, n = 2k + `1' \end{cases}, k = 0,1,2,\cdots$$

21. 解：

由

$$p'_{n+1}(x) - p'_{n-1}(x) = (2n+1)p_n(x), n \geq 1$$

得

$$p'_n(x) - p'_{n-2}(x) = (2n-1)p_{n-1}(x), n \geq 2$$

所以

$$p'_{n-2}(x) - p'_{n-4}(x) = (2n-5)p_{n-3}(x), n \geq 3$$
$$p'_{n-4}(x) - p'_{n-6}(x) = (2n-9)p_{n-5}(x), n \geq 3$$
$$\cdots$$

等式相加

$$p'_n(x) = (2n-1)p_{n-1}(x) + (2n-5)p_{n-3}(x) + (2n-9)p_{n-5}(x) + \cdots, n \geq 2$$
$$p'_1(x) = 1 = (2 \times 1 - 1)p_0(x)$$

综上

$$p'_n(x) = (2n-1)p_{n-1}(x) + (2n-5)p_{n-3}(x) + (2n-9)p_{n-5}(x) + \cdots, n \geq 1$$

22. 解：(1)

$m < n$时，

$$x^m = \sum_{i=0}^{m} A_i p_i(x)$$

$$\int_{-1}^{1} x^m p_n(x)dx = \int_{-1}^{1} \left[\sum_{i=0}^{m} A_i p_i(x)\right] p_n(x)dx = \int_{-1}^{1} \sum_{i=0}^{m} A_i p_i(x) 1 p_n(x)dx = \sum_{i=0}^{m} A_i \int_{-1}^{1} p_i(x)p_n(x)dx$$

注意到$\int_{-1}^{1} p_i(x)p_n(x)dx = 0, i \neq n$

故

$$\int_{-1}^{1} x^m p_n(x)dx = 0$$

$m \geq n$时，

由

$$np_n(x) - xp'_n(x) + p'_{n-1}(x) = 0$$

$$n\int_{-1}^{1} x^m p_n(x)dx = \int_{-1}^{1} x^m[xp'_n(x) - p'_{n-1}(x)]dx = \int_{-1}^{1} x^{m+1} p'_n(x)dx - \int_{-1}^{1} x^m p'_{n-1}(x)dx$$

$$\int_{-1}^{1} x^{m+1} p'_n(x)dx = \int_{-1}^{1} x^{m+1} dp_n(x) = x^{m+1}p_n(x)|_{-1}^{1} - (m+1)\int_{-1}^{1} x^m p_n(x)dx = p_n(1) - (-1)^{m+1}p_n(-1)$$

$$-(m+1)\int_{-1}^{1} x^m p_n(x)dx = 1 - (-1)^{m+1+n} - (m+1)\int_{-1}^{1} x^m p_n(x)dx$$

$$\int_{-1}^{1} x^m p'_{n-1}(x)dx = \int_{-1}^{1} x^m dp_{n-1}(x) = x^m p_{n-1}(x)|_{-1}^{1} - m\int_{-1}^{1} x^{m-1}p_{n-1}(x)dx = 1 - (-1)^{m+n-1} - m\int_{-1}^{1} x^{m-1}p_{n-1}(x)dx$$

记

$$\int_{-1}^{1} x^m p_n(x)dx = f(m,n)$$

则

$$nf(m,n)=1-(-1)^{m+1+n}-(m+1)f(m,n)-[1-(-1)^{m+n-1}-mf(m-1,n-1)]$$
$$nf(m,n)=-(m+1)f(m,n)+mf(m-1,n-1)$$

$$f(m,n)=\frac{m}{m+n+1}f(m-1,n-1)=\frac{m}{m+n+1}\cdot\frac{m-1}{m+n-1}f(m-2,n-2)$$

$$=\frac{m}{m+n+1}\cdot\frac{m-1}{m+n-1}\frac{m-2}{m+n-3}f(m-3,n-3)$$

$$=\cdots\frac{m!}{(m-n)!}\frac{1}{\frac{(m+n+1)!!}{(m-n+1)!!}}\int_{-1}^{1}x^{m-n}dx=\frac{m!}{(m-n)!}\frac{(m-n+1)!!}{(m+n+1)!!}\frac{1+(-1)^{m-n}}{m-n+1}$$

$$=\frac{m!}{(m-n)!}\frac{(m-n-1)!!}{(m+n+1)!!}[1+(-1)^{m-n}]=\frac{m!}{(m-n)!!}\frac{1+(-1)^{m-n}}{(m+n+1)!!}$$

综上

$$\int_{-1}^{1}x^m p_n(x)dx=\begin{cases}0,m<n\\ \frac{m!}{(m-n)!!}\frac{1+(-1)^{m-n}}{(m+n+1)!!},m\geq n\end{cases}$$

(2)

$$\int_{-1}^{1}xp_m(x)p_n(x)dx$$

依题

$$xp_m(x)=\sum_{i=0}^{m+1}A_ip_i(x)$$

$$\int_{-1}^{1}xp_m(x)p_n(x)dx=\int_{-1}^{1}\left[\sum_{i=0}^{m+1}A_ip_i(x)\right]p_n(x)dx=\sum_{i=0}^{m+1}A_i\int_{-1}^{1}p_i(x)p_n(x)dx$$

所以，当$m+1<n$时，由正交性可知上式等于 0

同理，对于

$$xp_n(x)=\sum_{i=0}^{n+1}B_ip_i(x)$$

所以，当$n+1<m$时，由正交性可知上式等于 0

故只需讨论，$m-n=0,\pm1$

$m-n=-1$时

$$原式=\int_{-1}^{1}xp_{n-1}(x)p_n(x)dx$$

由递推公式（1）

$$np_n-x(2n-1)p_{n-1}(x)+(n-1)p_{n-2}(x)=0$$

得

$$xp_{n-1}(x)=\frac{n}{2n-1}p_n+\frac{n-1}{2n-1}p_{n-2}(x)$$

代入并由正交性：

$$原式=\frac{n}{2n-1}\|p_n\|^2=\frac{n}{2n-1}\frac{2}{2n+1}=\frac{2n}{4n^2-1}$$

$m-n=0$时

$$原式=\int_{-1}^{1}x{p_n}^2(x)dx$$

对于$f(x)=x{p_n}^2(x),f(-x)=-x{p_n}^2(-x)=-x[(-1)^2p_n(x)]^2=-x{p_n}^2(x)=-f(x)$

因此

$$原式 = 0$$

$m-n=1$时

$$原式 = \int_{-1}^{1} x p_n(x) p_{n+1}(x) dx$$

由递推公式（1）

$$(n+1)p_{n+1} - x(2n+1)p_n(x) + np_{n-1}(x) = 0$$

得

$$xp_n(x) = \frac{n+1}{2n+1} p_{n+1} + \frac{n}{2n+1} p_{n-1}(x)$$

代入并由正交性：

$$原式 = \frac{n+1}{2n+1} \|p_{n+1}\|^2 = \frac{n+1}{2n+1}\frac{2}{2n+3} = \frac{2(n+1)}{(2n+1)(2n+3)}$$

综上

$$原式 = \begin{cases} \frac{2n}{4n^2-1}, m-n=-1 \\ 0, m-n \neq \pm 1 \\ \frac{2(n+1)}{(2n+1)(2n+3)}, m-n=1 \end{cases}$$

23. 解：

考虑到

$$\frac{d}{dx}[(1-x^2)p_n{}'(x)] + n(n+1)p_n(x) = 0$$

$$\int_{-1}^{1} (1-x^2)[p_n{}'(x)]^2 dx = \int_{-1}^{1} (1-x^2) p_n{}'(x) dp_n(x) = (1-x^2)p_n'(x)p_n(x)|_{-1}^{1} - \int_{-1}^{1} p_n(x) d[(1-x^2)p_n'(x)]\, dx$$

$$= n(n+1)\int_{-1}^{1} p_n{}^2(x) dx = \frac{2n(n+1)}{2n+1}$$

24. 解：（1）

$$f(x) = x^3, -1 < x < 1$$

$$f(x) = C_3 p_3(x) + C_1 p_1(x)$$

$$p_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, p_1(x) = x$$

故

$$f(x) = \frac{2}{5}p_3(x) + \frac{3}{5}p_1(x)$$

(2)

$$f(x) = x^4, -1 < x < 1$$

$$f(x) = C_4 p_4(x) + C_2 p_2(x) + C_0 p_0(x)$$

$$p_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}, p_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, p_0(x) = 1$$

则

$$\begin{cases} \frac{35}{8}C_4 = 1 \\ \frac{3}{2}C_2 - \frac{15}{4}C_4 = 0 \\ C_0 - \frac{1}{2}C_2 + \frac{3}{8}C_4 = 0 \end{cases}$$

得

$$\begin{cases} C_0 = \frac{1}{5} \\ C_2 = \frac{4}{7} \\ C_4 = \frac{8}{35} \end{cases}$$

故

$$f(x) = \frac{8}{35}p_4(x) + \frac{4}{7}p_2(x) + \frac{1}{5}p_0(x)$$

(3)

$$f(x) = |x|, -1 < x < 1$$

$$f(x) = \sum_{n=0}^{+\infty} C_n p_n(x)$$

$$C_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}$$

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} |x|p_n(x)dx, n \geq 1$$

$$\int_{-1}^{1} |x|p_n(x)dx = \int_0^1 xp_n(x)dx - \int_{-1}^{0} xp_n(x)dx$$

$$\int_{-1}^{0} xp_n(x)dx \overset{x=-t}{\Longrightarrow} -\int_0^1 tp_n(-t)\,dt = (-1)^{n-1}\int_0^1 x\,p_n(x)dx$$

$$\int_{-1}^{1} |x|p_n(x)dx = [1+(-1)^n]\int_0^1 xp_n(x)dx = \begin{cases} 2\int_0^1 xp_{2k}(x)dx \quad n = 2k \\ 0, n = 2k-1 \end{cases}, k = 1,2,\cdots$$

由课本 P280 例 1

$$\int_0^1 xp_{2k}(x)dx = \frac{1}{2k+2}\int_0^1 p_{2k-1}(x)dx$$

由递推公式(6)

$$\int_0^1 p_{2k-1}(x)dx = \frac{1}{4k-1}[p_{2k-2}(0) - p_{2k}(0)] =$$

注意到

$$w(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{+\infty} p_n(x)t^n, |t| < 1$$

则

$$w(t,0) = \frac{1}{\sqrt{1+t^2}} = \sum_{n=0}^{+\infty} p_n(0)t^n, |t| < 1$$

$$\frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left[-\frac{1}{2}-(n-1)\right]}{n!}(t^2)^n = \sum_{n=0}^{+\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}t^{2n} = \sum_{n=0}^{+\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}t^{2n}$$

所以

$$p_n(0) = \begin{cases} 0, \ n = 2k+1 \\ (-1)^k\frac{(2k-1)!!}{(2k)!!}, n = 2k \end{cases} k = 0,1,2,\cdots$$

$$p_{2k}(0) = (-1)^k\frac{(2k-1)!!}{(2k)!!}$$

$$p_{2k-2}(0) = (-1)^{k-1}\frac{(2k-3)!!}{(2k-2)!}$$

$$p_{2k}(0) - p_{2k-2}(0) = (-1)^k\frac{(2k-1)(2k-3)!!}{(2k)!!} - (-1)^{k-1}\frac{2k(2k-3)!!}{2k(2k-2)!!} = (-1)^k\frac{(4k-1)(2k-3)!!}{(2k)!!}$$

所以

$$\int_0^1 xp_{2k}(x)dx = \frac{1}{2k+2}\int_0^1 p_{2k-1}(x)dx = \frac{1}{(2k+2)(4k-1)}[p_{2k-2}(0)-p_{2k}(0)] = (-1)^{k+1}\frac{(2k-3)!!}{(2k+2)!!}$$

$$C_{2k} = (4k+1)\int_0^1 xp_{2k}(x)dx = (-1)^k\frac{(2k-3)!!\,(4k+1)}{(2k+2)!!} = (-1)^{k+1}\frac{(4k+1)\frac{(2k-2)!}{(2k-2)!!}}{(2k+2)!!}$$

$$= (-1)^k\frac{(4k+1)(2k-2)!}{2^{k+1}(k+1)!\,2^{k-1}(k-1)!} = (-1)^{k+!}\frac{(4k+1)}{2^{2k}(k-1)!}\frac{(2k-2)!}{(k+1)!}$$

$$f(x) = \frac{1}{2} + \sum_{k=1}^{+\infty}(-1)^{k+1}\frac{(4k+1)}{2^{2k}(k-1)!}\frac{(2k-2)!}{(k+1)!}p_{2k}(x)$$

25. 解：

根据球坐标系下与方位角φ无关的 Laplace 方程通解（球内形式）

$$u(r,\theta) = \sum_{n=0}^{+\infty} A_n r^n p_n(cos\theta)$$

$$u(a,\theta) = \sum_{n=0}^{+\infty} A_n a^n p_n(cos\theta) = cos^2\theta = \frac{2}{3}p_2(cos\theta) + \frac{1}{3}p_0(cos\theta)$$

$$A_0 = \frac{1}{3}, A_2 = \frac{2}{3a^2}, A_n = 0(n \neq 0,2)$$

$$u(r,\theta) = \frac{1}{3} + \frac{2r^2}{3a^2}p_2(cos\theta) = \frac{1}{3} + \frac{2r^2}{3a^2}\left(\frac{3}{2}cos^2\theta - \frac{1}{2}\right) = \frac{r^2}{a^2}cos^2\theta - \frac{r^2}{3a^2} + \frac{1}{3}$$

26. 解：

根据球坐标系下与方位角φ无关的 Laplace 方程通解（球内形式）

$$u(r,\theta) = \sum_{n=0}^{+\infty} A_n r^n p_n(cos\theta)$$

$$u(1,\theta) = \sum_{n=0}^{+\infty} A_n p_n(cos\theta) = 3cos2\theta + 1 = 6cos^2\theta - 2 = 4p_2(cos\theta)$$

$$A_2 = 4, A_n = 0(n \neq 2)$$

$$u(r,\theta) = 4r^2p_2(cos\theta) = 4r^2\left(\frac{3}{2}cos^2\theta - \frac{1}{2}\right) == 2r^2(3cos^2\theta - 1)$$

27. 解：

根据球坐标系下与方位角φ无关的 Laplace 方程通解（球外形式）

$$u(r,\theta) = \sum_{n=0}^{+\infty} B_n r^{-(n+1)} p_n(cos\theta)$$

$$u(1,\theta) = \sum_{n=0}^{+\infty} B_n p_n(cos\theta) = cos^2\theta = \frac{2}{3}p_2(cos\theta) + \frac{1}{3}p_0(cos\theta)$$

$$B_0 = \frac{1}{3}, B_2 = \frac{2}{3}, B_n(n \neq 0,2)$$

$$u(r,\theta) = \frac{1}{3r} + \frac{2}{3r^3}p_2(cos\theta) = \frac{1}{3r} + \frac{2}{3r^3}\left(\frac{3}{2}cos^2\theta - \frac{1}{2}\right) = \frac{1}{3}r^{-1} + r^{-3}cos^2\theta - \frac{1}{3}r^{-3}$$

28. 解：设球的半径为a

(1) 将半球补成一完整的球，则温度函数$u(r,\theta)$满足：

$$\begin{cases}\Delta u = 0, (0 \le r < a, 0 \le \theta \le \pi \\ u|_{r=a} = u_0, 0 < \theta \le \dfrac{\pi}{2} \\ u|_{\theta=0} = 0 \\ u|_{r=a} = -u_0, -\dfrac{\pi}{2} \le \theta < 0\end{cases}$$

根据球坐标系下与方位角φ无关的 Laplace 方程通解（球内形式）

$$u(r,\theta) = \sum_{n=0}^{+\infty} A_n r^n p_n(cos\theta)$$

$$u(a,\theta) = \sum_{n=0}^{+\infty} A_n a^n p_n(cos\theta) = \begin{cases} u_0, 0 \le \theta < \dfrac{\pi}{2} \\ 0, \theta = \dfrac{\pi}{2} \\ -u_0 \dfrac{\pi}{2} < \theta \le \pi \end{cases}$$

令$cos\theta = x$

则方程转化为求函数$\begin{cases} u_0, 0 < x \le 1 \\ 0, x = 0 \\ -u_0, -1 \le x < 0\end{cases}$的傅里叶-勒让德展开

该函数为奇函数，故只有奇数项系数不为 0

$$C_1 = 3u_0 \int_0^1 x dx = \frac{3}{2} u_0$$

由习题 24.（3）

$n \ge 1$ 时 C_{2n+1}

$$= (4n+3)\int_0^1 u_0 p_{2n+1}(x)dx$$

$$= (4n+3)u_0 \int_0^1 p_{2n+1}(x)dx = (4n+3)u_0 \cdot \frac{1}{4n+3}[p_{2n}(0) - p_{2n+2}(0)] = u_0(-1)^n \frac{(4n+3)(2n-1)!!}{(2n+2)!!}$$

所以

$$A_1 = \frac{3u_0}{2a}, A_{2n+1} = u_0(-1)^n \frac{(4n+3)(2n-1)!!}{(2n+2)!!}\frac{1}{a^{2n+1}}, n = 1,2,\cdots$$

因此

$$u(r,\theta) = \frac{3u_0}{2a}cos\theta + u_0 \sum_{n=1}^{+\infty}(-1)^n \frac{(4n+3)(2n-1)!!}{(2n+2)!!}\left(\frac{r}{a}\right)^{2n+1} p_{2n+1}(cos\theta)$$

(2) 将半球补成一完整的球，绝热指底面温度保持不变，则温度函数$u(r,\theta)$满足：

$$\begin{cases}\Delta u = 0, 0 \le r \le a, 0 \le \theta \le \pi \\ u|_{r=a} = u_0\end{cases}$$

根据球坐标系下与方位角φ无关的 Laplace 方程通解（球内形式）

$$u(r,\theta) = \sum_{n=0}^{+\infty} A_n r^n p_n(cos\theta)$$

$$u(a,\theta) = \sum_{n=0}^{+\infty} A_n a^n p_n(cos\theta) = u_0 = u_0 p_0(cos\theta)$$

$$A_0 = u_0, A_n = 0 (n = 1,2,\cdots)$$

所以

$$u(r,\theta) = u_0$$

29. 解：将半空心球补成全空心球，（注意到$\theta = \frac{\pi}{2}$时，$f\left(\frac{\pi}{2}\right) = \frac{A}{2}$, 说明底面温度没有发生跳变）

则温度函数$u(r,\theta)$满足

$$\begin{cases} \Delta u = 0, \dfrac{R}{2} \le r \le R, 0 \le \theta \le \pi \\ u|_{r=R} = u|_{r=\frac{R}{2}} = A\sin^2\dfrac{\theta}{2} = \dfrac{A}{2}(1 - \cos\theta) \\ u|_{\theta=\frac{\pi}{2}} = \dfrac{A}{2} \end{cases}$$

根据球坐标系下与方位角φ无关的Laplace 方程通解

$$u(r,\theta) = \sum_{n=0}^{+\infty} [A_n r^n + B_n r^{-(n+1)}] p_n(\cos\theta)$$

$$u\left(\frac{R}{2},\theta\right) = \sum_{n=0}^{+\infty} \left[A_n \left(\frac{R}{2}\right)^n + B_n \left(\frac{R}{2}\right)^{-(n+1)}\right] p_n(\cos\theta) = \frac{A}{2}(1 - \cos\theta)$$

$$u(R,\theta) = \sum_{n=0}^{+\infty} [A_n R^n + B_n R^{-(n+1)}] p_n(\cos\theta) = \frac{A}{2}(1 - \cos\theta)$$

比较系数得

$$\begin{cases} A_0 + \dfrac{2B_0}{R} + \dfrac{A_1}{2} R\cos\theta + \dfrac{4B_1}{R^2}\cos\theta = \dfrac{A}{2}(1 - \cos\theta) \\ A_0 + \dfrac{B_0}{R} + A_1 R\cos\theta + \dfrac{B_1}{R^2}\cos\theta = \dfrac{A}{2}(1 - \cos\theta) \end{cases}$$

$$\begin{cases} A_n = 0, n = 2,3,\cdots \\ B_n = 0. n = 2,3,\cdots \end{cases}$$

得

$$\begin{cases} A_0 + \dfrac{2B_0}{R} = \dfrac{A}{2} \\ A_0 + \dfrac{B_0}{R} = \dfrac{A}{2} \end{cases} \Rightarrow \begin{cases} A_0 = \dfrac{A}{2} \\ B_0 = 0 \end{cases}$$

$$\begin{cases} \dfrac{A_1}{2} R + \dfrac{4B_1}{R^2} = -\dfrac{A}{2} \\ A_1 R + \dfrac{B_1}{R^2} = -\dfrac{A}{2} \end{cases} \Rightarrow \begin{cases} A_1 = -\dfrac{3A}{7R} \\ B_1 = -\dfrac{A}{14} R^2 \end{cases}$$

则

$$u(r,\theta) = \frac{A}{2} - \left(\frac{3r}{7R} + \frac{R^2}{14r}\right) A\cos\theta$$

第四章作业参考解答

刘炜昊

2021 年 4 月 30 日

目录

1　课本 P_{303} T_1

1.1　课本 P_{303} $T_{1(1)}$

用傅里叶变换解下列定解问题：

$$\begin{cases} \Delta_2 u = 0\ (-\infty < x < +\infty, y > 0), \\ u(x,0) = f(x), \\ \text{当} x^2 + y^2 \to +\infty \text{ 时}, u(x,y) \to 0; \end{cases}$$

Sol:

由于 x 的取值范围无界，可对 x 作 Fourier 变换，记 $\tilde{u}(\lambda, y) = F[u(x,y)]$，并记 $\tilde{f}(\lambda) = F[f(x)]$，因此得到如下 ODE：

$$\begin{cases} \dfrac{\mathrm{d}^2\tilde{u}}{\mathrm{d}y^2} - \lambda^2\tilde{u} = 0 \\ \tilde{u}(\lambda, 0) = \tilde{f}(\lambda) \end{cases}$$

得到其通解为 $\tilde{u}(\lambda, y) = A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y}$，考虑边界条件和有界性条件有：

$$\tilde{u}(\lambda, y) = \begin{cases} \tilde{f}(\lambda)e^{-\lambda y},\ \lambda \geq 0 \\ \tilde{f}(\lambda)e^{\lambda y},\ \lambda < 0 \end{cases} = \tilde{f}(\lambda)e^{-|\lambda| y}$$

因此作反变换得到：

$$F^{-1}\left[e^{-|\lambda| y}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|\lambda| y}e^{i\lambda x}\mathrm{d}\lambda = \frac{1}{\pi}\int_0^{+\infty} e^{-\lambda y}\cos\lambda x\mathrm{d}\lambda$$

$$= \frac{1}{\pi}\cdot\left.\frac{\begin{vmatrix} \cos\lambda x & e^{-\lambda y} \\ -x\sin\lambda x & -ye^{-\lambda y} \end{vmatrix}}{(-y)^2 + x^2}\right|_0^{+\infty} = \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}$$

$$\begin{aligned} \Rightarrow\ u(x,y) &= F^{-1}[\tilde{f}(\lambda)] * F^{-1}\left[e^{-\lambda|y}\right] \\ &= f(x) * \left(\frac{y}{\pi}\cdot\frac{1}{x^2+y^2}\right) \\ &= \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi - x)\mathrm{d}\xi}{\xi^2 + y^2} \end{aligned}$$

1.2 课本 P_{303} $T_{1(2)}$

用傅里叶变换解下列定解问题：

$$\begin{cases} u_t = a^2 u_{xx} + f(t,x)\ (t>0, -\infty < x < +\infty), \\ u(0,x) = 0; \end{cases}$$

Sol:

由于 x 的取值范围无界，可对 x 作 Fourier 变换，记 $\tilde{u}(t,\lambda) = F[u(t,x)]$，并记 $\tilde{f}(t,\lambda) = F[f(t,x)]$，因此得到如下 ODE：

$$\begin{cases} \dfrac{\mathrm{d}\tilde{u}}{\mathrm{d}t} + a^2\lambda^2\tilde{u} = \tilde{f}(t,\lambda) \\ \tilde{u}(0,\lambda) = 0 \end{cases}$$

得到其通解为

$$\tilde{u}(t,\lambda) = e^{-a^2\lambda^2 t}\int_0^t e^{a^2\lambda^2\tau}\tilde{f}(\tau,\lambda)\mathrm{d}\tau = \int_0^t e^{a^2\lambda^2(\tau-t)}\tilde{f}(\tau,\lambda)\mathrm{d}\tau$$

因此作反变换得到：

$$\begin{aligned} F^{-1}\left[\tilde{u}(t,\lambda)\right] &= \int_{-\infty}^{+\infty}\left[\int_0^t e^{a^2\lambda^2(\tau-t)}\tilde{f}(\tau,\lambda)\mathrm{d}\tau\right]e^{i\lambda x}\mathrm{d}\lambda \\ &= \int_0^t\left[\int_{-\infty}^{+\infty} e^{a^2\lambda^2(\tau-t)}e^{i\lambda x}\tilde{f}(\tau,\lambda)\mathrm{d}\lambda\right]\mathrm{d}\tau \\ &= \int_0^t F^{-1}\left[e^{a^2\lambda^2(\tau-t)}\tilde{f}(\tau,\lambda)\right]\mathrm{d}\tau \\ &= \int_0^t F^{-1}\left[e^{-a^2\lambda^2(t-\tau)}\right] * F^{-1}\left[\tilde{f}(\tau,\lambda)\right]\mathrm{d}\tau \\ &= \int_0^t \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{2a\sqrt{\pi(t-\tau)}} * f(\tau,x)\mathrm{d}\tau \\ &= \frac{1}{2a\sqrt{\pi}}\int_0^t\int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}\frac{f(\tau,\xi)}{\sqrt{(t-\tau)}}\mathrm{d}\xi\mathrm{d}\tau \end{aligned}$$

以上已用到公式：

$$\int_{-\infty}^{+\infty} e^{-a^2\lambda^2 t}e^{i\lambda x}\mathrm{d}\lambda = \int_{-\infty}^{+\infty} e^{-a^2\lambda^2 t}\cos\lambda x\mathrm{d}\lambda = \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}}$$

1.3 课本 P_{303} $T_{1(3)}$

用傅里叶变换解下列定解问题:

$$\begin{cases} u_t = a^2 u_{xx}(0 < x < +\infty, t > 0), \\ u(t,0) = \varphi(t), u(0,x) = 0, \quad \text{（用正弦变换）}. \\ u(t,+\infty) = u_x(t,+\infty) = 0 \end{cases}$$

Sol:

由于 x 的取值范围在半直线上，且，可对 x 作正弦变换，记 $\tilde{u}(t,\lambda) = \int_0^{+\infty} u\, t, x) \sin \lambda x \mathrm{d}x$，那么则有

$$\begin{aligned} F_s\left[\frac{\partial^2 u}{\partial x^2}\right] &= \int_0^{+\infty} \frac{\partial^2 u}{\partial x^2} \sin \lambda x \mathrm{d}x = \frac{\partial u}{\partial x} \sin \lambda x\bigg|_0^{+\infty} - \lambda \int_0^{+\infty} \frac{\partial u}{\partial x} \cos \lambda x \mathrm{d}x \\ &= -\lambda u \cos \lambda x|_0^{+\infty} - \lambda^2 \int_0^{+\infty} u \sin \lambda x \mathrm{d}x \\ &= -\lambda \varphi(t) - \lambda^2 \tilde{u}. \end{aligned}$$

因此得到如下 ODE:

$$\begin{cases} \dfrac{\mathrm{d}\tilde{u}}{\mathrm{d}t} + a^2\lambda^2 \tilde{u} = a^2 \lambda \varphi(t) \\ \tilde{u}(0,\lambda) = 0 \end{cases}$$

得到其通解为

$$\tilde{u}(t,\lambda) = e^{-a^2\lambda^2 t} \int_0^t e^{a^2\lambda^2\tau} a^2 \lambda \varphi(\tau) \mathrm{d}\tau = \int_0^t e^{a^2\lambda^2(\tau - t)} a^2 \lambda \varphi(\tau) \mathrm{d}\tau$$

因此作反变换得到:

$$\begin{aligned}
u(t,x) &= \frac{2}{\pi}\int_0^{+\infty} e^{-a^2\lambda^2 t}\int_0^t a^2\lambda\varphi(\tau)e^{a^2\lambda^2\tau}\mathrm{d}\tau\sin\lambda x\mathrm{d}\lambda \\
&= \frac{2}{\pi}\int_0^t a^2\varphi(\tau)d\tau\int_0^{+\infty}\lambda\sin\lambda x e^{-a^2\lambda^2(t-\tau)}\mathrm{d}\lambda \\
&= \frac{1}{\pi}\int_0^t \frac{\varphi(\tau)}{t-\tau}\mathrm{d}\tau\int_0^{+\infty}\sin\lambda x\mathrm{d}e^{-a^2\lambda^2(t-\tau)}\mathrm{d}\lambda \\
&= \frac{1}{\pi}\int_0^t \frac{\varphi(\tau)}{t-\tau}\mathrm{d}\tau\int_0^{+\infty}x\cos\lambda x e^{-a^2\lambda^2(t-\tau)}\mathrm{d}\lambda \\
&= \frac{1}{\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\mathrm{d}\tau\cdot\frac{1}{2}\int_{-\infty}^{+\infty}\cos\lambda x e^{-a^2x^2(t-\tau)}\mathrm{d}\lambda \\
&= \frac{1}{2\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\mathrm{d}\tau\cdot\int_{-\infty}^{+\infty}\mathrm{Re}\left\{e^{i\lambda x}\right\}e^{-a^2(t-\tau)}\mathrm{d}\lambda \\
&= \frac{1}{2\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\mathrm{d}\tau\cdot\mathrm{Re}\left\{\int_{-\infty}^{+\infty}e^{-a^2\lambda^2(t-\tau)(\lambda-\frac{ix}{2a^2(t-\tau)})^2}\cdot e^{-\frac{x^2}{4a^2(t-\tau)}}\mathrm{d}\lambda\right\} \\
&= \frac{1}{2\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\cdot e^{-\frac{x^2}{4a^2(t-\tau)}}\sqrt{\frac{\pi}{a^2(t-\tau)}}\mathrm{d}\tau \\
&= \frac{x}{2a\sqrt{\pi}}\int_0^t (t-\tau)^{-\frac{3}{2}}\varphi(\tau)e^{-\frac{x^2}{4a^2(t-\tau)}}\mathrm{d}\tau
\end{aligned}$$

其中

$$\int_{-\infty}^{+\infty}e^{-a^2(t-\tau)(\lambda-\frac{ix}{2a^2(t-\tau)})^2}\mathrm{d}\lambda = \int_{-\infty}^{+\infty}e^{-a^2(t-\tau)\lambda^2}\mathrm{d}\lambda = \sqrt{\frac{\pi}{a^2(t-\tau)}}.$$

2 课本 P_{303} T_2

2.1 课本 P_{303} $T_{2(1)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = 1(x > 0, y > 0) \\ u(0, y) = y + 1, u(x, 0) = 1; \end{cases}$$

Sol:

对 x 作 Laplace 变换，记 $\tilde{u}(p, y) = L[u(x, y)]$，则有关系:

$$L\left[\frac{\partial u}{\partial x}\right] = p\tilde{u} - u(0, y) = p\tilde{u} - y - 1$$

因此得到如下 ODE:

$$\begin{cases} \dfrac{\partial}{\partial y}(p\tilde{u} - y - 1) = \dfrac{1}{p} \\ \tilde{u}(p, 0) = \dfrac{1}{p} \end{cases}$$

因此得到: $\dfrac{\partial u}{\partial y} = \dfrac{1}{p^2} + \dfrac{1}{p}$，那么 $\tilde{u}(p, y) = \dfrac{p+1}{p^2}y + f(p)$，$f \in \mathbb{C}^1$.
由 $\tilde{u}(p, 0) = f(p) = \dfrac{1}{p}$，得到 $\tilde{u}(p, y) = \dfrac{p+1}{p^2}y + \dfrac{1}{p}$.
因此作反变换得到:

$$u(x, y) = L^{-1}[\tilde{u}(p, y)] = L^{-1}\left[\frac{p+1}{p^2}y + \frac{1}{p}\right] = xy + y + 1.$$

PS: 也可对 y 作 Laplace 变换，记 $\tilde{u}(x, p) = L[u(x, y)]$，则有关系:

$$L\left[\frac{\partial u}{\partial y}\right] = p\tilde{u} - u(x, 0) = p\tilde{u} - 1$$

因此得到如下 ODE:

$$\begin{cases} \dfrac{\partial}{\partial x}(p\tilde{u}-1) = \dfrac{1}{p} \\ \tilde{u}(0,p) = \dfrac{1}{p} + \dfrac{1}{p^2} \end{cases}$$

因此得到: $\dfrac{\partial u}{\partial x} = \dfrac{1}{p^2}$, 那么 $\tilde{u}(p,y) = \dfrac{x}{p^2}y + f(p)$, $f \in \mathbb{C}^1$.
由 $\tilde{u}(0,p) = f(p) = \dfrac{1}{p} + \dfrac{1}{p^2}$, 得到 $\tilde{u}(x,p) = \dfrac{x+1}{p^2} + \dfrac{1}{p}$.
因此作反变换得到:

$$u(x,y) = L^{-1}[\tilde{u}(x,p)] = L^{-1}\left[\frac{x+1}{p^2} + \frac{1}{p}\right] = y + xy + 1.$$

PS: 也可以不用 Laplace 变换解题，直接用第一章的知识解题也可.

2.2 课本 P_{303} $T_{2(2)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_t = a^2 u_{xx}(t > 0, 0 < x < l) \\ u_x(t,0) = 0, u(t,l) = u_0(\text{常数}) \\ u(0,x) = u_1(\text{常数})\text{；} \end{cases}$$

Sol:

对 t 作 Laplace 变换，记 $\tilde{u}(p,x) = L[u(t,x)]$，得到如下 ODE:

$$\begin{cases} p\tilde{u} - u_1 = a^2 \tilde{u}_{xx} \\ \tilde{u}_x(p,0) = 0, \tilde{u}(p,l) = \dfrac{u_0}{p} \end{cases}$$

其中以上的方程 $p\tilde{u} - u_1 = a^2\tilde{u}_{xx}$ 等价于以下方程

$$\tilde{u} - \frac{u_1}{p} = \frac{a^2}{p}\frac{\partial^2}{\partial x^2}\left(\tilde{u} - \frac{u_1}{p}\right)$$

得到通解: $\tilde{u}(p,x) = \dfrac{u_1}{p} + A\,\mathrm{sh}\,\dfrac{\sqrt{p}}{a}x + B\,\mathrm{ch}\,\dfrac{\sqrt{p}}{a}x$

代入边界条件有:

$$\begin{cases} \tilde{u}_x(p,0) = A\cdot\dfrac{\sqrt{p}}{a} = 0 \Rightarrow A = 0 \\ \tilde{u}(p,l) = \dfrac{u_1}{p} + B\,\mathrm{ch}\,\dfrac{\sqrt{p}}{a}l = \dfrac{u_0}{p} \Rightarrow B = \dfrac{u_0 - u_1}{p\,\mathrm{ch}\,\frac{\sqrt{p}}{a}l} \end{cases}$$

$$\Rightarrow \tilde{u}(p,x) = \frac{u_1}{p} + \frac{u_0 - u_1}{p}\cdot\frac{\mathrm{ch}\,\frac{\sqrt{p}}{a}x}{\mathrm{ch}\,\frac{\sqrt{p}}{a}l}$$

对应的奇点（令分母为 0）的 p 值为:

$$p = 0,\ p_k = \left[\frac{(2k+1)\pi a i}{2l}\right]^2, k \in \mathbb{Z}_+$$

因此作 Laplace 反变换得到：

$$\begin{aligned}
u(t,x) &= L^{-1}[\tilde{u}(p,x)] = u_1 + \sum_{k=0}^{+\infty} \mathrm{Res}\left[\frac{u_0-u_1}{p}\cdot\frac{\mathrm{ch}\,\frac{\sqrt{p}}{a}x}{\mathrm{ch}\,\frac{\sqrt{p}}{a}l}, p_k\right] \\
&= u_1 + (u_0-u_1) + (u_0-u_1)\sum_{k=0}^{+\infty}\left[\frac{p-p_k}{p}\cdot\frac{\mathrm{ch}\,\frac{\sqrt{p}}{a}x}{\mathrm{ch}\,\frac{\sqrt{p}}{a}l}, p_k\right]\Bigg|_{p\to p_k} \\
&= u_0 - (u_0-u_1)\sum_{k=0}^{+\infty}\left\{\left[\frac{2l}{(2k+1)\pi a}\right]^2\cdot\cos\frac{(2k+1)\pi x}{2l}\cdot e^{-\left[\frac{(2k+1)\pi a}{2l}\right]^2 t}\cdot\lim_{p\to p_k}\frac{p-p_k}{\mathrm{ch}\,\frac{\sqrt{p}}{a}l}\right\}
\end{aligned}$$

其中

$$\lim_{p\to p_k}\frac{p-p_k}{\mathrm{ch}\,\frac{\sqrt{p}}{a}l} = \lim_{p\to p_k}\frac{1}{\frac{l}{2a\sqrt{p}}\,\mathrm{sh}\,\frac{\sqrt{p}}{a}l} = \frac{a^2(2k+1)\pi}{l^2\sin(k\pi+\frac{\pi}{2})} = (-1)^k\cdot\frac{a^2(2k+1)\pi}{l^2}$$

因此得到

$$u(t,x) = u_0 + \frac{4}{\pi}(u_0-u_1)\sum_{k=0}^{+\infty}\frac{(-1)^{k+1}}{2k+1}\cdot\cos\frac{(2k+1)\pi x}{2l}\cdot e^{-\left[\frac{(2k+1)\pi a}{2l}\right]^2 t}$$

2.3 课本 P_{303} $T_{2(3)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_t = a^2 u_{xx} - hu(x>0, t>0, h>0, h\text{为常数}) \\ u(0,x) = b \ \text{（常数）}, u(t,0) = 0 \\ \lim\limits_{x\to+\infty} u_x = 0; \end{cases}$$

Sol:

对 t 作 Laplace 变换，记 $\tilde{u}(p,x) = L[u(t,x)]$，得到如下 ODE:

$$\begin{cases} p\tilde{u} - b = a^2\tilde{u}_{xx} - h\tilde{u} \\ \tilde{u}_x(p,0) = 0, \lim\limits_{x\to+\infty} \tilde{u}(p,x) = 0 \end{cases}$$

其中以上的方程 $p\tilde{u} - b = a^2\tilde{u}_{xx} - h\tilde{u}$ 等价于以下方程

$$\tilde{u} - \frac{b}{p+h} = \frac{a^2}{p+h}\frac{\partial^2}{\partial x^2}\left(\tilde{u} - \frac{b}{p+h}\right)$$

得到通解：$\tilde{u}(p,x) = \dfrac{b}{p+h} + Ae^{\dfrac{\sqrt{p+h}}{a}x} + Be^{-\dfrac{\sqrt{p+h}}{a}x}$

代入边界条件有:

$$\begin{cases} \lim\limits_{x\to+\infty} \tilde{u}_x = 0 \Rightarrow A = 0 \\ \tilde{u}(p,0) = \dfrac{b}{p+h} + B = 0 \Rightarrow B = -\dfrac{b}{p+h} \end{cases}$$

$$\Rightarrow \tilde{u}(p,x) = \frac{b}{p+h}\cdot\left(1 - e^{-\frac{\sqrt{p+h}}{a}x}\right)$$

因此作 Laplace 反变换得到:

$$u(t,x) = L^{-1}[\tilde{u}(p,x)] = be^{-ht}\left[1 - \mathrm{erfc}\left(\frac{x}{2a\sqrt{t}}\right)\right]$$

其中余误差函数 $\mathrm{erfc}(x)$ 与 $\mathrm{erf}(x)$ 的关系为:

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} \mathrm{d}y$$

因此有

$$u(t,x) = be^{-ht}\mathrm{erf}\left(\frac{x}{2a\sqrt{t}}\right) = \frac{2b}{\sqrt{\pi}} e^{-ht} \int_0^{\frac{x}{2a\sqrt{t}}} e^{-y^2} \mathrm{d}y$$

2.4 课本 P_{304} $T_{2(4)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_{tt} = a^2 u_{xx}(x > 0, t > 0) \\ u(0,x) = 0, u_x(0,x) = b \\ u(t,0) = 0, \lim\limits_{x\to+\infty} u_x = 0; \end{cases}$$

Sol:

对 t 作 Laplace 变换，记 $\tilde{u}(p,x) = L[u(t,x)]$，得到如下 ODE:

$$\begin{cases} p^2\tilde{u} - b = a^2\tilde{u}_{xx} \\ \tilde{u}(p,0) = 0, \lim\limits_{x\to+\infty} \tilde{u}_x = 0 \end{cases}$$

得到通解为：$u(\tilde{p},x) = Ae^{\frac{p}{a}x} + Be^{-\frac{p}{a}x} + \frac{b}{p^2}$

代入边界条件有:

$$\begin{cases} \tilde{u}(p,0) = A + B + \frac{b}{p^2} = 0 \\ \lim\limits_{x\to+\infty} \tilde{u}_x = \lim\limits_{x\to+\infty} (Ae^{\frac{p}{a}x} + Be^{-\frac{p}{a}x}) = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = -\frac{b}{p^2} \end{cases} \Rightarrow \tilde{u}(p,x) = \frac{b}{p^2}(1 - e^{-\frac{x}{a}p})$$

因此作 Laplace 反变换得到:

$$u(t,x) = L^{-1}[\tilde{u}(p,x)] = bt - b\left(t - \frac{x}{a}\right) h\left(t - \frac{x}{a}\right)$$

其中 $h(t)$ 为单位函数，

$$h(t) = \begin{cases} 0, \ t \le 0 \\ 1, \ t > 0 \end{cases}$$

2.5 课本 P_{304} $T_{2(5)}$

用拉普拉斯变换解下列定解问题：

$$\begin{cases} u_t = u_{xx}(t>0, 0<x<4) \\ u(t,0) = u(t,4) = 0 \\ u(0,x) = 6\sin\frac{\pi x}{2} + 3\sin \pi x; \end{cases}$$

Sol:

对 t 作 Laplace 变换，记 $\tilde{u}(p,x) = L[u(t,x)]$，得到如下 ODE：

$$\begin{cases} p\tilde{u} - 6\sin\frac{\pi x}{2} - 3\sin\pi x = \tilde{u}_{xx} \\ \tilde{u}(p,0) = 0, \tilde{u}(p,4) = 0 \end{cases}$$

得到齐通解：$\tilde{\tilde{u}}(p,x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x}$ 和特解 $\tilde{u^*}(p,x) = C\sin\frac{\pi x}{2} + D\sin\pi x$

其中特解代入原方程得：

$$p(C\sin\frac{\pi x}{2} + D\sin\pi x) - 6\sin\frac{\pi x}{2} - 3\sin\pi x = -\frac{\pi^2}{4}C\sin\frac{\pi x}{2} - \pi^2 D\sin\pi x$$

由于 $\sin\frac{\pi x}{2}$ 和 $\sin\pi x$ 相互正交，因此可得

$$\begin{cases} pC - 6 = -\frac{\pi^2}{4}C \\ pD - 3 = -\pi^2 D \end{cases} \Rightarrow \begin{cases} C = \frac{6}{p+\frac{\pi^2}{4}} \\ D = \frac{3}{p+\pi^2} \end{cases}$$

因此通解为 $\tilde{u}(p,x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x} + \frac{6}{p+\frac{\pi^2}{4}}\sin\frac{\pi x}{2} + \frac{3}{p+\pi^2}\sin\pi x$

代入边界条件有：

$$\begin{cases} \tilde{u}(p,0) = A + B = 0 \\ \tilde{u}(p,4) = Ae^{4\sqrt{p}} + Be^{-4\sqrt{p}} = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 0 \end{cases} \Rightarrow \tilde{u}(p,x) = \frac{6}{p+\frac{\pi^2}{4}}\sin\frac{\pi x}{2} + \frac{3}{p+\pi^2}\sin\pi x$$

因此作 Laplace 反变换得到：

$$u(t,x) = L^{-1}[\tilde{u}(p,x)] = 6e^{-\frac{\pi^2}{4}t}\sin\frac{\pi x}{2} + 3e^{-\pi^2 t}\sin\pi x$$

2.6 课本 P_{304} $T_{2(6)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_t = u_{xx}(0<x<2) \\ u_x(t,0) = u_x(t,2) = 0 \\ u(0,x) = 4\cos\pi x - 2\cos 3\pi x; \end{cases}$$

Sol:

对 t 作 Laplace 变换，记 $\tilde{u}(p,x) = L[u(t,x)]$，得到如下 ODE:

$$\begin{cases} p\tilde{u} - 4\cos\pi x + 2\cos 3\pi x = \tilde{u}_{xx} \\ \tilde{u}_x(p,0) = 0, \tilde{u}_x(p,2) = 0 \end{cases}$$

得到齐通解: $\tilde{\bar{u}}(p,x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x}$ 和特解 $\tilde{u^*}(p,x) = C\cos\pi x + D\cos 3\pi x$
其中特解代入原方程得:

$$p(C\cos\pi x + D\cos 3\pi x) - 4\cos\pi x + 2\cos 3\pi x = -\pi^2 C\cos\pi x - 9\pi^2 D\cos 3\pi x$$

由于 $\cos\pi x$ 和 $\cos 3\pi x$ 相互正交，因此可得

$$\begin{cases} pC - 4 = -\pi^2 C \\ pD + 2 = -9\pi^2 D \end{cases} \Rightarrow \begin{cases} C = \dfrac{4}{p+\pi^2} \\ D = -\dfrac{2}{p+9\pi^2} \end{cases}$$

因此通解为 $\tilde{u}(p,x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x} + \dfrac{4}{p+\pi^2}\cos\pi x - \dfrac{2}{p+9\pi^2}\cos 3\pi x$
代入边界条件有:

$$\begin{cases} \tilde{u}_x(p,0) = \sqrt{p}(A-B) = 0 \\ \tilde{u}_x(p,2) = \sqrt{p}(Ae^{2\sqrt{p}} + Be^{-2\sqrt{p}}) = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 0 \end{cases} \Rightarrow \tilde{u}(p,x) = \frac{4\cos\pi x}{p+\pi^2} - \frac{2\cos 3\pi x}{p+9\pi^2}$$

因此作 Laplace 反变换得到:

$$u(t,x) = L^{-1}[\tilde{u}(p,x)] = 4e^{-\pi^2 t}\cos\pi x - 2e^{-9\pi^2 t}\cos 3\pi x$$

2.7 课本 P_{304} $T_{2(7)}$

用拉普拉斯变换解下列定解问题：

$$\begin{cases} u_{tt} = a^2\Delta_3 u(t>0, r>0) \\ u|_{r=0}\text{有界}, r=\sqrt{x^2+y^2+z^2} \\ u|_{t=0}=0, u_t|_{t=0}=(1+r^2)^{-2}; \end{cases}$$

[**提示:** 利用在复变函数部分第 5 章中已求得的积分

$$\int_0^{+\infty}\frac{\cos\omega x}{1+x^2}=\frac{\pi}{2}e^{-\omega}.\ ]$$

Sol:

由于初始条件只与 r 有关，故 $u=u(t,r)$，那么得到:$a^2\Delta_3 u=a^2(u_{rr}+\frac{2}{r}u_r)$，因此令 $v(t,r)=ru$，化简得到如下定解问题：

$$\begin{cases} v_{tt} = a^2 v_{rr}(t>0, r>0) \\ v|_{r=0}\text{有界}, r=\sqrt{x^2+y^2+z^2} \\ v|_{t=0}=0, v_t|_{t=0}=\dfrac{r}{(1+r^2)^2}; \end{cases}$$

对 $v(t,r)$ 做 Laplace 变换，设 $\tilde{v}(p,r)=L[v(t,r)]$，因此可得到：

$$\begin{cases} p^2\tilde{v}(p,r)-\dfrac{r}{(1+r^2)^2}=a^2\tilde{v}(p,r) \\ |\tilde{v}(p,0)|<+\infty \end{cases}$$

对 $\tilde{v}(p,r)$ 作正弦变换，记 $\tilde{V}(p,\omega)=F_s[\tilde{v}(p,r)]=\int_0^{+\infty}\tilde{v}(p,r)\sin\omega r\mathrm{d}r$，则有

$$\begin{aligned} F_s[\tilde{v}_{rr}] &= int_0^{+\infty}\tilde{v}_{rr}\sin\omega r\mathrm{d}r \\ &= \tilde{v}_r\sin\omega r|_0^{+\infty}-\omega\int_0^{+\infty}\tilde{v}_r\cos\omega r\mathrm{d}r \\ &= -\omega\int_0^{+\infty}\cos\omega r\mathrm{d}\tilde{v} \\ &= -\omega\left[\tilde{v}\cos\omega r|_0^{+\infty}+\omega\int_0^{+\infty}\tilde{v}\sin\omega r\mathrm{d}r\right] \\ &= -\omega^2\tilde{V} \end{aligned}$$

另外还有

$$
\begin{aligned}
F_s\left[\frac{r}{(1+r^2)^2}\right] &= \int_0^{+\infty}\frac{r}{(1+r^2)^2}\sin\omega r\mathrm{d}r \\
&= -\frac{1}{2}\int_0^{+\infty}\sin\omega r\mathrm{d}\left(\frac{1}{1+r^2}\right) \\
&= -\frac{1}{2}\left[\frac{\sin\omega r}{1+r^2}\bigg|_0^{+\infty} - \omega\int_0^{+\infty}\frac{\cos\omega r}{1+r^2}\mathrm{d}r\right] \\
&= \frac{\omega}{2}\int_0^{+\infty}\frac{\cos\omega r}{1+r^2}\mathrm{d}r \\
&= \frac{\pi}{4}\omega e^{-\omega}
\end{aligned}
$$

因此可以得到：$p^2\tilde{V} - \dfrac{\pi}{4}\omega e^{-\omega} = -a^2\omega^2\tilde{V}$，那么

$$
\tilde{V}(p,\omega) = \frac{1}{p^2+a^2\omega^2}\cdot\frac{\pi}{4}\omega e^{-\omega} = \left(\frac{1}{p-a\omega i} - \frac{1}{p+a\omega i}\right)\cdot\frac{\pi}{8ai}e^{-\omega}
$$

因此

$$
\begin{aligned}
v(t,r) &= L^{-1}[F_s^{-1}[\tilde{V}(p,\omega)]] = F_s^{-1}[L^{-1}[\tilde{V}(p,\omega)]] \\
&= F_s^{-1}\left[\frac{e^{a\omega it} - e^{-a\omega it}}{8ai}\pi e^{-\omega}\right] \\
&= F_s^{-1}\left[\frac{\pi}{4a}e^{-\omega}\sin a\omega t\right] \\
&= \frac{2}{\pi}\int_0^{+\infty}\frac{\pi}{4a}e^{-\omega}\sin a\omega t\sin\omega r\mathrm{d}\omega \\
&= \frac{1}{4a}\int_0^{+\infty}e^{-\omega}[\cos\omega(r-at) - \cos\omega(r+at)]\mathrm{d}\omega \\
&= \frac{1}{4a}\left[\frac{1}{1+(r-at)^2} - \frac{1}{1+(r+at)^2}\right] \\
&= \frac{t}{[1+(r+at)^2][1+(r-at)^2]}
\end{aligned}
$$

Sol*（未用到 Laplace 变换，直接使用 d'Alembert 公式）：

由于初始条件只与 r 有关，故 $u = u(t,r)$，那么得到:$a^2\Delta_3 u = a^2(u_{rr} + \dfrac{2}{r}u_r)$，因此令 $v(t,r) = ru$，化

简得到如下定解问题:

$$\begin{cases} v_{tt} = a^2 v_{rr}(t>0, r>0) \\ v|_{r=0}\text{有界}, r = \sqrt{x^2+y^2+z^2} \\ v|_{t=0} = 0, v_t|_{t=0} = \dfrac{r}{(1+r^2)^2}; \end{cases}$$

由于 v 满足的问题是半直线问题，可考虑使用奇延拓得到的全直线问题:

$$\begin{cases} V_{tt} = a^2 V_{xx}(t>0, x>0) \\ V|_{x=0}\text{有界} \\ V|_{t=0} = 0, V_t|_{t=0} = \dfrac{x}{(1+x^2)^2}; \end{cases}$$

可利用 d'Alembert 公式得到:

$$\begin{aligned} V(t,x) &= \frac{1}{2a}\int_{x-at}^{x+at} \frac{\xi}{(1+\xi^2)^2} = -\frac{1}{4a}\cdot\frac{1}{1+\xi^2}\bigg|_{x-at}^{x+at} \\ &= \frac{1}{4a}\cdot\left[\frac{1}{1+(x+at)^2} - \frac{1}{1-(x+at)^2}\right] \\ &= \frac{xt}{[1+(x+at)^2][1+(x-at)^2]} \end{aligned}$$

此时把解 $V(t,x)$ 中限制在 $x>0$（对应 $r>0$），可得到原半无界弦振动的定解问题的解为:

$$v(t,r) = \frac{rt}{[1+(r+at)^2][1+(r-at)^2]}$$

因此原三位波动方程定解问题的解为:

$$u(t,r) = \frac{v(t,r)}{r} = \frac{t}{[1+(r+at)^2][1+(r-at)^2]}$$

3 写在最后

本答案仅基于本人对数理方程的粗略理解，为方便同学们复习及纠正自己的答案而编写，很多题目也仅用了一种方法，仅提供参考，具体到对每个同学的意见，在批改作业的过程中也已经写了批注。

此外，考虑到本参考答案可能会存在一定的错误，后期可能会有修正版，可以**点击此处**查看最新版，对这些可能存在的错误，还请同学们海涵，同时也很感谢朱健同学在我编写答案的过程中提供的帮助。

2020-2021 春季学期数理方程 B 助教
本科 18 级 地球和空间科学学院 刘炜昊
2021 年 4 月 于合肥

数理方程第五章参考答案

1. 解：

(1) 由$\delta(x)$的性质，有

$$\int_{-\infty}^{+\infty} x\delta(x)dx = x|_{x=0} = 0$$

又当$x \neq 0$时，$\delta(x) = 0, x\delta(x) = 0$

所以当$x = 0$时，$x\delta(x) = 0$

所以

$$x\delta(x) \equiv 0$$

(2) 对于任意函数$g(x) \in K$, 有

$$\int_{-\infty}^{+\infty} g(x)f(x)\delta(x-a)dx = f(a)g(a)$$

$$= f(a)\int_{-\infty}^{+\infty} g(x)\delta(x-a)dx = \int_{-\infty}^{+\infty} g(x)f(a)\,\delta(x-a)dx$$

所以

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

(3) 对于任意函数$g(x) \in K$, 有

$a > 0$时

$$\int_{-\infty}^{+\infty} g(x)\delta(ax)\,dx \xRightarrow{t=ax} \int_{-\infty}^{+\infty} g\left(\frac{t}{a}\right)\frac{1}{a}\delta(t)dt = \frac{1}{a}g(0)$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} g(x)\delta(x)dx = \int_{-\infty}^{+\infty} g(x)\frac{1}{a}\delta(x)dx$$

所以

$$\delta(ax) = \frac{1}{a}\delta(x)$$

$a < 0$时

$$\int_{-\infty}^{+\infty} g(x)\delta(ax)\,dx \xRightarrow{t=ax} \int_{-\infty}^{+\infty} g\left(\frac{t}{a}\right)\frac{1}{-a}\delta(t)dt = \frac{1}{-a}g(0) =$$

$$= \frac{1}{-a}\int_{-\infty}^{+\infty} g(x)\delta(x)dx = \int_{-\infty}^{+\infty} g(x)\frac{1}{-a}\delta(x)dx$$

所以

$$\delta(ax) = \frac{1}{-a}\delta(x)$$

综上

$$\delta(ax) = \frac{1}{|a|}\delta(x), a \neq 0$$

(4) 对于任意函数$f(x) \in K$, 有

$$\int_{-\infty}^{+\infty} f(x)\,\delta'(-x)dx \xRightarrow{-x=t} \int_{-\infty}^{+\infty} f(-t)\delta'(t)dt = -[f(-t)]'|_{t=0} = f'(-t)|_{t=0} = f'(0)$$

$$\int_{-\infty}^{+\infty} f(x)[-\delta'(x)]dx = -\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = f'(0)$$

所以

$$\delta'(-x) = -\delta'(x)$$

（5）对于任意函数$f(x)\in K$，有

$$\int_{-\infty}^{+\infty} f(x)x\delta'(x)dx=\int_{-\infty}^{+\infty} xf(x)\delta'(x)dx=-[xf(x)]'|_{x=0}=-f(0)$$

$$\int_{-\infty}^{+\infty} f(x)[-\delta(x)]\,dx=-\int_{-\infty}^{+\infty} f(x)\delta(x)dx=-f(0)$$

所以

$$x\delta'(x)=-\delta(x)$$

2. 解：

对于任意函数$f(x,y)\in K$，有

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x-x_0,y-y_0)dxdy$$

做变量替换

$$\begin{cases}x=x(\xi,\eta)\\ y=y(\xi,\eta)\end{cases}$$

所以

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x-x_0,y-y_0)dxdy=$$

$$\iint_D f[x(\xi,\eta),y(\xi,\eta)]\,\delta[x(\xi,\eta)-x_0,y(\xi,\eta)-y_0]\frac{1}{|J|}d\xi d\eta$$

$$\iint_D f[x(\xi,\eta),y(\xi,\eta)]\,\delta[x(\xi,\eta)-x_0,y(\xi,\eta)-y_0]\frac{1}{|J|}d\xi d\eta$$

$$=\iint_D f[x(\xi_0,\eta_0),y(\xi_0,\eta_0)]\,\delta[x(\xi,\eta),y(\xi,\eta)]\frac{1}{|J|}d\xi d\eta$$

$$\iint_D f[x(\xi_0,\eta_0),y(\xi_0,\eta_0)]\,\delta[x(\xi,\eta),y(\xi,\eta)]\frac{1}{|J|}d\xi d\eta=\iint_D f(\xi_0,\eta_0)\,\delta(\xi,\eta)\frac{1}{|J|}d\xi d\eta$$

$$=\iint_D f(\xi,\eta)\,\delta(\xi-\xi_0,\eta-\eta_0)\frac{1}{|J|}d\xi d\eta$$

因此

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x-x_0,y-y_0)dxdy=\iint_D f(\xi,\eta)\,\delta(\xi-\xi_0,\eta-\eta_0)\frac{1}{|J|}d\xi d\eta$$

故

$$\delta(x-x_0,y-y_0)=\frac{1}{|J|}\delta(\xi-\xi_0,\eta-\eta_0)$$

对于变量替换

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases},r\geq 0,0\leq\theta<2\pi$$

$$|J|=r$$

$$\delta(x-x_0,y-y_0)=\frac{1}{r}\delta(r-r_0,\theta-\theta_0)$$

3. 解：

（1）由分离变量法，设$u(t,x)=T(t)X(x)$

完成分量变量手续得关于x,t的方程

$$\begin{cases} X''(x)+\lambda=0 \\ X(0)=X(l)=0 \end{cases}$$

$$T'(t)+\lambda a^2 T(t)=0$$

由关于x的方程得固有值$\lambda_n=\left(\frac{n\pi}{l}\right)^2$，固有函数$X_n(x)=sin\frac{n\pi x}{l},n=1,2,\cdots$

所以$T_n(t)=exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\}$

$$u(t,x)=\sum_{n=1}^{+\infty}A_n exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\}sin\frac{n\pi x}{l}$$

$$u(0,x)=\sum_{n=1}^{+\infty}A_n sin\frac{n\pi x}{l}=\delta(x-\xi)$$

$$A_n=\frac{2}{l}\int_0^l \delta(x-\xi)sin\frac{n\pi x}{l}dx=\frac{2}{l}sin\frac{n\pi\xi}{l}$$

$$u(t,x)=\frac{2}{l}\sum_{n=1}^{+\infty}exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\}sin\frac{n\pi\xi}{l}sin\frac{n\pi x}{l}$$

(2) 由分离变量法，设$u(t,x)=T(t)X(x)$

完成分量变量手续得关于x,t的方程

$$\begin{cases} X''(x)+\lambda=0 \\ X'(0)=X'(l)=0 \end{cases}$$

$$T''(t)+\lambda a^2 T(t)=0$$

由关于x的方程得固有值$\lambda_n=0,\left(\frac{n\pi}{l}\right)^2$，固有函数$X_0(x)=1$,

$$X_n(x)=cos\frac{n\pi x}{l},n=1,2,\cdots$$

所以$T_0(t)=A_0+B_0 t$,　$T_n(t)=A_n cos\frac{n\pi a}{l}t+B_n sin\frac{n\pi a}{l}t$

$$u(t,x)=A_0+B_0 t+\sum_{n=1}^{+\infty}A_n cos\frac{n\pi a}{l}tcos\frac{n\pi x}{l}+B_n sin\frac{n\pi a}{l}t\,cos\frac{n\pi x}{l}$$

$$u(0,x)=A_0+\sum_{n=1}^{+\infty}A_n cos\frac{n\pi x}{l}=0\Longrightarrow A_0=A_n=0$$

$$u_t(0,x)=B_0+\sum_{n=1}^{+\infty}\frac{n\pi a}{l}B_n cos\frac{n\pi x}{l}=\delta(x-\xi)$$

相当于对$\delta(x-\xi)$进行余弦级数展开，第一项为$\frac{1}{l}\int_0^l\delta(x-\xi)dl=\frac{1}{l}$

$$B_0=\frac{1}{l}$$

$$\frac{n\pi a}{l}B_n=\frac{2}{l}\int_0^l\delta(x-\xi)cos\frac{n\pi x}{l}dx=\frac{2}{l}cos\frac{n\pi\xi}{l}$$

$$B_n=\frac{2}{n\pi a}cos\frac{n\pi\xi}{l}$$

$$u(t,x)=\frac{t}{l}+\sum_{n=1}^{+\infty}\frac{2}{n\pi a}cos\frac{n\pi\xi}{l}sin\frac{n\pi at}{l}cos\frac{n\pi x}{l}$$

4. （1）

解：

设

$$\begin{cases}\bar{x}=x\\ \bar{y}=\frac{1}{\beta}y\end{cases}$$

$$u_{xx}=u_{\bar{x}\bar{x}}$$

$$u_{yy}=\frac{1}{\beta^2}u_{\bar{y}\bar{y}}$$

所以

$$u_{xx}+\beta^2u_{yy}=u_{\bar{x}\bar{x}}+u_{\bar{y}\bar{y}}=0$$

由二维拉普拉斯方程基本解

$$\tilde{u}_{xx}+\beta^2\tilde{u}_{yy}=\tilde{u}_{\bar{x}\bar{x}}+\tilde{u}_{\bar{y}\bar{y}}=\delta(x,y)=\frac{1}{\beta}\delta(\bar{x},\bar{y})$$

$$u(\bar{x},\bar{y})=\frac{1}{2\pi\beta}ln\bar{r},\bar{r}=\sqrt{\bar{x}^2+\bar{y}^2}=\sqrt{x^2+\left(\frac{y}{\beta}\right)^2}$$

所以所得基本解为

$$u(x,y)=\frac{1}{4\pi\beta}ln\left[x^2+\left(\frac{y}{\beta}\right)^2\right]$$

书上答案给的是

$$u(x,y)=\frac{1}{4\pi\beta}ln[\beta^2x^2+y^2]$$

其实是

$$u(x,y)=\frac{1}{4\pi\beta}ln\left[x^2+\left(\frac{y}{\beta}\right)^2\right]=\frac{1}{4\pi\beta}ln(\beta^2x^2+y^2)-\frac{1}{4\pi\beta}ln\beta^2$$

其中

$$-\frac{1}{4\pi\beta}ln\beta^2$$

为常数项，不能反应基本解在原点处无界的性质，故可以舍去

(2)

解：

$$\Delta_2\Delta_2u=0$$

设

$$v(x,y)=\Delta_2u$$

则有

$$\Delta_2v=0$$

则有基本解方程

$$\Delta_2 v = \delta(x,y)$$

$$v(x,y) = \frac{1}{2\pi} lnr$$

$$\Delta_2 u = \frac{1}{2\pi} lnr$$

该方程右边只有r，所以设$u = u(r)$

则有

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \frac{1}{2\pi} lnr$$

该方程为一欧拉方程，做变量替换

$$r = e^t$$

则

$$\frac{d^2u}{dt^2} = \frac{te^{2t}}{2\pi}$$

$$u = \frac{\frac{1}{4}te^{2t} - \frac{1}{4}e^{2t}}{2\pi} + Dt + C$$

$$u(r) = \frac{1}{8\pi} r^2 lnr - \frac{1}{8\pi} r^2 + Dlnr + C$$

$$\Delta_2 C = 0$$

$$\Delta_2 Dlnr = 2\pi D\delta(x,y)$$

$$\Delta_2 - \frac{1}{8\pi} r^2 = -\frac{1}{2\pi}$$

其中

$$\Delta_2 C = 0, \Delta_2 - \frac{1}{8\pi} r^2 = -\frac{1}{2\pi}$$

为常数，无法体现基本解在原点出无界的性质。

$$\Delta_2 Dlnr = 2\pi D\delta(x,y)$$

$$\Delta_2\Delta_2 Dlnr = \Delta_2 2\pi D\delta(x,y) = 2\pi D\Delta_2\delta(x,y) \neq 0$$

无法体现基本解满足齐次方程。

故基本解为

$$u = \frac{1}{8\pi} r^2 lnr$$

（3）

解：

$$\Delta_3\Delta_3 u = 0$$

设

$$v(x,y,z) = \Delta_3 u$$

则有

$$\Delta_3 v = 0$$

则有基本解方程

$$\Delta_3 v = \delta(x,y,z)$$

$$v=-\frac{1}{4\pi r}$$

$$\Delta_3 u=-\frac{1}{4\pi r}$$

该方程右边只有r，所以设$u=u(r)$

则有

$$\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}=-\frac{1}{4\pi r}$$

该方程为一欧拉方程，做变量替换

$$r=e^t$$

则

$$\frac{d^2u}{dt^2}+\frac{du}{dt}=-\frac{e^t}{4\pi}$$

对应的非齐次方程特解为（特解主要体现非齐次项影响，对应原方程体现基本解的影响，故该特解符合条件，不需要再进行Δ_3之后的分析）

$$u=-\frac{1}{8\pi}e^t$$

所以所得方程基本解为

$$u=-\frac{1}{8\pi}r$$

5. 解：

依题

$$\Delta_3 u+k^2u=\delta(x,y,z)$$

两边做三维傅里叶变换

设

$$\tilde{u}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}u(x,y,z)e^{i\lambda x+i\mu y+ivz}dxdydz$$

所以

$$(-\lambda^2-\mu^2-v^2)\tilde{u}+k^2\tilde{u}=1$$

$$\tilde{u}=\frac{1}{k^2-(\lambda^2+\mu^2+v^2)}$$

$$u(x,y,z)=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{k^2-(\lambda^2+\mu^2+v^2)}e^{-(i\lambda x+i\mu y+ivz)}d\lambda d\mu dv$$

记

$$\vec{\rho}=(\lambda,\ \mu,\ v),\vec{r}=(x,y,z)$$

则

$$\rho=\sqrt{\lambda^2+\mu^2+v^2}$$
$$r=\sqrt{x^2+y^2+z^2}$$

设

θ为$\vec{\rho},\vec{r}$的夹角，将v轴作为$\vec{r}$的方向，做球坐标代换

$$\begin{cases}\lambda=\rho sin\theta cos\varphi\\ \mu=\rho sin\theta sin\varphi\\ v=\rho cos\theta\end{cases}$$

$$u=\frac{1}{(2\pi)^3}\int_0^{2\pi}d\varphi\int_0^{\pi}sin\theta d\theta\int_0^{+\infty}\frac{\rho^2}{k^2-\rho^2}e^{-i\rho rcos\theta}d\rho$$

$$u=\frac{1}{4\pi^2}\int_0^{+\infty}\frac{\rho^2}{k^2-\rho^2}d\rho\int_0^{\pi}sin\theta\, e^{-i\rho rcos\theta}d\theta$$

$$\int_0^{\pi}sin\theta e^{-i\rho rcos\theta}d\theta=-\int_0^{\pi}e^{-i\rho rcos\theta}dcos\theta\xRightarrow{t=cos\theta}\int_{-1}^{1}e^{-i\rho rt}dt=2\frac{sin\rho r}{\rho r}$$

$$u=\frac{1}{2\pi^2 r}\int_0^{+\infty}\frac{\rho sin\rho r}{k^2-\rho^2}d\rho$$

计算

$$\int_0^{+\infty}\frac{\rho sin\rho r}{k^2-\rho^2}d\rho$$

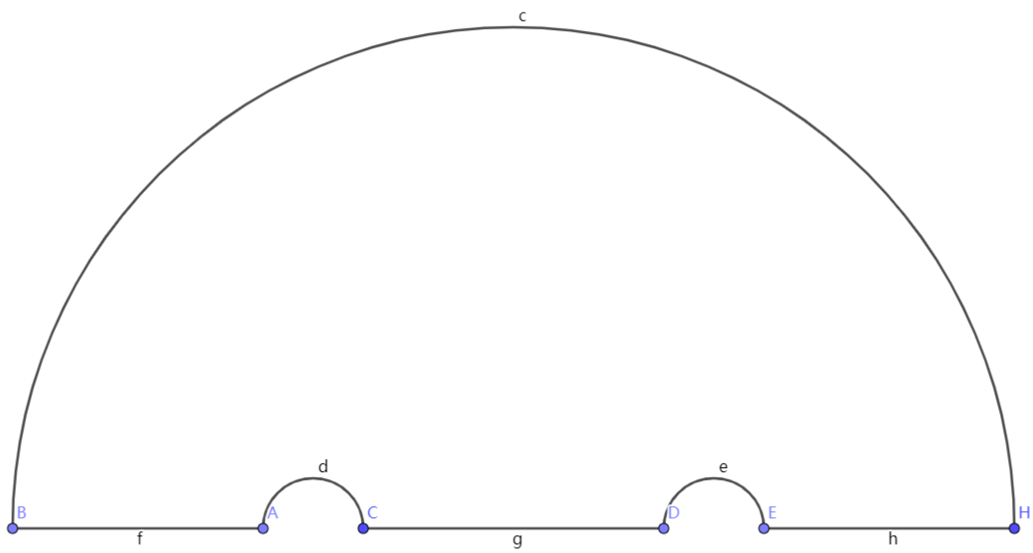
利用复变函数的方法计算该积分，首先

$$\int_0^{+\infty}\frac{\rho sin\rho r}{k^2-\rho^2}d\rho=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\rho sin\rho r}{k^2-\rho^2}d\rho$$

设

$$f(z)=\frac{ze^{irz}}{k^2-z^2}$$

取围道如图：其中设大半圆的半径为R，小半圆的半径为a. 围道方向为逆时针方向。



$$\int_c f(z)dz+\int_f f(z)dz+\int_d f(z)dz+\int_g f(z)dz+\int_e f(z)dz+\int_h f(z)dz=0$$

令$R\to+\infty,a\to 0$

则由约当引理

$$\int_c f(z)dz=0$$

由小圆弧引理

$$\int_d f(z)dz=i(0-\pi)\lim_{z\to-k}(z+k)f(z)=i\pi\frac{e^{-ikr}}{2}$$

$$\int_{e} f(z)dz = i(0-\pi)\lim_{z\to k}(z-k)f(z) = i\pi\frac{e^{ikr}}{2}$$

X 轴部分

$$\int_{f} f(z)dz = \int_{-\infty}^{-k} f(x)dx$$

$$\int_{g} f(z)dz = \int_{-k}^{k} f(x)dx$$

$$\int_{h} f(z)dz = \int_{k}^{+\infty} f(x)dx$$

所以

$$\int_{-\infty}^{+\infty} f(x)dx + i\pi coskr = 0$$

$$\int_{-\infty}^{+\infty}\frac{xe^{irx}}{k^2-x^2}dx = \int_{-\infty}^{+\infty}\frac{x}{k^2-x^2}(cosrx+isinrx)dx = i\int_{-\infty}^{+\infty}\frac{xsinrx}{k^2-x^2}dx = -i\pi coskr$$

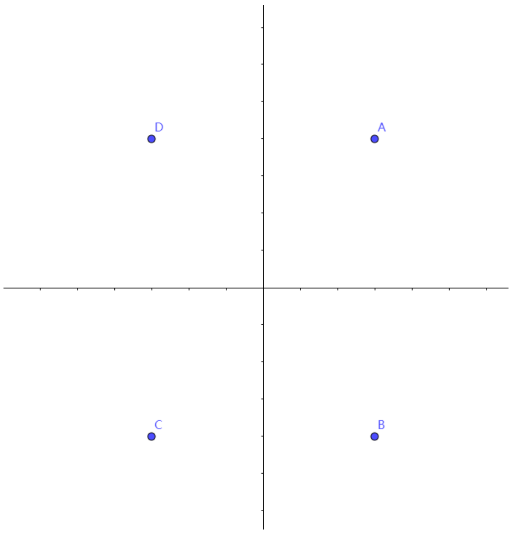
故

$$\int_{0}^{+\infty}\frac{\rho sin\rho r}{k^2-\rho^2}d\rho = -\frac{\pi coskr}{2}$$

$$u = \frac{1}{2\pi^2 r}\int_{0}^{+\infty}\frac{\rho sin\rho r}{k^2-\rho^2}d\rho = \frac{1}{2\pi^2 r}\cdot\left(-\frac{\pi coskr}{2}\right) = -\frac{coskr}{4\pi r}$$

6. 解：

(1) 如图所示 xoy 截面，在 A 点(ξ,η,ζ)处放置一大小为ε的电荷，为了使$x=0,y=0$处电势均为 0，需要在 A 点关于平面 x=0 对称点 B$(\xi,-\eta,\zeta)$处放置一大小为$-\varepsilon$的电荷，以此类推，B 点关于平面 y=0 对称点 C$(-\xi,-\eta,\zeta)$处放置一大小为ε的电荷，C 点关于平面 x=0 对称点 D$(-\xi,\eta,\zeta)$处放置一大小为$-\varepsilon$的电荷。



由三维形式点电荷的 Green 函数

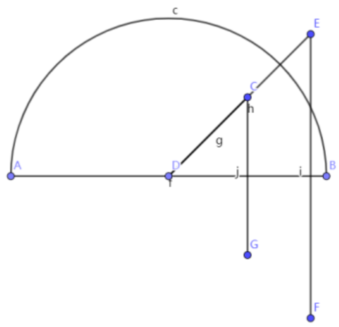
$$G = \frac{1}{4\pi}\left(\frac{1}{r_A}+\frac{1}{r_C}-\frac{1}{r_B}-\frac{1}{r_D}\right)$$

其中

$$r_A = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$$
$$r_B = \sqrt{(x-\xi)^2+(y+\eta)^2+(z-\zeta)^2}$$
$$r_C = \sqrt{(x+\xi)^2+(y+\eta)^2+(z-\zeta)^2}$$

$$r_D = \sqrt{(x+\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

(2) 如图所示 xoz 截面，在 C 点(ξ, η, ζ)处放置一大小为ε的电荷，为了使$x = 0$和球面处电势均为 0，需要在 C 点关于平面 z=0 对称点 G 处放置一大小为$-\varepsilon$的电荷，C 点关于球面对称点 E 处放置一大小为$-\frac{a}{\rho_0}\varepsilon$的电荷，E 点关于平面 z=0 对称点 F 处放置一大小为$\frac{a}{\rho_0}\varepsilon$的电荷，其中$\rho_0 = |DC| = \sqrt{\xi^2 + \eta^2 + \zeta^2}$。



由三维形式点电荷的 Green 函数

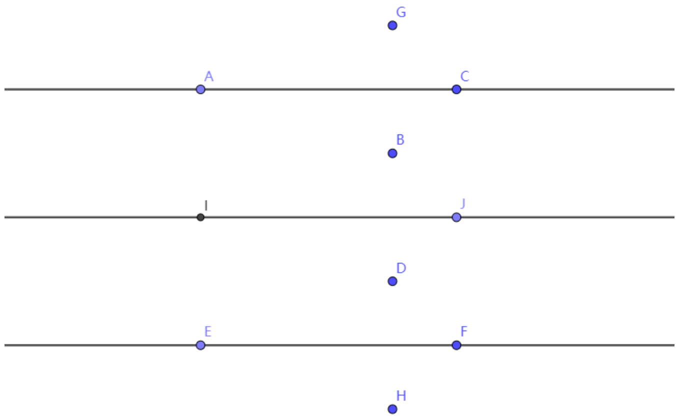
$$G = \frac{1}{4\pi}\left[\frac{1}{r(M,C)} + \frac{a}{\rho_0}\frac{1}{r(M,F)} - \frac{1}{r(M,G)} - \frac{a}{\rho_0}\frac{1}{r(M,E)}\right]$$

其中$M(x, y, z)$.

(3) 如图所示，xoz 截面，在$0 < z < H$间 B 点(ξ, η, ζ)放置一大小为ε的电荷，先只考虑 B 点处电荷，为了使得$z = 0$处电势为 0，需要在 D 点$(\xi, \eta, -\zeta)$（B 点关于 z=0 的对称点）放置一大小为$-\varepsilon$的电荷，为了使$z = H$需要在 G 点$(\xi, \eta, \zeta + 2H)$（B 点关于 z=H 的对称点）放置一大小为$-\varepsilon$的电荷。

加入新电荷之后，原来的平衡体系进一步被打破，需要加入新的电荷平衡。

故在$(\xi, \eta, 2nH + \zeta)$处放置一大小为ε的电荷，故在$(\xi, \eta, 2nH - \epsilon)$处放置一大小为$-\varepsilon$的电荷。$n = \cdots, -2, -1, 0, 1, 2, \cdots$



由三维形式点电荷的格林函数

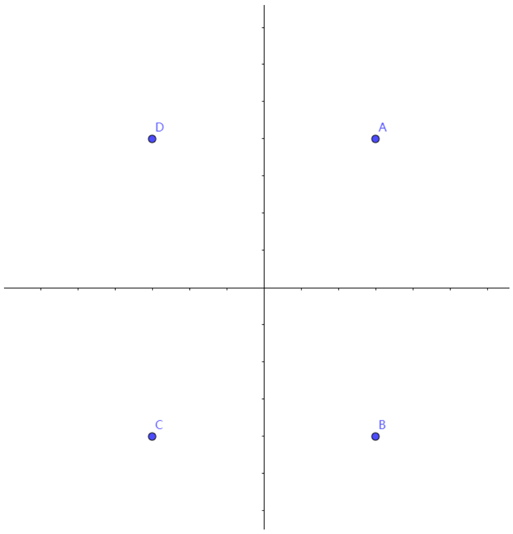
$$G = \frac{1}{4\pi}\sum_{n=-\infty}^{n=+\infty}\left(\frac{1}{r_n}-\frac{1}{r_n'}\right)$$

其中

$$r_n = \sqrt{(x-\xi)^2+(y-\eta)^2+[z-(2nH+\zeta)]^2}$$
$$r_n' = \sqrt{(x-\xi)^2+(y-\eta)^2+[z-(2nH-\zeta)]^2}$$

7. 解：

（1） 如图在 A 点(ξ,η)处放置一大小为ε的电荷，为了使$x=0, y=0$处电势均为 0，需要在 A 点关于 x=0 对称点 B$(\xi,-\eta)$处放置一大小为$-\varepsilon$的电荷，以此类推，B 点关于 y=0 对称点 C$(-\xi,-\eta)$处放置一大小为ε的电荷，C 点关于 x=0 对称点 D$(-\xi,\eta)$处放置一大小为$-\varepsilon$的电荷



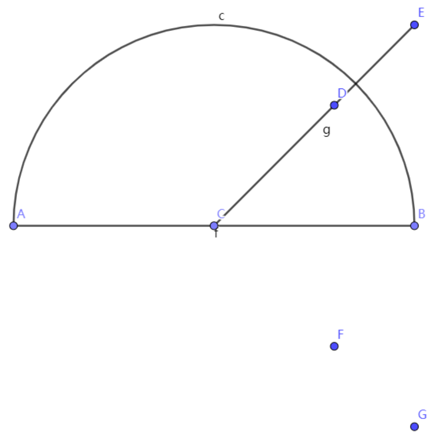
由二维形式点电荷的 Green 函数

$$G = \frac{1}{2\pi}\left(ln\frac{1}{r_A}+ln\frac{1}{r_C}-ln\frac{1}{r_B}-ln\frac{1}{r_D}\right)$$

其中

$$r_A=\sqrt{(x-\xi)^2+(y-\eta)^2}$$
$$r_B=\sqrt{(x-\xi)^2+(y+\eta)^2}$$
$$r_C=\sqrt{(x+\xi)^2+(y+\eta)^2}$$
$$r_D=\sqrt{(x+\xi)^2+(y-\eta)^2}$$

（2） 如图所示在 D 点(ξ,η)处放置一大小为ε的电荷，为了使$x=0$和圆边处电势均为 0，需要在 D 点关于 x=0 对称点 F 处放置一大小为$-\varepsilon$的电荷，D 点关于球面对称点 E 处放置一大小为$-\frac{1}{\rho_0}\varepsilon$的电荷，E 点关于 x=0 对称点 G 处放置一大小为$\frac{1}{\rho_0}\varepsilon$的电荷，其中$\rho_0=|DC|=\sqrt{\xi^2+\eta^2}$。



由二维形式点电荷的 Green 函数
$$G=\frac{1}{2\pi}\left(ln\frac{1}{r(M,D)}+\frac{1}{\rho_0}ln\frac{1}{r(M,G)}-ln\frac{1}{r(M,F)}-\frac{1}{\rho_0}ln\frac{1}{r(M,E)}\right)$$
其中$M(x,y)$

8. 解：依题基本解$U(t,x)$满足
$$\begin{cases}U_t=a^2U_{xx}+bU(t>0,-\infty<x<+\infty)\\ U|_{t=0}=\delta(x)\end{cases}$$

由傅里叶变换，设$\widetilde{U}(t,\lambda)=\int_{-\infty}^{+\infty}U(t,x)e^{i\lambda x}dx$

所以
$$\begin{cases}\dfrac{d\widetilde{U}}{dt}=(b-\lambda^2a^2)\widetilde{U}\\ \widetilde{U}|_{t=0}=1\end{cases}$$

故
$$\widetilde{U}(t,\lambda)=exp\{(b-\lambda^2a^2)t\}=e^{bt}e^{-\lambda^2a^2t}$$

所以
$$U(t,x)=e^{bt}F^{-1}\left[e^{-\lambda^2a^2t}\right]$$

$$F^{-1}\left[e^{-\lambda^2a^2t}\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-i\lambda x-\lambda^2a^2t}d\lambda$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}exp\left\{-a^2t\left(\lambda+\frac{ix}{2a^2t}\right)^2-\frac{x^2}{4a^2t}\right\}d\lambda$$

$$=\frac{exp\left\{-\frac{x^2}{4a^2t}\right\}}{2\pi a\sqrt{t}}\int_{-\infty}^{+\infty}exp\left\{-\left[a\sqrt{t}\left(\lambda+\frac{ix}{2a^2t}\right)\right]^2\right\}da\sqrt{t}\left(\lambda+\frac{ix}{2a^2t}\right)$$

$$=\frac{exp\left\{-\frac{x^2}{4a^2t}\right\}}{2\pi a\sqrt{t}}\sqrt{\pi}=\frac{1}{2a\sqrt{\pi t}}e^{--\frac{x^2}{4a^2t}}$$

$$U(t,x)=\frac{1}{2a\sqrt{\pi t}}e^{--\frac{x^2}{4a^2t}+bt}$$

9. 解：(1)

该问题基本解满足

$$\begin{cases}U_t=-aU_x,(t>0,-\infty<x<+\infty)\\ U|_{t=0}=\delta(x)\end{cases}$$

设

$$\widetilde{U}(t,\lambda)=\int_{-\infty}^{+\infty}U(t,x)e^{i\lambda x}dx$$

$$\begin{cases}\frac{d\widetilde{U}}{dt}=ia\lambda\widetilde{U}\\ \widetilde{U}|_{t=0}=1\end{cases}$$

$$\widetilde{U}=e^{-ia\lambda t}$$

$$U(t,x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\lambda at}e^{-i\lambda x}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i(-x+at)\lambda}d\lambda=\delta(-x+at)=\delta(x-at)$$

$$u(t,x)=U(t,x)*\varphi(x)+\int_0^t U(t-\tau,x)*f(\tau,x)d\tau$$

$$=\delta(x-at)*\varphi(x)+\int_0^t\delta[x-a(t-\tau)]*f(\tau,x)d\tau$$

$$=\varphi(x-at)+\int_0^t f[\tau,x-a(t-\tau)]d\tau$$

(2) 方程基本解满足

$$\begin{cases}\frac{\partial^2U}{\partial t^2}+2\frac{\partial U}{\partial t}=a^2\frac{\partial^2U}{\partial x^2}-2U\\ U|_{t=0}=0,U_t|_{t=0}=\delta(x)\end{cases}$$

设

$$\widetilde{U}(t,\lambda)=\int_{-\infty}^{+\infty}U(t,x)e^{i\lambda x}dx$$

则有

$$\begin{cases}\frac{d^2\widetilde{U}}{dt^2}+2\frac{d\widetilde{U}}{dt}=-(\lambda^2a^2+2)\widetilde{U}\\ \widetilde{U}|_{t=0}=0,\widetilde{U}_t|_{t=0}=1\end{cases}$$

$$\widetilde{U}=\frac{1}{\sqrt{\lambda^2a^2+1}}e^{-t}sin\sqrt{\lambda^2a^2+1}t$$

$$U(t,x)=F^{-1}\left[\frac{1}{\sqrt{\lambda^2a^2+1}}e^{-t}sin\sqrt{\lambda^2a^2+1}t\right]=e^{-t}F^{-1}\left[\frac{sinat\sqrt{\lambda^2+\frac{1}{a^2}}}{a\sqrt{\lambda^2+\frac{1}{a^2}}}\right]$$

$$=\frac{e^{-t}}{2a}J_0\left(\frac{1}{a}\sqrt{a^2t^2-x^2}\right)h(at-|x|)$$

$$=\frac{e^{-t}}{2a}J_0\left(\frac{1}{a}\sqrt{a^2t^2-x^2}\right)(-at<x<at)$$

注：课本给的公式有误，应该是

$$F^{-1}\left[\frac{sina\sqrt{\lambda^2+b^2}}{\sqrt{\lambda^2+b^2}}\right]=\frac{1}{2}J_0\left(b\sqrt{a^2-x^2}\right)h(a-|x|)$$

$$u(t,x)=U(t,x)*\psi(x)=\int_{-at}^{at}\frac{e^{-t}}{2a}J_0\left(\frac{1}{a}\sqrt{a^2t^2-\xi^2}\right)\psi(x-\xi)\,d\xi$$

10. 解：

由二维波动方程的基本解

$$U(t,x,y)=\begin{cases}\dfrac{1}{2\pi a}\displaystyle\iint_{D_{at}}\frac{\delta(\xi,\eta)}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta, x^2+y^2\le a^2t^2\\ 0, x^2+y^2>a^2t^2\end{cases}$$

其中

$$D_{at}:(\xi-x)^2+(\eta-y)^2<(at)^2$$

$$U(t,x,y)=\frac{1}{2\pi a}\iint_{D_{at}}\frac{\delta(\xi,\eta)}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta$$

$$=\begin{cases}\dfrac{1}{2\pi a}\dfrac{1}{\sqrt{a^2t^2-x^2-y^2}}, x^2+y^2\le a^2t^2\\ 0, x^2+y^2>a^2t^2\end{cases}$$

其中

$$D_{at}:(\xi-x)^2+(\eta-y)^2<(at)^2$$

记$\mathcal{D}_{at}$: $(\xi-x)^2+(\eta-y)^2<[a(t-\tau)]^2$

$\mathfrak{D}_{at}: x^2+y^2\le[a(t-\tau)]^2$

$$\iint_{\mathcal{D}_{at}}\frac{\delta(\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta*f(\tau,x,y)$$

$$=\iint_{\mathfrak{D}_{at}}\iint_{\mathfrak{D}_{at}}\frac{\delta(\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta f(\tau,x-\mathscr{x},y-\mathscr{y})d\mathscr{x}d\mathscr{y}$$

$$=\iint_{\mathfrak{D}_{at}}\iint_{\mathfrak{D}_{at}}\frac{\delta(\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}}f(\tau,x-\mathscr{x},y-\mathscr{y})d\mathscr{x}d\mathscr{y}d\xi d\eta$$

$$=\iint_{\mathcal{D}_{at}}\frac{f(\tau,\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta$$

这一步是$\delta(\xi,\eta)$将$\mathcal{D}_{at}$区域中$f(\tau,x-x,y-y)$中的$f(\tau,\xi,\eta)$筛选出来了。

因此

$$u(t,x,y)=\frac{1}{2\pi a}\int_0^t d\tau\iint\limits_{\mathcal{D}_{at}}\frac{f(\tau,\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta$$

$$\mathcal{D}_{at}:\ (\xi-x)^2+(\eta-y)^2<[a(t-\tau)]^2$$

11. 解：

考虑一维波动方程：

$$\begin{cases}u_{tt}=a^2u_{xx},(-\infty<x<+\infty,t>0)\\ u|_{t=0}=\varphi(x),u_t|_{t=0}=\psi(x)\end{cases}$$

此问题可以看成是二维波动方程

$$\begin{cases}u_{tt}=a^2\Delta_2 u,(-\infty<x,y<+\infty,t>0)\\ u|_{t=0}=\varphi(x,y),u_t|_{t=0}=\psi(x,y)\end{cases}$$

自变量限制在$y=0$时的特殊情形。

由二维情形解的表达式：

$$u(t,x,y)=\frac{1}{2\pi a}\iint\limits_{D_{at}}\frac{\psi(\xi,\eta)d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}$$

$$+\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint\limits_{D_{at}}\frac{\varphi(\xi,\eta)d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\right]$$

$$D_{at}:(\xi-x)^2+(\eta-y)^2<a^2t^2$$

则一维情形：

$$u(t,x)=\frac{1}{2\pi a}\iint\limits_{D}\frac{\psi(\xi)d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-\eta^2}}+\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint\limits_{D}\frac{\varphi(\xi)d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-\eta^2}}\right]$$

$$D:(\xi-x)^2+\eta^2<a^2t^2$$

$$\iint\limits_{D}\frac{\psi(\xi)d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-\eta^2}}=\int_{x-at}^{x+at}\psi(\xi)d\xi\int_{-\sqrt{(at)^2-(\xi-x)^2}}^{+\sqrt{(at)^2-(\xi-x)^2}}\frac{1}{\sqrt{(at)^2-(\xi-x)^2-\eta^2}}d\eta$$

对于

$$\int_{-\sqrt{(at)^2-(\xi-x)^2}}^{+\sqrt{(at)^2-(\xi-x)^2}}\frac{1}{\sqrt{(at)^2-(\xi-x)^2-\eta^2}}d\eta=arcsin\frac{\eta}{\sqrt{(at)^2-(\xi-x)^2}}\Big|_{-\sqrt{(at)^2-(\xi-x)^2}}^{\sqrt{(at)^2-(\xi-x)^2}}$$

$$=\pi$$

所以

$$\iint\limits_{D}\frac{\psi(\xi)d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-\eta^2}}=\pi\int_{x-at}^{x+at}\psi(\xi)d\xi$$

$$\frac{\partial}{\partial t}\left[\iint\limits_{D}\frac{\varphi(\xi)d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-\eta^2}}\right]=\pi\frac{\partial}{\partial t}\left[\int_{x-at}^{x+at}\varphi(\xi)d\xi\right]=\pi a[\varphi(x+at)+\varphi(x-at)]$$

故

$$u(t,x)=\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi+\frac{[\varphi(x+at)+\varphi(x-at)]}{2}$$

12. 解：

（1） 依题

已知一维热传导方程的基本解为

$$U(t,x)=\frac{1}{2a\sqrt{\pi t}}exp\left\{-\frac{x^2}{4a^2t}\right\}$$

则

$$u(t,x)=U(t,x)*e^{-x^2}=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}exp\left\{-\frac{\xi^2}{4a^2t}\right\}e^{-(x-\xi)^2}d\xi$$

$$\int_{-\infty}^{+\infty}exp\left\{-\frac{\xi^2}{4a^2t}\right\}e^{-(x-\xi)^2}d\xi$$

$$=\int_{-\infty}^{+\infty}exp\left\{-\frac{\xi^2(4a^2t+1)-8a^2tx\xi+4a^2tx^2}{4a^2t}\right\}d\xi$$

$$=e^{-x^2}\int_{-\infty}^{+\infty}exp\left\{-\frac{\xi^2(4a^2t+1)-8a^2tx\xi}{4a^2t}\right\}d\xi$$

$$=e^{-x^2}e^{\frac{4a^2tx}{4a^2t+1}x^2}\int_{-\infty}^{+\infty}exp\left\{-\frac{4a^2t+1}{4a^2t}\left(\xi-\frac{4a^2tx}{4a^2t+1}\right)^2\right\}d\xi$$

$$=e^{-\frac{x^2}{4a^2t+1}}\frac{2a}{\sqrt{4a^2t+1}}\sqrt{\pi t}$$

$$u(t,x)=\frac{e^{-\frac{x^2}{4a^2t+1}}}{\sqrt{4a^2t+1}}$$

（2） 依题

$$u(t,x,y)=\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint_{D_{at}}\frac{\xi^2(\xi+\eta)d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\right]$$

$$D_{at}:(\xi-x)^2+(\eta-y)^2<a^2t^2$$

$$\iint_{D_{at}}\frac{\xi^3d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}$$

$$\iint_{D_{at}}\frac{\xi^3d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}$$

$$=\int_{x-at}^{x+at}\xi^3d\xi\int_{y-\sqrt{(at)^2-(\xi-x)^2}}^{y+\sqrt{(at)^2-(\xi-x)^2}}\frac{1}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}d\eta$$

$$\int_{y-\sqrt{(at)^2-(\xi-x)^2}}^{y+\sqrt{(at)^2-(\xi-x)^2}}\frac{1}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}d\eta$$

$$\xRightarrow{\eta-y=\rho}\int_{-\sqrt{(at)^2-(\xi-x)^2}}^{\sqrt{(at)^2-(\xi-x)^2}}\frac{1}{\sqrt{(at)^2-(\xi-x)^2-\rho^2}}d\rho=\pi$$

$$\iint_{D_{at}} \frac{\xi^3 d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} = \pi 2xat[x^2+a^2t^2]$$

$$\iint_{D_{at}} \frac{\xi^2 \eta d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}$$

$$= \int_{x-at}^{x+at} \xi^2 d\xi \int_{-\sqrt{(at)^2-(\xi-x)^2}}^{+\sqrt{(at)^2-(\xi-x)^2}} \frac{\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} d\eta$$

$$\int_{y-\sqrt{(at)^2-(\xi-x)^2}}^{y+\sqrt{(at)^2-(\xi-x)^2}} \frac{\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} d\eta$$

$$\xRightarrow{\eta-y=\rho} \int_{-\sqrt{(at)^2-(\xi-x)^2}}^{\sqrt{(at)^2-(\xi-x)^2}} \frac{\rho+y}{\sqrt{(at)^2-(\xi-x)^2-\rho^2}} d\rho$$

$$\int_{-\sqrt{(at)^2-(\xi-x)^2}}^{\sqrt{(at)^2-(\xi-x)^2}} \frac{\rho}{\sqrt{(at)^2-(\xi-x)^2-\rho^2}} d\rho = 0$$

$$\int_{-\sqrt{(at)^2-(\xi-x)^2}}^{\sqrt{(at)^2-(\xi-x)^2}} \frac{y}{\sqrt{(at)^2-(\xi-x)^2-\rho^2}} d\rho = \pi y$$

$$\iint_{D_{at}} \frac{\xi^2 \eta d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} = \frac{2\pi ayt(3x^2+a^2t^2)}{3}$$

所以

$$u(t,x,y) = \frac{\partial}{\partial t}\left[xt(x^2+a^2t^2) + \frac{yt(3x^2+a^2t^2)}{3}\right] = x^3 + 3a^2xt + x^2y + a^2yt^2$$

（3） 依题

$$u(t,x,y) = \frac{1}{2\pi a} \iint_{D_{at}} \frac{\xi+\eta d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} + \int_0^t U(t-\tau,x,y) * (x+y) d\tau$$

$$D_{at}: (\xi-x)^2 + (\eta-y)^2 < a^2t^2$$

$$U(t-\tau,x,y) = \frac{1}{2\pi a} \iint_{\mathcal{D}_{at}} \frac{\delta(\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}} d\xi d\eta$$

其中$\mathcal{D}_{at}: (\xi-x)^2 + (\eta-y)^2 < [a(t-\tau)]^2$

$$\int_0^t U(t-\tau,x,y) * (x+y) d\tau$$

$$= \frac{1}{2\pi a} \int_0^t \iint_{\mathcal{D}_{at}} \frac{\delta(\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}} d\xi d\eta$$

$$* (x+y) d\tau$$

$$\iint_{\mathcal{D}_{at}} \frac{\delta(\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}} d\xi d\eta * (x+y)$$

$$= \iint_{D} \iint_{\mathcal{D}_{at}} \frac{\delta(\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}} (x-x_0+y$$
$$-y_0) d\xi d\eta dx_0 dy_0$$

$$\iint_{D} \iint_{\mathcal{D}_{at}} \frac{\delta(\xi,\eta)}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}} (x-x_0+y-y_0) d\xi d\eta dx_0 dy_0$$

$$= \iint_{\mathcal{D}_{at}} \frac{\xi+\eta}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}} d\xi d\eta$$

$$\iint_{\mathcal{D}_{at}} \frac{\xi d\xi d\eta}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}} = 2\pi x a(t-\tau)$$

$$\iint_{\mathcal{D}_{at}} \frac{\eta d\xi d\eta}{\sqrt{a^2(t-\tau)^2-(\xi-x)^2-(\eta-y)^2}} = 2\pi y a(t-\tau)$$

$$\iint_{\mathcal{D}_{at}} \frac{\xi d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}$$
$$= \int_{x-at}^{x+at} \xi d\xi \int_{-\sqrt{(at)^2-(\xi-x)^2}}^{+\sqrt{(at)^2-(\xi-x)^2}} \frac{1}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} d\eta$$
$$= 2\pi x a t$$

$$\iint_{\mathcal{D}_{at}} \frac{\eta d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}$$
$$= \int_{x-at}^{x+at} d\xi \int_{-\sqrt{(at)^2-(\xi-x)^2}}^{+\sqrt{(at)^2-(\xi-x)^2}} \frac{\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} d\eta$$
$$= 2\pi y a t$$

$$\frac{1}{2\pi a} \iint_{\mathcal{D}_{at}} \frac{\xi+\eta d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} + \int_0^t U(t-\tau,x,y) * (x+y) d\tau$$
$$= \frac{1}{2\pi a} [2\pi x a t + 2\pi y a t]$$
$$+ (x+y) \int_0^t (t-\tau) d\tau = (x+y)\left(t+\frac{t^2}{2}\right)$$

故

$$u(t,x) = (x+y)\left(t+\frac{t^2}{2}\right)$$

（4） 此题既可以效仿（3）的做法，直接由三维波动方程的公式直接做，也可以用特

解法求解。设$u=v+u^*$
观察方程的非齐次项，为$x+y+z$
且该非齐次项与初始位置，初始速度均一致。
且非齐次项的Δ_3之后的结果为 0。
因此我们可以设所求方程的特解为：

$$u^*(t,x,y,z)=f(t)(x+y+z)$$

则由题目已知得到三个关于$f(t)$的条件

$$\begin{cases} f''(t)=0 \\ f(0)=1 \\ f'(0)=1 \end{cases}$$

则可以解得

$$f(t)=\frac{1}{2}t^2+t+1$$

$$u^*(t,x,y,z)=\left(\frac{1}{2}t^2+t+1\right)(x+y+z)$$

之后所得方程为

$$\begin{cases} v_{tt}=a^2\Delta_3 v \\ v|_{t=0}=0 \\ v_t|_{t=0}=0 \end{cases}$$

由三维波动方程解的公式可知

$$v\equiv 0$$

所以原方程解为

$$u=\left(\frac{1}{2}t^2+t+1\right)(x+y+z)$$

与（3）过程做对比，说明恰当的特解选取对于解决问题而言效果十分显著。