1. (a) 【5】$t = 3.76/1.88 = 2$, 原假设下 t 统计量服从$t_{n-7} = t_{50}$.

   (b) 【5】$F = \frac{n-p}{p-1} \times \frac{R^2}{1-R^2} = \frac{57-7}{7-1} \times \frac{0.81}{1-0.81} = 35.526$. 分布为$F_{p-1,n-p} = F_{6,50}$.

   (c) 【5】$r_{y\hat{y}} = \sqrt{R^2} = 0.9$.

   (d) 【5】简单回归的b估计的方差$v = \sigma^2/s_{xx}$, 而全模型下的b估计的方差为 $v \times \frac{1}{1-R_x^2} = 1.88^2 = 3.534$, 所以$v = (1-R_x^2) \times 4.534 = 1.767$.

2. (a) $z_i = 0$时，模型为 $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \epsilon_i \sim (0, \sigma^2), i = 1, ..., n_0$, 此即第一个模型。
   $z_i = 1$时，模型为 $y_i = (\beta_0 + \gamma_0) + (\beta_1 + \gamma_1)x_i + \epsilon_i, \epsilon_i \sim (0, \sigma^2), i = n_0 + 1, ..., n$，此即第二个模型，其中$\alpha_0 = \beta_0 + \gamma_0, \alpha_1 = \beta_1 + \gamma_1$.

   (b) 由(a)知，$\alpha_1 - \beta_1 = \gamma_1$, 故$H_0 : \alpha_1 = \beta_1 \Leftrightarrow$ 在合并后的模型中 $H_0 : \gamma_1 = 0$

3. (a) 【10】给定 X 条件下 $E(\widetilde{\beta}|X) = E((AX)^{-1}AY) = (AX)^{-1}AX\beta = \beta$.

   (b) 【10】 由GM定理：

$$var(\widetilde{Y}) = (AX)^{-1}AA'(AX)^{-1} \geq var(\widehat{Y}) = \sigma^2 P_X$$

$\frac{2+3}{1+2+3+4}$

   因为无偏，M=var,

$$m(\widetilde{Y}) = trM(\widetilde{Y}) = tr(var(\widetilde{Y}) \geq tr(var(\widehat{Y}) = m(\widehat{Y}).$$

4. (a) $M(\widetilde{\beta}) = E(\widetilde{\beta} - \beta)(\widetilde{\beta} - \beta)' = var(\widetilde{\beta}) + \mathbf{bb}'$, 其中$\mathbf{b}$为$\widetilde{\beta}$的偏差：

$$\mathbf{b} = E(\widetilde{\beta}) - \beta = \begin{pmatrix} E(\widetilde{\beta}_1) - \beta_1 \\ E(\widetilde{\beta}_2) - \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\beta_2 \end{pmatrix},$$

   其中我们用到了事实$E(\widetilde{\beta}_1) = E[(X_1'X_1)^{-1}X_1'Y] = E[X_1'X_1)^{-1}X_1'(X_1\beta_1 + X_2\beta_2) = \beta_1 + X_1'X_1)^{-1}X_1'X_2\beta_2 = \beta_1$ (因为$X_1'X_2 = 0$)。 另外, $var(\widetilde{\beta}_1) = var(X_1'X_1)^{-1}X_1'Y) = \sigma^2(X_1'X_1)^{-1}$, $\widetilde{\beta}_2 = 0$, 故

$$var(\widetilde{\beta}) = var \begin{pmatrix} \widetilde{\beta}_1 \\ \widetilde{\beta}_2 \end{pmatrix} = \begin{pmatrix} \sigma^2(X_1'X_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

   两者相加即得：

$$M(\widetilde{\beta}) = \begin{pmatrix} \sigma^2(X_1'X_1)^{-1} & 0 \\ 0 & \beta_2\beta_2' \end{pmatrix}$$

   (b) $M(\widehat{\beta}) = var(\widehat{\beta}) = \sigma^2(X'X)^{-1}$, 因为$X_1'X_2 = 0$, 故

$$M(\widehat{\beta}) = \sigma^2 \begin{pmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{pmatrix}^{-1} = \begin{pmatrix} \sigma^2(X_1'X_1)^{-1} & 0 \\ 0 & \sigma^2(X_2'X_2)^{-1} \end{pmatrix}$$

   由引理, 条件 $||X_2\beta_2||^2 = \beta_2'X_2'X_2\beta_2 \leq \sigma^2$等价于 $\beta_2\beta_2' \leq \sigma^2(X_2'X_2)^{-1}$, 所以$M(\widetilde{\beta}) \leq M(\widehat{\beta})$。

5. (a) 【8】$Y = X\gamma + \epsilon, \epsilon \sim N(0, \sigma^2 I_4)$, 其中 $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$

$\hat{\alpha} = \frac{y_1+y_2+y_3+y_4}{4}, \hat{\beta} = \frac{y_1+y_2-y_3-y_4}{4}, var(\hat{\alpha}) = var(\hat{\beta}) = \frac{\sigma^2}{4}$. 总方差=$\frac{\sigma^2}{2}$.

(b) 【8】$\hat{Y} = (\hat{\alpha}+\hat{\beta}, \hat{\alpha}+\hat{\beta}, \hat{\alpha}-\hat{\beta}, \hat{\alpha}-\hat{\beta})' = (\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}, \frac{y_3+y_4}{2}, \frac{y_3+y_4}{2})'$。

原假设下$\hat{\beta} = \frac{y_1+y_2}{2}$, 拟合值 $\hat{Y}_0 = (\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}, 0, 0)'$，所以

$$F = \frac{||\hat{Y} - \hat{Y}_0||^2/1}{||Y - \hat{Y}||^2/(4-2)} = \frac{2(y_3+y_4)^2}{(y_1-y_2)^2 + (y_3-y_4)^2}$$

H_0下，$F \sim F_{1,2}$.

(c) 【4】方案2总方差 = $(\frac{1}{m} + \frac{1}{n})\sigma^2 \leq \frac{\sigma^2}{2}$, 因为 $(m+n)(1/m+1/n) \geq 4$, 所以 $m+n \geq 8$.

6. （20分）(a) 模型写成向量形式 $Y = \mathbf{1}a + \mathbf{x}b + \mathbf{z}c + \epsilon$, 令 $\mathbf{z}^{\perp} = \mathbf{z} - P_{\mathbf{1}}\mathbf{z} = \mathbf{z} - \mathbf{1}\bar{z}$ 以及

$$\mathbf{x}^{\perp} = \mathbf{x} - P_{(\mathbf{1},\mathbf{z})}\mathbf{x} = \mathbf{x} - P_{\mathbf{1}}\mathbf{x} - P_{\mathbf{z}^{\perp}}\mathbf{x} = \mathbf{x} - \mathbf{1}\bar{x} - P_{\mathbf{z}-\mathbf{1}\bar{z}}\mathbf{x} = \mathbf{x} - \mathbf{1}\bar{x} - (\mathbf{z} - \mathbf{1}\bar{z})\hat{\gamma}$$

带入 $\hat{b} = \mathbf{x}^{\perp\prime}Y/||\mathbf{x}^{\perp}||^2$ 即得。

(b) 由 (a) 的结论知

$$var(\hat{b}|x,z) = \sigma^2 / \sum_{i=1}^{n}(x_i - \bar{x} - (z_i - \bar{z})\hat{\gamma})^2$$

而由 $\hat{\gamma} = s_{xz}/s_{zz}$

$$\sum_{i=1}^{n}(x_i - \bar{x} - (z_i - \bar{z})\hat{\gamma})^2 = s_{xx} - 2\hat{\gamma}s_{xz} + \hat{\gamma}^2 s_{zz} = s_{xx} - \frac{s_{xz}^2}{s_{zz}} = s_{xx}(1 - r_{xz}^2).$$

(c) 相比于独立情形，$\hat{b}$的方差增大了$VIF = 1/(1 - r_{xz}^2)$倍。

(d) x, z高度相关（比如$r_{xz}^2 \approx 1$）而且误差方差较小时， 回归方程的显著性检验会比较显著，而$H_0: b = 0$的检验 $t_x = \sqrt{s_{xx}(1 - r_{xz}^2)}\frac{\hat{b}}{\hat{\sigma}}$ 可能会不显著，同样$H_0: c = 0$ 的检验 $t_z = \sqrt{s_{zz}(1 - r_{xz}^2)}\frac{\hat{c}}{\hat{\sigma}}$可能会不显著.