$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}=\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{C}=\boldsymbol{B}\cdot\boldsymbol{C}\times\boldsymbol{A}=\boldsymbol{B}\times\boldsymbol{C}\cdot\boldsymbol{A}=\boldsymbol{C}\cdot\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{C}\times\boldsymbol{A}\cdot\boldsymbol{B} \tag{V.1}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{C}\times\boldsymbol{B})\times\boldsymbol{A}=\boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}) \tag{V.2}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})+\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A})+\boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B})=0 \tag{V.3}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C}) \tag{V.4}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{D})\boldsymbol{C}-(\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{D} \tag{V.5}$$

$$\nabla(fg)=\nabla(gf)=f\nabla g+g\nabla f \tag{V.6}$$

$$\nabla\cdot(f\boldsymbol{A})=f\nabla\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\nabla f \tag{V.7}$$

$$\nabla\times(f\boldsymbol{A})=f\nabla\times\boldsymbol{A}+\nabla f\times\boldsymbol{A} \tag{V.8}$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot\nabla\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\times\boldsymbol{B} \tag{V.9}$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{A}(\nabla\cdot\boldsymbol{B})-\boldsymbol{B}(\nabla\cdot\boldsymbol{A})+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{V.10}$$

$$\boldsymbol{A}\times(\nabla\times\boldsymbol{B})=(\nabla\boldsymbol{B})\cdot\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{V.11}$$

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A} \tag{V.12}$$

$$\nabla^2 f=\nabla\cdot\nabla f \tag{V.13}$$

$$\nabla^2\boldsymbol{A}=\nabla(\nabla\cdot\boldsymbol{A})-\nabla\times\nabla\times\boldsymbol{A} \tag{V.14}$$

$$\nabla\times\nabla f=0 \tag{V.15}$$

$$\nabla\cdot\nabla\times\boldsymbol{A}=0 \tag{V.16}$$

$$\nabla\cdot\boldsymbol{R}=3,\quad \boldsymbol{R}=\text{position vector}. \tag{V.17}$$

$$\boldsymbol{I}\cdot\boldsymbol{A}=\boldsymbol{A}\cdot\boldsymbol{I}=\boldsymbol{A} \tag{T.1}$$

$$\boldsymbol{I}\cdot\boldsymbol{AB}=\boldsymbol{AB} \tag{T.2}$$

$$\boldsymbol{I}\cdot\nabla\boldsymbol{B}=\nabla\boldsymbol{B} \tag{T.3}$$

$$\boldsymbol{AB}:\boldsymbol{CD}=(\boldsymbol{B}\cdot\boldsymbol{C})(\boldsymbol{A}\cdot\boldsymbol{D})=\boldsymbol{D}\cdot\boldsymbol{AB}\cdot\boldsymbol{C} \tag{T.4}$$

$$\boldsymbol{F}:\boldsymbol{AB}=(\boldsymbol{F}\cdot\boldsymbol{A})\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{F}\cdot\boldsymbol{A} \tag{T.5}$$

$$\boldsymbol{I}:\boldsymbol{AB}=\boldsymbol{A}\cdot\boldsymbol{B} \tag{T.6}$$

$$\boldsymbol{I}:\nabla\boldsymbol{B}=\nabla\cdot\boldsymbol{B} \tag{T.7}$$

$$\boldsymbol{I}:\boldsymbol{F}=F_i^i=\operatorname{tr}\boldsymbol{F} \tag{T.8}$$

$$\nabla\cdot(a\boldsymbol{I})=\nabla a \tag{T.9}$$

$$\nabla\cdot\boldsymbol{AB}=\boldsymbol{A}\cdot\nabla\boldsymbol{B}+(\nabla\cdot\boldsymbol{A})\boldsymbol{B} \tag{T.10}$$

$$\boldsymbol{I}\times\boldsymbol{A}=\boldsymbol{A}\times\boldsymbol{I} \tag{T.11}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{I}=\boldsymbol{I}\times(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{BA}-\boldsymbol{AB} \tag{T.12}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}=\boldsymbol{C}\times(\boldsymbol{B}\times\boldsymbol{A})=\boldsymbol{B}(\boldsymbol{C}\cdot\boldsymbol{A})-\boldsymbol{A}(\boldsymbol{C}\cdot\boldsymbol{B})=\boldsymbol{C}\cdot(\boldsymbol{AB}-\boldsymbol{BA}) \tag{T.13}$$

$$\boldsymbol{A}\times(\boldsymbol{BC})=(\boldsymbol{A}\times\boldsymbol{B})\boldsymbol{C} \tag{T.14}$$

$$\nabla\cdot(\boldsymbol{I}\times\boldsymbol{C})=\nabla\times\boldsymbol{C} \tag{T.15}$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B})=\nabla\cdot(\boldsymbol{BA}-\boldsymbol{AB}) \tag{T.16}$$

$$\boldsymbol{A}\times(\nabla\times\boldsymbol{B})=(\nabla\boldsymbol{B})\cdot\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\boldsymbol{B} \tag{T.17}$$

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+\boldsymbol{A}\cdot\nabla\boldsymbol{B}+\boldsymbol{B}\cdot\nabla\boldsymbol{A}=(\nabla\boldsymbol{A})\cdot\boldsymbol{B}+(\nabla\boldsymbol{B})\cdot\boldsymbol{A} \tag{T.18}$$

$$\nabla(\boldsymbol{A}\times\boldsymbol{B})=\nabla\boldsymbol{A}\times\boldsymbol{B}-\nabla\boldsymbol{B}\times\boldsymbol{A} \tag{T.19}$$

$$\nabla\times(\boldsymbol{AB})=(\nabla\times\boldsymbol{A})\boldsymbol{B}-\boldsymbol{A}\times\nabla\boldsymbol{B} \tag{T.20}$$

$$\boldsymbol{A}\times\nabla\boldsymbol{B}-(\boldsymbol{A}\times\nabla\boldsymbol{B})^{\mathrm{T}}=[(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-\nabla\boldsymbol{B}\cdot\boldsymbol{A}]\times\boldsymbol{I} \tag{T.21}$$

$$\nabla\boldsymbol{B}\times\boldsymbol{A}-(\nabla\boldsymbol{B}\times\boldsymbol{A})^{\mathrm{T}}=[(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\boldsymbol{B}]\times\boldsymbol{I} \tag{T.22}$$

$$(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-\nabla\boldsymbol{B}\cdot\boldsymbol{A}=-(\boldsymbol{A}\times\nabla)\times\boldsymbol{B} \tag{T.23}$$

$$(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\boldsymbol{B}=\boldsymbol{A}\times(\nabla\times\boldsymbol{B})-(\boldsymbol{A}\times\nabla)\times\boldsymbol{B} \tag{T.24}$$

$$(\boldsymbol{A}\times\nabla)\cdot\boldsymbol{B}=\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) \tag{T.25}$$

$$\boldsymbol{A}\times\nabla\boldsymbol{B}+(\nabla\boldsymbol{B}\times\boldsymbol{A})^{\mathrm{T}}=[\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})]\boldsymbol{I}-(\nabla\times\boldsymbol{B})\boldsymbol{A} \tag{T.26}$$

$$\nabla\boldsymbol{B}\times\boldsymbol{A}+(\boldsymbol{A}\times\nabla\boldsymbol{B})^{\mathrm{T}}=[\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})]\boldsymbol{I}-\boldsymbol{A}(\nabla\times\boldsymbol{B}) \tag{T.27}$$

$$\begin{aligned}\boldsymbol{A}\times\nabla\boldsymbol{B}+\nabla\boldsymbol{B}\times\boldsymbol{A}&=\boldsymbol{I}\times[(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-\nabla\boldsymbol{B}\cdot\boldsymbol{A}]\\&\quad+[\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})]\boldsymbol{I}-\boldsymbol{A}(\nabla\times\boldsymbol{B})\\&=\boldsymbol{I}\times[(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\boldsymbol{B}]\\&\quad+[\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})]\boldsymbol{I}-(\nabla\times\boldsymbol{B})\boldsymbol{A}\end{aligned} \tag{T.28}$$

$$(\boldsymbol{A}\times\boldsymbol{C})\cdot\boldsymbol{F}=\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{F})=-\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{F}) \tag{T.29}$$

$$\boldsymbol{F}\cdot(\boldsymbol{A}\times\boldsymbol{C})=(\boldsymbol{F}\times\boldsymbol{A})\cdot\boldsymbol{C}=-(\boldsymbol{F}\times\boldsymbol{C})\cdot\boldsymbol{A} \tag{T.30}$$

$$\boldsymbol{A}\cdot\boldsymbol{F}\times\boldsymbol{C}=(\boldsymbol{A}\cdot\boldsymbol{F})\times\boldsymbol{C}=\boldsymbol{A}\cdot(\boldsymbol{F}\times\boldsymbol{C})=-\boldsymbol{C}\times(\boldsymbol{A}\cdot\boldsymbol{F}) \tag{T.31}$$

$$\boldsymbol{A}\times\boldsymbol{F}\cdot\boldsymbol{C}=\boldsymbol{A}\times(\boldsymbol{F}\cdot\boldsymbol{C})=(\boldsymbol{A}\times\boldsymbol{F})\cdot\boldsymbol{C}=-(\boldsymbol{F}\cdot\boldsymbol{C})\times\boldsymbol{A} \tag{T.32}$$

$$\boldsymbol{A}\cdot\nabla\boldsymbol{B}\times\boldsymbol{C}+\boldsymbol{C}\times\nabla\boldsymbol{B}\cdot\boldsymbol{A}=\boldsymbol{C}\times[\boldsymbol{A}\times(\nabla\times\boldsymbol{B})] \tag{T.33}$$

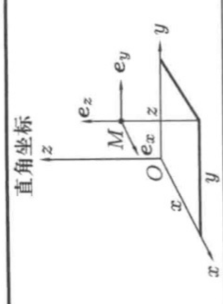
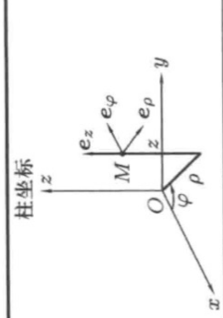
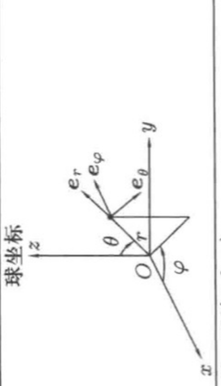
$$\boldsymbol{A}\cdot\nabla\boldsymbol{B}\times\boldsymbol{C}-\boldsymbol{C}\cdot\nabla\boldsymbol{B}\times\boldsymbol{A}=[(\nabla\cdot\boldsymbol{B})\boldsymbol{I}-\nabla\boldsymbol{B}]\cdot(\boldsymbol{A}\times\boldsymbol{C}) \tag{T.34}$$

$$\boldsymbol{A}\times\nabla\boldsymbol{B}\cdot\boldsymbol{C}-\boldsymbol{C}\times\nabla\boldsymbol{B}\cdot\boldsymbol{A}=(\boldsymbol{A}\times\boldsymbol{C})\cdot[(\nabla\cdot\boldsymbol{B})\boldsymbol{I}-\nabla\boldsymbol{B}] \tag{T.35}$$

$$\boldsymbol{A}\cdot\nabla\boldsymbol{B}\cdot\boldsymbol{C}-\boldsymbol{C}\cdot\nabla\boldsymbol{B}\cdot\boldsymbol{A}=(\boldsymbol{A}\times\boldsymbol{C})\cdot(\nabla\times\boldsymbol{B}) \tag{T.36}$$

$$\nabla\boldsymbol{R}=\boldsymbol{I},\qquad \boldsymbol{R}=\text{position vector}. \tag{T.37}$$

坐标系

	直角坐标	柱坐标	球坐标
			
定义	$U=U(x,y,z)$ $\boldsymbol{A}=A_x\boldsymbol{e}_x+A_y\boldsymbol{e}_y+A_z\boldsymbol{e}_z$ $A_x=A_x(x,y,z)$ $A_y=A_y(x,y,z)$ $A_z=A_z(x,y,z)$	$U=U(\rho,\varphi,z)$ $\boldsymbol{A}=A_\rho\boldsymbol{e}_\rho+A_\varphi\boldsymbol{e}_\varphi+A_z\boldsymbol{e}_z$ $A_\rho=A_x\cos\varphi+A_y\sin\varphi$ $A_\varphi=-A_x\sin\varphi+A_y\cos\varphi$	$U=U(r,\theta,\varphi)$ $\boldsymbol{A}=A_r\boldsymbol{e}_r+A_\theta\boldsymbol{e}_\theta+A_\varphi\boldsymbol{e}_\varphi$ $A_r=A_\rho\sin\theta+A_z\cos\theta$ $A_\theta=A_\rho\cos\theta-A_z\sin\theta$ $A_\varphi=-A_x\sin\varphi+A_y\cos\varphi$
梯度	$\nabla U=(\partial U/\partial x)\boldsymbol{e}_x$ $+(\partial U/\partial y)\boldsymbol{e}_y$ $+(\partial U/\partial z)\boldsymbol{e}_z$	$(\nabla U)_\rho=\partial U/\partial\rho$ $(\nabla U)_\varphi=[\partial U/\partial\varphi]/\rho$ $(\nabla U)_z=\partial U/\partial z$	$(\nabla U)_r=\partial U/\partial r$ $(\nabla U)_\theta=[\partial U/\partial\theta]/r$ $(\nabla U)_\varphi=[\partial U/\partial\varphi]/(r\sin\theta)$
拉普拉斯算符	$\Delta U=\dfrac{\partial^2U}{\partial x^2}+\dfrac{\partial^2U}{\partial y^2}+\dfrac{\partial^2U}{\partial z^2}$	$\Delta U=\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial U}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2U}{\partial\varphi^2}+\dfrac{\partial^2U}{\partial z^2}$	$\Delta U=\dfrac{1}{r}\dfrac{\partial^2}{\partial r^2}(rU)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial U}{\partial\theta}\right)$ $+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2U}{\partial\varphi^2}$
散度	$\nabla\cdot\boldsymbol{A}=\dfrac{\partial A_x}{\partial x}+\dfrac{\partial A_y}{\partial y}+\dfrac{\partial A_z}{\partial z}$	$\nabla\cdot\boldsymbol{A}=\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho A_\rho)+\dfrac{1}{\rho}\dfrac{\partial A_\varphi}{\partial\varphi}+\dfrac{\partial A_z}{\partial z}$	$\nabla\cdot\boldsymbol{A}=\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2A_r)+\dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta A_\theta)$ $+\dfrac{1}{r\sin\theta}\dfrac{\partial A_\varphi}{\partial\varphi}$
旋度	$\nabla\times\boldsymbol{A}=(\partial A_z/\partial y-\partial A_y/\partial z)\boldsymbol{e}_x$ $+(\partial A_x/\partial z-\partial A_z/\partial x)\boldsymbol{e}_y$ $+(\partial A_y/\partial x-\partial A_x/\partial y)\boldsymbol{e}_z$	$(\nabla\times\boldsymbol{A})_\rho=(\partial A_z/\partial\varphi)/\rho-\partial A_\varphi/\partial z$ $(\nabla\times\boldsymbol{A})_\varphi=\partial A_\rho/\partial z-\partial A_z/\partial\rho$ $(\nabla\times\boldsymbol{A})_z=[\partial(\rho A_\varphi)/\partial\rho-\partial A_\rho/\partial\varphi]/\rho$	$(\nabla\times\boldsymbol{A})_r=[\partial(\sin\theta A_\varphi)/\partial\theta-\partial A_\theta/\partial\varphi]/(r\sin\theta)$ $(\nabla\times\boldsymbol{A})_\theta=[\partial A_r/\partial\varphi-\sin\theta\partial(rA_\varphi)/\partial r]/(r\sin\theta)$ $(\nabla\times\boldsymbol{A})_\varphi=[\partial(rA_\theta)/\partial r-\partial A_r/\partial\theta]/r$

$$\operatorname{grad}\phi\equiv\vec\nabla\phi=\frac{\partial\vec\phi}{\partial u^i}\vec e^{\,i}$$

$$F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha$$

$$\vec\nabla\cdot\vec A=\vec\nabla\cdot\left(JA_i\frac{\vec e_i}{J}\right)=\frac{\vec e_i}{J}\cdot\vec e^{\,j}\frac{\partial(JA_i)}{\partial u^j}=\frac{1}{J}\frac{\partial(JA_i)}{\partial u^i}\qquad J=\sqrt{g}$$

$$\vec\nabla\times\vec A=\frac{\varepsilon^{ijk}}{J}\frac{\partial A_j}{\partial u^i}\vec e_k$$

$$F_{\alpha\beta}=\begin{bmatrix}0&E_x/c&E_y/c&E_z/c\\-E_x/c&0&-B_z&B_y\\-E_y/c&B_z&0&-B_x\\-E_z/c&-B_y&B_x&0\end{bmatrix}.$$

$$\vec\nabla\cdot(\vec f\vec g\times\vec r)=\vec\nabla\cdot\vec f\,(\vec g\times\vec r)+(\vec f\cdot\vec\nabla)(\vec g\times\vec r)$$

$$x^i=(ct,\vec x),\quad u^i=\frac{dx^i}{d\tau},\quad A^i=\left(\frac{\varphi}{c},\vec A\right),\quad j^i=(\rho c,\vec j)$$

$$S=-mc\int ds-\frac{1}{c}\int j^iA_i\,d\Omega-\frac{\varepsilon_0c}{4}\int F^{\mu\nu}F_{\mu\nu}\,d\Omega$$

$$F^{\alpha\beta}=\begin{bmatrix}0&-E_x/c&-E_y/c&-E_z/c\\E_x/c&0&-B_z&B_y\\E_y/c&B_z&0&-B_x\\E_z/c&-B_y&B_x&0\end{bmatrix},$$

$$\mu_0j^\nu-\partial_\mu F^{\mu\nu}=0$$

$$\hat T^{00}=\frac{1}{2}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right),$$

$$\hat T^{0i}=\frac{1}{c}\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B},$$

$$\begin{cases}\vec\nabla\cdot\vec E=\dfrac{\rho}{\varepsilon_0}\\\vec\nabla\times\vec B=\mu_0\varepsilon_0\dfrac{\partial\vec E}{\partial t}+\mu_0\vec j\end{cases}$$

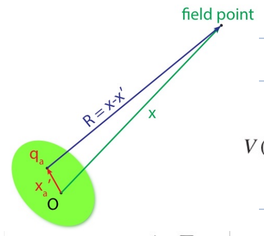
$$\begin{cases}\vec\nabla\times\vec E=-\dfrac{\partial\vec B}{\partial t}\\\vec\nabla\cdot\vec B=0\end{cases}$$

$$F^{\alpha\beta}F_{\alpha\beta}=2\left(B^2-\frac{1}{c^2}E^2\right).$$

$$E_x=E'_x,\qquad E_y=\frac{E'_y+VB'_z}{\sqrt{1-V^2/c^2}},\qquad E_z=\frac{E'_z-VB'_y}{\sqrt{1-V^2/c^2}}.$$

$$B_x=B'_x,\qquad B_y=\frac{B'_y-(V/c^2)E'_z}{\sqrt{1-V^2/c^2}},\qquad B_z=\frac{B'_z+(V/c^2)E'_y}{\sqrt{1-V^2/c^2}}.$$

$$\hat T^{ij}=-\left[\frac{1}{\mu_0}\boldsymbol{BB}+\epsilon_0\boldsymbol{EE}-\left(\frac{1}{2}\mu_0B^2+\frac{1}{2}\epsilon_0E^2\right)\mathsf{I}\right].$$



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau',$$

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = (3x^2-1)/2$$
$$P_3(x) = (5x^3-3x)/2$$
$$P_4(x) = (35x^4-30x^2+3)/8$$
$$P_5(x) = (63x^5-70x^3+15x)/8$$
$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l.$$

$$\phi^{(0)} = \phi(x) = \frac{1}{4\pi\epsilon_0}\frac{\sum_i q_i}{r}.$$
$$\phi^{(1)} = -\frac{1}{4\pi\epsilon_0}\sum_a q x'\cdot\nabla\frac{1}{r} = -\frac{\boldsymbol{p}}{4\pi\epsilon_0}\cdot\nabla\frac{1}{r} = \frac{\boldsymbol{p}\cdot\boldsymbol{x}}{4\pi\epsilon_0 r^3}.$$
$$\boldsymbol{E}^{(1)} = \frac{3(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n}-\boldsymbol{p}}{4\pi\epsilon_0 r^3}.$$
$$\phi^{(2)} = \frac{1}{8\pi\epsilon_0}\sum_a q x_i' x_j' \frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r}. \quad \phi^{(2)} = \frac{D_{ij}}{24\pi\epsilon_0}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r}. \quad D_{ij} = \sum_a 3q x_i' x_j',$$
$$\phi^{(2)} = \frac{1}{24\pi\epsilon_0}\left(D_{ij} - r'^2\delta_{ij}\right)\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r}. \quad D_{ij} = \sum_a q(3x_i' x_j' - r'^2\delta_{ij}),$$

$$f = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

Energy and power

Energy: $U = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$ 场:

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$ $\quad \vec{g} = \varepsilon_0\vec{E}\times\vec{B} = \frac{1}{c^2}\vec{S}$

Larmor formula: $P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2a^2}{c^3}$ $\quad \vec{E} = \frac{q\vec{R}}{4\pi\varepsilon_0 R^3}\ \frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}$

D and H

Definitions: $\begin{cases}\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}\\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\end{cases}$ $\quad \vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{S}$

Linear media: $\begin{cases}\mathbf{P} = \epsilon_0\chi_e\mathbf{E}, & \mathbf{D} = \epsilon\mathbf{E}\\ \mathbf{M} = \chi_m\mathbf{H}, & \mathbf{H} = \frac{1}{\mu}\mathbf{B}\end{cases}$

球坐标:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0.$$
$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$
$$V(r,\theta) = R(r)\Theta(\theta).$$
$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0.$$
$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1), \qquad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1).$$
$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).$$
$$\int_{-1}^{1}P_l(x)P_{l'}(x)\,dx = \int_0^{\pi}P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \begin{cases}0, & \text{if } l'\neq l,\\ \frac{2}{2l+1}, & \text{if } l' = l.\end{cases}$$

Problem 4.37 A point dipole $\mathbf{p}$ is imbedded at the center of a sphere of linear dielectric material (with radius R and dielectric constant ϵ_r). Find the electric potential inside and outside the sphere.

In view of Eq. 4.39, the *net* dipole moment at the center is $\mathbf{p}' = \mathbf{p} - \frac{\chi_e}{1+\chi_e}\mathbf{p} = \frac{1}{1+\chi_e}\mathbf{p} = \frac{1}{\epsilon_r}\mathbf{p}$. We want the potential produced by $\mathbf{p}'$ (at the center) and σ_b (at R). Use separation of variables:

$$\left\{\begin{array}{lll}\textit{Outside:} & V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) & \text{(Eq. 3.72)}\\ \textit{Inside:} & V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{\epsilon_r r^2} + \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) & \text{(Eqs. 3.66, 3.102)}\end{array}\right\}.$$

$$V \text{ continuous at } R \Rightarrow \left\{\begin{array}{ll}\frac{B_l}{R^{l+1}} = A_l R^l, & \text{or}\quad B_l = R^{2l+1}A_l\ (l\neq 1)\\ \frac{B_1}{R^2} = \frac{1}{4\pi\epsilon_0}\frac{p}{\epsilon_r R^2} + A_1 R, & \text{or}\quad B_1 = \frac{p}{4\pi\epsilon_0\epsilon_r} + A_1 R^3\end{array}\right\}.$$

$$\left.\frac{\partial V}{\partial r}\right|_{R+} - \left.\frac{\partial V}{\partial r}\right|_{R-} = -\sum(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) + \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{\epsilon_r R^3} - \sum lA_l R^{l-1}P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_b$$
$$= -\frac{1}{\epsilon_0}\mathbf{P}\cdot\hat{\mathbf{r}} = -\frac{1}{\epsilon_0}(\epsilon_0\chi_e\mathbf{E}\cdot\hat{\mathbf{r}}) = \chi_e\left.\frac{\partial V}{\partial r}\right|_{R-} = \chi_e\left\{-\frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{\epsilon_r R^3} + \sum lA_l R^{l-1}P_l(\cos\theta)\right\}.$$

$$-(l+1)\frac{B_l}{R^{l+2}} - lA_l R^{l-1} = \chi_e lA_l R^{l-1}\ (l\neq 1);\ \text{or}\ -(2l+1)A_l R^{l-1} = \chi_e lA_l R^{l-1} \Rightarrow A_l = 0\ (\ell\neq 1).$$

For $l=1$: $-2\frac{B_1}{R^3} + \frac{1}{4\pi\epsilon_0}\frac{2p}{\epsilon_r R^3} - A_1 = \chi_e\left(-\frac{1}{4\pi\epsilon_0}\frac{2p}{\epsilon_r R^3} + A_1\right) - B_1 + \frac{p}{4\pi\epsilon_0\epsilon_r} - \frac{A_1 R^3}{2} = -\frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r} + \chi_e\frac{A_1 R^3}{2}$;

$$-\frac{p}{4\pi\epsilon_0\epsilon_r} - A_1 R^3 + \frac{p}{4\pi\epsilon_0\epsilon_r} - \frac{A_1 R^3}{2} = -\frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r} + \chi_e\frac{A_1 R^3}{2} \Rightarrow \frac{A_1 R^3}{2}(3+\chi_e) = \frac{1}{4\pi\epsilon_0}\frac{\chi_e p}{\epsilon_r}.$$
$$\Rightarrow A_1 = \frac{1}{4\pi\epsilon_0}\frac{2\chi_e p}{R^3\epsilon_r(3+\chi_e)} = \frac{1}{4\pi\epsilon_0}\frac{2(\epsilon_r-1)p}{R^3\epsilon_r(\epsilon_r+2)};\quad B_1 = \frac{p}{4\pi\epsilon_0\epsilon_r}\left[1 + \frac{2(\epsilon_r-1)}{(\epsilon_r+2)}\right] = \frac{p}{4\pi\epsilon_0\epsilon_r}\frac{3\epsilon_r}{\epsilon_r+2}.$$
$$\boxed{V(r,\theta) = \left(\frac{p\cos\theta}{4\pi\epsilon_0 r^2}\right)\left(\frac{3}{\epsilon_r+2}\right)\ (r\geq R).}$$

Meanwhile, for $r\leq R$, $V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{\epsilon_r r^2} + \frac{1}{4\pi\epsilon_0}\frac{pr\cos\theta}{R^3}\frac{2(\epsilon_r-1)}{\epsilon_r(\epsilon_r+2)}$

$$= \boxed{\frac{p\cos\theta}{4\pi\epsilon_0 r^2\epsilon_r}\left[1 + 2\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)\frac{r^3}{R^3}\right]}\ (r\leq R).$$

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b \equiv -\nabla\cdot\mathbf{P}.$$

① Green's Function: $\hat{L}\phi = S(\vec{x})$ $\qquad \hat{L}$: linear operator

$\hat{L}\,G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$

$\int\delta(\vec{x}-\vec{x}')S(\vec{x}')dv' = S(\vec{x}) = \int\hat{L}\,G(\vec{x},\vec{x}')S(\vec{x}')dv' = \hat{L}\left(\int G(\vec{x},\vec{x}')S(\vec{x}')dv'\right)$ solution

② 傅立叶: $F(\omega) = \int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx \quad (-\infty<\omega<\infty)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}d\omega \quad (-\infty<x<\infty)$$

性质: $f'(x) \longleftrightarrow i\omega F(\omega)$ $\qquad$ 卷积: $f_1(x)*f_2(x) = \int_{-\infty}^{\infty}f_1(\xi)f_2(x-\xi)d\xi$

$x^n f(x) \longleftrightarrow i^n\frac{d}{d\omega}F(\omega)$ $\qquad f_1(x)*f_2(x) \longleftrightarrow F_1(\omega)F_2(\omega)$

$\int_{x_0}^{x}f(x)dx \longleftrightarrow \frac{F(\omega)}{i\omega}$ $\qquad$ 奇函数: $h(x)*\cos\omega x = F(\omega)\cos\omega x$

$h(x)*\sin\omega x = F(\omega)\sin\omega x$

$f(x+\varepsilon) \longleftrightarrow e^{i\omega\varepsilon}F(\omega)$ $\qquad f(x)*\cos\omega x = i\,H(\omega)\sin\omega x$

$f(x)*\sin\omega x = -i\,H(\omega)\cos\omega x$

柱坐标:

$\nabla^2 V = 0$

$\longrightarrow \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$

$V = I(s)\cdot K(\phi)$

$\longrightarrow \frac{s}{I}\frac{d}{ds}\left(s\cdot\frac{dI}{ds}\right) + \frac{1}{K}\frac{d^2K}{d\phi^2} = 0$

$$\begin{cases}\frac{s}{I}\frac{d}{ds}\left(s\cdot\frac{dI}{ds}\right) = m^2 \longrightarrow I = A\cdot s^m + B\cdot s^{-m}\\ \frac{1}{K}\frac{d^2K}{d\phi^2} = -m^2 \longrightarrow K = C\sin m\phi + D\cos m\phi\end{cases}$$

$V = \underbrace{(A\cdot s^m + B\cdot s^{-m})(C\sin m\phi + D\cos m\phi)}_{m\geq 1}\ \underbrace{+\,a_0 + b_0\ln s}_{m=0}$ 偶:

bounded 电子:

$\ddot{x} + \omega_0^2 x + \gamma\dot{x} = \frac{q}{m}Ee^{-i\omega t}.$ $\longrightarrow x = \frac{(q/m)\mathbf{E}_0 e^{-i\omega t}}{\omega_0^2-\omega^2-i\gamma\omega} = \frac{(q/m)\mathbf{E}_0 e^{-i(\omega t-\delta)}}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}},$

$\tan\delta = \frac{\gamma\omega}{\omega_0^2-\omega^2}.$ $\quad \ddot{p} = q\ddot{x} = -\frac{(q^2/m)\omega^2\mathbf{E}_0 e^{-i(\omega t-\delta)}}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$ $\quad \langle P\rangle = \frac{\mu_0 q^4}{12\pi m^2 c}\frac{\omega^4 E_0^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}.$

$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\mu_0}{32\pi^2 c}\left(\frac{q^2}{m}\right)^2\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}E_0^2\sin^2\theta.$ $\quad \sigma = \frac{\mu_0^2 q^4}{6\pi m^2}\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2},$

General Magnetics

$\vec A=\frac{\mu_0}{4\pi}\int\frac{\vec j}{R}\,dV'\qquad \vec B=\frac{\mu_0}{4\pi}\int\frac{\vec j\times\vec R}{R^3}\,dV'\qquad \vec R=\vec r-\vec r'$

Taylor Expansion: $\vec A=\vec A^{(0)}+\vec A^{(1)}+\vec A^{(2)}+\cdots$

$\vec A^{(0)}=0$

$\vec m=\frac12\int\vec r'\times\vec j\,dV'\longrightarrow\begin{cases}\vec A^{(1)}=\frac{\mu_0}{4\pi}\frac{\vec m\times\vec r}{r^3}\\ \vec B^{(1)}=\frac{\mu_0}{4\pi}\frac{3(\vec m\cdot\vec n)\vec n-\vec m}{r^3}\end{cases}$

$\int(f\vec j\cdot\nabla'g+g\vec j\cdot\nabla'f)\,dV'=0$

$f=x_i',\ g=x_l'\longrightarrow\int(x_i'j_l+x_l'j_i)\,dV=0$

$\vec x\cdot\int\vec x'\vec j\,dV'=x_l\int x_l'j_i\vec e_i\,dV'=\frac12x_l\int(x_l'j_i-x_i'j_l)\vec e_i\,dV'$

$=\frac12x_l\int\epsilon_{kli}(\vec x'\times\vec j)_k\vec e_i\,dV$

$=-\frac12\vec x\times\int(\vec x'\times\vec j)\,dV$

$\vec F=\int\vec j\times\vec B(\vec x')\,dV'=\vec F^{(0)}+\vec F^{(1)}+\vec F^{(2)}+\cdots$

$\vec F^{(0)}=\int\vec j\times\vec B(0)\,dV'=\int\vec j\,dV'\times\vec B(0)=0$

$\vec F^{(1)}=\int\vec j\times(\vec x'\cdot\nabla)\vec B(0)\,dV'$

$=(\vec m\times\nabla)\times\vec B=\nabla(\vec m\cdot\vec B)-\vec m(\nabla\cdot\vec B)$

$=\nabla(\vec m\cdot\vec B)\longrightarrow U=-\vec m\cdot\vec B,\ \vec F=-\nabla U$

$=\vec m\times(\nabla\times\vec B)+(\vec m\cdot\nabla)\vec B$

$=\mu_0\vec m\times\vec j_{外}+(\vec m\cdot\nabla)\vec B\qquad \vec j_{外}$:外部电流

$\vec K=\int\vec x'\times\vec f\,dV'=\int\vec x'\times(\vec j\times\vec B)\,dV'$

$\vec K^{(0)}=\int\vec x'\times(\vec j\times\vec B(0))\,dV'=\int[(\vec x'\cdot\vec B)\vec j-\vec B(\vec x'\cdot\vec j)]\,dV'$

$=\int(\vec x'\cdot\vec B)\vec j\,dV'$

$=\vec m\times\vec B(0)$

$\int\nabla\times\vec c\,dv=-\oint\vec c\times d\vec s$

$\vec A=\frac{\mu_0}{4\pi}\int\frac{\vec M\times\vec e_R}{R^2}\,dv'=-\frac{\mu_0}{4\pi}\int\vec M\times\nabla\frac1R\,dv'$

$=\frac{\mu_0}{4\pi}\int\left(\frac1R(\nabla\times\vec M)-\nabla\times\frac{\vec M}{R}\right)dv'$

$=\frac{\mu_0}{4\pi}\int\frac1R\nabla\times\vec M\,dv'+\frac{\mu_0}{4\pi}\oint\frac1R\vec M\times d\vec s'$

$=\frac{\mu_0}{4\pi}\left(\int\frac{\vec j_b}{R}\,dv'+\oint\frac{\vec K_b}{R}\,ds\right)\qquad\vec j_b=\nabla\times\vec M,\quad\vec K_b=\vec M\times\vec n$

$\vec j=\vec j_0+\vec j_b\qquad\nabla\times\vec B=\mu_0\vec j_0+\mu_0\nabla\times\vec M\longrightarrow\nabla\times(\vec B-\mu_0\vec M)=\mu_0\vec j_0$

$\vec H=\frac{\vec B}{\mu_0}-\vec M\longrightarrow\begin{cases}\nabla\times\vec H=\vec j_0\\ \nabla\cdot\vec H=-\nabla\cdot\vec M\neq0\end{cases}$

Linear Media $\quad M=\chi_mH\longrightarrow\vec B=\mu_0(\vec H+\vec M)=(1+\chi_m)\mu_0\vec H$

$=\mu_r\mu_0\vec H$

Para $\quad\mu_r=1+\chi_m>1$

Dia $\quad\mu_r=1+\chi_m<1\qquad\chi_m\sim10^{-5}$

Boundary Connection $\begin{cases}\nabla\cdot\vec B=0\longrightarrow\oint\vec B\cdot d\vec s=0\qquad B_\perp\text{ 连续}\\ \nabla\times\vec H=\vec j_0\longrightarrow\iint\nabla\times\vec H\cdot d\vec s=\oint\vec H\cdot d\vec l=\end{cases}$

电子散射:

$$\sigma=\left(\frac{q^2}{4\pi\epsilon_0mc^2}\right)^2\int_0^\pi\sin^3\theta\,\mathrm d\theta\int_0^{2\pi}\mathrm d\varphi=\frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0mc^2}\right)^2.\quad r_e$$

free 电子:

Plane Waves:

$\vec B=\frac1c\hat n\times\vec E$

$\vec S=\frac1{\mu_0}\vec E\times\vec B=\frac1{\mu_0c}E^2\hat n=\frac{c}{\mu_0}B^2\hat n$

$w=\frac12\varepsilon_0E^2+\frac1{2\mu_0}B^2=\varepsilon_0E^2\longrightarrow\vec S=cw\hat n$

Monochromatic Plane Wave

$\vec E=-\nabla\phi-\frac{\partial\vec A}{\partial t}=-i\vec k\phi+i\omega\vec A\qquad\Big|\quad\nabla\to i\vec k$

$\vec B=\nabla\times\vec A=i\vec k\times\vec A=\vec k\times\vec E/\omega\qquad\Big|\quad\partial_t\to-i\omega$

$w=\varepsilon_0\frac{\vec E+\vec E^*}{2}\cdot\frac{\vec E+\vec E^*}{2}=\frac14\varepsilon_0(E^2+E^{*2}+2\vec E\cdot\vec E^*)$

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in Linear Media:

$\begin{cases}\nabla\times\vec E=-\frac{\partial\vec B}{\partial t}\\ \nabla\cdot\vec D=\rho_f\\ \nabla\cdot\vec B=0\\ \nabla\times\vec H=\vec j_0+\frac{\partial\vec D}{\partial t}\end{cases}\qquad\Bigg|\qquad\begin{aligned}\varepsilon_1E_1^\perp&=\varepsilon_2E_2^\perp\\ E_1^\parallel&=E_2^\parallel\\ B_1^\perp&=B_2^\perp\\ \frac1{\mu_1}B_1^\parallel&=\frac1{\mu_2}B_2^\parallel\end{aligned}$

damping: $\vec k=\vec k_r+i\vec k_i\longrightarrow e^{-\vec k_i\cdot\vec r}$

$\ddot x+\gamma\dot x+\omega_0^2x=-\frac{e\vec E_0}{m}e^{-i\omega t}\longrightarrow\vec x=-\frac em\frac1{\omega_0^2-\omega^2-i\omega\gamma}\vec E_0e^{-i\omega t}$

$v\ll c,\ |x|\ll\lambda.$

$\longrightarrow\vec p=-ex=\frac{e^2}{m}\frac1{\omega_0^2-\omega^2-i\omega\gamma}\vec E_0e^{-i\omega t}$

$\vec P=\frac{Ne^2}{m}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}\vec E_0e^{-i\omega t}=\chi_e\varepsilon_0\vec E$

$\chi_e=\frac{Ne^2}{m\varepsilon_0}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}$

$\varepsilon_r=1+\chi_e=1+\frac{Ne^2}{m\varepsilon_0}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}$

$|\chi_e|\ll1\qquad\mu_r\to1$

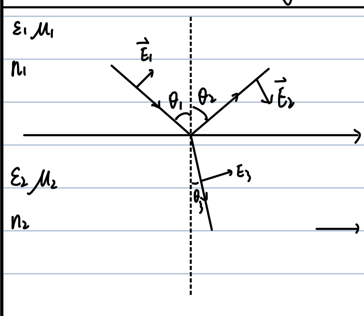
$\longrightarrow n\approx\sqrt{\varepsilon_r}\approx1+\frac{Ne^2}{2m\varepsilon_0}\sum_j\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}$

$=1+\frac{Ne^2}{2m\varepsilon_0}\sum_j\frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+\omega^2\gamma_j^2}+\frac{Ne^2}{2m\varepsilon_0}\sum_j\frac{f_j\omega\gamma_j}{(\omega_j^2-\omega^2)^2+\omega^2\gamma_j^2}\ i$

$=n_r+in_i$

$n_r=1+\frac{Ne^2}{2m\varepsilon_0}\sum_j\frac{f_j(\omega_j^2-\omega^2)}{(\omega_j^2-\omega^2)^2+\omega^2\gamma_j^2}$

$n_i=\frac{Ne^2}{2m\varepsilon_0}\sum_j\frac{f_j\omega\gamma_j}{(\omega_j^2-\omega^2)^2+\omega^2\gamma_j^2}$



$\theta_1=\theta_2$

$n_1\sin\theta_1=n_2\sin\theta_3$

$\alpha=\frac{\varepsilon_1n_2}{\varepsilon_2n_1}\qquad\beta=\frac{\cos\theta_1}{\cos\theta_3}$

$\longrightarrow\begin{cases}E_2=\frac{\alpha-\beta}{\alpha+\beta}E_1\\ E_3=\frac{2\alpha\beta}{\alpha+\beta}E_1\end{cases}$

$\left.\begin{aligned}\varepsilon_1(E_1-E_2)\sin\theta_1&=\varepsilon_2E_3\sin\theta_3\\ (E_1+E_2)\cos\theta_1&=E_3\cos\theta_3\end{aligned}\right\}$

Conductors:

$$\begin{cases} \vec\nabla\cdot\vec E = 0 \\ \vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \\ \vec\nabla\cdot\vec B = 0 \\ \vec\nabla\times\vec B = \mu\varepsilon\frac{\partial\vec E}{\partial t} + \mu\sigma\vec E \end{cases} \longrightarrow \begin{cases} \nabla^2\vec E = \mu\varepsilon\frac{\partial^2\vec E}{\partial t^2} + \mu\sigma\frac{\partial\vec E}{\partial t} \\ \nabla^2\vec B = \mu\varepsilon\frac{\partial^2\vec B}{\partial t^2} + \mu\sigma\frac{\partial\vec B}{\partial t} \end{cases}$$

$$\longrightarrow k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

$$k = \beta + i\alpha \longrightarrow \begin{cases} \beta = \sqrt{\frac{\mu\varepsilon}{2}}\,\omega\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{\frac12} \\ \alpha = \sqrt{\frac{\mu\varepsilon}{2}}\,\omega\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{\frac12} \end{cases}$$

$\longrightarrow$ skin depth: good conductor: $\sigma/\varepsilon\omega \gg 1$

$\longrightarrow \alpha = \beta = \sqrt{\frac{\mu\sigma\omega}{2}}$

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\mu\sigma\omega}}$$

$$k = (1+i)\sqrt{\frac{\mu\sigma\omega}{2}} = \sqrt{\mu\sigma\omega}\, e^{i\frac{\pi}{4}}$$

$$\vec B = \frac{\vec k}{\omega}\times\vec E = \sqrt{\frac{\mu\sigma}{\omega}}\,\vec e_k\times(\vec E e^{i\frac{\pi}{4}})$$ 差 $\frac{\pi}{4}$ 相位

Wave Guide:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\mu\epsilon\omega^2-k^2\right)\right]\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{B}\end{pmatrix} = 0.$$

$$\left.\begin{aligned} \vec\nabla\times\vec E &= -\partial_t\vec B \\ \vec\nabla\times\vec B &= \partial_t\vec E \end{aligned}\right\} \longrightarrow \begin{cases} E_x = \frac{\mathrm{i}}{\mu\epsilon\omega^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right), \\ E_y = \frac{\mathrm{i}}{\mu\epsilon\omega^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right), \\ B_x = \frac{\mathrm{i}}{\mu\epsilon\omega^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\mu\epsilon\omega\frac{\partial E_z}{\partial y}\right), \\ B_y = \frac{\mathrm{i}}{\underbrace{\mu\epsilon\omega^2-k^2}_{\neq 0}}\left(k\frac{\partial B_z}{\partial y}+\mu\epsilon\omega\frac{\partial E_z}{\partial x}\right). \end{cases}$$

$$\longrightarrow \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\mu\epsilon\omega^2-k^2\right)\right]\begin{pmatrix}E_z\\ B_z\end{pmatrix} = 0.$$

1° TE: $E_z = 0$

$$\begin{cases} \partial_x B_z = 0. & x = 0,\ a \\ \partial_y B_z = 0 & y = 0,\ b \end{cases} \longrightarrow B^z_{mn} = B_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\, e^{i(kz-\omega t)}$$

$$k = \sqrt{\mu\varepsilon\omega - \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \qquad m,n = 0,1,2\dots$$

2° TM: $B_z = 0$

$$\begin{cases} \partial_y E_z = 0 & x = 0,\ a \\ \partial_x E_z = 0 & y = 0,\ b \end{cases} \longrightarrow E^z_{mn} = E_0\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\, e^{i(kz-\omega t)}$$

$$k = \sqrt{\mu\varepsilon\omega - \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \qquad m,n = 1,2\dots$$

$$V_{ph} = \frac{\omega}{k} \qquad V_g = \frac{d\omega}{dk}$$

$$\phi_1(x,t) = \int\frac{1}{4\pi\epsilon_0 R}\rho\left(x', t-\frac{R}{c}\right)\mathrm{d}V', \qquad A_1(x,t) = \int\frac{\mu_0}{4\pi R}j\left(x', t-\frac{R}{c}\right)\mathrm{d}V',$$

$$t_r = t - \frac{|\vec x - \vec x_p(t_r)|}{c} \qquad \vec\beta = \frac{1}{c}\frac{\partial\vec x_p}{\partial t_r}$$

$$\vec\nabla t_r = -\frac{1}{c}\frac{\hat R}{1-\vec\beta\cdot\hat R} \qquad \frac{\partial R^2}{\partial t_r} = 2R\frac{\partial R}{\partial t_r} = 2\vec R\cdot\frac{\partial\vec R}{\partial t_r} = -2\vec R\cdot\vec V_p$$

$$\frac{\partial t_r}{\partial t} = \frac{1}{1-\vec\beta\cdot\hat R} \qquad \vec\nabla(R^2)\Big|_{t_r} = 2\vec R\,\vec\nabla R\Big|_{t_r} = 2(\vec R\cdot\vec\nabla)\vec R\Big|_{t_r} = 2\vec R$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{R-\vec R\cdot\vec\beta} \qquad \vec A = \frac{\mu_0 q}{4\pi c}\frac{\vec\beta}{R-\vec R\cdot\vec\beta}$$

$$\vec R = |\vec x - \vec X(T_0)| \qquad T_0 = t - \frac{|\vec x - \vec x_p(T_0)|}{c}$$

$$E = \frac{q}{4\pi\varepsilon_0 c}\frac{\vec R\times[(\vec R-R\vec\beta)\times\dot{\vec\beta}]}{(R-\vec R\cdot\vec\beta)^3} + \frac{q}{4\pi\varepsilon_0}(1-\beta^2)\frac{\vec R-R\vec\beta}{(R-\vec R\cdot\vec\beta)^3}$$

$$\vec B = \frac{q}{4\pi\varepsilon_0 c}\hat R\times\left\{(1-\beta^2)\frac{\vec R-R\vec\beta}{(R-\vec R\cdot\vec\beta)^3} + \frac{1}{c}\frac{\vec R\times[(\vec R-R\vec\beta)\times\dot{\vec\beta}]}{(R-\vec R\cdot\vec\beta)^3}\right\}$$

Radiation:

只考虑:

$$E = \frac{q}{4\pi\varepsilon_0 c}\frac{\vec R\times[(\vec R-R\vec\beta)\times\dot{\vec\beta}]}{(R-\vec R\cdot\vec\beta)^3} \qquad \vec B = \frac{q}{4\pi\varepsilon_0 c^2}\hat R\times\frac{\vec R\times[(\vec R-R\vec\beta)\times\dot{\vec\beta}]}{(R-\vec R\cdot\vec\beta)^3}$$

$$\vec S = \frac{1}{\mu_0}\vec E\times\vec B = \frac{1}{\mu_0 c}E^2\hat R = \frac{q^2}{16\pi^2\varepsilon_0 c R^6}\frac{\left|\vec R\times[(\vec R-R\vec\beta)\times\dot{\vec\beta}]\right|^2}{(1-\hat R\cdot\vec\beta)^6}\hat R$$

$$dE = \vec S\cdot\hat R\cdot dt = \vec S\cdot\hat R\frac{\partial t}{\partial t'}dt'$$

$$d\mathcal{E} = \vec S\cdot\hat R\frac{\partial t}{\partial t'}dt' R^2 d\Omega$$

$$dp = \frac{d\mathcal{E}}{dt'} = \vec S\cdot\hat R\frac{\partial t}{\partial t'}R^2 d\Omega$$

$$\frac{dp}{d\Omega} = \vec S\cdot\hat R\frac{\partial t}{\partial t'}R^2 = \frac{\mu_0 q^2 c}{16\pi^2}\frac{\left|\hat R\times[(\hat R-\vec\beta)\times\dot{\vec\beta}]\right|^2}{(1-\hat R\cdot\vec\beta)^5}$$

1°. $\beta \ll 1$ $\qquad \frac{dp}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c}\dot v^2\sin^2\theta \qquad \theta = \langle\vec a, \hat R\rangle$

$$p = \int dp = \frac{\mu_0 q^2 a^2}{6\pi c}$$

2° $\beta \sim 1$

(1) $\vec V \parallel \vec a$: $\qquad \frac{dp}{d\Omega} = \frac{q^2\mu_0 c}{16\pi c}\frac{\dot\beta^2\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad \theta = \langle\vec v, \hat R\rangle = \langle\vec a, \hat R\rangle$

$$p = \frac{q^2\mu_0}{6\pi c}\gamma^6 a^2 \leftarrow \vec F = \frac{d\vec p}{dt} = \gamma^3 m\dot{\vec v}$$

(2) $\vec V \perp \vec a$: $\qquad \frac{dp}{d\Omega} = \frac{q^2\mu_0 c}{16\pi^2}\frac{\dot\beta^2[(1-\beta\cos\theta)^2+\sin^2\theta\cos^2\phi(\beta^2-1)]}{(1-\beta\cos\theta)^5}$

$$p = \frac{q^2\mu_0}{6\pi c}\gamma^4 a^2 \leftarrow \vec F = \frac{d\vec p}{dt} = \gamma m\vec a$$

$$p = \frac{q^2\mu_0}{6\pi c}(\gamma^6 a_\parallel^2 + \gamma^4 a_\perp^2)$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 e^{ikr}}{4\pi cr}\ddot{\boldsymbol{p}}\sin\theta\boldsymbol{e}_\phi, \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{\mu_0 e^{ikr}}{4\pi r}\ddot{\boldsymbol{p}}\sin\theta\boldsymbol{e}_\theta.$$

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{\mu_0}{32\pi^2 c}|\ddot{\boldsymbol{p}}|^2\sin^2\theta = \frac{\mu_0\omega^4}{32\pi^2 c}|\boldsymbol{p}|^2\sin^2\theta,$$

$$P = \frac{\mu_0|\ddot{\boldsymbol{p}}|^2}{12\pi c} = \frac{\mu_0\omega^4|\boldsymbol{p}_0|^2}{12\pi c},$$

$$\boldsymbol{A} = -\frac{\mathrm{i}k\mu_0 e^{\mathrm{i}kr}}{4\pi r}\left(\frac16\dot{\boldsymbol{D}} - \boldsymbol{n}\times\boldsymbol{m}\right). \qquad \boldsymbol{B} = \frac{k^2\mu_0 e^{\mathrm{i}kr}\boldsymbol{n}}{4\pi r}\times\left(\frac16\dot{\boldsymbol{D}} - \boldsymbol{n}\times\boldsymbol{m}\right).$$

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle_m = \left\langle\frac{c}{\mu_0}|\boldsymbol{B}_m|^2 r^2\right\rangle = \left\langle\frac{c}{\mu_0}\left|\frac{\mu_0(\ddot{\boldsymbol{m}}\times\boldsymbol{n})\times\boldsymbol{n}}{4\pi c^2}\right|^2\right\rangle = \frac{\mu_0|\ddot{\boldsymbol{m}}|^2}{32\pi^2 c^3}\sin^2\theta.$$

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle_D = \left\langle\frac{c}{\mu_0}|\boldsymbol{B}_D|^2 r^2\right\rangle = \left\langle\frac{c}{\mu_0}\left|\frac{\mu_0\dddot{\boldsymbol{D}}\times\boldsymbol{n}}{24\pi c^2}\right|^2\right\rangle = \frac{\mu_0}{1152\pi^2 c^3}|\dddot{\boldsymbol{D}}\times\boldsymbol{n}|^2.$$

$$\langle P\rangle_m = \int\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle\mathrm{d}\Omega = \iint\frac{\mu_0|\ddot{\boldsymbol{m}}|^2}{32\pi^2 c^3}\sin^3\theta\mathrm{d}\theta\mathrm{d}\varphi = \frac{\mu_0|\ddot{\boldsymbol{m}}|^2}{12\pi c^3}.$$

$$\langle P\rangle_D = \frac{\mu_0}{1152\pi^2 c^3}\int_0^{2\pi}\mathrm{d}\varphi\int_0^\pi 36q^2 l^4\omega^6\cos^2\theta\sin^3\theta\mathrm{d}\theta = \frac{\mu_0 q^2 l^4\omega^6}{60\pi c^3}.$$

电子散射: $v \ll c$, $r \ll \lambda$.

$$\ddot{\boldsymbol{r}} = \frac{q}{m}\widehat{\boldsymbol{E}}_0 e^{-\mathrm{i}\omega t}, \text{ with } \widehat{\boldsymbol{E}}_0 = \boldsymbol{E}_0 e^{\mathrm{i}\alpha}. \longrightarrow \ddot{\boldsymbol{p}} = q\ddot{\boldsymbol{r}} = \frac{q^2}{m}\widehat{\boldsymbol{E}}_0 e^{-\mathrm{i}\omega t}.$$

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{\mu_0}{32\pi^2 c}|\ddot{\boldsymbol{p}}|^2\sin^2\theta = \frac{\mu_0}{32\pi^2 c}\frac{q^4}{m^2}E_0^2\sin^2\theta, \quad \langle S\rangle = \frac{1}{\mu_0 c}\frac{E_0^2}{2} = \frac{1}{2\mu_0 c}E_0^2.$$

$$\mathrm{d}\sigma = \frac{\mu_0^2 q^4}{16\pi^2 m^2}\sin^2\theta\mathrm{d}\Omega = \left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2\sin^2\theta\mathrm{d}\Omega.$$