

2017-2018 第二学期, 期中.

11) $\alpha_1 = (2, 1, 3, -1)^T$ $\alpha_2 = (3, -1, 2, 0)^T$

$\alpha_3 = (4, 2, 6, -2)^T$ $\alpha_4 = (4, -3, 1, 1)^T$

$\text{rank}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

$A = \begin{bmatrix} 2 & 3 & 4 & 4 \\ 1 & -1 & 2 & -3 \\ 3 & 2 & 6 & 1 \\ -1 & 0 & -2 & 1 \end{bmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & 0 & 10 \\ 0 & 5 & 0 & 10 \\ 0 & -1 & 0 & -2 \end{pmatrix}$ $\text{rank } A = 2$

12) $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 4 & -2 \\ 3 & 0 & 1 \end{pmatrix}$ $\det(AB) = \underline{0}$

$AB = \begin{pmatrix} 6 & 12 & -5 \\ 5 & -4 & 4 \\ 5 & 8 & -3 \end{pmatrix}$ ~~$\det AB = \begin{vmatrix} 5 & -4 & 4 \\ 0 & 12 & -7 \\ 1 & & \end{vmatrix}$~~

$\det AB = \det \begin{pmatrix} 6 & 12 & -5 \\ 0 & -14 & \frac{49}{6} \\ 0 & -2 & \frac{7}{6} \end{pmatrix} = 6 \left(-14 \cdot \frac{7}{6} + 2 \times \frac{49}{6} \right) = 0$

(或由 Binet-Cauchy 公式 (Pg 5 130 4.3.6) $\det AB = 0$ 显然)

13) $A = \begin{pmatrix} 3 & 0 & 4 & 0 \\ 2 & 2 & 2 & 2 \\ 0 & -1 & 0 & 0 \\ 5 & 3 & -2 & 2 \end{pmatrix}$ 第4行各元素代数余子式之和 = 0

代数余子式 $A_{41} = (-1)^5 (-56) = 56$

$A_{42} = (-1)^{4+2} 0 = 0$

$A_{43} = (-1)^{4+3} 42 = -42$

$A_{44} = (-1)^{4+4} (-14) = -14$

和为 0



14) A 伴随矩阵 A^* $(A^*)^T = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$ 则 $A = \begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & 2 \end{pmatrix}$

$$A^* = ((A^*)^T)^T = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

$$\det A^* = (-1)^{4+1} (-1)^{3+1} (-1)^{2+1} (-1)^{1+1} \cdot 1 \cdot 2 \cdot (-1) \cdot 4 = -8 \neq 0 \Rightarrow A^{-1} \text{ 存在}$$

$$AA^* = \det A \cdot I = (\det A^*) = (\det A)^{n-1} \Rightarrow \det A = -2$$

$$\Rightarrow A^* = \det A \cdot A^{-1} \quad A^{-1} = \frac{A^*}{\det A} = -\frac{1}{2} \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$$

$$\Rightarrow A = -2 \begin{pmatrix} \frac{1}{2} & 1 \\ 2 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -4 & 1 \end{pmatrix}$$

(弄明白 $A, A^{-1}, A^*, \det A, \det A^{-1}, \det A^*$ 的转换关系)

15) $\alpha_1 = (a, 0, c)$

$\alpha_2 = (b, c, 0)$

$\alpha_3 = (0, a, b)$

线性无关, 则 a, b, c 满足 $abc \neq 0$

线性无关 $\Leftrightarrow \det \begin{pmatrix} a & 0 & c \\ b & c & 0 \\ 0 & a & b \end{pmatrix} \neq 0$

$$\det A = acb + c(ab) = 2abc \neq 0$$



二 判断题

(1) $A \in \mathbb{R}^{3 \times 5}$ $\text{rank } A = 3$ 则 $\exists b \in \mathbb{R}^3$ s.t. $Ax = b$ 有唯一解

错误

$$A = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 5} \quad \text{rank } A = 3 \quad \text{行满秩}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

记 $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$

$$\text{则 } A \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \\ 0 \end{pmatrix} = b, \quad A \begin{pmatrix} b_1 - b_3 \\ b_2 \\ 0 \\ 0 \\ b_3 \end{pmatrix} = b, \quad A \begin{pmatrix} b_1 - b_2 \\ 0 \\ 0 \\ b_2 \\ b_3 \end{pmatrix} = b$$

由唯一性 $\Rightarrow b_3 = b_2 = 0$

再由 $A \begin{pmatrix} 0 \\ b_2 - b_1 \\ b_3 \\ b_1 \\ 0 \end{pmatrix} = b \Rightarrow b_1 = 0$ 而 $b = 0$ 时齐次方程组 V 为解空间
 $\dim V = n - \text{rank } A = 2$ 解不唯一

(2) 如果 $A, B \in \mathbb{R}^{m \times n}$ $\text{tr}((A-B)(A-B)^T) = 0$ 则 $A = B$

证 $\text{tr } A A^T = \sum_{i=1}^m |A A^T|_{ii} = \sum_{i=1}^m \sum_{j=1}^n a_{ij} a_{ij}$

$A A^T \in \mathbb{R}^{m \times m}$

$$= \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 = 0 \Rightarrow A = 0$$



(3) 设向量组 $\alpha_1, \dots, \alpha_r$ 线性无关 $\beta = \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \dots + \lambda_r \alpha_r$ 且 $\lambda_i \neq 0$

则 $\alpha_1, \dots, \alpha_{i-1}, \beta, \alpha_{i+1}, \dots, \alpha_r$ 线性无关

正确 习题 5 16 题

14) $A \in \mathbb{R}^{m \times n}$ $B \in \mathbb{R}^{n \times m}$ $\det AB = \det BA$

错误 $A = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\det AB = 1$

$\det BA = 0$

(由 Binet-Cauchy 公式, 结论显然不对)

三 已知方程组

$$\begin{cases} x_1 + x_2 - 2x_4 = -6 \\ 4x_1 - x_2 - x_3 - x_4 = 1 \\ 3x_1 - x_2 - x_3 = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 + ax_2 - x_3 - x_4 = -5 \\ bx_1 - x_3 - 2x_4 = -3 \\ x_3 - 2x_4 = 1-c \end{cases}$$

同解

求 a, b, c 值

解, 先求左边方程组的解

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 4 & -1 & -1 & -1 \\ 3 & -1 & -1 & 0 \end{pmatrix} X = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix} \quad (AX=b)$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 0 & -2 & -6 \\ 4 & -1 & -1 & -1 & 1 \\ 3 & -1 & -1 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & -2 & -6 \\ 0 & -5 & -1 & 7 & 25 \\ 0 & -4 & -1 & 6 & 21 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 0 & -2 & -6 \\ 0 & -4 & -1 & 6 & 21 \\ 0 & 1 & 0 & -1 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & -1 & 2 & 5 \end{pmatrix}$$



$$\text{令 } x_4 = t$$

$$x_3 = 2t - 5$$

$$x_2 = t - 4$$

$$x_1 = t - 2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 5 \\ 0 \end{pmatrix} \quad (*)$$

$$\text{有特解 } \begin{pmatrix} -2 \\ -4 \\ -5 \\ 0 \end{pmatrix} \text{ 代入右边方程} \Rightarrow \begin{cases} c = 6 \\ b = 4 \\ a = 2 \end{cases}$$

最后代入 a, b, c 值判右边方程, 证明其解也为 (*) 即可

$$\text{四 } n(>1) \text{ 阶方程 } A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 2 & & \\ \vdots & & \ddots & \\ 1 & & & 2 \end{pmatrix} \quad \text{求 } \det A \text{ 及 } A^{-1}$$

$$\det A = \begin{vmatrix} -\frac{1}{2} \times (n-1) & 1 & \cdots & 1 \\ 0 & 2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 2 \end{vmatrix} = -\frac{1}{2}(n-1) \cdot 2^{n-1} = -(n-1)2^{n-2} \neq 0$$

下求 A^{-1}

$$\left[\begin{array}{cccc|cccc} 0 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & & & 0 & 1 & 0 & 0 \\ \vdots & & \ddots & & 0 & 0 & \ddots & 0 \\ 1 & & & & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{消去行例}} \left[\begin{array}{cccc|cccc} -\frac{n-1}{2} & 1 & \cdots & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & & & 0 & 1 & 0 & 0 \\ \vdots & & \ddots & & 0 & 0 & \ddots & 0 \\ 1 & & & & 0 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{\text{利用 } -\frac{n-1}{2} \text{ 消去第一列}} \left[\begin{array}{cccc|cccc} -\frac{n-1}{2} & 1 & \cdots & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & & & 0 & 1 & 0 & 0 \\ \vdots & & \ddots & & 0 & 0 & \ddots & 0 \\ 1 & & & & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{将左边单位化}} A^{-1}$$



$$A^{-1} = \begin{pmatrix} -\frac{2}{n-1} & \frac{1}{1(n-1)} & - & - & \frac{1}{1(n-1)} \\ \frac{1}{n-1} & \frac{1}{2} - \frac{1}{2(n-1)} & - & - & -\frac{1}{2(n-1)} \\ & & \ddots & & \\ & & & \ddots & \\ \frac{1}{n-1} & -\frac{1}{2(n-1)} & - & - & \frac{1}{2} - \frac{1}{2(n-1)} \end{pmatrix}$$

五 $R^{2 \times 2}$ 为实域上所有 2×2 阶矩阵组成的集合，按矩阵加法数乘

构成线性空间

11) 证明 $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \right\}$

构成 $R^{2 \times 2}$ 一组基

12) 求基 S 到自然基的 $(E_{ij})_{1 \leq i, j \leq 2}$ 的过渡矩阵 T

13) 求 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ 在基 S 下的坐标

解 1) 相应 S 中矩阵为 $A \ B \ C \ D$

$$A = (E_{11}, E_{12}, E_{21}, E_{22}) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{所以} \quad (A, B, C, D) = (E_{11}, E_{12}, E_{21}, E_{22})^T$$

$$C = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$



再证明 $\det T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} = 16 \neq 0$

所以 S 为基, T 为其过渡矩阵

(3) 记相应系数为 a, b, c, d

则
$$\begin{cases} a - b - c - d = 1 \\ a + b + c - d = 2 \\ a - b + c + d = 3 \\ a + b - c + d = 4 \end{cases} \Rightarrow \begin{cases} a = \frac{5}{2} \\ b = \frac{1}{2} \\ c = 0 \\ d = 1 \end{cases} \text{ 为系数}$$

(或若求 T^{-1} 来计算亦可)

证 $A \in F^{m \times n} \quad B \in F^{n \times m} \quad \overset{\text{证明}}{\vee} \quad n + \text{rank}(I_m - AB) = m + \text{rank}(I_n - BA)$

P.116 习题 4 42

考虑 $\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix}$ 即可

