A Companion to *Classical Electrodynamics* 3^{rd} Edition by J.D. Jackson

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A lot of things can be said about *Classical Electrodynamics*, the third edition, by David J. Jackson. It's seemingly exhaustive, well researched, and certainly popular. Then, there is a general consensus among teachers that this book is the definitive graduate text on the subject. In my opinion, this is quite unfortunate. The text often assumes familiarity with the material, skips vital steps, and provides too few examples. It is simply not a good introductory text. On the other hand, Jackson was very ambitious. Aside from some notable omissions (such as conformal mapping methods), Jackson exposes the reader to most of classical electro-magnetic theory. Even Thomas Aquinas would be impressed! As a reference, Jackson's book is great!

It is obvious that Jackson knows his stuff, and in no place is this more apparent than in the problems which he asks at the end of each chapter. Sometimes the problems are quite simple or routine, other times difficult, and quite often there will be undaunting amounts of algebra required. Solving these problems is a time consuming endeavor for even the quickest reckoners among us. I present this *Companion to Jackson* as a motivation to other students. These problems *can* be done! And it doesn't take Feynman to do them.

Hopefully, with the help of this guide, lots of paper, and your own wits; you'll be able to wrestle with the concepts that challenged the greatest minds of the last century.

Before I begin, I will recommend several things which I found useful in solving these problems.

- Buy Griffiths' text, *an Introduction to Electrodynamics*. It's well written and introduces the basic concepts well. This text is at a more basic level than Jackson, and to be best prepared, you'll have to find other texts at Jackson's level. But remember Rome wasn't build in a day, and you have to start somewhere.

- Obtain other texts on the level (or near to it) of Jackson. I recommend Vanderlinde's *Electromagnetism* book or Eyges' *Electromagnetism* book. Both provide helpful insights into what Jackson is talking about. But even more usefully, different authors like to borrow each others' problems and examples. A problem in Jackson's text might be an example in one of these other texts. Or the problem might be rephrased in the other text; the rephrased versions often provide insight into what Jackson's asking! After all half the skill in writing a hard physics problem is wording the problem vaguely enough so that no one can figure out what your talking about.

- First try to solve the problem without even reading the text. More often than not, you can solve the problem with just algebra or only a superficial knowledge of the topic. It's unfortunate, but a great deal of physics problems tend to be just turning the crank. Do remember to go back and actually read the text though. Solving physics problems is meaningless if you don't try to understand the basic science about what is going on.

- If you are allowed, compare your results and methods with other students. This is helpful. People are quick to tear apart weak arguments and thereby help you strengthen your own understanding of the physics. Also, if you are like me, you are a king of stupid algebraic mistakes. If ten people have one result, and you have another, there's a good likelihood that you made an algebraic mistake. Find it. If it's not there, try to find what the other people could have done wrong. Maybe, you are both correct!

- Check journal citations. When Jackson cites a journal, find the reference, and read it. Sometimes, the problem is solved in the reference, but always, the references provide vital insight into the science behind the equations.

A note about units, notation, and diction is in order. I prefer SI units and will use these units whenever possible. However, in some cases, the use of Jacksonian units is inevitable, and I will switch without warning, but of course, I plan to maintain consistency within any particular problem. I will set $c = 1$ and $\hbar = 1$ when it makes life easier; hopefully, I will inform the reader when this happens. I have tried, but failed, to be regular with my symbols. In each case, the meaning of various letters should be obvious, or else if I remember, I will define new symbols. I try to avoid the clumsy $d^3\vec{x}$ symbols for volume elements and the $d^2\vec{x}$ symbols for area elements; instead, I use dV and dA. Also, I will use $\hat{x}$,$\hat{y}$, and $\hat{z}$ instead of $\hat{i}$,$\hat{j}$, and $\hat{k}$. The only times I will use ijk's will be for indices.

Please, feel free to contact me, rmagyar@eden.rutgers.edu, about any typos or egregious errors. I'm sure there are quite a few.

Now, the fun begins...

Problem 1.1

a.

In Jackson's own words, "A conductor by definition contains charges capable of moving freely under the action of applied electric fields". That implies that in the presence of electric fields, the charges in the conductor will be accelerated. In a steady configuration, we should expect the charges not to accelerate. For the charges to be non-accelerating, the electric field must vanish everywhere inside the conductor, $\vec{E} = 0$. When $\vec{E} = 0$ everywhere inside the conductor [1], the divergence of $\vec{E}$ must vanish. By Gauss's law, we see that this also implies that the charge density inside the conductor vanishes: $0 = \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$.

b.

The charge density *within* the conductor is zero, *but* the charges must be located somewhere! The only other place in on the surfaces. We use Gauss's law in its integral form to find the field outside the conductor.

$$\int \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \sum q_i$$

Where the sum is over *all* enclosed charges. Evidently, the field outside the conductor depends on the surface charges and also those charges concealed deep within the cavities of the conductor.

c.

We assume that the surface charge is static. Then, $\vec{E}$ at the surface of a conductor must be normal to the surface; otherwise, the tangential components of the E-field would cause charges to flow on the surface, and that would contradict the static condition we already assumed. Consider a small area.

$$\int \nabla \cdot \vec{E} dV = \int \vec{E} \cdot d\vec{A} = \int \frac{\rho}{\epsilon_0} dV$$

But $\rho = 0$ everywhere except on the surface so ρ should more appropriately be written $\sigma\delta(f(\vec{x}))$. Where the function $f(\vec{x})$ subtends the surface in question. The last integral can then be written $\int \frac{\sigma}{\epsilon_0} \hat{n} \cdot d\vec{A}$. Our equation can be rearranged.

$$\int \vec{E} \cdot d\vec{A} = \int \frac{\sigma}{\epsilon_0} \hat{n} \cdot d\vec{A} \rightarrow \int \left(\vec{E} - \frac{\sigma}{\epsilon_0} \hat{n} \right) \cdot d\vec{A} = 0$$

[1] excluding of course charge contained within any cavities

And we conclude

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$$

Problem 1.3

a. In spherical coordinates, a charge Q distributed over spherical shell of radius, R.

The charge density is zero except on a thin shell when r equals R. The charge density will be of the form, $\rho \propto \delta(r-R)$. The delta function insures that the charge density vanishes everywhere except when $r=R$, the radius of the sphere. Integrating ρ over that shell, we should get Q for the total charge.

$$\int A\delta(r-R)dV = Q$$

A is some constant yet to be determined. Evaluate the integral and solve for A.

$$\int A\delta(r-R)dV = \int A\delta(r-R)r^2 d(\cos\theta)d\phi dr = 4\pi R^2 A = Q$$

So $A = \frac{Q}{4\pi R^2}$, and

$$\rho(\vec{r}) = \frac{Q}{4\pi R^2}\delta(r-R)$$

b. In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

$$\int A\delta(r-b)dA = \lambda$$

Since we are concerned with only the charge density per unit length in the axial direction, the integral is only over the plane perpendicular to the axis of the cylinder. Evaluate the integral and solve for A.

$$\int B\delta(r-b)dA = \int B\delta(r-b)rd\theta dr = 2\pi bB = \lambda$$

So $B = \frac{\lambda}{2\pi b}$, and

$$\rho(\vec{r}) = \frac{\lambda}{2\pi b}\delta(r-b)$$

c. In cylindrical coordinates, a charge, Q, spread uniformly over a flat circular disk of negligible thickness and radius, R.

$$\int A\Theta(r-R)\delta(z)dV = Q$$

The Θ function of x vanishes when x is negative; when x is positive, Θ is unity.

$$\int A\Theta(R-r)\delta(z)dV = \int A\Theta(R-r)\delta(z)r\,d\theta dz dr = \pi R^2 A = Q$$

So $A = \frac{Q}{\pi R^2}$, and

$$\rho(\vec{r}) = \frac{Q}{\pi R^2}\Theta(R-r)\delta(z)$$

d. The same as in part c, but using spherical coordinates.

$$\int A\Theta(R-r)\delta\left(\theta - \frac{\pi}{2}\right) dV = Q$$

Evaluate the integral and solve for A.

$$\begin{aligned}\int A\Theta(R-r)\delta\left(\theta - \frac{\pi}{2}\right) dV = \int A\Theta(R-r)\delta\left(\theta - \frac{\pi}{2}\right) r^2 d(\cos\theta) d\phi dr \\ = 2\pi R^2 A = Q\end{aligned}$$

So $A = \frac{Q}{2\pi R^2}$, and

$$\rho(\vec{r}) = \frac{Q}{2\pi R^2}\Theta(R-r)\delta\left(\theta - \frac{\pi}{2}\right)$$

Problem 1.5

We are given the time average potential for the Hydrogen atom.

$$\Phi = q\frac{e^{-\alpha r}}{r}\left(1+\frac{1}{2}\alpha r\right)$$

Since this potential falls off faster than $\frac{1}{r}$, it is reasonable to suspect that the total charge described by this potential is zero. If there were any excess charge (+ of −) left over, it would have to produce a $\frac{1}{r}$ contribution to the potential.

Theoretically, we could just use Poisson's equation to find the charge density.

$$\rho = -\epsilon_0 \nabla^2 \Phi = -\frac{\epsilon_0}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)$$

But life just couldn't be that simple. We must be careful because of the singular behavior at $r = 0$. Try $\Phi' = -\frac{q}{r} + \Phi$. This *trick* amounts to adding a positive charge at the origin. We will have to subtract this positive charge from our charge distribution later.

$$\Phi' = q\left(\frac{e^{-\alpha r}-1}{r}\right) + \frac{1}{2}q\alpha e^{-\alpha r}$$

which has no singularities. Plug into Poisson's equation to get

$$\rho' = -\frac{\epsilon_0}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = -\frac{1}{2}\epsilon_0 q\alpha^3 e^{-\alpha r}$$

The total charge density is then

$$\rho(\vec{r}) = \rho'(\vec{r}) + q\delta(\vec{r}) = -\frac{1}{2}\epsilon_0 q\alpha^3 e^{-\alpha r} + q\delta(\vec{r})$$

Obviously, the second terms corresponds to the positive nucleus while the first is the negative electron cloud.

Problem 1.10

The average value of the potential over the spherical surface is

$$\bar{\Phi} = \frac{1}{4\pi R^2} \int \Phi dA$$

If you imagine the surface of the sphere as discretized, you can rewrite the integral as an infinite sum: $\frac{1}{a} \int dA \to \sum_{area}$. Then, take the derivative of $\bar{\Phi}$ with respect to R.

$$\frac{d\bar{\Phi}}{dR} = \frac{d}{dR} \sum \Phi = \sum \frac{d\Phi}{dR}$$

You can move the derivative right through the sum because derivatives are linear operators. Convert the infinite sum back into an integral.

$$\frac{d\bar{\Phi}}{dR} = \sum \frac{d\Phi}{dR} = \frac{1}{4\pi R^2} \int \frac{d\Phi}{dR} dA$$

One of the recurring themes of electrostatics is $\frac{d\Phi}{dR} = -E_n$. Use it.

$$\frac{d\bar{\Phi}}{dR} = \frac{1}{4\pi R^2} \int \frac{d\Phi}{dR} dA = -\frac{1}{4\pi R^2} \int E_n dA = 0$$

By Gauss's law, $\int E_n dA = 0$ since $q_{included} = 0$. And so we have the mean value theorem:

$$\frac{d\bar{\Phi}}{dR} = 0 \to \bar{\Phi}_{surface} = \Phi_{center}$$

q.e.d.

Problem 1.12

$$\int \rho\Phi' dV + \int \sigma\Phi' dA = \int \rho'\Phi dV + \int \sigma'\Phi dA$$

Green gave us a handy relationship which is useful here. Namely,

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV' = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] dA$$

Let $\phi = \Phi$ and $\psi = \Phi'$.

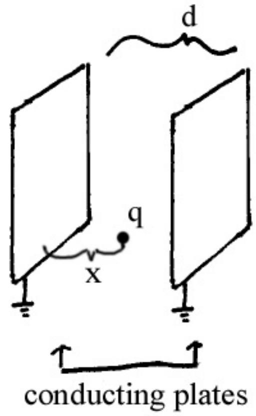
$$\int_V \left(\Phi\nabla^2\Phi' - \Phi'\nabla^2\Phi\right) dV' = \oint_S \left[\Phi\frac{\partial\Phi'}{\partial n} - \Phi'\frac{\partial\Phi}{\partial n}\right] dA$$

Use Gauss's law, $\nabla^2\Phi = \frac{\rho}{\epsilon_0}$, to replace the Laplacian's on the left side of the equal sign with charge densities. From problem 1.1, we know $\frac{\partial\Phi}{\partial n} = \frac{\sigma}{\epsilon_0}$. Replace the derivatives on the right side by surface charge densities.

$$\frac{1}{\epsilon_0}\int_V \left(\Phi\rho' - \Phi'\rho\right) d^3x' = \frac{1}{\epsilon_0}\oint_S \left[\Phi\sigma' - \Phi'\sigma\right] dA$$

With a tiny bit of rearrangement, we get Green's reciprocity theorem:

$$\int \rho\Phi' dV + \int \sigma\Phi' dA = \int \rho'\Phi dV + \int \sigma'\Phi dA$$



Problem 1.13

Two infinite grounded parallel conducting planes are separated by a distance d. A charge, q, is placed between the plates.
We will be using the Green's reciprocity theorem

$$\int \rho\Phi' dV + \int \sigma\Phi' dA = \int \rho'\Phi dV + \int \sigma'\Phi dA$$

For the unprimed case, we have the situation at hand. ρ and σ vanish at all points except at the two plates' surfaces and at the point charge. The potential at the two grounded plates vanishes.
We need to choose another situation *with the same surfaces* for which we know the potential. The easiest thing that comes to mind is the parallel plate capacitor. We take the first plate to be at $x = 0$ and the second at $x = d$. The charge density vanishes everywhere except on the two plates. The electrostatic potential is simple, $\Phi'(x) = \Phi_0 \frac{x}{d}$ which we know is true for the parallel plate capacitor.
Plugging into Green's reciprocity theorem, we have

$$\left(q \times \Phi_0 \frac{x}{d}\right) + \left(0 + q'\Phi_0 \frac{d}{d}\right) = (0) + (0)$$

With a little algebra, this becomes

$$q' = -\frac{x}{d} q$$

on plate two. By symmetry, we can read off the induced charge on the other plate, $q' = -\frac{d-x}{d} q = -(1 - \frac{x}{d})q$.

Bonus Section: A Clever Ruse

This tricky little problem was on my qualifying exam, and I got it wrong. The irony is that I was assigned a similar question as an undergrad. I got it wrong back then, thought, "Whew, I'll never have to deal with this again," and never looked at the solution. This was a most foolish move.

Calculate the force required to hold two hemispheres (radius R) each with charge $Q/2$ together.

Think about a Gaussian surface as wrapping paper which covers both hemispheres of the split orb. Now, pretend one of the hemispheres is not there. Since Gauss's law only cares about how much charge is enclosed, the radial field caused by one hemisphere is

$$\vec{E} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r}$$

Because of cylindrical symmetry, we expect the force driving the hemispheres apart to be directed along the polar axis. The non polar components cancel, so we need to consider only the polar projection of the electric field. The assumption is that we can find the polar components of the electric field by taking z part of the radial components. So we will find the northward directed electric field created by the southern hemisphere and affecting the northern hemisphere and integrate this over the infinitesimal charge elements of the northern hemisphere. Using $dq = \frac{Q}{4\pi R^2}dA$, we have

$$F_z = \int_{north} E_z dq = \int \left(\frac{1}{4\pi\epsilon_0}\frac{Q}{2R^2}\cos\theta\right)\frac{Q}{4\pi R^2}dA$$

where θ is the angle the electric field makes with the z-axis.

$$F_z = -\frac{1}{4\pi\epsilon_0}\frac{Q^2}{8\pi R^4}2\pi R^2\int_0^1 \cos\theta d(\cos\theta) = -\frac{Q^2}{32\pi\epsilon_0 R^2}$$

The conclusions is that we have to push down on the upper hemisphere if the bottom is fixed, and we want both shells to stay together.

Problem 2.1

a.

Jackson asks us to use the method of images to find the potential for a point charge placed a distance, d, from a infinitely large zero potential conducting x-z sheet located at $y = 0$.

$$\Phi(\vec{r}) = \frac{\frac{1}{4\pi\epsilon_0}q}{|\vec{r_o} - \vec{r_q}'|} + \frac{\frac{1}{4\pi\epsilon_0}q_I}{|\vec{r_o} - \vec{r_I}'|}$$

The first term is the potential contribution from the actual charge q and the second term is the contribution from the image charge q_I. Let the coordinates x, y, and z denote the position of the field in question, while the coordinates x_0, y_0, and z_0 denote the position of the actual charge. Choose a coordinate system so that the *real* point charge is placed on the positive y-axis. x_0 and y_0 vanish in this coordinate system. Now, apply boundary conditions $\Phi(y = 0) = 0$.

$$\Phi(y = 0) = \frac{\frac{1}{4\pi\epsilon_0}q}{\sqrt{x^2 + z^2 + y_0^2}} + \frac{\frac{1}{4\pi\epsilon_0}q'}{\sqrt{(x - x_I')^2 + y - {y_I'}^2 + (z - z_I')^2}} = 0$$

We can have $\Phi = 0$ for all points on the x-z plane only if $q' = -q$, $x_I' = 0$, $z_I' = 0$, and $y_I' = -y_0$. Label $y_0 = d$.

$$\Phi(x, y, z) = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\sqrt{x^2 + (y - d)^2 + z^2}} - \frac{1}{\sqrt{x^2 + (y + d)^2 + z^2}}\right)$$

To find the surface charge density induced on the sheet, we use the formula from problem 1.1.

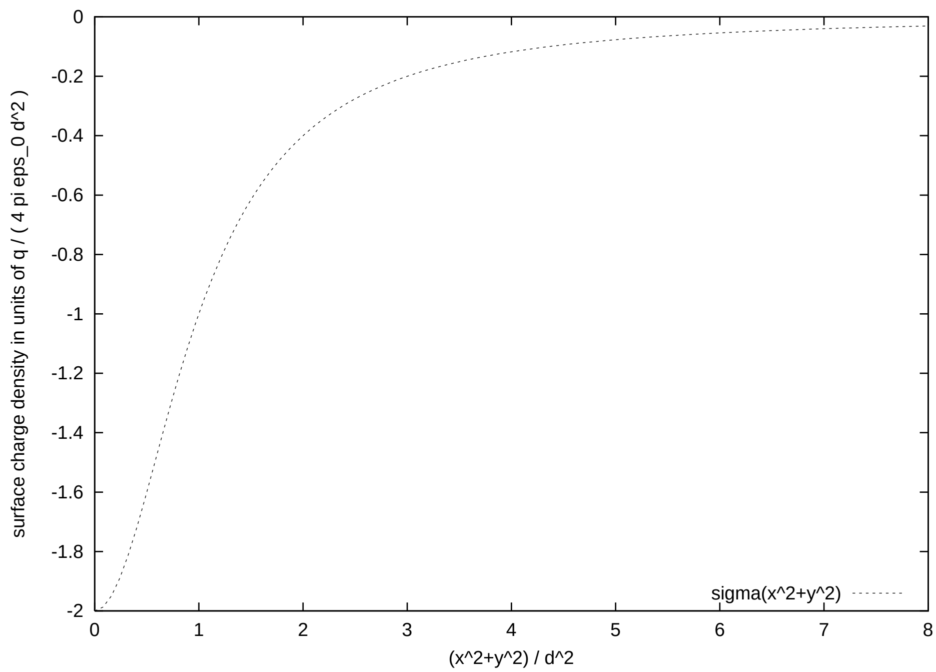
$$\sigma = \epsilon_0 E_n = -\epsilon_0 \frac{\partial\Phi}{\partial y}|_{y=0}$$

$$\sigma = -\frac{1}{4\pi\epsilon_0}q\frac{2d}{(x^2 + d^2 + z^2)^{\frac{3}{2}}} = -\frac{q}{4\pi\epsilon_0 d^2}\left(\frac{2}{(1 + x^2/d^2 + y^2/d^2)^{\frac{3}{2}}}\right)$$

b.

The force between the charge and its *image* is given by Coulomb's law.

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{|\vec{r_q} - \vec{r_I}'|^2}\hat{y} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}\hat{y}$$



Where the effective distance between the charge and image is $|\vec{r_q}' - \vec{r_I}'| = 2d$. The force is obviously attractive because of the minus sign.

c.

Now, we use the method Jackson suggests. First, we square our equation for σ.

$$\sigma^2 = \frac{q^2}{16\pi^2} \frac{4d^2}{(x^2 + d^2 + z^2)^3}$$

Jackson tell us that the force can be computed from the following integral:

$$\vec{F} = \int \frac{\sigma^2}{2\epsilon_0} d\vec{A}$$

So we do this integral.

$$\vec{F} = \int_0^\infty \int_0^{2\pi} \frac{q^2}{32\pi^2\epsilon_0} \frac{rd^2}{(r^2 + d^2)^3} d\theta dr \hat{y}$$

where $r^2 = x^2 + z^2$. Let $u = r^2 + d^2$ and $du = 2rdr$.

$$\vec{F} = \int_{-d^2}^\infty \frac{q^2}{16\pi\epsilon_0} \frac{1}{2} \frac{d^2}{u^3} du \hat{y} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d^2} \hat{r}$$

Which is the same as in part b.

d.

$$W = \int \mathbf{F} \cdot \mathbf{r} = -\int_d^\infty \frac{1}{4\pi\epsilon_0} \frac{q^2}{4r^2} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4r}|_d^\infty = -\frac{q^2}{16\pi\epsilon_0 d}$$

The image charge is allowed to move in the calculation.

e.

$$U = -\frac{1}{2} \frac{1}{4\pi\epsilon_0} \frac{q^2}{|r - r'|} = -\frac{q^2}{8\pi\epsilon_0 d}$$

Here we find the energy without moving the image charge so our result is different from part d.

f.

Use the result from part d. Take $d \approx 1$ Angstrom so $W = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4d} = 5.77 \times 10^{-19}$ joules or 3.6 eV.

Problem 2.2

I botched this one up the first time I did it. Hopefully, this time things will turn out better!

a. the potential inside the sphere

As implied by definition of conducting $V = 0$ on the surface. We must place an image charge outside the sphere on the axis defined by the real charge q and the center of the sphere. Use a Cartesian coordinate system and set the x-axis to be the axis defined by the charge, its image, and the center of the sphere.

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-x_1)^2+y^2+z^2}} + \frac{q'}{\sqrt{(x-x_2')^2+y^2+z^2}}\right]$$

The charge q is positioned at x_1 and its image q' is at x_2' [2]. For the real charge outside the sphere and its image inside, Jackson finds $q_{in} = -\frac{a}{x_{out}}q_{out}$ and $x_{in} = \frac{a^2}{x_{out}}$. We let $x_{in} = x_1$ and $x_{out} = x_2'$, and the second equations tells us: $x_2' = \frac{a^2}{x_1}$. Let $q_{in} = q$ and $q_{out} = q'$. Care must be taken because the first equation depends on $x_out = x_2$. $q = -\frac{a}{x_2}q' = -\frac{x_1}{a}q'$. So $q' = -\frac{a}{x_1}q$. Incidentally, even if I had no help from Jackson's text, this is a good guess because dimensionally it works. This image charge distribution does satisfy the boundary conditions.

$$\Phi(a) = \frac{1}{4\pi\epsilon_0}q\left[\frac{1}{\sqrt{x_1^2+a^2}} - \frac{a}{x_1}\frac{1}{\sqrt{(\frac{a^2}{x_1})^2+a^2}}\right] = 0$$

A more rigorous determination in not necessary because this function is unique. Therefore, for a real charge q placed within a conducting sphere of radius a, we find the potential to be:

$$\Phi(x,y,z) = \frac{1}{4\pi\epsilon_0}q\left[\frac{1}{\sqrt{(x-x_1)^2+y^2+z^2}} - \frac{a}{x_1}\frac{1}{\sqrt{(x-\frac{a^2}{x_1})^2+y^2+z^2}}\right]$$

where $x_1 < a$ for the charge inside the sphere and $x_1 \neq 0$. The charge should not be placed at the center of the sphere. I am sure that a limiting method

[2] I've been a little redundant with the subscript and the prime, but I felt clarity was better than brevity at this point.

could reveal the potential for a charge at the center, but that is not necessary. Use Gauss's law to get

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r} - \frac{1}{4\pi\epsilon_0}\frac{q}{a}$$

b. the induced surface charge density

The surface charge density will simply be the same as calculated by Jackson for the inverse problem. For a charge outside a conducting sphere, the surface charge density is such.

$$\sigma = -\frac{1}{4\pi\epsilon_0} q \frac{a}{x_1} \frac{1 - \frac{a^2}{x_1^2}}{(1 + \frac{a^2}{x_1^2} - 2\frac{a}{x_1}\cos\gamma)^{\frac{3}{2}}}$$

where γ is the angle between the x-axis and the area element. Jackson's result comes from taking $\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}$, but our potential is functionally the same. Thus, our surface charge distribution will be the same.

$$\sigma = -\frac{1}{4\pi\epsilon_0} q \frac{a}{x_1} \frac{1 - \frac{a^2}{x_1^2}}{(1 + \frac{a^2}{x_1^2} - 2\frac{a}{x_1}\cos\gamma)^{\frac{3}{2}}}$$

c. the magnitude and direction of the force acting on q.

The force acting on q can be obtained by Coulomb's law.

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{|r - r'|^2} = \frac{1}{4\pi\epsilon_0} q \left(-\frac{a}{x_1} q\right) \frac{1}{\left(\frac{a^2}{x_1} - x_1\right)^2} = -\frac{1}{4\pi\epsilon_0} q^2 \frac{a x_1}{(a^2 - x_1^2)^2}$$

d.

If the sphere is kept at a fixed potential Φ, we must add an image charge at the origin so that the potential at R is Φ. If the sphere has a total charge Q on its inner and outer surfaces, we figure out what image charge would create a surface charge equal to Q and place this image at the origin.

2.28

I will do a simple derivation. We have some crazy n-sided regular polyhedron. That means that each side has the same area and each corner has the same set of angles. If one side is at potential Φ_i but all the other sides are at zero potential. The potential in the center of the polygon will be some value, call it Φ_i'. By symmetry, we could use this same approach for any side; A potential Φ_i always produces another potential Φ_i' at the center. Now, we use linear superposition. Let all the sides be at Φ_i. Then, the potential at the center is

$$\Phi_{center} = \sum_{i=1}^{n} \Phi_i'$$

If all the Φ_i are equal, then so are all the Φ_i'. Then, $\Phi_c = n\Phi_i'$, and we can solve for $\Phi_i' = \frac{\Phi_c}{n}$. If each surface is at some potential, Φ_i, then the entire interior is at that potential, and $\Phi_i = \Phi_c$ according to the mean value theorem. Therefore, $\Phi_i' = \frac{\Phi_i}{n}$ is the contribution from each side.
For a set of arbitrary potentials for each side, we can use the principle of linear superposition again.

$$\Phi_c = \frac{1}{n} \sum_{i=1}^{n} \Phi_i$$

q.e.d.

Problem 3.3

Note ρ is used where I usually use r'.

a.

For a charged ring at $z = 0$ on the r-ϕ plane, Jackson derived the following:

$$\Phi(r,\theta) = \begin{cases} q\sum_{L=0}^{\infty} \frac{\rho^L}{r^{L+1}} P_L(0)P_L(\cos\theta) & , r \geq R \\ q\sum_{L=0}^{\infty} \frac{r^L}{\rho^{L+1}} P_L(0)P_L(\cos\theta) & , r < R \end{cases}$$

But

$$P_L(0) = \begin{cases} 0 \ \ , for\ L\ odd \\ (-1)^{\frac{L}{2}} \frac{(L+1)!!}{(L+1)L!!} = f(L) \ \ , for\ L\ even \end{cases}$$

We can replace L by 2ℓ because every other term vanishes.
Since $\sigma \propto (R^2 - \rho^2)^{-\frac{1}{2}}$ on the disk, the total charge on the disk is

$$Q = \int_0^R \frac{2\pi\kappa\rho}{\sqrt{R^2 - \rho^2}} d\rho$$

Let $u = R^2 - \rho^2$, $du = -2\rho d\rho$, so

$$Q = -\int_{R^2}^{0} \frac{\pi\kappa}{\sqrt{u}} du = 2\pi\kappa u^{\frac{1}{2}}|_0^{R^2} = 2\pi\kappa R$$

And $\kappa = \frac{Q}{2\pi R}$. Now, we solve for a disk made up of infinitely many infinitesimally small rings. Each contributes to the potential

$$\delta\Phi(r,\theta) = \sigma \sum_{\ell=0}^{\ell} \frac{\rho^{2\ell}}{r^{2\ell+1}} f(2\ell) P_{2\ell}(\cos\theta) dA, \ \ r \geq R$$

where $f(2\ell) = P_{2\ell}(0)$. And integrating over the disk gives the total potential.

$$\begin{aligned} \Phi(r,\theta) &= \int \kappa(R^2 - \rho^2)^{-\frac{1}{2}} \sum_{\ell=0}^{\ell} \frac{\rho^{2\ell}}{r^{2\ell+1}} f(2\ell) P_{2\ell}(\cos\theta) \rho d\rho d\phi \\ &= 2\pi\kappa \sum \int_0^R (R^2 - \rho^2)^{-\frac{1}{2}} \frac{\rho^{2\ell}}{r^{2\ell+1}} f(2\ell) P_{2\ell}(\cos\theta) \rho d\rho \end{aligned}$$

Consider the integral over ρ.

$$\int_0^R \frac{\rho^{2\ell+1}}{\sqrt{R^2 - \rho^2}} d\rho = \frac{1}{R} \int_0^R \frac{\rho^{2\ell+1}}{\sqrt{1 - \frac{\rho^2}{R^2}}} d\rho$$

Let $\frac{\rho}{R} = \sin\theta$, $d\rho = R\cos\theta d\theta$.

$$I_1 = \frac{1}{R}\int_0^{\frac{\pi}{2}} R\frac{(R)^{2\ell+1}\sin^{2\ell+1}\theta}{\cos\theta}\cos\theta d\theta = R^{2\ell+1}\frac{2^\ell \ell!}{(2\ell+1)!!}$$

Using

$$\int_0^{\frac{\pi}{2}} \sin^{2\ell+1}\theta d\theta = \frac{2^\ell \ell!}{(2\ell+1)!!}$$

So

$$\Phi = 2\pi\kappa \sum \frac{2^\ell \ell!}{(2\ell+1)!!} f(2\ell)\frac{R^{2\ell+1}}{r^{2\ell+1}} P_{2\ell}(\cos\theta)$$

but we know $f(2\ell)$.

$$\Phi = \frac{4Q}{R}\sum (-1)^\ell \frac{(2\ell+1)!!}{(2\ell+1)(2\ell)!!}\frac{2^\ell \ell!}{(2\ell+1)!!}\left(\frac{R}{r}\right)^{2\ell}\left(\frac{R}{r}\right)P_{2\ell}(\cos\theta)$$

Since $(2\ell)!! = 2^\ell \ell!$,

$$\Phi = \frac{4Q}{R}\sum (-1)^\ell \frac{1}{2\ell+1}\left(\frac{R}{r}\right)^{2\ell}\left(\frac{R}{r}\right)P_{2\ell}(\cos\theta)\,, r \geq R$$

The potential on the disk at the origin is V.

$$V = \int_0^{2\pi}\int_0^R \sigma\rho d\rho d\phi = \int \frac{2Q}{\pi R}\frac{2\pi\rho}{|\rho|\sqrt{R^2-\rho^2}}d\rho$$

Using $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{|a|}\right)$,

$$V = \frac{2Q}{\pi R}2\pi\sin^{-1}\left(\frac{x}{|R|}\right)|_0^R = \frac{2Q\pi}{R}$$

And $\kappa = \frac{2Q}{\pi R} = \frac{V}{\pi^2}$. Then,

$$\Phi = \frac{2V}{\pi}\left(\frac{R}{r}\right)\sum (-1)^\ell \frac{1}{2\ell+1}\left(\frac{R}{r}\right)^\ell P_{2\ell}(\cos\theta), r \geq R$$

A similar integration can be carried out for $r < R$.

$$\Phi = \frac{2\pi Q}{R} - \frac{4Q}{R}\sum (-1)^\ell \frac{1}{2\ell+1}\left(\frac{r}{R}\right)^{2\ell}\left(\frac{r}{R}\right)P_{2\ell}(\cos\theta)\,, r \leq R$$

b.

I can't figure out what I did here. I'll get back to this.

c.

$C = \frac{Q}{V}$, but from part a $Q = \frac{2VR}{\pi}$ so

$$C = \frac{2VR}{\pi}\left(\frac{1}{V}\right) = \frac{2R}{\pi}$$

Problem 3.9

$V = 0$ at $z = 0, L$. Because of cylindrical symmetry, we will try cylindrical coordinates. Then, we have

$$\nabla^2 \Phi = 0 \rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Try $\Phi(r, \phi, z) = R(r)Z(z)Q(\phi)$. Separating the Laplace equation in cylindrical coordinates, we find three differential equations which must be satisfied.

$$\frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0$$

has the solution

$$Z = A\sin(kz) + B\cos(kz)$$

The solution must satisfy boundary conditions that $\Phi = 0$ at $z = 0, L$. Therefore, B must vanish.

$$Z = A\sin(kz)$$

where $k = \frac{n\pi}{L}$.
Similarly, we have for Q

$$\frac{\partial^2 Q}{\partial \phi^2} - m^2 Q = 0$$

which has the solution

$$Q = C\sin(m\phi) + D\cos(m\phi)$$

m must be an integer for Q to be single valued.
The radial part must satisfy the frightening equation. Note the signs. This is not the typical Bessel equations, but have no fear.

$$\frac{\partial^2 R}{\partial x^2} + \frac{1}{x}\frac{\partial R}{\partial x} - \left(1 + \frac{m^2}{x^2}\right) R = 0$$

where $x = kr$. The solutions are just modified Bessel functions.

$$R(x) = EI_m(x) + FK_m(x)$$

m must be an integer for R to be single valued. I_m and K_m are related to other Bessel and Neumann functions via

$$I_m(kr) = i^{-m} J_m(ikr)$$

$$K_m(kr) = \frac{\pi}{2} i^{m+1} H_m^{(1)}(ikr)$$

The potential is finite at $r = 0$ so

$$H_m^{(1)}(0) = J_m(0) + iN_m(0) = 0$$

But $K_m \neq 0$ so $F = 0$.
We can now write Φ in a general form.

$$\Phi = RZQ = \sum A \sin\left(\frac{n\pi}{L}z\right)\left(C\sin(m\phi) + D\cos(m\phi)\right) E I_m\left(\frac{n\pi}{L}r\right)$$

Let A and E be absorbed into C and D.

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{L}z\right) I_m\left(\frac{n\pi}{L}r\right)\left(C_{mn}\sin(m\phi) + D_{mn}\cos(m\phi)\right)$$

Now, we match boundary conditions. At $r = b$, $\Phi(\phi, z) = V(\phi, z)$. So

$$\Phi(\phi, z) = \sum_{m,n} \sin\left(\frac{n\pi}{L}z\right) I_m\left(\frac{n\pi}{L}b\right)\left(C_{mn}\sin(m\phi) + D_{mn}\cos(m\phi)\right) = V$$

The $I_m\left(\frac{n\pi}{L}b\right)$ are just a set of constants so we'll absorb them into C'_{mn} and D'_{mn} for the time being. The coefficients, C'_{mn} and D'_{mn}, can be obtained via Fourier analysis.

$$C'_{mn} = \kappa \int_0^{2L}\int_0^{2\pi} \Phi(\phi, z)\sin\left(\frac{n\pi}{L}z\right)\sin(m\phi)\, d\phi dz$$

$$D'_{mn} = \kappa \int_0^{2L}\int_0^{2\pi} \Phi(\phi, z)\sin\left(\frac{n\pi}{L}z\right)\cos(m\phi)\, d\phi dz$$

κ is determined by orthonormality of the various terms.

$$\kappa^{-1} = \int_0^{2L} \sin^2\left(\frac{n\pi}{L}z\right) dz \int_0^{2\pi} \sin^2(m\phi)\, d\phi =$$
$$\frac{L}{\pi n m}\left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]\Big|_{x=0}^{2\pi n} \times \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]\Big|_{x=0}^{2\pi m}$$

So $\kappa = \frac{1}{L\pi}$. Finally, we have

$$C_{mn} = \frac{1}{L\pi}\frac{1}{I_m\left(\frac{n\pi}{L}b\right)}\int_0^{2L}\int_0^{2\pi} \Phi(\phi, z)\sin\left(\frac{n\pi}{L}z\right)\sin(m\phi)\, d\phi dz$$

And

$$D_{mn} = \frac{1}{L\pi} \frac{1}{I_m\left(\frac{n\pi}{L}b\right)} \int_0^{2L} \int_0^{2\pi} \Phi(\phi, z) \sin\left(\frac{n\pi}{L}z\right) \cos\left(m\phi\right) d\phi dz$$

Problem 3.14

a.

$$Q = \int_{-d}^{d} \kappa(d^2 - z^2)dz = \frac{4}{3}\kappa d^3$$

so $\kappa = \frac{3Q}{4d^3}$ and

$$\lambda = \frac{3Q}{4d^3}(d^2 - z^2)$$

For use later, we will write this in spherical coordinates.

$$\rho(r,\theta,\phi) = \frac{3Q}{4d^3}(d^2 - r^2)\frac{1}{\pi r^2}\delta(\cos^2\theta - 1)$$

For the inside of the spherical shell, the Green's function is:

$$G(x,x') = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)}{(2\ell+1)\left[1-(\frac{a}{b})^{2\ell+1}\right]} \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

Where a and b denote the inner and outer radii. Here $a = 0$ so

$$G(x,x') = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)}{(2\ell+1)} r_<^\ell \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

because of azimuthal symmetry on $m = 0$ terms contribute.

$$G(x,x') = 4\pi \sum_{\ell=0}^{\infty} P_\ell(\cos\theta')P_\ell(\cos\theta) r_<^\ell \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

The potential can be obtained through Green's functions techniques.

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int d^3x'\ \rho(x')G(x,x')$$

And explicitly

$$\Phi = \frac{1}{4\pi\epsilon_0}\int d\phi' d(\cos\theta') r'^2 dr'\ \frac{3Q}{d^3 r^2}(d^2 - r'^2)\delta(\cos^2\theta' - 1)$$
$$\sum_{\ell=0}^{\infty} P_\ell(\cos\theta')P_\ell(\cos\theta) r_<^\ell \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

The integrations over ϕ' and θ' are easy and fun!

$$\Phi = \frac{3Q}{16\pi\epsilon_0 d^3}\sum_{\ell=0}^{\infty}(P_\ell(1)+P_\ell(-1))P_\ell(\cos\theta)\int_0^b dr'\ r'^2(d^2-r'^2)r_<^\ell\left(\frac{1}{r_>^{\ell+1}}-\frac{r_>^\ell}{b^{2\ell+1}}\right)$$

Integration over r' must be done over several regions: $r < d$ and $r' > r$, $r < d$ and $r' < r$, $r > d$ and $r' > r$, and $r > d$ and $r' < r$. When the smoke clears, we find:

$$\Phi(r,\theta,\phi) = \frac{3Q}{16\pi\epsilon_0 d^3}\sum_{\ell=0}^{\infty}(P_\ell(1)+P_\ell(-1))P_\ell(\cos\theta)\mathcal{I}(r,\ell)$$

where

$$\mathcal{I}(r,\ell) = \left(\frac{1}{r^{\ell+1}}-\frac{r^\ell}{b^{2\ell+1}}\right)\left(d^2\frac{r^{\ell+1}}{\ell+1}-\frac{r^{\ell+3}}{\ell+3}\right)$$
$$+r^\ell\left[-\frac{d^{2-\ell}}{\ell}+\frac{d^{2-\ell}}{\ell-2}-\frac{d^{\ell+3}}{(\ell+1)b^{2\ell+1}}-\frac{d^{\ell+3}}{(\ell+3)b^{2\ell+1}}\right]$$
$$-r^\ell\left[-\frac{d^2}{\ell r^\ell}+\frac{r^{2-\ell}}{\ell-2}-\frac{d^2 r^{\ell+1}}{(\ell+1)b^{2\ell+1}}-\frac{r^{\ell+3}}{(\ell+3)b^{2\ell+1}}\right]$$

Presumably, this can be reduced, but I never got around to that. For $r < d$ and

$$\Phi(r,\theta,\phi) = \frac{3Q}{16\pi\epsilon_0 d^3}\sum_{\ell=0}^{\infty}(P_\ell(1)+P_\ell(-1))P_\ell(\cos\theta)\left(\frac{1}{r^{\ell+1}}-\frac{r^\ell}{b^{2\ell+1}}\right)\left(\frac{2d^{\ell+3}}{(\ell+1)(\ell+3)}\right)$$

The term $P_\ell(1)+P_\ell(-1)$ is zero for odd ℓ and $2P_\ell(1)$ for even ℓ. So we can rewrite our answer.

b.

$$\sigma = -\epsilon_0\nabla\Phi\cdot\hat{n}$$

$$\sigma = -\frac{3Q}{8\pi}\sum_{\ell=0}^{\infty}P_\ell(\cos\theta)\frac{(2\ell+1)}{(\ell+1)(\ell+3)}\frac{2}{b^2}\left(\frac{d}{b}\right)^\ell$$

c.

In this limit, the term $\left(\frac{d}{b}\right)^{\ell}$ except when $\ell = 0$. Then,

$$\sigma = -\frac{3Q}{8\pi} P_0(\cos\theta) \frac{1}{3} \frac{2}{b^2} = -\frac{Q}{4\pi b^2}$$

This is what we would have expected if a point charge were located at the origin and the sphere were at zero potential. When $d << b$, r will most likely be greater than d for the region of interest so it will suffice to take the limit of the second form for Φ. Once again, only $\ell = 0$ terms will contribute.

$$\Phi = \frac{Q}{4\pi\epsilon_0} \left(\frac{b - r}{br}\right)$$

This looks like the equation for a spherical capacitor's potential as it should!

Problem 4.6

a.

Recall that the quadrapole tensor is

$$Q_{ij} = \int \left(3x'_i x'_j - r'^2\delta_{ij}\right)\rho(\vec{r}')dV'$$

For the external field, Gauss's law tells us that a vanishing charge density means $\nabla\cdot\vec{E} = 0$. $\frac{\partial E_z}{\partial z} + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$. The problem is cylindrically symmetric so $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = -\frac{1}{2}\frac{\partial E_z}{\partial z}$.

According to Jackson's equation 4.23, the energy for a quadrapole is

$$W = -\frac{1}{6}\sum_{i=1}^{3}\sum_{j=1}^{3}\int \left(3x_i x_j - r^2\delta_{ij}\right)\rho\frac{\partial E_j}{\partial x_i}d^3x$$

When $i \neq j$, there is no contribution to the energy. You can understand this by recalling that the curl of $\vec{E}$ is zero for static configurations, i.e. $\nabla\times\vec{E} = 0$. When $x_i = z$ and $x_j = z$, the integral is clearly $qQ_{33} = qQ_{nucleus}$, and the energy contribution is $W_3 = -\frac{q}{6}Q\frac{\partial E_z}{\partial z}$. Jackson hints on page 151 that in nuclear physics $Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$. For x_i or x_j equals x or y, $W = (-\frac{1}{2})(-\frac{1}{2})(-\frac{q}{6}Q\frac{\partial E_z}{\partial z}) = -\frac{q}{24}Q\frac{\partial E_z}{\partial z}$. Thus,

$$W = -\left(\frac{1}{6}+\frac{1}{12}\right)qQ\left(\frac{\partial E_z}{\partial z}\right) = -\frac{q}{4}Q\left(\frac{\partial E_z}{\partial z}\right)$$

q.e.d.

b.

We are given $Q = 2\times 10^{-28}$ m^2, W/h = 10 MHz, $a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e q^2} = 0.529\times 10^{-10}$ m, $\frac{q}{4\pi\epsilon_0 a_0^3} = 9.73\times 10^2$ N/(mC), and from part a,

$$W = -\frac{q}{4}Q\left(\frac{\partial E_z}{\partial z}\right)$$

Solve for $\left(\frac{\partial E_z}{\partial z}\right)$,

$$\left(\frac{\partial E_z}{\partial z}\right) = \frac{W}{h}\left(\frac{-h}{q}\right)\frac{4}{1}\frac{1}{Q}$$

Plugging in numbers, 8.27×10^{20} N/(mC). In units of $\frac{q}{4\pi\epsilon_0 a_0^3}$:

$$\left(\frac{\partial E_z}{\partial z}\right) = 8.5\times 10^{-2}\left(\frac{q}{4\pi\epsilon_0 a_0^3}\right)\frac{N}{m\cdot C}$$

c.

For the nucleus, the total charge is Zq where q is the charge of the electron. The charge density is the total charge divided by the volume for points inside the nucleus. Outside the nucleus the charge density vanishes. The volume of an general ellipsoid is given by the high school geometry formula, $V = \frac{4}{3}\pi abc$. In our case, a is the semi-major axis, and b and c are the semi-minor axes. By cylindrical symmetry $b = c$.

$$\rho = \begin{cases} \frac{3Zq}{4\pi ab^2}, r \leq \frac{b}{a}\sqrt{a^2 - z^2} \\ 0, r > \frac{b}{a}\sqrt{a^2 - z^2} \end{cases}$$

The nuclear quadrapole moment is defined $Q = \frac{1}{q}\int \left(3z^2 - R^2\right)\rho dV$. Because of the obvious symmetry, we'll do this in cylindrical coordinates where $R^2 = z^2 + r^2$ and $dV = rd\theta drdz$.

$$Q = \frac{3Z}{4\pi ab^2}\int_{-a}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-z^2}}\int_{0}^{2\pi}\left(3z^2 - z^2 - r^2\right) rd\theta drdz$$

The limits on the second integral are determined because the charge density vanishes outside the limits.

$$Q = \frac{3Z}{4\pi ab^2}\int_{-a}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-z^2}}\left(2z^2 - r^2\right) rdrdz$$

Substitute $r^2 = u$, and integrate over du.

$$Q = \frac{3Z}{4\pi ab^2}\int_{-a}^{a}\int_{0}^{b^2 - \frac{z^2b^2}{a^2}}\left(2z^2 - u\right) dudz$$

$$= \frac{3Z}{4\pi ab^2}\int_{-a}^{a}\left[2z^2\left(b^2 - \frac{z^2b^2}{a^2}\right) - \frac{1}{2}\left(b^2 - \frac{z^2b^2}{a^2}\right)^2\right] dz$$

Simplify.

$$Q = \frac{3Z}{4ab^2}\int_{-a}^{a}\left[2z^2b^2 - \frac{2z^4b^2}{a^2} - \frac{1}{2}b^4 + \frac{b^4z^2}{a^2} - \frac{1}{2}b^4z^4a^4\right] dz$$

Evaluate the next integral.

$$Q = \frac{3Z}{4ab^2}\left[\frac{4a^3b^2}{3} - \frac{4b^2a^3}{5} - b^4a + \frac{2b^4a}{3} - \frac{1}{5}ab^4\right]$$

Simplify and factor.

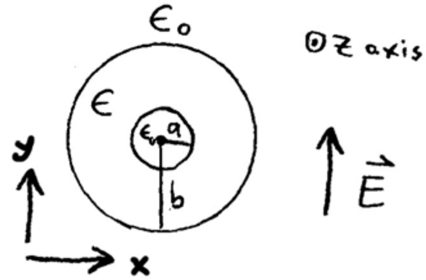
$$Q = \frac{2}{5}Za^2 - \frac{2}{5}zb^2 = \frac{2Z}{5}(a+b)(a-b) = \frac{8Z}{5}\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)$$

Plug in $R = \frac{a+b}{2}$. R is the mean radius, 7×10^{-15} meters.

$$Q = \frac{8Z}{5}R\left(\frac{a-b}{2}\right)$$

So finally, I can get what Jackson desires.

$$\left(\frac{a-b}{2}\right) = \frac{5Q}{8ZR} \rightarrow \frac{a-b}{R} = \frac{5Q}{8Z}\frac{2}{R^2}$$



Problem 4.8

Since the total charge is zero outside the cylinder and inside the shell, we can use regular Poisson's equation there:

$$\nabla^2 \Phi = 0$$

And solve for Φ, the total electro-static potential from which we get the field according to $\vec{E} = -\nabla\Phi$.

Within the dielectric, we need to think twice. There will be some sort of induced source, so the validly or using the total potential here is seriously in doubt. We can try to find, instead, a screened potential from which we could get the electric field.

$$\nabla^2 \phi = \rho_{ind.} \to \nabla^2 (\phi - \phi') = 0 \to \nabla^2 \phi_{screened} = 0$$

where $\nabla^2 \phi' = \rho_{inc.}$. The gradient of $\phi_{screened}$ gives the screened electric field according to $\vec{D} = -\nabla\Phi_{screened}$. The electric field and the screened electric field are related via: $\vec{D} = \epsilon\vec{E}$.

Symmetry in this problem leads me to choose cylindrical coordinates in which the Poisson equation is

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Because of translational symmetry along the $z$axis, Φ is independent of z, and we need only consider the problem in the r-θ plane.

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

Try a separation of variables, i.e. $\Phi(r,\theta) = R(r)\Theta(\theta)$.

$$\frac{r^2}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

This will give us two equations. The first isn't too hard to solve.

$$\frac{\partial^2 \Theta}{\partial \theta^2} = -m^2\Theta$$

This will have solutions proportional to $e^{+im\theta}$, $e^{-im\theta}$, or perhaps some linear combination of the both. We'll employ the convenient linear superposition

$$\Theta(\theta) = A_m \cos(m\theta + \alpha_m)$$

Notice that this choice is really just one possibility. For example,

$$A_m \cos(m\theta+\alpha_m) = A_m \cos(m\theta)\cos(\alpha_m) - A_m \sin(m\theta)\sin(\alpha_m) = B_m \cos(m\theta) + C_m \sin(m\theta)$$

which would we could also use.
The second equation is a bit trickier.

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{m^2}{R^2}R = 0$$

Let's guess that

$$R(r) = r^{\pm m},\ m \neq 0$$

and

$$R(r) = \ln r + C,\ m = 0$$

are solutions. In fact, it is not that hard to show that these are in fact solutions.
The general solution is a linear combination of these solutions. The boundary conditions will determine just what this linear combination is.
The first *boundary* condition is that sufficiently far from the shell, we will only measure the uniform electric field. This uniform field can be reproduced by a potential, $\Phi = -E_0 r\cos\phi$. Check it. $\vec{E} = -\nabla\Phi = E_0\hat{y}$
At the inner and outer surfaces of the cylindrical shell, we have two other boundary conditions. Namely, at $x = a$ and b, $\vec{E}_{||}$ and $\vec{D}_{\perp}$ are continuous. Recall that $E_{||} = -\frac{\partial\Phi}{\partial\theta}$ and $\epsilon E_{\perp} = -\epsilon\frac{\partial\Phi}{\partial r}$.

To make our lives easier, we will limit the possible forms of the solution outside and inside the cylindrical region. Outside, we need to have the electric field at infinity, but we certainly don't want the field to diverge. The logarithmic and r^n with $n > 1$ terms diverge as r goes to infinity; clearly, these terms are unphysical.

$$\Phi_{out} = -E_0 r \cos\theta + \sum_{m=1}^{\infty} A_m \frac{1}{r^m} \cos(m\theta + \alpha_m)$$

In between the cylindrical shells, we don't have any obvious physical constraints.

$$\Phi_{mid} = \sum_{m=1}^{\infty} B_{-m} \frac{1}{r^m} \cos(m\theta + \beta_{-m}) + \sum_{m=1}^{\infty} B_m r^m \cos(m\theta + \beta_m) + C \ln r$$

Inside, we have to eliminate the diverging terms at the origin.

$$\Phi_{in} = \sum_{m=1}^{\infty} D_m r^m \cos(m\theta + \delta_m)$$

Now, it's almost time to match boundary conditions. They were $\Phi = -E_0 r \cos\phi$ as $x \to \infty$, and at $x = a$ or b; $E_{||} = -\frac{\partial\Phi}{\partial\theta}$ and $\epsilon E_{\perp} = -\epsilon\frac{\partial\Phi}{\partial r}$ are continuous.

The only thing which breaks pure cylindrical symmetry is the external field. Even then, its only effect is to first order in a Legendre polynomial expansion. That means the external potential only depends on a trig. function of the angle, θ, and not some integer multiple of that angle. Such is the symmetry of our problem. Terms with $m > 1$ violate this symmetry, so it must be that $A_m = B_m = B_{-m} = D_m = 0$.

I am left with the following forms:

Outside:

$$\Phi_{out} = -E_0 r \cos\theta + A_1 \frac{1}{r} \cos(\theta + \alpha_1)$$

In between the cylinders:

$$\Phi_{mid} = B_{-1} \frac{1}{r} \cos(\theta + \beta_{-1}) + B_1 r \cos(\theta + \beta_1)$$

Inside the cylinders:

$$\Phi_{in} = D_1 r \cos(\theta + \delta_1)$$

Because each region has the same symmetry with respect to the external field, we can drop the phases.
For the outside region, we find the electro-static potential

$$\Phi_{out} = \left(-E_0 r + A_1 \frac{1}{r}\right)\cos(\theta)$$

And likewise in between, we have the screened potential

$$\Phi_{mid} = \left(B_{-1}\frac{1}{r} + B_1 r\right)\cos(\theta)$$

And inside, we have another full electro static potential

$$\Phi_{in} = D_1 r\cos(\theta)$$

Let's apply the next boundary condition, $\frac{\partial\Phi}{\partial\theta}$. Match first the electric fields that graze the surfaces.

$$-E_0 b + A_1\frac{1}{b} = \frac{1}{\epsilon}B_1 b + \frac{1}{\epsilon}B_{-1}\frac{1}{b}$$

And

$$\frac{1}{\epsilon}B_1 a + \frac{1}{\epsilon}\frac{B_{-1}}{a} = D_1 a$$

For the final boundary condition, we make sure the screened electric field, $\epsilon\frac{\partial\Phi}{\partial r}$, across a surface is continuous:

$$-\epsilon_0\left(E_0 + \frac{A_1}{b^2}\right) = (B_1 - B_{-1}\frac{1}{b^2})$$

$$(-\frac{B_{-1}}{a^2} + B_1) = \epsilon_0 D_1$$

Let $\frac{\epsilon}{\epsilon_0} = \kappa$, the capacitance. For historical reasons, I will let $B \to \epsilon B$. And solve for this rescaled B. I'll write the problem as a matrix equation:

$$\mathbf{M}\mathbf{v} = \mathbf{c}$$

with

$$\mathbf{v} = \begin{pmatrix} A_1 \\ B_1 \\ B_{-1} \\ D_1 \end{pmatrix}$$

and the the solution is just

$$\mathbf{v} = \mathbf{M}^{-1}\mathbf{c}$$

How, the hell, do we get the inverse, $\mathbf{M}$? We could get lucky and guess it, but I don't recommend this technique. It can be very frustrating. If we are smart, we'll use Maple or Mathematica. If we are a little less smart, we might try to figure out a scheme to get this inverse. One scheme is to repeatedly multiply both sides of the equation by simple almost diagonal matrices. Doing so, we can try to make the first matrix equation look more like the second. It seems like a reasonable requirement that we multiply by only invertible matrices, but well, you'll have to consult a math book for more about that. I used Mathematica instead.

$$\begin{pmatrix} 1/b & -b & -1/b & 0 \\ 0 & a & 1/a & -a \\ -1/b^2 & -\kappa & \kappa/b^2 & 0 \\ 0 & \kappa & -\kappa/a^2 & -1 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \\ B_{-1} \\ D_1 \end{pmatrix} = \begin{pmatrix} E_0 b \\ 0 \\ E_0 \\ 0 \end{pmatrix}$$

Then, presto!

$$\mathbf{M}^{-1} = \frac{1}{\frac{a}{b^3} - \frac{1}{ab} - \frac{2ak}{b^3} - \frac{2k}{ab} + \frac{ak^2}{b^3} - \frac{k^2}{ab}}$$

$$\begin{pmatrix} -\frac{k}{a} - \frac{ak}{b^2} - \frac{k^2}{a} + \frac{ak^2}{b^2} & -\frac{2k}{b} & -\frac{a}{b} + \frac{b}{a} + \frac{ak}{b} + \frac{bk}{a} & \frac{2ak}{b} \\ \frac{1}{ab^2} + \frac{k}{ab^2} & \frac{1}{b^3} - \frac{k}{b^3} & \frac{1}{ab} + \frac{k}{ab} & -\frac{a}{b^3} + \frac{ak}{b^3} \\ -\frac{a}{b^2} + \frac{ak}{b^2} & -\frac{1}{b} - \frac{k}{b} & -\frac{a}{b} + \frac{ak}{b} & \frac{a}{b} + \frac{ak}{b} \\ \frac{2k}{ab^2} & \frac{k}{b^3} + \frac{k}{a^2 b} - \frac{k^2}{b^3} + \frac{k^2}{a^2 b} & \frac{2k}{ab} & -\frac{a}{b^3} + \frac{1}{ab} + \frac{ak}{b^3} + \frac{k}{ab} \end{pmatrix}$$

Do the matrix multiplication, and you'll get the results.

$$A_1 = E_0 b^2 + 2E_0 b^2 \frac{a^2(1-\kappa)^2 - b^2(1+\kappa)^2}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2}$$

$$B_1 = \frac{-2E_0 b^2 (1+\kappa)^2}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2}$$

$$B_{-1} = \frac{2E_0 a^2 b^2 (1-\kappa)^2}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2}$$

$$D_1 = \frac{-4E_0 b^2 \kappa}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2}$$

Recall, that I scaled the B type terms by ϵ, so the original Bs would be:

$$B_1 = \epsilon \frac{-2E_0 b^2 (1+\kappa)^2}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2}$$

$$B_{-1} = \epsilon \frac{2E_0 a^2 b^2 (1-\kappa)^2}{b^2(1+\kappa)^2 - a^2(1-\kappa)^2}$$

For the rest of this problem, I will consider only the unscreened electric fields so these factors of ϵ will disappear $\vec{E} = \frac{1}{\epsilon}\vec{D}$.

b. Sketch the lines of force for a typical case of $b \simeq 2a$.

I will use units of $\frac{1}{E_0}$ and make plots for $\kappa = \frac{1}{2}$ and $\kappa = 2$.

c. Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

For the dielectric cylinder, I shrink the inner radius down to nothing; $a \to 0$.

$$A_1 = \frac{\kappa - 1}{\kappa + 1} b^2 E_0$$

$$B_1 = \frac{-2E_0}{1+\kappa}$$

$$B_{-1} = 0$$

$$D_1 = \frac{-4E_0}{(1+\kappa)^2}$$

For the cylindrical cavity, I place the surface of the outer shell at infinity, $b \to \infty$. In this limit A_1 is ill-defined, so we'll ignore it.

$$B_1 = \frac{-2E_0}{1+\kappa}$$

$$B_{-1} = 2E_0 a^2 \frac{1-\kappa}{(1+\kappa)^2}$$

$$D_1 = \frac{-4E_0}{(1+\kappa)^2}$$

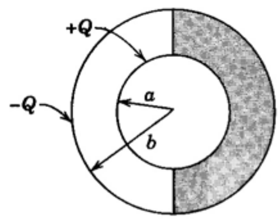
To tidy things up, redefine

$$E_1 = \frac{2E_0}{1+\kappa}$$

and write the potential

$$\Phi_2 = \left(-E_1 r + E_1 a^2 \frac{1-\kappa}{1+\kappa}\frac{1}{r}\right)\cos\theta$$

$$\Phi_3 = -\frac{2E_1}{1+\kappa} r\cos\theta$$



Problem 4.10

a.

We use what I like to call the D law, that is $\nabla \cdot \vec{D} = \rho_{free}$. The divergence theorem tells us

$$\oint \vec{D} \cdot d\vec{A} = Q$$

Only, the radial components of $\vec{D}$ will contribute to this integral if we take the surface of integration to be a sphere with the same center as the two shells. Use the D theorem and that $\vec{D} = \epsilon \vec{E}$.

$$\epsilon_0 E_r 2\pi r^2 + \epsilon E_r 2\pi r^2 = Q$$

This gives a radial electric field:

$$E_r = \left(\frac{2}{1 + \frac{\epsilon}{\epsilon_0}} \right) \frac{Q}{4\pi\epsilon_0 r^2}$$

We are not done. Let's do the same thing, but with a different surface of integration. This time we will integrate only over one hemi-sphere. The total enclosed charge is zero.

$$\left(\frac{2}{1 + \frac{\epsilon}{\epsilon_0}} \right) \frac{\epsilon Q}{2} - \left(\frac{2}{1 + \frac{\epsilon}{\epsilon_0}} \right) \frac{\epsilon Q}{2} + \int_{ring} \vec{D} \cdot d\vec{A} = 0$$

which tells us

$$\int_{ring} \vec{D} \cdot d\vec{A} = 0$$

or the integral over the non radial components of D vanish. By symmetry, the non-radial components must be parallel to the area vectors. So in order for the integral to vanish, the integrand must vanish. The non-radial components must vanish. We could do the same thing for any pie shaped wedge, and we would get the same result: the non-radial D components vanish. So we conclude

$$\vec{E} = \left(\frac{2}{1+\frac{\epsilon}{\epsilon_0}}\right)\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

This has the form of Coulomb's law but with an effective total charge, $Q_{eff} = \frac{2\epsilon_0}{\epsilon+\epsilon_0}Q$.

b.

$\sigma_i = \epsilon_i E_r$ in this case. On the inner surface,

$$\sigma_{dielectric} = \left(\frac{\epsilon}{\epsilon_0+\epsilon}\right)\frac{Q}{2\pi a^2}$$

And

$$\sigma_{air} = \left(\frac{\epsilon_0}{\epsilon_0+\epsilon}\right)\frac{Q}{2\pi a^2}$$

c.

Find the polarization charge density by subtracting the effective charge density from the total contained charge density: $Q_{eff} = Q + Q_{pol.}$. This gives $Q_{polarization} = \left(\frac{\epsilon_0-\epsilon}{\epsilon_0+\epsilon}\right)Q$. The total charge density is obtained by averaging the polarization charge over the half the inner sphere's surface which is in contact with the dielectric. $\sigma_{polarization} = \frac{Q_{polarization}}{2\pi a^2}$. Therefore, the polarization charge density is:

$$\sigma_{polarization} = -\left(\frac{\epsilon_0-\epsilon}{\epsilon_0+\epsilon}\right)\frac{Q}{2\pi a^2}$$

An alternative way of finding this result is to consider the polarization, $\vec{P} = (\epsilon-\epsilon_0)\vec{E}$. Jackson argues that $\sigma_{polarization} = \vec{P}\cdot\vec{n}$. But $\vec{P}$ points from the dielectric outward at $r = a$, and $\sigma_{polarization} = -P_r = (\epsilon_0-\epsilon)E_r = \left(\frac{\epsilon_0-\epsilon}{\epsilon_0+\epsilon}\right)\frac{Q}{2\pi a^2}$ as before.

Problem 5.1

Biot-Savart's law tells us how to find the magnetic field at some point $P(\vec{r})$ produced by a wire element at some other point $P_2(\vec{r'})$. At $P(\vec{r})$:

$$d\vec{B} = \frac{\mu_0}{4\pi} I d\vec{\ell'} \times \left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} \right)$$

The total $\vec{B}$-field at a point P is the sum of the $d\vec{B}$ elements from the entire loop. So we integral $d\vec{B}$ around the closed wire loop.

$$\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi} I \oint_\Gamma d\ell' \times \left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} \right)$$

There is a form of Stokes' theorem which is useful here: $\oint d\ell' \times A = \int dS' \times \nabla' \times A$. I'll look up a definitive reference for this someday; this maybe on the inside cover of Jackson's book.

$$\oint_\Gamma d\ell' \times \left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} \right) = \int dS' \times \nabla' \times \left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} \right)$$

With the useful identity, $\nabla' f(x - x') = \nabla f(x' - x)$, we have

$$dS' \times \nabla' \times \left(\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|^3} \right) = dS' \times \nabla \times \left(\frac{\vec{r'} - \vec{r}}{|\vec{r'} - \vec{r}|^3} \right)$$

Now, with the use the vector identity, $\vec{A} \times (\vec{B} \times \vec{C}) = (A \cdot C)\vec{B} - (A \cdot B)\vec{C}$, I can write the triple cross product under the integral as two terms. The integral becomes

$$\vec{B} = -\frac{\mu_0}{4\pi} I \int \nabla \cdot \left(\frac{\vec{r'} - \vec{r}}{|\vec{r'} - \vec{r}|^3} \right) d\vec{S'} + \frac{\mu_0}{4\pi} I \int \vec{\nabla} \left[\left(\frac{\vec{r'} - \vec{r}}{|\vec{r'} - \vec{r}|^3} \right) \cdot dS' \right]$$

But $\left(\frac{\vec{r'} - \vec{r}}{|\vec{r'} - \vec{r}|^3} \right) = \nabla(\frac{1}{|\vec{r} - \vec{r'}|})$ and $\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r'}|} \right) \propto \delta(r' - r)$. The first integral vanishes on the surface where r' does not equal r. Since I am free to choose any area which is delimited by the closed curve Γ, I choose a surface so that r' does not equal r on the surface, and the first term vanishes. We are left with

$$\vec{B} = \frac{\mu_0}{4\pi} I \vec{\nabla} \int \left[\left(\frac{\vec{r'} - \vec{r}}{|\vec{r'} - \vec{r}|^3} \right) \cdot dS' \right]$$

I take ∇ outside of the integral because the integral does not depend on r', as the integration does.
An element of solid angle is an element of the surface area, $a\vec{A} \cdot \hat{R}$, of a sphere divided by the square of that sphere's radius, R^2, so that the solid angle has dimension-less units (so called steradians). To get a solid angle, we integral over the required area.

$$\Omega = \int_A \frac{d\vec{A} \cdot (\hat{R})}{R^2} = \int \frac{\vec{R} \cdot d\vec{A}}{R^3}$$

And in our notation, this is

$$\Omega = \int \left[\left(\frac{\vec{r'} - \vec{r}}{|\vec{r'} - \vec{r}|^3} \right) \cdot d\vec{S}' \right]$$

Thus,

$$\vec{B} = \frac{\mu_0}{4\pi} I \vec{\nabla} \Omega$$

Where Ω is the solid angle viewed from the observation point subtended by the closed current loop.

Problem 5.17

a.

Well, this problem is not too bad. Jackson solved this for a charge distribution located in a dielectric ϵ_1 above a semi-infinite dielectric plane with ϵ_2.

$$q^* = -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right) q$$
$$q^{**} = \left(\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\right) q$$

With some careful replacements, we can generalize these to solve for the image currents. We will consider point currents, whatever the Hell they are. Physically, a point current makes no sense and violates the conservation of charge, but mathematically, it's useful to pretend such a thing could exist. Associate each component of $\vec{J}(x, y, z)$ with q. Set $\epsilon_1 \to \mu_1 = 1$ and $\epsilon_2 \to \mu_2 = \mu$. These replacements give us the images modulo a minus sign.
For $z > 0$, we have to be careful about the overall signs of the image currents. We can find the signs by considering the limiting cases of diamagnetism and paramagnetism. That is when $\mu \to 0$, we have paramagnetism, and when $\mu \to \infty$, we have diamagnetism. Let's work with the diamagnetic case. The image current will reduce the effect of the real current. Using the right hand rule, we'd expect parallel wires to carry the current in the same direction for this case. Therefore, we must have

$$\vec{J}^*_{\parallel} = \left(\frac{\mu - 1}{\mu + 1}\right) \vec{J}_{\parallel}$$

The perpendicular part of the image current, on the other hand, must flow in the opposite direction of the real current.

$$J^*_{\perp} = -\left(\frac{\mu - 1}{\mu + 1}\right) J_{\perp}$$

We can understand this using an argument about mirrors. For the parallel components, the image currents must be parallel and in the same direction for the diamagnetic case. Think of a mirror and the image of your right hand in a mirror. If you move your right hand to the right, its image also moved to the right. If it's a dirty mirror, then a dim image of our hand moves to the right. That's exactly what we'd expect. The same direction current,

but smaller magnitude. For the perpendicular point current, we need an opposite sign (anti-parallel) image current. This is not much more difficult to visualize. Think of the image of your right hand in mirror. Move your hand away from you, and watch its image move toward you!
If we had considered the paramagnetic case, the image currents would reverse direction. This is because we now want the images to contribute to the fields caused by the real currents. The sign flip changes two competing currents to two collaborating currents.
b.
For $z < 0$: Once again, we associate $\vec{J}$ with q to find $\vec{J}^{**}$. Set $\mu_1 = 1$ and $\mu_2 = \mu$. Notice that all the signs are positive. For the components of the current parallel to the surface, this is exactly as expected. For the z component, we have a reflection of a reflection or simply a weakened version of the original current as our image; therefore, the sign is positive.

$$\vec{J}^{**} = \left(\frac{2\mu}{\mu+1}\right)\vec{J}$$

To get a better understanding of the physics involved here, I will derive these results using the boundary conditions. We are solving

$$\nabla \times \vec{H} = \vec{J}$$

Which has a formal integral solution

$$\vec{H} = \frac{\mu_0}{4\pi\mu}\int d^3\vec{r'}\,\vec{J}(r') \times \frac{|\vec{r} - \vec{r'}|}{|r - r'|^3}$$

But our Js are point currents, that is $\vec{J} \propto \delta(\vec{r'} - \vec{a})$, so we can do the integral and write

$$\vec{H} \propto \frac{1}{\mu}\vec{I} \times (\vec{r} - \vec{a}')$$

The cross product is what causes all the trouble. We will choose $\vec{r} = \pm\hat{k}$, $\vec{a}' = \hat{j}$, $\vec{I} = I_x\hat{i} + I_y\hat{j} + I_z\hat{k}$, $\vec{I}^{*} = I_x^{*}\hat{i} + I_y^{*}\hat{j} + I_z^{*}\hat{k}$, and $\vec{I}^{**} = I_x^{**}\hat{i} + I_y^{**}\hat{j} + I_z^{**}\hat{k}$. Notice that I have not made any assumptions about the signs of the various image current components. Then,

$$\vec{H} \propto -(I_y + I_z)\hat{i} + I_x\hat{j} + I_x\hat{k}$$

$$\vec{H}^* \propto (I_y^* - I_z^*)\hat{\imath} - I_x^*\hat{\jmath} + I_x^*\hat{k}$$

And

$$\vec{H}^{**} \propto \frac{1}{\mu}(I_y^{**} - I_z^{**})\hat{\imath} - \frac{1}{\mu}I_x^{**}\hat{\jmath} + \frac{1}{\mu}I_x^{**}\hat{k}$$

We have the boundary conditions: 1.

$$\vec{B}_2 \cdot \hat{n} = \vec{B}_1 \cdot \hat{n} \rightarrow \mu_2 \vec{H}_2 \cdot \hat{n} = \mu_1 \vec{H}_1 \cdot \hat{n}$$

And 2.

$$\vec{H}_2 \times \hat{n} = \vec{H}_1 \times \hat{n}$$

Note $\hat{n} = \hat{k}$. From the first condition:

$$I_x + I_x^* = I_x^{**}$$

From the other condition, we find for the $\hat{\imath}$ component

$$-\frac{1}{\mu}I_x^{**} = -(I_x - I_x^*)$$

Solve simultaneously to find

$$I_x^{**} = \frac{2\mu}{\mu+1}I_x$$

And

$$I_x^* = \frac{\mu-1}{\mu+1}I_x$$

By symmetry, we know that these equations still hold with the replacement $x \rightarrow y$. We have one more condition left from the $\hat{\jmath}$ component.

$$-\frac{1}{\mu}(I_y^{**} - I_z^{**}) = (I_z + I_z^*) - (I_y - I_y^*)$$

To make life easier, we'll put I_y to zero. Then,

$$\frac{1}{\mu}I_z^{**} = I_z + I_z^*$$

I'm not sure how to get a unique solution out of this, but if I assume that I_z^{**} has the same form as I_x^{**}, I find

$$I_z^* = \frac{1-\mu}{\mu+1}I$$

Problem 6.11

a.

The momentum density for a plane wave is $\vec{\mathcal{P}} = \frac{1}{c^2}\vec{S}$ with the Poynting vector, $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$. The total momentum is the momentum density integrated over the volume in question.

$$\vec{p} = \int \vec{\mathcal{P}} dV \sim \vec{\mathcal{P}} A \; dx \; \hat{x}$$

The last step is true assuming $\vec{\mathcal{P}}$ does not vary much over the volume in question. Be aware that $dV = Adx$, the volume element in question. By Newton's second law, the force exerted in one direction (say x) is

$$F_x = \frac{dp_x}{dt} = \frac{d}{dt}(\mathcal{P}Adx) = \mathcal{P}A\frac{dx}{dt} = \mathcal{P}Ac$$

c is the speed of light. After all, electro-magnetic waves are just light waves. We want pressure which is force per unit area.

$$\vec{P} = \frac{\vec{F}}{A} = \vec{\mathcal{P}}c = \frac{1}{c}\vec{S} = \frac{1}{c\mu_0}\vec{E} \times \vec{B}$$

Take the average over time, and factor of one half comes in. We also know that $B_0 = \frac{E_0}{c}$. Then,

$$P = \frac{1}{2}\frac{1}{\mu_0 c^2}(E_0)^2$$

But $c^2 = \frac{1}{\mu_0 \epsilon_0}$, so

$$P = \frac{1}{2}\epsilon_0 (E_0)^2$$

We already know from high school physics or Jackson equation 6.106 that the energy density is $\frac{1}{2}(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \rightarrow \frac{1}{2}\epsilon_0 (E_0)^2$ and wait that's the same as the pressure!

$$P = \frac{1}{2}\epsilon_0 (E_0)^2 = u$$

This result generalizes quite easily to the case of a non-monochromatic wave by the superposition principle and Fourier's theorem.

b.

Energy Flux from the Sun: 1.4 kW/m^2

Mass/Area of Sail: 0.001 kg/m^2
The force on the sail is the radiation pressure times the sail area. In part a, we discovered that the electro-magnetic radiation pressure is the same as the energy density. Thus, $F = PA = uA$. Now, by Newton's law $F = ma$. The energy density is Φ/c where Φ is the energy flux given off by the sun. The acceleration of the sail is $\frac{\Phi}{c}\frac{A}{m} = 14000 \div (3 \times 10^8) \times 1000 = 4.6 \times 10^{-3}$ m /sec^2.
According to my main man, Hans C. Ohanian, the velocity of the solar wind is about 400 km/sec. I'll guess-timate the density of solar wind particles as one per cubic centimeter ($\rho =$ particles/volume $*$ mass/particle $= 1.7 \times 10^{-21}$ kg/m^3). Look in an astronomy book for a better estimate.

$$\Delta p = \mathcal{P} A v \Delta t$$

Clearly, $\mathcal{P} = \rho v$. The change in the momentum of the sail is thus $a = \frac{1}{m}\frac{\Delta p}{\Delta t} = \mathcal{P}Av/m = \rho\frac{A}{m}v^2$. Numerically, we find $a = 2.7 \times 10^{-9}$ m/sec^2.
Evidently, we can crank more acceleration out of a radiation pressure space sail ship than from a solar wind powered one.

Problem 6.15

a.

$$\vec{E} = \rho_0 \vec{J} + R(\vec{H} \times \vec{J}) + a_{2a}\vec{H}^2\vec{J} + a_{2b}(\vec{H} \cdot \vec{J})\vec{H}$$

In Jackson section 6.10, Jackson performs a similar expansion for $\vec{p}$. We'll proceed along the same lines.
The zeroth term is, $a_0\vec{J}$ (Ohm's law). This is the simplest combination of terms which can still give us a polar vector.
Because $\vec{E}$ is a polar vector, the vector terms on the right side on the equation must be polar. $\vec{H}$ is axial so it alone is not allowed, but certain cross products and dot products produce polar vectors and are allowed. They are
First Order: $a_1(\vec{H} \times \vec{J})$
Second Order:$a_{2a}(\vec{H} \cdot \vec{H})\vec{J} + a_{2b}(\vec{H} \cdot \vec{J})\vec{H}$
When $\vec{H} = 0$, $\vec{E} = \rho_0\vec{J}$ so $a_1 = \rho_0$. And

$$\vec{E} = \rho_0 \vec{J} + R(\vec{H} \times \vec{J}) + a_{2a}\vec{H}^2\vec{J} + a_{2b}(\vec{H} \cdot \vec{J})\vec{H}$$

I let $a_1 = R$.

b.

Under time reversal, we have a little problem. $\vec{E}$ and ρ_0 are even but $\vec{J}$ is odd. But then again if you think about it things really aren't that bad. Ohm's law is a dissipative effect and we shouldn't expect it to be invariant under time reversal.

Problem 6.16

a.

A magnetic dipole, $\vec{m} = \frac{q\hbar}{2m_p}$, creates a magnetic field, $\vec{B}$.

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}\right]$$

Along the meridian plane, $\vec{n}\cdot\vec{m} = 0$ so

$$\vec{B} = -\frac{\mu_0}{4\pi}\frac{\vec{m}}{|\vec{x}|^3}$$

Suppose this field is acting on a magnetic mono-pole with charge $g = \frac{2\pi\hbar}{q}n$. Where n is some quantum number which we'll suppose to be 1. The force is

$$F = -\frac{\mu_0}{4\pi}\frac{g|m|}{|x|^3}\hat{m}$$

And the magnitude

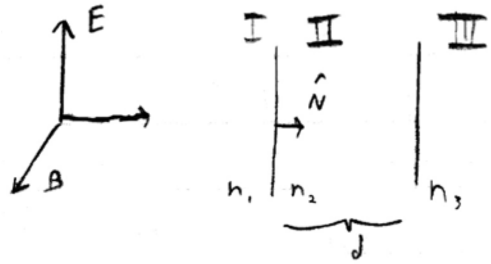
$$|F| = \frac{\mu_0}{4\pi}\frac{gq\hbar}{2m_p r^3} = \frac{\mu_0\hbar^2}{4m_p r^3} = 2\times 10^{-11}\, Newtons$$

where we use $r = 0.5$ Angstroms.

b.

The electrostatic force at the same separation is given by Coulomb's law. $|F| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = 9.2\times 10^{-8}$ Newtons where I have used $\frac{1}{4\pi\epsilon_0} = 8.988\times 10^9$ Nm^2/C, $q = 1.602\times 10^{-19}$ C, and $r = 0.5$ Angstroms. The fine structure is approximately $\alpha = \frac{1}{137}$ times the Coulomb force, so we expect this contribution to be about 7×10^{-10} Newtons. The hyperfine interaction is smaller by a factor of $\frac{m_e}{m_p} = \frac{1}{1836}$, so $F_{fs} \simeq 4\times 10^{-13}$ Newtons. I guess, because we can measure the hyperfine structure and the monopole interaction is larger, we should be able to see the effects of magnetic mono-poles on nuclei if those monopoles should exist. Unless of course monopoles are super-massive. Or perhaps, mono-poles are endowed with divine attributes which make them terribly hard to detect.

Problem 7.2



a.

Look at the diagram. We have three layers of material labeled I,II, and III respectively. Each layer has a corresponding index of refraction, n_1, n_2, and n_3. An electro-magnetic wave is incident from the left and travels through the layers in the sequence $I \rightarrow II \rightarrow III$. Because this is an electro-magnetic wave, we know $\vec{k} \times \hat{N} = 0$ and $\vec{B} \cdot \vec{E} = 0$. That is the E and B fields are perpendicular to the motion of the wave and are mutually perpendicular. These are non-permeable media so $\mu_1 = \mu_2 = \mu_3 = 1$ and $n_i = n(\epsilon_i)$ only.
To find the effective coefficient of reflection, we will consider closely what is going on. The first interface can reflect the wave and contribute directly to the effective reflection coefficient, or the interface can transmit the wave. The story's not over yet because the second interface can also reflect the wave. If the wave is reflected, it will travel back to the first interface where it could be transmitted back through the first interface. Or the wave could bounce back. The effective reflection coefficient will be an infinite series of terms. Each subsequent term corresponds to a certain number of bounces between surface A and surface B before the wave is finally reflected to the left.

$$r = r_{12} + t_{12} r_{23} t_{21} e^{2ik_2 d} + t_{12} r_{23} r_{21} t_{21} e^{4ik_2 d} + ...$$

The first term corresponds to the reflection at the n_1-n_2 interface. The second term is a wave which passes through the n_1-n_2 interface, reflects off the n_2-n_3 interface, and then transits through the n_1-n_2 interface. Higher order terms correspond to multiple internal reflections.
Note the phase change over the internal reflection path. It is $2k_2 d$ for one round trip from the n_1-n_2 to the n_2-n_3 interface and back. k_2 is $\frac{n_2 \omega}{c}$. The phase shift makes sense because the term in the exponent is really $i\vec{k} \cdot \vec{r}$

and the distance is $\vec{r} = \vec{d}$ for the first leg. On the return leg, the sign of both $\vec{k}$ and $\vec{d}$ change because the wave is propagating backward and over the same distance in the opposite direction as before. The total phase change is the product of these two changes and so $ikd + i(-k)(-d) = 2ikd$. If you are motivated, you could probably show this with matching boundary conditions. I think this *heuristic* argument suffices.
I'll write this series in a suggestive form:

$$r = r_{12} + \left[t_{12}r_{23}t_{21}e^{2ik_2d}\right] \times \left[1 + r_{23}r_{21}e^{2ik_2d} + (r_{23}r_{21})^2e^{4ik_2d} + ...\right]$$

The second term in the brackets is a geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, x < 1$$

and I can do the sum exactly.

$$r = r_{12} + \left[t_{12}r_{23}t_{21}e^{2ik_2d}\right]\left[\frac{1}{1 - r_{23}r_{21}e^{2ik_2d}}\right]$$

You can obtain for yourself with the help of Jackson page 306:

$$r_{ij} = \frac{n_i - n_j}{n_i + n_j} = \frac{k_i - k_j}{k_i + k_j}$$

And

$$t_{ij} = \frac{2n_i}{n_i + n_j} = \frac{2k_i}{k_i + k_j}$$

With these formulae, I'll show the following useful relationships:

$$r_{12} = \frac{n_1 - n_2}{n_1 + n_2} = -\frac{n_2 - n_1}{n_1 + n_2} = -r_{21}$$

And

$$t_{12}t_{21} = \left(\frac{2n_1}{n_1 + n_2}\right)\left(\frac{2n_1}{n_1 + n_2}\right) = \frac{4n_1n_2}{(n_1 + n_2)^2}$$

$$t_{12}t_{21} = 1 - \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2} = 1 + r_{12}r_{21}$$

Plug these into equation .

$$r = r_{12} + \frac{(1 + r_{12}r_{21})e^{2ik_2d}}{1 + r_{12}r_{23}e^{2ik_2d}} = \frac{r_{12} + r_{23}e^{2ik_2d}}{1 + r_{12}r_{23}e^{2ik_2d}}$$

The reflection coefficient is $R = |r|^2$.

$$R = \frac{r_{12}^2 + r_{23}^2 + 2r_{12}r_{23}\cos(2k_2d)}{1 + 2r_{12}r_{23}\cos(2k_2d) + (r_{12}r_{23})^2}$$

And it follows from $R + T = 1$ that

$$T = \frac{1 - r_{12}^2 - r_{23}^2 + (r_{12}r_{23})^2}{1 + 2r_{12}r_{23}\cos(2k_2d) + (r_{12}r_{23})^2}$$

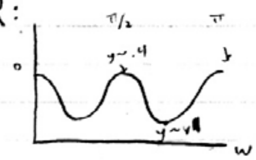
$R + T = 1$ is reasonable if we demand that energy be conserved after a long period has elapsed.
Now, here are all the crazy sketches Jackson wants:

To sketch this recall $k_2 = \frac{n_2 w}{c}$ and let $d = c$

1. $n_1 = 1$, $n_2 = 2$, $n_3 = 3$

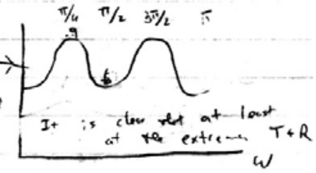
$r_{12} = \frac{-1}{2}$, $r_{23} = \frac{-1}{5}$, $k_2 = \frac{2}{c} w$

R:



π/4 out of phase

T:

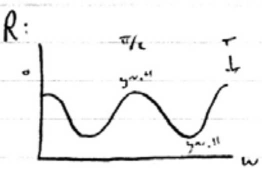


$$R = \frac{1/4 + 1/25 + 1/5 \cos 4w}{1 + \frac{1}{100} + \frac{1}{5}\cos 4w} \qquad T = \frac{1 - 1/4 - 1/25 + (1/10)^2}{1 + \frac{1}{100} + \frac{1}{5}\cos 4w}$$

2. $n_1 = 3$, $n_2 = 2$, $n_3 = 1$

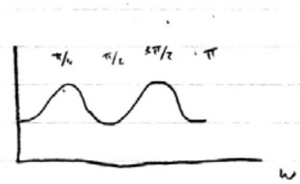
$r_{12} = \frac{1}{5}$, $r_{23} = \frac{1}{2}$, $k_2 = \frac{2}{c} w$

R:



same as above

T:

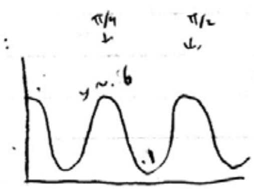


$$R = \frac{1/4 + 1/25 + \frac{1}{5}\cos 4w}{1 + \frac{1}{100} + \frac{1}{5}\cos 4w} \qquad T = \frac{1 - 1/4 - 1/25 + (1/10)^2}{1 + \frac{1}{100} + \frac{1}{3}\cos 4w}$$

3. $n_1 = 2$, $n_2 = 4$, $n_3 = 1$

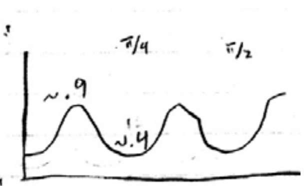
$r_{12} = \frac{-1}{3}$, $r_{23} = \frac{3}{5}$, $k_2 = \frac{4}{c} w$

R:



sorry this graph's are terrible but the intuition are right

T:



$$R = \frac{1/9 + 9/25 - \frac{2}{5}\cos 8w}{1 + 1/25 - \frac{2}{5}\cos 8w} \qquad T = \frac{1 - 1/9 - 9/25 + 1/25}{1 - \frac{2}{5}\cos 8w + 1/25}$$

Problem 7.12

a. The continuity equation states

$$\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}$$

From Ohm's law, $\vec{J}_f = \sigma \vec{E}$ so

$$\nabla \cdot \vec{J}_f = \nabla \cdot (\sigma \vec{E}_f) = \sigma \nabla \cdot \vec{E}_f$$

The last step is true if σ is uniform. According to Coulomb's law, $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$. We now have

$$\nabla \cdot \vec{J} = \sigma \nabla \cdot \vec{E} = \frac{\sigma}{\epsilon_0}\rho = -\frac{\partial \rho}{\partial t}$$

From now on, I'll drop the subscript f. We both know that I mean free charge and current. From the last equality,

$$\sigma \rho + \epsilon_0 \frac{\partial \rho}{\partial t} = 0 \tag{1}$$

Assume that $\rho(t)$ can be written as the time Fourier transform of $\rho(\omega)$. I.e.

$$\rho(t) = \frac{1}{\sqrt{2\pi}} \int \rho(\omega) e^{-i\omega t} d\omega$$

Plug $\rho(t)$ into equation 1.

$$\frac{1}{\sqrt{2\pi}} \int \left(\sigma \rho(\omega) e^{-i\omega t} + \epsilon_0 \rho(\omega) \frac{\partial}{\partial t} e^{-i\omega t} \right) d\omega = 0$$

For the integral to vanish the integrand must vanish so

$$\left[\sigma - i\omega\epsilon_0\right] \rho(\omega) e^{-i\omega t} = 0$$

For all t. We conclude that

$$\left[\sigma(\omega) - i\omega\epsilon_0\right] \rho(\omega) = 0$$

b.

From part a, $\sigma(\omega) - i\epsilon_0\omega = 0$. Let $\omega = -i\alpha$ so that

$$\sigma(\omega) = \frac{\epsilon_0\omega_p^2\tau}{1-\alpha\tau}$$

And the result from part a becomes $\sigma(\omega) - \epsilon_0\alpha = 0 \to$

$$\frac{\epsilon_0\omega_p^2\tau}{1-\alpha\tau} - \epsilon_0\alpha = 0 \to \frac{\epsilon_0\omega_p^2\tau - \epsilon_0\alpha + \alpha^2\tau\epsilon_0}{1-\alpha\tau} = 0$$

The numerator must vanish. Divide the numerator by $\tau\epsilon_0$,

$$\alpha^2 - \tau^{-1}\alpha + \omega_p^2 = 0$$

Solve for α.

$$\alpha = \frac{1}{2}\left[\tau^{-1} \pm \sqrt{\tau^{-2} - 4\omega_p^2}\right]$$

If $\omega_p >> 1$, we can write α in an approximate form,

$$\alpha \sim (2\tau)^{-1} \pm i\omega_p$$

The imaginary part corresponds to the oscillations at ω_p, the plasma frequency. The real part is the decay in amplitude $\frac{1}{2\tau}$.

Problem 7.16

a.

Consider the second Maxwell equation, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Take the curl of both sides. Plug in the fourth Maxwell equation for $\nabla \times \vec{B}$.

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\mu_0 \vec{J} + \mu_0 \frac{\partial \vec{D}}{\partial t}\right)$$

When $\vec{J} = 0$ this becomes

$$\nabla \times \left(\nabla \times \vec{E}\right) + \frac{\partial^2}{\partial t^2}\vec{D} = 0$$

Assume a solution of the form, $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, and try it.

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \frac{\partial^2}{\partial t^2}\vec{D} = 0$$

Use $\left[\vec{k} \times (\vec{k} \times \vec{E})\right]_i = k_i(\vec{k} \cdot \vec{E}) - k^2 E_i =$ to write the double curl out in expanded form.

$$k_i(\vec{k} \cdot \vec{E}) - k^2 E_i + \mu_0 \omega^2 \frac{\partial^2}{\partial t^2} D_i = 0 \tag{2}$$

Because $D_i = \epsilon_{ij} E_j$,

$$k_i(\vec{k} \cdot \vec{E}) - k^2 E_i + \mu_0 \omega^2 \frac{\partial^2}{\partial t^2} \epsilon_{ij} E_j = 0$$

Note $\vec{D}$ is not necessarily parallel to $\vec{E}$.

b.

We will write the result in part a as a matrix equation. The non-diagonal elements of ϵ_{ij} vanish so we replace $\epsilon_{ij} \to \epsilon_{ii}\delta_{ij}$. Define a second rank tensor, $\overleftrightarrow{T}$, as

$$T_{ij} = k_i k_j - \left(k^2 - \frac{\omega_i^2}{c^2}\epsilon_{ii}^2\right)\delta_{ij}$$

The result in equation 2 can be written $\overleftrightarrow{T} \cdot \vec{E} = 0$. In order for there to be a nontrivial solution $\det \overleftrightarrow{T} = 0$. Divide $\overleftrightarrow{T}$ by k^2 and use $\frac{k_i}{|k|} = n_i$ to make things look cleaner

$$T_{ij}/k^2 = n_i n_j - \left(1 - \frac{\omega^2}{k^2 c^2}\epsilon_{ii}^2\right)\delta_{ij}$$

We remember the relations $v = \frac{\omega}{k}$ and $v_i = \frac{c}{\sqrt{\epsilon_{ii}}}$. So we have

$$T_{ij}/k^2 = n_i n_j - \left(1 - \frac{v^2}{v_i^2}\right)\delta_{ij}$$

At this point, we can solve det $\overset{\leftrightarrow}{T} = 0$ for the allowed velocity values.

$$det \begin{pmatrix} n_1^2 - (1 - \frac{v^2}{v_1^2}) & n_2 n_1 & n_3 n_1 \\ n_1 n_2 & n_2^2 - (1 - \frac{v^2}{v_2^2}) & n_3 n_2 \\ n_1 n_3 & n_2 n_3 & n_3^2 - (1 - \frac{v^2}{v_3^2}) \end{pmatrix} = 0$$

Or written out explicitly, we have

$$\begin{aligned} &\left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) \\ +n_1^2 &\left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) \\ +n_2^2 &\left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) \\ +n_3^2 &\left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) = 0 \end{aligned}$$

Multiplying out the determinant,

$$\begin{aligned} &\left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) + n_1^2\left(\frac{v^2}{v_2^2} - 1\right)\left(\frac{v^2}{v_3^2} - 1\right) \\ &+n_2^2\left(\frac{v^2}{v_3^2} - 1\right)\left(\frac{v^2}{v_1^2} - 1\right) + n_3^2\left(\frac{v^2}{v_1^2} - 1\right)\left(\frac{v^2}{v_2^2} - 1\right) = 0 \end{aligned}$$

which can be written in a nicer form.

$$1 + \frac{n_1^2 v_1^2}{v^2 - v_1^2} + \frac{n_2^2 v_2^2}{v^2 - v_2^2} + \frac{n_3^2 v_3^2}{v^2 - v_3^2} = 0$$

Use $n_1^2 + n_2^2 + n_3^3 = 1$ to replace the number one in the above equation.

$$\begin{aligned} &\frac{n_1^2(v^2 - v_1^2)}{v^2 - v_1^2} + \frac{n_2^2(v^2 - v_2^2)}{v^2 - v_2^2} + \frac{n_3^2(v^2 - v_3^2)}{v^2 - v_3^2} \\ &\quad + \frac{n_1^2 v_1^2}{v^2 - v_1^2} + \frac{n_2^2 v_2^2}{v^2 - v_2^2} + \frac{n_3^2 v_3^2}{v^2 - v_3^2} = 0 \end{aligned}$$

Simplify. In the end, you'll obtain a relationship for the v values.

$$n_3^2(v^2-v_1^2)(v^2-v_2^2)+n_1^2(v^2-v_3^2)(v^2-v_2^2)+n_2^2(v^2-v_3^2)(v^2-v_1^2)=0$$

This is quadratic in v^2 so we expect two solutions for v^2. Divide by $(v^2-v_1^2)(v^2-v_2^2)(v^2-v_3^2)$ and write in the compact form which Jackson likes:

$$\sum_{i=1}^{3}\frac{n_i^2}{v^2-v_i^2}=0$$

c.

Divide the equation 2 by k^2 to find the equations which the eigenvectors must satisfy:

$$\vec{E}_1-\vec{n}(\vec{n}\cdot\vec{E}_1)=\frac{v_1^2}{c^2}\vec{D}_1 \tag{3}$$

And

$$\vec{E}_2-\vec{n}(\vec{n}\cdot\vec{E}_2)=\frac{v_2^2}{c^2}\vec{D}_2 \tag{4}$$

Dot the first equation by E_2 and the second by E_1.

$$\vec{E}_2\cdot\vec{E}_1-(\vec{E}_2\cdot\vec{n})(\vec{n}\cdot\vec{E}_1)=\frac{v_1^2}{c^2}\vec{E}_2\cdot\vec{D}_1$$

And

$$\vec{E}_2\cdot\vec{E}_1-(\vec{E}_2\cdot\vec{n})(\vec{n}\cdot\vec{E}_1)=\frac{v_2^2}{c^2}\vec{E}_1\cdot\vec{D}_2$$

Comparing these, we see that

$$v_1^2\vec{E}_2\cdot\vec{D}_1=v_2^2\vec{E}_1\cdot\vec{D}_2$$

Well, we already know that in general $v_1\neq v_2$. So $\vec{E}_2\cdot\vec{D}_1$ and $\vec{E}_1\cdot\vec{D}_2$ must either vanish or be related in such a way as to preserve the equality. However, $\vec{E}_2\cdot\vec{D}_1=\vec{E}_1\cdot\vec{D}_2$ because ϵ_{ij} is diagonal. Then, $\epsilon_{ij}E_{1i}E_{2j}\delta_{ij}=\epsilon_{ij}E_{2j}E_{1i}\delta_{ij}$. Therefore, we must conclude that $\vec{E}_2\cdot\vec{D}_1=\vec{E}_1\cdot\vec{D}_2=0$.
Dot product equation 3 into 4 and find

$$\vec{E}_2\cdot\vec{E}_1+(\hat{n}\cdot\vec{E}_1)(\hat{n}\cdot\vec{E}_2)-2(\hat{n}\cdot\vec{E}_1)(\hat{n}\cdot\vec{E}_2)=\frac{v_1^2v_2^2}{c^4}\vec{D}_2\cdot\vec{D}_1$$

The left hand side can be rewritten as $\vec{E}_2\cdot\vec{E}_1-(\vec{E}_2\cdot\vec{n})(\vec{n}\cdot\vec{E}_1)\propto\vec{E}_1\cdot\vec{D}_2$ which we have shown to vanish. Therefore, the left hand side is zero, and the right hand side,$\vec{D}_1\cdot\vec{D}_2=0$. The eigenvectors are perpendicular.

Problem 7.22

The Kramer-Krönig relation states:

$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega'}{\omega'^2 - \omega^2} \Im\left(\frac{\epsilon(\omega')}{\epsilon_0}\right) d\omega'$$

a. $\Im\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = \lambda\left[\Theta(\omega - \omega_1) - \Theta(\omega - \omega_2)\right]$.
Plug this into the Kramer-Kronig relationship.

$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1 + \frac{2\lambda}{\pi} \int_{\omega_1}^{\omega_2} \frac{\omega'}{\omega'^2 - \omega^2} d\omega' + 0$$

Notice that the real part of $\epsilon(\omega)$ depends on an integral over the entire frequency range for the imaginary part!
Here, we will use a clever trick.

$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1 + \frac{\lambda}{\pi} \int_{\omega_1^2}^{\omega_2^2} \frac{d(\omega'^2)}{\omega'^2 - \omega^2}$$

And this integral is easy to do!

$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1 + \frac{\lambda}{\pi} \ln\left(\omega'^2 - \omega^2\right)\Big|_{\omega_1^2}^{\omega_2^2} = 1 + \frac{\lambda}{\pi} \ln\left(\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\right)$$

b.
Do the same thing.

$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\lambda\gamma\omega'^2}{\left((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\right)\left(\omega'^2 - \omega^2\right)} d\omega'$$

The integral can be evaluated using complex analysis, but I'll avoid this. The integral is just a Hilbert transformation and you can look it up in a table.

$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1 + \frac{\lambda(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

Problem 9.3

A radiating something or other

$V(t) = V_0 \cos(\omega t)$. There should be a diagram showing a sphere split across the equator. The top half is kept at a potential $V(t)$ while the bottom half is $-V(t)$. From Jackson 9.9,

$$\lim_{kr\to\infty} \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(-ik)^n}{n!}\int \vec{J}(\vec{x}')(n\cdot\vec{x}')^n dV'$$

If $kd = k2R << 1$ (as it is), the higher order terms in this expansion fall off rapidly. In our case, it is sufficient to consider just the first term.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')dV'$$

Integrating by parts and substituting $\nabla\cdot\vec{J} = i\omega\rho$, we find

$$\vec{A} = \frac{-i\mu_0\omega}{4\pi}\vec{p}\frac{e^{ikr}}{r}$$

I solved the static situation (but neglected to include it) earlier.

$$\Phi = V\left[\frac{3}{2}\left(\frac{R}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{R}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16}\left(\frac{R}{r}\right)^6 P_5(\cos\theta) + ...\right]$$

Written a different way, this is

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3} + ...$$

Choose the z-axis so that $\vec{p}\cdot\vec{x} = pr\cos\theta$. Compare like terms between the two expressions for Φ to find the dipole moment in terms of known variables.

$$V\frac{3}{2}\left(\frac{R}{r}\right)^2\cos\theta = \frac{1}{4\pi\epsilon_0}\frac{pr\cos\theta}{r^3}$$

So

$$|p_0| = \frac{3}{2}4\pi\epsilon_0 V_0 R^2$$

The time dependent dipole moment is

$$\vec{p}(t) = 6\pi\epsilon_0 V_0 R^2\cos(\omega t)\hat{z}$$

And with this,

$$\vec{A} = \frac{-i\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}p_0\hat{z}$$

In the radiation zone, Jackson said that

$$\vec{H} = \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\frac{e^{ikr}}{r}$$

And

$$\vec{E} = \sqrt{\frac{\mu_0}{\epsilon_0}}\vec{H}\times\hat{n}$$

Some texts will use $z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$; I don't. First, find the magnetic field. Use $\omega = kc$.

$$\vec{B} = \mu_0\vec{H} = -\frac{\mu_0 p_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)e^{ikr}\hat{\phi}$$

Then, find the electric field.

$$\vec{E} = \frac{\mu_0 p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)e^{ikr}\hat{\theta}$$

The power radiated per solid angle can be obtained from the Poynting vector.

$$\frac{dP}{d\Omega} = \frac{r^2}{2\mu_0}|\vec{E}^*\times\vec{B}| = \frac{\mu_0}{2c}\left[\frac{p_0^2\omega^2}{16\pi^2}\sin^2\theta\right]\hat{r}$$

Notice how the complex conjugation and absolute signs get rid of the pesky wave factors.
Integrate over all solid angles to find the total radiated power.

$$P_{Total} = \int\frac{\mu_0}{2c}\left[\frac{p_0^2\omega^2}{16\pi^2}\sin^2\theta\right]d\Omega = \frac{\mu_0 p_0^2\omega^4}{16\pi c}\int_0^\pi \sin^3\theta d\theta$$

The final integral is quite simple, but I'll solve it anyway.

$$\int_0^\pi \sin^3\theta d\theta = \frac{-1}{3}\cos\theta\left(\sin^2\theta+2\right)|_{\theta=0}^{\pi}$$

Putting all this together, the final result is

$$P_{Total} = \frac{3\pi\epsilon_0 V_0^2 R^4\omega^4}{c^3}$$

Problem 9.10

a.

The magnetization is

$$\vec{M} = \frac{1}{2}(\vec{r} \times \vec{J})$$

$\vec{J}$ can be broken up into J_r and J_z components. We take the cross product of the two components with $\vec{r}$.

$$\vec{r} \times J_r = 0$$

And

$$\vec{r} \times J_z = -iv_0 \left(\frac{-x}{z}\hat{y} + \frac{y}{z}\hat{x} \right) a_0 \rho$$

To make things easier, we'll use angles. $\tan\theta = \frac{r}{z}$, $\sin\phi = \frac{y}{r}$, $\cos\phi = \frac{x}{r}$ Then,

$$\vec{r} \times \vec{J} = -ia_0 \rho v_0 \left(\tan\theta \sin\phi \hat{x} - \tan\theta \cos\phi \hat{y} \right)$$

Don't forget $v_0 = \alpha c$, so

$$\vec{M} = -i\frac{\alpha c a_0}{2} \tan\theta (\sin\phi \hat{x} - \cos\phi \hat{y})\rho$$

Let $\vec{\chi} = \frac{-i\alpha c a_0}{2} \tan\theta (\sin\phi \hat{x} - \cos\phi \hat{y})$ then

$$\vec{M} = \rho \vec{\chi}$$

Now, we take the divergence.

$$\nabla \cdot \vec{M} = (\nabla \cdot \vec{\chi})\rho + \vec{\chi} \cdot \nabla \rho$$

We'll consider each term separately to show that they all vanish. First of all,

$$\nabla \cdot \chi \sim \nabla \cdot (\frac{y}{z}\hat{x} - \frac{x}{z}\hat{y}) = 0$$

Now since $\rho \sim re^{\frac{-3r}{2a_0}} \cos\theta = ze^{\frac{-3r}{2a_0}}$, its gradient is

$$\nabla \rho = ze^{\frac{-3r}{2a_0}} \left[\frac{-3x}{2a_0 r}\hat{x} + \frac{-3y}{2a_0 r}\hat{y} + \left(\frac{1}{z} - \frac{3z}{2a_2 r} \right) \hat{z} \right]$$

Which is orthogonal to χ

$$\chi \cdot \nabla \rho \sim \frac{yx}{r} - \frac{xy}{r} = 0$$

Both terms in the divergence vanish, and $\nabla \vec{M} = 0$.
The dipole moment is

$$\vec{p} = \int (x\hat{x} + y\hat{y} + z\hat{z})\, \rho(\vec{x}) dV$$

Don't forget

$$\rho(\vec{x}) = \kappa z e^{\frac{-3}{2a_0}\sqrt{x^2+y^2+z^2}}$$

where $\kappa = \frac{2q}{\sqrt{6}a_0^4}\frac{1}{\sqrt{4\pi}}\sqrt{\frac{3}{4\pi}}$. Putting this together,

$$\vec{p} = \int z(x\hat{x} + y\hat{y} + z\hat{z})\kappa e^{\frac{-3r}{2a_0}} dV$$

Obviously,

$$\int_{-\infty}^{\infty} u e^{-f(u)} du = 0$$

if $f(u)$ is even. Thus, the integrals over the x and y coordinates vanish. We are left with

$$\vec{p} = \kappa \hat{z} \int_{-\infty}^{\infty} z^2 e^{\frac{-3r}{2a_0}} dV = 2\pi\kappa\hat{z} \left(\int r^4 e^{-3r} 2a_0 dr \right) \left(\int \cos^2\theta d(\cos\theta) \right)$$

Use

$$\int_0^{\infty} r^n e^{-\beta r} dr = \frac{n!}{\beta^{n+1}} \rightarrow \int_0^{\infty} r^4 e^{-\beta r} dr = \frac{4!}{\beta^5}$$

And

$$\int_{-1}^{1} \cos^2\theta d(\cos\theta) = \frac{2}{3}$$

To get

$$\vec{p} = \kappa\hat{z}\left(\frac{24}{3^5}2^5 a_0^5\right)\left(\frac{2}{3}\right)(2\pi)$$

Plug in κ explicitly.

$$\vec{p} = 1.49 q a_0 \hat{z}$$

Now, for the magnetic moment,

$$\vec{m} = \int \vec{M} dV = \frac{-ia_0 v_0}{2} \int \rho(\frac{y}{z}\hat{x} - \frac{x}{z}\hat{y})$$

Well, ρ is even but y and x are odd so $\vec{m}$ is zero. The magnetic dipole and electric quadrapole terms vanish because of their dependence on m.
We suspect that electric octo-pole and every other pole thereafter might persist because of symmetry, but we won't worry about that.
b.

$$P = \frac{c^2 z_0 k^4}{12\pi} \vec{p}^2$$

where $z_0 = \frac{1}{\epsilon c}$. Now, $\hbar\alpha = \frac{q^2}{4\pi\epsilon_0 c}$. With some fiddling,

$$P_{Jackson} = 3.9 \times 10^{-2} (\hbar\omega_0) \left(\frac{\alpha^4 c}{a_0}\right)$$

c. $\hbar\omega\Gamma = P$. Using numbers, $\Gamma = 6.3 \times 10^8$ seconds^{-1}.
d.
For a Bohr transition, a dipole transition,

$$\vec{p} = q(2a_0 - a_0)\hat{z}e^{-i\omega t} = qa_0 e^{-i\omega t}\hat{z}$$

which gives an emitted power of $P_{Bohr} = 0.018(\hbar\omega_0)\left(\frac{\alpha^4 c}{a_0}\right)$. And the ratio:

$$\frac{P_{Bohr}}{P_{Jackson}} \simeq 0.45$$

The grader claims that this is incorrect citing a *correct* value of 0.55. You decide, and tell me what you conclude.

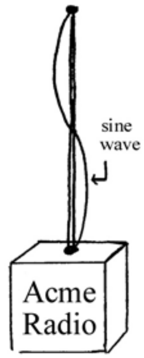


Figure 1:

Problem 9.16

a.

Assume the antenna is center fed.

$$\vec{J} = I_0 \sin\left(\frac{1}{2}kd - k|z|\right)\delta(x)\delta(y)e^{-i\omega t}\hat{z},\, |z| \leq \frac{1}{2}d$$

Note $J(\pm\frac{d}{2}) = 0$ as makes sense. Jackson makes some arguments to justify this current density for a center fed antenna. I'll take his word for it, but if you're not convinced, consult Jackson page 416 in the third edition.

The vector potential due to an oscillating current is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r},t)\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}d^3\vec{r}'$$

In the radiation zone,

$$\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \rightarrow \frac{e^{ikr}}{r}e^{-ik\frac{\vec{r}\cdot\vec{r}'}{r}}$$

The vector potential with the current density can be explicitly written

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{z}\int_{-\frac{d}{2}}^{\frac{d}{2}} I_0 \sin\left(\frac{1}{2}kd - k|z|\right)e^{-ikz'\cos\theta}dz'\,\hat{z}$$

Bear in mind that this expression for the vector potential includes all multipoles. The integral can be done quite easily. Use Euler's theorem, $e^{-ix} =$

$\cos x - i\sin x \rightarrow \sin x = \frac{e^x - e^{-x}}{2i}$, to write:

$$\mathcal{I}_1 = \int \frac{1}{2i}\left(e^{i\frac{1}{2}kd - ik|z'|} - e^{-i\frac{1}{2}kd + ik|z|}\right) e^{-ikz'\cos\theta} dz' \hat{z}$$

Written out in full,

$$\begin{aligned}\mathcal{I}_1 = & \frac{1}{2i}e^{i\frac{1}{2}kd}\int_0^{\frac{d}{2}} e^{(-ik-ik\cos\theta)z'}dz' - \frac{1}{2i}e^{-i\frac{1}{2}kd}\int_0^{\frac{d}{2}} e^{(ik-ik\cos\theta)z'}dz' \\ & + \frac{1}{2i}e^{i\frac{1}{2}kd}\int_{-\frac{d}{2}}^{0} e^{(ik-ik\cos\theta)z'}dz' - \frac{1}{2i}e^{-i\frac{1}{2}kd}\int_{-\frac{d}{2}}^{0} e^{(-ik-ik\cos\theta)z'}dz'\end{aligned}$$

Each integral can be solved quite easily by " u " substitution.

$$\begin{aligned}\mathcal{I}_1 = & \frac{1}{2i}e^{i\frac{1}{2}kd}\frac{1 - e^{(-ik-ik\cos\theta)\frac{d}{2}}}{(k+ik\cos\theta)} + \frac{1}{2i}e^{i\frac{1}{2}kd}\frac{1 - e^{(-k+ik\cos\theta)\frac{d}{2}}}{(ik-ik\cos\theta)} \\ & + \frac{1}{2i}e^{-i\frac{1}{2}kd}\frac{1 - e^{(-ik+ik\cos\theta)\frac{d}{2}}}{(ik-ik\cos\theta)} + \frac{1}{2i}e^{-i\frac{1}{2}kd}\frac{1 - e^{(ik+ik\cos\theta)\frac{d}{2}}}{(ik+ik\cos\theta)}\end{aligned}$$

The result is a mess. Use Maple or have patience. It takes a bit of algebra to get the neat result,

$$I_1 = \frac{2}{k}\left[\frac{\cos(\frac{1}{2}kd\cos\theta) - \frac{1}{2}\cos(kd)}{sin^2\theta}\right]$$

And then,

$$\vec{A}(\vec{r}) = \frac{2\mu_0}{4\pi}\frac{e^{ikr}}{kr}\left[\frac{\cos(\frac{1}{2}kd\cos\theta) - \frac{1}{2}\cos(kd)}{\sin^2\theta}\right]\hat{z}$$

Who cares about the vector potential? We want E and B fields. Fortunately, we know how to write the E and B fields in the radiation zone in terms of the vector potential.

$$\vec{B} = ik\hat{r}\times\vec{A} \rightarrow |\vec{B_0}| = k\sin\theta|\vec{A_0}|\hat{\phi}$$

$$\vec{E} = ick(\hat{r}\times\vec{A})\times\hat{r} \rightarrow |\vec{E_0}| = ck\sin\theta|\vec{A_0}|\hat{\theta}$$

The time averaged angular distribution of power is

$$\frac{dP}{d\Omega} = \frac{r^2}{2\mu_0}|\vec{E}\times\vec{B}^*| = \frac{1}{2\mu_0}ck^2\frac{I_0^2}{k^2}\left(\frac{2\mu_0}{4\pi}\right)^2\left[\frac{\cos(\frac{1}{2}kd\cos\theta) - \frac{1}{2}\cos kd}{\sin\theta}\right]^2$$

$$\frac{dP}{d\Omega} = \frac{2\mu_0 I_0^2 c}{16\pi^2}\left[\frac{cos(\frac{1}{2}kd\cos\theta) - \frac{1}{2}\cos kd}{\sin\theta}\right]^2$$

In this problem $\lambda = d$ so $kd = \frac{2\pi}{\lambda}d = 2\pi$.

$$\frac{dP}{d\Omega} = \frac{2\mu_0 I^2 c}{16\pi^2}\left[\frac{\cos(\pi\cos\theta) - \frac{1}{2}\cos\pi}{\sin\theta}\right]^2$$

Well, $\cos\pi = -1$, and of course, $\cos\alpha + \frac{1}{2} = 2\cos^2(\frac{\alpha}{2})$.

$$\frac{dP}{d\Omega} = \frac{8\mu_0 I^2 c}{16\pi^2}\left[\frac{\cos^4(\frac{1}{2}\pi\cos\theta)}{\sin^2\theta}\right]$$

b.

Integrate the result from part a over all solid angles.

$$P_{total} = \int \frac{dP}{d\Omega}d\Omega = \frac{\mu_0 I^2 c}{2\pi^2}\int\left[\frac{\cos^4(\frac{1}{2}\pi\cos\theta)}{\sin^2\theta}\right]d\Omega$$

Integrating over ϕ,

$$P_{total} = \frac{\mu_0 I^2 c}{\pi}\int\left[\frac{\cos^4(\frac{1}{2}\pi\cos\theta)}{\sin^2\theta}\right]\sin\theta d\theta$$

Obviously[3], the integral equals about 0.84.

$$P = (0.84)\frac{I_0^2\mu_0 c}{\pi}$$

We learned in high school that $P = I^2 R$ and it does take much to show $R = \frac{P}{I^2} = \frac{\mu_0 c}{8\pi}(6.7) \approx 100\Omega$. Here, Ω stands for Ohms. Actually, Jackson seems to define the radiative resistance as 2 times this, but typically Jackson is hard to follow so I'll ignore this factor without a better explanation about its origin.

[3]Solve the integral numerically.

Problem 9.17

Until further notice: the units in this problem are inconsistent. check them!

a.

For a linear antenna:

$$\vec{J}(\vec{r}) = \hat{z}\sin(kz)\delta(x)\delta(y)I_0$$

Use the multi-pole expansion.

$$\lim_{kr\to\infty}\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(-ik)^n}{n!}\int \vec{J}(\vec{x}')(n\cdot\vec{x}')^n dV'$$

For n=1 in the expansion, we find the electric dipole contribution:

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{r}')dV' = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\hat{z}I_0\int_{-\frac{d}{2}}^{\frac{d}{2}}\sin(kz')dz' = 0$$

When n=2 in the expansion, we get a term proportional to the integral of $\vec{J}(\vec{n}\cdot\vec{r}')$. Using the vector identities, this can be rewritten in terms of the magnetic dipole and electric quadrapole contributions. The magnetic dipole contribution is:

$$\vec{A} = -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{ik}{2}\int(\vec{r}'\times\vec{J}(\vec{r}'))\times\vec{n}dV' = 0$$

The electric quadrapole contribution is:

$$\begin{aligned}
\vec{A} &= -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{ik}{2}\int\left[(\vec{n}\cdot\vec{r}')\vec{J}(\vec{r}') + (\vec{n}\cdot\vec{J}(\vec{r}'))\vec{r}'\right]dV' \\
&= -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{ik}{2}\hat{z}I_0\int_{\frac{d}{2}}^{\frac{d}{2}}\left[z'\cos\theta\sin(kz') + \cos\theta\sin(kz')z'\right]dz' \\
&= -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}ik\hat{z}I_0\cos\theta\int z'\sin(kz')dz' = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}ik\hat{z}I_0\cos\theta\frac{\partial}{\partial k}\int\cos(kz')dz' \\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}2ik\hat{z}I_0\cos\theta\frac{\partial}{\partial k}\left(\frac{\sin(\frac{kd}{2})}{k}\right) \\
&= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}2ik\hat{z}I_0\cos\theta\left(\frac{d}{2k}\cos\left(\frac{kd}{2}\right) - \frac{1}{k^2}\sin\left(\frac{kd}{2}\right)\right)
\end{aligned}$$

b.

We were given $kd = 2\pi$ so

$$\vec{A} = -\frac{\mu_0}{4\pi} id \frac{e^{ikr}}{r} \hat{z} I_0 \cos\theta$$

And the power per solid angle

$$\frac{dP}{d\Omega} = \frac{r^2}{2\mu_0} |\vec{E} \times \vec{B}|$$

But $\vec{B} = ik\vec{A}\sin\theta$ and $\vec{E} = ic\vec{A}\sin\theta$ so

$$\frac{dP}{d\Omega} = \frac{cr^2k^2}{2\mu_0}|\vec{A}|^2 \sin^2\theta = \frac{c\mu_0 k^2 d^2 I_0^2}{32\pi^2} \cos^2\theta \sin^2\theta = \frac{c\mu_0 I_0^2}{8} \cos^2\theta \sin^2\theta$$

c.

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{c\mu_0 I_0^2}{8}(2\pi) \int_{-1}^{1} \cos^2(\theta)\sin^2(\theta) d(\cos\theta) = \frac{c\mu_0 \pi I_0^2}{15}$$

Evaluate the integral as follows:

$$\int_{-1}^{1} \cos^2(\theta)\sin^2(\theta) d(\cos\theta) = \int_{-1}^{1} \cos^2(\theta)\left(1 - \cos^2(\theta)\right) d(\cos\theta)$$

Let $\cos\theta = x$.

$$\int_{-1}^{1} (x^2 - x^4) dx = \frac{x^3}{3} - \frac{x^5}{5} \Big|_{x=-1}^{1} = \frac{4}{15}$$

In circuit analysis, we can write the power dissipated as

$$P = RI_0^2$$

Plug in the power radiated and solve for R.

$$R = \frac{c\mu_0\pi}{15} = 155\Omega$$

No paradox because interference of higher multi-poles is possible.

Bonus Section: Broadcasting Westward

A professor posed once posed this question to me:

Suppose you had a city on the western shore of a large lake and that you are commissioned to design an antenna arrangement which would broadcast westward over the suburbs and waste as little power as possible by not broadcasting over the lake. Can this be done? How?

Obviously, by asking *how*, I have given you the answer to the first part. It's a bit difficult to understand the solution without diagrams so I'll put some diagrams here later. Position two antenna along the east-west axis and separate them by a distance $\frac{\lambda}{4}$. Now, delay the westward antenna by $\frac{\lambda}{4}$.

Here's what happens. The signal first appears at the eastern antenna. It propagates outward in all directions. When the pulse has traveled $\frac{\lambda}{4}$ westward, it passes the other antenna. At this moment, the second antenna emits the delayed signal. Both signals propagate in phase westward and so constructively interfere. Things are different on the eastward direction. By the time the second pulse reaches the first antenna the two signals are $\frac{\lambda}{2}$ out of phase and will destructively interfere. Thus, the eastward signal will be greatly diminished. According to the prof. who asked me this question, this is roughly the set up atop the Sears tower in Chicago.

Problem 10.1

a.

O.K. This problem's a monster, a veritable *Ungeheuer*! The basic idea behind the problem is simple, and a college freshman with knowledge of high school algebra and a vague idea of how to manipulate vectors could quite conceivably solve this. Notwithstanding, the algebra is horrible, and algebra has been know to topple even the greatest physicists.

First, we will drop the vector notation. It should be obvious that all the n's and all the ϵ's are unit vectors.

a. An unpolarized beam is scattered by a conducting sphere of radius a. From the text,

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\hat{\epsilon}^*_{out} \cdot \hat{\epsilon}_0 - \frac{1}{2}(n \times \hat{\epsilon}^*_{out}) \cdot (n_0 \times \hat{\epsilon}_0)\right]^2$$

It is a bit easier to work with dot products instead of cross products. Use the vector identity,$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$, to get

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[(\hat{\epsilon}^*_{out} \cdot \hat{\epsilon}_0)\left[1 - \frac{1}{2}(n \cdot n_0)\right] + \frac{1}{2}(n \cdot \hat{\epsilon}_0)(\hat{\epsilon}^*_{out} \cdot n_0)\right]^2 \tag{5}$$

Look at Jackson's diagram which I have included for convenience here. Notice that $n_0 \cdot n = \cos\theta$.

First, construct an orthonormal basis. The most obvious unit vectors to use are one **parallel to the incident wave vector**,

$$n_0$$

one **perpendicular to the scattering plane**,

$$\frac{n \times n_0}{\sin\theta}$$

and the third **orthogonal to the first two**,

$$\frac{n - (n \cdot n_0)n_0}{\sin\theta}$$

In case it is not obvious, the Graham-Schmidt process gave me the third vector. You can check for yourself to see that these vectors are orthogonal and normalized ($\hat{v} \cdot \hat{v} = 1$). The vector identity given earlier is useful for this.

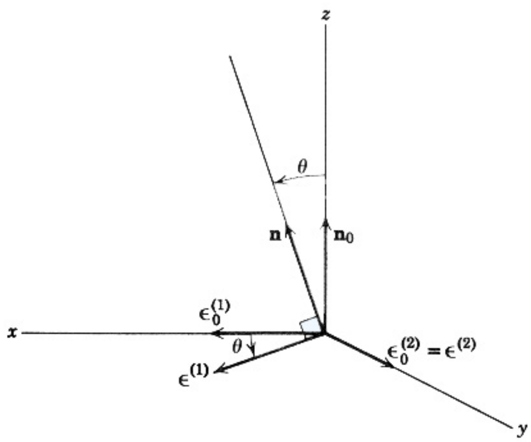


Figure 10.1 Polarization and propagation vectors for the incident and scattered radiation.

Figure 2: Jackson's insightful diagram.

The most general *incident* scattering wave polarization can be written in terms of these three unit vectors.

$$\epsilon_0 = A\left(\frac{n \times n_0}{\sin\theta}\right) + B\left(n_0\right) + \Gamma\left(\frac{n - (n \cdot n_0)n_0}{\sin\theta}\right)$$

And the most general *scattered* wave polarization vector can be expressed in terms of the same orthogonal basis.

$$\epsilon^*_{out} = \epsilon^*_{\perp(1)}\left(\frac{n \times n_0}{\sin\theta}\right) + \epsilon^*_{||(1)}\left(n_0\right) + \epsilon^*_{||(2)}\left(\frac{n - (n \cdot n_0)n_0}{\sin\theta}\right)$$

The parallel and perpendicular symbols refer to the polarizations orientation with respect to the scattering plane. We will use the following later:

$$\hat{\epsilon}^*_{\perp} = \epsilon^*_{\perp(1)}\left(\frac{n \times n_0}{\sin\theta}\right)$$

And

$$\epsilon^*_{||} = \epsilon^*_{||(1)}\left(n_0\right) + \epsilon^*_{||(2)}\left(\frac{n - (n \cdot n_0)n_0}{\sin\theta}\right)$$

Proceed by determining the coefficients for the incident wave. We do this be doting the incident wave vector by our basis vectors. Remember that

Jackson gives us $n_0 \cdot \hat{\epsilon}_0 = 0$. We realize immediately that $B = 0$. The other components are

$$\Gamma = \hat{\epsilon}_0 \cdot \left[\frac{n - (n \cdot n_0)n_0}{\sin\theta}\right] = \frac{1}{\sin\theta}\left[n \cdot \hat{\epsilon}_0 - (n \cdot n_0)(n_0 \cdot \hat{\epsilon}_0)\right] = \frac{1}{\sin\theta} n \cdot \hat{\epsilon}_0$$

And

$$A = \hat{\epsilon}_0 \cdot \frac{n \times n_0}{\sin\theta}$$

Calculate the scattering cross section for an arbitrarily polarized beam is done with the average of the incoming polarization and then the sum of the outgoing polarizations. That means that the total cross section is the sum of the cross sections for the two final polarization states. These states correspond to polarizations perpendicular and parallel to the scattering plane.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Tot} = \left(\frac{d\sigma}{d\Omega}\right)_{||} + \left(\frac{d\sigma}{d\Omega}\right)_{\perp}$$

In order to evaluate the cross sections, it will be helpful to know the following first: $\epsilon_{||}^* \cdot \epsilon_0$, $\epsilon_{\perp}^* \cdot \epsilon_0$, $n \times \epsilon_{||}^*$, $n \times \epsilon_{\perp}^*$, and $n_0 \times \epsilon_0$. Rewrite the incident polarization by putting Γ and A in explicitly.

$$\epsilon_0 = \frac{1}{\sin^2\theta}(n \cdot \epsilon_0)[n - (n_0 \cdot n)n_0] + \frac{1}{\sin^2\theta}[(n_0 \times n) \cdot \epsilon_0](n_0 \times n)$$

Now, take the relevant dot products.

$$(n \cdot \hat{\epsilon}_0) = \frac{n \cdot \hat{\epsilon}_0}{\sin^2\theta}[1 - (n_0 \cdot n)^2] = \frac{n \cdot \hat{\epsilon}_0}{\sin^2\theta}(1 - \cos^2\theta) = n \cdot \hat{\epsilon}_0$$

And

$$\epsilon_{||}^* \cdot \epsilon_0 = \frac{1}{\sin^2\theta}[-(n \cdot \hat{\epsilon}_0)(\hat{\epsilon}^* \cdot n_0)(n_0 \cdot n)] = \frac{1}{\sin\theta}(n \cdot \hat{\epsilon}_0)(n_0 \cdot n)$$

And

$$\epsilon_{\perp}^* \cdot \epsilon_0 = \frac{1}{\sin^2\theta}[\hat{\epsilon}_{\perp}^* \cdot (n_0 \times n)][(n_0 \times n) \cdot \hat{\epsilon}_0] = \frac{1}{\sin\theta}[(n_0 \times n) \cdot \hat{\epsilon}_0]$$

We can also find

$$\epsilon_{||}{}^* \cdot n_0 = -\sin\theta$$

$$\epsilon_\perp{}^* \cdot (n_0 \times n) = \sin\theta$$

Now, we have all the dot products needed to find the cross sections.
For the parallel case, the scattering cross section is equation 5 with only $\hat{\epsilon}_{||}$ in the final polarization.

$$\left(\frac{d\sigma}{d\Omega}\right)_{||} = k^4 a^6 \left[(\epsilon_{||}^* \cdot \epsilon_0)[1 - \frac{1}{2}(n \cdot n_0)] + \frac{1}{2}(n \cdot \epsilon_0)(\epsilon_{||}^* \cdot n_0)\right]^2$$

$$= k^4 a^6 \left[\frac{1}{\sin\theta}(n \cdot \hat{\epsilon}_0)(n_0 \cdot n)[1 - \frac{1}{2}(n_0 \cdot n)] + \frac{1}{2}(n \cdot \hat{\epsilon}_0)[-\sin\theta]\right]^2$$

$$= k^4 a^6 \left[(n \cdot \hat{\epsilon}_0)\frac{\cos\theta - \frac{1}{2}(\cos^2\theta + \sin^2\theta)}{\sin\theta}\right]^2$$

$$= k^4 a^6 \left[(n \cdot \hat{\epsilon}_0)\frac{\cos\theta - \frac{1}{2}}{\sin\theta}\right]^2$$

For the perpendicular case, we do the same as above but instead of $\hat{\epsilon}_{||}$, we have $\hat{\epsilon}_\perp$ in the final polarization.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\perp} = k^4 a^6 \left[(\epsilon_{\perp}^* \cdot \epsilon_0)[1 - \frac{1}{2}(n \cdot n_0)] + \frac{1}{2}(n \cdot \hat{\epsilon}_0)(\hat{\epsilon}_{\perp}^* \cdot n_0)\right]^2$$

$$= k^4 a^6 \left[(\hat{\epsilon}_{\perp}^* \cdot \hat{\epsilon}_0)[1 - \frac{1}{2}(n \cdot n_0)]\right]^2$$

$$= k^4 a^6 \left[\frac{1}{\sin\theta}\left[(n_0 \times n) \cdot \hat{\epsilon}_0\right]\left[1 - \frac{1}{2}n_0 \cdot n\right]\right]^2$$

$$= k^4 a^6 \left[[(n_0 \times n) \cdot \hat{\epsilon}_0]\frac{1 - \frac{1}{2}\cos\theta}{\sin\theta}\right]^2$$

We add these to get the total cross section.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Tot} = \frac{k^4 a^6}{\sin^2\theta}\left[[n \cdot \hat{\epsilon}_0]^2[\cos\theta - \frac{1}{2}]^2 + [(n_0 \times n) \cdot \hat{\epsilon}_0]^2[1 - \frac{1}{2}\cos\theta]^2\right]$$

Multiply out the squares.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Tot} = \frac{k^4 a^6}{\sin^2\theta}\left[[n \cdot \hat{\epsilon}_0]^2[(\cos^2\theta - 1) - \cos\theta + \frac{5}{4}]\right]$$
$$+\frac{k^4 a^6}{\sin^2\theta}\left[[(n_0 \times n) \cdot \hat{\epsilon}_0]^2[\frac{1}{4}(\cos^2\theta - 1) + \frac{5}{4} - \cos\theta]\right]$$

And then, with some algebra,

$$\left(\frac{d\sigma}{d\Omega}\right)_{Tot} = k^4a^6\left[-[n\cdot\hat{\epsilon}_0]^2 - \frac{1}{4}[(n_0\times n)\cdot\hat{\epsilon}_0]^2 + \frac{\frac{5}{4}-\cos\theta}{1-\cos^2\theta}([n\cdot\hat{\epsilon}_0]^2 + [(n_0\times n)\cdot\hat{\epsilon}_0]^2)\right]$$

Recall that we were given

$$\epsilon_0^*\cdot\epsilon_0 = 1 \rightarrow 1 = [\epsilon_{0||}]^2 + [\epsilon_{0\perp}]^2$$

This means that

$$\frac{1}{\sin^2\theta}[n\cdot\hat{\epsilon}_0]^2 + \frac{1}{\sin^2\theta}[(n_0\times n)\cdot\hat{\epsilon}_0]^2 = 1$$

Finally, we can report the total cross section.

$$\left(\frac{d\sigma}{d\Omega}\right)_{Tot} = k^4a^6\left[\frac{5}{4} - [\epsilon_0\cdot n]^2 - \frac{1}{4}[n\cdot(n_0\times\epsilon_0)]^2 - n_0\cdot n\right] \qquad (6)$$

Where we replaced $\cos\theta$ with $n_0\cdot n$.

b.

It is a simple matter of geometry to determine the following dot and cross products. I'll give you a diagram someday, but for now, you've got to draw this one yourself.

$$\epsilon_0\cdot n = \sin\phi\sin\theta$$

$$n\cdot(n_0\times\epsilon_0) = \epsilon_0\cdot(n\times n_0) = \epsilon_0\cdot\hat{v}\sin\theta = \sin\theta\cos\phi$$

Once we have these products, part b is simply a matter of trigonometric formulae and algebraic manipulations. Consider the term in brackets from equation 6, and write the newly revealed angles in.

$$\left[\frac{5}{4} - [\epsilon_0\cdot n]^2 - \frac{1}{4}[n\cdot(n_0\times\epsilon_0)]^2 - n_0\cdot n\right] =$$
$$\left[\frac{5}{4} - \sin^2\phi\sin^2\theta - \frac{1}{4}\sin^2\theta\cos^2\phi - \cos\theta\right]$$

I'm going to fly through this algebra. To start off, I will use $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$. It should be clear what's going on.

$$\left[\frac{5}{4} - \sin^2\phi\sin^2\theta - \frac{1}{4}\sin^2\theta\cos^2\phi - \cos\theta\right]$$

$$= \left[\frac{5}{4} - \frac{1}{2}(1 - \cos 2\phi)\sin^2\theta - \frac{1}{8}\sin^2\theta(1 - \cos 2\phi) - \cos\theta\right]$$

$$= \left[\frac{5}{8}(1 + \cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]$$

$$= \left[\frac{5}{4} - \frac{1}{2}(1 + \cos 2\phi)\sin^2\theta - \frac{1}{8}\sin^2\theta(1 - \cos 2\phi) - \cos\theta\right]$$

$$= \left[\frac{5}{4} - \frac{5}{8}\sin^2\theta - \frac{3}{8}\sin^2\theta\cos 2\phi - \cos\theta\right]$$

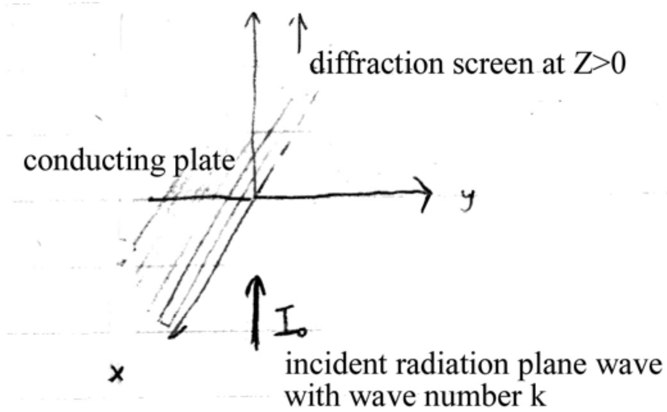
$$= \left[\frac{5}{4} - \frac{5}{8}(1 - \cos^2\theta) - \frac{3}{8}\sin^2\cos 2\phi - \cos\theta\right]$$

$$= \left[\frac{5}{8}(1 + \cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]$$

Then, we have what Jackson wants.

$$\left(\frac{d\sigma}{d\Omega}\right) = k^4 a^6 \left[\frac{5}{8}(1 + \cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]$$

Problem 10.11



a.

$$\Psi = \sqrt{I_0}\left(\frac{1+i}{2i}\right)e^{ikZ-i\omega t}\sqrt{\frac{2}{\pi}}\int_{-\Xi}^{\infty} e^{iu^2}du$$

where $\Xi = X(\frac{k}{2Z})^{\frac{1}{2}}$**.**

$$\Psi(r_0) = \frac{k}{2\pi i}\sqrt{I_0}\int_{Aperture}\frac{e^{ikr_p}}{r_p}dA'$$

r_0 is the observation point, and $r_p = \sqrt{(x'-X)^2+(y'-Y)^2+(z'-Z)^2}$ is the distance from the area point at the aperture to the observation point. The small letters denote the aperture values while the large letters denote values at the observation point. $dA' = dx'dy'$ in this case because the screen is in the xy plane.

I proceed first by evaluating the integral over the y coordinate.

$$I_1 = \int_{\infty}^{\infty}\frac{e^{ikr_p}}{r_p}dy'$$

I exploit the symmetry of the integral about $y = 0$, and replace $\rho^2 = (x'-X)^2+(z'-Z)^2$.

$$I_1 = 2\int_0^{\infty}\frac{e^{ik\sqrt{(y'-Y)^2+\rho^2}}}{\sqrt{(y'-Y)^2+\rho^2}}dy'$$

Substitute $\nu = \sqrt{(y' - Y)^2 + \rho^2}$.

$$I_1 = 2\int_\rho^\infty \frac{e^{ik\nu}}{\sqrt{\nu^2 - \rho^2}} d\nu$$

Remember from basic calculus,

$$\int_0^\infty \frac{\sin(Ax)}{\sqrt{x^2 - 1}} dx = \frac{\pi}{2} J_0(A)$$

$$\int_0^\infty \frac{\cos(Ax)}{\sqrt{x^2 - 1}} dx = -\frac{\pi}{2} N_0(A)$$

J_0 is a Bessel function and N_0 is a Neumann function. I will use these to reduce the integral to a more tractable form. By Euler's handy formula, $e^{ix} = \cos x + i \sin x$. so we can write

$$I_1 = 2\int_\rho^\infty \frac{e^{ik\nu}}{\sqrt{\nu^2 - \rho^2}} d\nu = 2\int_\rho^\infty \frac{[\cos(k\nu) + i\sin(k\nu)]}{\sqrt{\nu^2 - \rho^2}} d\nu$$

Let $\xi = \nu/\rho$ and $d\xi = \frac{1}{\rho} d\nu$.

$$I_1 = 2\int_1^\infty \frac{[\cos(k\rho\xi) + i\sin(k\rho\xi)]}{\sqrt{\xi^2 - 1}} d\xi = 2[-\frac{\pi}{2} N_0(k\rho) + i\frac{\pi}{2} J_0(k\rho)]$$

And so the first part of the surface integral is done.
Now, I will attempt to integrate over dx'. Don't forget ρ is a function of x'.

$$I_2 = \int_0^\infty -\pi N_0(k\rho) + i\pi J_0(k\rho) dx' = i\pi \int J_0(k\rho) + iN_0(k\rho) dx'$$

In the limit $\sqrt{kZ} >> 1 \rightarrow kZ >> 1$ and $\rho k >> 1$, the Bessel function and its friend can be approximated by the following:

$$J_0(A) \simeq \sqrt{\frac{2}{\pi A}} \cos(A - \frac{\pi}{4})$$

$$N_0(A) \simeq -\sqrt{\frac{2}{\pi A}} \sin(A - \frac{\pi}{4})$$

And the integral reduces to

$$I_2 = \int_0^\infty i\pi\sqrt{\frac{2}{\pi k\rho}}[\cos(k\rho - \frac{\pi}{4}) + i\sin(k\rho - \frac{\pi}{4})]dx'$$

Which easily reduces to

$$I_2 = \int_0^\infty i\sqrt{\frac{2\pi}{k\rho}}e^{i(\rho k - \frac{\pi}{4})}dx' = i\sqrt{2\pi}\int_0^\infty e^{i(\rho k - \frac{\pi}{4})}\sqrt{\frac{1}{k\rho}}dx'$$

Lest I loose track of all the coefficients, I'll rewrite Ψ.

$$\Psi = \frac{k}{2\pi i}\sqrt{I_0}i\sqrt{2\pi}\int_0^\infty \frac{e^{i(\rho k - \frac{\pi}{4})}}{\sqrt{\rho k}}dx' = k\sqrt{\frac{I_0}{2\pi}}e^{-i\frac{\pi}{4}}\int_0^\infty \frac{e^{[ik\sqrt{(x'-X)^2+(z'-Z)^2}]}}{\sqrt{k\sqrt{(x'-X)^2+(z'-Z)^2}}}dx'$$

I have written ρ in explicitly to remind us that ρ depends on x'. Now, I label the integral as I_3 and tackle this integration.

$$I_3 = \int_0^\infty \frac{e^{[ik\sqrt{(x'-X)^2+(z'-Z)^2}]}}{\sqrt{k\sqrt{(x'-X)^2+(z'-Z)^2}}}dx'$$

So far, I haven't make use of the fact that $z' = 0$. I'll do that now.
If $(x' - X) << Z$, we can expand $\sqrt{(x'-X)^2 + Z^2} \simeq Z + \frac{(x'-X)^2}{2Z}$. So

$$I_3 = \frac{e^{ikZ}}{\sqrt{kZ}}\int_{-\Xi}^\infty \sqrt{\frac{2Z}{k}}e^{iu^2}du$$

where $u = \sqrt{\frac{k}{2Z}}(x' - X)$, and the limits of integration have been changed accordingly, $\Xi = X\sqrt{k/(2Z)}$. This gives the result:

$$\Psi = k\sqrt{\frac{I_0}{2\pi}}e^{-i\frac{\pi}{4}}\frac{e^{ikZ}}{\sqrt{kZ}}\sqrt{\frac{2Z}{k}}\int_{-\Xi}^\infty e^{iu^2}du$$

A little work with an Argand diagram should convince you that

$$e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \sqrt{2}\left(\frac{1+i}{2i}\right)$$

and then, Ψ reduces to Jackson's result.

$$\Psi = \sqrt{I_0}\left(\frac{1+i}{2i}\right)e^{ikZ-i\omega t}\sqrt{\frac{2}{\pi}}\int_{-\Xi}^{\infty} e^{iu^2}du$$

where $\Xi = X(\frac{k}{2Z})^{\frac{1}{2}}$. Note: I didn't assume time dependence from the start, but if I did the derivation would be the same. I would have simply factored the $e^{-i\omega t}$ out from the start. So I just put it back here.

b.

We need to rewrite I_4 is a suggestive way.

$$I_4 = \int_{-\Xi}^{\infty} e^{iu^2}du$$

Everybody should know the friendly Fresnel Integrals:

$$C(\lambda) = \int_0^{\lambda}\cos(\frac{\pi x^2}{2})dx$$

$$S(\lambda) = \int_0^{\lambda}\sin(\frac{\pi x^2}{2})dx$$

And using Euler's handy relationship,

$$\int_0^{\lambda} e^{i\pi\frac{x^2}{2}}dx = C(\lambda) + iS(\lambda)$$

In our case.

$$\int_{-\Xi}^{\infty} e^{iu^2}du = \sqrt{\frac{\pi}{2}}[C(\infty) + iS(\infty) - C(-\Xi) - iS(-\Xi)]$$

I will use the symmetry of $C(x)$ and $S(x)$, namely, $C(x) = -C(-x)$ and $S(x) = -S(-x)$ to get rid of all the unwanted minus signs.

$$\int_{-\Xi}^{\infty} e^{iu^2}du = \sqrt{\frac{\pi}{2}}\left[C(\infty) + iS(\infty) + C(\Xi) + iS(\Xi)\right]$$

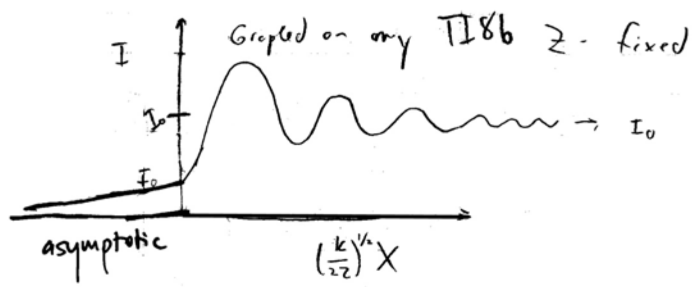
To find $C(x)$ and $S(x)$ at infinity, we need $\lim_{t\to\pm\infty} C(t) = \pm\frac{1}{2}$ and $\lim_{t\to\pm\infty} S(t) = \pm\frac{1}{2}$. I_4 is evidently representable by $\frac{1}{2}(1+i) + C(\Xi) + iS(\Xi)$. The intensity is given by $|\Psi|^2$ so

$$\mathcal{I} = |\sqrt{\frac{2}{\pi}}\sqrt{I_0}\left(\frac{1+i}{2i}\right)e^{ikZ-i\omega t}|^2\left[\frac{1+i}{2} + C(\Xi) + iS(\Xi)\right]^2 =$$
$$\left(\frac{2}{\pi}I_0\right)\frac{\pi}{2}\left[\left(C(\Xi) + \frac{1}{2}\right)^2 + \left(S(\Xi) + \frac{1}{2}\right)^2\right]$$

And finally, we have what Jackson wants.

$$\mathcal{I} = \frac{I_0}{2}\left[\left(C(\Xi) + \frac{1}{2}\right)^2 + \left(S(\Xi) + \frac{1}{2}\right)^2\right]$$

As $\Xi \to \infty+$, $\mathcal{I} \to I_0$, and we have a bright spot. As $\Xi \to \infty-$, $\mathcal{I} \to 0$, and we have a shadow. At $X = 0$, $\Xi = 0$, and $\mathcal{I} = \frac{I_0}{4}$.
The graph is coming soon!



Bonus Section: A Review of Lorentz Invariant Quantities

Relativistic notation is a mess, and admittedly Jackson doesn't do that poorly trying to straighten things out. In order to keep track of the pesky minus sign in the Minkowski metric, $\sigma^2 = t^2 - x^2 - y^2 - z^2$, we define two types of four tensor, covariant and contra-variant[4]. Contra-variant tensors are much like a regular old Euclidean tensor. The entries all have positive signs and transform as you'd expect (I'll get to that later). I keep the name and position of the indices straight by remembering how *contradictory* relativity first seemed, but since these vectors are easier to work with, I'll give them one thumb up and place their indices up. It's no surprise that the simpler named *covariant* vectors have tricky minus signs before the space coordinates. I'll put this covariant indices low because of this *covert* behavior. O.K. Enough silly semantics.

My purpose here is to review a bit of notation and to stress the usefulness of Lorentz invariants. First, accept $\frac{\partial x^\alpha}{\partial x^\beta} = \delta_{\alpha\beta}$. I think Goldstein discusses this in his sections on field theory, so I won't explain where this came from. Clearly, this is reasonable. A derivative of a constant is zero, and a derivative of a function by itself is one.

For a first rank tensor, the transformation rules are as follows: Covariant first ranked tensor,

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta$$

Contra-variant first ranked tensor,

$$B'_\gamma = \frac{\partial x^\epsilon}{\partial x'^\gamma} B_\epsilon$$

And the scalar product, $B'_\alpha A^{\alpha\prime}$,

$$B'_\alpha A'^\alpha = \frac{\partial x^\epsilon}{\partial x'^\alpha} B_\epsilon \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta = \frac{\partial x^\epsilon}{\partial x^\beta} B_\epsilon A^\beta = \delta_{\epsilon\beta} B_\epsilon A^\beta = B_\beta A^\beta$$

is invariant under Lorentz transformations. For example, the mass of a particle is a Lorentz invariant. $\wp'_\mu \wp^{\mu\prime} = \wp_\mu \wp^\mu = m^2$.

For second rank tensors, we can devise similar rules. First, for the completely covariant object

$$C'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\epsilon} C'^{\gamma\epsilon}$$

[4] For the more mathematically oriented, this should sound cacophonous.

And now for the completely contra-variant monster

$$D'_{\alpha\beta} = \frac{\partial x^{\zeta}}{\partial x'^{\alpha}} \frac{\partial x^{\eta}}{\partial x'^{\beta}} D'_{\zeta\eta}$$

And I suppose we could consider a mixed object, but I have grown tired of writing all these indices. The scalar product of two second rank tensors is invariant under Lorentz transformations:

$$D'_{\alpha\beta} C'^{\alpha\beta} = \frac{\partial x^{\zeta}}{\partial x'^{\alpha}} \frac{\partial x^{\eta}}{\partial x'^{\beta}} D'_{\zeta\eta} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\epsilon}} C'^{\gamma\epsilon} =$$
$$\frac{\partial x^{\zeta}}{\partial x^{\gamma}} \frac{\partial x^{\eta}}{\partial x^{\epsilon}} D_{\zeta\eta} C^{\gamma\epsilon} = \delta_{\zeta\gamma}\delta_{\eta\epsilon} D_{\zeta\eta} C^{\gamma\epsilon} = D_{\zeta\eta} C^{\zeta\eta}$$

For example, magnetic and electric dipoles can be expressed by a tensors, $M_{\mu\nu}$, and $F^{\alpha\beta}$ in the electro-magnetic field tensor. $U_{interaction} = \frac{1}{2} M_{\mu\nu} F^{\mu\nu} = \frac{1}{2} M'_{\mu\nu} F'^{\mu\nu}$; *id est* the interaction energy is invariant under Lorentz transformations.

Problem 11.5

as I will derive.
To make life easier on me, I'll omit the vector symbols on the vectors in this problem. They should be obvious anyway.
To solve this problem, I must use the relationship

$$\frac{dt'}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \left(1 + \frac{v \cdot u'}{c^2}\right)^{-1}$$

So first, I'll derive this. From Jackson page 531, we have $u' = c\frac{dx'}{dt'}$ which can be rearranged to give $dx' = \frac{u'}{c}dt'$. Dot multiply both sides of this equation for dx' by $\beta = \frac{v}{c}$. Then, we have $\beta \cdot dx' = \beta \cdot \frac{u'}{c}dt' = \frac{v \cdot u'}{c^2}dt'$. According to Jackson in section 11.4, we have the relationship $dt = \gamma(dt' + \beta \cdot dx')$. As usual, $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. Plug the equation for $\beta \cdot dx'$ into the equation for dt, and then we get

$$dt = \gamma \left(1 + \frac{v \cdot u'}{c^2}\right) dt' \rightarrow \frac{dt'}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \left(1 + \frac{v \cdot u'}{c^2}\right)^{-1}$$

As I intended to prove.
From Jackson 11.31, we have the velocity addition equation for parallel components of velocity

$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + \frac{v \cdot u'}{c^2}}$$

Take the derivative with respect to dt.

$$a_{\parallel} = \frac{du_{\parallel}}{dt} = \frac{a'_{\parallel}}{1 + \frac{v \cdot u'}{c^2}} \frac{dt'}{dt} - \left(u'_{\parallel} + v\right) \left(1 + \frac{v \cdot u'_{\parallel}}{c^2}\right)^{-2} \left(\frac{v}{c^2}\right) a'_{\parallel} \frac{dt'}{dt}$$

With some rearrangement,

$$\begin{aligned} a_{\parallel} = \frac{du_{\parallel}}{dt} &= \left(1 + \frac{v \cdot u'}{c^2}\right) \left(1 + \frac{v \cdot u'}{c^2}\right)^{-2} a'_{\parallel} \frac{dt'}{dt} \\ &- \left(\frac{u' \cdot v}{c^2} + \frac{v^2}{c^2}\right) \left(1 + \frac{v \cdot u'_{\parallel}}{c^2}\right)^{-2} a'_{\parallel} \frac{dt'}{dt} \end{aligned}$$

And the inclusion of $\frac{dt'}{dt}$, we have

$$a_{\|} = \frac{1 - \frac{v^2}{c^2}}{(1 + \frac{v \cdot u'_{\|}}{c^2})^2} a'_{\|} \frac{dt'}{dt} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}{\left(1 + \frac{v \cdot u'}{c^2}\right)^3} a'_{\|}$$

In 11.31, Jackson also reported that for the perpendicular components of velocity, the addition law is

$$u_{\perp} = \frac{u'_{\perp}}{\gamma \left(1 + \frac{v \cdot u'}{c^2}\right)}$$

Once again, we take the derivative with respect to dt.

$$a_{\perp} = \frac{du_{\perp}}{dt} = \frac{a'_{\perp}}{\gamma \left(1 + \frac{v \cdot u'}{c^2}\right)} \frac{dt'}{dt} - \frac{u'_{\perp}}{\gamma \left(1 + \frac{v \cdot u'}{c^2}\right)^2} \left(\frac{v \cdot a'}{c^2}\right) \frac{dt'}{dt}$$

After we plug in $\frac{dt'}{dt}$ explicitly,

$$a_{\perp} = \frac{1 - \frac{v^2}{c^2}}{\left(1 + \frac{u \cdot v'}{c^2}\right)^3} \left[a'_{\perp} \left(1 + \frac{v \cdot u'}{c^2}\right) - \frac{u'_{\perp}}{c^2} (v \cdot a')\right]$$

Using the vector identity, $\vec{A} \times (\vec{B} \times \vec{C}) = (A \cdot C)\vec{B} - (A \cdot B)\vec{C}$, we can write $v \times (a' \times u') = (v \cdot u')a'_{\perp} - (v \cdot a')u'_{\perp} +$ canceling $a'_{\|}$ and $u'_{\|}$ components. Possibly, this might not be so obvious to you. Well, v cross anything must be perpendicular to v. Therefore, the only vector components on the right side of the triple product must be perpendicular to v or be pair ed in such a way as to cancel. So finally, I can report.

$$a_{\perp} = \frac{1 - \frac{v^2}{c^2}}{\left(1 + \frac{u \cdot v'}{c^2}\right)^3} \left[a'_{\perp} + \frac{1}{c^2} v \times (a' \times u')\right]$$

And I have given Jackson what he wants.

Problem 11.6

One set of twins is born in the year 2080. In the year 2100, NASA decides to do an experiment. The government furtively seizes one twin, throws him aboard a rocket bound for a distant star, and sends the rocket into space. The rocket accelerates at the acceleration of gravity, g, in its own rest frame. Although the twin is lonely, he won't be too uncomfortable. The ship accelerates in a straight-line path for 5 years (by its own clocks), decelerates at the same rate for five years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth. The twin in the rocket is then 40 years old.

According to the twin on the rocket, the trip to the distant star and back lasts 20 years.

$$a(t') = \begin{cases} g, t' < 5 \\ -g, 5 < t' < 15 \\ g, 15 < t' < 20 \end{cases}$$

a. What year is it on earth?

How much time will pass on the earth during this trip? Will the space bound twin ever see his brother again?

Consider the first leg. $a = g$ while $t' = 0$ to 5. Let t' denote the time on the rocket and t denote the time on the earth. There is a simple relationship from the Lorentz transformation equations between these times $t' = \frac{t}{\gamma(t')}$. For infinitesimal intervals, we have $dt' = \frac{dt}{\gamma(t')}$. The total time elapsed on the rocket is:

$$T_{rocket} = \int_0^5 dt' = 5$$

$$T_{earth} = \int_0^? dt = \int_0^5 \gamma(t')dt'$$

To get $\gamma(t')$, we need to sum over possible velocities so I'll use rapidity[5] which is easier to work with.

First, we need dt' in terms of rapidity. $\beta = \frac{v}{c} \to d\beta = \frac{dv'}{dt'}\frac{dt'}{c} = \frac{g}{c}dt'$. This gives us $dt' = \frac{c}{g}d\beta$. Now, I need to figure out what $\gamma(\theta)$ is. Jackson, in one of his rare instructive moments, taught us that $\beta = \tanh(\theta)$. I can exploit

[5] The use of rapidity is not my own clever innovation. My prof. suggested this.

the additive properties of rapidity, θ, to write

$$\beta = \tanh(\sum \theta) \rightarrow \Delta\beta = \tanh(\Delta\theta)$$

And so in the infinitesimal limit, I have $d\beta = \tanh(d\theta)$. I can compare this to the first equation for $d\beta = \frac{g}{c}dt'$ to get an expression for dt'.

$$dt' = \frac{c}{g}\tanh(d\theta)$$

Now, I expand $\tanh(d\theta) = d\theta - \frac{d\theta^3}{3}$ and keep only the first term.

$$dt' = \frac{c}{g}d\theta$$

Finally,

$$\beta(t') = \tanh(\int d\theta) = \tanh(\int_0^{t'} \frac{g}{c}dt'') = \tanh(\frac{g}{c}t')$$

By the relationships given on Jackson 11.20, $\gamma(t') = \cosh(\frac{g}{c}t')$ and that $\beta(t')\gamma(t') = \sinh(\frac{g}{c}t')$. Putting all this together,

$$\begin{aligned} T_{earth} &= \int \gamma(t')dt' = \int_0^5 \cosh\left(\frac{g}{c}t'\right) dt' \\ &= \frac{c}{g}\sinh\left(\frac{g}{c}t'\right) |_0^5 = \frac{c}{g}\sinh\left(\frac{5g}{c}\right) \end{aligned}$$

Take $g = 10$ m/sec^2 and $c = 3 \times 10^8$ m/sec. Don't forget that 5 is in years, and the equation is wrong unless I covert 1 year $= 1.5768 \times 10^8$ seconds. Then, $T_{earth} = 91$ years. By symmetry, the next three legs must each take just as long. So the total time elapsed on the earth while the rocket twin makes a round trip is $4 \times 91 = 364$ years. The twin will come back home in the year $2100 + 365 = 2464$, and his brother will be dead.

b. How far away from the earth did the ship travel?

I will use a similar technique.

$$x_{earth} = \int c\beta(t')dt = \int_0^{10} c\beta(t')\gamma('t)dt'$$

$$\begin{aligned} x_{earth} = \int_0^5 c\sinh\left(\frac{g}{c}t'\right) dt' - \int_0^5 c\sinh\left(\frac{-g}{c}t'\right) dt' \\ = \frac{2c^2}{g}\cosh\left(\frac{g}{c}t\right) |_0^5 \approx 168 \end{aligned}$$

So the distance to the turning point is about 168 light years. This makes sense because if we assume that the ship was traveling at pretty much the speed of light for 2×91 years, the ship would have gone 182 light years. But obviously, the ship was going a little bit slower.

Problem 11.7

a.

The question is about the non-synchrony of events as witnessed in different Lorentz frame.

For simplicity and for symmetry reasons, we'll consider a Lorentz frame moving along the y-axis parallel to the starting line or perpendicular to the race path, x-axis.

In S, the race frame: $y_A = \frac{1}{2}d$ and $y_B = -\frac{1}{2}d$. The starting time for the first runner is $t_B = 0$, and for the second runner is $t_A = T$.

In S', the arbitrary Lorentz frame moving with velocity u along the y-axis relative to the race frame:

$$T' = \frac{Tc^2 - ud}{c^2\sqrt{1 - \frac{u^2}{c^2}}}$$

If the handicap is not real, we can find a frame in which $T' = 0$. This will be the case when $\Delta t = \frac{ud}{c^2}$. Since we have a range of possible frame speeds from 0 to c, we can find a corresponding range of possible delays, $T = 0$ to $\frac{d}{c}$. For larger time delays, the runner is given a true handicap.

b.

To find the Lorentz frame in which the time delay vanishes, we use the condition on T and solve for u.

$$u = \frac{c^2 T}{d}$$

Obtaining the Lorentz transformations is straightforward. We can write these as matrices.

$$\begin{pmatrix} t'_A \\ y'_A \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t_A \\ y_A \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{cT}{d} \\ -\gamma\frac{cT}{d} & \gamma \end{pmatrix} \begin{pmatrix} T \\ \frac{1}{2}d \end{pmatrix}$$

And

$$\begin{pmatrix} t'_B \\ y'_B \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t_B \\ y_B \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{cT}{d} \\ -\gamma\frac{cT}{d} & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2}d \end{pmatrix}$$

With $\gamma = \frac{1}{\sqrt{1 - \frac{c^2T^2}{d^2}}}$.

$$y'_A = \frac{\frac{1}{2}d - \frac{c^2T^2}{d}}{\sqrt{1 - \frac{c^2T^2}{d^2}}}$$

$$t'_A = \frac{\frac{1}{2}T}{\sqrt{1 - \frac{c^2T^2}{d^2}}}$$

$$y'_B = \frac{-\frac{1}{2}d}{\sqrt{1 - \frac{c^2T^2}{d^2}}}$$

$$t'_B = \frac{\frac{1}{2}T}{\sqrt{1 - \frac{c^2T^2}{d^2}}}$$

To find the transformations in the true handicap case is also straightforward. I define $T = \frac{d}{c} + \epsilon$, where ϵ is the part of the handicap that will never be transformed entirely away.

$$t'_A = \frac{\frac{d}{c} + \epsilon - \frac{1}{2}\frac{ud}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$y'_A = \frac{\frac{1}{2}d - u\left(\frac{d}{c} + \epsilon\right)}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$t'_B = \frac{\frac{1}{2}\frac{ud}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$y'_B = \frac{-\frac{1}{2}d}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Problem 11.13

In K, we have a wire with charge density, λ, but no current density. Because of the obvious cylindrical symmetry, we'll use cylindrical coordinates. In K', we have a nonzero current density, $\vec{J'} \neq 0$, and a charge density, λ'. The velocity of frame K' with respect to K is $\vec{v} = v\hat{z}$. Watch out because in this problem I start off using Jackson units, but then switch rather abruptly to S.I. units.

a.

In K,

$$E_r = \frac{q_0}{2\pi\epsilon_0 r}$$

$E_\theta = 0$ and $E_z = 0$ by Gauss's law, and $\vec{B} = 0$ from the fact that there is no current (real or displacement) in this frame. We can use the Lorentz transformations for the fields to get from one frame to another. They are:

$$E' = \gamma(E + \beta \times B) - \frac{\gamma^2}{\gamma+1}\beta(\beta \cdot E)$$

$$B' = \gamma(B - \beta \times E) - \frac{\gamma^2}{\gamma+1}\beta(\beta \cdot B)$$

After applying these transformations, we find

$$E_r' = \frac{\gamma q_0}{2\pi\epsilon_0 r}$$

And

$$B_\theta' = \gamma(-\beta E_r) = -\beta\gamma\frac{q_0}{2\pi\epsilon_0 r}$$

The other components vanish by symmetry or explicit calculations whichever you prefer. $E_\theta' = 0$,$E_z' = 0$, $B_r' = 0$, and $B_z' = 0$.

Now, I switch rather abruptly to S.I. so that I can compare my results to Griffiths. We have

$$E_r' = \frac{\gamma q_0}{2\pi\epsilon_0 r}$$

And

$$B_\theta' = -\frac{1}{c}\beta\gamma\frac{q_0}{2\pi\epsilon_0 r}$$

Fortunately, these compare well to Griffiths' results. It makes sense that β is negative if you look at the diagrams which I should scan someday.

b.
In K, we have

$$J^4 = \begin{pmatrix} c\rho \\ J_z \\ J_r \\ J_\theta \end{pmatrix} = \begin{pmatrix} \frac{c}{2\pi}\frac{q_0}{r}\delta(r) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And the appropriate Lorentz transformation also can be written in matrix form.

$$\mathbf{L} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To transform into the K' frame is easy: $J^{4\prime} = \mathbf{L}J^4$. Doing the matrix math gives

$$\vec{J}^{4\prime} = J^{4\prime} = \begin{pmatrix} c\rho' \\ J'_z \\ J'_r \\ J'_\theta \end{pmatrix} = \begin{pmatrix} \frac{c\gamma}{2\pi}\frac{q_0}{r}\delta(r) \\ -\frac{c\beta\gamma}{2\pi}\frac{q_0}{r}\delta(r) \\ 0 \\ 0 \end{pmatrix}$$

c.
For K', we have found $\vec{J}^{4\prime}$. From the results in part b, we deduce $\lambda' = \gamma\lambda$ and$\vec{J}' = -\beta\gamma c q_0 \delta(r)\hat{z} = -\frac{\beta\gamma q_0}{c\mu_0\epsilon_0}\delta(r)\hat{z}$. For E,

$$E'_r = \frac{\lambda'}{2\pi\epsilon_0 r} = \frac{\gamma q_0}{2\pi\epsilon_0 r}$$

For B,

$$B'_\theta = \frac{\mu_0 J'_z}{2\pi r} = \frac{\mu_0}{2\pi r}\left(\frac{-\beta\gamma q_0}{c\mu_0\epsilon_0}\right) = -\frac{1}{c}\beta\gamma\frac{q_0}{2\pi\epsilon_0 r}$$

Where I have used the relation, $c = \frac{1}{c\mu_0\epsilon_0}$. Once again by symmetry, $E_\theta = 0$,$E_z = 0$, $B_r = 0$, and $B_z = 0$.

Problem 11.15

In S: Given: $\vec{E} = E_0\hat{x}$, $|B| = 2E_0$,$\vec{B}$ lies in the $x - y$ plane.
Find an S' frame where E and B are parallel. That is a frame where $\vec{E}' \times \vec{B}' = 0$. From Jackson, we have the field transformation.

$$\vec{E}' = \gamma \left(\vec{E} + \beta \times \vec{B}\right) - \frac{\gamma^2}{\gamma + 1}\beta \left(\beta \cdot \vec{E}\right)$$

$$\vec{B}' = \gamma \left(\vec{B} - \beta \times \vec{E}\right) - \frac{\gamma^2}{\gamma + 1}\beta \left(\beta \cdot \vec{B}\right)$$

I want a frame where $\vec{E}' \times B' = 0$; in all its glory, this can be written

$$\begin{aligned}\frac{1}{\gamma^2}\vec{E}' \times B' &= \\ \vec{E} \times \vec{B} - \vec{E} \times (\beta \times \vec{E}) + (\beta \times \vec{B}) \times \vec{B} + (\beta \times \vec{B}) \times (\beta \times \vec{E}) & \\ + \frac{\gamma^2}{\gamma + 1}(\beta \times \beta)(\beta \cdot \vec{E})(\beta \cdot \vec{B}) & \\ -\frac{\gamma}{\gamma + 1}\left[(\vec{E} \times \beta)(\beta \cdot \vec{B}) + (\beta \times \vec{B}) \times \beta(\beta \cdot \vec{B})(\beta \times \vec{B})(\beta \cdot \vec{E}) - \beta \times (\beta \times \vec{E})(\beta \cdot \vec{E})\right] & \\ &= 0\end{aligned}$$

To simplify I must use the following identities:

1.

$$\beta \times \beta = 0$$

2.

$$\vec{E} \times (\beta \times \vec{E}) = \beta(\vec{E}^2) - \vec{E}(\beta \cdot \vec{E})$$

3.

$$(\beta \times \vec{B}) \times \vec{B} = -\vec{B} \times (\beta \times \vec{B}) = -\beta(\vec{B}^2) + \vec{B}(\beta \cdot \vec{B})$$

4.

$$(\beta \times \vec{B}) \times \beta = -\beta \times (\beta \times \vec{B}) = -\beta(\vec{B} \cdot \beta) - \vec{B}(\beta^2)$$

5.

$$\beta \times (\beta \times \vec{E}) = \beta(\beta \cdot \vec{E}) - \vec{E}(\beta^2)$$

6.

$$(\beta \times \vec{B}) \times (\beta \times \vec{E}) = \beta \cdot (\vec{B} \times \vec{E})\beta - [\beta \cdot (\beta \times \vec{B})]\vec{E} = [\beta \cdot (\vec{B} \times \vec{E})]\beta$$

Using these relationships, I have

$$\begin{aligned}\vec{E} \times \vec{B} + [\beta \cdot (\vec{B} \times \vec{E})]\beta - (\vec{E}^2 + \vec{B}^2)\beta + (\beta \cdot \vec{E})\vec{E} + (\beta \cdot \vec{B})\vec{B} \\ -\frac{\gamma}{\gamma+1}\left[-(\beta \times \vec{E})(\beta \cdot \vec{B}) + (\beta \times \vec{B})(\beta \cdot \vec{E})\right] \\ +\frac{\gamma}{\gamma+1}\left[(\beta \cdot \vec{B})[-\beta(\beta \cdot \vec{B}) + \vec{B}\beta^2] + (\beta \cdot \vec{E})[\beta(\beta \cdot \vec{E}) - \vec{E}\beta^2]\right] \\ = 0\end{aligned}$$

This is still exceptionally complicated.
There is a whole plane of possible Lorentz transformations which bring us to a frame where E and B are perpendicular. I won't bother to show this general relationship because Jackson only asks for *a* frame. That means one! I'll choose the particularly simple case where β is along the z axis. In this case $\beta \cdot \vec{E} = 0$ and $\beta \cdot \vec{B} = 0$. The equation reduces to

$$(\vec{E} \times \vec{B})\hat{z} + \beta^2(\vec{E} \times \vec{B})\hat{z} - \beta(\vec{E}^2 + \vec{B}^2)\hat{z} = 0$$

Upon rearrangement,

$$\frac{\vec{\beta}}{1+\beta^2} = \frac{|\vec{E} \times \vec{B}|}{\vec{E}^2 + \vec{B}^2}\hat{z}$$

Since $|\vec{E} \times \vec{B}| = 2E_0^2 \sin\theta$ and $\vec{E}^2 + \vec{B}^2 = 5\vec{E}^2$,

$$\frac{\beta}{1+\beta^2} = \frac{2}{5}\sin\theta$$

For $\theta << 1$, $\sin\theta \approx \theta$. We can't choose $\beta > 1$; it follows that $\beta \approx \frac{2}{5}\theta$ and $\gamma = \sqrt{1 - \frac{4}{25}\theta^2}$.

$$E' = \gamma(E + \beta \times B) + 0 = (1 - \frac{4}{5}\theta^2)^{-\frac{1}{2}}[E_0 + \frac{4}{5}E_0\theta]$$

Take advantage of the invariant quantity $\vec{E} \cdot \vec{B} = \vec{E}' \cdot \vec{B}'$.

$$\vec{E} \cdot \vec{B} \approx 2E_0^2 = E_0(1 - \frac{4}{25}\theta^2)^{-\frac{1}{2}}(1 + \frac{4}{5}\theta)B' \sin\frac{\pi}{2}$$

So

$$B' = 2\frac{\sqrt{1 - \frac{4}{25}\theta^2}}{1 - \frac{4}{5}\theta}E_0$$

As $\theta \to 0$, we get $|E| = E_0$ and $|B| = 2E_0$ as expected.
For $\theta \to \frac{\pi}{2}$, we the sin term goes to 1 and we get a quadratic equation.

$$\frac{\beta^2}{1+\beta^2} = \frac{2}{5}\sin\frac{\pi}{2} = \frac{2}{5} \to \beta^2 - \frac{5}{2}\beta + 1 = 0$$

which has two roots, 2 and $\frac{1}{2}$. The first is clearly unreasonable because $\beta > 1$ means that the frame velocity exceeds the speed of light. Good luck getting into that reference frame.
Therefore, $|\beta| = \frac{1}{2}$. In this case, $\vec{E}' \to 0$ and $\vec{B}' \to \sqrt{3}E_0$.

Problem 11.22

Take $c = 1$. At 3 K, the typical energy E_1 for a background photon is 2.5×10^{-4} eV. Assume the momentum for the background photon and the incident photon are anti-parallel. Exploit conservation of energy and the Lorentz invariant properties of four vectors squared.

$$\begin{pmatrix} 2m_e \\ 0 \end{pmatrix}^2 = \begin{pmatrix} E_2 + E_1 \\ p_2 + p_1 \end{pmatrix}^2$$

So

$$4m_e^2 = E_1^2 + E_2^2 + 2E_1E_2 - (p_1^2 + p_2^2 + 2p_1 \cdot p_2)$$

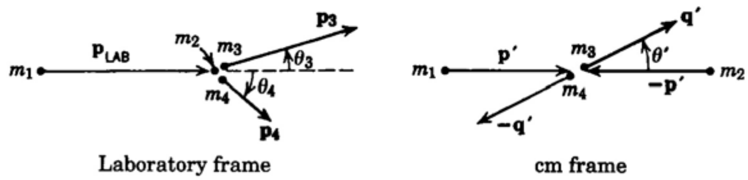
But $|p_i| = E_i$ for photons because they have no mass. Since p_1 and p_2 are anti-parallel $p_1 \cdot p_2 = -|p_1||p_2|$. So

$$4m_e^2 = E_1^2 + E_2^2 + 2E_1E_2 - (E_1^2 + E_2^2 - 2E_1E_2)$$

Which can easily be rearranged to give

$$E_2 = \frac{m_e^2}{E_1}$$

a.
$E_2 = 1.044 \times 10^{15}$ eV or 1.67×10^{-4} joules.
b.
Assume $E_1 = 500$ eV or 0.5 KeV. $E_2 = 5.22 \times 10^8$ eV.
An aside: From thermodynamics, we expect pair production effect to become significant when $k_bT \sim m_ec^2$. This corresponds to a temperature of about 10^9 K. So we should expect that E_2 must be very high if E_1 is very low, on the order of a few or even a few hundred thousand Kelvin, as it is in part a and then b.



Problem 11.23

$$\wp_1 + \wp_2 \rightarrow \wp_3 + \wp_4$$

a.

I am going to take advantage of the fact that the product of two Lorentz four vectors is invariant or the same in all Lorentz frames. A prime denotes that a quantity is measured in the center of momentum frame. no prime means that the quantity is measured in the lab frame. Sometimes, I get a bit carried away and use both subscripts and prime

$$W^2 = \begin{pmatrix} E_1 + m_2 \\ \vec{p_1} \end{pmatrix}_{lab}^2 = E_1^2 + m_2^2 + 2E_1 m_2 - |\vec{p_1}|^2$$

but $E_i^2 - p_i^2 = m_i^2$ consequently

$$W^2 = m_1^2 + m_2^2 + 2E_1 m_2$$

And in the center of momentum frame,

$$W'^2 = \begin{pmatrix} E_1' + E_2' \\ 0 \end{pmatrix}_{CM}^2 \rightarrow W' = E_1' + E_2'$$

So W' is the total energy in the center of mass frame. W^2 is an invariant scalar so $W'^2 = W^2$.
$p' = \beta_2' \gamma_2' m_2$, but $|\vec{p_2}| = 0$ because this particle is at rest. Therefore, in the center of momentum frame, particle two's velocity will be $-\beta_{CM}$ and $\gamma = \gamma_{CM}$. From part b of this problem or through gruesome algebra, we have results for β_{CM} and γ_{CM}. Thus,

$$p'_{CM} = -\beta'_{CM}\gamma'_{CM} m_2 = \frac{p_{lab}}{m_2 + E_{lab}} \frac{m_2 + E_{lab}}{W} m_2 = \frac{m_2 p_{lab}}{W}$$

Or if you the gratuitous details of the algebraic approach *in lingua latina*:
Ex principis invarianta quanta habemus

$$\wp_{lab}^2 = \wp_{CM}^2 \to (E_{lab} + m_2)^2 - p_{lab}^2 = (E_1' + E_2')^2 = E_1'^2 + E_2'^2 + 2E_1'E_2'$$

Eo quod $E_i' = \sqrt{m_1^2 + p_i'^2}$, sequitur

$$(E_{lab} + m_2)^2 - p_{lab}^2 = m_1^2 + m_2^2 + 2p'^2 + 2E_1'E_2'$$

Torqutum

$$2m_2E_{lab} = 2p'^2 + 2E_1'E_2' \to 2m_2E_{lab} - 2p'^2 = 2E_1'E_2'$$

Atque

$$4m_2^2E_{lab}^2 - 8p'^2m_2E_{lab} + 4p'^2 = 4E_1^2E_2^2$$

Quoniam $E_i' = \sqrt{m_1^2 + p_i'^2}$;

$$4m_2^2(m_1^2 + p_{lab}^2) - 8p'^2m_2E_{lab} + 4p'^4 = 4(m_1^2 + p_1'^2)(m_2^2 + p_2'^2)$$

Et

$$m_2^2p'^2 - 2m_2p'^2E_{lab} = m_1^2p'^2 + m_2^2p'^2$$

Mota posita litterarum

$$m_1p'^2 + m_2p'^2 + 2m_2E_{lab}p'^2 = m_2^2p'^2$$

igitur

$$W^2p'^2 = m_2^2p_{lab}^2$$

tandem

$$p'^2 = \frac{m_2^2p_{lab}^2}{W^2}$$

atque

$$p' = \frac{m_2p_{lab}}{W}$$

b.
The definition of β_{CM}:

$$\beta_{CM} = \frac{\sum p_{lab}}{\sum E_{lab}} = \frac{p_1}{E_1 + m_2}$$

Notice that in the center of momentum frame $\beta_{CM} = 0$ as expected since β_{CM} describes the velocity of the center of momentum frame relative to the frame in which the momentum and energies are measured.

$$\gamma_{CM} = \frac{1}{\sqrt{1-\beta^2}} = \left[1 - \left(\frac{p_1}{E_1 + m_2}\right)^2\right]^{-\frac{1}{2}} = \frac{E_1 + m_2}{\sqrt{(E_1+m_2)^2 - p_1^2}}$$

but $(E_1 + m_2)^2 - p_1^2 = E_1^2 + m_2^2 + 2E_1m_2 - p_1^2$ and $E_1^2 - p_1^2 = m_1^2$ so the terms in the square root are just $m_1^2 + m_2^2 + 2E_1m_2 = W^2$ and

$$\gamma_{CM} = \frac{E_1 + m_2}{W}$$

c.

Start with $W^2 = m_1^2 + m_2^2 + 2E_1m_2 = (m_1+m_2)^2 - 2m_1m_2 + 2E_1m_2$. Define $T = E_1 - m_1$.

$$W^2 = (m_1+m_2)^2 + 2m_2T$$

Write this in a suggestive form.

$$W = (m_1+m_2)\sqrt{1 + \frac{2m_2T}{(m_1+m_2)^2}}$$

Since $2m_2T << m_1 + m_2$, we can use $\sqrt{1+x} \simeq 1 + \frac{1}{2}x$.

$$W \simeq (m_1+m_2)\left(1 + \frac{m_2T}{(m_1+m_2)^2}\right) = m_1 + m_2 + \frac{m_2}{m_1+m_2}T$$

where $T = E_1 - m_1$. For $v << c$ the usual expansion applies $E_1 = m_1 + \frac{p^2}{2m_1} + O(p^4)$. So $T = m_1 + \frac{p_1^2}{2m_1} + ... - m_1$.

$$W \simeq m_1 + m_2 + \left(\frac{m_2}{m_1+m_2}\right)\frac{p_1^2}{2m_1}$$

And $W^{-1} \simeq (m_1+m_2)^{-1}$ because the final term in W is so small. Using this we have

$$p' = \frac{m_2p_1}{W} = \left(\frac{m_2}{m_1+m_2}\right)p_1$$

And

$$\beta_{CM} = \frac{p_1}{m_2+E_1} = \frac{p_1}{m_1+m_2}$$

Where I have ignored first order and higher corrections to E_1, i.e. $E_1 \simeq m_1$.

Problem 11.26

a.

The diagrams appropriate to this problem are shown for Jackson problem 11.23 m_1 is the incident particle's mass before the collision, and m_2 is the struck particle's mass before the collision. m_3 and m_4 designate the respective particle masses after the collision. For an elastic collision, $m_3 = m_1$ and $m_4 = m_2$. The unprimed quantities are in the lab frame while the primed quantities are in the center of momentum frame. Note also that $p_{lab} = p_1$ because $p_2 = 0$, the other particle is at rest initially in the lab.

For the first part of this problem, I wanted to do things the brute force way to give you some idea of the kind of awkward algebra involved. But as is typically the case, the algebra[6] "compelled" me not to take this route.

For the first relationship, I'll use an elegant approach which takes advantage of the invariant properties of four-vectors squared. First of all, consider the scalar product in the center of momentum frame. Note $E'_2 = E'_4$ because of conservation of momentum.

$$\wp'_2\wp'_4 = E'_2E'_4 - pq\cos\theta' = m_2^2 + p_2'^2 - p_2'^2\cos\theta' = m_2^2 + p_2^2(1-\cos\theta')$$

In the lab,

$$\wp_2\wp_4 = m_2E_4 - 0 = m_2^2 + m_2\Delta E$$

Using $E_4 = \Delta E + m_2$. Because of Lorentz invariance, $\wp'_2\wp'_4 = \wp_2\wp_4$, we get

$$\Delta E = \frac{p_2^2}{m_2}(1-\cos\theta')$$

For the second relationship,

$$\wp_1 + \wp_2 = \wp_3 + \wp_4 \to \wp_1 - \wp_4 = \wp_3 - \wp_2$$

Square it.

$$(\wp_1 - \wp_4)^2 = (\wp_3 - \wp_2)^2$$

Multiply out the squares. Don't forget $\wp^2 = m^2$.

$$m_1^2 + m_4^2 - 2(E_1E_4 - p_1p_4\cos\Theta') = m_3^2 + m_2^2 - 2(E_2E_3 - p_2p_3)$$

[6] In Arabic, al-jabra means roughly to compel, so this is really just a terrible pun

But m_2 is at rest before the collision so $p_2 = 0$ and $E_2 = m_2$. The collision is elastic, that is $m_1 = m_3$ and $m_2 = m_4$, so we get

$$E_1 E_4 - p_1 p_4 \cos\Theta' = m_2 E_3 \rightarrow p_1 p_4 \cos\Theta' = E_1 E_4 - m_2 E_3$$

As before $E_4 = \Delta E + m_2$ so $E_3 = E_1 - \Delta E$.

$$p_1 p_4 \cos\Theta' = E_1(\Delta E + m_2) - m_2(E_1 - \Delta E) = \Delta E(E_1 + m_2)$$

Square this, and substitute the total center of momentum energy, $W^2 = (E_1 + m_2)^2 - p_1^2$.

$$p_1^2 p_4^2 \cos^2\Theta' = \Delta E^2 (W^2 + p_1^2)$$

Play around with some algebra.

$$p_1^2(E_4^2 - m_4^2)\cos^2\Theta' = p_1^2(\Delta E^2 + 2m_2\Delta E)\cos^2\Theta' = \Delta E^2(W^2 + p_1^2)$$

Divide by ΔE.

$$2m_2 p_1^2 \cos^2\Theta' = \Delta E(W^2 + p_1^2 - p_1^2\cos^2\Theta') = \Delta E(W^2 + p_1^2 \sin^2\Theta')$$

Solve for ΔE.

$$\Delta E = \frac{2m_2 p_1^2 \cos^2\Theta'}{W^2 + p_1^2 \sin^2\Theta'}$$

For the third relation, we need Q^2. Q^2 is defined as follows

$$Q^2 = -(\wp_1 - \wp_3)^2 = (p_1 - p_3)^2 - (E_1 - E_2)^2$$

In the center of momentum frame, $|p_1'| = |p_3'| = p'$ so

$$E_1' - E_3' = \sqrt{m^2 + p_1'^2} - \sqrt{m^2 + p_3'^2} = 0$$

And

$$(p_1' - p_3')^2 = p_1'^2 + p_3'^2 - 2p_1'p_3'\cos\theta' \rightarrow \tag{7}$$

$$(p_1' - p_3')^2 = 2p'^2(1 - \cos\theta') \tag{8}$$

Thus,

$$Q^2 = 2p'^2(1 - \cos\theta')$$

Use the first relation, $\Delta E = \frac{p'^2}{m_2}(1 - \cos\theta')$, from problem 11.23. Substitute Q^2 and get

$$\Delta E = \frac{Q^2}{2m_2}$$

b.

$$\Delta E_{Max} \simeq 2m_e\gamma^2\beta^2$$

where γ and β are characteristic of the incident particle and $\gamma << \frac{m_1}{m_e}$. Give this result a simple interpretation by considering the relevant collision in the rest frame of the incident particle and then transforming back to the laboratory.

For $m_1 >> m_2$, we start with the first expression for the energy change.

$$\Delta E = \frac{m_2 p_1^2}{W^2}(1 - \cos\theta')$$

This attains its greatest value when $\theta' = \pi$ and $1 - \cos\theta' = 2$, then

$$\Delta E_{Max} = \frac{2m_2 p_1^2}{W^2} = \frac{2m_e\gamma^2\beta^2 m_1^2}{W^2}$$

$W^2 = m_1^2 + m_e^2 + 2m_e E_1$ can be written in a suggestive manner.

$$W^2 = m_1^2\left(1 + \frac{m_e^2}{m_1^2} + 2\frac{m_e}{m_1}\frac{\gamma}{\frac{m_1}{m_e}}\right)$$

Because $m_1 >> m_e$, the second term in the brackets is small and can be ignored to first order. Because $\gamma << \frac{m_1}{m_e}$, the third term is similarly small and can be ignored. So we have $W^2 \simeq m_1^2$. We conclude

$$\Delta E_{Max} \simeq 2m_e\gamma^2\beta^2$$

When Jackson asks for a simple interpretation, I'm not sure what the Hell he wants. His question is vague. Maybe, he wants us to make some statement about m_1 being almost stationary. You'll have to bullshit your way through this.

c.

For electron collisions, $m_1 = m_2 = m_e$. Use the formula for ΔE from the beginning of part b, but substitute in W explicitly.

$$\begin{aligned}\Delta E_{Max} &= \frac{2m_2p_1^2}{2m_2E_1 + m_1^2 + m_2^2} = \frac{m_ep_1^2}{m_eE_1 + m_e^2} \\ &= \frac{p^2}{E+m} = \frac{\gamma^2\beta^2m^2}{\gamma m + m} = \left(\frac{\gamma^2\beta^2}{\gamma+1}\right)m_e\end{aligned}$$

With a little trivial algebra, we find $\beta^2 = \frac{\gamma^2-1}{\gamma^2}$. Substitute in for β and get the desired result.

$$\Delta E_{Max} = (\gamma - 1)\, m_e$$

Problem 12.3

a.

Take $\vec{F} = e\vec{E}_0\hat{x}$. Since the initial velocity is non-vanishing but perpendicular to the field, we have $\vec{v_0} = v_0\hat{y}$. The relativistic force law is $\frac{d\wp_i}{dt} = F_i$, so we have two equations which must be satisfied.

$$\frac{d}{dt}\left(\frac{mv\hat{x}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = eE_0\hat{x}$$

And

$$\frac{d}{dt}\left(\frac{mv\hat{y}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = 0$$

Integrate these.

$$\frac{mv_x}{\sqrt{1-\frac{v^2}{c^2}}} = eE_0t + C_1$$

$C_1 = 0$ because the initial x velocity is zero.

$$\frac{mv_y}{\sqrt{1-\frac{v^2}{c^2}}} = C_2$$

To find C_2, we must invoke initial conditions. At time initial, $v_y = v_0$ and $v^2 = v_0^2$. Then,

$$\frac{mv_0}{\sqrt{1-\frac{v_0^2}{c^2}}} = C_2 \rightarrow C_2 = \frac{mv_0c}{\sqrt{c^2 - v_0^2}}$$

We can get v_x and v_y as functions of time.

$$v_x^2 = \frac{e^2E_0^2t^2}{m^2}(1 - v_x^2 - v_y^2)$$

$$v_y^2 = \frac{C_2^2}{m^2}(1 - v_x^2 - v_y^2)$$

Dividing these two, we get a relationship between v_x and v_y.

$$\frac{v_x^2}{v_y^2} = \frac{e^2E_0^2t^2}{A^2}$$

Now, solve for v_x and v_y.

$$v_x^2 = \frac{c^2 e^2 E_0^2 t^2}{c^2 m^2 + e^2 E_0^2 t^2 + C_2^2}$$

$$v_y^2 = \frac{C_2^2}{c^2 m^2 + e^2 E_0^2 t^2 + C_2^2}$$

Define $\gamma_0 = (1 - \frac{v_0^2}{c^2})^{-\frac{1}{2}}$ and $a = \frac{qE}{mc}$. We now have separate equations for v_x and v_y.

$$v_x = \frac{cat}{\sqrt{a^2 t^2 + \gamma_0^2}}$$

And

$$v_y = \frac{\gamma_0 v_0}{\sqrt{a^2 t^2 + \gamma_0^2}}$$

These two can be integrated over time to get the expressions for $x(t)$ and $y(t)$.

$$x(t) = c \int_0^t \frac{at'}{\sqrt{a^2 t'^2 + \gamma_0^2}} dt' = \frac{c}{a} \left[\sqrt{a^2 t^2 + \gamma_0^2} - \gamma_0 \right]$$

And

$$y(t) = \gamma_0 v_0 \int_0^t \frac{1}{\sqrt{a^2 t'^2 + \gamma_0^2}} dt' = \frac{\gamma_0 v_0}{a} \ln \left[\frac{\sqrt{a^2 t^2 + \gamma_0^2} + at}{\gamma_0} \right]$$

Problem 12.4

Consider the design of an $\vec{E} \times \vec{B}$ velocity selector. Take $c = 1$ as usual.
In S, $\vec{E} = E\hat{y}$, $\vec{B} = B\hat{z}$, $\vec{u_0} = u_0\hat{x}$, and $u_0 = \frac{E}{B}$ where u_0 is the average selected velocity. The aperture admittance is Δx, and the length of the selector is L. Let $L = u_0\bar{t} \rightarrow \bar{t} = \frac{L}{u_0}$. $\bar{t}$ is the average time per particle in the selector.
Go to S', a frame moving $\vec{u} = \frac{E}{B}\hat{x}$. We are moving along with the particles as they pass through the selector. Note that in this frame, the following transformations hold:

$$E' = \gamma(E - uB) = \gamma(E - E) = 0$$

And

$$B' = \gamma(B - uE)\hat{z} = \gamma(B - \frac{E^2}{B})\hat{z} = \frac{\gamma}{B}(B^2 - E^2)\hat{z}$$

Which can be further simplified because $\gamma = (1 - u^2)^{-\frac{1}{2}} = (1 - \frac{E^2}{B^2})^{-\frac{1}{2}} = \frac{B}{\sqrt{B^2 - E^2}}$, so

$$B' = \sqrt{B^2 - E^2}\hat{z} = \frac{B}{\gamma}\hat{z}$$

Particles which have $\vec{\beta} = u_0\hat{x}$ in the lab will be at rest in S' and so will be unaffected by the field. Also in S', the time it takes for a particle to travel from one aperture to the other is given by $t' = \frac{L}{\gamma u}$. (More appropriately, this is the time it takes one aperture to move away and the other one to arrive!) The γ comes in because in this frame the selector is moving so the distance is contracted. A particle with non-zero velocity in S' will be deflected in an arc. I'll draw this for clarity someday.

$$\Delta x' = r_0'(1 - \cos\omega_B' t')$$

Since $\Delta x'$ is perpendicular to u, $\Delta x' = \Delta x$. Jackson told us $\omega_B' \simeq \frac{qB'}{m} = \frac{qB}{\gamma m}$ and $r_0' = \frac{p'_\perp}{qB'} = \frac{mv'}{qB'} = \frac{m}{qB'}v' = \frac{v'}{\omega_B'}$.
We expect the deflection to be small because the aperture we are considering is small. Thus, we approximate $\cos\omega_B' t' \simeq 1 - \frac{(\omega_B' t')^2}{2}$. So

$$\Delta x' = \frac{\omega_B^2 t'^2}{2} r_0'$$

Get v' in terms of the variables we know, namely Δu and u_0. Assuming that v is small compared to u, we can use the approximation that $uv \simeq u^2$ so that $1 - uv \simeq \gamma^2$.

$$v' = \frac{v-u}{1-uv} = \gamma^2 \Delta u$$

Now, plug in v' to the expression for Δx.

$$\Delta x = \frac{q^2 B^2}{2\gamma^2 m^2} \frac{L^2}{\gamma^2 u^2} \frac{\gamma m}{qB} \gamma^2 \Delta u$$

Simplify and substitute $B = \frac{E}{u}$. We have the following expression for the deflection:

$$\Delta x = \frac{qEL^2}{2\gamma m u^3} \Delta u$$

With $\gamma u m = p$ and some rearrangement, you should get

$$\Delta u = \frac{2p}{qL^2 E} u^2 \Delta x$$

Depending on how you define Δu there may be a factor of two missing.

Problem 12.5

a.

$c = 1$, $\vec{E} = E\hat{x}$, $\vec{B} = B\hat{y}$. Since $|E| < |B|$, we can transform the E field away by choosing a suitable Lorentz frame. Try $\vec{u} = \frac{\vec{E}\times\vec{B}}{B^2} \rightarrow \vec{u} = \frac{E}{B}\hat{z}$. The appropriate Lorentz transformations are:

$$E' = \gamma(E + \beta \times B) - \frac{\gamma^2}{\gamma+1}\beta(\beta \cdot E)$$

$$B' = \gamma(B - \beta \times E) - \frac{\gamma^2}{\gamma+1}\beta(\beta \cdot B)$$

Note $\beta \cdot E = 0$ and $\beta \cdot B = 0$. With these, the fields transform as such.

$$E' = \gamma(E + \beta \times B) = \gamma(E - uB)\hat{y} = \gamma(E - \frac{E}{B}B) = 0$$

$$B' = \gamma(B - \beta \times E) = \gamma(B - uE)\hat{y} = \frac{\gamma}{B}(B^2 - E^2)\hat{y}$$

But $\gamma = (1 - \frac{E^2}{B^2})^{-\frac{1}{2}} = \frac{B}{\sqrt{B^2-E^2}}$ so

$$B' = \sqrt{B^2 - E^2}\hat{y}$$

Which can be expressed as in Jackson

$$B'_{perp} = \sqrt{\frac{B^2 - E^2}{B^2}}\vec{B}$$

In this frame, we have a particle moving in a uniform static B field. Jackson solved this for us.

$$\vec{x}(t) = \vec{x}_0 + v_{\|}t\vec{\epsilon}_3 + ia(\vec{\epsilon}_1 - i\vec{\epsilon}_2)e^{-i\omega_B t}$$

Matching initial conditions requires

$$\vec{x}(t) = v_y t\hat{y} + a\cos\omega_B t\hat{x} + a\sin\omega_B t\hat{z}$$

where $\omega_B = \frac{eB'}{\gamma m}$ and $a = \frac{p_x^2+p_z^2}{eB}$.

Now, I simply transform back to the lab to get what Jackson wants.

$$u = -\frac{E}{B}\hat{z}$$

$$\gamma = \frac{B}{\sqrt{B^2 - E^2}}$$

Jackson only wants parametric equation so I won't bother with the complication that $t = f(t')$. The equation of motion along the $\hat{y}$ direction is easy because the fields do not accelerate the particle along this direction. The $\hat{x}$ part is unaffected by the Lorentz transformation. The $\hat{z}$ component is not much more difficult. Just multiply by appropriate length contraction γ on the z position in S' and add an additional term to account for the motion of the frame. The γ factor on the latter term is necessary because the time is given in the other frame. The final result is

$$\vec{x}_{lab}(t) = a\cos(\omega_B t)\hat{x} + v_{y,t=0}t\hat{y} + (\gamma a \sin(\omega_B t) + \gamma u t)\hat{z}$$

b.

I didn't do.

Problem 12.14

$$\mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha$$

a.
The Euler-Lagrange theorem says

$$\frac{\partial \mathcal{L}}{\partial \phi^\alpha} = \partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta \phi^\alpha)}$$

So we have $\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c}J_\alpha$ and

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{8\pi}\frac{\partial}{\partial(\partial^\beta A^\alpha)}(g_{\sigma\mu}g_{\tau\nu}\partial^\mu A^\nu \partial^\sigma A^\tau)$$

Recall that the rule for differentiation is $\frac{\partial}{\partial(\partial^\kappa A^\lambda)}(\partial^\eta A^\gamma) = \delta_{\kappa\eta}\delta_{\lambda\gamma}$.

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{8\pi}g_{\sigma\mu}g_{\tau\nu}[\delta_{\beta\mu}\delta_{\alpha\nu}\partial^\mu A^\nu + \delta_{\beta\sigma}\delta_{\alpha\tau}\partial^\sigma A^\tau]$$

Using the Dirac deltas, we get

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{8\pi}[g_{\sigma\beta}g_{\tau\alpha}\partial^\beta A^\alpha + g_{\beta\mu}g_{\alpha\nu}\partial^\beta A^\alpha] = -\frac{1}{8\pi}[2\partial_\beta A_\alpha]$$

The Euler-Lagrange equation of motion is, in our case,

$$\partial^\beta \partial_\beta A_\alpha = \frac{4\pi}{c}J_\alpha \tag{9}$$

If we are in the Lorentz gauge, $\partial_\mu A^\mu = 0$, and we can write equation 9 as $\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c}J_\alpha$ because $F_{\beta\alpha} = \partial_\beta A_\alpha - \partial_\alpha A_\beta = \partial_\beta \partial_\alpha$. We have the inhomogeneous Maxwell equations!

b.
The other Lagrangian is

$$\mathcal{L} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha$$

Write $F_{\alpha\beta}$ explicitly as $\partial_\alpha A_\beta - \partial_\beta A_\alpha$.

$$\mathcal{L} = -\frac{1}{16\pi}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - \frac{1}{c}J_\alpha A^\alpha$$

The difference between this Lagrangian and the one in part a is

$$\Delta\mathcal{L} = -\frac{1}{16\pi}[\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\beta A_\alpha \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha + \partial_\beta A_\alpha \partial^\beta A^\alpha - 2\partial_\alpha A_\beta \partial^\alpha A^\beta]$$

$$\Delta\mathcal{L} = \frac{1}{16\pi}[\partial_\beta A_\alpha \partial^\alpha A^\beta + \partial_\alpha A_\beta \partial^\beta A^\beta] = \frac{1}{8\pi}\partial_\alpha A_\beta \partial^\beta A^\alpha$$

And by using the rule for differentiating a product.

$$\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\beta A^\alpha = \frac{1}{8\pi}\partial_\alpha (A_\beta \partial^\beta A^\alpha) - \frac{1}{8\pi} A_\beta \partial^\beta \partial_\alpha A^\alpha$$

A careful reader will notice that I have switched the order of differentiation on the last term. This is allowed because derivatives commute, i.e. $[\partial_\gamma, \partial_\eta] = 0$. In the Lorentz gauge, $\partial_\alpha A^\alpha = 0$, and the last term vanishes, $\frac{1}{8\pi} A_\beta \partial^\beta \partial_\alpha A^\alpha = 0$. The remaining term, $\frac{1}{8\pi}\partial_\alpha (A_\beta \partial^\beta A^\alpha)$, is just a four divergence.

Problem 13.4

a.

The quantum mechanical energy loss formula is:

$$\frac{dE}{dx} = 4\pi NZ\frac{z^2e^4}{m_ec^2\beta^2}\left[\ln\left(\frac{2\gamma^2\beta^2m_ec^2}{\hbar\langle\omega\rangle}\right) - \beta^2\right]$$

This formula gives results in units of energy per distance. Numerically, $4\pi\frac{z^2e^4}{m_ec^2} = 5.1\times10^{-25}$ Mev cm^2, and $\frac{2m_ec^2}{\hbar\langle\omega\rangle/Z} = \frac{2m_ec^2}{12} = 8.5\times10^4$. The m_ec^2 must be given in eV.

Another formula can be constructed which has units of energy times area per mass. I do that by dividing the first result by ρ, the density. ρ is equal to $NA\ m_{nucleon}$.

$$\frac{dE}{dx}/\rho = 4\pi\frac{Z}{Am_n}\frac{z^2e^4}{m_ec^2\beta^2}\left[\ln\left(\frac{2\gamma^2\beta^2mc^2}{\hbar\langle\omega\rangle}\right) - \beta^2\right]$$

β and γ can be determined for the muon and the electron using the relationship $\beta = \frac{p}{E}$, $E = T + m$, $E^2 = p^2 + m^2$ (These formulas require that I use units so that $c = 1$ and $\hbar = 1$).

b.

Aluminum has $Z = 13$, $Z = 27$, and density, $\rho = 2.7$ gm/cm^3. Copper has $Z = 29$, $A = 64$, and $\rho = 9.0$. Lead has $Z = 82$, $A = 208$, and $\rho = 11$. For air, we use Nitrogen, $Z = 14$, $A = 28$, and $\rho = 1.3\times10^{-3}$.

The energy loss per densities should be roughly the same because the electron densities are similar if the atomic densities are the same. By dividing out the density, we give out answer in a form that is independent or the atomic density.

Incident Protons with Various Energies. (Energy Loss in Mev/cm)

	10 Mev	100 MeV	1000MeV
air	5×10^{-2}	8×10^{-3}	3×10^{-3}
Al	100	17	5.2
Cu	310	52	16
Pb	330	55	17

Incident Muons with Various Energies. (Energy Loss in Mev/cm)

	10 Mev	100 MeV	1000MeV
air	9×10^{-3}	2.6×10^{-3}	2.7×10^{-3}
Al	19	5.4	5.6
Cu	58	17	17
Pb	61	18	18

Incident Protons. (Energy Loss in Mev cm^2 / gm)

	10 Mev	100 MeV	1000MeV
air	37	6.1	1.9
Al	37	6.3	1.9
Cu	34.8	5.8	1.8
Pb	30	5.0	1.6

Incident Muons. (Energy Loss in Mev cm^2 / gm)

	10 Mev	100 MeV	1000MeV
air	6.8	2.0	2.1
Al	7.0	2.0	2.1
Cu	6.5	1.9	1.9
Pb	5.6	1.6	1.6

Problem 13.9

As usual, I'm going to take $c = 1$. We are asked to consider the Cherenkov radiation for Plexiglas or Lucite. I think by index of $r$etraction Jackson meant index of $r$efraction, i.e. $n = 1.5$. From Jackson 13.50, we have: $\cos\theta_c = \frac{1}{\beta\sqrt{\epsilon(\omega)}} = \frac{1}{\beta n}$. The last equality is true because from Jackson 13.47, $v = \frac{c}{\sqrt{\epsilon(\omega)}}$ but also $v = \frac{c}{n}$, so $\sqrt{\epsilon(\omega)} = n$. To solve for β, use $\beta = \frac{p}{E}$. Since $E = T + m$, this gives us $\beta = \frac{\sqrt{(m+T)^2 - m^2}}{m+T} = \frac{\sqrt{T^2+2Tm}}{m+T}$.
To find the number of photons within some energy range emitted per unit length, consult the Particle Physics Data book to find

$$\frac{d^2N}{d\lambda dx} = \frac{-2\pi\alpha z^2}{\lambda^2}\sin^2\theta_c$$

This can also be derived from Jackson 13.48.

$$\frac{d^2E}{dxd\omega} = \frac{z^2e^2}{c^2}\omega\left(1 - \frac{1}{\beta^2n^2}\right)$$

Now, in cgs units, $e^2 = \alpha\hbar c$, so I can write

$$\frac{d^2E}{dxd\omega} = \frac{z^2\alpha\hbar}{c}\omega\left(1 - \frac{1}{\beta^2n^2}\right)$$

Notice that $\frac{1}{\beta^2n^2} = \cos^2\theta_c$. Thus, the term in parenthesis can be reduced using elementary trigonometric relations to $\sin^2\theta_c$. Now, I make a dubious step.

$$E = N\hbar\omega \rightarrow d^2E = -d^2N\hbar\omega$$

So we have

$$\frac{d^2N}{dxd\omega} = \frac{-z^2\alpha}{c}\sin^2\theta_c$$

Then, $\omega = \frac{2\pi c}{\lambda}$ so $d\omega = \frac{-2\pi c}{\lambda^2}d\lambda$. And finally, we get

$$\frac{d^2N}{dxd\lambda} = \frac{2\pi\alpha z^2}{\lambda^2}\sin^2\theta_c$$

which is the same as equation .

Integrate over λ.

$$\frac{dN}{dx} = \int_{\lambda_1}^{\lambda_2} \frac{-2\pi\alpha z^2}{\lambda^2} \sin^2\theta_c d\lambda = 2\pi\alpha z^2 \sin^2\theta_c \left(\frac{\lambda_2 - \lambda_1}{\lambda_2\lambda_1}\right)$$

Using $\lambda_1 = 4000|\vec{r} - \vec{r}'|A$ and $\lambda_2 = 6000|\vec{r} - \vec{r}'|A$, we have a numerical expression.

$$\frac{dN}{dx} \simeq 382.19 \sin^2\theta_c$$

in units of MeV/cm. θ_c is related to n and β from the results in part a.
I have lot's of cool Maple plots which I plan on including but for now, I'll just give you the final numbers.
For an incident electron with $T = 1$ MeV, the number of Cherenkov photons is about 187. The critical angle is 0.78 rad.
For an incident proton with $T = 500$ MeV, the number of Cherenkov photons is about 79. The critical angle is 0.50 rad.
For an incident proton with $T = 5$ TeV, the number of Cherenkov photons is about 208. The critical angle is 0.83 rad.

Problem 14.5

a.

The total energy for the particle is constant.

$$E = \frac{mv^2}{2} + V(r) \tag{10}$$

At r_{min}, the velocity will vanish and $E = V(r_{min})$.
From Jackson equation 14.21, we have the power radiated per solid angle for an accelerated charge.

$$\frac{dP}{d\Omega} = \frac{Z^2q^2}{4\pi c^3}|\dot{v}|^2 \sin^2\theta$$

From Newton's second law, $m|\dot{v}| = |\frac{dV}{dr}|$ so

$$\frac{dP}{d\Omega} = \frac{Z^2q^2}{4\pi c^3 m^2}|\frac{dV}{dr}|^2 \sin^2\theta$$

The total power is $\frac{dP}{d\Omega}$ integrated over all solid angles.

$$P_{total} = \int \frac{dP}{d\Omega}d\Omega = \frac{Z^2q^2}{4\pi c^3 m^2}|\frac{dV}{dr}|^2 \int_0^\pi \sin^2\theta d\theta \int_0^{2\pi} d\phi$$

Evaluating the integrals, $\int_0^\pi \sin^2\theta d\theta = \frac{4}{3}$ and $\int_0^{2\pi} d\phi = 2\pi$ gives

$$P_{total} = \frac{2}{3}\frac{Z^2q^2}{c^3m^2}|\frac{dV}{dr}|^2$$

The total work is the power integrated over the entire trip:

$$W_{total} = \int P_{total}dt = 2 \times \frac{2}{3}\frac{Z^2q^2}{c^3m^2}\int |\frac{dV}{dr}|^2 dt$$

The factor of two comes because the particle radiates as it accelerates to and from the potential. We can solve equation 10 for v.

$$v = \frac{dr}{dt} = \sqrt{\frac{2}{m}[V_{min} - V(r)]}$$

And from this equation, we find, $dt = \frac{dr}{\sqrt{\frac{2}{m}[V_{min} - V(r)]}}$. So

$$W_{total} = \frac{4}{3}\frac{Z^2q^2}{c^3m^2}\int_0^\infty |\frac{dV}{dr}|^2 \frac{dr}{\sqrt{\frac{2}{m}[V_{min} - V(r)]}}$$

The integral can be split into two integrals.

$$W_{total} = \frac{4Z^2q^2}{3c^3m^2}\sqrt{\frac{m}{2}}$$

$$\times \left[\int_0^{r_{min}} \mid \frac{dV}{dr} \mid^2 \frac{dr}{\sqrt{[V_{min} - V(r)]}} + \int_{r_{min}}^{\infty} |\frac{dV}{dr}|^2 \frac{dr}{\sqrt{[V_{min} - V(r)]}}\right]$$

The region for the first integral is excluded because the particle will never go there, thus, the first integral vanishes. We are left with

$$\Delta W = \frac{4}{3}\frac{Z^2q^2}{c^3m^2}\sqrt{\frac{m}{2}}\int_{r_{min}}^{\infty} \mid \frac{dV}{dr} \mid^2 \frac{dr}{\sqrt{[V_{min} - V(r)]}} \tag{11}$$

quod erat demonstrandum.

b.

First, $\frac{dV_c}{dr} = -\frac{zZq^2}{r^2} = -\frac{V_c}{r}$. Also, we can solve for dr.

$$dV_c = -\frac{V_c}{r}dr \rightarrow dr = -\frac{r^2 dV}{zZq^2}$$

Plug $V_c(r)$ and dr into equation 11:

$$\Delta W = -\frac{4}{3}\frac{Z^2q^2}{c^3m^2}\sqrt{\frac{m}{2}}\int_a^0 \frac{V_c}{r}^2 \frac{\frac{r^2}{zZq^2}dV_c}{\sqrt{[\frac{mv_0^2}{2} - V_c]}} = -\frac{4}{3}\frac{Z}{zm^2c^3}\sqrt{\frac{m}{2}}\int_a^0 \frac{V_c^2 dV_c}{\sqrt{a - V_c}}$$

The limits of integration have been changed $V(r_{min}) = \frac{mv_0^2}{2} = a$ and $V(\infty) = 0$.

The integral can be evaluated using your favorite table of integrals.

$$\int \frac{x^2dx}{\sqrt{A-x}} = -\sqrt{A-x}\left(\frac{16A^2}{15} + \frac{8Ax}{15} + \frac{2x^2}{5}\right)$$

So the integral equals $-\frac{16a^2}{15}\sqrt{a}$, and finally, we have

$$\Delta W = \frac{4}{3}\frac{Z}{zm^2c^3}\sqrt{\frac{m}{2}}\frac{16}{15}\left(\frac{mv_0^2}{2}\right)^{\frac{5}{2}} = \frac{8}{45}\frac{Zmv_o^5}{zc^3}$$