

HW12 参考答案

ch8

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* 是 $\min\{x_1, x_2\}$, \oplus 是 $\max\{x_1, x_2\}$

证明:

$$\forall x_1, x_2 \in R, \min\{x_1, x_2\} \leq x_1, \min\{x_1, x_2\} \leq x_2$$

故 $\min\{x_1, x_2\}$ 为 x_1, x_2 的下界。

若 c 是 x_1, x_2 的下界, 则 $c \leq x_1, c \leq x_2$ 而 $\min\{x_1, x_2\}$ 为 x_1 或 x_2

$$\text{故 } c \leq \min\{x_1, x_2\}$$

因此 $\min\{x_1, x_2\}$ 为 x_1, x_2 的最大下界。

同理 $\max\{x_1, x_2\}$ 为 x_1, x_2 的最小上界。

故 $\langle R, \leq \rangle$ 是格。

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由定义可得, $a * b = a, a * c = a, b * c = b, a \oplus b = b, a \oplus c = c, b \oplus c = c$

将上式代入(1)(2)可得等式显然成立。

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充分性:

$$\begin{aligned}(a \oplus b) * (b \oplus c) * (c \oplus a) &= (((a \oplus b) * b) \oplus ((a \oplus b) * c)) * (c \oplus a) = (b \oplus a * c \oplus b * c) * (c \oplus a) \\ &= b * c \oplus b * a \oplus a * c \oplus a * c \oplus b * c \oplus b * c * a = (b * c) \oplus (b * a) \oplus (a * c)\end{aligned}$$

必要性:

(\Leftarrow): 格中任意元素 a, b, c

$$\text{令 } \begin{cases} a' = (a * b) \oplus (a * c) \\ b' = b * c \\ c' = a \end{cases}$$

a', b', c' 仍在格中

$$\text{且有 } (a' * b') \oplus (b' * c') \oplus (c' * a') = (a' \oplus b') * (b' \oplus c') * (c' \oplus a') \quad (*)$$

将 a', b', c' 代入,

$$\begin{aligned} & (a' * b') \oplus (b' * c') \oplus (c' * a') \\ &= [(a * b) \oplus (a * c) * (b * c)] \oplus [b * c * a] \oplus [a * ((a * b) \oplus (a * c))] \\ &\because a * b \leq a \quad a * c \leq a \\ &\therefore (a * b) \oplus (a * c) \leq a \\ &\therefore a * [(a * b) \oplus (a * c)] = (a * b) \oplus (a * c) \\ &\therefore (a' * b') \oplus (b' * c') \oplus (c' * a') \\ &= [(a * b) \oplus (a * c) * (b * c)] \oplus [b * (c * a)] \oplus [(a * b) \oplus (a * c)] \\ &= [(a * b) \oplus (a * c) * (b * c)] \oplus [(a * b) \oplus (a * c)] \quad (a * b * c \leq a * c) \\ &= (a * b) \oplus (a * c) \quad (1) \end{aligned}$$

$$\begin{aligned} & \text{同理, } (a' \oplus b') * (b' \oplus c') * (c' \oplus a') \\ &= [(a * b) \oplus (a * c) \oplus (b * c)] * [(b * c) \oplus a] * [a \oplus (a * b) \oplus (a * c)] \\ &= [(a * b) \oplus (a * c) \oplus (b * c)] * [(b * c) \oplus a] * a \quad (a \geq a * b, a \geq a * c) \\ &= [(a * b) \oplus (a * c) \oplus (b * c)] * a \\ &= [(a \oplus b) * (a \oplus c) * (b \oplus c)] * a \quad (2) \\ &= a * (b \oplus c) \end{aligned}$$

$$\text{由 } (*) (1) (2), (a * b) \oplus (a * c) = a * (b \oplus c)$$

$$\text{由对偶性, 可证 } (a \oplus b) * (a \oplus c) = a \oplus (b * c)$$

\therefore 该格为分配格

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- 证明 f 是 $A \rightarrow B$ 的映射:

$(x \oplus a) * b \leq b$ 显然成立。

$x \oplus a \geq a$ 显然成立, $a < b$ 因此 $(x \oplus a) * b \geq a$

综上, f 是 $A \rightarrow B$ 的映射。

- 证明 f 是同态映射:

对于 $\forall x, y \in A, f(x), f(y) \in B,$

$$f(x) \oplus f(y) = ((x \oplus a) * b) \oplus ((y \oplus a) * b) = ((x \oplus a) \oplus (y \oplus a)) * b = ((x \oplus y) \oplus a) * b = f(x \oplus y)$$

$$f(x) * f(y) = ((x \oplus a) * b) * ((y \oplus a) * b) = ((x \oplus a) * (y \oplus a)) * b = ((x * y) \oplus a) * b = f(x * y)$$

证毕。

