量子力学 B
2021 秋季学期
作业 8 *(截止期: 12 月 1 号周三课上)*

1. 如三维粒子的波函数为$\varphi = A(x+y+2z)\,exp(-\alpha r)$，其中$A$为归一化常数，$\alpha > 0$。求$\hat{\ell}_z$及$\hat{\ell}_x$的可能测值和相应概率。

2. 对于$\hat{\vec{\ell}}^2$和$\hat{\ell}_z$的共同本征态$|\ell, m_z\rangle$，计算$\hat{\ell}_x^2, \hat{\ell}_y^2$的期望值以及$\Delta\ell_x, \Delta\ell_y$，并验证不确定性关系。

3. 考虑以$\hat{\vec{\ell}}^2$和$\hat{\ell}_z$的共同本征态$|\ell, m_z\rangle$为基的 Hilbert 空间中，$\ell = 1$的子空间。

a. 写出算符$\hat{\ell}_z$和$\hat{\ell}_\pm$的矩阵表示。
b. 求算符$\hat{\ell}_x$和$\hat{\ell}_y$的矩阵表示。
c. 求算符$\hat{\ell}_x^2$ 的矩阵表示及本征值。
d. 在($\hat{\vec{\ell}}^2$,$\hat{\ell}_z$)确立的表象下分别求$\hat{\ell}_z$及$\hat{\ell}_x$的本征态。
e. 求由($\hat{\vec{\ell}}^2$,$\hat{\ell}_z$)确立的表象到由($\hat{\vec{\ell}}^2$,$\hat{\ell}_x$)确立的表象之间的变换矩阵 S，并验证 b 中关于$\hat{\ell}_x$的结论。
f. 在坐标空间中写出($\hat{\vec{\ell}}^2$,$\hat{\ell}_x$)的共同本征态波函数。

4. 证明：
a. $\hat{j}_\pm \hat{j}_\mp = \hat{j}^2 - \hat{j}_z^2 \pm \hbar \hat{j}_z$
b. $\langle jm|\hat{j}_\mp \hat{j}_\pm|jm\rangle = j(j+1)\hbar^2 - m(m\pm 1)\hbar^2$

1. 如三维粒子的波函数为$\varphi = A(x+y+2z)\,exp(-\alpha r)$，其中$A$为归一化常数，$\alpha > 0$。求$\hat{\ell}_z$及$\hat{\ell}_x$的可能测值和相应概率。

解：$\varphi = A\left(\frac{x}{r}+\frac{y}{r}+2\frac{z}{r}\right) r\exp(-\alpha r)$

$$\propto \sqrt{\frac{2\pi}{3}}(Y_{1,-1}-Y_{1,1}) + i\sqrt{\frac{2\pi}{3}}(Y_{1,-1}+Y_{1,1}) + 2\sqrt{\frac{4\pi}{3}}Y_{1,0}$$

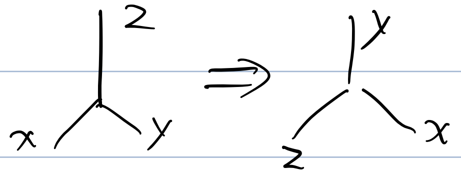
$$= 2\sqrt{\frac{4\pi}{3}}Y_{1,0} + \sqrt{\frac{4\pi}{3}}e^{i\frac{\pi}{4}}Y_{1,-1} + \sqrt{\frac{4\pi}{3}}e^{-i\frac{\pi}{4}}Y_{1,1}$$

归一化后得到

$$\varphi \propto \frac{2}{\sqrt{6}}Y_{1,0} + \frac{e^{i\frac{\pi}{4}}}{\sqrt{6}}Y_{1,-1} + \frac{e^{-i\frac{\pi}{4}}}{\sqrt{6}}Y_{1,1}$$

则测量 $\hat{l}_z$ 可能得到 $-\hbar$, 0, $\hbar$，概率分别为

$\frac{1}{6}$, $\frac{2}{3}$, $\frac{1}{6}$



令 $\varphi' = A(z+x+2y)\exp(-\alpha r)$

$$\propto i\sqrt{\frac{2\pi}{3}}(Y_{1,-1}-Y_{1,1}) + \sqrt{\frac{4\pi}{3}}Y_{1,0} + 2\sqrt{\frac{2\pi}{3}}(Y_{1,-1}+Y_{1,1})$$

$$= \sqrt{\frac{2\pi}{3}}(i+2)Y_{1,-1} + \sqrt{\frac{2\pi}{3}}(2-i)Y_{1,1} + \sqrt{\frac{4\pi}{3}}Y_{1,0}$$

归一化后得到

$$\varphi' \propto \frac{i+2}{\sqrt{12}}Y_{1,-1} + \frac{2-i}{\sqrt{12}}Y_{1,1} + \frac{1}{\sqrt{6}}Y_{1,0}$$

则测量 $\hat{l}_x$ 可能得到 $-\hbar$, 0, $\hbar$，相应的概率分别为

$\frac{5}{12}$, $\frac{1}{6}$, $\frac{5}{12}$.

2. 对于$\hat{\vec{\ell}}^2$和$\hat{\ell}_z$的共同本征态$|\ell, m_z\rangle$，计算$\hat{\ell}_x^2, \hat{\ell}_y^2$的期望值以及$\Delta\ell_x, \Delta\ell_y$，并验证不确定性关系。

解：$\hat{l}_x = \frac{1}{2}(\hat{l}_+ + \hat{l}_-)$ $\quad$ $\hat{l}_y = \frac{1}{2i}(\hat{l}_+ - \hat{l}_-)$

$$\hat{l}_x^2 = \frac{1}{4}(\hat{l}_+^2 + \hat{l}_-^2 + \hat{l}_+\hat{l}_- + \hat{l}_-\hat{l}_+)$$

$$\hat{l}_y^2 = -\frac{1}{4}(\hat{l}_+^2 + \hat{l}_-^2 - \hat{l}_+\hat{l}_- - \hat{l}_-\hat{l}_+)$$

首先 $\langle l_x \rangle = \langle l, m_z | \hat{l}_x | l, m_z \rangle = 0$

$$\langle l_y \rangle = \langle l, m_z | \hat{l}_y | l, m_z \rangle = 0$$

$$\langle \hat{l}_x^2 \rangle = \frac{1}{4}\langle l, m_z | \hat{l}_+\hat{l}_- + \hat{l}_-\hat{l}_+ | l, m_z \rangle$$

$$= \frac{1}{2}\langle l, m_z | \hat{l}^2 - \hat{l}_z^2 | l, m_z \rangle$$

$$= \frac{\hbar^2}{2}[l(l+1) - m_z^2]$$

由对称性：

$$\langle \hat{l}_y^2 \rangle = \langle \hat{l}_x^2 \rangle = \frac{\hbar^2}{2}[l(l+1) - m_z^2]$$

则 $\Delta l_x \Delta l_y = \sqrt{\frac{\hbar^2}{2}[l(l+1)-m_z^2]}\sqrt{\frac{\hbar^2}{2}[l(l+1)-m_z^2]}$

$$= \frac{\hbar^2}{2}[l(l+1)-m_z^2]$$

而 $\langle [\hat{l}_x, \hat{l}_y] \rangle = \langle i\hbar \hat{l}_z \rangle = i\hbar m_z \hbar = i m_z \hbar^2$

而 $\frac{1}{2}|\langle [\hat{l}_x, \hat{l}_y] \rangle| = \frac{1}{2} m_z \hbar^2$

则 $\Delta l_x \Delta l_y - \frac{1}{2}|\langle [\hat{l}_x, \hat{l}_y] \rangle|$

$$= \frac{\hbar^2}{2}[l(l+1) - m_z^2 - m_z] \geq 0$$

即 $\Delta l_x \Delta l_y \geq \frac{1}{2}|\langle [\hat{l}_x, \hat{l}_y] \rangle|$ 满足不确定性关系。

3. 考虑以$\hat{\vec{\ell}}^2$和$\hat{\ell}_z$的共同本征态$|\ell, m_z\rangle$为基的 Hilbert 空间中，$\ell = 1$的子空间。

a. 写出算符$\hat{\ell}_z$和$\hat{\ell}_\pm$的矩阵表示。

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d. 在$(\hat{\vec{\ell}}^2,\hat{\ell}_z)$确立的表象下分别求$\hat{\ell}_z$及$\hat{\ell}_x$的本征态。

e. 求由$(\hat{\vec{\ell}}^2,\hat{\ell}_z)$确立的表象到由$(\hat{\vec{\ell}}^2,\hat{\ell}_x)$确立的表象之间的变换矩阵 S，并验证 b 中关于$\hat{\ell}_x$的结论。

f. 在坐标空间中写出$(\hat{\vec{\ell}}^2,\hat{\ell}_x)$的共同本征态波函数。

a. $l=1$ 时，m_z 可以取 $-1, 0, 1$.

$\hat{l}_z$ 对应的矩阵为
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \rightarrow |1,-1\rangle \\ \rightarrow |1,0\rangle \\ \rightarrow |1,1\rangle \end{matrix}$$

$$\hat{l}_+|1,-1\rangle = \sqrt{(1+1)(1-1+1)}\,|1,0\rangle = \sqrt{2}\,|1,0\rangle$$
$$\hat{l}_+|1,0\rangle = \sqrt{(1-0)(1+0+1)}\,|1,1\rangle = \sqrt{2}\,|1,1\rangle$$
$$\hat{l}_+|1,1\rangle = 0$$
$$\hat{l}_-|1,1\rangle = \sqrt{(1+1)(1-1+1)}\,|1,0\rangle = \sqrt{2}\,|1,0\rangle$$
$$\hat{l}_-|1,0\rangle = \sqrt{(1+0)(1-0+1)}\,|1,-1\rangle = \sqrt{2}\,|1,-1\rangle$$
$$\hat{l}_-|1,-1\rangle = 0$$

则 $\langle 1,-1|\hat{l}_+|1,-1\rangle = 0$ $\quad\langle 1,-1|\hat{l}_+|1,0\rangle = 0$

$\langle 1,-1|\hat{l}_+|1,1\rangle = 0$ $\quad\langle 1,0|\hat{l}_+|1,-1\rangle = \sqrt{2}\hbar$

$\langle 1,0|\hat{l}_+|1,0\rangle = 0$ $\quad\langle 1,0|\hat{l}_+|1,1\rangle = 0$

$\langle 1,1|\hat{l}_+|1,-1\rangle = 0$ $\quad\langle 1,1|\hat{l}_+|1,0\rangle = \sqrt{2}\hbar$

$\langle 1,1|\hat{l}_+|1,1\rangle = 0$ $\quad\langle 1,-1|\hat{l}_-|1,-1\rangle = 0$

$\langle 1,-1|\hat{l}_-|1,0\rangle = \sqrt{2}\hbar$ $\quad\langle 1,-1|\hat{l}_-|1,1\rangle = 0$

$\langle 1,0|\hat{l}_-|1,-1\rangle = 0$ $\quad\langle 1,0|\hat{l}_-|1,0\rangle = 0$

$\langle 1,0|\hat{l}_-|1,1\rangle = \sqrt{2}\hbar$ $\quad\langle 1,1|\hat{l}_-|1,-1\rangle = 0$

$\langle 1,1|\hat{l}_-|1,0\rangle = 0 \quad \langle 1,1|\hat{l}_-|1,1\rangle = 0$

则 $\hat{l}_+$ 的矩阵表示为 $\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\hbar$

$\hat{l}_-$ 的矩阵表示为 $\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}\hbar$

b. 由于 $\hat{l}_x = \frac{1}{2}(\hat{l}_+ + \hat{l}_-)$

$\hat{l}_y = \frac{1}{2i}(\hat{l}_+ - \hat{l}_-)$

则 $\hat{l}_x$, $\hat{l}_y$ 对应的矩阵表示分别为

$\frac{\sqrt{2}}{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \frac{\sqrt{2}}{2i}\hbar\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

c. $\hat{l}_x^2$ 的矩阵表示为 $\hat{l}_x$ 对应矩阵的平方.

即 $\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

令 $\det\left|\frac{\hbar^2}{2}\begin{pmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-1 \end{pmatrix}\right| = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = \frac{\hbar^2}{2} \quad \lambda_3 = \hbar^2$

即本征值为 $0, \frac{1}{2}\hbar^2, \hbar^2$

d. $\hat{l}_z$ 的本征态为 $|1,-1\rangle = (1,0,0)^T \quad |1,0\rangle = (0,1,0)^T$

$|1,1\rangle = (0,0,1)^T$

令 $\det\left|\frac{\sqrt{2}}{2}\hbar\begin{pmatrix}\lambda & -1 & 0\\ -1 & \lambda & -1\\ 0 & -1 & \lambda\end{pmatrix}\right| = 0 \Rightarrow \lambda_1 = -\hbar,\ \lambda_2 = 0\ \lambda_3 = \hbar$

$\lambda_1 = -\hbar$，$\lambda_2 = 0$，$\lambda_3 = \hbar$ 对应的归一化本征向量分别为

$|\alpha\rangle = \frac{1}{2}(1, -\sqrt{2}, 1)^T \qquad |\beta\rangle = \frac{1}{2}(-\sqrt{2}, 0, \sqrt{2})^T$

$|\gamma\rangle = \frac{1}{2}(1, \sqrt{2}, 1)^T$

e.

$$S = \frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}$$

验证：设 $\hat{l}_x$ 在 $|l, m_z\rangle$ 表象下对应矩阵为 l_x

$$S l_x S = \frac{\sqrt{2}\hbar}{8}\begin{pmatrix}1 & -\sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & -\sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & \sqrt{2} & 1\end{pmatrix}$$

$$= \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \longrightarrow \text{对角阵.}$$

f. $|\alpha\rangle = \frac{1}{\sqrt{2}}|-1\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$|\beta\rangle = \frac{1}{2}|-1\rangle + \frac{\sqrt{2}}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|\gamma\rangle = \frac{1}{2}|-1\rangle - \frac{\sqrt{2}}{2}|0\rangle + \frac{1}{2}|1\rangle$

其中 $|-1\rangle = Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}$.

$|0\rangle = Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$

$|1\rangle = Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}$

4. 证明：

a. $\hat{J}_{\pm}\hat{J}_{\mp} = \hat{J}^2 - \hat{J}_z^2 \pm \hbar\hat{J}_z$

b. $\langle jm|\hat{J}_{\mp}\hat{J}_{\pm}|jm\rangle = j(j+1)\hbar^2 - m(m\pm 1)\hbar^2$

证明：

a. $\hat{J}_{\pm}\hat{J}_{\mp} = (\hat{J}_x \pm i\hat{J}_y)(\hat{J}_x \mp i\hat{J}_y)$

$= \hat{J}_x^2 + \hat{J}_y^2 \mp i(J_xJ_y - J_yJ_x)$

$= \hat{J}^2 - \hat{J}_z^2 \mp i(i\hbar\hat{J}_z)$

$= \hat{J}^2 - \hat{J}_z \pm \hbar\hat{J}_z$

b. $\langle j,m|\hat{J}_{\mp}\hat{J}_{\pm}|jm\rangle$

$= \langle j,m|\hat{J}^2 - \hat{J}_z^2 \mp \hbar\hat{J}_z|j,m\rangle$

$= j(j+1)\hbar^2 - m^2\hbar^2 \mp m\hbar^2$

$= j(j+1)\hbar^2 - m(m\pm 1)\hbar^2$

hw8-answer

1. 解：(1) 球谐函数 Y_{lm}：

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$

$$\Rightarrow \begin{cases} \cos\theta = \sqrt{\frac{4\pi}{3}}\,Y_{1,0} = \frac{z}{r} \\ \sin\theta\cos\varphi = \sqrt{\frac{2\pi}{3}}\,(Y_{1,-1} - Y_{1,1}) = \frac{x}{r} \\ \sin\theta\sin\varphi = i\sqrt{\frac{2\pi}{3}}\,(Y_{1,-1} + Y_{1,1}) = \frac{y}{r} \end{cases}$$

$$\psi = A(x+y+2z)\exp(-\alpha r)$$

$$= A r \exp(-\alpha r)\sqrt{\frac{2\pi}{3}}\left[(i-1)Y_{1,1} + 2\sqrt{2}\,Y_{1,0} + (i+1)Y_{1,-1}\right] \qquad (*)$$

注意 Y_{lm} 为 $|lm\rangle$ 本征值为 $m\hbar$ 的本征态的坐标表象，

故 $\hat{l}_z$ 的可能测值及相应概率为

$$\begin{cases} \hbar: & P_1(l_z=\hbar) = |1-i|^2 = 2 \\ 0: & P_2(l_z=0) = |2\sqrt{2}|^2 = 8 \\ -\hbar: & P_3(l_z=-\hbar) = |1+i|^2 = 2 \end{cases} \xRightarrow{\text{归一化}} \begin{aligned} P_1 &= \frac{1}{6} \\ P_2 &= \frac{2}{3} \\ P_3 &= \frac{1}{6} \end{aligned}$$

(2) 把 (*) 式展到 $\hat{l}_x$ 的本征态上去

$$\hat{j}_z|jm\rangle = m\hbar|jm\rangle \Rightarrow \hat{l}_z: \begin{array}{c} \begin{array}{ccc} |1,1\rangle & |1,0\rangle & |1,-1\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar \end{array}$$

$$\begin{cases} \langle l'm'|\hat{l}_x|lm\rangle = \langle l'm'|\frac{1}{2}(\hat{l}_+ + \hat{l}_-)|lm\rangle \\ \hat{l}_+|1,1\rangle = 0 \qquad \hat{l}_-|1,1\rangle = \sqrt{2}\hbar|1,0\rangle \\ \hat{l}_+|1,0\rangle = \sqrt{2}\hbar|1,1\rangle \qquad \hat{l}_-|1,0\rangle = \sqrt{2}\hbar|1,-1\rangle \\ \hat{l}_+|1,-1\rangle = \sqrt{2}\hbar|1,0\rangle \qquad \hat{l}_-|1,-1\rangle = 0 \end{cases} \Rightarrow \hat{l}_x: \begin{pmatrix} 0 & \frac{\sqrt{2}}{2}\hbar & 0 \\ \frac{\sqrt{2}}{2}\hbar & 0 & \frac{\sqrt{2}}{2}\hbar \\ 0 & \frac{\sqrt{2}}{2}\hbar & 0 \end{pmatrix}$$

解 $\hat{l}_x$ 的本征问题：$\frac{\sqrt{2}}{2}\hbar\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\lambda\begin{pmatrix}a\\b\\c\end{pmatrix}$，$\lambda=\pm\hbar, 0$

$\lambda=0 \Rightarrow \frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} \to |l=1, m_x=0\rangle = \frac{\sqrt{2}}{2}|m_z=1\rangle - \frac{\sqrt{2}}{2}|m_z=-1\rangle$

$\lambda=\pm\hbar \Rightarrow \frac{1}{2}\begin{pmatrix}1\\\pm\sqrt{2}\\1\end{pmatrix} \to |l=1, m_x=\pm1\rangle = \frac{1}{2}|m_z=1\rangle \pm \frac{\sqrt{2}}{2}|m_z=0\rangle + \frac{1}{2}|m_z=-1\rangle$

即 $\begin{cases} Y'_{1,0} = \frac{\sqrt{2}}{2}(Y_{1,1} - Y_{1,-1}) \\ Y'_{1,\pm1} = \frac{1}{2}(Y_{1,1} \pm \sqrt{2}\, Y_{1,0} + Y_{1,-1}) \end{cases}$

变换矩阵 $S = S^+$

$\xrightarrow{\text{反逆}} \begin{cases} Y_{1,0} = \frac{\sqrt{2}}{2}(Y'_{1,1} - Y'_{1,-1}) \\ Y_{1,\pm1} = \frac{1}{2}(Y'_{1,1} \pm \sqrt{2}\, Y'_{1,0} + Y'_{1,-1}) \end{cases}$

代入 (*) 式，有

$$\begin{aligned}\varphi &\propto [(i-1)Y_{1,1} + 2\sqrt{2}\, Y_{1,0} + (i+1)Y_{1,-1}] \\ &= (i-1)\frac{1}{2}(Y'_{1,1} + \sqrt{2}\, Y'_{1,0} + Y'_{1,-1}) + 2\sqrt{2}\cdot\frac{\sqrt{2}}{2}(Y'_{1,1} - Y'_{1,-1}) \\ &\quad + (i+1)\frac{1}{2}(Y'_{1,1} - \sqrt{2}\, Y'_{1,0} + Y'_{1,-1}) \\ &= (2+i)Y'_{1,1} - \sqrt{2}\, i\, Y'_{1,0} + (-2+i)Y'_{1,-1}\end{aligned}$$

同理，$\hat{l}_x$ 的可能测值与相应概率为

$$\begin{cases} \hbar: & P_1 = |2+i|^2 = 5 \\ 0: & P_2 = |-\sqrt{2}i|^2 = 2 \\ -\hbar: & P_3 = |-2+i|^2 = 5 \end{cases} \xrightarrow{\text{归一}} \begin{aligned} P_1 &= \frac{5}{12} \\ P_2 &= \frac{1}{6} \\ P_3 &= \frac{5}{12} \end{aligned}$$

法二：原式为 $\psi_1=\varphi(x,y,z)$，对新波函数 $\psi_2=\varphi(z,x,y)$

测量 $\hat{l}_z^{(2)}$，结果与原 $\hat{l}_x^{(1)}$ 等效.

2.解: $\hat{l}_x = \frac{1}{2}(\hat{l}_+ + \hat{l}_-) \qquad \hat{l}_y = \frac{1}{2i}(\hat{l}_+ - \hat{l}_-)$

$$\hat{l}_x^2 = \frac{1}{4}(\hat{l}_+^2 + \hat{l}_-^2 + \hat{l}_+\hat{l}_- + \hat{l}_-\hat{l}_+)$$

$$\hat{l}_y^2 = -\frac{1}{4}(\hat{l}_+^2 + \hat{l}_-^2 - \hat{l}_+\hat{l}_- - \hat{l}_-\hat{l}_+)$$

首先 $\langle l_x \rangle = \langle l, m_z | \hat{l}_x | l, m_z \rangle = 0$

$\langle l_y \rangle = \langle l, m_z | \hat{l}_y | l, m_z \rangle = 0$

$$\langle \hat{l}_x^2 \rangle = \frac{1}{4}\langle l, m_z | \hat{l}_+\hat{l}_- + \hat{l}_-\hat{l}_+ | l, m_z \rangle$$

$$= \frac{1}{2}\langle l, m_z | \hat{l}^2 - \hat{l}_z^2 | l, m_z \rangle$$

$$= \frac{\hbar^2}{2}[l(l+1) - m_z^2]$$

由对称性:

$$\langle \hat{l}_y^2 \rangle = \langle \hat{l}_x^2 \rangle = \frac{\hbar^2}{2}[l(l+1) - m_z^2]$$

则 $\Delta l_x \Delta l_y = \sqrt{\frac{\hbar^2}{2}[l(l+1)-m_z^2]}\sqrt{\frac{\hbar^2}{2}[l(l+1)-m_z^2]}$

$$= \frac{\hbar^2}{2}[l(l+1)-m_z^2]$$

而 $\langle [\hat{l}_x, \hat{l}_y] \rangle = \langle i\hbar \hat{l}_z \rangle = i\hbar m_z \hbar = i m_z \hbar^2$

而 $\frac{1}{2}|\langle [\hat{l}_x, \hat{l}_y] \rangle| = \frac{1}{2} m_z \hbar^2$

则 $\Delta l_x \Delta l_y - \frac{1}{2}|\langle [\hat{l}_x, \hat{l}_y] \rangle|$

$$= \frac{\hbar^2}{2}[l(l+1) - m_z^2 - m_z] \geq 0$$

即 $\Delta l_x \Delta l_y \geq \frac{1}{2}|\langle [\hat{l}_x, \hat{l}_y] \rangle|$ 满足不确定性关系。

3. 解：以 $\{ |m_z=-1\rangle, |0\rangle, |1\rangle \}$ 为基

a. $$l_z = \hbar \begin{pmatrix} -1 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

由 $\hat{l}_\pm |l m\rangle = \hbar \sqrt{l(l+1)-m(m\pm1)}\, |l, m\pm1\rangle$

知 $\langle l m| \hat{l}_\pm | l m'\rangle = \sqrt{2-m(m\pm1)}\, \delta_{m, m'\pm1}$

$$\Rightarrow \quad \hat{l}_+ = \hbar \begin{pmatrix} 0 & & \\ \sqrt{2} & 0 & \\ & \sqrt{2} & 0 \end{pmatrix}, \quad \hat{l}_- = \hbar \begin{pmatrix} 0 & \sqrt{2} & \\ & 0 & \sqrt{2} \\ & & 0 \end{pmatrix}$$

b.

$$\hat{l}_x = \frac{1}{2}(\hat{l}_+ + \hat{l}_-) = \frac{\sqrt{2}}{2}\hbar \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}$$

$$\hat{l}_y = \frac{1}{2i}(\hat{l}_+ - \hat{l}_-) = \frac{\sqrt{2}\hbar}{2i} \begin{pmatrix} & -1 & \\ 1 & & -1 \\ & 1 & \end{pmatrix}$$

c. $$\hat{l}_x^2 = \left(\frac{\sqrt{2}}{2}\hbar \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \right)^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & & 1 \\ & 2 & \\ 1 & & 1 \end{pmatrix}$$

$$|\hat{l}_x^2 - \lambda I| = 0 \quad \Rightarrow \quad \lambda_1 = 0, \ \lambda_2 = \lambda_3 = \hbar^2$$

d. (上课笔记及第1.题已解过，不过基的顺序不同)

本征值	$\hat{L}_z$ 本征态	$\hat{L}_x$ 本征态
$-\hbar$	$(1,0,0)^T$	$\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right)^T$
0	$(0,1,0)^T$	$\left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)^T$
$\hbar$	$(0,0,1)^T$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)^T$

e.

$$S = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

f. $Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$

$$\langle \vec{r} | 1, m_x = 0\rangle = \frac{\sqrt{2}}{2}\left(Y_{1,1} - Y_{1,-1}\right)$$

$$\langle \vec{r} | 1, m_x = \pm 1\rangle = \frac{1}{2}\left(Y_{1,1} \pm \sqrt{2}\, Y_{1,0} + Y_{1,-1}\right)$$

4. 证明：

a. $\hat{J}_{\pm}\hat{J}_{\mp} = \hat{J}^2 - \hat{J}_z^2 \pm \hbar\hat{J}_z$

b. $\langle jm|\hat{J}_{\mp}\hat{J}_{\pm}|jm\rangle = j(j+1)\hbar^2 - m(m\pm 1)\hbar^2$

证明：

a. $\hat{J}_{\pm}\hat{J}_{\mp} = (\hat{J}_x \pm i\hat{J}_y)(\hat{J}_x \mp i\hat{J}_y)$

$= \hat{J}_x^2 + \hat{J}_y^2 \mp i(J_xJ_y - J_yJ_x)$

$= \hat{J}^2 - \hat{J}_z^2 \mp i(i\hbar\hat{J}_z)$

$= \hat{J}^2 - \hat{J}_z \pm \hbar\hat{J}_z$

b. $\langle j,m|\hat{J}_{\mp}\hat{J}_{\pm}|jm\rangle$

$= \langle j,m|\hat{J}^2 - \hat{J}_z^2 \mp \hbar\hat{J}_z|j,m\rangle$

$= j(j+1)\hbar^2 - m^2\hbar^2 \mp m\hbar^2$

$= j(j+1)\hbar^2 - m(m\pm 1)\hbar^2$