量子物理第五至七章习题答案

第五章

1. 因为

$$[B,C]=iA,$$

所以只需证明 $[B,C]B+B[B,C]=0$ 与 $[B,C]C+C[B,C]=0$ 即可。

$$[B,C]B+B[B,C]=BCB-CBB+BBC-BCB=0,$$

$$[B,C]C+C[B,C]=BCC-CBC+CBC-CCB=0,$$

证明完毕。

2.

(1)

$$B^2=A^\dagger AA^\dagger A=A^\dagger(I-A^\dagger A)A=A^\dagger A-(A^\dagger)^2A^2=A^\dagger A=B.$$

(2)

记 B 的特征向量为 $|0\rangle, |1\rangle$ ，对应特征值为 $0, 1$ ，因此

$$\begin{cases} B|0\rangle=0, \\ B|1\rangle=|1\rangle. \end{cases}$$

A 第 $i+1$ 行，第 $j+1$ 列 $(i,j=0,1)$ 的元素为 $\left\langle i\left|A\right|j\right\rangle$ ，而

$$A=IA=(AA^\dagger+A^\dagger A)A=AA^\dagger A+A^\dagger A^2=AA^\dagger A=AB,$$

因此

$$\begin{aligned}\left\langle 0\left|A\right|0\right\rangle &=\left\langle 0\left|AB\right|0\right\rangle \\ &=\left\langle A^\dagger|0\rangle, B|0\rangle\right\rangle \\ &=0,\end{aligned}$$

$$\begin{aligned}\left\langle 1\left|A\right|0\right\rangle &=\left\langle 1\left|AB\right|0\right\rangle \\ &=\left\langle A^\dagger|1\rangle, B|0\rangle\right\rangle \\ &=0,\end{aligned}$$

但是 $\left\langle 0\left|A\right|1\right\rangle, \left\langle 1\left|A\right|1\right\rangle$ 的元素目前无法得知。假设它们分别为 a,b ，因此

$$A = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix},$$

根据 $A^2 = 0,\ AA^\dagger + A^\dagger A = I$ 可得

$$\begin{cases} ab = 0, \\ b^2 = 0, \\ \left|a\right|^2 = 1, \\ \left|a\right|^2 + 2\left|b\right|^2 = 1, \end{cases}$$

因此

$$\begin{cases} a = e^{i\theta}, \\ b = 0, \end{cases}$$

其中 $\theta \in \mathbb{R}$ 是常数。因此

$$A = \begin{pmatrix} 0 & e^{i\theta} \\ 0 & 0 \end{pmatrix},$$

用狄拉克符号可以表示为

$$A = e^{i\theta} \left|0\right\rangle \left\langle 1\right| .$$

3. 记 $\mathbf{A} = \left(A^1, A^2, A^3\right), \mathbf{B} = \left(B^1, B^2, B^3\right), \sigma = \left(\sigma_1, \sigma_2, \sigma_3\right),$

其中

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

因此

$$\left(\mathbf{A} \cdot \sigma\right)\left(\mathbf{B} \cdot \sigma\right) = A^i \sigma_i B^j \sigma_j = A^i B^j \sigma_i \sigma_j,$$

$$\left(\mathbf{A} \cdot \mathbf{B}\right) I + i \left(\mathbf{A} \times \mathbf{B}\right) \cdot \sigma = (\delta_{ij} A^i B^j) I + i \epsilon_{ij}^k A^i B^j \sigma_k,$$

而

$$\left[\sigma_i, \sigma_j\right] = \sigma_i \sigma_j - \sigma_j \sigma_i = 2i \epsilon_{ij}^k \sigma_k,$$

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{jk}I,$$

因此

$$\begin{aligned}A^iB^j\sigma_i\sigma_j - (\delta_{ij}A^iB^j)I - i\epsilon_{ij}^kA^iB^j\sigma_k &= A^iB^j\left(\sigma_i\sigma_j - \delta_{ij}I - i\epsilon_{ij}^k\sigma_k\right)\\ &= A^iB^j\left(\sigma_i\sigma_j - \frac{1}{2}\{\sigma_i, \sigma_j\} - \frac{1}{2}[\sigma_i, \sigma_j]\right)\\ &= 0,\end{aligned}$$

即 $(\mathbf{A}\cdot\sigma)(\mathbf{B}\cdot\sigma) = (\mathbf{A}\cdot\mathbf{B})I + i(\mathbf{A}\times\mathbf{B})\cdot\sigma$ 。

4.
SU(2) 的一般形式为

$$U = \begin{pmatrix}\alpha & -\bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix},$$

其中 $|\alpha|^2 + |\beta|^2 = 1.$

(红色部分不需要抄)
SU(2) 与单位球面 $\mathbb{S}^3 = \{(w, x, y, z) : w^2 + x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^4$ 是同胚的，对应的同胚映射为

$$\begin{pmatrix}\alpha & -\bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix} \mapsto (\mathrm{Re}(\alpha), \mathrm{Im}(\alpha), \mathrm{Re}(\beta), \mathrm{Im}(\beta)).$$

$C_2 = \{I, -I\}$ 是 SU(2) 的正规子群，并且 $\mathrm{SU}(2)/C_2 \cong \mathrm{SO}(3)$ ，对应的同态映射为

$$U \mapsto \frac{1}{2}\begin{pmatrix}\mathrm{tr}(\sigma_1U\sigma_1U^{-1}) & \mathrm{tr}(\sigma_1U\sigma_2U^{-1}) & \mathrm{tr}(\sigma_1U\sigma_3U^{-1})\\ \mathrm{tr}(\sigma_2U\sigma_1U^{-1}) & \mathrm{tr}(\sigma_2U\sigma_2U^{-1}) & \mathrm{tr}(\sigma_2U\sigma_3U^{-1})\\ \mathrm{tr}(\sigma_3U\sigma_1U^{-1}) & \mathrm{tr}(\sigma_3U\sigma_2U^{-1}) & \mathrm{tr}(\sigma_3U\sigma_3U^{-1})\end{pmatrix},$$

该映射的核为 C_2 。因此，SU(2) 是 SO(3) 的覆叠空间 (Covering Space)， SO(3) 同胚于 $\mathbb{RP}^3$ 。

5.
因为 $\sigma_1^2 = I$ ，所以

$$\begin{aligned}e^{i\theta\sigma_1} &= \sum_{k=0}^{\infty}\frac{(i\theta\sigma_1)^k}{k!}\\ &= \sum_{k=0}^{\infty}\frac{(i\theta)^{2k}}{(2k)!}I + \frac{(i\theta)^{2k+1}}{(2k+1)!}\sigma_1\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!}I + \frac{i(-1)^k\theta^{2k+1}}{(2k+1)!}\sigma_1\\ &= (\cos\theta)I + (i\sin\theta)\sigma_1,\end{aligned}$$

因为 $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3, \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1, \sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2$，所以

$$\begin{aligned} e^{-i\frac{\alpha}{2}\sigma_1}\sigma_2 e^{i\frac{\alpha}{2}\sigma_1} &= \left(\cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}\sigma_1\right)\sigma_2\left(\cos\frac{\alpha}{2}I + i\sin\frac{\alpha}{2}\sigma_1\right) \\ &= \left(\cos\frac{\alpha}{2}\sigma_2 + \sin\frac{\alpha}{2}\sigma_3\right)\left(\cos\frac{\alpha}{2}I + i\sin\frac{\alpha}{2}\sigma_1\right) \\ &= \left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right)\sigma_2 + 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sigma_3 \\ &= (\cos\alpha)\sigma_2 + (\sin\alpha)\sigma_3, \end{aligned}$$

$$\begin{aligned} e^{-i\frac{\alpha}{2}\sigma_1}\sigma_3 e^{i\frac{\alpha}{2}\sigma_1} &= \left(\cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}\sigma_1\right)\sigma_3\left(\cos\frac{\alpha}{2}I + i\sin\frac{\alpha}{2}\sigma_1\right) \\ &= \left(\cos\frac{\alpha}{2}\sigma_3 - \sin\frac{\alpha}{2}\sigma_2\right)\left(\cos\frac{\alpha}{2}I + i\sin\frac{\alpha}{2}\sigma_1\right) \\ &= \left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right)\sigma_3 - 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sigma_3 \\ &= (\cos\alpha)\sigma_3 - (\sin\alpha)\sigma_2\,. \end{aligned}$$

所以题目错了。

6.
因为

$$\begin{aligned} S_1\,|\uparrow\rangle &= \frac{\hbar}{2}\,|\downarrow\rangle, \\ S_1\,|\downarrow\rangle &= \frac{\hbar}{2}\,|\uparrow\rangle, \\ S_2\,|\uparrow\rangle &= \frac{i\hbar}{2}\,|\downarrow\rangle, \\ S_2\,|\downarrow\rangle &= -\frac{i\hbar}{2}\,|\uparrow\rangle, \end{aligned}$$

所以

$$\begin{aligned} S_1^2\,|\uparrow\rangle &= \frac{\hbar^2}{4}\,|\uparrow\rangle, \\ S_2^2\,|\uparrow\rangle &= \frac{\hbar^2}{4}\,|\uparrow\rangle, \end{aligned}$$

$$
\begin{aligned}
\langle S_1\rangle &= \left\langle \uparrow \left| S_1 \right| \uparrow \right\rangle = \frac{\hbar}{2}\langle \uparrow \mid \downarrow \rangle = 0,\\
\langle S_1^2\rangle &= \left\langle \uparrow \left| S_1^2 \right| \uparrow \right\rangle = \frac{\hbar^2}{4}\langle \uparrow \mid \uparrow \rangle = \frac{\hbar^2}{4},\\
\langle S_2\rangle &= \left\langle \uparrow \left| S_2 \right| \uparrow \right\rangle = \frac{i\hbar}{2}\langle \uparrow \mid \downarrow \rangle = 0,\\
\langle S_2^2\rangle &= \left\langle \uparrow \left| S_2^2 \right| \uparrow \right\rangle = \frac{\hbar^2}{4}\langle \uparrow \mid \uparrow \rangle = \frac{\hbar^2}{4},
\end{aligned}
$$

$$
\begin{aligned}
(\Delta S_1)^2 &= \langle S_1^2\rangle - \langle S_1\rangle^2 = \frac{\hbar^2}{4},\\
(\Delta S_2)^2 &= \langle S_2^2\rangle - \langle S_2\rangle^2 = \frac{\hbar^2}{4}.
\end{aligned}
$$

7.
此时厄米算子为

$$
\sigma\cdot\mathbf{n} = \sigma_1 \sin\theta\cos\phi + \sigma_2\sin\theta\sin\phi + \sigma_3\cos\theta,
$$

记 σ_3 的特征向量为

$$
\alpha = (1,\ 0)^T, \beta = (0,\ 1)^T,
$$

可以计算得出

$$
\begin{aligned}
\sigma_1\alpha &= \beta,\\
\sigma_1\beta &= \alpha,\\
\sigma_2\alpha &= i\beta,\\
\sigma_2\beta &= -i\alpha,\\
\sigma_3\alpha &= \alpha,\\
\sigma_3\beta &= -\beta,
\end{aligned}
$$

令 $\sigma\cdot\mathbf{n}$ 的特征向量为 $\left(v^1, v^2\right)^T = v^1\alpha + v^2\beta$ ，因此

$$
(\sigma_1\sin\theta\cos\phi + \sigma_2\sin\theta\sin\phi + \sigma_3\cos\theta)(v^1\alpha + v^2\beta) = \lambda(v^1\alpha + v^2\beta),
$$

化简可得

$$
\begin{aligned}
v^1(\cos\theta - \lambda) + v^2(\sin\theta\cos\phi - i\sin\theta\sin\phi) &= 0,\\
v^1(\sin\theta\cos\phi + i\sin\theta\sin\phi) - v^2(\cos\theta + \lambda) &= 0,
\end{aligned}
$$

上述方程有非零解 $\left\{v^1, v^2\right\}$ 当且仅当

$$
\det\begin{pmatrix} \cos\theta - \lambda & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta - \lambda \end{pmatrix} = 0,
$$

可得

$$\lambda = \pm 1,$$

当 $\lambda = 1$ 时

$$v^2 = v^1 \tan\frac{\theta}{2} e^{i\phi},$$

根据归一化条件可得

$$\left|v^1\right|^2 + \left|v^2\right|^2 = 1,$$

因此

$$v^1 = \cos\frac{\theta}{2},$$
$$v^2 = \sin\frac{\theta}{2} e^{i\phi}.$$

当 $\lambda = 1$ 时

$$v^2 = -v^1 \cot\frac{\theta}{2} e^{i\phi},$$

根据归一化条件可得

$$\left|v^1\right|^2 + \left|v^2\right|^2 = 1,$$

因此

$$v^1 = -\sin\frac{\theta}{2},$$
$$v^2 = \cos\frac{\theta}{2} e^{i\phi}.$$

综上所述，本征态为

$$\chi_1 = \cos\frac{\theta}{2}\alpha + \sin\frac{\theta}{2} e^{i\phi}\beta,$$
$$\chi_2 = -\sin\frac{\theta}{2}\alpha + \cos\frac{\theta}{2} e^{i\phi}\beta,$$

而

$$\left|\uparrow\right\rangle = \alpha = \cos\frac{\theta}{2}\chi_1 - \sin\frac{\theta}{2}\chi_2,$$

因此，可能的测量值为 ± 1 ，相应的概率分别为 $\cos^2\frac{\theta}{2}, \sin^2\frac{\theta}{2}$ 。

8.
总角动量 J 的可能取值为

$$J = L + S, L + S - 1, \ldots, \left| L - S \right|,$$

当 $L = 2$, $S = 1$ 时

$$J = 3,\ 2,\ 1,$$

$$\hat{L} \cdot \hat{S} = \frac{1}{2}\left(J(J+1) - L(L+1) - S(S+1)\right)\hbar^2 = 2\hbar^2,\ -\hbar^2,\ -3\hbar^2.$$

9.
$^2\mathrm{D}_{3/2}$ 对应

$$L = 2,\ S = 1/2,\ J = 3/2,$$

因此

$$\hat{L} \cdot \hat{S} = \frac{1}{2}\left(J(J+1) - L(L+1) - S(S+1)\right)\hbar^2 = -\frac{3}{2}\hbar^2,$$

郎德因子为

$$g = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)} = \frac{4}{5},$$

测量到的磁矩值为

$$m = g\mu_B = \frac{4}{5}\mu_B.$$

10.
磁场为

$$B = \frac{\mu_0}{4\pi}\left(\frac{Ze}{m_e r^3}\right)L,$$

其中 $Z = 1$ 是氢原子的电荷数，r 是氢原子的轨道半径，L 是氢原子的角动量。2p 态对应

$$n = 2,\ l = 1,$$

因此氢原子的轨道半径为

$$r = n^2 a_0 = 4a_0,$$

其中 $a_0 = \dfrac{4\pi\epsilon_0\hbar^2}{m_e e^2}$ 是氢原子的经典半径。

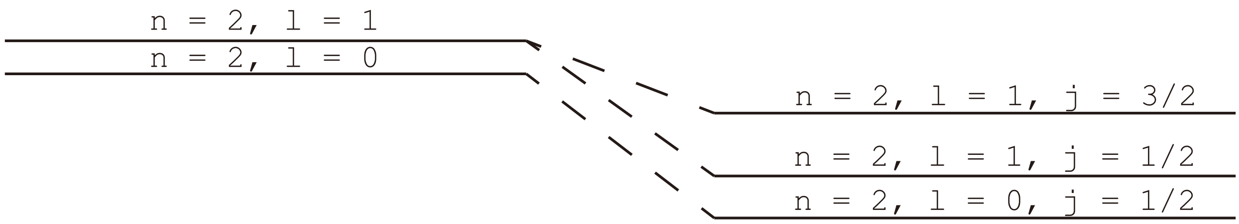
氢原子的角动量为

$$L = \sqrt{l(l+1)}\hbar = \sqrt{2}\hbar,$$

代入数据可得

$$B \approx 0.14\ \mathrm{T}.$$

11.



$n=2, l=1$ 能级间隔为

$$\Delta E = -\frac{E_n \alpha^2}{n\,l(l+1)} = 1.8 \times 10^{-4}\ \mathrm{eV},$$

其中 $E_n = -\dfrac{13.6\ \mathrm{eV}}{n^2}, \alpha = \dfrac{1}{137}$.

因为 $n=2, l=0$ 能级没有分裂，所以没有间隔。

12.

A 处于状态 $\chi^A = |\uparrow\rangle$，B 处于状态 $\chi^B = \dfrac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big)$，因此复合系统的状态为

$$\chi^A \otimes \chi^B = \frac{1}{\sqrt{2}}\Big(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle\Big),$$

总自旋算子 S 的本征态为

$$\begin{aligned}
\chi_{1,1} &= |\uparrow\uparrow\rangle, \\
\chi_{1,0} &= \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big), \\
\chi_{1,-1} &= |\downarrow\downarrow\rangle, \\
\chi_{0,0} &= \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big),
\end{aligned}$$

对应总自旋分别为 1, 1, 1, 0. 可以将复合系统分解为

$$\chi^A \otimes \chi^B = \frac{1}{\sqrt{2}} \chi_{1,1} + \frac{1}{2} \chi_{1,0} + \frac{1}{2} \chi_{0,0},$$

因此，测量到总自旋为 0 状态 $\chi_{0,0}$ 的概率为

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

13.

(1) 在非耦合表象中

$$\begin{aligned} \left| \uparrow\uparrow \right\rangle &\mapsto (1\ 0\ 0\ 0)^T, \\ \left| \uparrow\downarrow \right\rangle &\mapsto (0\ 1\ 0\ 0)^T, \\ \left| \downarrow\uparrow \right\rangle &\mapsto (0\ 0\ 1\ 0)^T, \\ \left| \downarrow\downarrow \right\rangle &\mapsto (0\ 0\ 0\ 1)^T, \end{aligned}$$

因此

$$P_{12} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$P_{12} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

记

$$P_{12} = \begin{pmatrix} A_1^1 & A_2^1 & A_3^1 & A_4^1 \\ A_1^2 & A_2^2 & A_3^2 & A_4^2 \\ A_1^3 & A_2^3 & A_3^3 & A_4^3 \\ A_1^4 & A_2^4 & A_3^4 & A_4^4 \end{pmatrix},$$

因此

$$\begin{aligned} A_2^1 = A_2^2 = A_2^4 = 0, \\ A_3^1 = A_3^3 = A_3^4 = 0, \end{aligned}$$

$$P_{12} = \begin{pmatrix} A_1^1 & 0 & 0 & A_4^1 \\ A_1^2 & 0 & 1 & A_4^2 \\ A_1^3 & 1 & 0 & A_4^3 \\ A_1^4 & 0 & 0 & A_4^4 \end{pmatrix}.$$

又因为 P_{12} 可以对两自旋系统的自旋状态实现交换操作，对于自旋状态

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix},$$

其中

$$|a|^2 + \left|b\right|^2 = 1, |c|^2 + \left|d\right|^2 = 1.$$

$$P_{12} \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \otimes \begin{pmatrix} a \\ b \end{pmatrix},$$

$$P_{12} \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} ac \\ bc \\ ad \\ bd \end{pmatrix},$$

因此

$$\begin{aligned} (A_1^1 - 1)ac + A_4^1 bd = 0, \\ A_1^2 ac + A_4^2 bd = 0, \\ A_1^3 ac + A_4^3 bd = 0, \\ A_1^4 ac + (A_4^4 - 1)bd = 0, \end{aligned}$$

对于任意满足 $|a|^2 + \left|b\right|^2 = 1, |c|^2 + \left|d\right|^2 = 1$ 的 $\{a, b, c, d\}$ 均成立。

取 $a = c = 1, b = d = 0$ 与 $a = c = 0, b = d = 1$ 可得

$$A_1^2 = A_4^2 = A_1^3 = A_4^3 = 0,$$

取 $a = b = c = d = 1/\sqrt{2}$ 可得

$$\begin{aligned} A_1^1 + A_4^1 = 1, \\ A_1^4 + A_4^4 = 1, \end{aligned}$$

因此

$$P_{12} = \begin{pmatrix} 1-\alpha & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 0 & 1-\beta \end{pmatrix},$$

其中

$$\begin{aligned} A_1^1 &= 1 - \alpha, \\ A_4^1 &= \alpha, \\ A_1^4 &= \beta, \\ A_4^4 &= 1 - \beta. \end{aligned}$$

$$
\begin{aligned}
-\alpha ac + \alpha bd &= 0, \\
\beta ac - \beta bd &= 0,
\end{aligned}
$$

根据 ac, bd 的任意性可得

$$\alpha = \beta = 0,$$

因此

$$P_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(2)
很显然

$$P_{12}^2 = I,$$

至于怎么用泡利算子表示我就不知道了，实在是凑不出来。

14.
(1)

$$E = 4 \times 2 \times (-13.6 \text{ eV}) = -108.8 \text{ eV}.$$

(2) 不会，滚你妈的。

15.
2s 组态对应

$$l_1 = 0,\ s_1 = \frac{1}{2},$$

3d 组态对应

$$l_2 = 2,\ s_2 = \frac{1}{2},$$

原子的角动量取值为

$$L = l_1 + l_2, l_1 + l_2 - 1,\ \ldots,\ \left| l_1 - l_2 \right| = 2,$$

原子的自旋取值为

$$S = s_1 + s_2, s_1 + s_2 - 1,\ \ldots,\ \left| s_1 - s_2 \right| = 1,\ 0,$$

原子的总角动量取值为

$$J = L + S, L + S - 1, \ldots, \left| L - S \right|,$$

因此，总共有 4 个组态：
当 $L = 2, S = 1$ 时，$J = 3, 2, 1$ ，对应组态为 ${}^3\mathrm{D}_3, {}^3\mathrm{D}_2, {}^3\mathrm{D}_1$ ；
当 $L = 2, S = 0$ 时，$J = 2$ ，对应组态为 ${}^1\mathrm{D}_2$ 。

16.
(1)

$$E = 4 \times (-13.6\ \mathrm{eV}) + (-13.6\ \mathrm{eV}) = -68.0\ \mathrm{eV}.$$

(2) 不会，滚你妈的。

第六章

1.
因为 $\sigma_3^2 = I$ ，所以

$$\begin{aligned} e^{i\theta\sigma_3} &= \sum_{k=0}^{\infty} \frac{(i\theta\sigma_3)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} I + \frac{(i\theta)^{2k+1}}{(2k+1)!} \sigma_3 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} I + \frac{i(-1)^k \theta^{2k+1}}{(2k+1)!} \sigma_3 \\ &= (\cos\theta) I + (i \sin\theta) \sigma_3. \end{aligned}$$

2.

$$\left\langle \psi_1 \right| = \cos\frac{\theta_1}{2} \left\langle 0 \right| + \sin\frac{\theta_1}{2} \left\langle 1 \right|,$$

$$\begin{aligned} \left| \left\langle \psi_1 \middle| \psi_2 \right\rangle \right|^2 &= \left| \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \left\langle 0 \middle| 0 \right\rangle + \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \left\langle 0 \middle| 1 \right\rangle + \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \left\langle 1 \middle| 0 \right\rangle + \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \left\langle 1 \middle| 1 \right\rangle \right|^2 \\ &= \left| \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \right|^2 \\ &= \cos^2\frac{\theta_1 - \theta_2}{2}. \end{aligned}$$

3.
因为

$$\rho(t) = \left| \psi(t) \right\rangle \left\langle \psi(t) \right|,$$

$$\left| \psi(t) \right\rangle = \left| \psi(0) \right\rangle e^{-iHt/\hbar},$$

所以

$$\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}.$$

因为

$$\begin{aligned}\frac{d}{dt}e^{-itA} &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{(-itA)^k}{k!} \\ &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{(-iA)^k}{k!}t^k \\ &= \sum_{k=1}^{\infty}\frac{(-iA)^k}{(k-1)!}t^{k-1} \\ &= (-iA)\sum_{k=1}^{\infty}\frac{(-iA)^{k-1}}{(k-1)!}t^{k-1} \\ &= (-iA)\sum_{k=0}^{\infty}\frac{(-iA)^k}{k!}t^k \\ &= (-iA)e^{itA},\end{aligned}$$

同理可得 $\frac{d}{dt}e^{itA} = ie^{itA}A$，所以

$$\begin{aligned}\frac{d\rho}{dt} &= \left(\frac{d}{dt}e^{-iHt/\hbar}\right)\rho(0)e^{iHt/\hbar} + e^{-iHt/\hbar}\rho(0)\left(\frac{d}{dt}e^{iHt/\hbar}\right) \\ &= -\frac{iH}{\hbar}e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar} + e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}\frac{iH}{\hbar},\end{aligned}$$

而

$$\left[\rho, H\right] = \left[e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}, H\right] = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}H - He^{-iHt/\hbar}\rho(0)e^{iHt/\hbar},$$

因此

$$\frac{d\rho}{dt} = \frac{i}{\hbar}\left[\rho, H\right].$$

第七章

1.

该量子门等效于酉矩阵

$$(I \otimes H)\mathrm{CNOT}(I \otimes H),$$

而

$$I \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

因此

$$(I \otimes H)\mathrm{CNOT}(I \otimes H) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

这与受控相位门等价。

2.

$$\begin{aligned}
\sigma_{x1}\sigma_{x2} \left|0\right\rangle_1 \left|0\right\rangle_2 &= \left(\sigma_x \left|0\right\rangle\right)_1 \otimes \left(\sigma_x \left|0\right\rangle\right)_2 = \left|1\right\rangle_1 \left|1\right\rangle_2, \\
\sigma_{x1}\sigma_{x2} \left|1\right\rangle_1 \left|1\right\rangle_2 &= \left(\sigma_x \left|1\right\rangle\right)_1 \otimes \left(\sigma_x \left|1\right\rangle\right)_2 = \left|0\right\rangle_1 \left|0\right\rangle_2, \\
\sigma_{y1}\sigma_{y2} \left|0\right\rangle_1 \left|0\right\rangle_2 &= \left(\sigma_y \left|0\right\rangle\right)_1 \otimes \left(\sigma_y \left|0\right\rangle\right)_2 = - \left|1\right\rangle_1 \left|1\right\rangle_2, \\
\sigma_{y1}\sigma_{y2} \left|1\right\rangle_1 \left|1\right\rangle_2 &= \left(\sigma_y \left|1\right\rangle\right)_1 \otimes \left(\sigma_y \left|1\right\rangle\right)_2 = - \left|0\right\rangle_1 \left|0\right\rangle_2, \\
\sigma_{z1}\sigma_{z2} \left|0\right\rangle_1 \left|0\right\rangle_2 &= \left(\sigma_z \left|0\right\rangle\right)_1 \otimes \left(\sigma_z \left|0\right\rangle\right)_2 = \left|0\right\rangle_1 \left|0\right\rangle_2, \\
\sigma_{z1}\sigma_{z2} \left|1\right\rangle_1 \left|1\right\rangle_2 &= \left(\sigma_z \left|1\right\rangle\right)_1 \otimes \left(\sigma_z \left|1\right\rangle\right)_2 = \left|1\right\rangle_1 \left|1\right\rangle_2.
\end{aligned}$$

因此

$$\sigma_{x1}\sigma_{x2}\left|\psi_{1,2}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_1\left|0\right\rangle_2 + \left|1\right\rangle_1\left|1\right\rangle_2\right),$$
$$\sigma_{y1}\sigma_{y2}\left|\psi_{1,2}\right\rangle = -\frac{1}{\sqrt{2}}\left(\left|0\right\rangle_1\left|0\right\rangle_2 + \left|1\right\rangle_1\left|1\right\rangle_2\right),$$
$$\sigma_{z1}\sigma_{z2}\left|\psi_{1,2}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_1\left|0\right\rangle_2 + \left|1\right\rangle_1\left|1\right\rangle_2\right),$$

$$\left\langle\sigma_{x1}\sigma_{x2}\right\rangle = 1,$$
$$\left\langle\sigma_{y1}\sigma_{y2}\right\rangle = -1,$$
$$\left\langle\sigma_{z1}\sigma_{z2}\right\rangle = 1.$$

3.
$1, 2, 3, 4$ 粒子的量子比特状态均为复数域 $\mathbb{C}$ 上的向量空间，分别记为 V_1, V_2, V_3, V_4 。

利用同构关系

$$(V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \cong V_1 \otimes V_2 \otimes V_3 \otimes V_4 \cong V_1 \otimes V_4 \otimes V_2 \otimes V_3 \cong (V_1 \otimes V_4) \otimes (V_2 \otimes V_3),$$

我们可以将 $(V_1 \otimes V_2) \otimes (V_3 \otimes V_4)$ 上的向量 $\left|\psi_{1,2}\right\rangle \otimes \left|\psi_{3,4}\right\rangle$ 一一映射到 $(V_1 \otimes V_4) \otimes (V_2 \otimes V_3)$ 上的向量。

因此

$$\begin{aligned}
\left|\psi_{1,2}\right\rangle \otimes \left|\psi_{3,4}\right\rangle &= \frac{1}{2}\left(\left|0000\right\rangle - \left|0011\right\rangle + \left|1100\right\rangle - \left|1111\right\rangle\right) \\
&\mapsto \frac{1}{2}\left(\left|0000\right\rangle - \left|0101\right\rangle + \left|1010\right\rangle - \left|1111\right\rangle\right) \\
&= \frac{1}{2}\left(\left|00\right\rangle \frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right) - \left|01\right\rangle \frac{1}{\sqrt{2}}\left(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\right) + \left|10\right\rangle \frac{1}{\sqrt{2}}\left(\left|\Psi^+\right\rangle - \left|\Psi^-\right\rangle\right) - \left|11\right\rangle \frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\right)\right) \\
&= \frac{1}{2}\left(\left|\Phi^-\right\rangle\left|\Phi^+\right\rangle + \left|\Phi^+\right\rangle\left|\Phi^-\right\rangle - \left|\Psi^-\right\rangle\left|\Psi^+\right\rangle - \left|\Psi^+\right\rangle\left|\Psi^-\right\rangle\right),
\end{aligned}$$

因此，如果 $2, 3$ 粒子测量得到的 Bell 态分别为 $\left|\Phi^+\right\rangle, \left|\Phi^-\right\rangle, \left|\Psi^+\right\rangle, \left|\Psi^-\right\rangle$ ，那么 $1, 4$ 粒子的状态还是纠缠态，分别对应 $\left|\Phi^-\right\rangle, \left|\Phi^+\right\rangle, \left|\Psi^-\right\rangle, \left|\Psi^+\right\rangle$ 。