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绪论

力学：物体位置、形状变化，$\vec{r}_a = \vec{r}_a(t)$，$(a = 1, 2, \cdots, n)$

- 描述 — 运动学 坐标系、参考系
- 解释 — 动力学 $\vec{F} = m\vec{a}$（惯）

$\vec{F} = -\frac{GMm}{r^3}\vec{r}$ $\vec{F} = -k\vec{r}$ $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

质点、两体

刚体、波（叠加原理）

- 动量：整体位置
- 角动量：整体方位
- 能量

复杂体系（质心系）质点在质心上，随质心平动 受到三个描述方向的自由度 [illegible]

理论力学：介绍另外一些与牛顿方程等价的动力学解释

牛顿方程
力

$$\begin{cases} \vec{r}(t+\varepsilon) = \vec{r}(t) + \varepsilon\vec{v}(t) \\ \vec{v}(t+\varepsilon) = \vec{v}(t) + \varepsilon\vec{a}(t) = \vec{v}(t) + \varepsilon\frac{\vec{F}(t)}{m} \end{cases}$$

考查相邻时刻状态之间的联系

$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$\nabla\cdot\vec{B} = 0$ $\nabla\times\vec{B} = \mu_0\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

分析力学
能量

- 最小作用原理（位形空间）$\frac{\partial L}{\partial q_a} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a}$ 拉格朗日方程
- 最小作用原理（相空间）

哈密顿方程

$$\begin{cases} \dot{q}_a = \frac{\partial H}{\partial p_a} \\ \dot{p}_a = -\frac{\partial H}{\partial q} \end{cases}$$

对于位置和速度同等对待
牛顿力学只给出速度变化的原因

给出一个圆的不同方法：

1. 一个点一个点描绘　　　　牛顿力学

2. 围出最大面积的闭曲线　　拉格朗日力学

3. $g=\frac{u^2}{2a}$　(u,g)　$y=ux-g$　　哈密顿力学

对称性是现代物理的基石之一

空间平移不变性 ⇒ 动量守恒
空间旋转不变性 ⇒ 角动量守恒
时间平移不变性 ⇒ 能量守恒

课程结构

· Lagrange 力学

多自由度微振动 → 场论

· Hamilton 力学

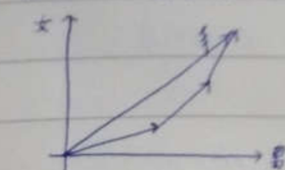
散射

刚体

CH1. 运动学

§1 坐标变换

有大小、有方向、满足三角形法则，称为矢量



$$\vec{r}=\sum_j x_j\hat{x}_j=\sum_j(\vec{r}\cdot\hat{x}_j)\hat{x}_j$$

$$=\sum_j x_j'\hat{x}_j'=\sum_j(\vec{r}\cdot\hat{x}_j')\hat{x}_j'$$

$$\begin{cases}\hat{x}_i'=\sum_j(\hat{x}_i'\cdot\hat{x}_j)\hat{x}_j=\sum_j\lambda_{ij}\hat{x}_j \\ \hat{x}_i=\sum_j(\hat{x}_i\cdot\hat{x}_j')\hat{x}_j'=\sum_j\lambda_{ji}\hat{x}_j'\end{cases}\qquad \lambda_{ij}\equiv\hat{x}_i'\cdot\hat{x}_j$$

$$x_i'=\vec{r}\cdot\hat{x}_i'=\sum_j(x_j\cdot\hat{x}_j)\cdot\hat{x}_i'=\sum_j\lambda_{ij}x_j\qquad x_i=\sum_j\lambda_{ji}x_j'$$

一、变换矩阵

$$\begin{array}{cc} \lambda=\begin{pmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\\ \lambda_{31} & \lambda_{32} & \lambda_{33}\end{pmatrix} & \leftarrow \hat{x}_1' \\ \begin{array}{c}\uparrow\\ \hat{x}_1\end{array} & \end{array}$$

$$\delta_{ij}=\hat{x}_i'\cdot\hat{x}_j'=\left(\sum_k\lambda_{ik}\hat{x}_k\right)\cdot\left(\sum_l\lambda_{jl}\hat{x}_l\right)=\sum_{k,l}\lambda_{ik}\lambda_{jl}\hat{x}_k\cdot\hat{x}_l=\sum_k\lambda_{ik}\lambda_{jk}$$

$(ij)=(1\,1)(2\,2)(3\,3)(1\,2)(2\,3)(3\,1)$　三个独立变量

$(2\,1)(3\,2)(1\,3)$

$$I=\lambda I\lambda^T=\lambda\lambda^T$$

1. 正交变换　$\lambda\in O(3)$：$\lambda\lambda^T=I=\lambda^T\lambda$

2. $\det(\lambda)=\begin{cases}+1 & \text{转动}\qquad \lambda\in SO(3)\\ -1 & \text{反演(+转动)，右手系→左手系}\end{cases}$

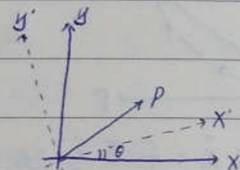
3. $\lambda, \mu \in SO(3) \Rightarrow \lambda\mu, \mu\lambda \in SO(3)$

一般 $\lambda\mu \neq \mu\lambda$ 转动次序不可交换

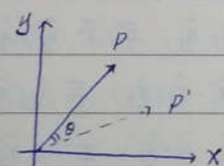
4. $\lambda \in SO(3)$ 描述的变换可以经由一次转动实现

转轴 $\hat{n}$, 转角 Θ 奇阶实正交阵必有 1 这一特征值

5. 两种观点 被动 轴动



主动 点动



二、求和约定

如果某一指标在同一单项式中重复出现，则对其求和

矩阵乘法 $(AB)_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \quad (i, j = 1, 2, 3)$

$\Leftrightarrow (AB)_{ij} = A_{ik} B_{kj}$ $k \leftrightarrow$ 求和(哑)指标, $ij \leftrightarrow$ 自由指标

$\lambda_{ik}\lambda_{jk} = \delta_{ij} = \lambda_{ki}\lambda_{kj}$

$\delta_{ii} = 3 \quad (A_{ii} = A_{11} + A_{22} + A_{33} = trA)$

$A_{ij}\delta_{jk} = A_{ik} \qquad A_{ij}\delta_{ji} = A_{ii} = A_{jj} = trA$

$A_{ij}B_{ji} = (AB)_{ii} = tr(AB)$ 先对 j 求和

$= (BA)_{jj} = tr(BA)$ 先对 i 求和

三、排列符号 $(i, j, k = 1, 2, 3)$

$\varepsilon = \begin{cases} \pm 1 \\ 0 \end{cases}$ (ijk) 偶排列 -1 ; (ijk) 奇排列 -1

$$\begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}$$

1. 完全反对称 $(+ \varepsilon_{123} = 1)$

2. $\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} = \varepsilon_{ikj}\varepsilon_{mkn} = \varepsilon_{kij}\varepsilon_{kmn}$

$\varepsilon_{ijk}\varepsilon_{mjk} = \delta_{im}\delta_{jj} - \delta_{ij}\delta_{jm} = 3\delta_{im} - \delta_{im} = 2\delta_{im}$

$\varepsilon_{ijk}\varepsilon_{ijk} = 6$

3. $\varepsilon_{ijk}A_{i1}A_{j2}A_{k3} = \det A = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k}$

$\varepsilon_{ijk}A_{il}A_{jm}A_{kn} = \varepsilon_{lmn}\det A$

§2 张量及其运算

一、张量

3 维 Euclid 空间的 n 阶张量 T

在任一给定直角坐标系下由 3^n 个分量 $T_{\overbrace{i\cdots j}^{n个}}$ 描述

在正交变换 $x_i \mapsto x_i' = \lambda_{ij}x_j$, $\lambda \in O(3)$, T

$$T'_{i\cdots j} = \underbrace{\lambda_{ik}\cdots\lambda_{jl}}_{n个} T_{k\cdots l}$$

1. 标量(零阶) ϕ: $\phi' = \phi$ 与坐标系选取无关

eg: $\pi, t, l, m, \rho_m, Q, \rho_e$

2. 矢量(一阶) $\vec{f}$: $f_i' = \lambda_{ij}f_j$

eg: $\vec{r}, d\vec{r}, \hat{x}_i \, (i = 1, 2, 3)$

3. 二阶张量 $\overleftrightarrow{T}$: $T'_{ij} = \lambda_{ik}\lambda_{jl}T_{kl}$ $(T' = \lambda T \lambda^T$ 相似变换)

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eg: $\overleftrightarrow{T} = \vec{A}\vec{B}$ $T_{ij} = A_i B_j \Rightarrow \hat{x}_i \hat{x}_j \ (i,j=1,2,3)$

eg: 单位张量 $\overleftrightarrow{I}$: $I_{ij} = \delta_{ij} = I'_{ij}$

对称张量 $T_{ij}=T_{ji}$ 反对称张量 $T_{ij} = -T_{ji}$

4. n阶赝张量: $T'_{i\cdots j} = \det\lambda \cdot \lambda_{il}\cdots\lambda_{jk} T_{l\cdots k}$

eg. $\det\lambda\cdot\varepsilon_{ijk} = \lambda_{il}\lambda_{jm}\lambda_{kn}\varepsilon_{lmn}$

二. 张量代数运算

0. 相等 若 $T_{ij}=S_{ij}$, 则 $T'_{ij}=S'_{ij}$

1. 线性组合 n阶 n阶 ⇒ n阶

$$(aT+bS)_{i\cdots j} = aT_{i\cdots j} + bS_{i\cdots j}$$

eg: $\vec v = \frac{d\vec r}{dt}$, $\vec a = \frac{d\vec v}{dt}$, $\vec p = m\vec v$, $\vec F = m\vec a$

$$\vec f = f_i \hat x_i \qquad \overleftrightarrow{T} = T_{ij}\hat x_i \hat x_j$$

$$f'_i \hat x'_i = (\lambda_{ik} f_k)(\lambda_{il}\hat x_l) = \lambda_{ik}\lambda_{il}(f_k \hat x_l) = \delta_{kl} f_k \hat x_l = f_i \hat x_i$$

$$T_{ij} = \frac{T_{ij}+T_{ji}}{2} + \frac{T_{ij}-T_{ji}}{2} = T_{(ij)} + T_{[ij]}$$

2. 张量积 n阶, m阶 ⇒ n+m阶

$$(\underset{\uparrow n}{T}\otimes \underset{\uparrow m}{S})_{\underbrace{i\cdots j}_{n}\underbrace{k\cdots l}_{m}} = T_{i\cdots j}S_{k\cdots l}$$

$\overleftrightarrow{T} = \vec f \vec g$ 并矢 不满足交换律

3. 缩并 n(≥2)阶 ⇒ (n−2)阶

$$T_{ijk} \Rightarrow \begin{cases} T_{ikk}=A_i \\ T_{kik}=B_i \\ T_{kki}=C_i\end{cases}$$

$$T_{ij} \Rightarrow T_{ii} = \mathrm{tr}\,T$$

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$\vec A\times\vec B = (\varepsilon_{ijk}A_jB_k)\hat x_i$ 轴(赝矢量)

对应于反对称张量

$\vec r\times\vec p$ $\vec r\times\vec F$ $d\vec B = \frac{\mu_0}{4\pi}\frac{Id\vec l\times\vec r}{r^3}$ $\vec m = \frac12 I\oint \vec r\times d\vec r$

$$\vec f\vec g = (f_i\hat x_i)(g_j\hat x_j) = (f_ig_j)\hat x_i\hat x_j$$

$$\vec f\cdot\vec g = (f_i\hat x_i)\cdot(g_j\hat x_j) = f_ig_j\,\hat x_i\cdot\hat x_j = f_ig_i$$

$$\vec f\times\vec g = (f_i\hat x_i)\times(g_j\hat x_j) = f_ig_j\,\hat x_i\times\hat x_j = (\varepsilon_{kij}f_ig_j)\hat x_k$$

$$\overleftrightarrow T\cdot\vec g = (T_{ij}\hat x_i\hat x_j)\cdot(g_k\hat x_k) = T_{ij}g_k\,\hat x_i\,\hat x_j\cdot\hat x_k = (T_{ij}g_j)\hat x_i$$

三. 张量不变性

Theorem 线性映射 $\overleftrightarrow T: \vec A\mapsto\vec B = \overleftrightarrow T(\vec A)$ 为二阶张量

$\vec B = B_i\hat x_i$

$\overleftrightarrow T(A_j\hat x_j) = A_j\overleftrightarrow T(\hat x_j) = A_jT_{ij}\hat x_i$ $T_{ij} = \hat x_i\cdot\overleftrightarrow T(\hat x_j)$

$B_i = T_{ij}A_j$

变换行为 $T'_{ij} = \underset{\uparrow\,\lambda_{ik}\hat x_k}{\hat x'_i}\cdot\overleftrightarrow T(\underset{\uparrow\,\lambda_{jl}\hat x_l}{\hat x'_j}) = \lambda_{ik}\lambda_{jl}T_{kl}$

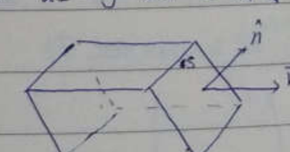
表示两个矢量之间的变换关系

$\overleftrightarrow I: \vec A\mapsto I(\vec A) = \vec A$

如: $\vec P = \varepsilon_0\overleftrightarrow{\chi}\cdot\vec E$

$\overleftrightarrow T: \vec a\mapsto\vec F = \overleftrightarrow T(\vec a) = m\vec a \Rightarrow \vec F = m\vec a = m\overleftrightarrow I\cdot\vec a$

如: $dI = \vec j\cdot d\vec S$ Δt内穿过ΔS的动量



$\Delta V = \Delta S\cos\theta\, v\Delta t = \vec v\cdot\hat n\Delta S\Delta t$

$\rho\vec v\Delta V = \rho\vec v\vec v\cdot\hat n\Delta S\Delta t$

$\rho\vec{v}\vec{v}\cdot\Delta\vec{S} = \overleftrightarrow{T}\cdot\Delta\vec{S}$ $(\overleftrightarrow{T} = \rho\vec{v}\vec{v})$ 动量流密度

$\oint \overleftrightarrow{T}\cdot d\vec{S}$

$T_{ij} = \rho \hat{x}_i \hat{x}_j = \hat{x}_i \cdot \Delta\hat{S} \cdot \hat{x}_j$ 有误

在$\hat{x}_i$上的投影　从与$\hat{x}_j$垂直流出

$-\frac{d}{dt}\int \rho\, dV = \oint_{\partial V} \vec{j}\cdot d\vec{S} \Longleftrightarrow -\partial_t \rho = \nabla\cdot\vec{j}$ 连续性方程

$-\frac{d}{dt}\int \vec{g}\, dV = \oint_{\partial V} \overleftrightarrow{T}\cdot d\vec{S} + \int \vec{f}\, dV$

$\varepsilon_0 \vec{E}\times\vec{B}$　$\frac{1}{2}\varepsilon_0(E^2+CB^2)\overleftrightarrow{I} - \varepsilon_0(\vec{E}\vec{E}+C\vec{B}\vec{B})$　$\rho\vec{E}+\vec{j}\times\vec{B}$

电磁场动量减少　电磁动动量流出　使粒子动量增加

$\Rightarrow -\partial_t \vec{g} = \nabla\cdot\overleftrightarrow{T} + \vec{f}$

Theorem　双线性函数

$\overleftrightarrow{T}: \vec{A}, \vec{B} \longmapsto \phi = \overleftrightarrow{T}(\vec{A}, \vec{B})$ 为二阶张量

$T_{ij} = \overleftrightarrow{T}(\hat{x}_i, \hat{x}_j)$

$dl^2 = dx_i dx_{i} = \delta_{ij} dx_i dx_j = \delta_{ij} dx_i' dx_j' = dl'^2$, $x_i' = \lambda_{ij} x_j$

$\Rightarrow \lambda\lambda^T = I$

正交变换另一种引入，易推广

四维空间亦有6个独立转动

$ds^2 = g_{\alpha\beta} dx_\alpha dx_\beta = dl^2 - (cdt)^2 = dl'^2 - (cdt')^2 = g_{\alpha\beta} dx_\alpha' dx_\beta' = ds'^2$

$x_\alpha' = \Lambda_{\alpha\beta} x_\beta$　$x_\alpha \to (x_0, x_1, x_2, x_3) = (ct, \vec{r})$

Lorentz 变换　$g_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

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在以上讨论中，将~~矢量作~~位矢作为矢量的原型

弯曲空间下矢量的定义

$\vec{X}_p: f \longmapsto \vec{X}_p(f) = X_i \frac{\partial}{\partial x_i} f|_p$ （以偏导为基）

用对应关系作为一切的基础

§3 转动矩阵的几何含义(主动)

Theorem　$\lambda \in SO(3)$ 的本征值可写为 $\lambda_1 = e^{i\theta}$, $\lambda_2 = e^{-i\theta}$, $\lambda_3 = +1$

证：① $\det\lambda = 1 = \lambda_1\lambda_2\lambda_3$

② 本征矢方程　$\lambda X = aX$

$X^+ \lambda^+ = a^* X^+$　　$\lambda^+ = (\lambda^*)^T$

$\Rightarrow X^+ \lambda^T \lambda X = a a^* X^+ X$

$\Rightarrow X^+ X = |a|^2 X^+ X$

$\Rightarrow |\lambda_1| = |\lambda_2| = |\lambda_3| = 1$

③ 久期方程　$\det(\lambda - aI) = 0$

是三次实系数的方程

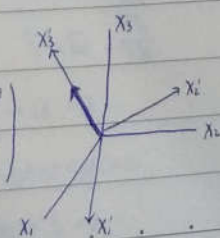
Theorem　$\lambda \in SO(3)$ 描述的变换可经由一次转动实现

转轴 $\hat{n}$: $\lambda\hat{n} = +1\cdot\hat{n}$

转角 θ: $\cos\theta = \frac{tr\lambda - 1}{2}$

证. $x_i' = \mu_{ij} x_j$　$(\mu\mu^T = I)$

$\lambda' = \mu\lambda\mu^T = \begin{pmatrix} \cos\theta' & -\sin\theta' & 0 \\ +\sin\theta' & \cos\theta' & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$$tr\lambda = \lambda_1+\lambda_2+\lambda_3 = 1+2\cos\theta$$

$$\parallel$$

$$tr\lambda' = 1+2\cos\theta'$$

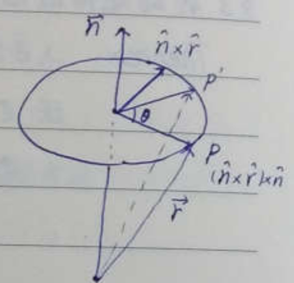
$$\Rightarrow \theta' = \pm\theta$$

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \qquad \cos\theta = \frac{(\lambda_{11}+\lambda_{22}+\lambda_{33})-1}{2}$$

$$\hat{n} = (\lambda_{32}-\lambda_{23},\ \lambda_{13}-\lambda_{31},\ \lambda_{21}-\lambda_{12})$$

$$\vec{r}' = \vec{r}'(\vec{r},\hat{n},\theta)$$

$$= \underset{\hat{n}\cdot\vec{r}}{\underset{\uparrow}{a}}\hat{n} + \underset{\sin\theta}{\underset{\uparrow}{b}}\hat{n}\times\vec{r} + \underset{\cos\theta}{\underset{\uparrow}{c}}(\hat{n}\times\vec{r})\times\hat{n}$$



1. 转动公式 $\vec{r}' = \vec{r}\cos\theta + \hat{n}(\hat{n}\cdot\vec{r})(1-\cos\theta) + \hat{n}\times\vec{r}\sin\theta$

转动用3个独立变量描述，但不能用一个矢量描述

(1) 其分量无意义

(2) 次序不可交换

2. 无限小转动 ($\theta\to d\theta$)

$$\vec{r}' = \vec{r} + \vec{n}\times\vec{r}\,d\theta \Rightarrow d\vec{r} = \vec{r}'-\vec{r} = \hat{n}\,d\theta\times\vec{r} = \vec{d\theta}\times\vec{r}$$

$$\Rightarrow \frac{d\vec{r}}{d\theta} = \hat{n}\times\vec{r}$$

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r} \qquad \vec{\omega} \equiv \frac{\vec{d\theta}}{dt} = \vec{n}\frac{d\theta}{dt}$$

$$x_i' = x_i + \varepsilon_{ikj}n_k x_j\,d\theta = \lambda_{ij}x_j$$

$$\lambda_{ij} = \delta_{ij} - \varepsilon_{ikj}n_k\,d\theta$$

$$\lambda = I + d\Omega$$

$$d\Omega = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} d\theta$$

无限小转动的对应张量分解

= 对称张量 + 反对称张量

都相等 $\qquad\qquad \Downarrow$

赝矢量

$$|\vec{C}(t)| = 常数 \Longleftrightarrow \frac{d\vec{C}}{dt} = \vec{\omega}\times\vec{C}$$

两边$\vec{C}$点乘

$$m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B} \Rightarrow \frac{d\vec{v}}{dt} = \vec{\omega}\times\vec{v} \quad \left(\vec{\omega} = -\frac{q\vec{B}}{m}\right)$$

§4 场

场也可以看作一种映射 $\vec{E}: P\longrightarrow\vec{E}(P)$

一. 标量场

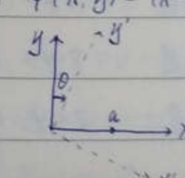
$\varphi: P\longmapsto\varphi(P)$

$\varphi'(P) = \varphi(P)$

$\Uparrow$

$\varphi'(\mathbb{X}') = \varphi(\mathbb{X})$

如. $\varphi(x,y) = (x-a)^2+y^2$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\varphi'(\mathbb{X}') = \varphi(\mathbb{X}) = \varphi(\lambda^{-1}\mathbb{X}')$$

$$\varphi'(x',y') = (x'\cos\theta+y'\sin\theta-a)^2+(-x'\sin\theta+y'\cos\theta)^2$$

$$= (x'-a\cos\theta)^2+(y'-a\sin\theta)^2$$

二、标量场的对称性

若 $\varphi'(X) = \varphi(X')$，即 $\varphi(\lambda X) = \varphi(X)$

则称 φ 在变换 $X' = \lambda X$ 下是不变/对称的

而 λ 称为 φ 的对称操作

$\forall\theta$

- n 次对称轴　$\theta_n = \frac{2\pi}{n}$
- 旋转对称轴　$\forall\theta$

三、梯度算子

$$\nabla \equiv \hat{x}_i \frac{\partial}{\partial x_i} = \hat{x}_i \partial_i \quad (\partial_i = \frac{\partial}{\partial x_i},\ \partial_x = \frac{\partial}{\partial x},\ \partial_\theta = \frac{\partial}{\partial \theta})$$

一阶线性偏微分矢量算子

$$\frac{\partial}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'} \frac{\partial}{\partial x_j}$$

$$x_j = \lambda_{kj} x_k' \Rightarrow \frac{\partial x_j}{\partial x_i'} = \lambda_{kj} \frac{\partial x_k'}{\partial x_i'} = \lambda_{kj} \delta_{ki} = \lambda_{ij}$$

$$\Rightarrow \partial_i' = \lambda_{ij} \partial_j$$

1. 三个定义

$d\varphi = \nabla\varphi \cdot d\vec{l}$　梯度　$\nabla\varphi = (\partial_i \varphi)\hat{x}_i$

$\nabla \cdot \vec{F} = \lim_{V\to 0} \frac{\oint_S \vec{F}\cdot d\vec{S}}{V}$　散度　$\nabla\cdot\varphi = \partial_i F_i$

$\hat{n}\cdot(\nabla\times\vec{F}) = \lim_{S\to 0} \frac{\oint_L \vec{F}\cdot d\vec{l}}{S}$　旋度　$\nabla\times\varphi = (\varepsilon_{ijk}\partial_j F_k)\hat{x}_i$

2. 恒等式

$$\nabla\times\nabla\varphi = 0, \quad \nabla\cdot(\nabla\times\vec{A}) = 0$$

$$\nabla\cdot(\nabla\times\vec{A}) = \partial_i(\nabla\times\vec{A})_i = (\partial_i \varepsilon_{ijk}\partial_j A_k)$$

$$= \varepsilon_{ijk}\partial_i\partial_j A_k = 0$$

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$$\nabla\times\nabla\varphi = \hat{x}_i \varepsilon_{ijk}\partial_j(\partial_k\varphi) = 0$$

3. 两个定理

$$\nabla\times\vec{F} = 0 \Rightarrow \vec{F} = -\nabla\varphi$$

$$\nabla\cdot\vec{F} = 0 \Rightarrow \vec{F} = \nabla\times\vec{A}$$

要求 单连通区域，或局部上

如：$\nabla\times\vec{F}(\vec{r}) = 0 \Rightarrow \vec{F}(\vec{r}) = -\nabla U(\vec{r})$

$$\Rightarrow U(\vec{r}) = \int_P^{\vec{r}} dU = \int_P^{\vec{r}} \nabla U\cdot d\vec{r} = -\int_P^{\vec{r}} \vec{F}\cdot d\vec{r}$$

恒力 $\vec{F}$　$U = -\vec{F}\cdot\int d\vec{r} = -\vec{F}\cdot\vec{r}$

弹性力 $\vec{F}$　$U = +k\int \vec{r}\cdot d\vec{r} = k\int r\,dr = \frac{1}{2}kr^2$

平方反比 $\vec{F}$　$U = -\alpha\int \frac{\vec{r}\cdot d\vec{r}}{r^3} = -\alpha\int\frac{dr}{r^2} = \frac{\alpha}{r}$

如：$\nabla\times\vec{F}(\vec{r},t) \Rightarrow \vec{F}(\vec{r},t) = -\nabla U(\vec{r},t)$

可以写出标量势，但是不是保守力

如：电磁势

$$\nabla\cdot\vec{B} = 0 \Rightarrow \vec{B} = \nabla\times\vec{A}$$

$$\nabla\times\vec{E} = -\partial_t\vec{B} = -\partial_t(\nabla\times\vec{A}) = -\nabla\times(\partial_t\vec{A})$$

$$\Rightarrow \nabla\times(\vec{E}+\partial_t\vec{A}) = 0,\quad \vec{E}+\partial_t\vec{A} = -\nabla\varphi$$

$$\vec{E} = -\nabla\varphi - \partial_t\vec{A},\quad \vec{B} = \nabla\times\vec{A}$$

$$\begin{cases} \vec{E} = -\nabla\varphi - \partial_t\vec{A} = -\nabla\varphi' - \partial_t\vec{A}' \\ \vec{B} = \nabla\times\vec{A} = \nabla\times\vec{A}' \end{cases}$$

$$\Rightarrow \begin{cases} \nabla\times(\vec{A}'-\vec{A}) = 0 \Rightarrow \vec{A}'-\vec{A} = \nabla\psi' = \nabla\psi \\ \nabla(\varphi'-\varphi) = -\partial_t(\vec{A}'-\vec{A}) = -\partial_t\nabla\psi' = -\nabla\partial_t\psi \end{cases}$$

$$\nabla(\varphi'-\varphi+\partial_t\psi') = 0 \Rightarrow \varphi' = \varphi - \partial_t\psi' - f(t) = \varphi - \partial_t\psi$$

四.推广

1.对矢量求导

$$f=f(\vec A):\ \frac{\partial f}{\partial \vec A} \underset{\text{直角坐标系}}{\equiv} \frac{\partial f}{\partial A_i}\hat x_i = \left(\frac{\partial f}{\partial A_1}, \frac{\partial f}{\partial A_2}, \frac{\partial f}{\partial A_3}\right)$$

$$f(\vec r, t):\ \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x_i}\dot x_i + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial \vec r}\cdot\dot{\vec r} + \frac{\partial f}{\partial t}$$

$$f_1 = \vec A(\vec r, t)\cdot\vec r \qquad \frac{\partial f}{\partial \vec r} = \vec A$$

$$f = f(\vec r, \dot{\vec r}, t) \qquad \frac{df}{dt} = \frac{\partial f}{\partial \vec r}\cdot\dot{\vec r} + \frac{\partial f}{\partial \dot{\vec r}}\cdot\ddot{\vec r} + \frac{\partial f}{\partial t}$$

2.对一组变量求导

$$f=f(q)=f(q_1,q_2,\cdots,q_n) \qquad \frac{\partial f}{\partial q} = \left(\frac{\partial f}{\partial q_1},\cdots,\frac{\partial f}{\partial q_n}\right)$$

$$q=(r,\theta,\phi) \qquad \frac{\partial f}{\partial q} = \left(\frac{\partial f}{\partial r},\frac{\partial f}{\partial \theta},\frac{\partial f}{\partial \phi}\right) \neq \nabla f$$

$$dU = \nabla U\cdot d\vec r + \frac{\partial U}{\partial t}dt$$

3.(单连通,局部)

$$F_i = -\frac{\partial \varphi}{\partial q_i} \Leftrightarrow \frac{\partial}{\partial q_j}F_i = \frac{\partial}{\partial q_i}F_j$$

场的引入:避免超距作用

§5 正交曲线坐标系

$$\vec r = \vec r(u_1, u_2, u_3)$$

$$d\vec r = \underbrace{\frac{\partial \vec r}{\partial u_1}}_{h_1\hat u_1}du_1 + \underbrace{\frac{\partial \vec r}{\partial u_2}}_{h_2\hat u_2}du_2 + \underbrace{\frac{\partial \vec r}{\partial u_3}}_{h_3\hat u_3}du_3 \qquad h_i \leftrightarrow \text{Lamé系数 (拉梅)}$$

$= (dx_i)\hat x_i$ 直角坐标

$= (dr)\hat r + (r\,d\theta)\hat\theta + (r\sin\theta\,d\phi)\hat\phi$ 球坐标

$= (ds)\hat s + (s\,d\phi)\hat\phi + (dz)\hat z$ 柱坐标

$$\vec v \equiv \frac{d\vec r}{dt}$$

$= \dot x_i\hat x_i$

$= \dot r\hat r + r\dot\theta\hat\theta + r\dot\phi\sin\theta\hat\phi$

$= \dot s\hat s + s\dot\phi\hat\phi + \dot z\hat z$

$v^2 = \dot x_i\dot x_i$

$= \dot r^2 + r^2\dot\theta^2 + r^2\dot\phi^2\sin^2\theta$

$= \dot s^2 + s^2\dot\phi^2 + \dot z^2$

§6 约束

$(x,y,z) \leftarrow$ 独立 $\qquad \frac{\partial x_i}{\partial x_j} = \delta_{ij}$

$(\dot x,\dot y,\dot z) \leftarrow$ 独立 $\qquad \frac{\partial \dot x_i}{\partial \dot x_j} = \delta_{ij}$

$(\vec r,\vec v) \leftarrow$ 独立 $\qquad \frac{\partial \dot x_i}{\partial x_j} = 0$

约束:是理想模型,与质点、刚体类似

主动力

约束力:应运而生

一、例子

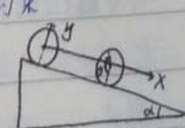
1.曲面:$f(\vec r,t)=0 \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial \vec r}\cdot\dot{\vec r} + \frac{\partial f}{\partial t} = 0$

2.曲线:$f_1(\vec r,t)=0,\ f_2(\vec r,t)=0$

需要两个方程来描述,是两个约束

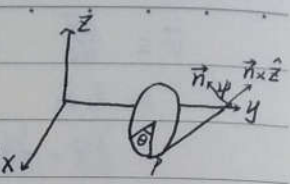
3.纯滚动 $\dot x - R\dot\theta = 0$

$\Rightarrow x = R\theta$



4.纯滚动 $\dot x = -R\dot\phi\sin\psi \quad \dot y = R\dot\phi\cos\psi$

$$d\vec{\theta} = d\phi\,\hat{n} + d\psi\,\hat{z}$$



$$0 = \vec{v}_p = \vec{u} + \vec{\omega}\times\vec{r}_p^{\,*}$$

$$= (\dot{x}\hat{x} + \dot{y}\hat{y}) + (\dot{\phi}\hat{n} + \dot{\psi}\hat{z})\times(-R\hat{z})$$

$$= (\dot{x}\hat{x} + \dot{y}\hat{y}) - R\dot{\phi}\,\hat{n}\times\hat{z}$$

$$= (\dot{x} + R\dot{\phi}\sin\psi)\hat{x} + (\dot{y} - R\dot{\phi}\cos\psi)\hat{y}$$

（这个约束条件导致的方程是不可积的）

（原本是四个坐标，仍需要四个坐标）

$f(x_1, x_2, \cdots, x_n) = 0 \Rightarrow df = 0 \Rightarrow \sum_i A_i dx_i = 0$

$\sum_i A_i(x_1, x_2, \cdots, x_n)\, dx_i = 0 \Rightarrow df(x_1, x_2, \cdots, x_n) = 0 \Rightarrow f(x_1, x_2, \cdots, x_n) = C$

$gA_i = \frac{\partial f}{\partial x_i}$ （非零 g）可积条件

$\Updownarrow$

$\partial_i(gA_j) = \partial_j(gA_i)$ 无旋

$\vec{A} = \vec{A}(\vec{r})$　积分曲线 $\vec{r} = \vec{r}(t)$
$\begin{cases} \frac{d\vec{r}}{dt} = g\vec{A}(\vec{r}) & \text{准线性} \\ \vec{r}(0) = \vec{r}_0 \end{cases}$

至少在局部，解存在且唯一

积分曲面 $f(\vec{r}) = C$
$\begin{cases} \frac{\partial f}{\partial \vec{r}} = g\vec{A}(\vec{r}) \\ f(\vec{r}_0) = C \end{cases}$

除非 $\vec{A}\cdot(\nabla\times\vec{A}) = 0$，在局部也无解

二. 约束分类

约束方程　$f(\vec{r}, \dot{\vec{r}}, t) = f(\vec{r}_1, \cdots, \vec{r}_n, \dot{\vec{r}}_1, \cdots, \dot{\vec{r}}_n, t) = 0$ (≥ 0)

1. $\begin{cases} \text{可解（单侧）} & f \geq 0 \\ \text{不可解（双侧）} & f = 0 \end{cases}$

"解"指解除．双侧：既不能进曲面内，也不能离开曲面

2. $\begin{cases} \text{稳定} & \text{不显含时间 } t \\ \text{不稳定} & \text{显含时间 } t \end{cases}$

3. $\begin{cases} \text{几何} & f(\vec{r}, t) = 0 \\ \text{微分} & f(\vec{r}, \dot{\vec{r}}, t) = 0 \end{cases}$
$\begin{cases} \text{可积、可转化为对位置的限制} \\ \text{不可积} \end{cases}$

4. $\begin{cases} \text{完整：几何 + 可积} \\ \text{非完整} \end{cases}$

三. 完整体系的运动学描述

$\begin{cases} n\ \text{粒子} & \vec{r} = (\vec{r}_1, \cdots, \vec{r}_n) \\ m\ \text{约束} & f_\alpha(\vec{r}, t) = 0\ (\alpha = 1, 2, \cdots, m) \end{cases}$ （独立）

$f = z = 0$　(x, y)

$f = r - R = 0$
$\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$ (θ, φ)

1. 自由度

$s = 3n - m$

为了完全确定完整体系的位形所需独立变量个数

2. 广义坐标

$q = (q_1, q_2, \cdots, q_s)$

$\begin{cases} \text{独立} \\ \text{能够完全确定位形} \end{cases}$

广义速度 $\dot{q} = (\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_s)$

3. 变换方程　$\vec{r} = \vec{r}(q, t)$ 来自不稳定约束

$$\dot{\vec{r}} = \frac{\partial\vec{r}}{\partial q_i}\dot{q}_i + \frac{\partial\vec{r}}{\partial t} = \dot{\vec{r}}(q, \dot{q}, t)$$

$$\frac{\partial \dot{\vec r}}{\partial q_k} = \frac{\partial^2 \vec r}{\partial q_k \partial q_i}\dot q_i + \frac{\partial \vec r}{\partial q_i}\frac{\partial \dot q_i}{\partial q_k} + \frac{\partial^2 \vec r}{\partial q_k \partial t} \cdots$$

$$\Rightarrow \frac{\partial \dot{\vec r}}{\partial q_k} = \frac{d}{dt}\frac{\partial \vec r}{\partial q_k}, \quad \frac{\partial \dot{\vec r}}{\partial t} = \frac{d}{dt}\frac{\partial \vec r}{\partial t}, \quad \frac{\partial \dot{\vec r}}{\partial \dot q_k} = \frac{\partial \vec r}{\partial q_k}$$

四. 位形空间

以 $q=(q_1, q_2, \cdots, q_s)$ 为直角坐标构建的 s 维空间

用一条曲线代替了原来的 n 条曲线

$\vec r = (\vec r_1, \vec r_2, \cdots, \vec r_n) = (x_1, y_1, z_1, \cdots, x_n, y_n, z_n)$ 3n 维(母空间)

$f(\vec r, t) = 0$ (3n−1)维超曲面

$f_\alpha(\vec r, t) = 0 \ (\alpha = 1, 2, \cdots, m)$ $s = (3n-m)$ 维位形空间曲面

参数方程 $\vec r = \vec r(q, t)$

s 个独立切矢量 $\vec\tau_i = \frac{\partial \vec r}{\partial q_i}$ $(i = 1, 2, \cdots, s)$

m 个独立法向量 $\vec n_\alpha = \frac{\partial f_\alpha}{\partial \vec r}$ $(\alpha = 1, 2, \cdots, m)$

独立约束，也指法向量独立。必须低一维。

广义坐标是局部的，严谨的坐标是一一对应的。

理论力学中很多定理不严谨处，往往是坐标失效处

五.(速度)相空间

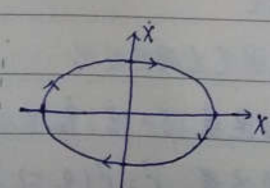
以 $(q, \dot q) = (q_1, q_2, \cdots, q_s, \dot q_1, \dot q_2, \cdots, \dot q_s)$ 为直角坐标构建的 2s 维空间

相点 → 相轨迹

$m\ddot x = -kx$; $\frac{d}{dt}\begin{pmatrix} x \\ \dot x \end{pmatrix} = \begin{pmatrix} \dot x \\ \ddot x \end{pmatrix} = \begin{pmatrix} \dot x \\ -\omega^2 x \end{pmatrix}$

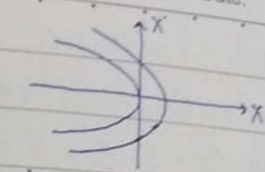
$\ddot x = -\omega x^2$

$(\omega \equiv \sqrt{\frac{k}{m}})$; $\frac{d\vec R}{dt} = \vec V(\vec R)$



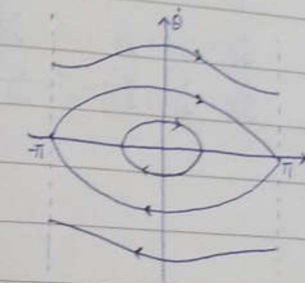
$E = \frac{1}{2}m\dot x^2 + \frac{1}{2}m\omega^2 x^2$ 椭圆

$\ddot x = -g$ $\frac{d}{dt}\begin{pmatrix} x \\ \dot x \end{pmatrix} = \begin{pmatrix} \dot x \\ -g \end{pmatrix}$



$\ddot\theta = -\omega_0^2 \sin\theta$ $\frac{d}{dt}\begin{pmatrix} \theta \\ \dot\theta \end{pmatrix} = \begin{pmatrix} \dot\theta \\ -\omega_0^2 \sin\theta \end{pmatrix}$

$\omega_0^2 = \frac{g}{l}$



相空间中任意两条轨迹不可能相交

相空间维数 { 受可积、几何约束 减二
受不可积约束 减一

六. 动能

1. 一般形式

$$T = \frac{1}{2}m_a\left(\frac{\partial \vec r_a}{\partial q_i}\dot q_i + \frac{\partial \vec r_a}{\partial t}\right)\cdot\left(\frac{\partial \vec r_a}{\partial q_j}\dot q_j + \frac{\partial \vec r_a}{\partial t}\right)$$

$$= \underbrace{\frac{1}{2}m_a \frac{\partial \vec r_a}{\partial q_i}\cdot\frac{\partial \vec r_a}{\partial q_j}}_{A_{ij}(q,t)}\dot q_i \dot q_j + \underbrace{m_a\left(\frac{\partial \vec r_a}{\partial q_i}\cdot\frac{\partial \vec r_a}{\partial t}\right)}_{B_i(q,t)}\dot q_i + \underbrace{\frac{1}{2}m_a\frac{\partial \vec r_a}{\partial t}\cdot\frac{\partial \vec r_a}{\partial t}}_{C(q,t)}$$

$T = \frac{1}{2}m_a \dot{\vec r}_a^2 = T(\dot{\vec r})$

$T = \frac{1}{2}A_{ij}\dot q_i \dot q_j + B_i \dot q_i + C = T_2 + T_1 + T_0 = T(q, \dot q, t)$

对称、正定

2. Euler 定理

$\dot q_k \frac{\partial T}{\partial \dot q_k} = 2T_2 + T_1$

稳定约束 $\vec r = \vec r(q, t_0) \Rightarrow T = T_2 \Rightarrow \dot q_k \frac{\partial T}{\partial \dot q_k} = 2T$

$$\vec{r}=\vec{r}(q,t),\quad \dot{\vec{r}}=\dot{\vec{r}}(q,\dot{q},t)$$

$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\phi}^2\sin^2\theta)=\frac{1}{2}mv^2$$

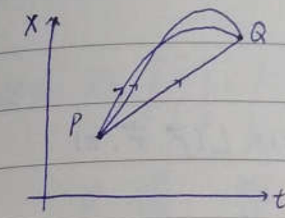
$$\frac{\partial T}{\partial \dot{r}}=\frac{\partial T}{\partial \dot{\vec{r}}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{r}}=\vec{p}\cdot\frac{\partial \vec{r}}{\partial r}=\vec{p}\cdot\hat{r}=m\dot{r}$$

$$\frac{\partial T}{\partial \dot{\theta}}=\frac{\partial T}{\partial \dot{\vec{r}}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{\theta}}=\vec{p}\cdot\frac{\partial \vec{r}}{\partial \theta}=\vec{p}\cdot r\hat{\theta}=\vec{p}\cdot(\hat{\phi}\times\vec{r})=\cancel{\vec{p}\cdot(\vec{r}\times}\hat{\phi}\cdot(\vec{r}\times\vec{p})=\hat{\phi}\cdot\vec{L}$$

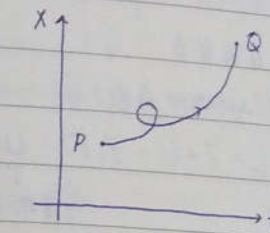
$$\frac{\partial T}{\partial \dot{\phi}}=\frac{\partial T}{\partial \dot{\vec{r}}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{\phi}}=\vec{p}\cdot\frac{\partial \vec{r}}{\partial \phi}=\vec{p}\cdot r\sin\theta\,\hat{\phi}=\vec{p}\cdot(\hat{z}\times\vec{r})=\hat{z}\cdot(\vec{r}\times\vec{p})=\hat{z}\cdot\vec{L}$$

CH2. Lagrange 力学

§2.0 运动遐想



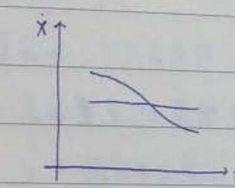
可能路径



不可能路径

一. 匀速直线运动

解释: 是最短路径



$$\int\sqrt{dx^2+c^2dt^2}=\int_{t_1}^{t_2}\sqrt{\dot{x}^2+c^2}\,dt$$

$$\Rightarrow\int_{t_1}^{t_2}(\dot{x}^2+c^2)dt\Rightarrow\int_{t_1}^{t_2}\dot{x}^2dt$$

$$0\le\int_{t_1}^{t_2}(\dot{x}-\langle\dot{x}\rangle)^2dt$$

$$=\int_{t_1}^{t_2}\dot{x}^2dt-2\int_{t_1}^{t_2}\dot{x}\langle\dot{x}\rangle dt+\int_{t_1}^{t_2}\langle\dot{x}\rangle^2dt$$

$$=\int_{t_1}^{t_2}\dot{x}^2dt-\int_{t_1}^{t_2}\langle\dot{x}\rangle^2dt$$

$$\int_{t_1}^{t_2}T\,dt=\int_{t_1}^{t_2}\frac{1}{2}m\dot{x}^2dt\quad 最小$$

二. 抛物运动

解释: 在自由下落参考系看, 匀速

$$x'=x+\frac{1}{2}gt^2,\quad \dot{x}'=\dot{x}+gt$$

$$\int_{t_1}^{t_2}\frac{1}{2}m\dot{x}'^2dt=\int_{t_1}^{t_2}\frac{1}{2}m(\dot{x}+gt)^2dt$$

$$=\int_{t_1}^{t_2}\frac{1}{2}m\dot{x}^2dt$$

$$+\int_{t_1}^{t_2}mgt\dot{x}\,dt\leftarrow mgtx\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}mgx\,dt$$

$$+\int\frac{1}{2}mg^2t^2dt\leftarrow 常数$$

$\int_{t_1}^{t_2}(T-U)dt=\int_{t_1}^{t_2}(\frac{1}{2}m\dot{x}^2-mgx)dt$ 最小

不仅解释了抛物运动，也将匀速直线运动作为特例包含

三. 猜想1: 自由体系

拉氏量(Lagrange 函数)

$$L=T-U=T(\dot{\vec{r}})-\underset{\substack{\uparrow\\ \text{内力. 外力}}}{U(\vec{r},t)}=L(\vec{r},\dot{\vec{r}},t)$$

作用量(action)

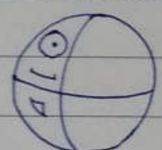
$$S=\int_{t_1}^{t_2}L(\vec{r},\dot{\vec{r}},t)\,dt$$

$[S]=[m][v]^2[t]=ML^2/T=L(ML/T)$ 具有角动量量纲

约束条件下，需要算上约束力的势能，但约束力不知道

四. 猜想2: 完整体系

“二维天才实验”



短程线. 测地线. 直线

角度. 圆. 三角形

惯性定律

约束力与位形曲面垂直，在其切线上无效应

可能路径：相同端点，位形曲面/满足约束

拉氏量: $L=T-U$ (主动力)

约束力: $f_\alpha(\vec{r},t)=0\Rightarrow\vec{N}_\alpha=\begin{pmatrix}\vec{N}_{\alpha 1}\\ \vdots\\ \vec{N}_{\alpha n}\end{pmatrix}$

$\vec{N}_\alpha \parallel \frac{\partial f_\alpha}{\partial\vec{r}}$ (不求和) 理想约束假设

牛顿力学给不出约束力方向

非完整约束下，约束力方向无统一规律

$f_\alpha=0\ (\alpha=1,2,\cdots,m)\Rightarrow\vec{N}=\begin{pmatrix}\vec{N}_1\\ \vdots\\ \vec{N}_m\end{pmatrix}$ $\vec{N}_\alpha\parallel\frac{\partial f_\alpha}{\partial\vec{r}}$ 成立

主动力: $\vec{F}=\begin{pmatrix}\vec{F}_1\\ \vdots\\ \vec{F}_n\end{pmatrix}=\begin{pmatrix}-\frac{\partial U}{\partial\vec{r}_1}\\ \vdots\\ -\frac{\partial U}{\partial\vec{r}_n}\end{pmatrix}=-\frac{\partial U}{\partial\vec{r}}$

$$\vec{F}\cdot\frac{\partial\vec{r}}{\partial q_k}=-\frac{\partial U}{\partial\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q_k}=-\frac{\partial U}{\partial q_k}$$

怎样证明: ① 把这条最小路径找出来

② 证明与牛顿方程同

极值 $\begin{cases}\text{函数}X:\ t\longrightarrow X(t) & \text{微分}\\ \text{泛函}S:\ X(t)\longrightarrow S[X(t)] & \text{变分}\end{cases}$

$\Delta X=X(t+\Delta t)-X(t)\approx k\Delta t$, $k=0\Rightarrow\Delta X=\frac{1}{2}a(\Delta t)^2$ $\begin{cases}a>0 & \text{最小}\\ a<0 & \text{最大}\\ a=0\Rightarrow\Delta X=\frac{1}{2}b(\Delta t)^3\cdots\end{cases}$

$\Delta t\to 0$: dt, $dx=\frac{dx}{dt}dt$

变分中，用 δX. $\frac{\delta f}{\delta X}$. δS 以示区别

§2.1 泛函与变分

一. 泛函

I: $q(t)=(q_1(t), q_2(t),\cdots,q_n(t))\longmapsto I[q(t)]=\int_{t_1}^{t_2}f(q,\dot{q},t)dt$

Theorem 若对任意 $\eta(t)$，都有 $\int_{t_1}^{t_2}G(t)\eta(t)dt=0$. 则 $G(t)\equiv 0$

推论 若对任意 $\eta_1(t)$. $\eta_2(t)$. $\cdots$. $\eta_n(t)$ 都有 $\int_{t_1}^{t_2}[G_1(t)\eta_1(t)+\cdots+G_n(t)\eta_n(t)]dt=0$. 则 $G_1(t)=\cdots=G_n(t)\equiv 0$

二. 泛函极值的含义

小量 $\eta(t)=\bar{q}(t)-q(t)$. $\dot{\eta}(t)=\dot{\bar{q}}(t)-\dot{q}(t)$

$\Rightarrow \Delta I[q(\bar{t})]=I[q+\eta]-I[q]\simeq\int_{t_1}^{t_2}G_k(t)\eta_k(t)dt$

若 $I[q(t)+\eta(t)]-I[q(t)]>0$

则 $I[q(t)-\eta(t)]-I[q(t)]<0$

不可能是极值路径

若 $q(t)$ 为极值路径. 则 $G_k(t)=0 \quad (k=1,2,\cdots,n)$

$\eta(t)\to 0$ 　路径变分 $\delta q(t)=\eta(t)$

速度变分 $\delta\dot{q}(t)=\dot{\eta}(t)$

$\delta\frac{dq}{dt}=\frac{d}{dt}\delta q$

泛函的(一阶)变分 $\delta I[q(t)]=I[q+\delta q]-I[q]$

$=\int_{t_1}^{t_2}f(q+\delta q,\dot{q}+\delta\dot{q},t)dt-\int_{t_1}^{t_2}f(q,\dot{q},t)dt$

$=\int_{t_1}^{t_2}[f(q+\delta q,\dot{q}+\delta\dot{q},t)-f(q,\dot{q},t)]dt$

函数变分 $\delta f(q,\dot{q},t)=f(q+\delta q,\dot{q}+\delta\dot{q},t)-f(q,\dot{q},t)$

$df(q,\dot{q},t)=f(q(t+dt),\dot{q}(t+dt),t+dt)-f(q(t),\dot{q}(t),t)$

"变商" $G_k(t)=\frac{\delta f}{\delta q_k}$

三. 变分法则

$\delta f(q,\dot{q},t)=\frac{\partial f}{\partial q_k}\delta q_k+\frac{\partial f}{\partial\dot{q}_k}\delta\dot{q}_k+\frac{\partial f}{\partial t}\delta t$ ($\delta t=0$ 等时变分)

$\delta f(g)=\frac{\partial f}{\partial g}\delta g$ 链式法则

$\delta(fg)=(\delta f)g+f(\delta g)$

$\delta\int f dt=\int\delta f dt$

$\delta\frac{df}{dt}=\frac{d}{dt}\delta f$

证: $\frac{df}{dt}=\dot{q}_i\frac{\partial f}{\partial q_i}+\ddot{q}_i\frac{\partial f}{\partial\dot{q}_i}+\frac{\partial f}{\partial t}$

$\delta\frac{df}{dt}=\dot{q}_i(\frac{\partial}{\partial q_i}\frac{\partial f}{\partial q_k})\delta q_k+\ddot{q}_i(\frac{\partial}{\partial\dot{q}_i}\frac{\partial f}{\partial q_k})\delta q_k+(\frac{\partial}{\partial t}\frac{\partial f}{\partial q_k})\delta q_k$

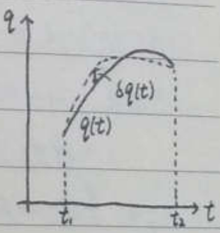
$+\dot{q}_i(\frac{\partial}{\partial q_i}\frac{\partial f}{\partial\dot{q}_k})\delta\dot{q}_k+\ddot{q}_i(\frac{\partial}{\partial\dot{q}_i}\frac{\partial f}{\partial\dot{q}_k})\delta\dot{q}_k+(\frac{\partial}{\partial t}\frac{\partial f}{\partial\dot{q}_k})\delta\dot{q}_k$

$+\frac{\partial f}{\partial q_i}\delta\dot{q}_i \quad +\frac{\partial f}{\partial\dot{q}_i}\delta\ddot{q}_i$

$=(\frac{d}{dt}\frac{\partial f}{\partial q_k})\delta q_k+(\frac{d}{dt}\frac{\partial f}{\partial\dot{q}_k})\delta\dot{q}_k+\frac{\partial f}{\partial q_k}\frac{d\delta q_k}{dt}+\frac{\partial f}{\partial\dot{q}_k}\frac{d\delta\dot{q}_k}{dt}$

四. 极值路径

$$\begin{cases}\delta I[q(t)]=\delta\int_{t_1}^{t_2}f(q,\dot{q},t)dt=0\\ \delta q_k(t_1)=0,\ \delta q_k(t_2)=0 \quad (k=1,2,\cdots,n)\end{cases}$$



$0=\delta I=\int_{t_1}^{t_2}[\frac{\partial f}{\partial q_k}\delta q_k+\underline{\frac{\partial f}{\partial\dot{q}_k}\delta\dot{q}_k}]dt$

逆用莱布尼兹法则, 分步积分

$\frac{d}{dt}(\frac{\partial f}{\partial\dot{q}_k}\delta q_k)-(\frac{d}{dt}\frac{\partial f}{\partial\dot{q}_k})\delta q_k$

$=\int_{t_1}^{t_2}[\frac{\partial f}{\partial q_k}-\frac{d}{dt}\frac{\partial f}{\partial\dot{q}_k}]\delta q_k dt+\underline{\frac{\partial f}{\partial\dot{q}_k}\delta q_k\Big|_{t_1}^{t_2}}$

不能交换次序　　端点项, 恒为0

$\Leftrightarrow \frac{\delta f}{\delta q_k}\equiv\frac{\partial f}{\partial q_k}-\frac{d}{dt}\frac{\partial f}{\partial\dot{q}_k}=0 \quad (k=1,2,\cdots,n)$ Eular-Lagrange方程

Theorem ① $\frac{\delta f}{\delta q_k}=0 \Leftrightarrow \frac{\delta(cf)}{\delta q_k}=0$

② $\frac{\delta f}{\delta q_k}=0 \Leftrightarrow \frac{\delta}{\delta q_k}[f+\frac{dF(q,t)}{dt}]=0$

$\int_{t_1}^{t_2}\frac{dF(q,t)}{dt}dt=F(q,t)\Big|_{t_1}^{t_2}$ 常数

Def $f(q,\dot{q},t)$ 的 Jacobi 积分 $h(q,\dot{q},t)\equiv\dot{q}_k\frac{\partial f}{\partial\dot{q}_k}-f$

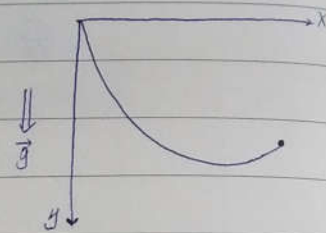
$\frac{df}{dt}=\frac{\partial f}{\partial q_k}\dot{q}_k+\frac{\partial f}{\partial\dot{q}_k}\ddot{q}_k+\frac{\partial f}{\partial t}\ \underline{\underline{极值路径}}\ \frac{d}{dt}(\frac{\partial f}{\partial\dot{q}_k}\dot{q}_k)+\frac{\partial f}{\partial t}$

Theorem 对极值路径

① $\frac{dh}{dt}=-\frac{\partial f}{\partial t}$, 若 $f=f(q,\dot{q},\cancel{t})$, 则 $h=const$

② 若 f 不显含 q_k, 则 $\frac{\partial f}{\partial\dot{q}_k}=const$

eg. 速降线(Branchistochrone)



$\frac{1}{2}mv^2 = mgy \Rightarrow v^2 = 2gy$

$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+y'^2}\,dx$, $y' = \frac{dy}{dx}$

$t = \int \frac{ds}{v} = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$

$f = \sqrt{\frac{1+y'^2}{2gy}} = f(y, y', x)$ 不显含x

$h = y'\frac{\partial f}{\partial y'} - f = \frac{y'^2}{\sqrt{2gy(1+y'^2)}} - \sqrt{\frac{1+y'^2}{2gy}} = \frac{-1}{\sqrt{2gy(1+y'^2)}} = \frac{-1}{\sqrt{2g}A}$

$y + yy'^2 = A^2 \Rightarrow (\sqrt{y})^2 + (\sqrt{y}y')^2 = A^2$

不妨设 $\begin{cases} \sqrt{y} = A\sin\frac{\theta}{2} \Rightarrow y = A^2\sin^2\frac{\theta}{2} \Rightarrow y' = A^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{d\theta}{dx} \\ \sqrt{y}\,y' = A\cos\frac{\theta}{2} \Rightarrow 1 = \frac{A^2}{2}(1-\cos\theta)\frac{d\theta}{dx} = \frac{A^2}{2}\frac{d(\theta-\sin\theta)}{dx} \end{cases}$

$\Rightarrow \begin{cases} x = \frac{A^2}{2}(\theta-\sin\theta) = a(\theta-\sin\theta) \quad (a = \frac{A^2}{2}) \\ y = \frac{A^2}{2}(1-\cos\theta) = a(1-\cos\theta) \end{cases}$

单摆: ①振幅越大，周期越长

②小角度 $2\pi\sqrt{\frac{l}{g}}$，摆长小时周期短

利用这条曲线，摆长 $L=4a$ 时，实现周期与摆长无关

§2.2 最小作用原理 / Hamilton 原理

$L = T - U = L(\vec r, \dot{\vec r}, t)$

总动能 主动力贡献 $\vec F = -\frac{\partial U}{\partial \vec r}$, $U(\vec r, t) = \sum_{a<b} U_{ab}(r_{ab}) + U_{外}(\vec r, t)$

$L(q, \dot q, t) = L(\vec r(q,t), \dot{\vec r}(q, \dot q, t), t)$

一、两种表述

$\begin{cases} 0 = \delta S = \delta\int_{t_1}^{t_2} L(\vec r, \dot{\vec r}, t)\,dt \\ \delta\vec r_a(t_1) = 0 = \delta\vec r_a(t_2) \quad (a = 1, 2, \cdots, n) \end{cases}$

$(\delta f_\alpha(\vec r, t) = \frac{\partial f_\alpha}{\partial \vec r}\cdot\delta\vec r = 0 \quad (\alpha = 1, 2, \cdots, n)$

三维空间

$\begin{cases} 0 = \delta S = \delta\int_{t_1}^{t_2} L(q, \dot q, t)\,dt \\ \delta q_k(t_1) = 0 = \delta q_k(t_2) \quad (k = 1, 2, \cdots, n) \end{cases}$

位形空间

二、证明

$\frac{\delta L}{\delta q_k} \equiv \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_k} = 0 \quad (k = 1, 2, \cdots, s)$

$\frac{\partial L}{\partial q_k} = \frac{\partial L}{\partial \vec r}\cdot\frac{\partial \vec r}{\partial q_k} + \frac{\partial L}{\partial \dot{\vec r}}\cdot\frac{\partial \dot{\vec r}}{\partial q_k}$

$\frac{\partial L}{\partial \dot q_k} = \frac{\partial L}{\partial \dot{\vec r}}\cdot\frac{\partial \dot{\vec r}}{\partial \dot q_k} = \frac{\partial L}{\partial \dot{\vec r}}\cdot\frac{\partial \vec r}{\partial q_k}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot q_k} = (\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec r}})\cdot\frac{\partial \vec r}{\partial q_k} + \frac{\partial L}{\partial \dot{\vec r}}\cdot\frac{d}{dt}\frac{\partial \vec r}{\partial q_k}$

$\Rightarrow \frac{\delta L}{\delta q_k} = (\frac{\partial L}{\partial \vec r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec r}})\cdot\frac{\partial \vec r}{\partial q_k}$

$\begin{cases} \frac{\delta T}{\delta q_k} = (\frac{\partial T}{\partial \vec r} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\vec r}})\cdot\frac{\partial \vec r}{\partial q_k} = -\dot{\vec p}\cdot\frac{\partial \vec r}{\partial q_k} \\ -\frac{\delta U}{\delta q_k} = (-\frac{\partial U}{\partial \vec r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec r}})\cdot\frac{\partial \vec r}{\partial q_k} \end{cases}$

若 $U = U(\vec r, t)$ 与 $\dot{\vec r}$ 无关，则 $-\frac{\delta U}{\delta q_k} = -\frac{\partial U}{\partial q_k} = \vec F\cdot\frac{\partial \vec r}{\partial q_k}$

$\frac{\delta L}{\delta q_k} \equiv \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_k} = 0 \quad (k = 1, 2, \cdots, s)$

$\Leftrightarrow (\vec F - \dot{\vec p})\cdot\frac{\partial \vec r}{\partial q_k} = 0$

$\begin{cases} \vec F + \vec N - \dot{\vec p} = 0 \Rightarrow (\vec F + \vec N - \dot{\vec p})\cdot\frac{\partial \vec r}{\partial q_k} = 0 \\ f_\alpha = 0 \end{cases}$

1. 二者在确定系统位形上等价

拉格朗日方程是牛顿方程在位形曲面上的投影

2. "最小"其实是指驻值

如费马原理，也是驻值而非"最小"

3. 拉格朗日方程与坐标选取无关

牛顿方程是矢量方程，直角坐标系下最简单

eg. 直角坐标 $L=\frac{1}{2}m\dot{\vec{r}}^2-U(\vec{r})$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{r}}}=\frac{\partial L}{\partial\vec{r}}\qquad m\ddot{\vec{r}}=0-\frac{\partial U}{\partial\vec{r}}=\vec{F}$$

球坐标 $L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\phi}^2\sin^2\theta)-U(r,\theta,\phi)$

$$\begin{cases}\frac{\partial L}{\partial r}=mr\dot{\theta}^2+mr\dot{\phi}^2\sin^2\theta-\frac{\partial U}{\partial r}\\ \frac{d}{dt}\frac{\partial L}{\partial\dot{r}}=\frac{d}{dt}(m\dot{r})=m\ddot{r}\end{cases}\qquad\hat{r}\cdot(\vec{F}-m\vec{a})=0$$

$$\begin{cases}\frac{\partial L}{\partial\phi}=-\frac{\partial U}{\partial\phi}=-\frac{\partial U}{\partial\vec{r}}\cdot\frac{\partial\vec{r}}{\partial\phi}=\vec{F}\cdot\hat{\phi}\,r\sin\theta=\vec{F}\cdot(\hat{z}\times\vec{r})=\hat{z}\cdot(\vec{r}\times\vec{F})=\hat{z}\cdot\vec{\tau}\\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}}=\frac{d}{dt}(mr^2\dot{\phi}\sin\theta=\frac{d}{dt}(\hat{z}\cdot\vec{l})\end{cases}$$

$$\begin{cases}\frac{\partial L}{\partial\theta}=mr^2\dot{\phi}^2\sin\theta\cos\theta-\frac{\partial U}{\partial\theta}=mr^2\dot{\phi}^2r\sin\theta\cos\theta+\hat{\phi}\cdot\vec{\tau}\\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}=\frac{d}{dt}(mr^2\dot{\theta})=\frac{d(\hat{\phi}\cdot\vec{l})}{dt}\end{cases}$$

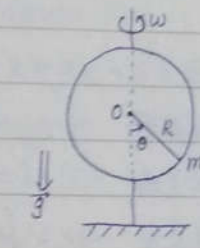
三. 术语

1. 与 q_k 关联的
   - 广义主动力 $Q_k\equiv\vec{F}_\alpha\cdot\frac{\partial\vec{r}_\alpha}{\partial q_k}$
     若 $\vec{F}=-\frac{\partial U}{\partial\vec{r}}$，则 $Q_k=-\frac{\partial U}{\partial q_k}$
   - 广义约束力 $Q_k'\equiv\vec{N}_\alpha\cdot\frac{\partial\vec{r}_\alpha}{\partial q_k}$
     理想约束 $Q_k'=0$
2. 与 q_k 共轭的广义动量 $p_k=\frac{\partial L}{\partial\dot{q}_k}\Rightarrow\dot{p}_k=\frac{\partial L}{\partial q_k}$
   若 L 不显含 q_k(循环坐标)，则 p_k 守恒
3. $L(q,\dot{q},t)$ 的 Jacobi 积分(能量函数) $h(q,\dot{q},t)=\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}-L$
   (1) $\frac{dh}{dt}=-\frac{\partial L}{\partial t}$；若 L 不显含 t，则 h 守恒
   (2) $L=T-U=T_2+T_1+(T_0-U)=L_2+L_1+L_0$
   $\Rightarrow h=L_2-L_0=T_2-T_0+U$

若 $\vec{r}=\vec{r}(q)$，则 $T=T_2$，因此 $h=T+U=E$

注意(1)(2)相互独立

eg. $r=R$，$\dot{\phi}=\omega(\phi=\omega t)$，$U=-mgR\cos\theta$



$L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\phi}^2\sin^2\theta)-U$

$=\frac{1}{2}mR^2\dot{\theta}^2+\frac{1}{2}mR^2\omega^2\sin^2\theta+mgR\cos\theta$

$$\begin{cases}\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}=\frac{d}{dt}(mR^2\dot{\theta})=mR^2\ddot{\theta}\\ \frac{\partial L}{\partial\theta}=mR^2\omega^2\sin\theta\cos\theta-mgR\sin\theta\end{cases}$$

$\Rightarrow\ddot{\theta}=-(\omega_0^2-\omega^2\cos\theta)\sin\theta=f(\theta)\quad(\omega_0^2=\frac{g}{R})$

平衡位置 θ_e: $\dot{\theta}_e=0=\ddot{\theta}_e\Rightarrow f(\theta_e)=0$

上式令 $\ddot{\theta}=0$，$\theta_1=0$，$\theta_2=\pi$，$\theta_3=\arccos\frac{\omega_0^2}{\omega^2}\ (\omega>\omega_0)$

稳定性 令 $\xi=\theta-\theta_e$(小量) $\Rightarrow\ddot{\xi}=\ddot{\theta}$

$\ddot{\xi}\approx\xi f'(\theta_e)\quad\ddot{\xi}=-\Omega^2\xi\ (\Omega^2>0$ 稳定$)$

$f'(\theta)=-\omega_0^2\cos\theta+\omega^2\cos2\theta$

$\ddot{\xi}=-\Omega^2\xi\quad\Omega^2=\omega_0^2\cos\theta_e-\omega^2\cos2\theta_e$

① $\theta_2=\pi\Rightarrow\Omega^2=-\omega_0^2-\omega^2<0$ 不稳定

② $\theta_1=0\Rightarrow\Omega^2=\omega_0^2-\omega^2\begin{cases}<0\ (\omega>\omega_0) & \text{不稳定}\\ >0\ (\omega<\omega_0) & \text{稳定}\end{cases}$

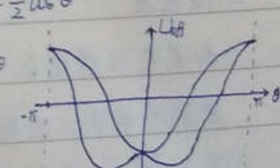
③ $\cos\theta_3=\frac{\omega_0^2}{\omega^2}\Rightarrow\Omega^2=\frac{\omega^4-\omega_0^4}{\omega^2}>0\ (\omega>\omega_0)$ 稳定

④ $\omega=\omega_0\Rightarrow\theta_3=\theta_1=0\quad\ddot{\theta}=-\omega_0^2(1-\cos\theta)\sin\theta\approx-\frac{1}{2}\omega_0^2\theta^3$

$h=L_2-L_0=\frac{1}{2}mR^2\dot{\theta}^2-\frac{1}{2}mR^2\omega^2\sin^2\theta-mgR\cos\theta$

$=\frac{1}{2}mR^2\dot{\theta}^2+U_{有效}(\theta)$

有效势能 圆环系中惯性力势能+重力势能.



对最小作用量原理的理解

$S=\int L\,dt=\int(T-U)\,dt$

运动能力 potential 潜力

使得粒子活跃程度较低的路径

海森堡 $\Delta q\Delta p\geq\frac{\hbar}{2}$ $\hbar=\frac{h}{2\pi}=1.05\times10^{-34}\,J\cdot s$

几率幅 $\psi(\vec{r},t)\Rightarrow$ 几率 $\int|\psi(\vec{r},t)|^2dV=\int\psi^*\psi\,dV$

期望值 $\langle x\rangle\equiv\int x|\psi(x,y,z,t)|^2dV=\int\psi^*x\psi\,dV$

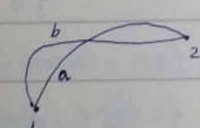
$\langle f\rangle\equiv\int\psi^*f\psi\,dV$

$\langle p_x\rangle\equiv m\frac{d\langle x\rangle}{dt}$

Schrödinger $i\hbar\frac{\partial\psi}{\partial t}=[-\frac{\hbar^2}{2m}\nabla^2+U]\psi$

$\frac{d\langle\vec{p}\rangle}{dt}=\langle-\nabla U\rangle$

费曼 粒子沿着每一条路径到达



$\psi(1\to2)=\sum_a\psi_a(1\to2)$

$\psi_a(1\to2)\propto\exp(i\frac{S_a}{\hbar})$

相位相干抵消，只在 $\dot{S}_a$ 一阶导为0处保留，为真实路径

五. Lagrange 函数的性质

1. 一般形式 $L=T-U=L_2+L_1+L_0=T_2+T_1+(T_0-U)$

$h=T_2-T_0$，若 $\vec{r}=\vec{r}(q,t)$，则 $h=T+U=E$

2. 惯性系

如果在非惯性系，要考虑惯性力的势能

更改参考系，拉格朗日函数可能变也可能不变

若不变，对各项的解读可能变化

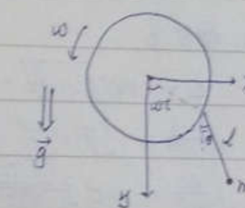
3. L 不确定性

标度变换 $L'=CL$

规范变换 $L'=L(q,\dot{q},t)+\frac{dF(q,t)}{dt}$

eg. $x=R\sin\omega t+l\sin\theta$

$y=R\cos\omega t+l\cos\theta$



$\Rightarrow\dot{x}=\omega R\cos\omega t+l\dot{\theta}\cos\theta$

$\dot{y}=\omega R\sin\omega t-l\dot{\theta}\sin\theta$

$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+mgy$

$=\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta+\underbrace{mgR\cos\omega t+\frac{1}{2}mR^2\omega^2}_{t\text{的函数}}+\underbrace{mRl\omega\dot{\theta}\cos(\theta-\omega t)}_{=mRl\omega[\frac{d\sin(\theta-\omega t)}{dt}+\omega\cos(\theta-\omega t)]}$

规范变换，对拉格朗日函数进行简化

$L'=\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta+mRl\omega^2\cos(\theta-\omega t)$

二者对应的广义动量，雅克比函数往往不同

$L'=L+\frac{\partial F}{\partial q_i}\dot{q}_i+\frac{\partial F}{\partial t}$

广义动量 $p_k\longrightarrow p_k'=\frac{\partial L'}{\partial\dot{q}_k}=p_k+\frac{\partial F}{\partial q_k}$

雅克比函数 $h\mapsto h'=\dot{q}_k\frac{\partial L'}{\partial\dot{q}_k}-L'=h-\frac{\partial F}{\partial t}$

4. 可加性

$L_{A+B}=(T_A+T_B)-(U_A+U_B+U_{AB})$

$=L_A+L_B+L_{AB}\quad(L_{AB}=-U_{AB})$

$\lim L_{A+B}=L_A(q_A,\dot{q}_A)+L_B(q_B,\dot{q}_B)$

$L=\frac{1}{2}m_1\dot{\vec{r}}_1^2+\frac{1}{2}m_2\dot{\vec{r}}_2^2-U(|\vec{r}_1-\vec{r}_2|)=\frac{1}{2}M\dot{\vec{r}}_c^2+\frac{1}{2}\mu\dot{\vec{r}}^2-U(r)$

$M=m_1+m_2\quad\vec{r}_c=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}\quad\mu=\frac{m_1m_2}{m_1+m_2}\quad\vec{r}=\vec{r}_1-\vec{r}_2$

如果拉格朗日函数坐标不相耦合，等效拆开

非完整约束体系的最小作用原理，尚无定论

因为约束力的方向难定论

六. 相对论

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - U(\vec{r}, t)$$

$$\begin{cases} \frac{\partial L}{\partial \vec{r}} = -\frac{\partial U}{\partial \vec{r}} = \vec{F} \\ \frac{\partial L}{\partial \vec{v}} = -mc^2\frac{-\vec{v}/c^2}{\sqrt{1-v^2/c^2}} = \gamma m\vec{v} \quad (\gamma = \frac{1}{\sqrt{1-v^2/c^2}}) \end{cases}$$

$$\Rightarrow \vec{F} = \frac{d\vec{p}}{dt}$$

写出 L 的方法 ① 凑

② 根据哈密顿函数逆变换

③ 对称性

构造一个与坐标系无关的标量，描述粒子状态，$d\tau = \frac{dt}{\gamma}$，而 mc^2 用于量纲，负号使在低速时回到动能减势能

§2.3 与速度有关的作用

$$0 = \vec{F}(\vec{r}, \dot{\vec{r}}, t) + \vec{N} - \dot{\vec{p}}$$

$$\Rightarrow 0 = [\vec{F}(\vec{r}, \dot{\vec{r}}, t) + \vec{N} - \dot{\vec{p}}]\cdot\frac{\partial \vec{r}}{\partial q_k}$$

假设 $L = T(\dot{\vec{r}}) - U(\vec{r}, \dot{\vec{r}}, t)$

$$\frac{\delta L}{\delta \vec{r}}\cdot\frac{\delta \vec{r}}{\delta q_k} = \left(\frac{\delta T}{\delta \vec{r}} - \frac{\delta U}{\delta \vec{r}}\right)\cdot\frac{\partial \vec{r}}{\partial q_k} = \Big[\underbrace{\left(\frac{\partial T}{\partial \vec{r}} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{r}}}\right)}_{0-\dot{\vec{p}}} + \underbrace{\left(-\frac{\partial U}{\partial \vec{r}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{r}}}\right)}_{\Downarrow \; \vec{F}}\Big]\cdot\frac{\partial \vec{r}}{\partial q_k}$$

一. 势力与广义势能

$U = U(\vec{r}, \dot{\vec{r}}, t)$ 时：$\vec{F} = -\frac{\delta U}{\delta \vec{r}} = -\frac{\partial U}{\partial \vec{r}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{r}}}$

$U = U(q, \dot{q}, t)$：$Q_k = \vec{F}\cdot\frac{\partial \vec{r}}{\partial q_k} = -\frac{\delta U}{\delta q_k} = -\frac{\partial U}{\partial q_k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_k}$

$$* \; U(\vec{r}, \dot{\vec{r}}, t) \Longleftrightarrow U'(\vec{r}, \dot{\vec{r}}, t) = U(\vec{r}, \dot{\vec{r}}, t) - \frac{dF(\vec{r}, t)}{dt}$$

eg. Lorentz力 $\vec{F} = e(\vec{E} + \vec{v}\times\vec{B})$ $\quad \vec{E} = -\nabla\varphi - \partial_t\vec{A}$

$$\vec{B} = \nabla\times\vec{A}$$

$$\vec{v}\times(\nabla\times\vec{A})_i = \varepsilon_{ijk}\dot{x}_j(\nabla\times\vec{A})_k = \varepsilon_{kij}\varepsilon_{kmn}\dot{x}_j\partial_m A_n$$

$$= \dot{x}_j\partial_i A_j - \dot{x}_j\partial_j A_i$$

$$\frac{\vec{F}}{e} = -\nabla\varphi - \partial_t\vec{A} + \vec{v}\times(\nabla\times\vec{A})_i$$

$$= -\frac{\partial\varphi}{\partial\vec{r}} - \frac{\partial\vec{A}}{\partial t} + \frac{\partial(\vec{v}\cdot\vec{A})}{\partial\vec{r}} - \dot{x}_j\frac{\partial\vec{A}}{\partial x_j}$$

$$= -\frac{\partial(\varphi - \vec{v}\cdot\vec{A})}{\partial\vec{r}} + \frac{d(-\vec{A})}{dt}$$

1. 广义势能 $U = e(\varphi - \vec{v}\cdot\vec{A})$

$$U' = U - \frac{dF}{dt}$$

$$= U - \frac{\partial F}{\partial\vec{r}}\cdot\vec{v} - \frac{\partial F}{\partial t}$$

$$= e\left(\varphi - \frac{1}{e}\frac{\partial F}{\partial t}\right) - e\vec{v}\cdot\left(\vec{A} + \frac{1}{e}\frac{\partial F}{\partial\vec{r}}\right)$$

$$U' = U - \frac{dF(\vec{r}, t)}{dt} \Longleftrightarrow \begin{cases} \varphi' = \varphi - \partial_t\psi \quad (\psi = \frac{1}{e}F) \\ \vec{A}' = \vec{A} + \nabla\psi \end{cases}$$

2. Laragrange函数

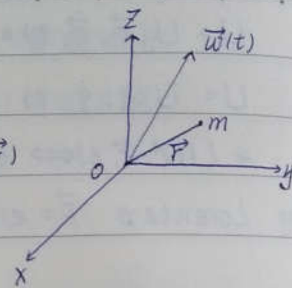
$$L = T - U = \frac{1}{2}mv^2 - e(\varphi - \vec{v}\cdot\vec{A})$$

$$= \frac{1}{2}mv^2 + e\vec{v}\cdot\vec{A} - e\varphi = L_2 + L_1 + L_0$$

3. 广义动量，雅克比函数

$$\begin{cases} P_k = m\dot{x}_k + eA_k \\ h = L_2 - L_0 = \frac{1}{2}mv^2 + e\varphi \end{cases}$$

eg. 转动惯性力 $\vec v = \vec v' + \vec\omega\times\vec r$



$L = T - U = \frac12 m(\vec v' - \vec\omega\times\vec r)^2 - U$

$= \frac12 m v'^2 - U - \frac12 m(\vec\omega\times\vec r)^2 - m\vec v'\cdot(\vec\omega\times\vec r)$

$= T - U - U'$

$U' = \underbrace{-\frac12 m(\vec\omega\times\vec r)^2}_{\text{与位置有关}} - \underbrace{m\vec v'\cdot(\vec\omega\times\vec r)}_{\text{与}\cancel{\text{位置}}\text{速度有线性关系}}$

对比电磁场 $U = e\varphi - \vec v\cdot\vec A$

如果不作 $L=T-U$ 的假设，即使是耗散力也可能找到 L

但对于耗散力，没有找到 L 的普遍方法

二. 耗散力

$0 = \frac{\delta L}{\delta q_k} + D_k = \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_k} + D_k$ （$D_k \equiv \vec F_\alpha^\circ\cdot\frac{\partial \vec r_\alpha}{\partial q_k}$ 广义耗散力）

设 $\vec F^\circ = -g(v)\hat v$ 其中 $g(v)>0 \Rightarrow \vec F^\circ\cdot\vec v = -g(v)v<0$

$D_k = -g(v)\frac{\vec v}{v}\cdot\frac{\partial\vec v}{\partial \dot q_k} = -g(v)\frac{\partial v}{\partial \dot q_k}$

则 $D_k = -\frac{\partial \mathcal F}{\partial \dot q_k}$ * $\mathcal F = \int_0^v g(z)dz = \mathcal F(v) = \mathcal F(q,\dot q,t)$

$-\frac{\partial\mathcal F}{\partial \dot x} = -\frac{\partial \mathcal F}{\partial v}\frac{\partial v}{\partial \dot q_k}$

$0 = \frac{\delta L}{\delta q_k} - \frac{\partial\mathcal F}{\partial \dot q_k}$ （L, $\mathcal F$ ← 耗散函数）

eg. $L = \frac12 m\dot x^2 - \frac12 m\omega_0^2 x^2$ $\mathcal F = \frac12\beta\dot x^2$

$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot x} - \frac{\partial\mathcal F}{\partial\dot x} = -m\omega_0^2 x - m\ddot x - \beta\dot x$

$\Rightarrow \ddot x + 2\gamma\dot x + \omega_0^2 x = 0$

$\vec F^\circ \dot x = -\frac{\partial\mathcal F}{\partial \dot x}\dot x = -2\mathcal F$ 单位时间消耗的能量

如果不执着于 $L=T-U$

$L = e^{2\gamma t}(\frac12 m\dot x^2 - \frac12 m\omega_0^2 x^2)$

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$\frac{\partial L}{\partial x} = -e^{2\gamma t} m\omega_0^2 x$

$\frac{d}{dt}\frac{\partial L}{\partial \dot x} = \frac{d}{dt}(e^{2\gamma t} m\dot x) = e^{2\gamma t} m\ddot x + e^{2\gamma t}2\gamma m\dot x$

三. d'Alembert 原理与虚功原理

$0 = \vec F + \vec N - \dot{\vec p} \Rightarrow 0 = (\vec F + \vec N - \dot{\vec p})\cdot\delta\vec r$

Def 虚位移 $\delta\vec r = \frac{\partial \vec r}{\partial q_k}\delta q_k$ （满足约束）

Def 虚功 $\begin{cases} \text{主动力} & \delta A = \vec F\cdot\delta\vec r = Q_k\delta q_k \\ \text{约束力} & \delta A' = \vec N\cdot\delta\vec r = Q_k'\cdot\delta q_k \\ \text{惯性力} & -\dot{\vec p}\cdot\delta\vec r = -\dot{\vec p}\cdot\frac{\partial\vec r}{\partial q_k}\delta q_k = \frac{\delta T}{\delta q_k}\delta q_k \end{cases}$

假设: 理想约束 $\delta A' = 0 \Leftrightarrow Q_k' = 0\ (k=1,2,\cdots,s)$

1. d'Alembert 原理 $(\vec F - \dot{\vec p})\cdot\delta\vec r = 0$

虚功原理 ($\dot{\vec p}=0$) $\vec F\cdot\delta\vec r = 0$

2. Lagrange 方程

$\begin{cases} \text{动力学} & \frac{\delta T}{\delta q_k} + Q_k = 0 \\ \text{静态} & Q_k = 0 \end{cases} \xrightarrow[Q_k = -\frac{\delta U}{\delta q_k}]{\vec F = -\frac{\delta U}{\delta \vec r}} \begin{cases} \frac{\delta L}{\delta q_k} = 0 \\ \frac{\partial U}{\partial q_k} = 0 \end{cases}$

$-\dot{\vec p}\cdot\delta\vec r = \frac{\partial T}{\partial q_k}\delta q_k - (\frac{d}{dt}\frac{\partial T}{\partial \dot q_k})\delta q_k = \frac{\partial T}{\partial q_k}\delta q_k + \frac{\partial T}{\partial \dot q_k}\delta\dot q_k + \frac{d}{dt}(\frac{\partial T}{\partial \dot q_k}\delta q_k)$

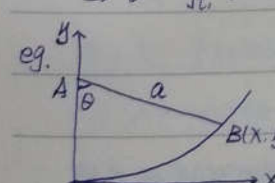
$0 = (\vec F + \vec N - \dot{\vec p})\cdot\delta\vec r = (\vec F - \dot{\vec p})\cdot\delta\vec r$

$0 = \int_{t_1}^{t_2}(\vec F - \dot{\vec p})\delta\vec r\,dt = \int_{t_1}^{t_2}(\delta A + \delta T)dt + [\frac{\partial T}{\partial\dot q_k}\delta q_k]\Big|_{t_1}^{t_2}$

$\vec F = -\frac{\delta U}{\delta\vec r} \Rightarrow \delta A = -\frac{\delta U}{\delta q_k} = -\delta U + \frac{d}{dt}[\ ]$

$\Rightarrow 0 = \int_{t_1}^{t_2}[\delta T - \delta U]dt = \delta\int_{t_1}^{t_2}(T-U)dt$

eg.



一个自由度

$x = a\sin\theta$, $y = y(\theta) \Rightarrow y_c = y(\theta) + \frac a2\cos\theta$

1° $\vec F = -mg\hat y$ $\delta\vec r_c = \delta x_c\hat x + \delta y_c\hat y$

$\Rightarrow \vec F\cdot\delta\vec r_c = -mg\,\delta y_c = -mg\,\delta\left[y(\theta)-\frac{a}{2}\cos\theta\right]=0$

$\Rightarrow \frac{\partial}{\partial\theta}\left[y+\frac{a}{2}\cos\theta\right]=0 \qquad y+\frac{a}{2}\cos\theta=C=\frac{a}{2}$

$\Rightarrow \begin{cases} x=a\sin\theta \\ y=\frac{a}{2}(1-\cos\theta) \end{cases}$

2° $U=mgy_c \Rightarrow \frac{\partial U}{\partial\theta}=0 \Rightarrow y_c=C=\frac{a}{2}$

§2.4 Lagrange 乘子法

问题：对自由度为 s 的体系，求其某个约束 $f(\vec r,t)=0$ 约束力？

方法：设想解除该约束，使其成为 (s+1) 个自由度的体系

$\vec r=\vec r(q_1,q_2,\cdots,q_s,q_{s+1},t) \qquad \vec N \parallel \frac{\partial f}{\partial\vec r} \Rightarrow \vec N=\lambda\frac{\partial f}{\partial\vec r}$

一、带乘子的 Lagrange 方程

$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial\dot q_k}-\frac{\partial L}{\partial q_k}=\lambda\frac{\partial f}{\partial q_k} \quad (k=1,2,\cdots,s,s+1) \\ f(q,t)=0 \end{cases}$$

1. Lagrange 乘子 λ

$\vec N=\lambda\frac{\partial f}{\partial\vec r} \qquad Q'=\lambda\frac{\partial f}{\partial q_k}$

2. 约束力虚功

$\delta A'=\vec N\cdot\delta\vec r=\vec N\cdot\frac{\partial\vec r}{\partial q_k}\delta q_k=Q'_k\delta q_k=\lambda\delta f=0$

3. 约束方程

约束方程在偏导数之后才能代入

否则会丢失约束力的信息

4. $\tilde L=L+\lambda f=\tilde L(q,\lambda,\dot q,\dot\lambda,t)$

是形式上的记法，之前有关 Lagrange 函数的结论仍成立

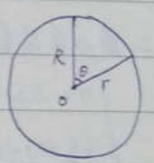
$\tilde L=T-(U-\lambda f)$

$\tilde p_k=\frac{\partial\tilde L}{\partial\dot q_k}=p_k \qquad \tilde p_\lambda=\frac{\partial\tilde L}{\partial\dot\lambda}=0 \qquad \tilde h=\dot q_k\frac{\partial\tilde L}{\partial\dot q_k}+\dot\lambda\frac{\partial\tilde L}{\partial\dot\lambda}-\tilde L=h-\lambda f$

雅克比函数在真实路径下与原来同

5. $\tilde L=L+\lambda_\alpha f_\alpha$

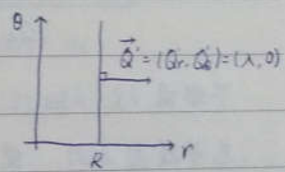
eg. $q=(r,\theta) \qquad f=r-R=0$



$L=\frac12 m\dot r^2+\frac12 mr^2\dot\theta^2-mgr\cos\theta$

$$\begin{cases} \frac{d}{dt}(mr^2\dot\theta)-mgr\sin\theta=\lambda\cdot 0=Q'_\theta \\ m\ddot r-mr\dot\theta^2+mg\cos\theta=\lambda\cdot 1=Q'_r=N \\ r=R \end{cases}$$

$$\Rightarrow \begin{cases} \lambda=mg\cos\theta-mR\dot\theta^2 \\ R\dot\theta\frac{d\dot\theta}{dt}=g\sin\theta\frac{d\theta}{dt} \end{cases}$$

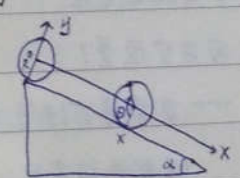


$R\dot\theta^2=-2g\cos\theta+C=2g(1-\cos\theta)$

$\Rightarrow \lambda=mg(3\cos\theta-2)=N \qquad \cos\theta=\frac23$

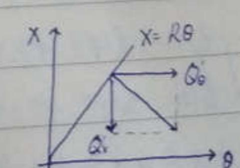
尽可能不要同时处理两个约束的约束力

eg. $q=(x,\theta),\ f=x-R\theta=0$



$L=\frac12 m\dot x^2+\frac14 mR^2\dot\theta^2+mgx\sin\alpha$

$$\begin{cases} m\ddot x-mg\sin\alpha=\lambda\cdot 1=Q'_x \\ \frac12 mR\ddot\theta-0=\lambda\cdot(-R)=Q'_\theta \\ \ddot x=R\ddot\theta \end{cases}$$



$$\Rightarrow \begin{cases} \ddot x=\frac23 g\sin\alpha \\ \lambda=-\frac12 mR\ddot\theta=-\frac12 m\ddot x \end{cases}$$

约束力与位形垂直，虚功为0

$f_1 = x - R\theta = 0 \quad f_2 = y = 0$

$U = -mg(\hat{x}\sin\alpha - \hat{y}\cos\alpha)\cdot\vec{r}$

$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{4}mR^2\dot{\theta}^2 + mg(x\sin\alpha - y\cos\alpha)$

约束力，是为使维持~~每~~某个约束而受的力

二.最小作用量原理

$$\begin{cases} 0 = \delta S = \delta\int_{t_1}^{t_2} L(q,\dot{q},t)\,dt = \int_{t_1}^{t_2}\frac{\delta L}{\delta q_k}\delta q_k\,dt + \left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} \\ \delta q_k(t_1) = 0 = \delta q_k(t_2) \quad (k=1,2,\cdots,s+1) \\ \delta f = \frac{\partial f}{\partial q_k}\delta q_k = 0 \Rightarrow \int_{t_1}^{t_2}\lambda(t)\frac{\partial f}{\partial q_k}\delta q_k\,dt = 0 \quad (\forall\lambda) \end{cases}$$

$$\Rightarrow \int_{t_1}^{t_2}\underbrace{\left[\frac{\delta L}{\delta q_k} + \lambda\frac{\partial f}{\partial q_k}\right]}_{G_k}\delta q_k\,dt = 0 \quad (\forall\lambda)$$

不妨设 $\delta q_1, \delta q_2, \cdots, \delta q_s$ 独立

总可找到 $\lambda(t)$，使 $G_{s+1} = 0 \Rightarrow G_k = 0 \ (k=1,2,\cdots,s)$

§2.5 对称与守恒

一.何为守恒量?

—力学量 $\Gamma(q,\dot{q},t)$: $\frac{d\Gamma}{dt} = 0 \Rightarrow \Gamma(q,\dot{q},t) = C = \Gamma(q^{(0)},\dot{q}^{(0)},0)$

(也称运动常数、初次积分)

自由度为 s 的体系至多有 2s 个独立的守恒量

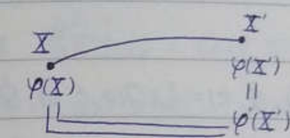
$$\begin{cases} q_k(t) = f_k(q^{(0)},\dot{q}^{(0)},t) \\ \dot{q}_k(t) = g_k(q^{(0)},\dot{q}^{(0)},t) \end{cases} \Rightarrow \begin{cases} q_k^{(0)} = f_k(q,\dot{q},-t) \\ \dot{q}_k^{(0)} = f_k(q,\dot{q},-t) \end{cases}$$

自由度为 s 的体系至多有 2s-1 个不显含 t 的独立的守恒量

(考虑相空间、含时间 t 的拓展相空间)

二.何为对称性(不变性)?

$X' = X'(X)$

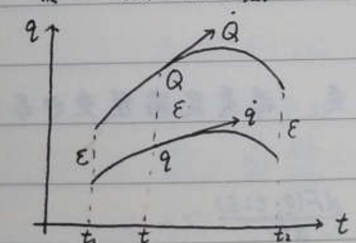


$$\begin{cases} \text{变换} \quad \varphi'(X') = \varphi(X) \\ \text{不变} \quad \varphi'(X') = \varphi(X') \end{cases}$$

如: $\begin{cases} X = x\cos\theta + y\sin\theta = X(x,y;\theta) \\ Y = -x\sin\theta + y\cos\theta = Y(x,y;\theta) \end{cases}$

三.单参数点变换

$$\begin{cases} q_k \longmapsto Q_k = Q_k(q,t;\varepsilon) \quad \text{不妨设 } Q_k|_{\varepsilon=0} = q_k \\ \dot{q}_k \longmapsto \dot{Q}_k = \frac{\partial Q_k}{\partial q_i}\dot{q}_i + \frac{\partial Q_k}{\partial t} = \dot{Q}_k(q,\dot{q},t;\varepsilon) \end{cases}$$



$\varepsilon\to0 \quad \begin{cases} q_k \longmapsto Q_k = q_k + \varepsilon S_k \\ \dot{q}_k \longmapsto \dot{Q}_k = \dot{q}_k + \varepsilon\dot{S}_k \end{cases} \quad S_k = \frac{\partial Q_k}{\partial\varepsilon}\Big|_{\varepsilon=0} = S_k(q,t)$

$L(q,\dot{q},t) \Rightarrow L_\varepsilon(q,\dot{q},t) \equiv L(Q,\dot{Q},t)$

$\quad = L(Q(q,t,\varepsilon),\dot{Q}(q,\dot{q},t,\varepsilon),t)$

$$\begin{cases} L_\varepsilon = L + \frac{dF(q,t,\varepsilon)}{dt} \quad \text{对应守恒量} \\ L_\varepsilon = \lambda(\varepsilon)L \quad \text{不对应守恒量} \end{cases}$$

$\varepsilon\to0$ 时称无穷小变换

四. 何为对称变换?

体系 $L=L(q,\dot{q},t)$

Def $L_\varepsilon(q,\dot{q},t)=L(Q,\dot{Q},t)=L(Q(q,t,\varepsilon),\dot{Q}(q,\dot{q},t,\varepsilon),t)$

Def 若 $L_\varepsilon(q,\dot{q},t)=L(q,\dot{q},t)$, 则称L是不变的

如 抛物运动

$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz$

变换 $\begin{cases} X=x\cos\theta+y\sin\theta \\ Y=-x\sin\theta+y\cos\theta \\ Z=z \end{cases} \Rightarrow L_\varepsilon=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz$

变换 $\begin{cases} X=x+\varepsilon \\ Y=y \\ Z=z \end{cases}$ $\begin{cases} X=x \\ Y=y+\varepsilon \\ Z=z \end{cases}$

这样的变换下作用量不变, 将真实路径变为另一条真实路径

Def 若 $L_\varepsilon(q,\dot{q},t)=L(q,\dot{q},t)+\frac{dF(q,t;\varepsilon)}{dt}$

则 称L是规范的不变的. 而相应的变换称对称变换

对称变换将真实路径变为真实路径

但 将真实路径变为真实路径的未必是对称变换

如: $X=x$, $Y=y$, $Z=e^\varepsilon z$

五. Noether 定理(位形空间)

若 $q_k\mapsto Q_k=Q_k(q,t;\varepsilon)$ 是体系 $L(q,\dot{q},t)$ 的对称变换.

则 力学量 $\Gamma(q,\dot{q},t)=\frac{\partial L}{\partial \dot{q}_k}S_k-G=p_kS_k-G$ 为运动常数

其中 $S_k\equiv\left.\frac{\partial Q_k}{\partial\varepsilon}\right|_{\varepsilon=0}=S_k(q,t)$

$G\equiv\left.\frac{\partial F}{\partial\varepsilon}\right|_{\varepsilon=0}=G(q,t)$

如: $X=x\cos\theta+y\sin\theta$, $Y=-x\sin\theta+y\cos\theta$, $Z=z$

$L_\varepsilon=L$, $S_x=y$, $S_y=-x$, $S_z=0$

$\Gamma=p_xS_x+p_yS_y=yp_x-xp_y=-l_z$

证: (1) $Q_k|_{\varepsilon=0}=q_k$

对任意变换

$\left.\frac{\partial L_\varepsilon}{\partial\varepsilon}\right|_{\varepsilon=0}=\frac{\partial L}{\partial Q_k}\left.\frac{\partial Q_k}{\partial\varepsilon}\right|_{\varepsilon=0}+\frac{\partial L}{\partial\dot{Q}_k}\left.\frac{\partial\dot{Q}_k}{\partial\varepsilon}\right|_{\varepsilon=0}$

$=\frac{\partial L}{\partial q_k}S_k+\frac{\partial L}{\partial\dot{q}_k}\dot{S}_k$

(2) 对于对称变换

$\left.\frac{\partial L_\varepsilon}{\partial\varepsilon}\right|_{\varepsilon=0}=\lim_{\varepsilon\to0}\frac{L_\varepsilon-L_{\varepsilon=0}}{\varepsilon}=\frac{dG}{dt}$

$\varepsilon\to0$: $F=\varepsilon G$ $L_\varepsilon=L+\varepsilon\frac{dG}{dt}$

(3) 对于真实运动

$\frac{\partial L}{\partial q_k}=\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_k}=\dot{p}_k$

$\Rightarrow \dot{p}_kS_k+p_k\dot{S}_k=\dot{G}$

$\frac{d}{dt}(p_kS_k-G)=0$

如: $L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\phi}^2\sin^2\theta)-U(r,\cancel{\theta},\cancel{\phi})$

$\begin{cases} r\mapsto R=r \\ \theta\mapsto\Theta=\theta \\ \phi\mapsto\Phi=\phi+\varepsilon \end{cases}$ $\Rightarrow L_\varepsilon=L$ $\Gamma=p_\phi=mr^2\dot{\phi}\sin^2\theta$

六. Noether 定理(3维 Euclid 空间)

若 $\vec{r}_a\mapsto\vec{R}_a=\vec{R}_a(\vec{r},t;\varepsilon)$ 是体系的对称变换.

即 $L_\varepsilon(\vec{r}, \dot{\vec{r}}, t) \equiv L(\vec{R}, \dot{\vec{R}}, t) = L(\vec{R}(\vec{r}, t; \varepsilon), \dot{\vec{R}}(\vec{r}, \dot{\vec{r}}, t; \varepsilon), t)$

$$= L(\vec{r}, \dot{\vec{r}}, t) + \frac{dF(\vec{r}, t; \varepsilon)}{dt}$$

$q_k \longleftrightarrow \vec{r}_a = \vec{r}_a(q_k, t)$

$\downarrow$

$Q_k \longleftrightarrow \vec{R}_a = \vec{R}_a(Q_k, t)$

$\varepsilon \to 0$: $\vec{R}_a = \vec{r}_a + \varepsilon \vec{\eta}_a$

$$\vec{\eta}_a = \frac{\partial \vec{R}_a}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial \vec{r}_a}{\partial Q_k}\Big|_{Q=q} \frac{\partial Q_k}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial \vec{r}_a}{\partial q} S_k$$

$$\frac{\partial L}{\partial \dot{q}_k} S_k = \frac{\partial L}{\partial \dot{\vec{r}}_a} \frac{\partial \dot{\vec{r}}_a}{\partial \dot{q}_k} S_k = \frac{\partial L}{\partial \dot{\vec{r}}_a} \frac{\partial \vec{r}_a}{\partial q_k} S_k = \frac{\partial L}{\partial \dot{\vec{r}}_a} S_k$$

则 力学量 $\Gamma = \frac{\partial L}{\partial \dot{\vec{r}}_a} \cdot \vec{\eta}_a - G$ 为运动常数

其中 $\vec{\eta}_a \equiv \frac{\partial \vec{R}_a}{\partial \varepsilon}\Big|_{\varepsilon=0}$

$$G = \frac{\partial F}{\partial \varepsilon}\Big|_{\varepsilon=0} = G(\vec{r}, t)$$

注意不能将 $\frac{\partial L}{\partial \dot{\vec{r}}_a}$ 轻易写为动量

七. 孤立体系（不采用求和约定）

$L = T - U \quad U = \frac{1}{2}\sum_{a,b} U_{ab}(r_{ab})$

1. 空间平移

$\vec{r}_a \longmapsto \vec{R}_a = \vec{r}_a + \varepsilon \hat{n}$

$T = \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^2 \quad \vec{r}_{ab} = \vec{R}_{ab}$

$L_\varepsilon = L \quad \vec{\eta}_a = \hat{n}$

$\Gamma = \sum_a \vec{p}_a \cdot \vec{\eta}_a = \sum_a \vec{p}_a \cdot \hat{n} = \hat{n} \cdot \vec{P}$

$\Rightarrow \vec{P} \equiv \sum_a m_a \dot{\vec{r}}_a$ 守恒

2. 空间转动

$\vec{r}_a \longmapsto \vec{R}_a = \vec{r}_a + \hat{n}\, \overset{\varepsilon}{d\theta} \times \vec{r}_a$ 无穷小转动

$L_\varepsilon = L \quad \vec{\eta}_a = \hat{n} \times \vec{r}_a$

$\Gamma = \sum_a \vec{p}_a \cdot \vec{\eta}_a = \sum_a \vec{p}_a \cdot (\hat{n} \times \vec{r}_a) = \hat{n} \cdot \sum_a (\vec{r}_a \times \vec{p}_a) = \hat{n} \cdot \vec{L}$

$\Rightarrow \vec{L} \equiv \sum_a \vec{r}_a \times m_a \vec{v}_a$ 守恒

注意 力矩. 角动量定义含位矢. 与原点有关

此处角动量对任一点均守恒

3. 速度变换(推动)

$\vec{r}_a \longmapsto \vec{R}_a = \vec{r}_a + \varepsilon \hat{n} t \Rightarrow \dot{\vec{R}}_a = \dot{\vec{r}}_a + \varepsilon \hat{n}$

$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$

$\vec{R}_{ab} = \vec{R}_a - \vec{R}_b = \vec{r}_{ab} \Rightarrow U_\varepsilon = U$

$T_\varepsilon = \sum_a \frac{1}{2} m_a \dot{\vec{R}}_a^2 = \sum_a \frac{1}{2} m_a (\dot{\vec{r}}_a + \varepsilon \hat{n})^2$

$$= \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^2 + \underbrace{\varepsilon \hat{n} \cdot \sum_a m_a \dot{\vec{r}}_a}_{\frac{d}{dt}[\varepsilon \cdot \hat{n} \sum_a m_a \vec{r}_a]} + \varepsilon^2 \sum_a \frac{1}{2} m_a$$

规范不变, $\vec{\eta}_a = \hat{n} t$, $G = \hat{n} \cdot \sum_a m_a \vec{r}_a$

$\Gamma = \sum_a \vec{p}_a \cdot \vec{\eta}_a - G$

$= \hat{n} \cdot t \sum_a m_a \dot{\vec{r}}_a - \hat{n} \cdot \sum_a m_a \vec{r}_a$

$= \hat{n} \cdot [(\sum_a m_a \dot{\vec{r}}_a) t - \sum_a m_a \vec{r}_a]$

$= \hat{n} \cdot M [\vec{V}_c t - \vec{r}_c]$

$\Rightarrow \vec{r}_c = \vec{r}_{c0} + \vec{V}_c t$

$\dot{\vec{r}}_c = \frac{d}{dt}(\vec{V}_c t) = \dot{\vec{V}}_c t + \vec{V}_c \Rightarrow \dot{\vec{V}}_c = 0$

$\Rightarrow$ 质心匀速运动

八. 非孤立体系

$L = T - U$, $T = \sum_a \frac{1}{2} m_a v_a^2$, $U = \frac{1}{2}\sum_{a,b} U_{ab}(r_{ab}) + U_{外}(\vec{r}, t)$

平移/转动. L 不变? $\Leftrightarrow$ $U_{外}$ 不变 $\Leftrightarrow$ "荷"不变

在"荷"均匀时. ⇔ "荷"区域是否几何对称性

讨论时间平移不变性:

避免不等时变分, 将时间参数化 $t=t(\sigma)$

则 $q=q(\sigma)=q(t(\sigma))$ $\quad \dot{q}=\frac{d\sigma}{dt}\frac{dq}{d\sigma}=\frac{q'}{t'}$

$$S=\int_{t_1}^{t_2} L\,dt=\int_{\sigma_1}^{\sigma_2}\tilde{L}\,d\sigma \qquad \tilde{L}=t'L(q,\frac{q'}{t'},t)=\tilde{L}(q,t,q',t',\sigma)$$

形式式地将 t 作为一个坐标

$$\tilde{P}_k=\frac{\partial\tilde{L}}{\partial q'_k}=t'\frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q'_k}=P_k$$

$$\tilde{P}_t=\frac{\partial\tilde{L}}{\partial t'}=L+t'\frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial t'}=L-P_k\dot{q}_k=-h$$

$$\tilde{h}=\tilde{P}_k q'_k+\tilde{P}_t t'-\tilde{L}\equiv 0$$

变换 $\begin{cases} Q_k=Q_k(q,t;\varepsilon) \\ \tau=\tau(q,t;\varepsilon) \end{cases}$

$$\tilde{L}_\varepsilon(q,t;q',t')=\tilde{L}(q,t;q',t')+\frac{dF(q,t,\varepsilon)}{d\sigma}$$

$$\Gamma=\tilde{P}_k S_k+\tilde{P}_t S_t-G$$

其中 $S_k=\frac{\partial Q_k}{\partial\varepsilon}\Big|_{\varepsilon=0}$ $\quad S_t=\frac{\partial\tau}{\partial\varepsilon}\Big|_{\varepsilon=0}$ $\quad G=\frac{\partial F}{\partial\varepsilon}\Big|_{\varepsilon=0}$

$$t'\frac{\partial\tau}{\partial t}L(Q,\frac{dQ}{d\tau},\tau)=t'L(q,\dot{q},t)+t'\frac{dF(q,t,\varepsilon)}{dt}$$

$$\Rightarrow \frac{d\tau}{dt}L(Q,\frac{dQ}{d\tau},\tau)=L(q,\dot{q},t)+\frac{dF(q,t,\varepsilon)}{dt}$$

$$\Gamma=p_k S_k-hS_t-G$$

$$L(q,\dot{q},t+\varepsilon)=L(q,\dot{q},t)$$

$$\Rightarrow \Gamma=-h$$

一般转动 $SO(3)$ $\quad dl^2=dx_i dx_i=dx^2+dy^2+dz^2$ $\quad$ 正交变换

狭义相对论 $SR\ O(3,1)$ $\quad ds^2=dl^2-c^2dt^2$ $\quad$ 洛伦兹变换

$x_0=ct$

Galileo $\quad x'_i=\lambda_{ij}x_j+D_i+V_i t$

$t'=t+t_0 \quad (dt'=dt)$

Poincare $\quad x'_\alpha=\underline{\Lambda_{\alpha\beta}x_\beta}+D_\alpha$

Lorentz

沿三个轴的转动.沿三个轴各洛伦兹变换.沿三个轴推动

狭义相对论要求物理理论必须满足其规定的对称性

对于物理规律, 此处对拉格朗日函数作了很强的限制

$\begin{cases}\text{乒乓球} \to \text{篮球} \quad \text{标度变换} \\ \text{填充二者间隔} \quad \text{规范变换}\end{cases}$

量子电动力学(QED)要求

(1) 满足狭义相对论

(2) 要求满足 $U(1)$ $\quad \psi'(x)=e^{i\alpha(x)}\psi(x)$

强相互作用 QCD $\quad SU(3)$

标准模型SM $\quad$ 更高对称性 $\quad SU(2)\times U(1)$

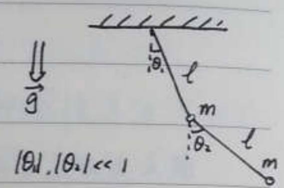
CH3. 微振动

§3.1 双摆

一. Lagrange 函数

$$L = \frac{1}{2}m[l^2\dot\theta_1^2 + l^2(\dot\theta_1 + \dot\theta_2)^2]$$

$$-mg[\frac{1}{2}l\theta_1^2 + \frac{1}{2}l(\theta_1^2 + \theta_2^2)]$$



合理近似：近似要自洽，保留到同一阶 $|\theta_1|, |\theta_2| \ll 1$

$$L = ml^2[(\dot\theta_1^2 + \dot\theta_1\dot\theta_2 + \frac{1}{2}\dot\theta_2^2) - \omega_0^2(\theta_1^2 + \frac{1}{2}\theta_2^2)] \qquad \omega_0 = \sqrt{\frac{g}{l}}$$

二. 运动方程

$$\begin{cases} 2\ddot\theta_1 + \ddot\theta_2 = -2\omega_0^2\theta_1 \\ \ddot\theta_1 + \ddot\theta_2 = -\omega_0^2\theta_2 \end{cases} \qquad 耦合 \Rightarrow 解耦$$

解耦：(1) 考虑对上式求二阶导，代入下式

解耦 并得到四阶微分方程

(2) 考虑更换广义坐标的选取

三. 解耦

$$① + \alpha\times② \quad (2+\alpha)\frac{d^2}{dt^2}[\theta_1 + \frac{1+\alpha}{2+\alpha}\theta_2] = -2\omega_0^2[\theta_1 + \frac{\alpha}{2}\theta_2]$$

使之系数相等

$$2 + 2\alpha = 2\alpha + \alpha^2 \Rightarrow \alpha^2 = 2 \Rightarrow \alpha = \pm\sqrt2$$

$$\frac{d^2}{dt^2}(\theta_1 \pm \frac{\theta_2}{\sqrt2}) = -(2\mp\sqrt2)\omega_0^2(\theta_1 \pm \frac{\theta_2}{\sqrt2})$$

$$\begin{cases} \xi_1 = \theta_1 + \frac{\theta_2}{\sqrt2} = 2\lambda_1\cos(\omega_1 t + \varphi_1) & \omega_1 \equiv \sqrt{2-\sqrt2}\,\omega_0 \\ \xi_2 = \theta_1 - \frac{\theta_2}{\sqrt2} = 2\lambda_2\cos(\omega_2 t + \varphi_2) & \omega_2 \equiv \sqrt{2+\sqrt2}\,\omega_0 \end{cases}$$

$$\Rightarrow \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\frac{\xi_1+\xi_2}{2}\\ \frac{\xi_1-\xi_2}{2}\end{pmatrix} = \begin{pmatrix}1\\ \sqrt2\end{pmatrix}\lambda_1\cos(\omega_1 t+\varphi_1) + \begin{pmatrix}1\\ -\sqrt2\end{pmatrix}\lambda_2\cos(\omega_2 t+\varphi_2)$$

四. 分析

$$\begin{cases} \lambda_1 \neq 0,\ \lambda_2 = 0 & \omega_1: \begin{pmatrix}1\\ \sqrt2\end{pmatrix} \\ \lambda_1 = 0,\ \lambda_2 \neq 0 & \omega_2: \begin{pmatrix}1\\ -\sqrt2\end{pmatrix} \end{cases} \quad 简正模$$

简正频率　本征矢

ξ_1，ξ_2 简正坐标

§3.2 简谐近似

一. 问题的描述

体系：自由度 s $q = (q_1, q_2, \cdots, q_s)$

假定：1. 外部约束稳定 $\vec r = \vec r(q, t)$

外场稳定 $U = U(\vec r, \dot{\vec r}, t) = U(q, \dot q, t)$

$$T = \frac{1}{2}m_{ij}\dot q_i\dot q_j \qquad m_{ij}(q) = m_a\frac{\partial\vec r_a}{\partial q_i}\frac{\partial\vec r_a}{\partial q_j}$$

2. 稳定平衡位置 $q^{(0)}$

$$U(q) = U(q^{(0)}) + \frac{\partial U}{\partial q_k}\Big|_{q^{(0)}}(q_k - q_k^{(0)}) + \frac{1}{2!}\frac{\partial^2 U}{\partial q_i\partial q_j}(q_i - q_i^{(0)})(q_j - q_j^{(0)}) + $$

$$\frac{\partial U}{\partial q_k}\Big|_{q^{(0)}} = 0 \qquad (k = 1, 2, \cdots, s)$$

$$K \equiv \left(\frac{\partial^2 U}{\partial q_i\partial q_j}\right)_{q^{(0)}}$$ 正定，对称（假定存在）

二. 谐振近似

$$\xi_k = q_k - q_k^{(0)}$$

$$T \approx \frac{1}{2}M_{ij}\dot\xi_i\dot\xi_j \qquad M_{ij} = m_{ij}(q^{(0)})$$

$$U \approx \frac{1}{2}K_{ij}\xi_i\xi_j$$

$$\Rightarrow L = \frac{1}{2}M_{ij}\dot\xi_i\dot\xi_j - \frac{1}{2}K_{ij}\xi_i\xi_j$$

$$\frac{\partial L}{\partial\dot\xi_n} = \frac{1}{2}M_{ij}(\delta_{in}\dot\xi_j + \dot\xi_i\delta_{jn}) = \frac{1}{2}(M_{nj}\dot\xi_j + M_{ni}\dot\xi_i) = M_{nj}\dot\xi_j$$

$$M_{ij}\ddot{\xi}_j + K_{ij}\xi_j = 0 \Rightarrow M\ddot{\xi} + K\xi = 0$$

§3.3 简正坐标与简正模(不采用求和约定)

$\begin{cases} M \text{ 对称, 正定} \\ K \text{ 对称} \end{cases}$ $\quad M\ddot{\xi} + K\xi = 0 \Rightarrow \ddot{\xi} = -(M^{-1}K)\xi$

一. 线性坐标变换 $\xi = A\eta$

$MA\ddot{\eta} + KA\eta = 0 \Rightarrow \ddot{\eta} = -A^{-1}(M^{-1}K)A\eta$

二. 同时对角化

M 对称 $\Rightarrow O_1^T M O_1 = M_d = diag\{m_1, m_2, \cdots, m_s\}$ $\quad O_1 \in O(3)$

M 正定 $\Rightarrow m_a > 0 \Rightarrow D^T O_1^T M O_1 D = I$ $\quad D = diag\{\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \cdots, \frac{1}{\sqrt{m_s}}\}$

K 对称 $\Rightarrow D^T O_1^T K O_1 D$ 对称 $\Rightarrow O_2^T D^T O_1^T K O_1 D O_2 = \Omega_d = diag\{\omega_1^2, \omega_2^2, \cdots, \omega_s^2\}$

$O_2^T D^T O_1^T M O_1 D O_2 = I$ $\quad O_2 \in O(3)$

存在 $A = O_1 D O_2$, $A^T M A = I$, $A^T K A = \Omega_d = diag\{\omega_1^2, \omega_2^2, \cdots, \omega_s^2\}$

三. 本征值与本征矢

$A^T M A = I$, $A^T K A = \Omega_d = diag\{\omega_1^2, \omega_2^2, \cdots, \omega_s^2\}$

$(A^T)^{-1} = MA \Rightarrow KA = MA\Omega_d \Rightarrow (KA^{(1)}, KA^{(2)}, \cdots, KA^{(s)}) = (\omega_1^2 MA^{(1)}, \cdots, \omega_s^2 MA^{(s)})$

A 每一列 X: $KX = \omega^2 MX$

久期方程: $\det(K - \omega^2 M) = 0 \Rightarrow \omega_a^2 \ (a = 1, 2, \cdots, s)$

本征矢方程: $(K - \omega_a^2 M)X = 0 \Rightarrow A^{(a)} = cX$

1. ω_a^2 实

$X = \alpha + i\beta$ (α, β 实列矩阵)

$X^\dagger K X = (\alpha^T - i\beta^T)K(\alpha + i\beta) = \underline{\overset{\text{实}}{\alpha^T K\alpha + \beta^T K\beta}} + \underline{i(\alpha^T K\beta - \beta^T K\alpha)} = 0$

$\omega^2 X^\dagger M X = \omega^2[\underline{\alpha^T M\alpha + \beta^T M\beta}]$ 正

特别地, K 正定 $\Rightarrow \omega_a^2 > 0$

2. 本征矢可取为实

3. $KA = MA\Omega_d$

未必正交, 但对问题求解无影响

如果正交化、归一化, 会有更良好的几何性质

四. 方程的解

$$M\ddot{\xi} + K\xi = 0 \xrightarrow{\xi = A\eta} MA\ddot{\eta} + KA\eta = 0 = MA(\ddot{\eta} + \Omega_d\eta) = 0$$

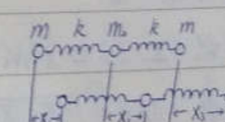
$$\ddot{\eta}_a + \omega_a^2\eta_a = 0 \quad \begin{cases} \omega_a^2 > 0 & \eta_a = \lambda_a\cos(\omega_a t + \varphi_a) \\ \omega_a^2 = 0 & \eta_a = \lambda_a t + \varphi_a \\ \omega_a^2 = -\Omega_a^2 < 0 & \eta_a = c_a\cosh\Omega_a t + d_a\sinh\Omega_a t \end{cases}$$

$$\xi = A\eta = \sum_a A^{(a)}\eta_a(t) = A^{(1)}\eta_1(t) + A^{(2)}\eta_2(t) + \cdots + A^{(s)}\eta_s(t)$$

五. 简正模(K 半正定)

$\omega_a^2(\geq 0)$: A^a 简正频率. 本征矢

eg. CO_2



$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m_0\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$$

$$M = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad r = \frac{m_0}{m}$$

$$U = \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 = \frac{1}{2}k(x_1^2 + 2x_2^2 + x_3^2) - k(x_1x_2 + x_2x_3)$$

$$K = m\omega_0^2\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad k \equiv m\omega_0^2$$

$$K-\omega^2 M = m\omega_0^2\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-a & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}$$

$$\det(K-\omega^2 M) = \lambda(1-\lambda)[r\lambda-(r+2)]$$

$\Rightarrow \lambda_1 = 1,\ \omega_1^2 = \omega_0^2,\ A^{(1)} = (1, 0, -1)$

$\lambda_2 = 1+\frac{2}{r},\ \omega_2^2 = (1+\frac{2m}{m_0})\omega_0^2,\ A^{(2)} = (1, -2\frac{m}{m_0}, 1)$

$\lambda_3 = 0,\ \omega_3^2 = 0,\ A^{(3)} = (1, 1, 1)$

$\omega_3^2 = 0$ 刚性运动(整体平移、整体转动)

$f(r) = f(|\vec{r}|)$ 在 $\vec{r} = \vec{R}$ 作展开 $(\vec{u} \equiv \vec{r}-\vec{R})$

$$\frac{\partial f}{\partial x_i} = \frac{\partial r}{\partial x_i}\frac{\partial f}{\partial r} = \frac{x_i}{r}f'(r)$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{x_i x_j}{r^2}f''(r) + \left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right)f'(r)$$

$$f(r) = f(R) + (\hat{R}\cdot\vec{u})f'(R) + \frac{1}{2}(\hat{R}\cdot\vec{u})^2 f''(R) + \frac{1}{2}\left[\frac{u^2-(\hat{R}\cdot\vec{u})^2}{R}\right]f'(R) + \cdots$$

角动量 $\vec{L} = \sum_a \vec{r}_a \times m_a\dot{\vec{u}}_a \approx \sum_a \vec{R}_a \times m_a\dot{\vec{u}}_a = \frac{d}{dt}(\sum_a \vec{R}_a \times m_a\vec{u}_a)$

六. 正交归一化

内积 $(X, Y) \equiv X^T M Y$

$\begin{cases} \text{模} & \|X\| = \sqrt{(X, X)} \\ \text{夹角} & \cos\theta = \dfrac{(X, Y)}{\|X\|\|Y\|} \end{cases}$

$(A^{(a)}, A^{(b)}) = \delta_{ab} \Leftrightarrow A^T M A = I$

不同本征值的本征矢必正交

重根本征矢可由 Gram-Smit 程序正交归一化

eg. 0—m—0—m—0 … —0—m—0—m—0

0 1 2 … n-2 n-1 n

平衡时, $x_a = ad$

$L = T - U = \cdots$

$$\frac{\partial L}{\partial \psi_a} = -k(\psi_a - \psi_{a-1}) + k(\psi_{a+1} - \psi_a)$$

$\ddot{\psi} = -K\psi$ where $K = \omega_0^2\begin{pmatrix} \ddots & \ddots & 0 & 0 & 0 & 0 \\ \ddots & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & \ddots \\ 0 & 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}$ 最近邻作用

次近邻作用

一、令 $\psi_a(t) = Ce^{i(kx_a-\omega t)}$, $k = \frac{2\pi}{\lambda}$

$\psi_{a\pm1} = \psi_a e^{\pm ikd}$

$-\omega^2\psi_a = -\omega_0^2(2-e^{ikd}-e^{-ikd})\psi_a$

$\omega^2 = \omega_0^2(2-2\cos kd) = 4\omega_0^2\sin^2\frac{kd}{2}$

$\Rightarrow \omega = 2\omega_0\sin\frac{kd}{2}$ 色散关系

二、边界条件

选择: $\psi_0 = 0 = \psi_n$

令 $\psi_a(t) = Ce^{-i\omega t}\sin kx_a = C\dfrac{e^{i(kx_a-\omega t)}-e^{-i(kx_a+\omega t)}}{2i}$

$\psi_n = 0 \Rightarrow \sin knd = \sin kL = 0 \Rightarrow k_\beta = \beta\frac{\pi}{nd} = \beta\frac{\pi}{L}\ (\beta = 1, 2, \ldots, n-1)$

$\Rightarrow \lambda_\beta = \frac{2\pi}{k_\beta} = \frac{2L}{\beta} \Rightarrow L = \frac{\beta}{2}\lambda_\beta$

$\omega_\beta = 2\omega_0\sin\frac{\beta\pi}{2n}$

$$A^{(\beta)} = C\begin{pmatrix} \sin k_\beta x_1 \\ \sin k_\beta x_2 \\ \vdots \\ \sin k_\beta x_{n-1} \end{pmatrix} = C\begin{pmatrix} \sin 1\cdot\frac{\beta\pi}{n} \\ \sin 2\cdot\frac{\beta\pi}{n} \\ \vdots \\ \sin(n-1)\cdot\frac{\beta\pi}{n} \end{pmatrix}$$

$\omega_\beta = 2\omega_0\sin\frac{\beta\pi}{2n}$

§3.4 连续体系（场论初步）

eg. 螺丝纹螺母

编号 $-n, \cdots, 0, \cdots, n$

摆长 l. $x_i = \beta\phi_i$. $\lambda^2 = \beta^2 + l^2$

质量 m, 弹性系数 k. $\omega^2 = \frac{k\beta^2}{m\lambda^2}$ $\Omega^2 = \frac{gl}{\lambda^2}$

间距 a

$T_i = \frac{1}{2}m(\dot{x}_i^2 + l^2\dot{\phi}^2) = \frac{1}{2}m(\beta^2 + l^2)\dot{\phi}^2$

$L = \frac{1}{2}m\lambda^2 \sum_{i=-n}^{n} \dot{\phi}_i^2 - \frac{1}{2}k\beta^2 \sum_{i=-n}^{n-1} (\phi_{i+1} - \phi_i)^2 - mgl \sum_{i=-n}^{n} (1-\cos\phi_i)$

$\frac{\partial L}{\partial \dot{\phi}_i} = m\lambda^2\dot{\phi}_i$

$\frac{\partial L}{\partial \phi_i} = k\beta^2[(\phi_{i+1} - \phi_i) - (\phi_i - \phi_{i-1})] - mgl\sin\phi_i$

$m\lambda^2\ddot{\phi}_i = k\beta^2[(\phi_{i+1} - \phi_i) - (\phi_i - \phi_{i-1})] - mgl\sin\phi_i$

$\ddot{\phi}_i - \omega^2[(\phi_{i+1} - \phi_i) - (\phi_i - \phi_{i-1})] + \Omega^2\sin\phi_i = 0$, $-(n-1) \le i \le (n-1)$

一. 例子

二. 连续极限

每一摆替换为 s 个长为 l. 质量 $\Delta m = \frac{m}{s}$. 间距 $\Delta x = \frac{a}{s}$

质量 $m \to \Delta m = \frac{m}{s}$ $\Rightarrow \rho = \frac{\Delta m}{\Delta x} = \frac{m}{a}$

间距 $a \to \Delta x = \frac{a}{s}$ $\Rightarrow Y = \frac{ak\beta^2}{\lambda^2}$

弹性系数 $k \to sk$

$\omega^2 \to \frac{Y}{\rho}\frac{1}{(\Delta x)^2} = v^2\frac{1}{(\Delta x)^2}$ $v = \sqrt{\frac{Y}{\rho}}$

L: $\frac{1}{2}m\lambda^2 \to \frac{1}{2}\rho\lambda^2\Delta x$

$\frac{1}{2}k\beta^2 \to \frac{1}{2}sk\beta^2 = \frac{1}{2}\frac{ak\beta^2}{\lambda^2}\frac{\lambda^2 s}{a} = \frac{1}{2}Y\lambda^2\frac{1}{\Delta x}$

$mgl \to \rho gl\Delta x$

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$L = \sum \Delta x \{ \frac{1}{2}\rho\lambda^2\dot{\phi}^2(x) - \frac{1}{2}Y\lambda^2[\frac{\phi(x+\Delta x) - \phi(x)}{\Delta x}]^2 - \rho gl[1-\cos\phi(x)] \}$

$\phi_i \to \phi(x)$ x - 平衡时悬挂点位置

$\ddot{\phi}(x) - v^2\frac{1}{\Delta x}[\frac{\phi(x+\Delta x) - \phi(x)}{\Delta x} - \frac{\phi(x) - \phi(x-\Delta x)}{\Delta x}] + \Omega^2\sin\phi(x) = 0$

$\phi_i \to \phi(x) \to \phi(t, x)$

1. Lagrange 密度

$$\mathcal{L} = \frac{1}{2}\rho\lambda^2(\partial_t\phi)^2 - \frac{1}{2}Y\lambda^2(\partial_x\phi)^2 - \rho g(1-\cos\phi)$$

$$= \mathcal{L}(\phi, \partial_t\phi, \partial_x\phi)$$

2. sine-Gordon 方程 (sG)

$$\frac{\partial^2\phi}{\partial t^2} - v^2\frac{\partial^2\phi}{\partial x^2} + \Omega^2\sin\phi = 0$$

(1) $g = 0$ ($\Omega = 0$)

令 $\psi(t, x) = \lambda\phi(t, x)$

$\mathcal{L} = \frac{1}{2}\rho(\partial_t\psi)^2 - \frac{1}{2}Y(\partial_x\psi)^2 = \mathcal{L}(\partial_t\psi, \partial_x\psi)$

$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$ 波动方程

(2) $|\phi| \ll 1$ $\psi = \lambda\phi$

$\mathcal{L} = \frac{1}{2}\rho[(\partial_t\psi)^2 - v^2(\partial_x\psi)^2 + \Omega^2\psi^2]$

$\frac{\partial^2\psi}{\partial t^2} - v^2\frac{\partial^2\psi}{\partial x^2} + \Omega^2\psi = 0$ Klen-Gordon 方程

$$\begin{cases} \frac{\delta L}{\delta q_k} \equiv \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0 \\ \frac{\delta \mathcal{L}}{\delta \psi} \equiv \frac{\partial \mathcal{L}}{\partial \psi} - \partial_t\frac{\partial \mathcal{L}}{\partial(\partial_t\psi)} - \partial_x\frac{\partial \mathcal{L}}{\partial(\partial_x\psi)} = 0 \end{cases}$$

$\Leftrightarrow \delta S = \delta\int \mathcal{L}\, dx\, dt = 0$

三. 符号

1. 时间空坐标

$x = (x_0, x_1, x_2, x_3) = (t, \vec{r})$ $\alpha, \beta, \gamma = 0, 1, 2, 3$ | $i, j, k = 1, 2, 3$ 求和

2. 场

$\psi_I = \psi_I(x) = \psi_I(t, \vec{r}) \quad (I = 1, 2, \cdots, N)$

$\psi = (\psi_1, \psi_2, \cdots, \psi_N) \qquad \partial\psi$

3. 作用量

$S = \int L\,dt = \int \mathcal{L}(\psi, \partial\psi, x)\,d^4x \qquad L = \int \mathcal{L}\,d^3x$

四. Hamilton 原理

$$\begin{cases} \psi_I \longmapsto \psi_I + \delta\psi_I \\ \partial_\alpha \psi_I \longmapsto \partial_\alpha \psi_I + \delta(\partial_\alpha \psi_I) \end{cases}$$

$$\begin{cases} 0 = \delta S = \delta \int_R \mathcal{L}(\psi, \partial\psi, x)\,d^4x \\ \delta\psi_I(x)|_{x\in\partial R} = 0 \end{cases}$$

$\delta\mathcal{L} = \mathcal{L}(\psi + \delta\psi, \partial\psi + \partial(\delta\psi), x)$

$- \mathcal{L}(\psi, \partial\psi, x)$

$= \frac{\partial\mathcal{L}}{\partial\psi_I}\delta\psi_I + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_I)}\partial_\alpha(\delta\psi_I) \rightarrow \partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_I)}\delta\psi_I\right] - \left[\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_I)}\right]\delta\psi_I$

$= \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\psi_I} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_I)}\right]}_{\frac{\delta\mathcal{L}}{\delta\psi_I}}\delta\psi_I + \partial_\alpha\underbrace{\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_I)}\delta\psi_I\right]}_{X_\alpha}$

四维矢量的散度(端点处)是零

Euler - lagrange 方程

$$\frac{\delta\mathcal{L}}{\delta\psi_I} \equiv \frac{\partial\mathcal{L}}{\partial\psi_I} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_I)} = \frac{\partial\mathcal{L}}{\partial\psi_I} - \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\psi_I)} - \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\psi_I)} = 0$$

$\mathcal{L}(\phi, \partial_t\phi, \partial_x\phi) = \rho\lambda^2\left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}v^2(\partial_x\phi)^2 - \Omega^2(1-\cos\phi)\right]$

$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} + \partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}$

$-\Omega^2\sin\phi = \frac{\partial^2\phi}{\partial t^2} - v^2\frac{\partial^2\phi}{\partial x^2} \qquad$ SG 方程

非线性方程. 不满足叠加原理.

若 $f(x, y, z(x, y))$

$$\begin{cases} 偏导 & \frac{\partial f}{\partial x} \\ 全偏导 & \left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \end{cases}$$

拉格朗日密度的不确定性

$\mathcal{L} \Longleftrightarrow \mathcal{L}' = \mathcal{L} + \partial_\alpha F_\alpha(\psi, x)$

五. Maxwell 方程组

1. 源已知 $\quad j_\alpha = (j_0, \vec{j}) = (\rho, \vec{j})$

$0 = \partial_t\rho + \nabla\cdot\vec{j} = \partial_\alpha j_\alpha \qquad Q = \int\rho\,dV$ 连续性方程

2. 电磁势 $\quad A_\alpha = (A_0, \vec{A}) = (-\varphi, \vec{A})$

$$\begin{cases} \vec{E} = -\nabla\varphi - \partial_t\vec{A} \\ \vec{B} = \nabla\times\vec{A} \end{cases} \Rightarrow \begin{cases} E_i = \partial_i A_0 - \partial_0 A_i \\ B_i = \varepsilon_{ijk}\partial_j A_k \end{cases}$$

3. Lagrange 密度 $\left(c^2 = \frac{1}{\varepsilon_0\mu_0}\right)$

$\mathcal{L} = \mathcal{L}(A, \partial A, x) = \frac{1}{2}\varepsilon_0(E^2 - c^2B^2) - (\rho\varphi - \vec{j}\cdot\vec{A})$

$-\rho(\varphi - \vec{v}\cdot\vec{A})$ 单位体积广义势能

$= \frac{1}{2}\varepsilon_0(E^2 - c^2B^2) + j_\alpha A_\alpha$

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial A_0} = \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} + \partial_j\frac{\partial\mathcal{L}}{\partial(\partial_j A_0)} \\ \frac{\partial\mathcal{L}}{\partial A_k} = \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 A_k)} + \partial_j\frac{\partial\mathcal{L}}{\partial(\partial_j A_k)} \end{cases}$$

$\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\rho A_\sigma)} = \delta_{\alpha\rho}\delta_{\beta\sigma}$

$\frac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} = 0$

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial(\partial_j A_0)} = \varepsilon_0 E_k\frac{\partial E_k}{\partial(\partial_j A_0)} = \varepsilon_0 E_k\delta_{jk} = \varepsilon_0 E_j \\ \frac{\partial\mathcal{L}}{\partial(\partial_0 A_k)} = \varepsilon_0 E_j\frac{\partial E_j}{\partial(\partial_0 A_k)} = -\varepsilon_0 E_k \\ \frac{\partial\mathcal{L}}{\partial(\partial_j A_k)} = -\varepsilon_0 c^2 B_i\frac{\partial B_i}{\partial(\partial_j A_k)} = -\varepsilon_0 c^2 B_i\varepsilon_{imn}\delta_{mj}\delta_{nk} = -\varepsilon_0 c^2\varepsilon_{ijk}B_i \end{cases}$$

$Y_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial A_\beta)}$ 反对称

$Y_{00} = 0 \quad Y_{j0} = \varepsilon_0 E_j = -Y_{0j}$

$Y_{ij} = -\varepsilon_0 c^2 \varepsilon_{ijk} B_k$

$\begin{cases} j_0 = \varepsilon_0 \partial_j E_j \\ j_k = -\varepsilon_0 \partial_t E_k - \varepsilon_0 c^2 \varepsilon_{jki} \partial_j B_i = [-\varepsilon_0 \partial_t \vec{E} + \frac{1}{\mu_0} \nabla\times\vec{B}] \end{cases}$

另外两个方程自然成立(因为是用电磁势描述)

六. Nother 定理

若 $\psi_I \mapsto \Psi_I = \Psi_I(\psi, x; \varepsilon)$ $\Psi|_{\varepsilon=0} = \psi$ 为体系 $\mathcal{L}(\psi, \partial\psi, x)$ 的对称变换. 即 $\mathcal{L}_\varepsilon(\psi, \partial\psi, x) = \mathcal{L}(\Psi, \partial\Psi, x) = \mathcal{L}(\psi, \partial\psi, x) + \partial_\alpha F_\alpha(\psi, x; \varepsilon)$

则 $\Gamma_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \eta_I - G_\alpha$

其中 $\eta_I(\psi, x) \equiv \frac{\partial \Psi}{\partial \varepsilon}\Big|_{\varepsilon=0}$, $G_\alpha(\psi, x) \equiv \frac{\partial F_\alpha}{\partial \varepsilon}\Big|_{\varepsilon=0}$

为守恒流. 即 $\partial_\alpha \Gamma_\alpha = 0$

而 $Q = \int \Gamma_0 \, dV$ 为守恒荷, 即 $\frac{dQ}{dt} = 0$

证明: $\frac{\partial \mathcal{L}}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial \mathcal{L}}{\partial \Psi_I}\Big|_{\varepsilon=0} \frac{\partial \Psi_I}{\partial \varepsilon}\Big|_{\varepsilon=0} + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Psi_I)}\Big|_{\Psi=\psi} \frac{\partial(\partial_\alpha \Psi_I)}{\partial \varepsilon}\Big|_{\varepsilon=0}$

$= \frac{\partial \mathcal{L}}{\partial \psi_I} \eta_I + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \partial_\alpha \eta_I$

$\frac{\partial \mathcal{L}}{\partial \varepsilon}\Big|_{\varepsilon=0} = \partial_\alpha G_\alpha$

对真实场 $\frac{\partial \mathcal{L}}{\partial \psi_I} \eta_I + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \partial_\alpha \eta_I = \partial_\alpha [\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \eta_I]$

$\partial_\alpha [\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \eta_I] = \partial_\alpha G_\alpha$

eg. $\mathcal{L} = \frac{i\hbar}{2}[\psi^* \partial_t \psi - \psi \partial_t \psi^*] - \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi - U \psi^* \psi$

$\psi \longmapsto \Psi = e^{i\theta} \psi$

其 Lagrange 方程为薛定谔方程, 其运动积分为几率

Theorem 若 $\mathcal{L} = \mathcal{L}(\psi, \partial\psi, \cancel{x})$ 不显含时空坐标

则 存在四个守恒流(能量动量张量)

$T_{\alpha\beta} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \partial_\beta \psi_I - \delta_{\alpha\beta} \mathcal{L} \quad \partial_\alpha T_{\alpha\beta} = 0$

四个守恒荷(能量. 动量)

$P_\beta = \int T_{0\beta} \, dV \quad (P_0, \vec{P}) = (W, \vec{P})$

$\Gamma_\alpha \leftrightarrow \Gamma'_\alpha = \Gamma_\alpha + X_\alpha \ (\partial_\alpha X_\alpha = 0)$ 存在不确定性

$X_\alpha = \partial_\beta F_{\alpha\beta} \quad (F_{\alpha\beta} = -F_{\beta\alpha})$

$\partial_\alpha X_\alpha = \partial_\alpha \partial_\beta F_{\alpha\beta} \equiv 0$

$\frac{\partial \mathcal{L}}{\partial x_\beta} = \frac{\partial \mathcal{L}}{\partial \psi_I} \partial_\beta \psi_I + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \underline{\partial_\beta \partial_\alpha \psi_I} \rightarrow \partial_\alpha(\partial_\beta \psi_I)$

$= \partial_\alpha [\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi)} \partial_\beta \psi_I]$

$\frac{\partial \mathcal{L}}{\partial x_\beta} = \partial_\alpha(\delta_{\alpha\beta} \mathcal{L})$

eg. 自由空间的电磁场

$\mathcal{L} \equiv \frac{1}{2}\varepsilon_0(E^2 - c^2 B^2) = \mathcal{L}(A, \partial A, \cancel{x})$

$T_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\rho)}(\partial_\beta A_\rho - \partial_\rho A_\beta) - \delta_{\alpha\beta} \mathcal{L} + Y_{\alpha\rho} \partial_\rho A_\beta$

$\cancel{Y_{\alpha\rho} \partial_\rho A_\beta}$ $\partial_\alpha(Y_{\alpha\beta} \partial_\rho A_\beta) = (\partial_\alpha Y_{\alpha\beta}) \partial_\rho A_\beta + Y_{\alpha\beta} \partial_\alpha \partial_\rho A_\beta = 0$

$T_{\alpha\beta} = Y_{\alpha\rho}(\partial_\beta A_\rho - \partial_\rho A_\beta) - \delta_{\alpha\beta} \mathcal{L}$

$\begin{cases} T_{00} = Y_{0\rho}(\partial_0 A_\rho - \partial_\rho A_0) - \mathcal{L} = -\varepsilon_0 E_k(-E_k) - \mathcal{L} \\ T_{0j} = Y_{0\rho}(\partial_j A_\rho - \partial_\rho A_j) = -\varepsilon_0 E_k \varepsilon_{jki} B_i \\ T_{i0} = Y_{i\rho}(\partial_0 A_\rho - \partial_\rho A_0) = -\varepsilon_0 c^2 \varepsilon_{ik l} B_l (-E_k) \end{cases}$

(上三式 ρ 只能取 k)(反对称)

$T_{ij} = Y_{i\rho}(\partial_j A_\rho - \partial_\rho A_j) - \delta_{ij} \mathcal{L} = Y_{i0}(\partial_j A_0 - \partial_0 A_j) + Y_{ik}(\partial_j A_k - \partial_k A_j) - \delta_{ij} \mathcal{L}$

$= \varepsilon_0 E_i E_j - \varepsilon_0 c^2 \varepsilon_{ikm} B_m(\varepsilon_{jkn} B_n) - \delta_{ij} \mathcal{L}$

电磁场(EMF)能量密度 $w = \frac{1}{2}\varepsilon_0(E^2 + c^2B^2) = T_{00}$

能流密度 $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \varepsilon_0 c^2 \vec{E}\times\vec{B}$ $S_i = T_{i0}$

动量密度 $\vec{g} = \varepsilon_0 \vec{E}\times\vec{B} = \vec{S}/c^2$ $g_i = -T_{0i}$

Maxwell应力张量 $\overleftrightarrow{T} = \varepsilon_0(\vec{E}\vec{E} + c^2\vec{B}\vec{B}) - w\overleftrightarrow{I}$

动量流密度 $-\overleftrightarrow{T}$

$\partial_\alpha T_{\alpha\beta} = 0$

$$\begin{cases} \beta=0 & \partial_t w + \nabla\cdot\vec{S} = 0 \\ \beta=k & \partial_t \vec{g} - \nabla\cdot\overleftrightarrow{T} = 0 \end{cases}$$

对于存在电流电荷不成立

更优美的记号 $\mathcal{L} = -\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}$

CH4. Hamilton 力学

§4.0 序言

一. $(q, \dot{q})$ 相空间

2s 个一阶微分方程 $\dot{\xi} = \Xi(\xi, t)$

$\xi_0 \longmapsto \xi(t)$

$$\Xi = \begin{cases} \Xi(\xi) & \text{不同相轨迹不相交} \\ \Xi(\xi, t) & \text{在特定时刻，不同相轨迹不相交} \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ g(q, \dot{q}, t) \end{pmatrix}$$

$\begin{cases} \text{一半方程并非运动学方程} \\ \text{另一半方程难以给出} \end{cases}$

且 q 与 $\dot{q}$ 并不能随意、独立变化

二. (q, p) 相空间

$p_k \equiv \frac{\partial L}{\partial \dot{q}_k} = p_k(q, \dot{q}, t) \quad (k = 1, 2, \cdots, s)$

$\Rightarrow \dot{q}_k = \dot{q}_k(q, p, t)$ 可以做反变换

$L = T - U$

$$\begin{cases} T = \frac{1}{2}A_{ij}\dot{q}_i\dot{q}_j + B_i\dot{q}_i + C = \frac{1}{2}\dot{q}^T A\dot{q} + B^T\dot{q} + C \qquad p = A\dot{q} + B \\ U = U(q, t) \end{cases}$$

$T \longrightarrow -mc^2\sqrt{1-\frac{v^2}{c^2}}$

$U \longrightarrow e(\varphi - \vec{v}\cdot\vec{A})$

状态决定状态的演化，可以定义速度场和流

1. $H = H(q, p, t)$

$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}$

$$\frac{d}{dt}\begin{pmatrix} q\\ p\end{pmatrix} = \begin{pmatrix} f(q,p,t)\\ g(q,p,t)\end{pmatrix}$$

2. q. p 地位对等. $\dot{q}_k = [q_k, H]$, $\dot{p}_k = [p_k, H]$

$\xi = \begin{pmatrix} q\\ p\end{pmatrix}$: $\dot{\xi}_\alpha = [\xi_\alpha, H]$

$\xi \mapsto \eta = \eta(\xi, t)$ $\quad q \mapsto Q = Q(q,p,t)$ $\quad p \mapsto P = P(q,p,t)$

若变换后仍满足原方程，称正则变换

可能使所有变量全部成为循环坐标

其中的流是不可压缩的 (Liouville)

哈密顿函数提供了一种量子化的方法

泊松括号提供了另一种量子化的途径

执 $\dot{\xi}_\alpha = [\xi_\alpha, H]$ Poisson → 对易子

三. 等价性

$x_0 \to y_0 = f(x_0) \to (x_0, y_0) \to$ 点 → 曲线

$u_0 = f'(x_0) \to g_0 = u_0 x_0 - f(x_0) \to (u_0, g_0) \to$ 直线

$$\begin{cases} u = f'(x) = u(x) \Longrightarrow x = x(u) \\ g = ux - f(x) = g(x) \Longrightarrow g(u) \end{cases}$$

$u'(x) \neq 0 = f''(x)$

§4.1 Legendre 变换

一. $f(x)$ 对 x 的 Legendre 变换

$g(u) = ux - f(x) \quad u = f'(x) = u(x)$

Hess 条件: $f''(x) \neq 0 \Rightarrow x = x(u)$

eg. $f(x) = \frac{1}{2}ax^2 \ (a>0)$

$u = f'(x) = ax \Rightarrow x = \frac{u}{a}$

$g = ax^2 - \frac{1}{2}ax^2 = \frac{1}{2}ax^2 = \frac{u^2}{2a}$

eg. $f(x) = -b\sqrt{1 - \frac{x^2}{a^2}} \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ y \le 0\right)$

$u = b\frac{x/a^2}{\sqrt{1 - x^2/a^2}} \Rightarrow g = b\frac{x^2/a^2}{\sqrt{1-x^2/a^2}} + b\sqrt{1-x^2/a^2} = \frac{b}{\sqrt{1-x^2/a^2}}$

$\left(\frac{au}{b}\right)^2 + 1 = \frac{1}{1 - x^2/a^2}$

$g(u) = \sqrt{a^2u^2 + b^2}$ 将圆下半变为双曲线的上支

$$\begin{cases} u & \text{斜率} \\ g & \text{截距负值} \end{cases}$$

$u = f'(x) \quad x = g'(u) \quad ux = f(x) + g(u)$

$\frac{\partial f}{\partial a} + \frac{\partial g}{\partial a} = 0$ 对共同参数偏导和为0

二. $f(x,y)$ 对 x 的 Legendre 变换

$$\begin{cases} g(x,y) = ux - f(x,y) \quad u = \frac{\partial f}{\partial x} = u(x,y) \\ \text{Hess 条件}: \frac{\partial^2 f}{\partial x^2} \neq 0 \Rightarrow x = x(u,y) \end{cases}$$

eg. $f(x,y) = \frac{1}{2}ax^2 - \frac{1}{2}by^2$

$u = ax \Rightarrow x = \frac{u}{a} \Rightarrow g = \frac{1}{2}ax^2 + \frac{1}{2}by^2$

$g(u,y) = \frac{u^2}{2a} + \frac{1}{2}by^2$

$y = y_0$: $u_0 \to g_0 \to (x, y_0, ux - y_0)$

$\frac{\partial g}{\partial u} = \frac{\partial u}{\partial u}x + u\frac{\partial x}{\partial u} - \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} = x$

$\frac{\partial g}{\partial y} = \frac{\partial u}{\partial y}x + u\frac{\partial x}{\partial y} - \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} - \frac{\partial f}{\partial y} = -\frac{\partial f}{\partial y}$

$u = \frac{\partial f}{\partial x} \quad x = \frac{\partial g}{\partial u} \quad \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} = 0$

三. $f(x,y)$ 对 x 的 Legendre 变换

$$\begin{cases} g(u,y) = u_i x_i - f(x,y) \quad u_i = \dfrac{\partial f}{\partial x_i} = u_i(x,y) \Rightarrow x_i = x_i(u,y) \\ \text{Hess 条件: } \det\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0 \end{cases}$$

eg. $f(x,y) = \frac{1}{2}ax^2 + \frac{1}{2}by^2 \to g(u,v) = \frac{u^2}{2a} + \frac{v^2}{2b}$

$u_k = \dfrac{\partial f}{\partial x_k} \quad x_k = \dfrac{\partial g}{\partial u_k}$

1. 新旧函数之和 = 新、旧自变量乘积之和

$f(x,y) + g(u,y) = u_i x_i \qquad \det\left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0 \neq \det\left(\dfrac{\partial^2 g}{\partial u_i \partial u_j}\right)$

2. 旧(新)自变量 = 新(旧)函数对新(旧)自变量的偏导数

$x_i = \dfrac{\partial g}{\partial u_i} \qquad u_i = \dfrac{\partial f}{\partial x_i}$

3. 新、旧函数对共同自变量(参数)的偏导数之和

$\dfrac{\partial f}{\partial y_k} + \dfrac{\partial g}{\partial y_k} = 0 \qquad \dfrac{\partial f}{\partial \lambda} + \dfrac{\partial g}{\partial \lambda} = 0$

$L(q, \dot{q}, t) \qquad p_k = \dfrac{\partial L}{\partial \dot{q}_k}$

$H(q, p, t) = p_k \dot{q}_k - L(q, \dot{q}, t) = h(q, \dot{q}, t)$

§4.2 Hamilton 方程

$(q, \dot{q}) \longrightarrow (q, p)$

$L(q, \dot{q}, t) \to p_k \dot{q}_k - L = H$

一. Hamilton 函数

$H(q, p, t) \equiv p_k \dot{q}_k - L(q_k, \dot{q}_k, t) \qquad \dot{q}_k = \dot{q}_k(q, p, t)$

1. 数值上 $H(q, p, t) = h(q, \dot{q}, t)$

2. $L = L_2 + L_1 + L_0 = \frac{1}{2}\dot{q}^T A \dot{q} + B^T \dot{q} + C \qquad p = A\dot{q} + B$

$\Rightarrow H = L_2 - L_0 = \frac{1}{2}(p-B)^T A^{-1} (p-B) - C$

$L = \frac{1}{2} A_{ij} \dot{q}_i \dot{q}_j + B_i \dot{q}_i + C \qquad p_k = A_{kj} \dot{q}_j + B_k$

$\dot{q} = A^{-1}(p - B) \qquad L_2 = \frac{1}{2}(p-B)^T A^{-1} (p-B) A^{-1} A A^{-1} (p-B)$

$A = \mathrm{diag}(A_1, A_2, \cdots, A_s) \Rightarrow H = \dfrac{(p_i - B_i)^2}{2A_i} - C$ 正交坐标系下，动量互不耦合

eg. $L = \frac{1}{2} m \dot{x}_i \dot{x}_i + e \dot{x}_i A_i - e\varphi \qquad H = \dfrac{(p_i - eA_i)^2}{2m} + e\varphi = T + e\varphi$

eg. $L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) - U(r) \qquad H = \dfrac{p_r^2}{2m} + \dfrac{1}{2mr^2}\left(p_\theta^2 + \dfrac{p_\phi^2}{\sin^2\theta}\right) + U(r)$

绕θ角动量

eg. $L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - U(\vec{r}) \qquad H = \sqrt{m^2c^4 + p^2c^2} + U = c\sqrt{p_r^2 + \frac{p_\theta^2}{r^2} + m^2c^2} + U$

二. Hamilton 方程

$\dot{q}_k = \dfrac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\dfrac{\partial H}{\partial q_k} \qquad (k = 1, 2, \cdots, s)$

$-\dfrac{\partial H}{\partial q_k} = \dfrac{\partial L}{\partial q_k} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_k} = \dfrac{d}{dt} p_k$

1. 若 q_k 为 H 的循环坐标，则 p_k 为运动常数

2. $\dfrac{dH}{dt} = \dfrac{\partial H}{\partial t}$ 若 H 不显含 t，则 H 为运动常数

$\dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t} = \dfrac{dh}{dt} = \dfrac{dH}{dt}$

$\dfrac{dH}{dt} = \dfrac{\partial H}{\partial q_k} \dot{q}_k + \dfrac{\partial H}{\partial p_k} \dot{p}_k + \dfrac{\partial H}{\partial t}$

eg. 中心力问题 $H \quad p_\phi = mr^2\dot{\phi}\sin^2\theta = \Rightarrow \dot{\phi} = \dfrac{l_z}{mr^2\sin^2\theta}$

代入Lagrange得 $L(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \dfrac{l_z^2}{2mr^2\sin^2\theta} - U(r)$

代入Hamilton对 $H(r, \theta, p_r, p_\theta) = \dfrac{p_r^2}{2m} + \dfrac{p_\theta^2}{2mr^2} + \dfrac{l_z^2}{2mr^2\sin^2\theta} + U(r)$

进一步 $H(r, p_r) = \dfrac{p_r^2}{2m} + \dfrac{l^2}{2mr^2} + U(r)$

eg. 简谐运动 $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 \quad (p_x = m\dot{x})$

$H = \dfrac{p_x^2}{2m} + \frac{1}{2}m\omega_0^2 x^2$

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p_x} = \dfrac{p_x}{m} \Rightarrow p_x = m\dot{x} \\ \dot{p}_x = -\dfrac{\partial H}{\partial x} = -m\omega_0^2 x \Rightarrow \ddot{x} = -\omega_0^2 x \end{cases}$$

在Lagrange力学中，不可能将相空间中的椭圆运动变为匀速圆周

$q = \sqrt{m\omega_0}\, x$

$L = \frac{\dot{q}^2}{2\omega_0} - \frac{1}{2}\omega_0 q^2 \quad (p = \frac{\dot{q}}{\omega_0}) \Rightarrow p = \frac{p_x}{\sqrt{m\omega_0}}$

$\Rightarrow H = \frac{1}{2}\omega(p^2 + q^2)$

$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \omega\begin{pmatrix} p \\ -q \end{pmatrix} = \omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \omega\Lambda\begin{pmatrix} q \\ p \end{pmatrix}$

$\frac{dx}{dt} = Ax \Rightarrow x = e^{\Lambda t} x_0$

$\begin{pmatrix} q \\ p \end{pmatrix} = e^{\omega\Lambda t}\begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$

$e^{\omega\Lambda t} = \sum_{n=0}^{\infty} \frac{(\omega t)^n}{n!}\Lambda^n$

$\Lambda^0 = I \quad \Lambda^1 = \Lambda \quad \Lambda^2 = \Lambda\Lambda = -I \quad \Lambda^3 = -\Lambda \quad \Lambda^4 = I$

$\Lambda^{2k} = (-1)^k I \quad \Lambda^{2k+1} = (-1)^k \Lambda$

$e^{\omega\Lambda t} = \sum_{k=0}^{\infty}(-1)^k \frac{(\omega t)^{2k}}{(2k)!} I + \sum_{k=0}^{\infty}(-1)^k \frac{(\omega t)^{2k+1}}{(2k+1)!}\Lambda = I\cos\omega_0 t + \Lambda\sin\omega_0 t$

$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix}\begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$

坐标变化前后，
- Lagrange力学下，相空间椭圆面积变化
- Hamilton力学下，相空间椭圆面积不变

$L' = L + \frac{dF(q,t)}{dt} = L + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t}$

$p' = \frac{\partial L'}{\partial \dot{q}} = p + \frac{\partial F}{\partial q}$

$H' = p'\dot{q} - L' = H - \frac{\partial F}{\partial t}$

§4.3 相空间中的运动

一、动力学含义

$$\begin{cases} q_k(t+\varepsilon) = q_k(t) + \varepsilon\dot{q}_k(t) = q_k(t) + \varepsilon\left(\frac{\partial H}{\partial p_k}\right)_t \\ p_k(t+\varepsilon) = p_k(t) + \varepsilon\dot{p}_k(t) = p_k(t) + \varepsilon\left(-\frac{\partial H}{\partial q_k}\right)_t \end{cases}$$

$$\begin{pmatrix} \dot{x} \\ \dot{p}_x \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_x} \\ -\frac{\partial H}{\partial x} \end{pmatrix}$$

1. 正则变量(q, p) Hamilton方程 — 正则方程

canon: a christion priest

a generally accept rule, standard or prinple

by which something is judged

canonical: in the simplest accepted form in mathemalies

2. $H(q,p,t)$生成的Hamilton矢量场 $\Delta_H \equiv \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} = \Delta_H(q,p,t)$

二、ξ记号

$\alpha, \beta, \gamma = 1, 2, \cdots, 2s \quad i, j, k = 1, 2, \cdots, s$

1. 正则变量

$\xi_\alpha = \begin{cases} \xi_k = q_k \\ \xi_{s+k} = p_k \end{cases} \quad \xi = \begin{pmatrix} q \\ p \end{pmatrix} \quad \frac{\partial f}{\partial \xi} = \begin{pmatrix} \frac{\partial f}{\partial \xi_1} \\ \vdots \\ \frac{\partial f}{\partial \xi_{2s}} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial q} \\ \frac{\partial f}{\partial p} \end{pmatrix}$

2. 正则方程

$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix}$

$\dot{\xi} = \Omega\frac{\partial H}{\partial \xi} \quad (\dot{\xi}_\alpha = \Omega_{\alpha\beta}\frac{\partial H}{\partial \xi_\beta})$

$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \Omega^T = -\Omega = \Omega^{-1}$

3. $H(\xi, t)$生成的Hamilton矢量场 $\Delta_H(\xi, t) = \Omega\frac{\partial H}{\partial \xi}$，$\dot{\xi} = \Delta_H$

三、$H(\xi, t)$生成的Hamilton体系

在ξ-相空间中，随时间的演化由$\dot{\xi} = \Delta_H(\xi, t)$决定

1. H可为任一函数

$H = C(0).\quad C_\alpha \xi_\alpha.\quad \frac{1}{2} C_{\alpha\beta} \xi_\alpha \xi_\beta$

eg. $H = q$ $\qquad H = p$ $\qquad H = xp_y - yp_x$

$\Delta_H = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\qquad \Delta_H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad \Delta_H = \begin{pmatrix} -y \\ x \\ -p_y \\ p_x \end{pmatrix}$

任给一个 Hamilton 函数，给定一个 Hamilton 体系

但任给一个 Lagrange 函数，未必决定一个体系

$X = X(\xi, t) \Rightarrow \dot{\xi} = X(\xi, t) \overset{?}{=\!=} \Omega \frac{\partial H}{\partial \xi}$

$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -\omega_0^2 q - 2\gamma p \end{pmatrix} \overset{?}{=\!=} \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix}$

$$\begin{cases} \frac{\partial}{\partial q}\frac{\partial H}{\partial p} = \frac{\partial}{\partial q} p = 0 \\ \frac{\partial}{\partial p}\frac{\partial H}{\partial q} = \frac{\partial}{\partial p}(\omega_0^2 q + 2\gamma p) = 2\gamma \end{cases}$$

但若取 $\begin{cases} Q = q \\ P = pe^{2\gamma t} \end{cases}$ 为状态参量，是一个 Hamilton 体系

2. $\dot{\xi} = X(\xi, t)$ 为 Hamilton 体系

$\Leftrightarrow$ 存在 $H(\xi, t)$，使得 $X = \Omega \frac{\partial H}{\partial \xi}$

$\Leftrightarrow$ 存在 $H(\xi, t)$，使 $Y \equiv \Omega X = -\frac{\partial H}{\partial \xi}$

$\Leftrightarrow \partial_\alpha Y_\beta = \partial_\beta Y_\alpha$

$\Omega_{\beta\gamma} \partial_\alpha X_\gamma = \Omega_{\alpha\gamma} \partial_\beta X_\gamma$

$\Omega_{\alpha\rho} \Omega_{\beta\sigma} \Omega_{\beta\gamma} \partial_\alpha X_\gamma = \Omega_{\alpha\gamma} \partial_\beta X_\gamma \Omega_{\alpha\rho} \Omega_{\beta\sigma}$

$\Omega_{\alpha\rho} \partial_\alpha X_\sigma = \Omega_{\beta\sigma} \partial_\beta X_\rho \Rightarrow -D_\rho X_\sigma = -D_\sigma X_\rho$

$\Leftrightarrow D_\alpha X_\beta = D_\beta X_\alpha \quad (D_\alpha \equiv \Omega_{\alpha\beta} \partial_\beta)$

四. 相空间中的最小原理

1. 相空间中的 Lagrange 函数

$\tilde{L}(\xi, \dot{\xi}, t) \equiv p_i \dot{q}_i - H(q, p, t) = \tilde{L}(q, p, \dot{q}, \dot{p}, t)$

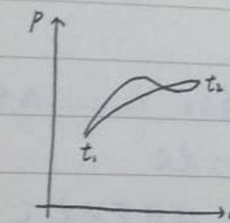
并不是 Legendre 变换

1. H 未必满足 Hess 条件

2. 自变量不对

2. 相空间中的 Hamilton 原理

$$\begin{cases} 0 = \delta \tilde{S} = \delta \int_{t_1}^{t_2} \tilde{L}(\xi, \dot{\xi}, t)\,dt \\ \delta \xi_\alpha(t_1) = 0 = \delta \xi_\alpha(t_2) \quad (\alpha = 1, 2, \cdots, s) \end{cases}$$



$\delta \tilde{L} = \frac{\partial \tilde{L}}{\partial \xi_\alpha} \delta \xi_\alpha + \frac{\partial \tilde{L}}{\partial \dot{\xi}_\alpha} \delta \dot{\xi}_\alpha$

$= \left(\frac{\partial \tilde{L}}{\partial \xi_\alpha} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\xi}_\alpha}\right) \delta \xi_\alpha + \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\xi}_\alpha} \delta \xi_\alpha\right)$

$0 = \delta \tilde{S} = \int_{t_1}^{t_2} \frac{\delta \tilde{L}}{\delta \xi_\alpha} \delta \xi_\alpha\, dt + \left(\frac{\partial \tilde{L}}{\partial \dot{\xi}_\alpha} \delta \xi_\alpha\right)\Big|_{t_1}^{t_2}$

$\frac{\delta \tilde{L}}{\delta \xi_\alpha} \equiv \frac{\partial \tilde{L}}{\partial \xi_\alpha} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\xi}_\alpha} = 0$

$$\begin{cases} \xi_\alpha = q_k & \frac{\delta \tilde{L}}{\delta \xi_\alpha} = \left(0 - \frac{\partial H}{\partial q_k}\right) - \frac{d}{dt} p_k = 0 \\ \xi_\alpha = p_k & \frac{\delta \tilde{L}}{\delta p_k} = \left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right) - \frac{d}{dt}(0) = 0 \end{cases}$$

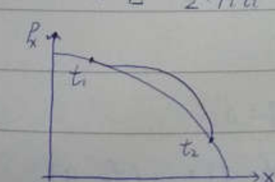
端点项 $\left(\frac{\partial \tilde{L}}{\partial \dot{q}_k} \delta q_k + \frac{\partial \tilde{L}}{\partial \dot{p}_k} \delta p_k\right)\Big|_{t_1}^{t_2}$

其实并不需要要求，因为系数为0

3. 规范变换 $\tilde{L} \Leftrightarrow \tilde{L}' = \tilde{L} + \frac{dF(\xi, t)}{dt}$

如：$F = -\frac{1}{2} p_i q_i \Rightarrow \frac{dF}{dt} = -\frac{1}{2}(p_i \dot{q}_i - \dot{p}_i q_i)$

$\Rightarrow \tilde{L}' = \frac{1}{2}(p_i \dot{q}_i - \dot{p}_i q_i) - H = \frac{1}{2} \dot{\xi} \Omega \xi - H$



$H = \frac{p_x^2}{2m} + mgx$

$\dot{x} = \frac{p_x}{m} \qquad \dot{p}_x = -mg$

$p_x = m\dot{x} \Rightarrow \delta p_x = m\delta \dot{x}$ 不独立 争议?

$$\begin{cases} \langle \dot{x} + \delta \dot{x} \rangle = \langle \dot{x} \rangle = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \\ p_x = m\dot{x} \Rightarrow \langle \dot{x} + \delta \dot{x} \rangle = \langle \dot{x} \rangle + \langle \frac{\delta p_x}{m} \rangle \neq \langle \dot{x} \rangle \end{cases}$$

Hamilton 力学中, p_k与q_k之间没有任何先验关系

$\dot{x}=\frac{p_x}{m}$ 是动力学方程, 并不一定成立

只有在真实路径, $\dot{x}=\frac{p_x}{m}$, $\dot{p}_x=-mg$ 才同时成立

这2s个方程地位对等, 2s个变量相互独立

§4.4 Poisson 括号

一. 定义

2s 维空间 $\xi_1, \xi_2, \cdots, \xi_{2s}$

力学量 $f(\xi,t)$ 与 $g(\xi,t)$ 的 Poisson 括号

$$[f,g]_\xi \equiv \frac{\partial f}{\partial \xi_\alpha}\Omega_{\alpha\beta}\frac{\partial g}{\partial \xi_\beta} = \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}$$

1. $f*g \equiv [f,g]$ —种乘法
2. $D_f g \equiv [f,g]$ —阶微分算子
3. 一种内积(此处不讨论)

$[q_k, g] = \frac{\partial g}{\partial p_k}$

$[p_k, g] = \frac{\partial g}{\partial q_k}$

$[\xi_\alpha, g] = \Omega_{\alpha\beta}\frac{\partial g}{\partial \xi_\beta}$

$\xi_\alpha \leftrightarrow D_\alpha = D_{\xi_\alpha} = \Omega_{\alpha\beta}\frac{\partial}{\partial \xi_\beta}$

二. 数学性质

1. 反对称 $[f,g]=-[g,f] \Rightarrow [f,f]=0$
2. 双线性 $[f, C_k g_k] = C_k[f, g_k]$ $[C_k f_k, g] = C_k[f_k, g]$
3. Jacobi 恒等式 $[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$

   $[[f,g],h]+[[g,h],f]+[[h,f],g]=0$

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满足前两性质, 称代数

满足三个性质, 称李代数

如: 矢量叉乘、矩阵 $[A,B]=AB-BA$

证明: $[f,[g,h]] = \frac{\partial f}{\partial \xi_\alpha}\Omega_{\alpha\rho}\frac{\partial}{\partial \xi_\rho}\left(\frac{\partial g}{\partial \xi_\beta}\Omega_{\beta\gamma}\frac{\partial h}{\partial \xi_\gamma}\right)$

$= \Omega_{\alpha\rho}\Omega_{\beta\gamma}\frac{\partial f}{\partial \xi_\alpha}\frac{\partial}{\partial \xi_\rho}\left(\frac{\partial g}{\partial \xi_\beta}\frac{\partial h}{\partial \xi_\gamma}\right)$

$[g,[h,f]] = \Omega_{\beta\rho}\Omega_{\gamma\alpha}\frac{\partial g}{\partial \xi_\beta}\frac{\partial}{\partial \xi_\rho}\left(\frac{\partial h}{\partial \xi_\gamma}\frac{\partial f}{\partial \xi_\alpha}\right)$

$[h,[f,g]] = \Omega_{\gamma\rho}\Omega_{\alpha\beta}\frac{\partial h}{\partial \xi_\gamma}\frac{\partial}{\partial \xi_\rho}\left(\frac{\partial f}{\partial \xi_\alpha}\frac{\partial f}{\partial \xi_\beta}\right)$

考虑 g 的二阶导

$\Omega_{\alpha\rho}\Omega_{\beta\gamma}\frac{\partial f}{\partial \xi_\alpha}\frac{\partial^2 g}{\partial \xi_\rho \partial \xi_\beta}\frac{\partial h}{\partial \xi_\gamma}$ β与ρ互换

$\Omega_{\gamma\rho}\Omega_{\alpha\beta}\frac{\partial h}{\partial \xi_\gamma}\frac{\partial^2 g}{\partial \xi_\rho \partial \xi_\beta}\frac{\partial f}{\partial \xi_\alpha}$

4. Leibnia 法则 $[f, gh] = [f,g]h + g[f,h]$

   $\Rightarrow$ 若 h 为常数或 $[f,h]=0$

   则 $[f,gh]=[f,g]h$

5. Chain 法则 $[f, g(h)] = \frac{\partial g}{\partial h}[f,h]$

   $[f, g(h_1, h_2, \cdots, h_n)] = \frac{\partial g}{\partial h_k}[f, h_k]$

   $\Rightarrow \begin{cases} [f, g^n] = n g^{n-1}[f,g] \\ [f, g(f)] = 0 \end{cases}$

   $[f,g]_\xi = \frac{\partial f}{\partial \xi_\alpha}[\xi_\alpha, \xi_\beta]\frac{\partial g}{\partial \xi_\beta}$

   $\delta_{\alpha\beta}$(度规) $M_{\alpha\beta}$. $\Omega_{\alpha\beta}$ 似内积(辛几何)

6. 基本 Poisson 括号 $[\xi_\alpha, \xi_\beta]_\xi = \Omega_{\alpha\beta}$

   $\begin{cases} [q_i, q_j] = 0 = [p_i, p_j] \\ [q_i, p_j] = \delta_{ij} = -[p_i, q_j] \end{cases}$

7. 对参数偏导数 $\partial_t[f,g]=[\partial_t f,g]+[f,\partial_t g]$

eg. $2s=6$ $\quad \xi=(\vec{r},\vec{p})=(x_1,x_2,x_3,p_1,p_2,p_3)$ $\quad L_i=\varepsilon_{ijk}x_j p_k$

$$[L_i,f]=\varepsilon_{ikl}[x_k p_l,f]=\varepsilon_{ikl}([x_k,f]p_l-x_k[p_l,f])$$

$$=\varepsilon_{ikl}\left(\frac{\partial f}{\partial p_k}p_l-x_k\frac{\partial f}{\partial x_l}\right)$$

$$[L_i,x_j]=-\varepsilon_{ikl}x_k\delta_{jl}=\varepsilon_{ijk}x_k$$

$$[L_i,p_j]=\varepsilon_{ijk}p_k$$

$$[L_i,L_j]=\varepsilon_{jmn}[L_i,x_m p_n]=\varepsilon_{jmn}([L_i,x_m]p_n+x_m[L_i,p_n])$$

$$=\varepsilon_{jmn}\varepsilon_{imk}x_k p_n+\varepsilon_{jmn}\varepsilon_{ink}x_m p_k$$

$$=x_i p_j-x_j p_i=\varepsilon_{ijk}L_k$$

	$\vec{r}$	$\vec{p}$	$\vec{L}$	$f=f(r,p,\vec{r}\cdot\vec{p},t)$
$\vec{r}$	r^2	$\vec{r}\cdot\vec{p}$	0	$\vec{A}=f_1\vec{r}+f_2\vec{p}+f_3\vec{L}$
$\vec{p}$		p^2	0	$[L_i,f]=0$
$\vec{L}$			$r^2p^2-(\vec{r}\cdot\vec{p})^2$	$[L_i,A_j]=\varepsilon_{ijk}A_k$

三. Poisson 括号应用于 Hamilton 体系: $\dot{\xi}=\Omega\frac{\partial H}{\partial \xi}$

1. 正则方程 $\dot{\xi}_\alpha=[\xi_\alpha,H]$

2. 力学量 $f=f(\xi,t)$ 的运动方程

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \xi_\alpha}\dot{\xi}_\alpha=\partial_t f+\frac{\partial f}{\partial \xi_\alpha}[\xi_\alpha,H]$$

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+[f,H]=\frac{df}{dt}(\xi,t)$$

3. 力学量 $f(\xi,t)$ 的 Taylor 展开

$$f(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left(\frac{d^n f}{dt^n}\right)_0$$

设 $f=f(\xi)$, $H=H(\xi)$ 不显含 t

$$\frac{df}{dt}=[f,H] \qquad \frac{d^2f}{dt^2}=[[f,H],H]$$

$$f(t)\equiv f(\xi(t),t)=f_0+[f,H]_0t+\frac{1}{2!}[[f,H],H]_0t^2+\frac{1}{3!}[[[f,H],H],H]t^3+\cdots$$

eg. $x(t)=x_0+[x,H]t+[[x,H],H]t^2$

$$H=\frac{p_x^2}{2m}+mgx$$

$$[x,H]=[x,\frac{p_x^2}{2m}]=\frac{2p_x}{2m}[x,p_x]=\frac{p_x}{m}$$

$$[[x,H],H]=[\frac{p_x}{m},mgx]=-g$$

$$\Rightarrow x=x_0+\frac{p_x}{m}t-\frac{1}{2}gt^2$$

$$f(t)=\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}(D_H^n f)_0:=\exp(-tD_H)f_0$$

$$f(t+\tau)=\sum_{n=0}^{\infty}\frac{(-\tau)^n}{n!}D_H^n f=\exp(-\tau D_H)f$$

eg. $H=\frac{1}{2}\omega(p^2+q^2)$

$D_H^0 q=q \qquad D_H^1 q=[H,q]=[\frac{1}{2}\omega p^2,q]$

$D_H^2 q=-\omega^2 q \qquad D_H^3 q=\omega^3 p$

$$q(t)=\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}(D_H^n q)_0=\sum_{k=0}^{\infty}\frac{(-t)^{2k}}{2k!}(D_H^{2k}q_0)+\sum_{k=0}^{\infty}\frac{(-t)^{2k+1}}{(2k+1)!}(D_H^{2k+1}q_0)$$

$$=q_0\cos\omega t+p_0\sin\omega t$$

4. 对时间的全微商 $\frac{d}{dt}[f,g]$

$$\frac{d}{dt}[f,g]=\partial_t[f,g]+[[f,g],H]$$

$$=[\partial_t f,g]-[[g,H],f]$$

$$+[f,\partial_t g]-[[H,f],g]$$

$$=[\partial_t f+[f,H],g]+[f,\partial_t g+[g,H]]$$

$$\frac{d}{dt}[f,g]=[\frac{df}{dt},g]+[f,\frac{dg}{dt}]$$

$$\frac{d^n}{dt^n}[f,g]=\sum_{k+l=n}\frac{n!}{k!\,l!}[\frac{d^kf}{dt^k},\frac{d^lg}{dt^l}]$$

5. 判断运动常数

$\partial_t f+[f,H]=0\Rightarrow f$ 为运动常数

eg. 中心力问题 $H=\frac{p^2}{2m}+U(r)$

$H.\ L_i(i=1,2,3)$

6. 生成新的运动常数

$f.g$ 为运动常数 $\Rightarrow [f.g]$ 也是运动常数

要求是哈密顿体系

eg. (s=3 自由度为3) $L_1.L_2$ 守恒 $\Rightarrow L_3=[L_1.L_2]$ 守恒

函数 $f.g$ 独立 $\Rightarrow$ 不存在 $h(f.g)=0$

四. 判断 Hamilton 体系

$\dot{\xi}=X(\xi.t)$ 为 Hamilton 体系

$D_\alpha X_\beta = D_\beta X_\alpha \Leftrightarrow [\xi_\alpha.X_\beta]=[\xi_\beta.X_\alpha]$

$\Leftrightarrow [\dot{\xi}_\alpha.\xi_\beta]_\xi+[\xi_\alpha.\dot{\xi}_\beta]_\xi=0$

$\Leftrightarrow \frac{d}{dt}[f.g]_\xi=[\frac{df}{dt}.g]_\xi+[f.\frac{dg}{dt}]_\xi \quad (\forall f.g)$

$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \xi}\dot{\xi} \quad \dot{\xi}\to X$

eg. 虫口模型 q: 猎物 p: 捕食者 $\xi:(q.p)$

$$\begin{cases}\dot{q}=aq-bqp\\ \dot{p}=dqp-cp\end{cases}$$

$[\dot{q}.p]_\xi+[q.\dot{p}]_\xi=[aq-bqp.\ p]_\xi+[q.\ dqp-cp]$

$=(a-a\cancel{b}p)+(dq-c)\neq 0$

$$\begin{cases}q=e^Q\\ p=e^P\end{cases}\quad \eta=(Q.P)$$

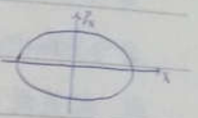
$[\dot{Q}.P]_\eta+[Q.\dot{P}]_\eta=0+0=0$

在新的变量描述下是哈密顿体系

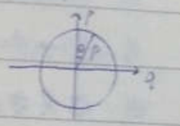
§4.5 正则变换 (CT: Canonical Transformation)

一. 简谐振子

$$(x.p_x)\begin{cases}\dot{x}=\frac{p_x}{m}\\ \dot{p}_x=-m\omega^2 x\end{cases}\quad H(x.p_x)=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2x^2=E$$

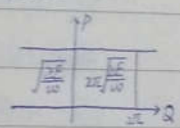


$$\xi(q.p)\begin{cases}q=\sqrt{m\omega}\,x\\ p=\frac{p_x}{\sqrt{m\omega}}\end{cases}\begin{cases}\dot{q}=\omega p\\ \dot{p}=-\omega q\end{cases}\quad H=\frac{1}{2}\omega(p^2+q^2)$$



相轨道所围的面积相等

$$\eta=(Q.P)\begin{cases}Q=\arctan\frac{q}{p}\\ P=\sqrt{p^2+q^2}\end{cases}\begin{cases}q=P\sin Q\\ p=P\cos Q\end{cases}\begin{cases}\dot{Q}=\omega\\ \dot{P}=0\end{cases}\quad K=\omega P=\sqrt{2\omega E}$$



$\xi\to\eta$ 不是正则变换

$\xi\to\eta$ 应用于 $H(q.p.t)$

$0\neq[\dot{Q}.P]_\eta+[Q.\dot{P}]_\eta$

$$\begin{cases}\dot{Q}=[Q.H]_\xi=\frac{\partial H}{\partial P}[Q.P]_\xi\\ \dot{P}=[P.H]_\xi=-\frac{\partial H}{\partial Q}[Q.P]_\xi\end{cases}$$

$[Q.P]_\xi=\frac{1}{\sqrt{p^2+q^2}}(p[Q.p]_\xi+q[Q.q]_\xi)$

$=\frac{1}{\sqrt{p^2+q^2}}(p\frac{p}{p^2+q^2}-q\frac{-q}{p^2+q^2})=\frac{1}{\sqrt{p^2+q^2}}$

$[Q.P]_\xi=\frac{1}{P}$

$$\begin{cases}\dot{Q}=\frac{1}{P}\frac{\partial H}{\partial P}\Rightarrow\frac{\partial K}{\partial P} & \frac{\partial}{\partial Q}\frac{\partial K}{\partial P}=\frac{1}{P}\frac{\partial^2 H}{\partial Q\partial P}\\ \dot{P}=-\frac{1}{P}\frac{\partial H}{\partial Q}\Rightarrow-\frac{\partial K}{\partial Q} & \frac{\partial}{\partial P}\frac{\partial K}{\partial Q}=\frac{1}{P}\frac{\partial^2 H}{\partial P\partial Q}-\frac{1}{P}\frac{\partial H}{\partial Q}\end{cases}$$

$\frac{\partial H}{\partial Q}=0\Rightarrow H=H(\cancel{Q}.P.t)=H(\sqrt{q^2+p^2}.t)$

$\eta=\xi\quad Q=\arctan\frac{q}{p}\quad P=\frac{1}{2}(p^2+q^2)$ 为正则变换

相空间下最简单的运动即为不动. 于是找到守恒量

二. CT 的定义

若 $H(\xi, t)$ 定义的体系以 $\eta = \eta(\xi, t)$ 为状态参量时，仍为 Hamilton 体系

则 称变换 $\xi \mapsto \eta = \eta(\xi, t)$ 对该 $H(\xi, t)$ 是类正则的

若 $\xi \mapsto \eta = \eta(\xi, t)$ 对 $H(\xi, t)$ 均为类正则的

则 称其为正则变换

如: $\begin{cases} Q = \arctan \frac{q}{p} \\ P = \frac{1}{2}(p^2+q^2) \end{cases} \Rightarrow \begin{cases} q = \sqrt{2P}\sin Q \\ p = \sqrt{2P}\cos Q \end{cases}$

验证 $[\dot{Q}, P]_\eta + [Q, \dot{P}]_\eta = 0$

$\begin{cases} \dot{Q} = \partial_t Q + [Q, H]_\xi = [Q, Q]_\xi \frac{\partial H}{\partial Q} + [Q, P]_\xi \frac{\partial H}{\partial P} \\ \dot{P} = \partial_t P + [P, H]_\xi = [P, Q]_\xi \frac{\partial H}{\partial Q} + [P, P]_\xi \frac{\partial H}{\partial P} \end{cases}$

$[Q, P]_\xi = \frac{\partial P}{\partial p}[Q, p]_\xi + \frac{\partial P}{\partial q}[Q, q]_\xi = p\frac{\partial Q}{\partial q} - q\frac{\partial Q}{\partial p}$

$= p\frac{p}{p^2+q^2} - q\frac{-q}{p^2+q^2} = 1$

$H(q, p, t) = H(\sqrt{2P}\sin Q, \sqrt{2P}\cos Q, t)$

$H = \frac{1}{2}\omega(p^2+q^2) \Rightarrow K = \omega P$

$[\dot{Q}, P]_\eta + [Q, \dot{P}]_\eta = [\frac{\partial H}{\partial P}, P]_\eta + [Q, \frac{\partial H}{\partial Q}]_\eta = \frac{\partial}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial}{\partial P}\frac{\partial H}{\partial Q} = 0$

1. 定义 $\Lambda_{\alpha\beta} \equiv [\xi_\alpha, \xi_\beta]_\eta = \Lambda_{\alpha\beta}(\xi, t)$

$\begin{cases} \partial_t \Lambda_{\alpha\beta},\ \partial_\gamma \Lambda_{\alpha\beta} \text{ 由变换决定} \\ \Lambda_{\alpha\beta} = -\Lambda_{\beta\alpha},\ \Lambda = (\Lambda_{\alpha\beta}) \text{ 可逆} \end{cases}$

$\frac{\partial \xi_\alpha}{\partial \eta_\rho}\Omega_{\rho\sigma}\frac{\partial \xi_\beta}{\partial \eta_\sigma}$

2. $\xi \mapsto \eta = \eta(\xi, t)$ 对某一 $H(\xi, t)$ 类正则

$\frac{d\Lambda_{\alpha\beta}}{dt} = [\dot{\xi}_\alpha, \xi_\beta]_\eta + [\xi_\alpha, \dot{\xi}_\beta]_\eta = \partial_t \Lambda_{\alpha\beta} + [\Lambda_{\alpha\beta}, H]$ $\quad \dot{\xi}_\alpha = \Omega_{\alpha\beta}\frac{\partial H}{\partial \xi_\beta}$

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3. $\xi \mapsto \eta = \eta(\xi, t)$ 为 CT，对任何 $H(\xi, t)$ 成立

(1) $H = C \Rightarrow \dot{\xi}_\alpha = 0 \Rightarrow \partial_t \Lambda_{\alpha\beta} = 0$

(2) $H = C_\gamma \xi_\gamma \Rightarrow \dot{\xi}_\alpha = \Omega_{\alpha\gamma} C_\gamma \Rightarrow C_\gamma[\Lambda_{\alpha\beta}, \xi_\gamma]_\xi = 0\ (\forall C_\gamma)$

$\forall \Rightarrow [\Lambda_{\alpha\beta}, \xi_\gamma]_\xi = 0 \quad (^*\gamma = 1, 2, \cdots, 2s)$

$\Rightarrow \Lambda_{\alpha\beta} = \Lambda_{\alpha\beta}[\xi, t]$ 常数 $\Rightarrow [\dot{\xi}_\alpha, \xi_\beta]_\eta = [\dot{\xi}_\beta, \xi_\alpha]_\eta$

(3) $H = \frac{1}{2}C_{\rho\sigma}\xi_\rho\xi_\sigma\ (C_{\rho\sigma} = C_{\sigma\rho}) \Rightarrow \dot{\xi}_\alpha = \Omega_{\alpha\rho}C_{\rho\sigma}\xi_\sigma$

$\Omega_{\alpha\rho}C_{\rho\sigma}[\xi_\sigma, \xi_\beta]_\eta = \Omega_{\beta\rho}C_{\rho\sigma}[\xi_\sigma, \xi_\alpha]_\eta$

$(\Omega C\Lambda)_{\alpha\beta} = (\Lambda^T C^T \Omega^T)_{\alpha\beta}$

$\cancel{\Omega C \Lambda} = \Omega C\Lambda = \Lambda C\Omega \Rightarrow \Omega\Omega C\Lambda\Omega = \Omega\Lambda C\Omega\Omega$

$\Rightarrow C(\Lambda\Omega) = (\Omega\Lambda)C \quad (\forall C)$

取 $C = I \Rightarrow \Lambda\Omega = \Omega\Lambda \Rightarrow C(\Omega\Lambda) = (\Omega\Lambda)C$

$\Rightarrow \Omega\Lambda = -aI \Rightarrow \Lambda = a\Omega$

$[f, g]_\eta = \frac{\partial f}{\partial \xi_\alpha}[\xi_\alpha, \xi_\beta]_\eta\frac{\partial g}{\partial \xi_\beta} = \frac{\partial f}{\partial \xi_\alpha}a\Omega_{\alpha\beta}\frac{\partial g}{\partial \xi_\beta} = a[f, g]_\xi$

$\Rightarrow \frac{d}{dt}[f, g]_\eta = a\frac{d}{dt}[f, g]_\xi = a[\frac{df}{dt}, g]_\xi + a[f, \frac{dg}{dt}]_\xi$

$= [\frac{df}{dt}, g]_\eta + [f, \frac{dg}{dt}]_\eta$

三. (受限) 正则变换的条件

1. $[f, g]_\eta = [f, g]_\xi \quad (\forall f, g)$

2. $[\xi_\alpha, \xi_\beta]_\eta = \Omega_{\alpha\beta}$ 或 $[\eta_\alpha, \eta_\beta]_\xi = \Omega_{\alpha\beta} \quad (1 \le \alpha < \beta \le 2s)$

3. (辛条件) $M\Omega M^T = \Omega$ 或 $M^T\Omega M = \Omega \quad (M_{\alpha\beta} \equiv \frac{\partial \eta_\alpha}{\partial \xi_\beta})$ M 辛矩阵

$M(\Omega M^T\Omega^T) = I \Rightarrow (\Omega M^T\Omega^T)M = I$

$(M\Omega M^T)_{\alpha\beta} = [\eta_\alpha, \eta_\beta]_\xi$

$(M^T\Omega M)_{\alpha\beta} = \frac{\partial \eta_\rho}{\partial \xi_\alpha}\Omega_{\rho\sigma}\frac{\partial \eta_\sigma}{\partial \xi_\beta} = \{\xi_\alpha, \xi_\beta\}_\eta \to$ Lagrange 括号

4.（可积条件） $P_k \delta q_k - P_k \delta Q_k = \delta F(\xi, t)$ 生成函数

$$\delta F = \frac{\partial F}{\partial q_k}\delta q_k + \frac{\partial F}{\partial p_k}\delta p_k$$

$$\begin{cases} \frac{\partial F}{\partial q_k} = p_k - \frac{\partial Q_i}{\partial q_k} P_i \\ \frac{\partial F}{\partial q_k} = -\frac{\partial Q_i}{\partial p_k} P_i \end{cases}$$

$$\Leftrightarrow \frac{1}{2}(p_k \delta q_k - q_k \delta p_k) - \frac{1}{2}(P_k \delta Q_k - Q_k \delta P_k) = \delta\left[F - \frac{1}{2} q_k p_k + \frac{1}{2} Q_k P_k\right]$$

$$\Leftrightarrow \frac{1}{2}\Omega_{\alpha\beta}\xi_\beta \delta\xi_\alpha - \frac{1}{2}\Omega_{\rho\sigma}\eta_\sigma \delta\eta_\rho = \delta G(\xi, t)$$

$$\Leftrightarrow \frac{\partial G}{\partial \xi_\alpha} = \frac{1}{2}\Omega_{\alpha\beta}\xi_\beta - \frac{1}{2}\Omega_{\rho\sigma}\eta_\sigma \frac{\partial \eta_\rho}{\partial \xi_\alpha} \equiv X_\alpha$$

$$\Leftrightarrow \partial_\beta X_\alpha - \partial_\alpha X_\beta = 0$$

$$\begin{cases} \partial_\beta X_\alpha = \frac{1}{2}\Omega_{\alpha\beta} - \frac{1}{2}\Omega_{\rho\sigma}\frac{\partial \eta_\sigma}{\partial \xi_\beta}\frac{\partial \eta_\rho}{\partial \xi_\alpha} - \frac{1}{2}\Omega_{\rho\sigma}\eta_\sigma \frac{\partial^2 \eta_\rho}{\partial \xi_\beta \partial \xi_\alpha} \\ \partial_\alpha X_\beta = \frac{1}{2}\Omega_{\beta\alpha} - \frac{1}{2}\Omega_{\rho\sigma}\frac{\partial \eta_\sigma}{\partial \xi_\alpha}\frac{\partial \eta_\rho}{\partial \xi_\beta} - \frac{1}{2}\Omega_{\rho\sigma}\eta_\sigma \frac{\partial^2 \eta_\rho}{\partial \xi_\alpha \partial \xi_\beta} \end{cases}$$

$$\Rightarrow \partial_\beta X_\alpha - \partial_\alpha X_\beta = \Omega_{\alpha\beta} - \Omega_{\rho\sigma}\frac{\partial \eta_\rho}{\partial \xi_\alpha}\frac{\partial \eta_\sigma}{\partial \xi_\beta}$$ 辛条件

eg.（规范变换）

$$\begin{cases} Q_i = q_i \\ P_i = p_i + \frac{\partial G(q, t)}{\partial q_i} \end{cases}$$

1° $[Q_i, Q_j] = [q_i, q_j] = 0$

$[Q_i, P_j] = [q_i, p_j + \frac{\partial G}{\partial q_j}]_\xi = \delta_{ij} + 0$

$[P_i, P_j] = [p_i + \frac{\partial G}{\partial q_i}, p_j + \frac{\partial G}{\partial q_j}] = [p_i, \frac{\partial G}{\partial q_j}] + [\frac{\partial G}{\partial q_i}, p_j]$

$= -\frac{\partial}{\partial q_i}\frac{\partial G}{\partial q_j} + \frac{\partial}{\partial q_j}\frac{\partial G}{\partial q_i}$

2° $p_k \delta q_k - (p_k + \frac{\partial G}{\partial q_k})\delta q_k = -\frac{\partial G}{\partial q_k}\delta q_k = -\delta G(q, t)$

eg.（点变换）

$$\begin{cases} Q_i = Q_i(q, t) \\ P_i = \frac{\partial q_k}{\partial Q_i} p_k \end{cases}$$

$p_k \delta q_k - \frac{\partial q_k}{\partial Q_i} p_k \frac{\partial Q_i}{\partial q_l}\delta q_l = p_k \delta q_k - \delta_{kl} p_k \delta q_l = 0 = \delta[0 + f(t)]$

如果由生成函数得到正则变换，需要对正则变换进行分类

四. 新的 Hamilton 函数 $K(\eta, t) = K(Q, P, t)$

将CT：$\xi \longrightarrow \eta = \eta(\xi, t)$ 应用于 $H(\xi, t)$

$\dot\eta_\alpha = [\eta_\alpha, K]_\eta = [\eta_\alpha, K']_\eta \Rightarrow [\eta_\alpha, K' - K]_\eta = 0$

1. $K'(\eta, t) = K(\eta, t) + f(t)$

2. 若 $\eta = \eta(\xi, t)$，则 $K(\eta, t) = H(\xi(\eta), t)$

$\dot\eta_\alpha = \partial_t \eta_\alpha + [\eta_\alpha, H]_\xi = 0 + [\eta_\alpha, H]_\eta$

3. 一般 $K = H + \frac{\partial Q_i}{\partial t} P_i + \frac{\partial F}{\partial t}$

$\dot\xi_\alpha = [\xi_\alpha, H]_\xi = \Omega_{\alpha\beta}\frac{\partial H}{\partial \xi_\beta} \Leftrightarrow \delta\int_{t_1}^{t_2} \tilde L_H(\xi, \dot\xi, t)dt = 0 \quad \tilde L_H = p_k \dot q_k - H(\xi, t)$

$\dot\eta_\alpha = [\eta_\alpha, H]_\eta = \Omega_{\alpha\beta}\frac{\partial K}{\partial \eta_\beta} \Longleftrightarrow \delta\int_{t_1}^{t_2} \tilde L_K(\eta, \dot\eta, t)dt = 0 \quad \tilde L_K = P_i \dot Q_i - K(\eta, t)$

$\Rightarrow \delta\int \tilde L(\xi, \dot\xi, t)dt = 0$

$\tilde L = \cancel{H}\tilde L_H - \tilde L_K = p_k \dot q_k - P_i \dot Q_i + (K - H)$

$$\tilde L = \underbrace{(p_k - \frac{\partial Q_i}{\partial q_k}P_i)}_{\frac{\partial F}{\partial q_k}}\dot q_k + \underbrace{(-\frac{\partial Q_i}{\partial p_k}P_i)}_{\frac{\partial F}{\partial p_k}}\dot p_k + \underbrace{\frac{\partial Q_i}{\partial t}P_i}_{\frac{\partial F}{\partial t} - \frac{\partial F}{\partial t}} + \frac{\partial F}{\partial t}(K - H)$$

$$\tilde L = \frac{dF(\xi, t)}{dt} + \underbrace{K - H - \frac{\partial Q_i}{\partial t}P_i - \frac{\partial F}{\partial t}}_{\tilde L'(\xi, \dot\xi, t)}$$

$\frac{\partial \tilde L'}{\partial \xi_\alpha} = \frac{d}{dt}\frac{\partial \tilde L'}{\partial \dot\xi_\alpha} = 0 \quad \tilde L' = \tilde L(\xi, \dot\xi, t) \Rightarrow \tilde L = f(t)$

$$p_k \cancel{Q_k} \dot q_k - P_i \dot Q_i + (K - H) = \frac{dF}{dt} = \frac{\partial F}{\partial q_k}\dot q_k + \frac{\partial F}{\partial p_k}\dot p_k + \frac{\partial F}{\partial t}$$

$$p_k dq_k - \cancel{P} P_i dQ_i + (K - H)dt = dF = \frac{\partial F}{\partial q_k}dq_k + \frac{\partial F}{\partial p_k}dp_k + \frac{\partial F}{\partial t}dt$$

eg. $Q_i = Q_i(q, t) \quad P_i = \frac{\partial q_k}{\partial Q_i}p_k \Rightarrow F = 0 \Rightarrow K = H + \frac{\partial Q_i}{\partial t}P_i$

eg. $Q_i = q_i, \quad P_i = p_i + \frac{\partial G(q, t)}{\partial q_i} \Rightarrow F = -G \Rightarrow K = H - \frac{\partial G}{\partial t}$

eg. $\begin{cases} Q = \arctan\frac{q}{p} - \omega t \\ P = \frac{1}{2}(p^2+q^2) \end{cases}$

$$\begin{cases} \frac{\partial F}{\partial q} = p - \frac{\partial Q}{\partial q}P = \frac{p}{2} \Rightarrow F = \frac{1}{2}qp + f(p,t) \\ \frac{\partial F}{\partial q} = -\frac{\partial Q}{\partial p}P = \frac{q}{2} \Rightarrow \frac{1}{2}q + \frac{\partial f}{\partial p} = \frac{q}{2} \Rightarrow \frac{\partial f}{\partial p} = 0 \Rightarrow f(t) = 0 \end{cases}$$

$$\Rightarrow F = \frac{1}{2}qP$$

$$K = H + \frac{\partial Q}{\partial t}P + \frac{\partial F}{\partial t} = H - \omega P \overset{}{\underset{H=\frac{1}{2}(p^2+q^2)=\omega P}{=\!=\!=}} 0$$

$$\begin{cases} \dot{Q} = 0 \\ \dot{P} = 0 \end{cases} \Rightarrow \begin{cases} Q = Q_0 \\ P = P_0 \end{cases}$$

五. CT数学推论

1. $|\det M| = 1$, 可证 $\det M = +1$

$M\Omega M^T = \Omega \Rightarrow (\det M)^2 \det\Omega = \det\Omega$

$s=1$ $\det M = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = [Q,P]_\xi = 1$

$s \geq 2$ $\det M \nLeftarrow CT$

如: $\begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}$ $\quad Q_1 = q_1 + q_2$

$[Q_1, P_1]_\xi = 1$

$[Q_2, P_2]_\xi = 1$ ×

2. M 辛矩阵 $\Rightarrow M^{-1}$ 亦是

3. M_1、M_2 辛矩阵 $\Rightarrow M_1M_2$ 亦是

4. CT保持相空间的体积不变.

$$\int d\eta_1 d\eta_2 \cdots d\eta_{2s} = \int \Big|\underbrace{\frac{\partial(\eta_1,\eta_2,\cdots,\eta_{2s})}{\partial(\xi_1,\xi_2,\cdots,\xi_{2s})}}_{\to 1}\Big| d\xi_1 d\xi_2 \cdots d\xi_{2s}$$

六. 物理推论

1. Hamilton 体系的演化从被动角度看为CT

$\xi_\alpha(t+\varepsilon) = \xi_\alpha(t) + \varepsilon\dot{\xi}_\alpha(t) = \xi_\alpha(t) + \varepsilon[\xi_\alpha, H]_\xi$

$\xi_\alpha \longrightarrow \eta_\alpha = \xi_\alpha + \varepsilon[\xi_\alpha, H]_\xi = \eta(\xi, t)$

$[\eta_\alpha, \eta_\beta]_\xi = [\xi_\alpha + \varepsilon[\xi_\alpha, H], \xi_\beta + \varepsilon[\xi_\beta, H]]$

$= \Omega_{\alpha\beta} + \varepsilon[\xi_\alpha, [\xi_\beta, H]] + \varepsilon[\xi_\beta, [H, \xi_\alpha]] + O(\varepsilon^2)$

$= \Omega_{\alpha\beta} - \varepsilon[H, [\xi_\alpha, \xi_\beta]] = \Omega_{\alpha\beta}$

$\xi_\alpha(t+\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!}\left(\frac{d^n\xi}{dt^n}\right)_t$

$\xi_\alpha \longrightarrow \eta_\alpha = \sum_{n=0}^{\infty} \frac{\tau^n}{n!}\frac{d^n\xi_\alpha}{dt^n} = \eta(\xi, t; \tau)$

$[\eta_\alpha, \eta_\beta] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\tau^{k+l}}{k!\,l!}[\xi_\alpha^{(k)}, \xi_\beta^{(l)}]_\xi$

$[\eta_\alpha, \eta_\beta]_\xi = \sum_{n=0}^{\infty}\frac{\tau^n}{n!}\underbrace{\sum_{k+l=n}\frac{n!}{k!\,l!}[\xi_\alpha^{(k)}, \xi_\beta^{(l)}]_\xi}_{\frac{d^n}{dt^n}[\xi_\alpha,\xi_\beta] = \Omega_{\alpha\beta}} = \Omega_{\alpha\beta}$

2. Liouville 体积定理(LV)

正则区域的体积不随时间变化

其中每一个按 $\dot{\xi}_\alpha = [\xi_\alpha, H]_\xi$ 演化

如: $H = \frac{p_x^2}{2m} + mgx$

$$\begin{cases} \dot{x} = \frac{p_x}{m} \\ \dot{p}_x = -mg \end{cases} \Rightarrow \begin{cases} x(t) = x_0 + \frac{p_{x0}}{m}t - \frac{1}{2}gt^2 \\ p_x(t) = p_{x0} - mgt \end{cases}$$

$$dx\,dp_x = \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial p_{x0}} \\ \frac{\partial p_x}{\partial x_0} & \frac{\partial p_x}{\partial p_{x0}} \end{vmatrix} dx_0\,dp_{x0} = \begin{vmatrix} 1 & \frac{t}{m} \\ 0 & 1 \end{vmatrix} dx_0\,dp_{x0}$$

相点数密度 $n \equiv \lim_{\Delta\Gamma\to 0} \frac{\Delta N}{\Delta \Gamma} = n(\xi, t)$

态密度 $\rho(\xi, t) = \frac{n}{N}$ $\int \rho(\xi, t) d\xi_1 d\xi_2 \cdots d\xi_{2s} = 1$

$C = \rho(\xi, t) = \rho(\xi_0, 0)$ 其中 $\xi = \xi(t, \xi_0)$

$0 = \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + [\rho, H]_\xi$

$[\rho, H] = \frac{\partial \rho}{\partial \xi_\alpha} \Omega_{\alpha\beta} \frac{\partial H}{\partial \xi_\beta}$

$= \frac{\partial}{\partial \xi_\alpha}\left(\rho \Omega_{\alpha\beta} \frac{\partial H}{\partial \xi_\beta}\right) - \rho \Omega_{\alpha\beta} \frac{\partial}{\partial \xi_\alpha} \frac{\partial H}{\partial \xi_\beta} \to 0$

流密度 $J_\alpha \equiv \rho \Omega_{\alpha\beta} \frac{\partial H}{\partial \xi_\beta}$ $(J = \rho \Delta_H)$

$0 = \partial_t \rho + \partial_\alpha J_\alpha$

刘维尔定理与遍历假设，是统计力学的根基

n 个粒子，看作体系中 n 个相点

§4.6 正则变换及其生成函数的分类

零、CT 的分类

$$M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \xrightarrow{s=2} \begin{pmatrix} \frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \frac{\partial Q_1}{\partial p_1} & \frac{\partial Q_1}{\partial p_2} \\ \frac{\partial Q_2}{\partial q_1} & \frac{\partial Q_2}{\partial q_2} & \frac{\partial Q_2}{\partial p_1} & \frac{\partial Q_2}{\partial p_2} \\ \frac{\partial P_1}{\partial q_1} & \frac{\partial P_1}{\partial q_2} & \frac{\partial P_1}{\partial p_1} & \frac{\partial P_1}{\partial p_2} \\ \frac{\partial P_2}{\partial q_1} & \frac{\partial P_2}{\partial q_2} & \frac{\partial P_2}{\partial p_1} & \frac{\partial P_2}{\partial p_2} \end{pmatrix}$$

对任一给定 CT，总可以在 ξ，η 中各选一半作为状态参量

若 $\left|\frac{\partial Q}{\partial p}\right| \neq 0$ $Q_i = Q_i(q, p, t) \Rightarrow p_j = p_j(q, Q, t)$

若 $\left|\frac{\partial P}{\partial p}\right| \neq 0$ $P_i = P_i(q, p, t) \Rightarrow p_j = p_j(q, Q, t)$

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$\det(\partial Q_i/\partial p_j) \neq 0 \Rightarrow p = p(q, Q, t) \Rightarrow (q, Q)$ 独立

$\det(\partial Q_i/\partial q_j) \neq 0 \Rightarrow q = q(p, Q, t) \Rightarrow (p, Q)$

$\det(\partial P_i/\partial p_j) \neq 0 \Rightarrow p = p(q, P, t) \Rightarrow (q, P)$

$\det(\partial P_i/\partial q_j) \neq 0 \Rightarrow q = q(p, P, t) \Rightarrow (p, P)$

eg. $Q_i = q_i$, $P_i = p_i$ Ⅰ.Ⅱ

eg. $Q_i = p_i$, $P_i = -q_i$ Ⅰ.Ⅳ

eg. $Q = \frac{q+p}{\sqrt{2}}$, $P = \frac{q+p}{\sqrt{2}}$ Ⅰ.Ⅱ.Ⅲ.Ⅳ

eg. $\begin{cases} Q_1 = q_1, & P_1 = p_1 \\ Q_2 = p_2, & P_2 = -q_2 \end{cases}$ 不属于上述任何一类

eg. $\begin{cases} Q_k = Q_k(q, t) \text{ (可逆)} \\ P_k = \frac{\partial q_i}{\partial Q_k} p_i \end{cases}$ Ⅱ.Ⅲ

一、Type Ⅰ CT

(q, Q) 独立，$\det(\partial Q_i/\partial p_j) \neq 0$

$p_k dq_k - P_k dQ_k + (K - H)dt = dF_1(q, Q, t) = \frac{\partial F_1}{\partial q_k} \dot q_k + \frac{\partial F_1}{\partial Q_k} \dot Q_k + \frac{\partial F_1}{\partial t} dt$

$F_1(q, Q, t) = F(q, p, t) = F(q, p(q, Q, t), t)$

$\begin{cases} \frac{\partial F_1}{\partial q_k} = p_k & ① \\ \frac{\partial F_1}{\partial Q_k} = -P_k & ② \\ K = H + \frac{\partial F_1(q, Q, t)}{\partial t} & ③ \end{cases}$

$CT \xrightarrow{①②} F_1(q, Q, t) \xrightarrow{③} K(Q, P, t)$

$F_1(q, Q, t) \Longrightarrow CT$

$\begin{cases} p_k = p_k(q, Q, t) = \frac{\partial F_1}{\partial q_k} \\ P_k = P_k(q, Q, t) = -\frac{\partial F_1}{\partial Q_k} \end{cases}$

eg. $F_1 = q_i Q_i$

$$\det\left(\frac{\partial^2 F_1}{\partial q_i \partial Q_j}\right) = \det\left(\frac{\partial q_i}{\partial q_j}\right) = 1$$

$$\begin{cases} p_k = \dfrac{\partial F_1}{\partial q_k} = Q_k \\ P_k = \dfrac{\partial F_1}{\partial Q_k} = -q_k \end{cases} \Rightarrow \begin{cases} Q_k = p_k \\ P_k = -q_k \end{cases}$$

eg. $\begin{cases} Q = q\cos\omega t - p\sin\omega t \\ P = q\sin\omega t + p\cos\omega t \end{cases}$

$$1^\circ \Rightarrow \begin{cases} p = \dfrac{q\cos\omega t - Q}{\sin\omega t} = \dfrac{\partial F_1}{\partial q} \\ P = \dfrac{q - Q\cos\omega t}{\sin\omega t} = -\dfrac{\partial F_1}{\partial Q} \end{cases}$$

$$F_1 = \int p\,dq - P\,dQ = \int_{q_0}^{q} \frac{q\cos\omega t - Q_0}{\sin\omega t}\,dq - \int_{Q_0}^{Q} \frac{q - Q\cos\omega t}{\sin\omega t}\,dQ$$

$$= \cdots = \frac{(q^2+Q^2)\cos\omega t - 2qQ}{2\sin\omega t}$$

$$2^\circ \Rightarrow \begin{cases} p = \dfrac{q\cos\omega t - \cancel{p\sin\omega t}^{\,Q}}{\sin\omega t} = \dfrac{\partial F_1}{\partial q} \Rightarrow F_1 = \int p\,dq + f(Q,t) \\ P = \dfrac{q - Q\cos\omega t}{\sin\omega t} = -\dfrac{\partial F_1}{\partial Q} \Rightarrow \dfrac{q - Q\cos\omega t}{\sin\omega t} = \dfrac{q}{\sin\omega t} - \dfrac{\partial f}{\partial Q} \end{cases}$$

$$\Rightarrow F_1 = \frac{(q^2+Q^2)\cos\omega t - 2qQ}{2\sin\omega t}$$

$$\frac{\partial F_1}{\partial t} = \frac{(q - Q\cos\omega t)^2 + Q^2\sin^2\omega t}{\sin^2\omega t} = -\frac{\omega}{2}(P^2+Q^2)$$

应用于 $H = \frac{\omega}{2}(p^2+q^2) = \frac{\omega}{2}(P^2+Q^2) \Rightarrow K = 0$

$$\begin{cases} \dot{Q} = 0 \\ \dot{P} = 0 \end{cases} \Rightarrow \begin{cases} Q = Q_0 = q_0 \\ P = P_0 = p_0 \end{cases}$$

二. Type Ⅱ CT

(q, P) 独立 $\det(\partial P_i/\partial p_j) \neq 0$

$$p_k dq_k - P_k dQ_k + \underline{d(Q_k P_k)} + (K-H)dt = dF(q,p,t) + \underline{d(Q_k P_k)}$$

$$F_2(q,P,t) = F(q,p,t) + Q_k P_k$$

$$p_k dq_k + Q_k dP_k + (K-H)dt = dF_2(q,P,t)$$

$$\begin{cases} p_k = \dfrac{\partial F_2}{\partial q_k} \\ Q_k = \dfrac{\partial F_2}{\partial P_k} \end{cases} \quad K = H + \frac{\partial F_2}{\partial t}$$

三. Type Ⅲ CT

(p, Q) 独立 $F_3(p,Q,t) = F(q,p,t) - q_k p_k$

四. Type Ⅳ CT

(p, P) 独立 $F_4(p,P,t) = F(q,p,t) - q_k p_k + Q_k P_k$

eg. 点变换

$$\begin{cases} Q_i = Q_i(q,t) \quad \text{可逆} \\ P_i = \dfrac{\partial q_k}{\partial Q_i} p_k \end{cases}$$

$$P_i \frac{\partial Q_i}{\partial q_i} = \frac{\partial Q_i}{\partial q_i}\frac{\partial q_k}{\partial Q_i} p_k = \delta_{ki} p_k \Rightarrow p_k = \frac{\partial Q_i}{\partial q_k} P_i$$

$$\begin{cases} p_k = \dfrac{\partial Q_i}{\partial q_k} P_i = \dfrac{\partial F_2}{\partial q_k} \\ Q_k = Q_k(q,t) = \dfrac{\partial F_2}{\partial P_k} \Rightarrow F_2 = \int Q_k dP_k + f(q,t) = Q_k P_k + f(q,t) \end{cases}$$

$$\Rightarrow \frac{\partial Q_i}{\partial q_k} P_i = \frac{\partial Q_i}{\partial q_k} P_i + \frac{\partial f}{\partial q_k} \Rightarrow \frac{\partial f}{\partial q_k} = 0 \Rightarrow f = 0$$

eg. $(r, \theta) \to (x, y) = (r\cos\theta, r\sin\theta)$

$(p_r, p_\theta) \to (p_x, p_y)$

$$F_2 = x p_x + y p_y = F_2(r,\theta,p_x,p_y) = r p_x\cos\theta + r p_y\sin\theta = \vec{r}\cdot\vec{p}$$

$$\begin{cases} p_r = \dfrac{\partial F_2}{\partial r} = \hat{r}\cdot\vec{p} \\ p_\theta = \dfrac{\partial F_2}{\partial r} = r\hat{\theta}\cdot\vec{p} = l \end{cases}$$

不同类生成函数的关系 - Legendre 变换

$$F_2(q,P,t) = F_1(q,Q,t) + Q_k P_k = -\left[Q_k\frac{\partial F_1}{\partial Q_k} - F_1\right]$$

要求满足 Hess 条件，正则变换需要同时属于两类

(如果不属于第一类正则变换，也就没有第一类生成函数)

eg $\begin{cases} Q_i = Q_i(q, t) & \text{可逆} \\ P_i = P_i(q, p, t) \end{cases}$

$$p_k \delta q_k - P_i \delta Q_i = [p_k - \frac{\partial Q_i}{\partial q_k} P_i] \delta q_k = \delta F(q, p, t)$$

$$\begin{cases} \frac{\partial F}{\partial q_k} = p_k - \frac{\partial Q_i}{\partial q_k} P_i \\ \frac{\partial F}{\partial p_k} = 0 \Rightarrow F = F(q, t) \end{cases}$$

$$\frac{\partial q_k}{\partial Q_j} \frac{\partial Q_i}{\partial q_k} P_i = (p_k - \frac{\partial F(q,t)}{\partial q_k}) \frac{\partial q_k}{\partial Q_j}$$

$$\Rightarrow P_j = \frac{\partial q_k}{\partial Q_j} (p_k - \frac{\partial F(q,t)}{\partial q_k})$$

$\begin{cases} F = 0 & \text{点变换} \\ Q = q & \text{规范变换} \end{cases}$

eg. 证辛矩阵行列式为1

辛矩阵 M ($M \Omega M^T = \Omega$) 定义CT: $\eta = M \xi$ (不妨设为 Type II)

(若不是, 可以初等变换为 Type II)

$$(q, p) \xrightarrow{\det M = \frac{\partial(Q,P)}{\partial(q,p)}} (Q, P)$$

$$(q,p) \xrightarrow{\begin{cases} q = q \\ P = P(q,p,t) \end{cases}} (q, P) \xrightarrow{\begin{cases} Q = Q(q,P,t) \\ P = P \end{cases}} (Q,P)$$

$$\det M_1 = \frac{\partial(q, P)}{\partial(q, p)} = \begin{vmatrix} I & 0 \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} \qquad \det M_2 = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial P} \\ 0 & I \end{vmatrix}$$

$$\frac{\partial(Q,P)}{\partial(q,p)} = \frac{\partial(Q,P)}{\partial(q,P)} \Big/ \frac{\partial(q,p)}{\partial(q,P)} = \left[\frac{\partial(Q_1, Q_2, \cdots, Q_s)}{\partial(q_1, q_2, \cdots, q_s)}\right]_P \Big/ \left[\frac{\partial(p_1, p_2, \cdots, p_s)}{\partial(P_1, P_2, \cdots, P_s)}\right]_q$$

$$= \det B / \det A$$

$$A_{ij} = \frac{\partial p_i}{\partial P_j} \quad B_{ij} = \frac{\partial Q_i}{\partial q_j} = \frac{\partial}{\partial q_j} \frac{\partial F_2}{\partial P_i} = \frac{\partial}{\partial P_i} \frac{\partial F_2}{\partial q_j} = \frac{\partial p_j}{\partial P_i} = A_{ji}$$

$$\Rightarrow \frac{\partial(Q,P)}{\partial(q,p)} = \det B / \det A = 1$$

§4.7 Hamilton-Jacobi 理论 (不采用求和约定)

$\begin{cases} Q = q\cos\omega t - p\sin\omega t \\ P = q\sin\omega t + p\cos\omega t \end{cases}$ $\begin{cases} Q = \arctan\frac{q}{p} - \omega t \\ P = \frac{1}{2}(q^2 + p^2) \end{cases}$ $H = \frac{1}{2}\omega(p^2 + q^2)$

$$H(q,p,t) \xrightarrow[p_k = \frac{\partial S}{\partial q_k},\ Q_k = \frac{\partial S}{\partial P_k}]{F_2(q,P,t) = S(q,P,t)} K(Q,P,t) = 0 = H(q, p, t) + \frac{\partial F_2(q,P,t)}{\partial t}$$

一. HJ 方程

$$H = H(q,p,t) \Rightarrow -\frac{\partial S}{\partial t} = H(q, \frac{\partial S}{\partial q}, t)$$

$S = S(q, P, t)$ Hamilton 主函数

1. Q_k, P_k ($k = 1, 2, \cdots, s$) 为 2s 个运动常数

$\begin{cases} p_k = \frac{\partial S}{\partial q_k} = p_k(q, P, t) \\ Q_k = \frac{\partial S}{\partial P_k} = Q_k(q, P, t) \end{cases}$ $\begin{matrix} CT ① \Rightarrow P_k = P_k(q, p, t) \text{ 代入②} \Rightarrow Q_k = Q_k(q, p, t) \\ ② \Rightarrow q_k = q_k(Q, P, t) \text{ 代入①} \Rightarrow p_k = p_k(Q, P, t) \end{matrix}$

Hess 条件 $\det(\frac{\partial^2 S}{\partial q_i \partial P_j}) \neq 0$

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_k}\dot{q}_k + \frac{\partial S}{\partial P_k}\dot{P}_k + \frac{\partial S}{\partial t} \xrightarrow{\text{真实路径}} p_k \dot{q}_k - 0 - H(q,p,t)$$

2. $S =$ (相空间中) 真实实路径的作用量 $S = \int_0^t (p_k \dot{q}_k - H) dt$

3. HJ 方程是 S 关于 (s+1) 个自变量 $(q_1, q_2, \cdots, q_s, t)$ 的一阶偏微分方程, 一般非线性

若 S 是解, $S(q, P, t) + C$ (相加常数) 也是解

eg. $H = p \Rightarrow -\frac{\partial S}{\partial t} = \frac{\partial S}{\partial q}$

$\begin{cases} X = q + t \\ Y = q - t \end{cases}$

$$\Rightarrow -(\frac{\partial S}{\partial X} - \frac{\partial S}{\partial Y}) = \frac{\partial S}{\partial X} + \frac{\partial S}{\partial Y} \Rightarrow \frac{\partial S}{\partial X} = 0 \Rightarrow S = S(X, Y)$$

$S = f(q - t)$

解不是依赖于若干任意常数, 而是依赖于若干任意函数.

$$\begin{cases} p = \frac{\partial S}{\partial q} = P \\ Q = \frac{\partial S}{\partial P} = q - t \end{cases} \Rightarrow \begin{cases} Q = q - t = q_0 \\ P = p = p_0 \end{cases}$$

eg. $H = \frac{p^2}{2} \Rightarrow -\frac{\partial S}{\partial t} = \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2$

$$S_a(q, P_a, t) = \frac{(q - P_a)^2}{2t} \Rightarrow \begin{cases} p = \frac{\partial S}{\partial q} = \frac{q - P_a}{t} \\ Q = \frac{\partial S}{\partial P_a} = \frac{P_a - q}{t} \end{cases} \Rightarrow \begin{cases} Q_a = -p \\ P_a = q - pt \end{cases}$$

$$S_b(q, P_b, t) = q\sqrt{2P_b} - P_b t \Rightarrow \begin{cases} p = \sqrt{2P_b} \\ Q_b = \frac{q}{\sqrt{2P_b}} - t \end{cases} \Rightarrow \begin{cases} Q_b = \frac{q}{p} - t \\ P_b = \frac{1}{2}p^2 \end{cases}$$

如: $\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \varphi}{\partial t^2} \Rightarrow \varphi(x,t) = f(x - vt) + g(x + vt)$

$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \Rightarrow \varphi(x,y) = f(x - iy) + g(x + iy) \quad (i^2 = -1)$

二. 偏微分方程的两类解

1. 一般积分: 依赖于若干任意函数
2. 完全积分: 依赖于若干任意常数 (数目与自变量个数一样)

$S(q, t; C_1, C_2, \cdots C_s) + C_0 = S(q, P, t) + C_0 \quad (P_k = C_k)$

不能取 C_0 为 P. 因为不满足 Hess 条件

$$\begin{cases} x = (x_1, x_2, \cdots x_n) & f(x) = g(y) \Rightarrow f(x) = C = g(y) \\ y = (y_1, y_2, \cdots y_m) & f(x) + g(y) = D \Rightarrow f(x) = C, \ -g(y) = D - C \end{cases}$$

$H = H(q, p, \cancel{t}) \Rightarrow -\frac{\partial S}{\partial t} = H\left(q, \frac{\partial S}{\partial q}\right)$

不妨令 $S(q,t) = W(q) + T(t)$

$\Rightarrow -\frac{dT}{dt} = H\left(q, \frac{\partial W}{\partial q}\right) := P_1 \qquad T = -P_1 t$

三. 不显含 t 的 HJ 方程 $H = H(q, p)$

$H\left(q, \frac{\partial W}{\partial q}\right) = P_1$

$S = W(q) - P_1 t$, $W(q) \longleftrightarrow$ Hamilton 特征函数

eg. $H = \frac{1}{2}\omega(p^2 + q^2) \Rightarrow \frac{1}{2}\omega\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}\omega q^2 = P \Rightarrow W = \int\sqrt{\frac{2P}{\omega} - q^2}\,dq$

$S = W - Pt$

$p = \sqrt{\frac{2P}{\omega} - q^2}$

$Q = \frac{1}{\omega}\int\frac{dq}{\sqrt{\frac{2P}{\omega} - q^2}} - t = \frac{1}{\omega}\arcsin\left(\sqrt{\frac{\omega}{2P}}q\right) - t$

$$\begin{cases} P = \frac{1}{2}\omega(p^2 + q^2) = E \\ Q = \frac{1}{\omega}\arcsin\frac{q}{\sqrt{q^2 + p^2}} - t = -t_0 \end{cases} \Rightarrow q = \sqrt{\frac{2E}{\omega}}\sin\omega(t - t_0)$$

分离变量 → 微扰动 → 猜

临时记号 $\bar{q} = (q_1, q_2, \cdots q_{s-1})$. $\bar{p} = (p_1, p_2, \cdots p_{s-1})$ $W(q) = \bar{W}(\bar{q}) + W_s(q_s)$

1. $H = H(\bar{q}, q_s, \bar{p}, p_s) \Rightarrow H\left(\bar{q}, \frac{\partial \bar{W}}{\partial \bar{q}}, \frac{\partial W_s}{\partial q_s}\right) = P_1$

$$\begin{cases} \frac{\partial W_s}{\partial q_s} = P_s \Rightarrow W_s = P_s q_s \\ H\left(\bar{q}, \frac{\partial \bar{W}}{\partial \bar{q}}, P_s\right) = P_1 \end{cases}$$

2. $H = H(\bar{q}, \bar{p}) + H_s(q_s, p_s)$

$H\left(\bar{q}, \frac{\partial \bar{W}}{\partial \bar{q}}\right) + H_s\left(q_s, \frac{\partial W_s}{\partial q_s}\right) = P_1$

$$\begin{cases} H_s\left(q_s, \frac{\partial W_s}{\partial q_s}\right) = P_s \\ H\left(\bar{q}, \frac{\partial \bar{W}}{\partial \bar{q}}\right) = P_1 - P_s \end{cases}$$

3. $f(\bar{q})H = H(\bar{q}, \bar{p}) + H_s(q_s, p_s)$

$f(\bar{q})P_1 = H\left(\bar{q}, \frac{\partial \bar{W}}{\partial \bar{q}}\right) + H_s\left(q_s, \frac{\partial W_s}{\partial q_s}\right)$

$$\begin{cases} H_s\left(q_s, \frac{\partial W_s}{\partial q_s}\right) = P_s \\ H\left(\bar{q}, \frac{\partial \bar{W}}{\partial \bar{q}}\right) = f(\bar{q})P_1 - P_s \end{cases}$$

4. $f(q_s)H = H(\bar{q}, \bar{p}) + H_s(q_s, p_s)$

四. 完全可分离体系

$W(q, P) = \sum_{i=1}^{s} W_i(q_i, P) \qquad S = \sum_{i=1}^{s} W_i(q_i, P) - P_1 t$

$$\begin{cases} P_k = \dfrac{\partial W_k}{\partial q_k} = P_k(q_k, P) \quad (k=1,2,\cdots,s) & (\mathrm{I}a) \\ Q_j = \sum\limits_i \dfrac{\partial W_i}{\partial P_j} = Q_j(q, P) \quad (j=2,3,\cdots,s) & (\mathrm{I}b) \\ Q_1 = \sum\limits_i \dfrac{\partial W_i}{\partial P_1} - t = Q_1(q, P, t) & (\mathrm{I}c) \end{cases}$$

$$\Rightarrow \begin{cases} P_k = P_k(q, p) & (\mathrm{II}a) \\ Q_j = Q_j(q, p) & (\mathrm{II}b) \\ Q_1 = Q_1(q, p, t) & (\mathrm{II}c) \end{cases}$$

(IIa)、(IIb)：$(2s-1)$个不显含t运动常数　相轨迹

(IIc)：1个显含t运动常数　时间箭头

- $\mathrm{I}a$　相轨迹在(q_k, p_k)平面上投影，由P决定，与Q无关
- $\mathrm{I}b$　位形空间轨迹
- $\mathrm{I}c$　时间箭头

eg. $\vec{B} = B\hat{z}$　带电e，在(x, y)平面

$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ 下不能分离

$\vec{A} = (B_y z, B_z x, B_x y) = B x\hat{y}$ 下可以分离

$$H = \sum \frac{(p_i - eA_i)^2}{2m} (= T) = \frac{p_x^2}{2m} + \frac{(p_y - eBx)^2}{2m}$$

是否完全可分离，取决于特定 Hamilton 函数

H 不显含 $t \Rightarrow S(x, y, t) = W(x, y) - P_1 t \quad (P_1 = H = T)$

H 不显含 $y \Rightarrow S = X(x) + P_2 y - P_1 t \quad (P_2 = p_y = m\dot{y} + eBx)$

$$P_1 = \frac{1}{2m}\left(\frac{\partial X}{\partial x}\right)^2 + \frac{(P_2 - eBx)^2}{2m} \Rightarrow X(x) = \int\sqrt{2mP_1 - (P_2 - eBx)^2}\,dx$$

$$p_x = \frac{\partial S}{\partial x} = \sqrt{2mP_1 - (P_2 - eBx)^2}$$

$$p_y = \frac{\partial S}{\partial y} = P_2$$

$$Q_2 = \frac{\partial S}{\partial P_2} = -\frac{\sqrt{2mP_1 - (P_2 - eBx)^2}}{eB} + y$$

$$Q_1 = \frac{\partial S}{\partial P_1} = \int\frac{m\,dx}{\sqrt{2mP_1 - (P_2 - eBx)^2}} - t$$

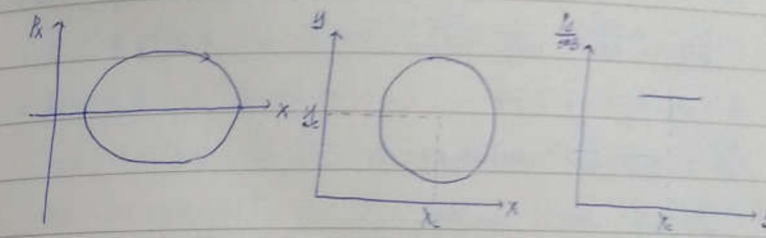
记 $\omega = \frac{eB}{m}$，$Q_1 = -t_c$，$Q_2 = y_c$，$2mP_1 = e^2B^2r^2$，$P_2 = eBx_c$

$p_x = eB\sqrt{r^2 - (x - x_c)^2}$　　$\dfrac{(x - x_c)^2}{r^2} + \dfrac{p_x^2}{(eBr)^2} = 1$

$p_y = eBx_c$

$y_c = y - \sqrt{r^2 - (x - x_c)^2}$　　$(x - x_c)^2 + (y - y_c)^2 = r^2$

$x = x_c + r\sin\omega(t - t_c)$



eg. 中心力问题

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r)$$

$$S = R(r) + \Theta(\theta) - P_1 t = R(r) + P_2\theta - P_1 t$$

$$Q_2 = \frac{\partial S}{\partial P_2} = -\int\frac{P_2\,dr}{r^2\sqrt{2m(P_1 - U) - \frac{P_2^2}{r^2}}} + \theta \equiv \theta_0$$

$$p_r = \frac{\partial S}{\partial r} = \sqrt{2m(P_1 - U) - \frac{P_2^2}{r^2}}$$

$$R = \int\sqrt{2m(P_1 - U) - \frac{P_2^2}{r^2}}\,dr$$

$$p_\theta = \frac{\partial S}{\partial \theta} = P_2 = l$$

$$\frac{1}{2m}\left(\frac{\partial R}{\partial r}\right)^2 + \frac{P_2^2}{2mr^2} + U = P_1$$

若 $U = -\frac{\alpha}{r}$，记 $p = \frac{p_\theta^2}{m\alpha}$，$\varepsilon = \sqrt{1 + \frac{2Ep_\theta^2}{m\alpha^2}}$

$$\theta - \theta_0 = \int\frac{p_\theta\,dr}{r^2\sqrt{2mE + \frac{2m\alpha}{r} - \frac{p_\theta^2}{r^2}}} \overset{u = \frac{1}{r}}{=\!=} -\int\frac{du}{\sqrt{\frac{\varepsilon^2}{p^2} - (u - \frac{1}{p})^2}} = \arccos\frac{u - \frac{1}{p}}{\varepsilon/p}$$

$$\Rightarrow u - \frac{1}{p} = \frac{\varepsilon}{p}\cos(\theta - \theta_0) \Rightarrow r = \frac{p}{1 + \varepsilon\cos(\theta - \theta_0)}$$

$$p_r^2 = p_\theta^2\left(\frac{1+\varepsilon}{p} - \frac{1}{r}\right)\left(\frac{\varepsilon - 1}{p} + \frac{1}{r}\right)$$

$$p_r = p_\theta \sqrt{\frac{\varepsilon^2}{p^2} - \left(\frac{1}{r} - \frac{1}{p}\right)^2} \Rightarrow J_r = \frac{1}{2\pi}\oint p_r dr = -|p_\theta| + \alpha\sqrt{\frac{m}{2E}}$$

eg. 球坐标

$$H = \frac{p_r^2}{2m} + \frac{1}{2mr^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + U(r,\theta,\phi)$$

$$S = R(r) + \Theta(\theta) + \Phi(\phi) - P_1 t$$

$$r^2\left(\frac{\partial R}{\partial r}\right)^2 + \left[\left(\frac{\partial\Theta}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial\Phi}{\partial\phi}\right)^2\right] = 2mr^2[P_1 - U(r,\theta,\phi)]$$

若 $r^2U = A(r) + V(\theta,\phi)$

$$\begin{cases} \left(\frac{\partial\Theta}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial\Phi}{\partial\phi}\right)^2 + 2mV(\theta,\phi) = P_2 \\ r^2\left(\frac{\partial R}{\partial r}\right)^2 + P_2 = 2mr^2P_1 - 2mA(r) \end{cases}$$

$$\Rightarrow \sin^2\theta\left(\frac{\partial\Theta}{\partial\theta}\right)^2 + \left(\frac{\partial\Phi}{\partial\phi}\right)^2 + 2m\sin^2\theta\, V(\theta,\phi) = P_2\sin^2\theta$$

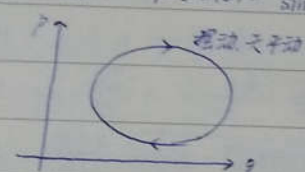
若 $\sin^2\theta\, V(\theta,\phi) = B(\theta) + C(\phi)$

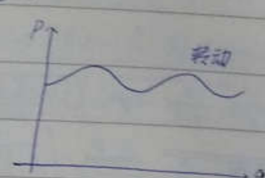
$$\begin{cases} \left(\frac{\partial\Phi}{\partial\phi}\right)^2 + 2mC(\phi) = P_3 \\ \sin^2\theta\left(\frac{\partial\Theta}{\partial\theta}\right)^2 + 2mB(\theta) + P_3 = P_2\sin^2\theta \end{cases}$$

$$U = \frac{A(r)}{r^2} + \frac{V(\theta,\phi)}{r^2}$$

$$= \frac{A(r)}{r^2} + \frac{1}{r^2}\left[\frac{B(\theta)}{\sin^2\theta} + \frac{C(\phi)}{\sin^2\theta}\right]$$

$$= a(r) + \frac{1}{r^2}\left[b(\theta) + \frac{c(\phi)}{\sin^2\theta}\right]$$





作用变量角变量理论

$$\begin{cases} J_r = \frac{1}{2\pi}\oint p_r dr = -|p_\theta| + \alpha\sqrt{\frac{m}{2E}} \\ J_\theta = \frac{1}{2\pi}\oint p_\theta d\theta = p_\theta \end{cases}$$

. 用之作为新的广义动量，称作用变量，与之共轭广义坐量称角变量.

早期量子力学

H原子 $\frac{1}{\lambda} = R_H\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$ $R_H = 10967758\,m^{-1}$ $(n_2 > n_1)$

电子应会向外辐射能量，半径减小，连续光谱，并落到原子核上

一. Bohr 模型

1. 定态模型

2. 角动量量子化 $p_\theta = n\hbar$ $(n = 1, 2, 3, \cdots)$

$m_e c^2 = 0.511\,MeV$ $k = \frac{e^2}{4\pi\varepsilon_0} = e_s^2 = 1.44\,MeV\cdot fm$

$\hbar c = 197\,MeV\cdot fm$ $\alpha = \frac{e_s^2}{\hbar c} = \frac{1}{137}$

$p_\theta = m_e r v = \sqrt{m_e r e_s^2} = n\hbar \Rightarrow r_n = \frac{n^2\hbar^2}{m_e e_s^2}$ $(\approx 0.529 n^2 \text{Å})$

$\frac{e_s^2}{r^2} = m_e\frac{v^2}{r} \Rightarrow v = \sqrt{\frac{e_s^2}{m_e r}} = \frac{e_s^2}{n\hbar} = \frac{c}{n}\frac{1}{\alpha}$

$E_n = -\frac{e_s^2}{2r_n} = -\frac{1}{n^2}\frac{m_e e_s^4}{2\hbar^2} = -\frac{13.6}{n^2}\,(eV)$

3. 跃迁假设 $h\nu = h\frac{c}{\lambda} = E_{n_2} - E_{n_1} = \frac{m_e e_s^4}{2\hbar^2}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$

缺陷: (1) 仅能精确到 α^2

(2) 仅适用氢原子和类氢原子

(3) 不能解释氢原子光谱的精细结构和超精细结构

二. Bohr-Sommerfeld

$$\begin{cases} J_r = k\hbar \quad (k = 1, 2, \cdots) \\ J_\theta = p_\theta = l\hbar \quad (l = 0, 1, 2, \cdots) \end{cases}$$ 允许是椭圆

$n = k + l$

$-\frac{\partial S}{\partial t} = H(\vec{r}, \nabla S, t)$ 将S去掉视作算子

$\partial_t \to -i\hbar\partial_t$ $\nabla \to -i\hbar\nabla$

$i\hbar\frac{\partial}{\partial t}\psi = H(\vec{r}, -i\hbar\nabla, t)\psi$

$$H=\frac{p^2}{2m}+U$$

$$i\hbar\frac{\partial\psi}{\partial t}=\left(\frac{-\hbar^2}{2m}\nabla^2+U\right)\psi$$

$$H=\sum_i\frac{(P_i-eA_i)^2}{2m}+e\varphi$$

$$i\hbar\frac{\partial\psi}{\partial t}=\left[\frac{(-i\hbar\nabla-e\vec A)^2}{2m}+e\varphi\right]\psi$$

CH5 刚体

§5.1 刚体运动学概念

一. 定点运动 —— 转动 $\vec v=\vec\omega\times\vec r$

1. 任一点 X ↔ $\vec x$ $\quad\frac{d|\vec x|^2}{dt}=2\vec x\cdot\dot{\vec x}=0\Rightarrow\dot{\vec x}\perp\vec x$ 或 $\dot{\vec x}=0$

2. 取平面，包含OX，法向$\dot{\vec x}$

   在面内任取 Y ↔ $\vec y$ $\Rightarrow\dot{\vec y}\parallel\dot{\vec x}$

   $0=\frac{1}{2}\frac{d(\vec x-\vec y)^2}{dt}=(\vec x-\vec y)\cdot(\dot{\vec x}-\dot{\vec y})=-\vec x\cdot\dot{\vec y}$

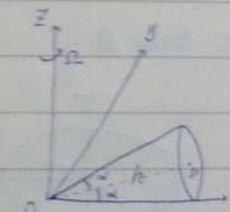
   $0=\vec y\cdot\dot{\vec y}$

3. 在面内任取三个共线点 X ↔ $\vec x$, Y ↔ $\vec y$, Z ↔ $\vec z$

   $\vec y-\vec x=\beta(\vec z-\vec x)\Rightarrow\dot{\vec y}-\dot{\vec x}=\beta(\dot{\vec z}-\dot{\vec x})$

   若$\dot{\vec z}\neq\dot{\vec x}$，令$\dot{\vec y}=0\Rightarrow\dot{\vec x}=\beta(\dot{\vec x}-\dot{\vec z})$

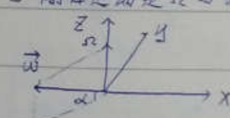
eg. 纯滚动 $\quad OD:\vec\Omega=\Omega\hat z$



$\vec\omega=-\omega\hat x$

1° $\omega h\sin\alpha=\Omega h\cos\alpha\Rightarrow\vec\omega=-\Omega\cot\alpha\,\hat x$

2° 刚体运动是$\vec\Omega$与绕刚体转动的合成



$\vec\omega=-\Omega\cot\alpha\,\hat x$

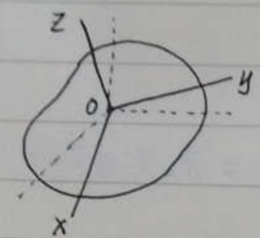
$\vec\beta=\frac{d\vec\omega}{dt}=\vec\Omega\times\vec\omega=-\Omega^2\cot\alpha\,\hat y$

二. 一般运动

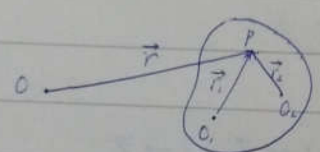
1. 空间坐标系 (space system)

   本体坐标系 (body system)

2. 一般 $S=6$



3. 平移与参考点有关，转动 $\vec{\omega}_2=\vec{\omega}_1$



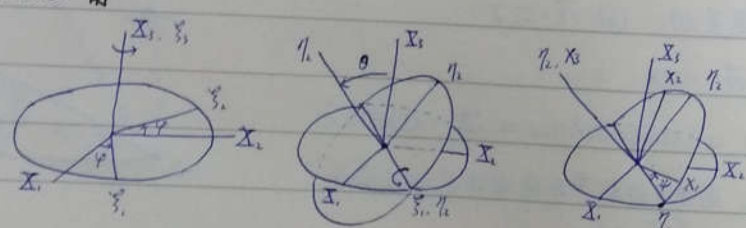
$$\vec{V}=\frac{d\vec{r}}{dt}=\begin{cases}\vec{V}_1+\vec{\omega}_1\times\vec{r}_1\\ \vec{V}_2+\vec{\omega}_2\times\vec{r}_2\end{cases}$$

$$\hookrightarrow \vec{V}_1+\vec{\omega}_1\times(\vec{r}_1-\vec{r}_2)$$

$$\vec{V}_1+\vec{\omega}_2\times\vec{r}_1=\vec{V}_1+\vec{\omega}_1\times\vec{r}_1+(\vec{\omega}_2-\vec{\omega}_1)\times\vec{r}_2$$

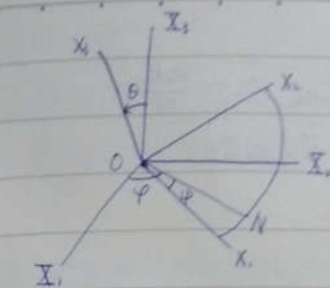
$$\Rightarrow 0=(\vec{\omega}_2-\vec{\omega}_1)\times\vec{r}_2$$

三. Euler 角



$$\lambda_\varphi=\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\qquad \lambda_\theta=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\qquad \lambda_\psi=\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}\xrightarrow[\text{进动}]{\hat{X}_3,\ \varphi}\lambda_\varphi\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}\xrightarrow[\text{章动}]{\hat{ON},\ \theta}\lambda_\theta\lambda_\varphi\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}\xrightarrow[\text{自转}]{\hat{x}_3,\ \psi}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\lambda_\psi\lambda_\theta\lambda_\varphi\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}$$



给定Euler角，可以唯一确定刚体位置

给定刚体位置，未必能唯一确定Euler角

四. Euler 运动学方程

$$\vec{\omega}=\dot{\varphi}\hat{X}_3+\dot{\theta}\,\widehat{ON}+\dot{\psi}\hat{x}_3$$

$$\omega^2=\dot{\varphi}^2+\dot{\theta}^2+\dot{\psi}^2+2\dot{\varphi}\dot{\psi}\cos\theta$$

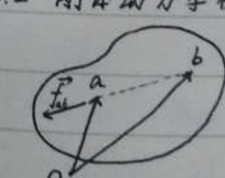
$$\vec{\omega}=\omega_i\hat{x}_i=\Omega_i\hat{X}_i$$

$$d\vec{\Theta}=\hat{n}\,d\Theta=\hat{X}_3d\varphi+\hat{ON}\,d\theta+\hat{x}_3d\psi$$

$$\begin{cases}\hat{X}_3=\hat{x}_1\sin\theta\sin\psi+\hat{x}_2\sin\theta\cos\psi+\hat{x}_3\cos\theta\\ \hat{ON}=\hat{x}_1\cos\psi-\hat{x}_2\sin\psi\\ \hat{x}_3=\hat{x}_3\end{cases}$$

$$\begin{cases}\omega_1=\dot{\varphi}\sin\theta\sin\psi+\dot{\theta}\cos\psi\\ \omega_2=\dot{\varphi}\sin\theta\cos\psi-\dot{\theta}\sin\psi\\ \omega_3=\dot{\varphi}\cos\theta+\dot{\psi}\end{cases}$$

§5.2 刚体动力学概述



$$\vec{F}_a=\vec{F}_a^{\text{外}}+\sum_{b\neq a}\vec{f}_{ab}$$

$$=m_a\ddot{\vec{r}}_a=\frac{d\vec{P}_a}{dt}$$

一. 质点组整体定理

$$\vec{F}^* = \sum_a \vec{F}_a^* = \frac{d\vec{P}}{dt} = m\ddot{\vec{r}}_c \quad (m = \sum_a m_a \quad \vec{r}_c = \sum_a \frac{m_a \vec{r}_a}{m})$$

$$\vec{\tau}^* = \sum_a (\vec{r}_a \times \vec{F}_a)^* = \frac{d\vec{L}}{dt} \quad (\vec{L} = \sum_a \vec{r}_a \times m_a \vec{v}_a)$$

$$P = \sum_a \vec{F}_a \cdot \vec{v}_a = \frac{dT}{dt} \quad (T = \sum_a \frac{1}{2} m_a v_a^2)$$

$$\vec{f}_{ab} \cdot \vec{v}_a + \vec{f}_{ba} \cdot \vec{v}_b$$

$$= \vec{f}_{ab} \cdot \vec{v}_{ab} \propto \frac{d}{dt}(\frac{1}{2} r_{ab}^2)$$

二. 质心系 $\begin{cases} \text{原点位于质心} \\ \text{随质心平移} \end{cases}$

$$\begin{cases} \vec{F} = \vec{F}_c + \vec{F}^* = \vec{F}^* + 0 \\ \vec{\tau} = \vec{\tau}_c + \vec{\tau}^* = \vec{r}_c \times \vec{F}^* + \sum_a \vec{r}_a^* \times \vec{F}_a \\ P = P_c + P^* = \vec{F}^* \cdot \vec{v}_c + \sum_a \vec{F}_a \cdot \vec{v}_a^* \end{cases}$$

$$\begin{cases} \vec{P} = \vec{P}_c + \vec{P}^* = m\vec{v}_c + 0 \\ \vec{L} = \vec{L}_c + \vec{L}^* = \vec{r}_c \times m\vec{v}_c + \sum_a \vec{r}_a^* \times m_a \vec{v}_a^* \\ T = T_c + T^* = \frac{1}{2} m v_c^2 + \sum_a \frac{1}{2} m_a v_a^{*2} \end{cases}$$

$$\Rightarrow P^* = 0, \quad \vec{\tau}^* = \frac{d\vec{L}^*}{dt}, \quad P^* = \frac{dT^*}{dt}$$

三. 刚体动力学

1. 一般运动 $\begin{cases} \text{平移} \quad \vec{F}^* = m\ddot{\vec{r}}_c \\ \text{转动} \quad \vec{\tau}^* = \dfrac{d\vec{L}^*}{dt} \end{cases}$

$$P^* = \sum_a \vec{F}_a \cdot \vec{v}_a^* = \sum_a \vec{F}_a \cdot (\vec{\omega} \times \vec{r}_a^*)$$

$$= \vec{\omega} \cdot \sum_a (\vec{r}_a^* \times \vec{F}_a) = \vec{\tau}^* \cdot \vec{\omega}$$

$$= \vec{\omega} \cdot \frac{d\vec{L}^*}{dt} = \vec{\omega} \cdot \frac{d\sum(\vec{r}_a^* \times m_a \vec{v}_a^*)}{dt} = \vec{\omega} \cdot \sum_a [\vec{r}_a^* \times m_a \ddot{\vec{r}}_a^*]$$

$$= \sum_a m_a (\vec{\omega} \times \vec{r}_a^*) \cdot \frac{d\vec{v}_a^*}{dt} = \sum_a m_a v_a^* \frac{dv_a^*}{dt} = \frac{dT^*}{dt}$$

2. 定点转动 $\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \vec{r}_c(t)$

解除约束，可求约束力

3. 定轴转动 $\tau_n = \frac{dL_n}{dt}$

§5.3 定点转动刚体的角动量与动能

一. 角动量

$$\vec{L} = \int (\vec{r} \times \vec{v}) dm = \int [\vec{r} \times (\vec{\omega} \times \vec{r})] dm = \int [r^2 \vec{\omega} - \vec{r}(\vec{r} \cdot \vec{\omega})] dm$$

$$= \int [r^2 \overleftrightarrow{I} - \vec{r}\vec{r}] dm \cdot \vec{\omega} = \overleftrightarrow{J} \cdot \vec{\omega}$$

角动量 $\vec{L} = \overleftrightarrow{J} \cdot \vec{\omega} \quad L_i = J_{ij} \omega_j$

$$T = \frac{1}{2} \int v^2 dm = \frac{1}{2} \int (\vec{\omega} \times \vec{r}) \cdot \vec{v} dm = \frac{1}{2} \vec{\omega} \cdot \int (\vec{r} \times \vec{v}) dm = \frac{1}{2} \vec{\omega} \cdot \vec{L}$$

动能 $T = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{J} \cdot \vec{\omega} = \frac{1}{2} J_{ij} \omega_i \omega_j$

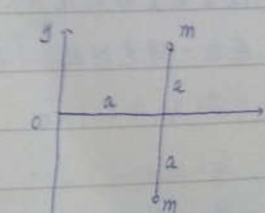
二. 绕O点的惯量张量(Inertia Tensor)

$$\overleftrightarrow{J} \equiv \int (r^2 \overleftrightarrow{I} - \vec{r}\vec{r}) dm = J_{ij} \hat{x}_i \hat{x}_j$$

$$J_{ij} = \hat{x}_i \cdot \overleftrightarrow{J} \cdot \hat{x}_j = \int (r^2 \delta_{ij} - x_i x_j) dm$$

$\begin{cases} J_{11} = \int (x_2^2 + x_3^2) dm & \text{绕}\hat{x}_1\text{轴转动惯量} \\ J_{12} = -\int x_1 x_2 dm & \text{惯量积} \end{cases}$

eg. $\overleftrightarrow{J} = (m_1 r_1^2 + m_2 r_2^2) \overleftrightarrow{I} - (m_1 \vec{r}_1 \vec{r}_1 + m_2 \vec{r}_2 \vec{r}_2)$



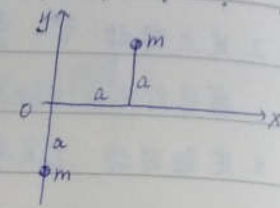
$$r_1^2 = r_2^2 = 2a^2$$

$$\begin{cases} \vec{r}_1 = a(\hat{x} + \hat{y}) \\ \vec{r}_2 = a(\hat{x} - \hat{y}) \end{cases} \Rightarrow \begin{cases} \vec{r}_1 \vec{r}_1 = a^2(\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{x}\hat{y} + \hat{y}\hat{x}) \\ \vec{r}_2 \vec{r}_2 = a^2(\hat{x}\hat{x} + \hat{y}\hat{y} - \hat{x}\hat{y} - \hat{y}\hat{x}) \end{cases}$$

$$\overleftrightarrow{J} = 4ma^2 \overleftrightarrow{I} - 2ma^2(\hat{x}\hat{x} + \hat{y}\hat{y})$$

$$\overleftrightarrow{J} = 2ma^2(\hat{x}\hat{x} + \hat{y}\hat{y} + 2\hat{z}\hat{z})$$

eg. $r_1^2 = 2a^2 \quad r_2^2 = a^2$

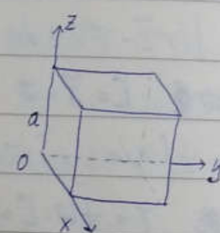


$$\begin{cases} \vec{r}_1 = a(\hat{x}+\hat{y}) \\ \vec{r}_2 = -a\hat{y} \end{cases} \Rightarrow \begin{cases} \vec{r}_1\vec{r}_1 = a^2(\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{x}\hat{y}+\hat{y}\hat{x}) \\ \vec{r}_2\vec{r}_2 = a^2\hat{y}\hat{y} \end{cases}$$

$$\overleftrightarrow{J} = 3ma^2 I - ma^2(\hat{x}\hat{x}+2\hat{y}\hat{y}+\hat{x}\hat{y}+\hat{y}\hat{x})$$

$$= ma^2\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

eg. $J_{11} = J_{xx} = \rho\int_0^a dx\int_0^a dy\int_0^a dz(y^2+z^2)$



$= \frac{2}{3}\rho a^5 = \frac{2}{3}ma^2 = J_{22} = J_{33}$

$J_{12} = J_{xy} = -\rho\int_0^a dx\int_0^a dy\int_0^a dz(xy)$

$= -\rho\frac{a^2}{2}\cdot\frac{a^2}{2}\cdot a = -\frac{1}{4}ma^2 = J_{23} = J_{31}$

$$\overleftrightarrow{J} = \frac{1}{4}ma^2\begin{pmatrix} \alpha & -1 & -1 \\ -1 & \alpha & -1 \\ -1 & -1 & \alpha \end{pmatrix} \qquad \alpha = \frac{8}{3}$$

绕(1,1,1)转动时 $\vec{L}\parallel\vec{\omega}$

三.惯量张量的物理含义

在任一特定时刻，$\overleftrightarrow{J}$由O点决定，J_{ij}由O点及轴取向决定

1. $\vec{\omega} = \hat{x}_j \Rightarrow L_i = J_{ij}$

$\vec{\omega} = \hat{x}_j \Rightarrow \vec{L} = \overleftrightarrow{J}\cdot\hat{x}_j \Rightarrow L_i = \hat{x}_i\cdot\vec{L} = \hat{x}_i\cdot\overleftrightarrow{J}\cdot\hat{x}_j = J_{ij}$

2. $\vec{\omega} = \omega\hat{n} \Rightarrow L_n = J_n\omega,\ T = \frac{1}{2}J_n\omega^2 \quad J_n = \hat{n}\cdot\overleftrightarrow{J}\cdot\hat{n} = \int|\hat{n}\times\vec{r}|^2 dm$

$L_n = \hat{n}\cdot\vec{L} = \hat{n}\cdot\overleftrightarrow{J}\cdot\vec{\omega} = \omega\hat{n}\cdot\overleftrightarrow{J}\cdot\hat{n}$

$\hookrightarrow \int[r^2-(\hat{n}\cdot\vec{r})^2]dm$

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$T = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{J}\cdot\vec{\omega} = \cancel{\frac{1}{2}\omega\hat{n}\cdot\overleftrightarrow{J}\cdot} \ \frac{1}{2}J_n\omega^2$

3. $\overleftrightarrow{J} = \overleftrightarrow{J}_c + \overleftrightarrow{J}^* \quad \overleftrightarrow{J}_c = m(r_c^2\overleftrightarrow{I} - \vec{r}_c\vec{r}_c)$

$J_{ij} = J_{ij}^* + m(r_c^2\delta_{ij} - x_{ci}x_{cj})$

$J_n = J_n^* + md_c^2$

$\vec{L} = \vec{L}_c + \vec{L}^* \qquad T = T_c + T^*$

$\overleftrightarrow{J}\cdot\vec{\omega} \quad \overleftrightarrow{J}_c\cdot\vec{\omega} \quad \overleftrightarrow{J}^*\cdot\vec{\omega} \qquad \frac{1}{2}J_n\omega^2 = \frac{1}{2}m\omega^2 d_c^2 + \frac{1}{2}J^*\omega^2$

eg. 立方体

$$\left.\begin{aligned} \vec{r}_c &= \frac{a}{2}(\hat{x}+\hat{y}+\hat{z}) \\ r_c^2 &= \frac{3}{4}a^2 \\ x_{ci}x_{cj} &= \frac{1}{4}a^2 \end{aligned}\right| \Rightarrow (J_c)_{ij} = \frac{1}{4}ma^2(3\delta_{ij}-1),\quad \overleftrightarrow{J}_c = \frac{1}{4}ma^2\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$\Rightarrow J_{*}^{*} = \frac{1}{6}ma^2\overleftrightarrow{I}$

与均质球表现完全相同的对称性

4. $J_{11} + J_{22} = \int(x_1^2+x_2^2+2x_3^2)dm \geq J_{33}$

若 $x_3 \equiv 0$（平面刚体），则 $J_{11} + J_{22} = J_{33}$

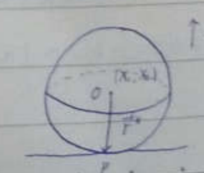
$J_{11} = \int(x_2^2+x_3^2)dm \geq 2|J_{23}|$

四.随时间的演化

$$\begin{cases} \text{本体系} \quad \left(\frac{d\overleftrightarrow{J}}{dt}\right)_{body} = 0 \qquad \frac{dJ_{ij}}{dt} = \left[\frac{d(\hat{x}_i\cdot\overleftrightarrow{J}\cdot\hat{x}_j)}{dt}\right]_{body} = 0 \\ \text{空间系} \quad \left(\frac{d\overleftrightarrow{J}}{dt}\right)_{space} = -\int(\vec{r}\vec{v}+\vec{v}\vec{r})dm \neq 0 \end{cases}$$

$\vec{\omega}$、ω_i、Ω_i 与 t 有关

eg. 水平面纯滚动 球 m. R. $\overleftrightarrow{J}^* = \alpha mR^2\overleftrightarrow{I}$



$$\begin{cases} \vec{F} = m\vec{a}_c \qquad F_1 = m\ddot{x}_1 \quad F_2 = m\ddot{x}_2 \\ \vec{\tau}^* = \frac{d\vec{L}^*}{dt} \end{cases}$$

$\vec{L}^* = \vec{J}^*\vec{\omega} = \alpha m R^2\vec{\omega}$

$\vec{\tau}^* = \vec{r}^* \times \vec{F} = -R\hat{x}_3 \times (F_1\hat{x}_1 + F_2\hat{x}_2)$

$\Rightarrow \begin{cases} F_1 = m\ddot{x}_1 \\ F_2 = m\ddot{x}_2 \\ \tau_1 = RF_2 = \alpha m R^2\dot{\omega}_1 \\ \tau_2 = -RF_1 = \alpha m R^2\dot{\omega}_2 \\ 0 = \alpha m R\dot{\omega}_3 \end{cases}$

$\vec{v}_p = \vec{v}_c^* + \vec{\omega}\times\vec{r}^*$

$= (\dot{x}_1\hat{x}_1 + \dot{x}_2\hat{x}_2) - (\omega_1\hat{x}_1 + \omega_2\hat{x}_2 + \omega_3\hat{x}_3)\times R\hat{x}_3$

$= (\dot{x}_1 - \omega_2 R)\hat{x}_1 + (\dot{x}_2 + \omega_1 R)\hat{x}_2$

$\vec{v}_p = 0 \Rightarrow \begin{cases} \dot{x}_1 = \omega_2 R \\ \dot{x}_2 = -\omega_1 R \end{cases}$

$\Rightarrow \begin{cases} F_1 = 0 = \ddot{x}_1 = \dot{\omega}_2 \\ F_2 = 0 = \ddot{x}_2 = \dot{\omega}_1 \\ \dot{\omega}_3 = 0 \end{cases}$

五.主轴系

主轴 $\hat{n}$：若 $\vec{\omega} = \omega\hat{n}$，则 $\vec{L} = \overleftrightarrow{J}\cdot\vec{\omega} \parallel \vec{\omega}$

$JY = bY$ $\begin{cases} \text{久期方程} \quad \det(J-bI)=0 \Rightarrow b = J_1, J_2, J_3 \text{ 主转动惯量} \\ \text{本征矢方程} \quad (J-J_iI)Y = 0 \Rightarrow Y = \hat{x}_i \end{cases}$

eg. $0 = \det(J-bI) \propto \begin{vmatrix} \alpha-k & -1 & -1 \\ -1 & \alpha-k & -1 \\ -1 & -1 & \alpha-k \end{vmatrix} = (\alpha-k)^3 - 3(\alpha-k) - 2 = (\alpha-k+1)^2(\alpha-k-2)$

$\alpha - k = -1$ 或 $\alpha - k = 2$ $\quad b = \frac{1}{2}ma^2k$

$k_1 = k_2 = \alpha - 1 = \frac{11}{6} \quad k_3 = \alpha - 2 = \frac{2}{6}$

$J_1 = J_2 = \frac{11}{12}ma^2, \quad J_3 = \frac{1}{6}ma^2$

$\hat{x}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \hat{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \hat{x}_2 = \hat{x}_3 \times \hat{x}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

1. 主轴系下，$L_1 = J_1\omega_1$，$L_2 = J_2\omega_2$，$L_3 = J_3\omega_3$

$T = \frac{1}{2}J_i\omega_i^2 = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$

$J_n = \hat{n}\cdot\overleftrightarrow{J}\cdot\hat{n} = J_i n_i^2 = J_1 n_1^2 + J_2 n_2^2 + J_3 n_3^2$

2. 定点转动刚体分类

J_1, J_2, J_3 各不相同	不对称陀螺
$J_1 = J_2 \neq J_3$	对称陀螺
$J_1 = J_2 = J_3$	球形陀螺

3. 若 x_3 轴为 n 次 $(n\geq 2)$，则 x_3 轴为主轴

而若 $n\geq 3$，与 x_3 轴垂直的任一轴均为主轴 $(J_1 = J_2)$

证明：$x_i' = \lambda_{ij}x_j \Rightarrow J' = \lambda J \lambda^T$

$\lambda = \begin{pmatrix} \cos\theta_n & \sin\theta_n & 0 \\ -\sin\theta_n & \cos\theta_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow J = \lambda J \lambda^T$

$(\theta_n = \frac{2\pi}{n})$

$J_{ij} = \lambda_{ik}\lambda_{jl}J_{kl}$

$(ij) = (13) \quad (1-c)J_{13} - sJ_{23} = 0 \quad ①$

$(ij) = (23) \quad sJ_{13} + (1-c)J_{23} = 0 \quad ②$

$(ij) = (11) \quad 2csJ_{12} - s^2(J_{11} - J_{22}) = 0 \quad ③$

(3)-(2) $2s^2 J_{23} + cs(J_{22} - J_{33}) = 0$ ⑥

①② $\begin{vmatrix} 1-c & -s \\ s & 1-c \end{vmatrix} = 2(1-c) > 0$

$\Rightarrow J_{13} = J_{23} = 0$

当 $n=3$

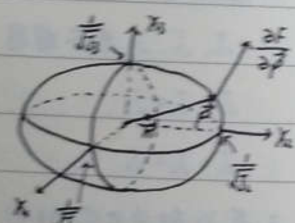
⑤⑥ $\begin{vmatrix} 2c & -s \\ 2s & c \end{vmatrix} = 2(c^2+s^2) = 2 > 0$

$\Rightarrow J_{23} = 0, \ J_{22} = J_{33}$

4. 若 $x_3 = 0$ 为刚体质量分布的对称平面，则 x_3 轴为主轴

二、~~三轴系~~ 惯量椭球

$\vec{\rho} = \frac{\hat{n}}{\sqrt{J_n}} \Rightarrow 1 = J_{ij}\rho_i\rho_j = J_1\rho_1^2 + J_2\rho_2^2 + J_3\rho_3^2$



1. $J_n = \frac{1}{\rho^2}$

2. $F(\vec{\rho}) = \vec{\rho} \cdot \overline{\overline{J}} \cdot \vec{\rho} = J_{ij}\rho_i\rho_j$

$\begin{cases} \text{椭球} & F(\vec{\rho}) = 1 \\ \text{法向} & \frac{\partial F}{\partial \vec{\rho}} = 2\overline{\overline{J}} \cdot \vec{\rho} \end{cases}$

3. 动力学：椭球在本体系不动

$\left.\begin{array}{l} \hat{n} = \frac{\vec{\omega}}{\omega} \\ \vec{\rho} = \frac{\vec{\omega}}{\sqrt{2T}} \end{array}\right\} \Rightarrow \begin{cases} \text{法向} & \frac{\partial F}{\partial \vec{\rho}} = \sqrt{\frac{2}{T}}\vec{L} \\ \text{距离} & h = \vec{\rho} \cdot \frac{\vec{L}}{L} = \frac{\sqrt{2T}}{L} \end{cases}$

§5.4 定点转动动力学

$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{space}$

↑

外力矩（定点之外）

一、角动量定理的分量方程

1. 空间系

$\hat{X}_i \cdot \vec{\tau} = \hat{X}_i \cdot \left(\frac{d\vec{L}}{dt}\right)_{space} = \left[\frac{d(\hat{X}_i \cdot \vec{L})}{dt}\right]_{space} - \left(\frac{d\hat{X}_i}{dt}\right)_{space} \cdot \vec{L}$

$\downarrow \qquad \qquad \downarrow$

$J_{ij}\Omega_j \qquad \qquad 0$

$\tau_i = J_{ij}\dot{\Omega}_j + \dot{J}_{ij}\Omega_j$

2. 本身系

$\hat{x}_i \cdot \vec{\tau} = \hat{x}_i \cdot \left(\frac{d\vec{L}}{dt}\right)_{space} = \left[\frac{d(\hat{x}_i \cdot \vec{L})}{dt}\right]_{space} - \left(\frac{d\hat{x}_i}{dt}\right)_{space} \cdot \vec{L}$

$\downarrow \qquad \qquad \downarrow \ \leftarrow \vec{\omega} \times \hat{x}_i$

$J_{ij}\dot{\omega}_j \qquad -(\vec{\omega} \times \hat{x}_i) \cdot \vec{L} = \hat{x}_i \cdot (\vec{\omega} \times \vec{L})$

$\tau_i = J_{ij}\dot{\omega}_j + (\vec{\omega} \times \vec{L})_i \qquad \vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$

3. 转动系

$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega}_{rot} \times \vec{L}$

二、Euler 动力学方程

$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$

$\vec{\tau} + \vec{\tau}' = \left(\frac{d\vec{L}_{body}}{dt}\right)_{body} = 0$

$\vec{\tau}' = \int \vec{r} \times [-\vec{\omega} \times (\vec{\omega} \times \vec{r}) - \dot{\vec{\omega}} \times \vec{r}]\, dm$ 惯性力

$= \int (\vec{\omega} \cdot \vec{r})(\vec{\omega} \times \vec{r})\, dm - \int \vec{r} \times (\dot{\vec{\omega}} \times \vec{r})\, dm$

$= \vec{\omega} \times \int [(\vec{\omega} \cdot \vec{r})\vec{r} - r^2\vec{\omega}]\, dm - \left[\frac{d}{dt}\int \vec{r} \times (\vec{\omega} \times \vec{r})\, dm\right]_{body}$

主轴系 $\begin{cases} \tau_1 = J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 \\ \tau_2 = J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 \\ \tau_3 = J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 \end{cases}$

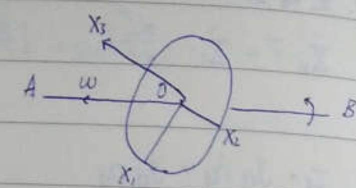
1. 关于 ω_i 的一阶非线性方程

2. $\omega_i(t) \Rightarrow \tau_i$

$\omega_1 = \omega_2 = 0 \quad \tau_1 = \tau_2 = 0 \quad \tau_3 = J_3 \dot{\omega}_3$

$J_1 = J_2 + J_3 \quad \tau_1 = \cdots \quad \tau_2 = \cdots \quad \tau_3 = J_3 \dot{\omega}_3$

eg. $\omega_1 = \omega \sin\theta \quad \omega_3 = \omega\cos\theta, \quad \omega_2 = 0$



$J_3 = \frac{1}{2} mR^2. \quad J_1 = J_2 = \frac{1}{2} J_3 = \frac{1}{4} mR^2$

$\tau_1 = \tau_3 = 0, \quad \tau_2 = -\frac{1}{4} mR^2 \omega^2 \sin\theta\cos\theta$

$L = 1m, \quad m = 100kg, \quad R = 1m, \quad \theta = 1^\circ \quad \omega = 100r/s$

$\tau_2 = LF \Rightarrow F = 1.7\times 10^5 N = 170\, mg$

§5.5 Euler 陀螺 —— 无外力矩作用，定点

$$\begin{cases} J_1 \dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 \\ J_2 \dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 \\ J_3 \dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 \end{cases}$$

$\vec{\tau} = 0 \Rightarrow \vec{L}$ 守恒

$\vec{\tau}\cdot\vec{\omega} = 0 \Rightarrow T$ 守恒

一、对称 Euler 陀螺

$$\begin{cases} \dot{\omega}_1 = \Omega\omega_2 \\ \dot{\omega}_2 = -\Omega\omega_1 \\ \omega_3 = 常数 \end{cases} \Rightarrow \begin{cases} \omega_1 = \omega_\perp \sin(\Omega t + \alpha) \\ \omega_2 = \omega_\perp \cos(\Omega t + \alpha) \end{cases} \quad \Omega = \frac{J_1 - J_3}{J_1}\omega_3$$

1. ω_3 和 $\omega_\perp$ 以及 $L_3 = J_3\omega_3$ 和 $L_\perp = J_1\omega_\perp$ 不变

2. Euler 角

$$\begin{cases} \omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases} \Rightarrow \begin{cases} \dot{\varphi} = \frac{\omega_1\sin\psi + \omega_2\cos\psi}{\sin\theta} = \frac{\omega_\perp}{\sin\theta}\cos(\Omega t + \alpha - \psi) \\ \dot{\theta} = \omega_1\cos\psi - \omega_2\sin\psi = \omega_\perp \sin(\Omega t + \alpha - \psi) \\ \dot{\psi} = \omega_3 - \dot{\varphi}\cos\theta \end{cases}$$

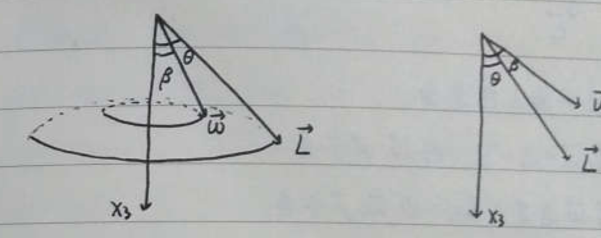
取 x_3 轴沿 $\vec{L}$ 方向

$\theta = const \Rightarrow \tan\theta = \frac{L_1}{L_3} = \frac{J_1\omega_\perp}{J_3\omega_3}$

$\psi = \Omega t + \alpha \Rightarrow \dot{\psi} = \Omega \Rightarrow \dot{\varphi} = \frac{\omega_\perp}{\sin\theta} = \frac{J_3\omega_3}{J_1\cos\theta}$

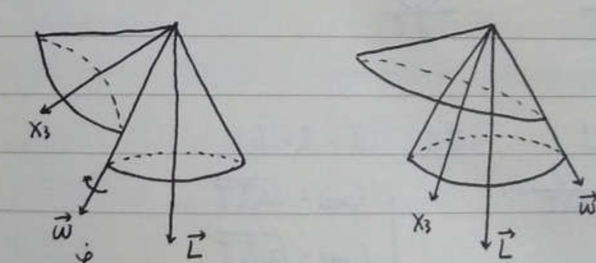
3. 本体系

$\tan\beta = \frac{\omega_\perp}{\omega_3}$



$J_1 > J_3 \qquad J_1 < J_3$

4. 空间系



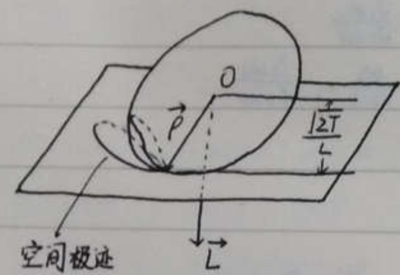
二、潘索（Poinsot）方法

$$\begin{cases} \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2 = T \\ J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 = L^2 \end{cases} \Rightarrow \begin{cases} F(\vec{\rho}) = J_1\rho_1^2 + J_2\rho_2^2 + J_3\rho_3^2 = 1 \\ J_1^2\rho_1^2 + J_2^2\rho_2^2 + J_3^2\rho_3^2 = \frac{L^2}{2T} \end{cases}$$

本体坐标系中，这两个椭球不动的，ρ 在其交线上运动

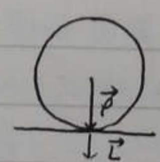
$$\begin{cases} \frac{\partial F}{\partial \vec{\rho}} = 2\overleftrightarrow{J}\cdot\vec{\rho} = \sqrt{\frac{2}{T}}\vec{L} & 法向为角动量方向 \\ \vec{\rho}\cdot\hat{L} = \frac{\sqrt{2T}}{L} & 平面到0点距离不变 \end{cases}$$ 切平面不变

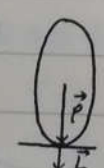
转动瞬轴上点不动 ⇒ 接触点速度为0



1. 椭球在不变平面上纯滚动
2. 特别地，若 $J_1=J_2=J_3$，则 $\vec{\omega}, \vec{p}$ 不变

　　若 $\vec{\omega}$ 沿某主轴，则 $\vec{\omega}, \vec{p}$ 不变





本体坐标系下

$$\begin{cases} L_1^2+L_2^2+L_3^2=L^2 \\ \dfrac{L_1^2}{2J_1T}+\dfrac{L_2^2}{2J_2T}+\dfrac{L_3^2}{2J_3T}=1 \end{cases}$$

要求 $(J_1>J_2>J_3)$

$$\begin{cases} L_{max}=\sqrt{2J_1T} \\ L_{mid}=\sqrt{2J_2T} \\ L_{min}=\sqrt{2J_3T} \end{cases}$$

三. 稳定性

考察极短的时间，$\alpha_1, \alpha_2, \alpha_3$ 为小量

$$\begin{cases} \omega_1=\omega_0+\alpha_1 \\ \omega_2=\alpha_2 \\ \omega_3=\alpha_3 \end{cases} \Rightarrow \begin{cases} J_1\dot{\alpha}_1=(J_2-J_3)\alpha_2\alpha_3\sim 0 \\ J_2\dot{\alpha}_2=-(J_1-J_3)(\omega_0+\alpha_1)\alpha_3 \\ J_3\dot{\alpha}_3=(J_1-J_2)(\omega_0+\alpha_1)\alpha_2 \end{cases}$$

$$\Rightarrow \ddot{\alpha}_2=-\frac{J_1-J_2}{J_3}\frac{J_1-J_3}{J_2}\omega_0^2\alpha_2=-\Omega^2\alpha_2$$

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$$\begin{cases} \omega_1=\alpha_1 \\ \omega_2=\omega_0+\alpha_2 \\ \omega_3=\alpha_3 \end{cases} \Rightarrow \begin{cases} J_1\dot{\alpha}_1=(J_2-J_3)(\omega_0+\alpha_2)\alpha_3 \\ J_2\dot{\alpha}_2=-(J_1-J_3)\alpha_1\alpha_3\sim 0 \\ J_3\dot{\alpha}_3=(J_1-J_2)(\omega_0+\alpha_2)\alpha_1 \end{cases}$$

两方程前系数均为正，不满足简谐条件

特殊地，对对称陀螺 $(J_1=J_2>J_3)$

$$\begin{cases} \omega_1=\omega_0+\alpha_1 \\ \omega_2=\alpha_2 \\ \omega_3=\alpha_3 \end{cases} \Rightarrow \begin{cases} J_1\dot{\alpha}_1=(J_1-J_3)\alpha_3\alpha_2 \\ J_1\dot{\alpha}_2=-(J_1-J_3)(\omega_0+\alpha_1)\alpha_3 \\ \dot{\alpha}_3=0 \end{cases}$$

§5.6 Lagrange 陀螺 —— 重力矩作用下的定点转动对称陀螺

一、Euler 动力学方程

$$\vec{\tau}=l\hat{x}_3\times(-mg\hat{X}_3)=mgl\sin\theta\,\hat{ON}，\quad \hat{ON}=\hat{x}_1\cos\psi-\hat{x}_2\sin\psi$$

$$\begin{cases} J_1\dot{\omega}_1=(J_1-J_3)\omega_2\omega_3+mgl\sin\theta\cos\psi \\ J_1\dot{\omega}_2=(J_3-J_1)\omega_3\omega_1-mgl\sin\theta\sin\psi \\ J_3\dot{\omega}_3=0 \end{cases}$$

二、Lagrange 函数

$$\begin{cases} \omega_1=\dot{\varphi}\sin\theta\sin\psi+\dot{\theta}\cos\psi \\ \omega_2=\dot{\varphi}\sin\theta\cos\psi-\dot{\theta}\sin\psi \\ \omega_3=\dot{\varphi}\cos\theta+\dot{\psi} \end{cases}$$

$$L=\frac{1}{2}J_1(\dot{\varphi}^2\sin^2\theta+\dot{\theta}^2)+\frac{1}{2}J_3(\dot{\varphi}\cos\theta+\dot{\psi})^2-mgl\cos\theta$$

三、运动常数

$$p_\psi=\frac{\partial L}{\partial\dot{\psi}}=J_3(\dot{\varphi}\cos\theta+\dot{\psi})=J_3\omega_3$$

$$p_\varphi=\frac{\partial L}{\partial\dot\varphi}=J_1\dot\varphi\sin^2\theta+J_3(\dot\varphi\cos\theta+\dot\psi)\cos\theta=J_1\dot\varphi\sin^2\theta+p_\psi\cos\theta$$

$$E=T+U=\frac{1}{2}J_1\dot\theta^2+\frac{1}{2}J_1\dot\varphi^2\sin^2\theta+\frac{p_\psi^2}{2J_3}+mgl\cos\theta$$

$$\Rightarrow\begin{cases}\dot\psi=\dfrac{p_\psi}{J_3}-\dfrac{(p_\varphi-p_\psi\cos\theta)\cos\theta}{J_1\sin^2\theta}\\[2mm]\dot\varphi=\dfrac{p_\varphi-p_\psi\cos\theta}{J_1\sin^2\theta}\end{cases}$$

1. $\Psi=\psi-t\,p_\psi\left(\frac{1}{J_3}-\frac{1}{J_1}\right)\Rightarrow\dot\Psi=\dfrac{p_\psi-p_\varphi\cos\theta}{J_1\sin^2\theta}$

可据此构造正则变换进行讨论

$$k=E-\frac{p_\psi^2}{2}\left(\frac{1}{J_3}-\frac{1}{J_1}\right)=\frac{1}{2}J_1\dot\theta^2+\frac{p_\varphi^2+p_\psi^2-2p_\varphi p_\psi\cos\theta}{2J_1\sin^2\theta}+mgl\cos\theta$$

2. $u=\cos\theta\quad\dot\theta^2=\dfrac{\dot u^2}{\sin^2\theta}=\dfrac{\dot u^2}{1-u^2}$

$$\Rightarrow\frac{k}{J_1}(1-u^2)=\frac{1}{2}\dot u^2+\frac{1}{2}(p_\varphi^2+p_\psi^2-2p_\varphi p_\psi u)\frac{1}{J_1^2}+\frac{mgl}{J_1}u(1-u^2)$$

$$\tau=\frac{t}{t_0}\quad\left(t_0=\sqrt{\frac{J_1}{2mgl}}\right)\qquad a_\varphi=\frac{t_0p_\varphi}{J_1}\quad a_\psi=\frac{t_0p_\psi}{J_1}\quad a_0=\frac{k}{mgl}$$

$$\Rightarrow\frac{1}{2}a_0(1-u^2)=\frac{1}{2}\left(\frac{du}{d\tau}\right)^2+\frac{1}{2}(a_\varphi^2+a_\psi^2-2a_\varphi a_\psi u)+\frac{1}{2}u(1-u^2)$$

$$\frac{d\varphi}{d\tau}=\frac{a_\varphi-a_\psi u}{1-u^2}\qquad\frac{d\psi}{d\tau}=\frac{a_\psi-a_\varphi u}{1-u^2}$$

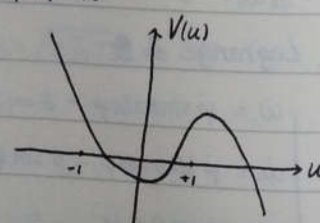
四. 章动角

$$\frac{1}{2}\left(\frac{du}{d\tau}\right)^2+V(u)=0$$

$$V(u)=-\frac{1}{2}\left[(a_0-u)(1-u^2)-(a_\varphi^2+a_\psi^2-2a_\varphi a_\psi u)\right]$$

$$\begin{cases}u\to\pm\infty\Rightarrow V\to\mp\infty\\ u=\pm1\Rightarrow V=\frac{1}{2}(a_\varphi\mp a_\psi)^2>0\end{cases}\quad\Rightarrow$$

物理意义要求 $[-1,1]$ 内有两根



$$\Rightarrow V(u)=-\frac{1}{2}(u-u_0)(u-u_2)(u-u_1)$$

$u_0>1\quad -1<u_2<u_1<1$

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CH6 散射

§6.1 中心力问题

$$\vec F=F(r)\hat r=-\frac{\partial U}{\partial r}\hat r,\quad U=U(r)=-\int F(r)dr$$

一. 运动常数

1. 角动量 $\vec l=\vec r\times m\vec v=2m\dfrac{d\vec S}{dt}$

(1) 方向守恒 ⇒ 平面运动 ⇒ 可用 (r,θ) 描述

(2) 大小守恒 ⇒ $l=mr^2\dot\theta\Rightarrow\dot\theta=\dfrac{l}{mr^2}$

2. 机械能 $E=\frac{1}{2}mv^2+U(r)=\frac{1}{2}m\dot r^2+\dfrac{l^2}{2mr^2}+U(r)$

二. 径向运动

$$E=\frac{1}{2}m\dot r^2+V(r)\qquad V(r)\equiv U(r)+\frac{l^2}{2mr^2}$$

$$m\ddot r=-F(r)+\underbrace{\frac{l^2}{mr^3}}_{\text{离心惯性力}}\qquad -m\vec\omega\times\vec r-2m\vec\omega\times\vec v=0\ \text{抵消了}$$

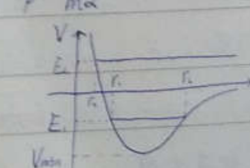
1. 定量求解

$$t=\pm\sqrt{\frac{m}{2}}\int_{r_0}^{r}\frac{dr}{\sqrt{E-V(r)}}=t(r)\Rightarrow r=r(t)\Rightarrow\theta(t)\Rightarrow r(\theta)$$

2. 定性分析

$$\frac{1}{2}m\dot r^2=E-V(r)\geq0$$

eg. $U=-\dfrac{\alpha}{r}\Rightarrow V=\dfrac{l^2}{2mr^2}-\dfrac{\alpha}{r}=\dfrac{\alpha}{2p}\left[\left(\dfrac{p}{r}-1\right)^2-1\right]\quad p=\dfrac{l^2}{m\alpha}$



1. $E=V_{min}\Rightarrow r=p$ ~~$\Rightarrow r_1\le r\le r_2\quad r_{1,2}=\frac{p}{1\pm\varepsilon}$~~

2. $p=\dfrac{l^2}{m\alpha}\quad\varepsilon=\sqrt{1+\dfrac{2pE}{\alpha}}$

$E=E_1,\ V_{min}<E_1<0\Rightarrow r_1\le r\le r_2,\ r_{1,2}=\dfrac{p}{1\pm\varepsilon}$

运动介于两个圆周之间，径向运动周期

3. $E=0\Rightarrow r\geq\dfrac{p}{2}$

4. $E=E_2>0 \Rightarrow r\geq r_0$

径向速度为0的点，称拱点

连接拱点与力心的线，称拱线

相邻近日点/远日点与力心的夹角，称进动角

eg $U=-\frac{\alpha}{r^3} \Rightarrow V=-\frac{\alpha}{r^3}+\frac{l^2}{2mr^2}$

1. $E=V_{max} \Rightarrow r=r_0$

2. $E=E_1<V_{max} \Rightarrow r\leq r_1$

3. $E=E_2>V_{max} \Rightarrow$ 落向力心

可以到达并穿过力心

三. 圆周运动($r=R$)的稳定性

$$\left.\frac{\partial V}{\partial r}\right|_{r=R}=-F(R)-\frac{l^2}{mR^3}=0 \Rightarrow F(R)=-\frac{l^2}{mR^3}<0$$

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=R}=-F'(R)+\frac{3l^2}{mR^4}=-F'(R)-\frac{3F(R)}{R}>0 \Rightarrow RF'(R)+3F(R)<0$$

$$E=\frac{1}{2}m\dot{r}^2+V(r) \xlongequal[r=R+\xi]{} \frac{1}{2}m\dot{\xi}^2+\frac{1}{2}V''(R)\xi^2+V(R) \Rightarrow \omega_r=\sqrt{\frac{V''(R)}{m}}$$

eg. $F=-\alpha r^n\ (\alpha>0)$

立方反比力在角动量一定时未必有圆周运动

$n>-3$

$V''(R)=(n+3)\alpha R^{n-1} \Rightarrow \omega_r=\sqrt{\frac{\alpha}{m}(n+3)R^{n-1}}$

$$\begin{cases} \vec{F}=-\alpha\vec{r} \Rightarrow \omega_r=2\sqrt{\frac{\alpha}{m}} \Rightarrow T_r=\frac{2\pi}{\omega_r}=\pi\sqrt{\frac{m}{\alpha}} \\ \vec{F}=-\frac{\alpha}{r^2}\hat{r} \Rightarrow \omega_r=\sqrt{\frac{\alpha}{mR^3}} \Rightarrow T_r=2\pi\sqrt{\frac{m}{\alpha}}R^{\frac{3}{2}} \end{cases}$$

四. 轨道方程

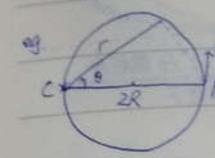
$E=\frac{1}{2}m\dot{r}^2+V(r) \quad V(r)=U(r)+\frac{l^2}{2mr^2} \quad \dot{r}=\frac{dr}{d\theta}\dot{\theta}=\frac{l}{mr^2}\frac{dr}{d\theta}$

$$E=\frac{l^2}{2mr^4}\left(\frac{dr}{d\theta}\right)^2+V(r) \Rightarrow \theta=\pm\int_{r_0}^{r}\frac{l\,dr}{r^2\sqrt{2m(E-V(r))}}=\theta(r)$$

1. 进动角 $\Delta\theta=2\int_{r_1}^{r_2}\frac{l\,dr}{r^2\sqrt{2m(E-V)}} \quad \dot{r}_{1,2}=0 \Rightarrow V(r_{1,2})=E$

2. 轨道方程 $u=\frac{1}{r} \Rightarrow \left(\frac{du}{d\theta}\right)^2+u^2=\frac{2m(E-U)}{l^2}$

eg.



$r=2R\cos\theta \Rightarrow u=\frac{1}{2R\cos\theta}$

$\Rightarrow \left(\frac{du}{d\theta}\right)^2=\left(\frac{\sin\theta}{2R\cos^2\theta}\right)^2=\frac{4R^2\sin^2\theta}{(2R\cos\theta)^4}=\frac{4R^2-r^2}{r^4}$

$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{4R^2}{r^4}=\frac{E-U}{2mR^2v_0^2}$

$\Rightarrow U=E-\frac{8mR^4v_0^2}{r^4} \quad U(r\to\infty)=0 \Rightarrow E=-\frac{8mR^4v_0^2}{r^4}$

3. Bimet 公式 $\frac{d^2u}{d\theta^2}+u=-\frac{m}{l^2}\frac{dU}{du}=-\frac{m}{l^2u^2}F$

$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}+2u\frac{du}{d\theta}=-\frac{2m}{l^2}\frac{dU}{du}\frac{du}{d\theta} \quad \frac{dU}{du}=\frac{dr}{du}\frac{dU}{dr}=\frac{1}{u^2}F$

$$\begin{cases} \frac{d^2u}{d\theta^2}+u=-\frac{m}{l^2u^2}F \\ u|_{\theta=0}=u_0 \\ \left.\frac{du}{d\theta}\right|_{\theta=0}=u_0' \end{cases} \xRightarrow{\theta\to-\theta} \begin{cases} \text{不变} \\ \text{不变} \\ \left.\frac{du}{d\theta}\right|_{\theta=0}=-u_0' \text{ 在 } u_0'=0 \text{ 不变} \end{cases}$$

4. 轨道关于拱线对称

5. 闭合条件 $n_1\Delta\theta=n_2 2\pi \Rightarrow \Delta\theta=\frac{n_2}{n_1}\cdot 2\pi$

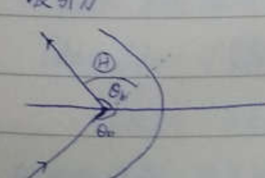
eg. $U=-\frac{\alpha}{r}=-\alpha u \Rightarrow \left(\frac{du}{d\theta}\right)^2+u^2=\frac{2mE}{l^2}+\frac{2m\alpha}{l^2}u=\frac{2E}{\alpha p}+\frac{2u}{p}$

$\left(\frac{du}{d\theta}\right)^2+\left(u-\frac{1}{p}\right)^2=\frac{1}{p^2}\left(1+\frac{2pE}{\alpha}\right)=\left(\frac{\varepsilon}{p}\right)^2$ 简谐运动方程

$u-\frac{1}{p}=\frac{\varepsilon}{p}\cos(\theta-\theta_0) \Rightarrow r=\frac{p}{\varepsilon\cos(\theta-\theta_0)+1} \xrightarrow{\theta_0=0} r=\frac{p}{\varepsilon\cos\theta+1}$

若 $U=\frac{\alpha}{r} \Rightarrow r=\frac{p}{\varepsilon\cos\theta-1}$

吸引力



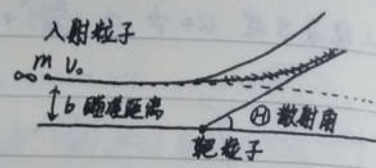
排斥力

$\sin\frac{\Theta}{2}=\frac{1}{\varepsilon}$

$\cos\theta_b=-\frac{1}{\varepsilon}$ $\quad$ $\cos\theta_b=\frac{1}{\varepsilon}$

§6.2 散射(Scattering)

一. 假设: $r\to\infty$ 时 $F\to 0$ 且 $U\to 0$

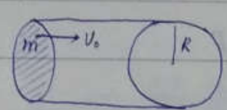


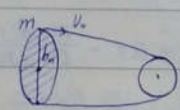
入射粒子能量 $E=\frac{1}{2}mv_0^2$

瞄准距离(碰撞参数 b)

$l=bmv_0=b\sqrt{2mE}$

二. 截面(Cross section)





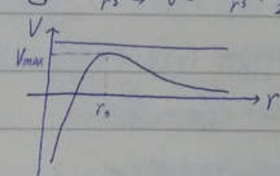
作用截面 $\sigma=\pi R^2$

俘获截面 $\sigma=\pi b_m^2=\pi R^2\left[1+\frac{2GM}{Rv_0^2}\right]$

$\frac{1}{2}mv_0^2=\frac{1}{2}mv_m^2-\frac{GMm}{R}$

$b_m\cdot mv_0=R\cdot mv_m\Rightarrow v_m=\frac{b_m}{R}v_0$

$U=-\frac{\alpha}{r^3}\Rightarrow V=-\frac{\alpha}{r^3}+\frac{l^2}{2mr^2}$



$r_0=\frac{3m\alpha}{l^2}$

$V_m=\frac{1}{2}\frac{\alpha}{r_0^3}$

要求 $E>V_m$

那么已知散射角, 可以知道相互作用吗? 那当然不可能了.

散射问题
- 正问题 $U(r)\Rightarrow r=r(\theta;b,E)\Rightarrow\Theta=\Theta(b;E)$
- 逆问题 $\Theta=\Theta(b;E)\not\Rightarrow U(r)$

eg. $U=\pm\frac{\alpha}{r}$, $\sin\frac{\Theta}{2}=\frac{1}{\varepsilon}\Rightarrow b=\frac{a}{2}\cot\frac{\Theta}{2}\quad(a=\frac{\alpha}{E})$

$\frac{1}{\sin^2(\Theta/2)}=1+\cot^2\frac{\Theta}{2}=1+\frac{2El^2}{m\alpha^2}=1+\frac{2Eb^2 2mE}{m\alpha^2}$

eg. 刚性球 $b=R\cos\frac{\Theta}{2}$

eg. 中心排斥力

$\theta_0=\int_{r_m}^{\infty}\frac{l\,dr}{\sqrt{2m(E-V)}}=\int_{r_m}^{\infty}\frac{l\,dr}{\sqrt{2m(E-U-\frac{l^2}{2mr^2})}}$

$\Theta=\pi-2\theta_0=\pi-2\int_{r_m}^{\infty}\frac{b\,dr}{\sqrt{1-\frac{U}{E}-\frac{b^2}{r^2}}}\qquad 1-\frac{U}{E}-\frac{b^2}{r_m^2}=0$

eg. $U=-\frac{\alpha}{r^3}\Rightarrow V=-\frac{\alpha}{r^3}+\frac{b^2E}{r^2}$ 落向力心条件

要求 $E>V_{max}$

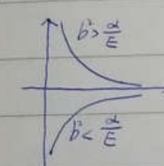
$0=V'(r_0)=+\frac{3\alpha}{r_0^4}-\frac{2b^2E}{r_0^3}\Rightarrow r_0=\frac{3\alpha}{2b^2E}$

$E>V_0=-\frac{\alpha}{r_0^3}+\frac{3}{2}\frac{\alpha}{r_0^3}=\frac{1}{2}\frac{\alpha}{r_0^3}$

$\frac{2E}{\alpha}>\left(\frac{2b^2E}{3\alpha}\right)^3=\frac{1}{3^3}\left(\frac{2E}{\alpha}\right)^3b^6$

$b^2<3\left(\frac{\alpha}{2E}\right)^{\frac{2}{3}}\Rightarrow\sigma=\pi b^2=3\pi\left(\frac{\alpha}{2E}\right)^{\frac{2}{3}}$

eg. $U=-\frac{\alpha}{r^2}\Rightarrow V=\frac{b^2E-\alpha}{r^2}\Rightarrow\sigma=\pi b^2=\pi\frac{\alpha}{E}$



初态动能, $E>0$

1. 入射流强度

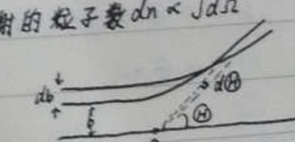
$J=\frac{N}{St}=\frac{\text{入射粒子总数}}{\text{横截面积}\times\text{时间}}$

2. 单位时间观测到的被散射的粒子数 $n\propto J\Rightarrow$ 总散射截面 $\sigma=\frac{n}{J}$

三. 微分散射截面

单位时间观测到由 $d\Omega=\sin\Theta\,d\Theta\,d\Phi$ 出射的粒子数 $dn\propto Jd\Omega$

$\Rightarrow\frac{d\sigma}{d\Omega}\equiv\frac{dn}{Jd\Omega}$ 微分散射截面



$(dn=Jd\sigma=J\frac{d\sigma}{d\Omega}d\Omega)$ 粒子数守恒

入射角度 出射角度

轴对称问题: 单位时间观测到由 $d\Omega=2\pi\sin\Theta\,d\Theta$ 出射粒子数

$dn=J|2\pi b\,db|=J\frac{d\sigma}{d\Omega}|2\pi\sin\Theta\,d\Theta|\Rightarrow\frac{d\sigma}{d\Omega}=\frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|\Rightarrow\sigma=\int\frac{d\sigma}{d\Omega}d\Omega$

eg. 刚性球

$$\frac{d\sigma}{d\Omega}=\frac{R\cos\frac{\Theta}{2}}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\left|-\frac{1}{2}R\sin\frac{\Theta}{2}\right|=\frac{1}{4}R^2$$

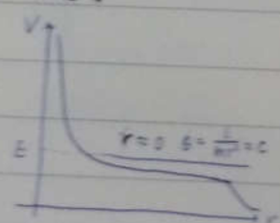
$$\sigma=\int\frac{d\sigma}{d\Omega}d\Omega=\pi R^2$$

eg. 平方反比力

$$\frac{d\sigma}{d\Omega}=\frac{\frac{a}{2}\cos\frac{\Theta}{2}/\sin\frac{\Theta}{2}}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\left|-\frac{1}{2}\cdot\frac{a}{2}\cdot\frac{1}{\sin^2\frac{\Theta}{2}}\right|=\frac{a^2}{16\sin^4\frac{\Theta}{2}}$$

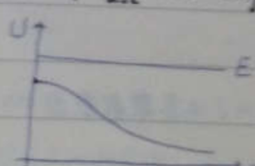
四. 几点说明

1. 中心力



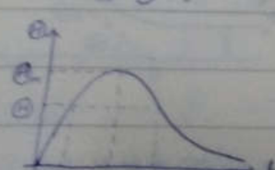
$$\Theta=|(2\theta_0-\pi)-2n\pi|$$

2. $\sigma=\int\frac{d\sigma}{d\Omega}d\Omega=2\pi\int\frac{d\sigma}{d\Theta}\sin\Theta\, d\Theta$ Θ 取值范围



$b=0 \Rightarrow \Theta\to 0$ 穿越心而过

$b\to\infty \Rightarrow \Theta\to 0$



3. $\Theta=\Theta(b;E) \Rightarrow b=b(\Theta;E)$

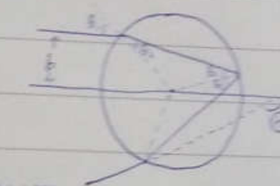
$$\frac{d\sigma}{d\Omega}=\sum_i\frac{b_i}{\sin\Theta}\left|\frac{db_i}{d\Theta}\right|\xrightarrow{\Theta\to\Theta_m}\infty$$ （彩虹散射）

eg. 彩虹散射

$\sin\theta_1=\frac{b}{R}=x \quad \sin\theta_2=\frac{\sin\theta_1}{n}=\frac{x}{n}$

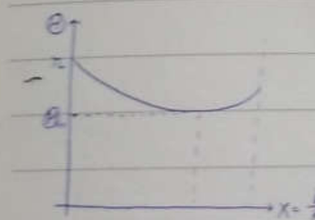
$\Theta=\pi+2\theta_1-4\theta_2=\pi+2\arcsin x-4\arcsin\frac{x}{n}$

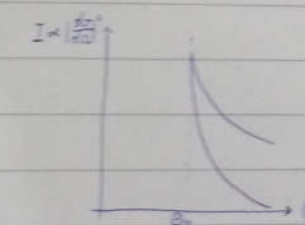
$\frac{d\Theta}{dx}=\frac{2}{\sqrt{1-x^2}}-\frac{4}{\sqrt{n^2-x^2}}=0$



$\Rightarrow 4-4x^2=n^2-x^2, \quad x_m=\sqrt{\frac{4-n^2}{3}}\approx 0.86\ (n=1.33)$

$\Theta_m\approx 2.4\,\text{rad}\approx 135.5^\circ$





Θ_m 附近最强　而各色光不同

在早晚，太阳光接近地平线，更易看到彩虹

· 物理量在不同参考系下的导数

1. 标量　$\frac{df}{dt}=\left(\frac{df}{dt}\right)'$

2. 矢量　$\frac{d\vec{C}}{dt}=\left(\frac{d\vec{C}}{dt}\right)'+\vec{\omega}\times\vec{C}$

$$\vec{C}(t)=C_i\hat{x}_i=C_i'\hat{x}_i'$$

$$\frac{d\vec{C}}{dt}=\frac{d(C_i\hat{x}_i)}{dt}=\frac{dC_i}{dt}\hat{x}_i+C_i\frac{d\hat{x}_i}{dt}$$ $\to 0$

$$=\frac{d(C_i'\hat{x}_i')}{dt}=\frac{dC_i'}{dt}\hat{x}_i'+C_i'\frac{d\hat{x}_i'}{dt}$$ $\to \vec{\omega}\times\vec{C}$

$$\left(\frac{d\vec{C}}{dt}\right)'=\left(\frac{d(C_i'\hat{x}_i')}{dt}\right)'=\left(\frac{dC_i'}{dt}\right)'\hat{x}_i'+C_i'\left(\frac{d\hat{x}_i'}{dt}\right)'$$ $\to 0$

$\vec{\tau}+\vec{\tau}'=\left(\frac{d\vec{L}}{dt}\right)_{rot}$　在非惯性系下

$\vec{\tau}=\left(\frac{d\vec{L}}{dt}\right)_{space}=\left(\frac{d\vec{L}}{dt}\right)_{rot}+\vec{\omega}_{rot}\times\vec{L}$　在惯性系下

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本来说……本来说，一个学期下来能至少有一节课是准时下课的。给同学们留个好印象。

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