量子物理第三章习题答案

1. 电子的动量为

$$p=\frac{h}{\lambda}=\frac{6.63\times10^{-34}\ \mathrm{J\cdot s}}{550\times10^{-9}\ \mathrm{m}}=1.21\times10^{-27}\ \mathrm{kg\cdot m/s},$$

电子的动能为

$$K=\frac{p^2}{2m_e}=\frac{\left(1.21\times10^{-27}\ \mathrm{kg\cdot m/s}\right)^2}{2\times9.11\times10^{-31}\ \mathrm{kg}}=7.98\times10^{-25}\ \mathrm{J}=4.98\times10^{-6}\ \mathrm{eV}.$$

2. 电子的动量为

$$p=\sqrt{2m_eK}=\sqrt{2\times9.11\times10^{-31}\ \mathrm{kg}\times50{,}000\times1.60\times10^{-19}\ \mathrm{J}}=1.21\times10^{-22}\ \mathrm{kg\cdot m/s},$$

对应物质波的波长 (分辨本领) 为

$$\lambda=\frac{h}{p}=\frac{6.63\times10^{-34}\ \mathrm{J\cdot s}}{1.21\times10^{-22}\ \mathrm{kg\cdot m/s}}=5.48\times10^{-12}\ \mathrm{m}=5.48\times10^{-3}\ \mathrm{nm}.$$

3. 因为

$$\int_{-\infty}^{\infty}\left|\psi(x,y,t=0)\right|^2dxdy=\int_{-\infty}^{\infty}\left(x^2+y^2\right)e^{-2\left(x^2+y^2\right)}dxdy,$$

转换为极坐标 $(x,y)=(r\cos\theta,r\sin\theta)$，其 Jacobi 矩阵为

$$DF=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix},$$

因此

$$\begin{aligned}\int_{-\infty}^{\infty}\left(x^2+y^2\right)e^{-2\left(x^2+y^2\right)}dxdy&=\int_0^{2\pi}\int_0^{\infty}r^2e^{-2r^2}\left|\det(DF)\right|drd\theta\\&=\int_0^{2\pi}\int_0^{\infty}r^3e^{-2r^2}drd\theta\\&=2\pi\int_0^{\infty}\frac{1}{2}te^{-2t}dt\qquad\left(t=r^2\right)\\&=\frac{\pi}{4},\end{aligned}$$

因此归一化的波函数为

$$\psi\left(x, y, t=0\right)=\frac{2}{\sqrt{\pi}}\left(x+iy\right)e^{-\left(x^2+y^2\right)},$$

几率密度为

$$\left|\psi\left(x, y, t=0\right)\right|^2=\frac{4}{\pi}\left(x^2+y^2\right)e^{-2\left(x^2+y^2\right)}.$$

4. 因为

$$\int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx=\int_{-\infty}^{\infty}e^{-2|x|}dx=2\int_{0}^{\infty}e^{-2x}dx=1,$$

因此归一化的波函数就是 $\psi(x)=e^{-|x|}$ 本身。

动量表象的波函数为

$$\begin{aligned}\phi(p)&=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-\frac{ipx}{\hbar}}dx\\&=\frac{1}{\sqrt{2\pi\hbar}}\left(\int_{-\infty}^{0}e^{\left(1-\frac{ip}{\hbar}\right)x}dx+\int_{0}^{\infty}e^{\left(-1-\frac{ip}{\hbar}\right)x}dx\right)\\&=\frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{1-\frac{ip}{\hbar}}+\frac{1}{1+\frac{ip}{\hbar}}\right)\\&=\frac{1}{\sqrt{2\pi\hbar}}\frac{2}{1+\frac{p^2}{\hbar^2}}.\end{aligned}$$

5. 因为

$$\int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx=\int_{-\infty}^{\infty}e^{-2x^2/\sigma^2}dx=\frac{\sigma}{2}\int_{-\infty}^{\infty}e^{-t^2/2}dt,$$

而标准正态分布 $N(0,1)$ 的几率密度函数满足

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2/2}dx=1,$$

因此

$$\frac{\sigma}{2}\int_{-\infty}^{\infty}e^{-t^2/2}dt=\sigma\sqrt{\frac{\pi}{2}},$$

因此归一化的波函数为

$$\psi(x) = \left(\frac{2}{\sqrt{2\pi}\sigma}\right)^{1/2} e^{-x^2/\sigma^2}.$$

动量表象的波函数为

$$\begin{aligned}
\phi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-\frac{ipx}{\hbar}} dx \\
&= \left(\frac{1}{\sqrt{2}\pi^{3/2}\hbar\sigma}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}-\frac{x^2}{\sigma^2}} dx \\
&= \left(\frac{1}{\sqrt{2}\pi^{3/2}\hbar\sigma}\right)^{1/2} e^{-\frac{\sigma^2 p^2}{4\hbar^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{\sigma^2}\left(x+i\frac{\sigma^2 p}{2\hbar}\right)^2} dx \\
&= \left(\frac{\sigma}{\sqrt{2}\pi^{3/2}\hbar}\right)^{1/2} e^{-\frac{\sigma^2 p^2}{4\hbar^2}} \int_{-\infty}^{\infty} e^{-\left(x+i\frac{\sigma^2 p}{2\hbar}\right)^2} dx,
\end{aligned}$$

令 $a \in \mathbb{R}$，

$$I(a) = \int_{-\infty}^{\infty} e^{-(x+ia)^2} dx,$$

因此

$$\begin{aligned}
I'(a) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial a} e^{-(x+ia)^2} dx \\
&= \int_{-\infty}^{\infty} -2i(x+ia) e^{-(x+ia)^2} dx \\
&= i \int_{-\infty}^{\infty} \frac{\partial}{\partial x} e^{-(x+ia)^2} dx \\
&= i \left[e^{-(x+ia)^2}\right]_{x\to-\infty}^{x\to\infty} \\
&= 0,
\end{aligned}$$

因此

$$I(a) \equiv I(0) = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$$

$$\phi(p) = \left(\frac{\sigma}{\sqrt{2\pi}\hbar}\right)^{1/2} e^{-\frac{\sigma^2 p^2}{4\hbar^2}}.$$

坐标算符 $\hat{x}$ 的期望值为

$$\langle x \rangle = \int_{-\infty}^{\infty} x \left| \psi(x) \right|^2 dx = \frac{2}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-2x^2/\sigma^2} dx = 0,$$

动量算符 $\hat{p}$ 的期望值为

$$\langle p \rangle = \int_{-\infty}^{\infty} p \left| \phi(p) \right|^2 dp = \frac{\sigma}{\sqrt{2\pi}\hbar} \int_{-\infty}^{\infty} p e^{-\frac{\sigma^2 p^2}{4\hbar^2}} dp = 0,$$

动能算符 $\hat{K} = \frac{1}{2m}\hat{p}^2$ 的期望值为

$$\langle K \rangle = \frac{1}{2m} \int_{-\infty}^{\infty} p^2 \left| \phi(p) \right|^2 dp = \frac{\sigma}{2\sqrt{2\pi}\hbar m} \int_{-\infty}^{\infty} p^2 e^{-\frac{\sigma^2 p^2}{2\hbar^2}} dp = \frac{\hbar^2}{2\sqrt{2\pi}\sigma^2 m} \int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}} dt,$$

而

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \left[t e^{-\frac{t^2}{2}} \right]_{t\to-\infty}^{t\to\infty} + \int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}} dt = \int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}} dt,$$

因此

$$\langle K \rangle = \frac{\hbar^2}{2\sigma^2 m}.$$

6. 因为

$$\int_{-\infty}^{\infty} \left| \psi(x) \right|^2 dx = \int_{-1}^{1} 1 dx = 2,$$

因此归一化的波函数为

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{2}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

动量表象的波函数为

$$\begin{aligned} \phi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-\frac{ipx}{\hbar}} dx \\ &= \frac{1}{2\sqrt{\pi\hbar}} \int_{-1}^{1} e^{-\frac{ipx}{\hbar}} dx \\ &= \sqrt{\frac{\hbar}{\pi}} \frac{1}{p} \sin \frac{p}{\hbar}. \end{aligned}$$

坐标算符 $\hat{x}$ 的期望值为

$$\langle x\rangle=\int_{-\infty}^{\infty} x\left|\psi(x)\right|^{2} dx=\frac{1}{2}\int_{-\infty}^{\infty} x dx=0,$$

算符 $\hat{x}^2$ 的期望值为

$$\left\langle x^2\right\rangle=\int_{-\infty}^{\infty} x^2\left|\psi(x)\right|^{2} dx=\frac{1}{2}\int_{-\infty}^{\infty} x^2 dx=\frac{1}{3},$$

动量算符 $\hat{p}$ 的期望值为

$$\langle p\rangle=\int_{-\infty}^{\infty} p\left|\phi(p)\right|^{2} dp=\frac{\hbar}{\pi}\int_{-\infty}^{\infty}\frac{1}{p}\sin^2\frac{p}{\hbar}dp=\frac{\hbar}{\pi}\int_{-\infty}^{\infty}\frac{1}{t}\sin^2 t dt,$$

因为

$$\lim_{t\to 0}\frac{1}{t}\sin^2 t=0,$$

所以积分 $\int_{-\infty}^{\infty}\frac{1}{t}\sin^2 t dt$ 是收敛的。根据 $f(t)=\frac{1}{t}\sin^2 t$ 是奇函数的性质可以得出

$$\langle p\rangle=0,$$

算符 $\hat{p}^2$ 的期望值为

$$\left\langle p^2\right\rangle=\int_{-\infty}^{\infty} p^2\left|\phi(p)\right|^{2} dp=\frac{\hbar}{\pi}\int_{-\infty}^{\infty}\sin^2\frac{p}{\hbar}dp=\frac{1}{\pi}\int_{-\infty}^{\infty}\sin^2 t dt,$$

积分 $\int_{-\infty}^{\infty}\sin^2 t dt$ 是不收敛的。所以

$$\left\langle p^2\right\rangle=\infty,$$

因此

$$\Delta x=\sqrt{\left\langle x^2\right\rangle-\langle x\rangle^2}=\frac{1}{\sqrt{3}},$$

$$\Delta p=\sqrt{\left\langle p^2\right\rangle-\langle p\rangle^2}=\infty,$$

满足不确定关系 $\Delta x\Delta p\geq\frac{\hbar}{2}$ 。

7. 根据题意可得

$$\Delta x = L,$$

根据不确定关系可得

$$\Delta p \geq \frac{\hbar}{2L},$$

而盒子是对称的，因此 $\langle p \rangle = 0$ ，

$$\left\langle p^2 \right\rangle = (\Delta p)^2 + \langle p \rangle^2 = (\Delta p)^2 \geq \frac{\hbar^2}{8L^2},$$

最小动能为

$$K = \frac{\left\langle p^2 \right\rangle_{\min}}{2m} = \frac{\hbar^2}{8mL^2}.$$

代入数据可得

(a) 0.95 eV,

(b) 0.05 MeV,

(c) 9.5×10^{-36} MeV.

8. 如果本征值不是实数，令 $E = u + iv, v \neq 0$ ，那么

$$\int_{-\infty}^{\infty} \left| \Psi(x,t) \right|^2 dx = \int_{-\infty}^{\infty} \left| \psi(x) e^{-\frac{iEt}{\hbar}} \right|^2 dx = e^{\frac{2vt}{\hbar}} \int_{-\infty}^{\infty} \left| \psi(x) \right|^2 dx = e^{\frac{2vt}{\hbar}},$$

因为 $v \neq 0$ ，所以 $\Psi(x,t)$ 不是常数，不满足归一化条件。

9. 记 $X = \left\{ \psi(x) : -\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x), \int_{-\infty}^{\infty} \left| \psi(x) \right|^2 dx < \infty \right\}$ 为所有“平方可积，并且满足定态薛定谔方程的函数”的集合。

现在证明 X 是复数域 $\mathbb{C}$ 上的向量空间：

(A) $(X, +)$ 是 Abel 群：

(I) 对于任意 $\psi(x), \theta(x) \in X$ ，都具有

$$-\frac{\hbar^2}{2m}(\psi(x) + \theta(x))'' + V(x)(\psi(x) + \theta(x)) = E(\psi(x) + \theta(x)),$$

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\psi(x)+\theta(x)\right|^2 dx &\le \int_{-\infty}^{\infty}\left(\left|\psi(x)\right|+\left|\theta(x)\right|\right)^2 dx\\ &\le \int_{-\infty}^{\infty}\left(2\max\left(\left|\psi(x)\right|,\left|\theta(x)\right|\right)\right)^2 dx\\ &\le \int_{-\infty}^{\infty}4\left(\left|\psi(x)\right|^2+\left|\theta(x)\right|^2\right)dx<\infty.\end{aligned}$$

所以 $\psi(x)+\theta(x)\in X$ 。

(II) 根据加法交换律， $\psi(x)+\theta(x)=\theta(x)+\psi(x)$ 。

(B)

(I) 对于任意 $\psi(x)\in X, z\in\mathbb{C}$ ，都具有

$$-\frac{\hbar^2}{2m}(z\psi(x))''+V(x)(z\psi(x))=E(z\psi(x)),$$

$$\int_{-\infty}^{\infty}\left|z\psi(x)\right|^2 dx=\left|z\right|^2\int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx<\infty.$$

所以 $z\psi(x)\in X$ 。

(II) 对于任意 $\psi(x),\theta(x)\in X, z, w\in\mathbb{C}$ ，都具有

$$z(w\psi(x))=(zw)\psi(x),$$

$$1\psi(x)=\psi(x),$$

$$(z+w)\psi(x)=z\psi(x)+w\psi(x),$$

$$z(\psi(x)+\theta(x))=z\psi(x)+z\theta(x).$$

证明完毕。

现在证明，当 $\psi(x)\in X$ 时，$\psi^*(x)\in X$ 。记 $\psi(x)=u(x)+iv(x)$ ，可得

$$\left(-\frac{\hbar^2}{2m}u''(x)+V(x)u(x)-Eu(x)\right)+i\left(-\frac{\hbar^2}{2m}v''(x)+V(x)v(x)-Ev(x)\right)=0,$$

因为 $\{1,\, i\}$ 是向量空间 $\mathbb{C}$ 上线性无关的一组基，所以

$$\begin{cases}-\frac{\hbar^2}{2m}u''(x)+V(x)u(x)=Eu(x),\\ -\frac{\hbar^2}{2m}v''(x)+V(x)v(x)=Ev(x),\end{cases}$$

而

$$\int_{-\infty}^{\infty} \left|u(x)\right|^2 dx \le \int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx < \infty,$$

$$\int_{-\infty}^{\infty} \left|v(x)\right|^2 dx \le \int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx < \infty,$$

因此 $u(x), v(x) \in X$ 。因为 $\psi^*(x) = u(x) - i v(x)$ ，所以 $\psi^*(x) \in X$ 。证明完毕。

因为 $\psi(x), \psi^*(x) \in X$ ，所以 $\psi(x) + \psi^*(x), i(\psi(x) - \psi^*(x)) \in X$ ，而这两者都是实函数，所以完成了题目的证明。

10. 令 $\int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx = 1$ 「如果不是，可以进行归一化」。那么

$$\begin{aligned}
E &= E\int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx \\
&= \int_{-\infty}^{\infty} \psi^*(x)\big(E\psi(x)\big)\, dx \\
&= \int_{-\infty}^{\infty} \psi^*(x)\left(-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x)\right) dx \\
&= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \psi^*(x)\psi''(x)dx + \int_{-\infty}^{\infty} V(x)\left|\psi(x)\right|^2 dx \\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \left|\psi'(x)\right|^2 dx + \int_{-\infty}^{\infty} V(x)\left|\psi(x)\right|^2 dx \\
&\ge \int_{-\infty}^{\infty} V(x)\left|\psi(x)\right|^2 dx \\
&\ge \inf V(x)\int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx \\
&= \inf V(x).
\end{aligned}$$

其中第 4 行到第 5 行利用了以下等式，对于 $f(x), g(x)$ 满足

$$\lim_{|x|\to\infty} f(x) = 0, \lim_{|x|\to\infty} g(x) = 0, \sup \left|g'(x)\right| < \infty,$$

都具有

$$\int_{-\infty}^{\infty} f(x)g''(x)dx = \big[f(x)g'(x)\big]_{x\to-\infty}^{x\to\infty} - \int_{-\infty}^{\infty} f'(x)g'(x)dx = -\int_{-\infty}^{\infty} f'(x)g'(x)dx.$$

11. 第 n 个定态的波函数为

$$\psi(x)=\begin{cases}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, & \text{for } n \text{ even},\\ \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}, & \text{for } n \text{ odd},\end{cases}$$

（这题利用动量表象计算动量期望值是会死人的，所以换了另一个办法）

(A) n 是偶数 (even)：

坐标算符 $\hat{x}$ 的期望值为

$$\langle x\rangle=\int_{-\infty}^{\infty}x\left|\psi(x)\right|^2dx=\frac{2}{L}\int_{-L/2}^{L/2}x\sin^2\frac{n\pi x}{L}dx=0,$$

算符 $\hat{x}^2$ 的期望值为

$$\left\langle x^2\right\rangle=\int_{-\infty}^{\infty}x^2\left|\psi(x)\right|^2dx=\frac{2}{L}\int_{-L/2}^{L/2}x^2\sin^2\frac{n\pi x}{L}dx=\frac{2L^2}{n^3\pi^3}\int_{-n\pi/2}^{n\pi/2}t^2\sin^2 t\,dt=\frac{1}{12}L^2\left(1-\frac{6}{n^2\pi^2}\right),$$

动量算符 $\hat{p}$ 的期望值为

$$\langle p\rangle=-i\hbar\int_{-\infty}^{\infty}\psi^*(x)\psi'(x)dx=-i\frac{n\pi\hbar}{L^2}\int_{-L/2}^{L/2}\sin\frac{2n\pi x}{L}dx=0,$$

算符 $\hat{p}^2$ 的期望值为

$$\left\langle p^2\right\rangle=-\hbar^2\int_{-\infty}^{\infty}\psi^*(x)\psi''(x)dx=\hbar^2\int_{-\infty}^{\infty}\left|\psi'(x)\right|^2dx=\frac{2n^2\pi^2\hbar^2}{L^3}\int_{-L/2}^{L/2}\cos^2\left(\frac{n\pi x}{L}\right)dx=\frac{n^2\pi^2\hbar^2}{L^2},$$

因此

$$\Delta x=\sqrt{\left\langle x^2\right\rangle-\langle x\rangle^2}=\frac{1}{2\sqrt{3}}\sqrt{1-\frac{6}{n^2\pi^2}}L,$$

$$\Delta p=\sqrt{\left\langle p^2\right\rangle-\langle p\rangle^2}=\frac{n\pi\hbar}{L},$$

$$\Delta x\Delta p=\frac{\hbar}{2}\sqrt{\frac{n^2\pi^2}{3}-2}\geq\frac{\hbar}{2}\sqrt{\frac{4\pi^2}{3}-2}\geq\frac{\hbar}{2}.$$

(B) n 是奇数 (odd)：

坐标算符 $\hat{x}$ 的期望值为

$$\langle x\rangle = \int_{-\infty}^{\infty} x\left|\psi(x)\right|^2 dx = \frac{2}{L}\int_{-L/2}^{L/2} x\cos^2\frac{n\pi x}{L}dx = 0,$$

算符 $\hat{x}^2$ 的期望值为

$$\left\langle x^2\right\rangle = \int_{-\infty}^{\infty} x^2\left|\psi(x)\right|^2 dx = \frac{2}{L}\int_{-L/2}^{L/2} x^2\cos^2\frac{n\pi x}{L}dx = \frac{2L^2}{n^3\pi^3}\int_{-n\pi/2}^{n\pi/2} t^2\cos^2 t\,dt = \frac{1}{12}L^2\left(1-\frac{6}{n^2\pi^2}\right),$$

动量算符 $\hat{p}$ 的期望值为

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty}\psi^*(x)\psi'(x)dx = i\frac{n\pi\hbar}{L^2}\int_{-L/2}^{L/2}\sin\frac{2n\pi x}{L}dx = 0,$$

算符 $\hat{p}^2$ 的期望值为

$$\left\langle p^2\right\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi^*(x)\psi''(x)dx = \hbar^2\int_{-\infty}^{\infty}\left|\psi'(x)\right|^2 dx = \frac{2n^2\pi^2\hbar^2}{L^3}\int_{-L/2}^{L/2}\sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{n^2\pi^2\hbar^2}{L^2},$$

因此

$$\Delta x = \sqrt{\left\langle x^2\right\rangle - \langle x\rangle^2} = \frac{1}{2\sqrt{3}}\sqrt{1-\frac{6}{n^2\pi^2}}L,$$

$$\Delta p = \sqrt{\left\langle p^2\right\rangle - \langle p\rangle^2} = \frac{n\pi\hbar}{L},$$

$$\Delta x\Delta p = \frac{\hbar}{2}\sqrt{\frac{n^2\pi^2}{3}-2} \geq \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3}-2} \geq \frac{\hbar}{2}.$$

综上所述 $n=1$ 最接近不等式极限，其值约为 $1.1\dfrac{\hbar}{2}$ 。

12. 因为

$$\left|\psi(x,0)\right|^2 = \int_{-\infty}^{\infty}\left|\phi_1(x)\right|^2 + \left|\phi_2(x)\right|^2 dx = 2,$$

因此归一化的波函数为

$$\psi(x,0) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right).$$

记 $\phi_1(x), \phi_2(x)$ 对应的定态能量分别为 E_1, E_2 ，因此

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\phi_1(x)e^{-\frac{iE_1t}{\hbar}} + i\phi_2(x)e^{-\frac{iE_2t}{\hbar}}\right),$$

$$\begin{aligned}\left|\Psi(x,t)\right|^2 &= \Psi^*(x,t)\Psi(x,t)\\ &= \frac{1}{2}\left(\phi_1(x)e^{\frac{iE_1t}{\hbar}} - i\phi_2(x)e^{\frac{iE_2t}{\hbar}}\right)\left(\phi_1(x)e^{-\frac{iE_1t}{\hbar}} + i\phi_2(x)e^{-\frac{iE_2t}{\hbar}}\right)\\ &= \frac{1}{2}\left(\phi_1(x)^2 + \phi_2(x)^2 - 2\phi_1(x)\phi_2(x)\sin\frac{(E_1-E_2)t}{\hbar}\right).\end{aligned}$$

根据 11 题的结论

$$\begin{cases}\phi_1(x) = \sqrt{\frac{2}{L}}\cos\frac{\pi x}{L},\\ \phi_2(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L},\end{cases}$$

坐标算符 $\hat{x}$ 的期望值为

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} x\left|\Psi(x,t)\right|^2 dx\\ &= \frac{1}{L}\int_{-L/2}^{L/2} x\left(\cos^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L} - 2\cos\frac{\pi x}{L}\sin\frac{2\pi x}{L}\sin\frac{(E_1-E_2)t}{\hbar}\right)dx\\ &= -\frac{2}{L}\left(\int_{-L/2}^{L/2} x\cos\frac{\pi x}{L}\sin\frac{2\pi x}{L}dx\right)\sin\frac{(E_1-E_2)t}{\hbar}\\ &= -\frac{4L}{\pi^2}\left(\int_{-\pi/2}^{\pi/2} y\cos^2 y\sin y\,dy\right)\sin\frac{(E_1-E_2)t}{\hbar}\\ &= -\frac{16L}{9\pi^2}\sin\frac{(E_1-E_2)t}{\hbar},\end{aligned}$$

算符 $\hat{x}^2$ 的期望值为

$$\begin{aligned}\left\langle x^2\right\rangle &= \int_{-\infty}^{\infty} x^2\left|\Psi(x,t)\right|^2 dx\\ &= \frac{1}{L}\int_{-L/2}^{L/2} x^2\left(\cos^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L} - 2\cos\frac{\pi x}{L}\sin\frac{2\pi x}{L}\sin\frac{(E_1-E_2)t}{\hbar}\right)dx\\ &= \frac{1}{L}\int_{-L/2}^{L/2} x^2\left(\cos^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L}\right)dx\\ &= \frac{L^2}{\pi^3}\int_{-\pi/2}^{\pi/2} y^2\left(\cos^2 y + \sin^2 2y\right)dy\\ &= \left(\frac{1}{12} - \frac{5}{16\pi^2}\right)L^2,\end{aligned}$$

动量算符 $\hat{p}$ 的期望值为

$$\begin{aligned}
\langle p\rangle &= -i\hbar\int_{-\infty}^{\infty}\Psi^*(x,t)\Psi'(x,t)dx \\
&= -\frac{i\hbar}{2}\int_{-L/2}^{L/2}\left(\phi_1(x)e^{\frac{iE_1t}{\hbar}}-i\phi_2(x)e^{\frac{iE_2t}{\hbar}}\right)\left(\phi_1'(x)e^{\frac{-iE_1t}{\hbar}}+i\phi_2'(x)e^{\frac{-iE_2t}{\hbar}}\right)dx \\
&= -\frac{i\hbar}{2}\int_{-L/2}^{L/2}\phi_1(x)\phi_1'(x)+\phi_2(x)\phi_2'(x)+i\left(\phi_1(x)\phi_2'(x)e^{\frac{i(E_1-E_2)t}{\hbar}}-\phi_1'(x)\phi_2(x)e^{\frac{-i(E_1-E_2)t}{\hbar}}\right)dx \\
&= \frac{\hbar}{2}\int_{-L/2}^{L/2}\left(\phi_1(x)\phi_2'(x)e^{\frac{i(E_1-E_2)t}{\hbar}}-\phi_1'(x)\phi_2(x)e^{\frac{-i(E_1-E_2)t}{\hbar}}\right)dx \\
&= \frac{\pi\hbar}{L^2}\int_{-L/2}^{L/2}\left(2\cos\frac{\pi x}{L}\cos\frac{2\pi x}{L}e^{\frac{i(E_1-E_2)t}{\hbar}}+\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}e^{\frac{-i(E_1-E_2)t}{\hbar}}\right)dx \\
&= \frac{4\hbar}{3L}\cos\frac{(E_1-E_2)t}{\hbar},
\end{aligned}$$

算符 $\hat{p}^2$ 的期望值为

$$\begin{aligned}
\left\langle p^2\right\rangle &= -\hbar^2\int_{-\infty}^{\infty}\Psi^*(x,t)\Psi''(x,t)dx \\
&= \hbar^2\int_{-\infty}^{\infty}\left|\Psi'(x,t)\right|^2dx \\
&= \frac{\hbar^2}{2}\int_{-L/2}^{L/2}\left(\phi_1'(x)e^{\frac{iE_1t}{\hbar}}-i\phi_2'(x)e^{\frac{iE_2t}{\hbar}}\right)\left(\phi_1'(x)e^{\frac{-iE_1t}{\hbar}}+i\phi_2'(x)e^{\frac{-iE_2t}{\hbar}}\right)dx \\
&= \frac{\hbar^2}{2}\int_{-L/2}^{L/2}\phi_1'(x)^2+\phi_2'(x)^2-2\phi_1'(x)\phi_2'(x)\sin\frac{(E_1-E_2)t}{\hbar}dx \\
&= \frac{\hbar^2}{2}\int_{-L/2}^{L/2}\phi_1'(x)^2+\phi_2'(x)^2dx \\
&= \frac{\pi^2\hbar^2}{L^3}\int_{-L/2}^{L/2}4\cos^2\frac{2\pi x}{L}+\sin^2\frac{\pi x}{L}dx \\
&= \frac{5\pi^2\hbar^2}{2L^2}.
\end{aligned}$$

系统能量的平均值为

$$\begin{aligned}
\langle H\rangle &= i\hbar\int_{-\infty}^{\infty}\Psi^*(x,t)\dot{\Psi}(x,t)dx \\
&= \frac{1}{2}\int_{-L/2}^{L/2}\left(\phi_1(x)e^{\frac{iE_1t}{\hbar}}-i\phi_2(x)e^{\frac{iE_2t}{\hbar}}\right)\left(E_1\phi_1(x)e^{\frac{-iE_1t}{\hbar}}+iE_2\phi_2(x)e^{\frac{-iE_2t}{\hbar}}\right)dx \\
&= \frac{1}{2}\int_{-L/2}^{L/2}E_1\phi_1(x)^2+E_2\phi_2(x)^2+i\phi_1(x)\phi_2(x)\left(E_2e^{\frac{i(E_1-E_2)t}{\hbar}}-E_1e^{\frac{-i(E_1-E_2)t}{\hbar}}\right)dx \\
&= \frac{1}{2}\int_{-L/2}^{L/2}E_1\phi_1(x)^2+E_2\phi_2(x)^2dx \\
&= \frac{1}{2}\left(E_1+E_2\right).
\end{aligned}$$

如果测量粒子的能量，得到的结果是 E_1 或 E_2 ，几率均为 $\frac{1}{2}$ 。(哪个婊子出的题目？？？)

13. 定态薛定谔方程为

$$\begin{cases} -\frac{\hbar^2}{2m}\psi''(x) = E\psi(x), & x \in (0,\, a), \\ \psi(x) \equiv 0, & x \in \mathbb{R}\backslash(0,\, a), \end{cases}$$

因此

$$\psi(x) = \begin{cases} P\sin kx + Q\cos kx, & x \in (0,\, a), \\ 0, & x \in \mathbb{R}\backslash(0,\, a), \end{cases}$$

并且根据连续性可得

$$\lim_{x\searrow 0}\psi(x) = \lim_{x\nearrow 0}\psi(x) = 0,$$

$$\lim_{x\nearrow a}\psi(x) = \lim_{x\searrow a}\psi(x) = 0,$$

因此

$$Q = 0,$$

$$ka = n\pi,$$

$$\psi(x) = \begin{cases} P\sin\frac{n\pi x}{a}, & x \in (0,\, a), \\ 0, & x \in \mathbb{R}\backslash(0,\, a), \end{cases}$$

归一化可得

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}, & x \in (0,\, a), \\ 0, & x \in \mathbb{R}\backslash(0,\, a)\,. \end{cases}$$

而

$$\int_{-\infty}^{\infty}\left|\psi(x,\,0)\right|^2 dx = A^2\int_0^a \sin^6\frac{\pi x}{a}dx = \frac{A^2 a}{\pi}\int_0^{\pi}\sin^6 t dt = \frac{5}{16}A^2 a = 1,$$

因此

$$A = \sqrt{\frac{16}{5a}},$$

$$\psi(x,\,0)=\sqrt{\frac{16}{5a}}\sin^3\frac{\pi x}{a}=\frac{1}{\sqrt{10}}\big(3\phi_1(x)-\phi_3(x)\big),$$

因此

$$\Psi(x,t)=\frac{1}{\sqrt{10}}\left(3\phi_1(x)e^{\frac{-iE_1t}{\hbar}}-\phi_3(x)e^{\frac{-iE_3t}{\hbar}}\right),$$

$$\begin{aligned}\langle x\rangle&=\int_{-\infty}^{\infty}x\left|\Psi(x,t)\right|^2dx\\&=\frac{1}{10}\int_0^a x\left(3\phi_1(x)e^{\frac{iE_1t}{\hbar}}-\phi_3(x)e^{\frac{iE_3t}{\hbar}}\right)\left(3\phi_1(x)e^{\frac{-iE_1t}{\hbar}}-\phi_3(x)e^{\frac{-iE_3t}{\hbar}}\right)dx\\&=\frac{1}{10}\int_0^a x\left(9\phi_1(x)^2+\phi_3(x)^2-6\phi_1(x)\phi_3(x)\cos\frac{(E_1-E_3)t}{\hbar}\right)dx\\&=\frac{a}{2},\end{aligned}$$

$$\begin{aligned}\langle p\rangle&=-i\hbar\int_{-\infty}^{\infty}\Psi^*(x,t)\Psi'(x,t)dx\\&=-\frac{i\hbar}{10}\int_0^a\left(3\phi_1(x)e^{\frac{iE_1t}{\hbar}}-\phi_3(x)e^{\frac{iE_3t}{\hbar}}\right)\left(3\phi_1'(x)e^{\frac{-iE_1t}{\hbar}}-\phi_3'(x)e^{\frac{-iE_3t}{\hbar}}\right)dx\\&=-\frac{i\hbar}{10}\int_0^a\left(9\phi_1(x)\phi_1'(x)+\phi_3(x)\phi_3'(x)-3\left(\phi_1(x)\phi_3'(x)e^{\frac{i(E_1-E_3)t}{\hbar}}+\phi_1'(x)\phi_3(x)e^{\frac{-i(E_1-E_3)t}{\hbar}}\right)\right)dx\\&=0.\end{aligned}$$

14. 原来的波函数为

$$\psi(x)=\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a},\qquad 0<x<a,$$

势阱宽度增加后，波函数的函数空间 (实数域 $\mathbb{R}$ 上的向量空间) 标准正交基为

$$\phi_n(x)=\sqrt{\frac{1}{a}}\sin\frac{n\pi x}{2a},\qquad 0<x<2a,$$

根据傅立叶级数的性质，$\psi(x)$ 可以被表示为

$$\psi(x)=\sum_{n=1}^{\infty}c_n\phi_n(x),$$

因此

$$\begin{aligned} c_n &= \left\langle \psi(x), \phi_n(x) \right\rangle \\ &= \int_0^{2a} \psi(x)\phi_n(x) dx \\ &= \frac{\sqrt{2}}{a}\int_0^{a} \sin\frac{\pi x}{a} \sin\frac{n\pi x}{2a} dx \\ &= \frac{\sqrt{2}}{\pi}\int_0^{\pi} \sin t \sin\frac{nt}{2} dt, \end{aligned}$$

而

$$\int_0^{\pi} \sin t \sin\frac{nt}{2} dt = \begin{cases} (-1)^{\frac{n+1}{2}}\frac{4}{n^2-4}, & n = 1, 3, 5, \ldots, \\ \frac{\pi}{2}, & n = 2, \\ 0, & n = 4, 6, 8, \ldots, \end{cases}$$

因此

$$c_n = \begin{cases} (-1)^{\frac{n+1}{2}}\frac{4\sqrt{2}}{(n^2-4)\pi}, & n = 1, 3, 5, \ldots, \\ \frac{\sqrt{2}}{2}, & n = 2, \\ 0, & n = 4, 6, 8, \ldots, \end{cases}$$

测量到状态 $\phi_n(x)$ 对应的能量为

$$E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2} = \frac{n^2\pi^2\hbar^2}{8ma^2},$$

对应概率为

$$\Pr\left\{\phi_n(x)\right\} = \left|c_n\right|^2 = \begin{cases} \frac{32}{(n^2-4)^2\pi^2}, & n = 1, 3, 5, \ldots, \\ \frac{1}{2}, & n = 2, \\ 0, & n = 4, 6, 8, \ldots, \end{cases}$$

因此最有可能的结果对应 $n = 2$ ，其概率为 $1/2$ ，相应的能量为 $E_2 = \frac{\pi^2\hbar^2}{2ma^2}$.

粒子能量的平均值为

$$
\begin{aligned}
\langle H \rangle &= \sum_{n=1}^{\infty} \left| c_n \right|^2 E_n \\
&= \frac{1}{2}\frac{\pi^2\hbar^2}{2ma^2} + \sum_{k=1}^{\infty} \frac{32}{((2k-1)^2-4)^2\pi^2} \frac{(2k-1)^2\pi^2\hbar^2}{8ma^2} \\
&= \frac{1}{2}\frac{\pi^2\hbar^2}{2ma^2} + \frac{4\hbar^2}{ma^2}\sum_{k=1}^{\infty} \frac{(2k-1)^2}{((2k-1)^2-4)^2} \\
&= \frac{1}{2}\frac{\pi^2\hbar^2}{2ma^2} + \frac{\hbar^2}{ma^2}\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2} + \frac{1}{(2k+1)^2} + \frac{2}{4k^2-4k-3},
\end{aligned}
$$

因为

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6},
$$

所以

$$
\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24},
$$

因此

$$
\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2} = 1 + 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = 1 + \frac{\pi^2}{6} - \frac{\pi^2}{24} = 1 + \frac{\pi^2}{8},
$$

$$
\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{(2k-3)^2} - 2 = \frac{\pi^2}{8} - 1,
$$

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{2}{4k^2-4k-3} &= \sum_{k=1}^{\infty} \frac{2}{(2k-3)(2k+1)} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{2k-3} - \frac{1}{2k+1} \\
&= \frac{1}{2}\left(\frac{1}{-1} - \frac{1}{3} + \frac{1}{1} - \frac{1}{5} + \frac{1}{3} - \frac{1}{7} + \cdots \right) \\
&= \frac{1}{2}\left(\frac{1}{-1} + \frac{1}{1} \right) = 0.
\end{aligned}
$$

$$
\langle H \rangle = \frac{1}{2}\frac{\pi^2\hbar^2}{2ma^2} + \frac{\hbar^2}{ma^2}\sum_{k=1}^{\infty} \frac{1}{(2k-3)^2} + \frac{1}{(2k+1)^2} + \frac{2}{4k^2-4k-3} = \frac{\pi^2\hbar^2}{2ma^2}.
$$

15. 对于散射态 $E > 0$ ，定态薛定谔方程为

$$\begin{cases} -\frac{\hbar^2}{2m}\psi''(x) = E\psi(x), & x < 0, \\ -\frac{\hbar^2}{2m}\psi''(x) = (E + V_0)\psi(x), & x \geq 0, \end{cases}$$

因此

$$\psi(x) = \begin{cases} e^{iax} + Re^{-iax}, & x < 0, \\ Te^{ibx}, & x \geq 0, \end{cases}$$

其中 R, T 分别为反射率、透射率,

$$a = \frac{\sqrt{2mE}}{\hbar},$$

$$b = \frac{\sqrt{2m(E + V_0)}}{\hbar},$$

因为势能是有界的，所以波函数的零阶与一阶导数均连续，即

$$\lim_{x \nearrow 0} \psi(x) = \lim_{x \searrow 0} \psi(x),$$

$$\lim_{x \nearrow 0} \psi'(x) = \lim_{x \searrow 0} \psi'(x),$$

因此

$$\begin{cases} 1 + R = T, \\ a(1 - R) = bT, \end{cases}$$

$$\begin{cases} R = \frac{a-b}{a+b}, \\ T = \frac{2a}{a+b}, \end{cases}$$

所以粒子有可能被反射，并且被反射回来的几率为

$$\left|R\right|^2 = \left|\frac{a - b}{a + b}\right|^2 = \left(\frac{\sqrt{E + V_0} - \sqrt{E}}{\sqrt{E + V_0} + \sqrt{E}}\right)^2.$$

16. 定态薛定谔方程为

$$\begin{cases}\psi(x) \equiv 0, & x < 0 \\ -\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x), & x \geq 0,\end{cases}$$

因此

$$\psi(x) = \begin{cases}0, & x < 0 \\ \sqrt{2}N_n e^{-\xi^2/2}H_n(\xi), & x \geq 0,\end{cases}$$

($\sqrt{2}$ 的出现是因为归一化) 因为波函数连续，所以

$$\lim_{x \searrow 0}\psi(x) = \lim_{x \nearrow 0}\psi(x) = 0,$$

并且当 n 是奇数时

$$\lim_{\xi \to 0} H_n(\xi) = 0,$$

当 n 是偶数时

$$\lim_{\xi \to 0} H_n(\xi) \neq 0,$$

所以根据连续性， n 只能是奇数，对应能量为

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad n = 1, 3, 5, \ldots.$$

17. 多维情况的定态薛定谔方程为

$$-\frac{\hbar^2}{2m}\Delta\psi + V\psi = E\psi,$$

其中 $\Delta\psi = \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2}$. 令 $\psi(x, y) = F(x)G(y)$ ，因此

$$-\frac{\hbar^2}{2m}(FG'' + F''G) + VFG = EFG,$$

$$-\frac{\hbar^2}{2m}\left(\frac{F''}{F} + \frac{G''}{G}\right) + V = E,$$

$$\left(-\frac{\hbar^2}{2m}\frac{F''}{F} + \frac{1}{2}m(2\omega)^2x^2\right) + \left(-\frac{\hbar^2}{2m}\frac{G''}{G} + \frac{1}{2}m\omega^2y^2\right) = E,$$

因此

$$-\frac{\hbar^2}{2m}\frac{F''}{F}+\frac{1}{2}m(2\omega)^2x^2=C,$$

$$-\frac{\hbar^2}{2m}\frac{G''}{G}+\frac{1}{2}m\omega^2y^2=D,$$

其中 $C,D>0$ 均为常数，并且 $C+D=E$ 。因此本征函数为

$$F_m(x)=A_me^{-\chi^2/2}H_m(\chi),$$

$$G_n(y)=B_ne^{-\xi^2/2}H_n(\xi),$$

其中 $\chi=\sqrt{\frac{2m\omega}{\hbar}}x,\xi=\sqrt{\frac{m\omega}{\hbar}}y$,本征能量为

$$C_m=\left(m+\frac{1}{2}\right)2\hbar\omega,$$

$$D_n=\left(n+\frac{1}{2}\right)\hbar\omega,$$

其中 $m,n=0,1,2,\ldots$ 。当 $E=\left(N+\frac{1}{2}\right)\hbar\omega$ 时

$$2\left(m+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)=N+\frac{1}{2},$$

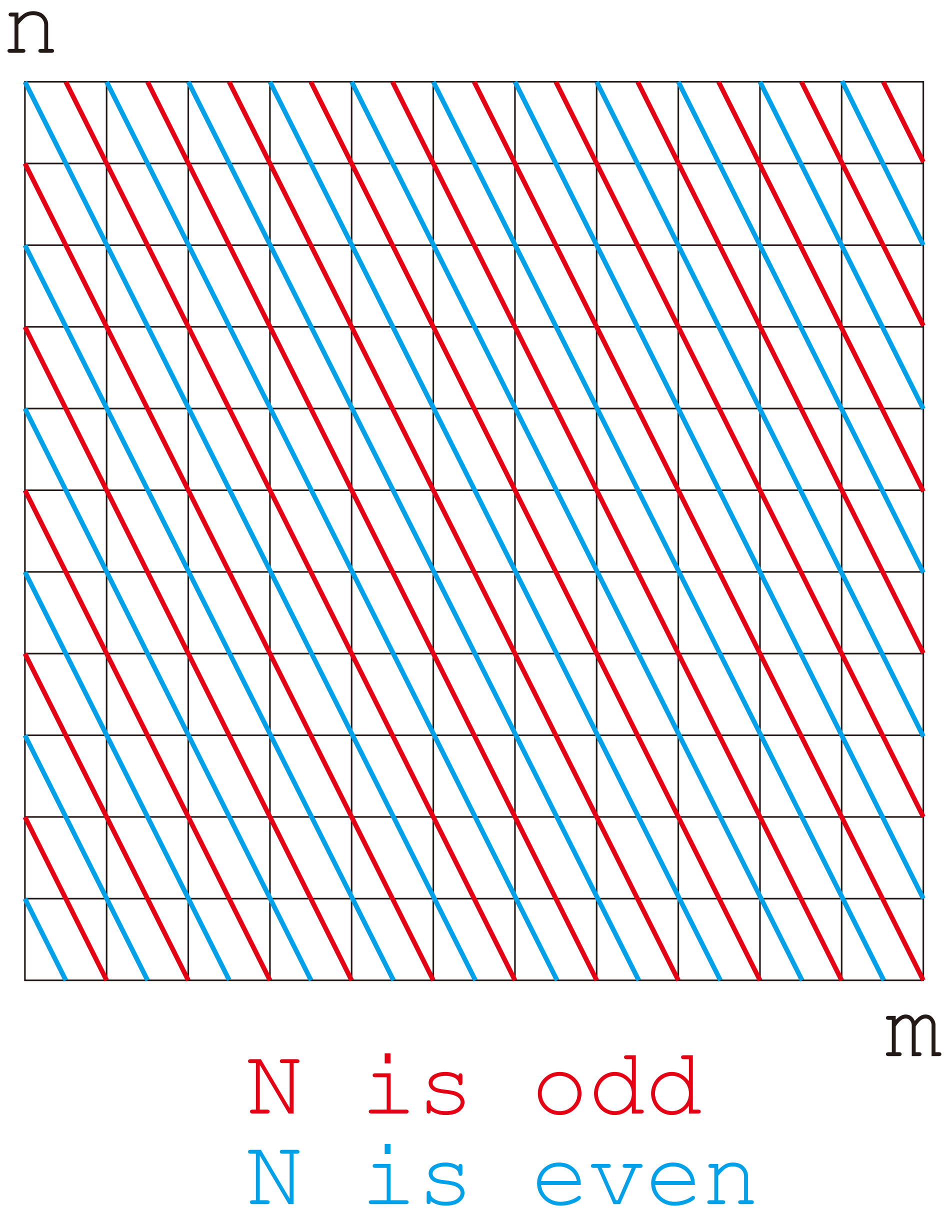
$$2m+n=N-1,$$

$N=0$ 显然无解。令 $N\geq 1$ ，绘制图像可以发现：

(A) 当 N 是奇数时，$2m+n$ 是偶数，对应图像中的红线，可以发现红线经过 $\frac{N+1}{2}$ 个格点。

(B) 当 N 是偶数时，$2m+n$ 是奇数，对应图像中的蓝线，可以发现蓝线经过 $\frac{N}{2}$ 个格点。

因此，能级的简并度为 $\lceil\frac{N}{2}\rceil$ ，其中 $\lceil\rceil$ 是向上取整。


n
m
N is odd
N is even