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Date. 9.3 No. 1

Newton力学：物体位形变化 $\vec{r}_a(t)$ — 机械运动

描述 — 运动学：坐标系、参考系（观测者）

解释 — 动力学：惯性系 $\vec{F}=m\vec{a}$

运动学方程（$\vec{v}, \vec{a}$ 定义式）
$$\begin{cases} \vec{r}(t+\varepsilon)=\vec{r}(t)+\varepsilon\vec{v}(t), \quad \vec{r}(t+2\varepsilon)=\vec{r}(t+2)+\varepsilon\vec{v}(t+2) \\ \vec{v}(t+\varepsilon)=\vec{v}(t)+\varepsilon\vec{a}(t)=\vec{v}(t)+\varepsilon\frac{\vec{F}(t)}{m} \end{cases}$$

牛顿：$(\vec{r}_0, \vec{v}_0)\xrightarrow[(\vec{F}已知)]{\vec{F}=m\vec{a}}(\vec{r}, \vec{v})$ → 任一时刻力学状态.

质点 $\vec{F}=-\frac{GMm}{r^2}\hat{r}$ → 中心力

$\vec{F}=-k\vec{x}$ → 多自由度微振动

→ 广义势能

两体 碰撞 → 散射（截面）

刚体 平面平行 → 一般运动.

波动 简谐波 → 场论（连续体系）

变换体系. 三大推论原理 → 对称与守恒.

最小原理
- 位形空间：Lagrange：$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=0$
- 相空间：Hamilton
$$\begin{cases} \dot{q}_k=\frac{\partial H}{\partial p_k}=[q_k, H] \\ \dot{p}_k=-\frac{\partial H}{\partial q_k}=[p_k, H] \end{cases}$$

$$\begin{tikzcd} L=T-U \arrow[r, leftrightarrow, bend right] & H=T+U \end{tikzcd}$$

几何上是切线（包络线），Legendre 变换

CH1 运动学

Date. 9.6 No. 3

CH1 运动学

§1 正交变换

一、位矢与坐标（在本讨论下，$\vec r$ 不变，而坐标系改变，相当于"被动"变换即坐标变换）

$$\vec r=\sum_{j=1}^{3}x_j\hat x_j=\sum_{j=1}^{3}(\vec r\cdot\hat x_j)\hat x_j.\quad \hat x_i\cdot\hat x_j=\delta_{ij}$$

$$\vec r=\sum_{j=1}^{3}x'_j\hat x'_j=\sum_{j=1}^{3}(\vec r\cdot\hat x'_j)\hat x'_j$$

在不同坐标系下表示同一个向量

二、坐标变换

$$\hat x'_i=\sum_{j=1}^{3}(\hat x'_i\cdot\hat x_j)\hat x_j$$

$$x'_i=\vec r\cdot\hat x'_i=\sum_j x_j(\hat x_j\cdot\hat x'_i)$$

定义 $\boxed{\lambda_{ij}\equiv\hat x'_i\cdot\hat x_j}$

$$\begin{cases}\hat x'_i=\sum_j\lambda_{ij}\hat x_j\\ x'_i=\sum_j\lambda_{ij}x_j\end{cases}\qquad\begin{cases}\hat x_i=\sum_j\lambda_{ji}\hat x'_j\\ x_i=\sum_j=\lambda_{ji}x'_j\end{cases}$$

[ssj 注1]：和线代B1的关系.

$$(\hat x'_1,\hat x'_2,\hat x'_3)=(\hat x_1,\hat x_2,\hat x_3)\begin{pmatrix}\lambda_{11}&\lambda_{21}&\lambda_{31}\\\lambda_{12}&\lambda_{22}&\lambda_{32}\\\lambda_{13}&\lambda_{23}&\lambda_{33}\end{pmatrix}$$

即 $\boxed{(\hat x'_1,\hat x'_2,\hat x'_3)=(\hat x_1,\hat x_2,\hat x_3)\lambda^T}$

对应的线性变换矩阵 $A=\lambda^T$，

A 是在基 $(\hat x_1,\hat x_2,\hat x_3)$ 下的变换矩阵

$\vec r$ 在基 $(\hat x_1,\hat x_2,\hat x_3)$ 和 $(\hat x'_1,\hat x'_2,\hat x'_3)$ 下的坐标分别记为 $X, X'\in F^3$，则有：

$$\boxed{X'=A^{-1}X=\lambda X}$$

在新基下的矩阵 P 将变为 P'

$$P'=A^{-1}PA$$

$$\Rightarrow\boxed{P'=\lambda P\lambda^T}$$

三、λ变换矩阵 (λ_{ij}) 满足的一般性质

$$\delta_{ij}=\hat x'_i\cdot\hat x'_j=\sum_{k,l=1}^{3}\lambda_{ik}\lambda_{jl}\hat x_k\cdot\hat x_l=\sum_{k,l=1}^{3}\lambda_{ik}\lambda_{jl}\delta_{kl}=\sum_{k=1}^{3}\lambda_{ik}\lambda_{jk}=(\lambda\lambda^T)_{ij}$$

$$\delta_{ij}=\hat x_i\cdot\hat x_j=\sum_{k,l=1}^{3}\lambda_{ki}\lambda_{lj}\hat x'_k\cdot\hat x'_l=(\lambda^T\lambda)_{ij}\quad\Rightarrow\quad\lambda\lambda^T=I=\lambda^T\lambda$$

⇒ 1、λ 为 3×3 的实正交矩阵，$\lambda\in O(3)$ orthogonal /ɔrˈθɒgənəl/

① 只有3个独立元素. ② 正交变换和正交矩阵（在一）建立一一对应关系

③ $\det\lambda=\begin{cases}+1,\ \text{转动} & \lambda\in SO(3)\\ -1,\ \text{反演}(+\text{转动})\end{cases}$

2、两次转动不能交换次序.（矩阵乘法不能交换）

3、若干次连续转动可以经由一次转动实现.

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四、求和约定：若某一指标在同一单项式中重复出现，默认对该指标求和

[例] $(AB)_{ij} = \sum_k A_{ik}B_{kj}$, $^{(i,j=1,2,3)}$ $\Leftrightarrow (AB)_{ij} = A_{ik}B_{kj}$

k：哑指标（求和指标） $\searrow$ 可随意替换不引起歧义

i,j：自由指标 $\searrow$ 可两边同时换.

[例] $\delta_{ii} = 3$, $A_{ii} = tr(A)$, $\boxed{A_{ij}B_{ji} = tr(AB)}$

$A_{ik}\delta_{kj} = A_{ij}$, $A_{ik}\delta_{ki} = A_{ii} = A_{kk}$.

$$\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix} = \lambda\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \quad \boxed{\begin{aligned}x_i' &= \lambda_{ij}x_j\\ x_i &= \lambda_{ji}x_j'\end{aligned}}$$

带撇的放在λ下标第1个位置.

$\lambda_{ik}\lambda_{jk} = \delta_{ij} = \lambda_{ki}\lambda_{kj}$

五、排列符号 $(i,j,k=1,2,3)$

赝张量

$$\varepsilon_{ijk} \triangleq \begin{cases}+1, & (i,j,k)\text{为}(1,2,3)\text{的偶置换}\\ -1, & \text{------ 奇置换}\\ 0, & \text{其它}\end{cases} = \begin{vmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{vmatrix}$$

1. 完全反对称性质

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik}$$

2. 行列式 $(A: 3\times 3)$

$$\boxed{\varepsilon_{ijk}A_{i1}A_{j2}A_{k3} = \det(A)}$$

$$\boxed{\varepsilon_{ijk}A_{il}A_{jm}A_{kn} = \varepsilon_{lmn}\det(A)}$$

3. $$\boxed{\begin{aligned}\varepsilon_{ijk}\varepsilon_{mnk} &= \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\\ &= \varepsilon_{ikj}\varepsilon_{mkn} = \varepsilon_{kij}\varepsilon_{kmn}\end{aligned}}$$

$$\boxed{\varepsilon_{ijk}\varepsilon_{mjk} = \delta_{im}\delta_{jj} - \delta_{ij}\delta_{jm} = 3\delta_{im} - \delta_{im} = 2\delta_{im}}$$

$$\boxed{\varepsilon_{ijk}\varepsilon_{ijk} = 3! = 6}$$

4. $$\boxed{\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{vmatrix}}$$

§2 张量及其运算

一、定义 若量 T 有 3^n 个分量 $T_{\underbrace{i\cdots j}_{n个}}$；

且在正交变换 $x_i \mapsto x_i' = \lambda_{ij} x_j$

$$\boxed{T'_{i\cdots j} = \lambda_{ik}\cdots\lambda_{jl}T_{k\cdots l},}$$

则称 T 为3维欧氏空间中的 n 阶张量.

1、零阶张量即为标量 ϕ：$\phi' = \phi$ （在变换下不改变）

（事实上，与空间的选择有关。上述仅在3维欧中不变）

2. 一阶张量即为矢量 $\vec{f}$：$f_i' = \lambda_{ij} f_j$ ，$x_i' = \lambda_{ij} x_j$

3. 二阶张量 $\overleftrightarrow{T}$：$T'_{ij} = \lambda_{ik}\lambda_{jl}T_{kl}$

即（$\boxed{T' = \lambda T \lambda^T}$ 由相似变换联系着）

eg.: 并矢 $\overleftrightarrow{T}=\vec{f}\vec{g}$: $T_{ij}=f_ig_j$

可以构造九个张量 $\hat{x}_i\hat{x}_j$ $(i,j=1,2,3)$

$$\hat{x}_1\hat{x}_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\quad \hat{x}_1\hat{x}_2=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\quad \hat{x}_2\hat{x}_1=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$$

eg. 单位张量 $\overleftrightarrow{I}_{ij}=\delta_{ij}=I'_{ij}$ （变换不变）.

（因 $\lambda_{ik}\lambda_{jl}I_{kl}=\lambda_{ik}\lambda_{jk}=\delta_{ij}=I'_{ij}$ ）

对称张量：$T_{ij}=T_{ji}$ 张量对称性也与坐标系无关.（6个独立）

反对称张量 $T_{ij}=-T_{ji}$ 张量反对称 ————.（3个独立）

eg. $\varepsilon_{ijk}A_{il}A_{jm}A_{kn}=\varepsilon_{lmn}\det A$

令 $A=\lambda$. 则有 $\det\lambda$,

$$\varepsilon_{lmn}\lambda_{li}\lambda_{mj}\lambda_{lk}=\varepsilon_{ijk}\det\lambda$$

$$\Rightarrow\quad \varepsilon_{ijk}=(\det\lambda)\cdot\lambda_{il}\lambda_{jm}\lambda_{kn}\varepsilon_{lmn}.=\varepsilon'_{ijk}.$$

称为 赝张量：$\boxed{T'_{i\cdots j}=(\det\lambda)\lambda_{ik}\cdots\lambda_{jl}T_{k\cdots l}}$

在"转动"下不变，"反演"下改变.

如赝矢量：$\vec{L}$, $\vec{B}$、$\vec{m}$（磁矩）、$\vec{\omega}$

二. 张量运算.

0、同阶张量相等 $T=S$ ：$T_{i\cdots j}=S_{i\cdots j}$ （存在一个坐标系即可）

推 → $T'_{i\cdots j}=S'_{i\cdots j}$ ↳ 每个分量都相等

物理学定律必然（在宏观上的）是用张量描述的（满足对所选坐标系无关）

1、同阶张量的线性组合运算

$$R=aT+bS:\quad R_{i\cdots j}=aT_{i\cdots j}+bS_{i\cdots j}$$

① 所有n阶张量的集合构成 3^n 维线性空间

② 基 $\hat{x}_i$、$\hat{x}_i\hat{x}_j\cdots\cdots$ 即 $\boxed{\vec{f}=f_i\hat{x}_i,\quad \overleftrightarrow{T}=T_{ij}\hat{x}_i\hat{x}_j}$

$$\overleftrightarrow{T}=T_{ij}\hat{x}_i\hat{x}_j \qquad \Rightarrow \boxed{T_{ij}=\hat{x}_i\cdot\overleftrightarrow{T}\cdot\hat{x}_j}$$

δ_{km}

$$\overleftrightarrow{T}=T'_{ij}\hat{x}'_i\hat{x}'_j=\lambda_{ik}\lambda_{jl}\lambda_{im}\lambda_{jn}T_{kl}\hat{x}_m\hat{x}_n=T_{kl}\hat{x}_k\hat{x}_l$$

δ_{lm}

注：二阶张量可由对称、反对称张量表示. $T_{ij}=\frac{T_{ij}+T_{ji}}{2}+\frac{T_{ij}-T_{ji}}{2}$

2、张量积（一般不满足交换律）

$$\boxed{\underset{n\times m}{R}=\underset{n}{T}\otimes\underset{m}{S}:\quad R_{i\cdots jk\cdots l}=T_{i\cdots j}S_{k\cdots l}}$$

3、缩并 n阶 $(n\geq 2)\rightarrow n-2$ 阶.

$$\boxed{T_{i\cdots k\cdots l\cdots j}\longrightarrow R_{i\cdots j}=\underset{\text{遍历}}{T_{i\cdots k\cdots k\cdots j}}}$$

[例] $T_{ij}\rightarrow\phi=T_{ii}\quad \phi'=T'_{ii}=\lambda_{ik}\lambda_{i}T_{kl}=\delta_{kl}T_{kl}=T_{kk}=\phi$

满足不变性且仅一个分量（自身）是0阶张量.

· eg. $T_{ij} = -T_{ji} \Rightarrow C_i = \varepsilon_{ijk} T_{jk}$, 轴

$$C_i' = \varepsilon_{ijk} T_{jk}' = \lambda_{jm} \lambda_{kn} \varepsilon_{ijk} T_{mn}$$

$$\lambda_{il} C_i' = (\det\lambda) \varepsilon_{lmn} T_{mn} = (\det\lambda) C_l$$

$$\Rightarrow \quad \lambda_{il} \lambda_{jl} C_i' = (\det\lambda) \lambda_{jl} C_l$$

$$\Rightarrow \quad C_j' = (\det\lambda) \lambda_{jl} C_l \quad \text{赝矢量}$$

eg. 有2个矢量$\vec{A}, \vec{B}$. $\Rightarrow \begin{cases} \vec{A}\vec{B} \xrightarrow{\text{缩并}} \vec{A}\cdot\vec{B} = A_i B_i \\ \vec{A}\vec{B} - \vec{B}\vec{A} \Rightarrow \vec{A}\times\vec{B} = (\varepsilon_{ijk} A_j B_k)\hat{x}_i \end{cases}$

eg. 就近点(叉)乘

$$\boxed{\vec{A}\vec{B} = (A_i \hat{x}_i)(B_j \hat{x}_j) = A_i B_j \hat{x}_i \hat{x}_j}$$

$$\boxed{\vec{A}\cdot\vec{B} = (A_i \hat{x}_i)\cdot(B_j \hat{x}_j) = A_i B_j \hat{x}_i \cdot \hat{x}_j = A_i B_j \delta_{ij} = A_i B_i}$$

$$\boxed{\vec{A}\times\vec{B} = (A_i \hat{x}_i)\times(B_j \hat{x}_j) = A_i B_j \underline{(\hat{x}_i \times \hat{x}_j)} = A_i B_j \underline{\varepsilon_{ijk} \hat{x}_k}}$$

$$\boxed{\begin{aligned} \vec{A}\cdot\overleftrightarrow{T} &= (A_i \hat{x}_i)\cdot(T_{jk} \hat{x}_j \hat{x}_k) = A_i T_{jk} \underbrace{\hat{x}_i \cdot \hat{x}_j}_{\text{就近点乘}} \hat{x}_k \\ &= A_i T_{jk} (\hat{x}_i \cdot \hat{x}_j) \hat{x}_k = A_i T_{jk} \delta_{ij} \hat{x}_k \\ &= (A_i T_{ik}) \hat{x}_k \end{aligned}}$$

$$\boxed{\begin{aligned} \overleftrightarrow{T}\cdot\vec{A} &= T_{jk} A_i \hat{x}_j \hat{x}_k \cdot \hat{x}_i = T_{jk} A_i \hat{x}_j (\hat{x}_k \cdot \hat{x}_i) \\ &= (T_{jk} A_k) \hat{x}_j = (T_{ki} A_i) \hat{x}_k \end{aligned}}$$

☆ 仅在$\overleftrightarrow{T}$为对称张量时，$\vec{A}\cdot\overleftrightarrow{T} = \overleftrightarrow{T}\cdot\vec{A}$

$\overleftrightarrow{S} = \vec{f} \times \overleftrightarrow{T} = (\varepsilon_{imn} f_m T_{nj})\hat{x}_i \hat{x}_j$ | $\overleftrightarrow{R} = \overleftrightarrow{T} \times \vec{f} = (\varepsilon_{jmn} T_{im} f_n)\hat{x}_i\hat{x}_j$

双点乘(可交换) $\overleftrightarrow{S} : \overleftrightarrow{T} = S_{ij} T_{ji} = \overleftrightarrow{T} : \overleftrightarrow{S}$ | $= \hat{x}_i \hat{x}_j (T_{im} f_n \varepsilon_{mnj})$

$\overleftrightarrow{S} \cdot \overleftrightarrow{T} = (S_{ik} T_{kj})\hat{x}_i \hat{x}_j$, $\overleftrightarrow{T} \cdot \overleftrightarrow{S} = (T_{ik} S_{kj})\hat{x}_i \hat{x}_j$

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$\vec{A} \cdot \overleftrightarrow{T} \cdot \vec{B} = A_i T_{jk} B_l \ \hat{x}_i \cdot \hat{x}_j \hat{x}_k \cdot \hat{x}_l$

$= A_i T_{jk} B_l \ \delta_{ij} \delta_{kl} = A_i T_{ik} B_k$

对 $\forall \overleftrightarrow{S}$,

$\overleftrightarrow{I} \cdot \overleftrightarrow{S} = \overleftrightarrow{S} \cdot \overleftrightarrow{I}$

三. 张量的不变性(判断)

Theorem 1. 任一线性映射 $\overleftrightarrow{T}$: $\vec{A} \mapsto \overset{矢量}{\vec{B}} = \overleftrightarrow{T}(\vec{A})$ 均为二阶张量

Theorem 2: 任一双线性函数 $\overleftrightarrow{T}$: $\vec{A}, \vec{B} \mapsto \phi = \overleftrightarrow{T}(\vec{A}, \vec{B})$ 均映为标量.

$$\overleftrightarrow{T}(a_1\vec{A}_1 + a_2\vec{A}_2) = a_1 \overleftrightarrow{T}(A_1) + a_2 \overleftrightarrow{T}(A_2)$$

[证明] 选定坐标系, $B_i \hat{x}_i = \overleftrightarrow{T}(A_j \hat{x}_j) = A_j \overleftrightarrow{T}(\hat{x}_j)$

$(\overleftrightarrow{T}(\hat{x}_j) = T_{ij} \hat{x}_i)$ 定义 $\boxed{T_{ij} = \hat{x}_i \cdot \overleftrightarrow{T}(\hat{x}_j)}$

原式 $= (T_{ij} A_j)\hat{x}_i \Rightarrow B_i = T_{ij} A_j$

对新坐标系, $T_{ij} = \hat{x}_i \cdot \overleftrightarrow{T}(\hat{x}_j) = \lambda_{ik} \hat{x}_k \cdot \overleftrightarrow{T}(\lambda_{jl} \hat{x}_l)$

$= \lambda_{ik} \lambda_{jl} \hat{x}_k \cdot \overleftrightarrow{T}(\hat{x}_l) = \lambda_{ik} \lambda_{jl} T_{kl}$

故 $\overleftrightarrow{T}$ 是二阶张量.

eg. 定义 $\overleftrightarrow{I}$: $\vec{A} \mapsto \vec{B} = \overleftrightarrow{I}(\vec{A})$ $\quad I_{ij} = \hat{x}_i \overleftrightarrow{I}(\hat{x}_j) = \delta_{ij}$

$\vec{A}, \vec{B} \mapsto \phi = \overleftrightarrow{I}(A, B) = \vec{A} \cdot \vec{B}$

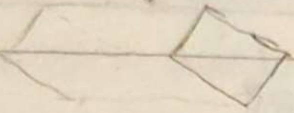
$I_{ij} = \hat{x}_i \hat{x}_j = \delta_{ij}$.

eg. $\vec{P} = \overleftrightarrow{\chi} \varepsilon_0 \cdot \vec{E}$

eg. 主动下心转动量可用二阶张量描述.

9. 动量流密度张量 $\boxed{\overleftrightarrow{T} = \rho\vec{v}\vec{v}}$

$d\vec{\sigma}\cdot\overleftrightarrow{T}$ 单位时间穿过 dS 之动量.



$$\frac{\vec{v}\rho(\vec{v}\Delta t\cdot\hat{n}\Delta S)}{\Delta t\Delta S} = \rho\vec{v}\vec{v}\cdot\hat{n}$$

$T_{ij} = \hat{x}_i\cdot\overleftrightarrow{T}\cdot\hat{x}_j$ 单位时间穿过 ⊥ $\hat{x}_i$ 之单位面积之动量在 $\hat{x}_j$ 方向之投影.

$$\boxed{-\partial_t w = \vec{f}\cdot\vec{v} + \nabla\cdot\vec{S}}$$

单位体积电荷受力.

$$-\frac{d}{dt}\int_V w\,dV = \int_V \vec{f}\cdot\vec{v}\,dV + \oint_{S=\partial V}\vec{S}\cdot d\vec{\sigma}$$

$\frac{1}{2}\varepsilon_0(E^2+c^2B^2)$ $\vec{E}\cdot\vec{j}$ $\frac{1}{\mu_0}\vec{E}\times\vec{B}$

$$\boxed{-\frac{d}{dt}\int_V \vec{g}\,dV = \int_V \vec{f}\,dV + \oint_{\partial V} d\vec{\sigma}\cdot\overleftrightarrow{T}}$$

$\varepsilon_0\vec{E}\times\vec{B}$ $w\overleftrightarrow{I} - \varepsilon_0(\vec{E}\vec{E} + c^2\vec{B}\vec{B})$

1. 其它张量定义：n重线性函数，将n个矢量映为标量（此映不变性）

故最关键之是定义矢量.

矢量具有原型：位矢：$d\vec{r}$ $dx_i' = \lambda_{ij}dx_j$

2. 正交变换是保持距离不变之线性变换.

I. ① $x_i' = \lambda_{ij}x_j$ ② $dl^2 = dx_i dx_i = \delta_{ij}dx_i dx_j \underline{\underline{成立}} \delta_{ij}dx_i' dx_j'$

i取1…n，定义了n维Euclid空间之矢量

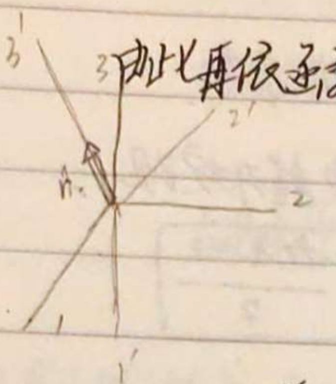
从而可定义n维张量.

Ⅱ.洛伦兹变换：Minkowski 空间.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2+dl^2$$

Ⅲ. 把矢量作为微分算子，在非平直空间.

先定义矢量（由函数定义） $f \mapsto x_i\frac{\partial f}{\partial x_i}$ $\vec{x} = x_i\frac{\partial}{\partial x_i}$



由此再依函数定义张量.

$x_i' = \mu_{ij}x_j$ 被动观点.

$$\lambda' = \mu\lambda\mu^T$$

1'、2' 任取，只要保证 1'、2'、3' 成右手系.

$$\lambda' = \begin{pmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ※注意矩阵元位置.

$$\mathrm{tr}\lambda' = \mathrm{tr}(\mu\lambda\mu^T) = \mathrm{tr}\lambda = \lambda_1+\lambda_2+\lambda_3 = 1+e^{i\theta}+e^{-i\theta}$$
$$\overset{\|}{2\cos\Theta+1} \qquad = 2\cos\theta+1$$

故 $\Theta = \theta$ （舍去 $\Theta = -\theta$，$\hat{n}$ 转 θ 与 $-\hat{n}$ 转 $-\theta$ 同义）

有限转动只能用张量不能用矢量描述

无限小转动可用二阶反对称张量描述，由于只有三个独立分量，有时可描述为，可用矢量描述无限小转动.

主动观点

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix} \quad \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

§3 转动矩阵的几何意义

Theorem. $\lambda \in SO(3)$ 的本征值总可表示为

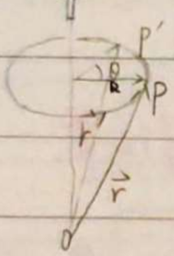
$$\lambda_1 = e^{i\theta} \quad \lambda_2 = e^{-i\theta} \quad \lambda_3 = +1 \ . \quad \theta \in [0, \pi]$$

（简证 $\lambda x = a x$, $x^+ \lambda^+ = a^* x^+ \Rightarrow |a| = 1$ ）

转轴是 λ_3 对应的本征矢方向，θ 是转动的角度.

即 $\lambda \in SO(3)$ 所描述的正交变换 可经由一次转动实现.

转轴方向 $\hat{n}$: $\lambda \hat{n} = \hat{n}$, 转角 $\boxed{\cos\theta = \frac{\operatorname{tr}\lambda - 1}{2}}$



设 $\vec{r}' = \vec{r}'(\vec{r}, \hat{n}, \theta)$

$$= a\hat{n} + b\,\hat{n}\times\vec{r} + c\,(\hat{n}\times\vec{r})\times\hat{n}$$

$$\hat{n}\cdot\vec{r}' = a\,\hat{n}\cdot\hat{n} \Rightarrow a = \hat{n}\cdot\vec{r}$$

$$\underbrace{(\hat{n}\times\vec{r})\cdot\vec{r}'}_{R^2\sin\theta} = b\underbrace{|\hat{n}\times\vec{r}|^2}_{R^2} \Rightarrow b = \sin\theta$$

$$\underbrace{[(\hat{n}\times\vec{r})\times\hat{n}]\cdot\vec{r}'}_{R^2\sin\theta} = c\,\underbrace{|(\hat{n}\times\vec{r})\times\hat{n}|}_{R^2}{}^2 \Rightarrow c = \cos\theta$$

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$\to$ 没有撇.

补注：这里的转动是主动转动，$\vec{r}' = x_i' \hat{x}_i$ ，$\vec{r} = x_i \hat{x}_i$

转动公式 $\boxed{\vec{r}' = \vec{r}\cos\theta + \hat{n}(\hat{n}\cdot\vec{r})(1-\cos\theta) + \hat{n}\times\vec{r}\sin\theta}$

$$x_i' = x_i\cos\theta + n_i n_j x_j(1-\cos\theta) + \varepsilon_{ikj} n_k x_j \sin\theta \equiv \lambda_{ij} x_j$$

故 $\boxed{\lambda_{ij} = \underbrace{\delta_{ij}\cos\theta + n_i n_j(1-\cos\theta)}_{\text{对称}} - \underbrace{\varepsilon_{ijk} n_k \sin\theta}_{\text{反对称}}}$

反对称部分可看出转轴 $\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \to \hat{n}\times\vec{?}$

当反对称部分为0时，说明是 0或π、　　接p14.1

无穷小转动公式：当 $\theta \to d\theta$. $\boxed{\vec{r}' = \vec{r} + \hat{n}\times\vec{r}\, d\theta}$

记 $\vec{d\theta} \equiv \hat{n}\, d\theta$，则 $d\vec{r} = \vec{d\theta}\times\vec{r}$

故： $\vec{r_1} = \vec{r_1} + \hat{n}_1\times\vec{r}\, d\theta_1$

$\vec{r_2} = \vec{r_1} + \hat{n}_2\times\vec{r_1}\, d\theta_2$

$= \vec{r} + \hat{n}_1\times\vec{r}\, d\theta_1 + \vec{n_2}\times\vec{r}\, d\theta_2 + \underbrace{\hat{n}_2\times(\hat{n}_1\times\vec{r})\, d\theta_1 d\theta_2}_{=0}$

$\Rightarrow \boxed{\frac{d\vec{r}}{d\theta} = \hat{n}\times\vec{r}}$, $\frac{d\vec{r}}{dt} = \frac{d\theta}{dt}\frac{d\vec{r}}{d\theta} = \vec{\omega}\times\vec{r}$, $\omega \equiv \hat{n}\frac{d\theta}{dt} = \frac{\vec{d\theta}}{dt}$

此时 $\lambda_{ij} = \underbrace{\delta_{ij}}_{\text{单位张量那样}} - \underbrace{\varepsilon_{ijk} n_k\, d\theta}$ 可用二阶反对称张量描述.

↓ 等价于一个轴矢量.

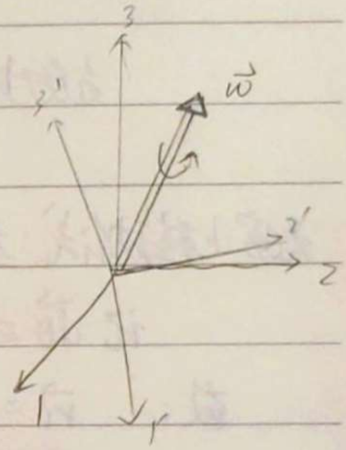
若存在: $|\vec{G}(t)| = \text{const} \Leftrightarrow \frac{d\vec{G}}{dt} = \vec{\omega}\times\vec{G}$

反推, $\frac{d\vec{G}}{dt}\cdot\vec{G} = \frac{d}{dt}(\frac{1}{2}G^2) \Rightarrow$ 模为0.

例: $m\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}$, $\vec{\omega} \equiv -\frac{q\vec{B}}{m} \Rightarrow \frac{d\vec{v}}{dt} = \vec{\omega}\times\vec{v}$.

§4 相对转动

$\vec{\omega}$: K 相对K 的角速度.



一、标量: $\frac{df}{dt} = (\frac{df}{dt})_{rot}$.

二、矢量:

$$\frac{d\vec{G}}{dt} = \frac{d(G_i\hat{x}_i)}{dt} = \frac{dG_i}{dt}\hat{x}_i + G_i\cancel{\frac{d\hat{x}_i}{dt}}^{0}$$

$$= \frac{d(G_i'\hat{x}_i')}{dt} = \frac{dG_i'}{dt}\hat{x}_i' + G_i'\frac{d\hat{x}_i'}{dt} = \omega\times\hat{x}_i'$$

$$(\frac{d\vec{G}}{dt})_{rot} = (\frac{dG_i'}{dt})_{rot}\hat{x}_i' + G_i'\cancel{(\frac{d\hat{x}_i'}{dt})_{rot}}^{0}$$

$\Downarrow$ 标量, $= \frac{dG_i'}{dt}$

$$\Rightarrow \boxed{\frac{d\vec{G}}{dt} = (\frac{d\vec{G}}{dt})_{rot} + \vec{\omega}\times\vec{G}}$$

故 $\frac{d\omega}{dt} = (\frac{d\omega}{dt})_{rot}$, 与系无关, 定义为 $\vec{\beta}$

故 $\vec{v} = \frac{d\vec{r}}{dt} = (\frac{d\vec{r}}{dt})_{rot} + \vec{\omega}\times\vec{r} = \vec{v}' + \vec{\omega}\times\vec{r}$

故 $a = \frac{d\vec{v}}{dt} = (\frac{d\vec{v}}{dt})_{rot} + \vec{\omega}\times\vec{v}$

$$= (\frac{d\vec{v}'}{dt})' + (\frac{d\vec{\omega}}{dt})'\times\vec{r} + \vec{\omega}\times(\frac{d\vec{r}}{dt})' + \vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= \vec{a}' + \vec{\beta}\times\vec{r} + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega\times r})$$

河补 p13 转动.

在与 $\hat{n}$ 垂直之平面内关于 r 有自然而然的坐标系：

$$\hat{n}\times\vec{r} \text{ 和 } -\hat{n}\times(\hat{n}\times\vec{r}) = \underbrace{\vec{r}-(\vec{r}\cdot\hat{n})\hat{n}}_{\vec{r}\text{扣除投影之分量}}$$

则 $\vec{r}$ 相当于作转动.

$$\vec{r} = (\vec{r}\cdot\hat{n})\hat{n} + (\vec{r}-(\vec{r}\cdot\hat{n})\hat{n})$$

$$\mapsto \vec{r} = (\vec{r}\cdot\hat{n})\hat{n} + \cos\theta\,(\vec{r}-(\vec{r}\cdot\hat{n})\hat{n}) + \sin\theta(\hat{n}\times\vec{r})$$

$$= \vec{r}\cos\theta + (\vec{r}\cdot\hat{n}\hat{n})(1-\cos\theta) + \sin\theta(\hat{n}\times\vec{r})$$

又 $\vec{r}' = (\hat{n}\hat{n}(1-\cos\theta) + \underset{\cos\theta}{\overleftrightarrow{I}} + \hat{n}\times\overleftrightarrow{I}\sin\theta)\cdot\vec{r} = \overleftrightarrow{\lambda}\cdot\vec{r}$

张量形式.

※ 常用的叉乘转点乘之技巧：

$$\hat{n}\times\vec{r} = \hat{n}\times(\overleftrightarrow{I}\cdot\vec{r}) = (\hat{n}\times\overleftrightarrow{I})\cdot\vec{r}$$

14.2

附补充：正交曲线坐标系

$$\boxed{\nabla\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}\hat{u}_3}$$

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]}$$

$$\boxed{\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}}$$

别漏了.

$$\boxed{\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]}$$

空间 (dq_1, dq_2, dq_3) 只是一个数学信息空间，$dq_1\hat{q}_1 + dq_2\hat{q}_2 + dq_3\hat{q}_3$ 无实际意义

应该是 $d\vec{r} = \frac{\partial\vec{r}}{\partial q}dq$

$$= \left|\frac{\partial\vec{r}}{\partial q}\right| dq \cdot \frac{\frac{\partial\vec{r}}{\partial q}}{\left|\frac{\partial\vec{r}}{\partial q}\right|} \triangleq h_q\, dq\, \hat{q}, \quad h_q = \left|\frac{\partial\vec{r}}{\partial q}\right|, \quad \hat{q} = \frac{\frac{\partial\vec{r}}{\partial q}}{\left|\frac{\partial\vec{r}}{\partial q}\right|}$$

$$\boxed{dr_{q_i} \triangleq \left|\frac{\partial\vec{r}}{\partial q_i}\right| dq_i}$$

$$\boxed{\nabla q_i = \frac{\hat{q}_i}{h_i}} \quad \boxed{\vec{\nabla}\times\left(\frac{\hat{q}_i}{h_i}\right) = 0} \quad \boxed{\vec{\nabla}\cdot\left(\frac{\hat{q}_1}{h_2h_3}\right) = 0}$$

§5 正交曲线坐标系

$\vec{r} = F(u_1, u_2, u_3)$

$d\vec{r} = \underline{\frac{\partial \vec{r}}{\partial u_1}} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$

$= \underline{h_1 \hat{u}_1} du_1 + h_2 \hat{u}_2 du_2 + h_3 \hat{u}_3 du_3$

$\left\{ \begin{array}{l} \text{Lame 系数} \\ \text{正交 } \hat{u}_i \cdot \hat{u}_j = \delta_{ij} \end{array} \right.$

$\Rightarrow \boxed{\vec{v} = \dot{\vec{r}} = \sum_{i=1}^{3} (h_i \dot{u}_i) \hat{u}_i}$, $v^2 = \sum_{i=1}^{3} h_i^2 \dot{u}_i^2$

球坐标中. $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}$

柱 $\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$

球 $\vec{v}^2 = \dot{x}_i \dot{x}_i = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta$

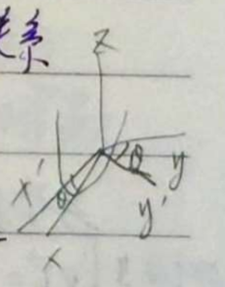
柱 $\vec{v}^2 = \dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2$

§6 场及其微分

一、标量场 φ $\left\{ \begin{array}{l} \varphi: P \mapsto \varphi(P) \\ \varphi'(P) = \varphi(P) \end{array} \right.$ 当有 $\underline{X}' = \lambda \underline{X}$ 时.

$\varphi'(\underline{X}') \triangleq \varphi(\underline{X}) = \varphi(\lambda^{-1}\underline{X}')$ 同一标量场在两坐标系下的表达式间的关系

eg. $\varphi(x,y) = (x-a)^2 + y^2$. $\left\{ \begin{array}{l} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{array} \right.$



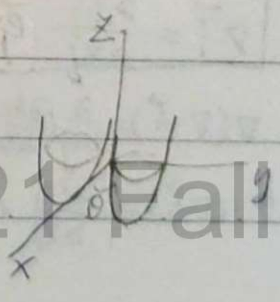
被动观点（注意此时是 $-\theta$ 代入 λ）

$\varphi'(x',y') = (x'\cos\theta + y'\sin\theta - a)^2 + (-x'\sin\theta + y'\cos\theta)^2$

$= (x' - a\cos\theta)^2 + (y' - a\sin\theta)^2$

主动观点：曲面转过 θ

$\varphi'(x,y) = (x - a\cos\theta)^2 + (y - a\sin\theta)^2$



即主动 $\varphi'(\underline{X}) = \varphi(\lambda^{-1}\underline{X})$ 　同一坐标系下两标量场（"曲面"）下的表达式之间的关系.

若标量场φ满足 $\varphi'(\underline{X}) = \varphi(\underline{X}')$ → x换成x' 　即 $\varphi(\lambda\underline{X}) = \varphi(\underline{X})$

则称φ在变换 $\underline{X} \mapsto \lambda\underline{X}$ 下是不变/对称的、

而λ所描述的变换称为φ的对称操作/对称变换、

eg. 若对物绕某轴转过角度 $\theta_n = \frac{2\pi}{n}$ $(n \geq 2)$ 下不变，则称该轴为n次对称轴

— — — — — 任何角度都不变，称之为旋转对称轴.

二、梯度算子 $\boxed{\nabla \triangleq \hat{x}_i \frac{\partial}{\partial x_i} = \hat{x}_i \partial_i}$

$$\partial_i' = \frac{\partial x_j}{\partial x_i'} \cdot \frac{\partial}{\partial x_j} = \lambda_{ij} \frac{\partial}{\partial x_j} = \lambda_{ij} \partial_i$$ 与矢量满足相同的规律

$$\left(x_j = \lambda_{kj} x_k' \Rightarrow \frac{\partial x_j}{\partial x_i'} = \lambda_{kj} \frac{\partial x_k'}{\partial x_i'} = \lambda_{kj} \delta_{ki} = \lambda_{ij}. \right)$$

1. 三个定义

$$\boxed{\begin{aligned} \nabla\varphi &= (\partial_i \varphi)\hat{x}_i = \frac{\partial\varphi}{\partial x_i}\hat{x}_i \\ \nabla\cdot\vec{F} &= \partial_i F_i = \frac{\partial F_i}{\partial x_i} \\ \nabla\times\vec{F} &= (\varepsilon_{ijk}\partial_j F_k)\hat{x}_i = \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j}\hat{x}_i \end{aligned}}$$

与坐标系无关定义: $d\varphi = \nabla\varphi \cdot d\vec{l}$

[ij注]

$$\boxed{\begin{array}{l|l} \nabla\vec{F} = \frac{\partial F_j}{\partial x_i}\hat{x}_i\hat{x}_j = & \partial_i F_j \hat{x}_i \hat{x}_j \\ \nabla\cdot\overleftrightarrow{T} = \frac{\partial T_{ij}}{\partial x_i}\hat{x}_j & \partial_i T_{ij} \hat{x}_j \end{array}}$$

$$\nabla\cdot\vec{F} = \lim_{V\to 0}\frac{\oint_{\partial V}\vec{F}\cdot d\vec{S}}{V}$$

$$\hat{n}\cdot(\nabla\times\vec{F}) = \lim_{S\to 0}\frac{\oint_{\partial S}\vec{F}\cdot d\vec{l}}{S}$$

故 $\nabla(\nabla\cdot\vec{A}) = \frac{\partial}{\partial x_j}\left(\frac{\partial A_k}{\partial x_k}\right)\hat{x}_j$

$\nabla\cdot(\nabla\vec{A}) = \frac{\partial^2 A_j}{\partial x_i^2}\hat{x}_j$

[ssj注] $\nabla r = \frac{\vec{r}}{r} = \vec{e}_r$, $\nabla \vec{r} = \overleftrightarrow{I}$, $\nabla f(r) = f'(r)\,\vec{e}_r$, $\nabla\cdot\vec{r} = 3$

$\nabla\times[f(r)\,\vec{r}] = 0$

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$$\begin{aligned}
\left[\nabla\times(\vec{A}\times\vec{B})\right]_i &= \varepsilon_{ijk}\,\partial_j(\vec{A}\times\vec{B})_k = \varepsilon_{ijk}\varepsilon_{mnk}\,\partial_j(A_mB_n)\\
&= (\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\,\partial_j(A_mB_n)\\
&= \partial_j(A_iB_j) - \partial_j(A_jB_i)\\
&= B_j(\partial_jA_i) + A_i\partial_jB_j - (\partial_jA_j)B_i - A_j\partial_jB_i\\
&= (\vec{B}\cdot\nabla)A_i + (\nabla\cdot\vec{B})A_i - (\nabla\cdot\vec{A})B_i - (\vec{A}\cdot\nabla)B_i
\end{aligned}$$

$$\nabla\times(\vec{A}\times\vec{B}) = (B\cdot\nabla)A - B(\nabla\cdot A) - (A\cdot\nabla)B + A(\nabla\cdot B)$$

2、两个恒等式.

$$\boxed{0 = \nabla\cdot(\nabla\times\vec{A})} = \overset{\nearrow 反对称}{\varepsilon_{ijk}}\,\underbrace{\partial_i\partial_j}_{\searrow 对称} A_k = 0$$

$$\boxed{0 = \nabla\times\nabla\varphi} : \quad (\nabla\times\nabla\varphi)_i = \varepsilon_{ijk}\,\partial_j\partial_k\varphi = 0$$

3. 两个定理.
$$\nabla\times\vec{F} = 0 \Rightarrow \vec{F} = -\nabla\varphi$$
$$\nabla\cdot\vec{F} = 0 \Rightarrow \vec{F} = \nabla\times\vec{A}$$

要求单连通空间 (或仅局域地成立)

eg. $\nabla\times\vec{F}(\vec{r}) = 0 \Rightarrow \vec{F}(\vec{r}) = -\nabla U(\vec{r})$

$\Rightarrow dU = \nabla U\cdot d\vec{r} = -\vec{F}\cdot d\vec{r}$

$\Rightarrow U(\vec{r}) = -\int_p^{\vec{r}} \vec{F}\cdot d\vec{r}$

如 $\vec{F}$恒力 , $U = -\vec{F}\cdot\vec{r}$

$\vec{F} = -k\vec{r}$ · $U = k\int\vec{r}\cdot d\vec{r} = k\int r\,dr = \frac{1}{2}kr^2$

$\nabla\times\vec{F}(\vec{r},t) = 0 \Rightarrow \vec{F}(\vec{r},t) = -\nabla(\vec{r}.t)$

如 $F = \vec{F}_0\cos\omega t$, $U = -\vec{F}_0\cdot\vec{r}\cos\omega t$

补 $\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$

4. $\nabla\cdot\vec{B}=0 \Rightarrow \vec{B}=\nabla\times\vec{A}$

$$\nabla\times\vec{E}=-\partial_t\vec{B}=-\partial_t(\nabla\times\vec{A})=-\nabla\times\partial_t A$$

$$\Rightarrow \quad \nabla\times(\vec{E}+\underbrace{\partial_t A)}_{-\nabla\varphi}=0$$

电磁势： $\boxed{\begin{array}{l}\vec{E}(\vec{r},t)=-\nabla\varphi(\vec{r},t)-\partial_t\vec{A}(\vec{r},t)\\ \vec{B}=\nabla\times\vec{A}\end{array}}$

$$\vec{B}=\nabla\times\vec{A}=\nabla\times\vec{A}' \quad\Rightarrow\quad \vec{A}'-\vec{A}=\nabla\psi'(\vec{r},t)$$

$$\vec{E}=-\nabla\varphi-\partial_t\vec{A}=-\nabla\varphi'-\partial_t\vec{A}'$$

$$\Rightarrow \nabla(\varphi'-\varphi)+\partial_t(\vec{A}'-\vec{A})=\nabla(\varphi'-\varphi+\partial_t\psi')=0$$

故 $\varphi'=\varphi-\partial_t\psi'+f(t)=\varphi-\partial_t\psi$ ，$\psi\triangleq\psi'-\int_0^t f(t)\,dt$

（注意到 $\nabla\psi=\nabla\psi'$）

规范变换： $\boxed{\varphi'=\varphi-\partial_t\psi\ ,\ \vec{A}'=\vec{A}+\nabla\psi}$

ψ：规范函数.

三. 符号及推广

1、对矢量的导数 $f(\vec{A})$： $\frac{\partial f}{\partial\vec{A}}\equiv\frac{\partial f}{\partial x_i}\hat{x}_i=\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3}\right)$

（直角坐标）

eg. $f(\vec{r})$，$\frac{\partial f}{\partial\vec{r}}=\nabla f$

$f=f(\vec{r},\dot{\vec{r}},t)$，$\frac{df}{dt}=\frac{\partial f}{\partial\vec{r}}\cdot\dot{\vec{r}}+\frac{\partial f}{\partial\dot{\vec{r}}}\ddot{\vec{r}}+\frac{\partial f}{\partial t}$.

$\boxed{f=\vec{A}\cdot\vec{v}\ ,\ \frac{\partial f}{\partial\vec{v}}=\vec{A}.}$

2、对一组变量的导数

$$f=f(q)=f(q_1,\cdots,q_n),\quad \frac{\partial f}{\partial q}=\left(\frac{\partial f}{\partial q_1},\cdots,\frac{\partial f}{\partial q_n}\right)$$

3、"无旋"

$$F_i=-\frac{\partial\varphi}{\partial q_i}\ \underset{\text{单连通,或局部上.}}{\xrightarrow{\hspace{6em}}}\ \frac{\partial}{\partial q_i}F_j=\frac{\partial}{\partial q_j}F_i$$

其余补充见后文习题部分

§7 约束

对三维空间自由粒子, 独立变量: (x,y,z), $(\dot{x},\dot{y},\dot{z})$

$$\frac{\partial x_i}{\partial x_j}=\delta_{ij},\quad \frac{\partial \dot{x}_i}{\partial \dot{x}_j}=\delta_{ij},\quad \frac{\partial \dot{x}_i}{\partial x_j}=0$$

约束: 对体系状态的约束

一、例子

1. 曲面 $f(\vec{r},t)=0 \Rightarrow 0=\frac{df}{dt}=\frac{\partial f}{\partial \vec{r}}\cdot\dot{\vec{r}}+\frac{\partial f}{\partial t}$.

当 $\frac{\partial f}{\partial t}=0$ 时, 即梯度·速度为0, 即粒子速度只能垂直曲面法向.

2. 曲线 $f_1(\vec{r},t)=0$, $f_2(\vec{r},t)=0$ (两个独立约束)

3. 沿斜面纯滚动 $0=\dot{x}-R\dot{\theta} \Rightarrow x=R\theta$

4. 在平面上纯滚动的竖直圆盘 $0=\vec{v}_p=\vec{v}_c+\vec{v}_p^*$, $\vec{v}_c=\dot{x}\hat{x}+\dot{y}\hat{y}$

$$d\theta=\hat{n}\,d\phi+\hat{z}\,d\psi \Rightarrow \boxed{\vec{\omega}=\dot{\phi}\hat{n}+\dot{\psi}\hat{z}}$$

$$\vec{v}_p^*=\vec{\omega}\times(-R\hat{z})=-R\dot{\phi}\,\hat{n}\times\hat{z}=-R\dot{\phi}(\hat{x}\sin\psi+\hat{y}\cos\psi)$$

$\sum A_i(x_1,\cdots x_n)dx_i=0 \overset{?}{\Rightarrow} df(x_1,\cdots,x_n)=0 \Rightarrow f(x_1,\cdots x_n)=C$

若 $gA_i=\frac{\partial f}{\partial x_i}$ $(g\neq0)$ $\Leftrightarrow$ $\boxed{\frac{\partial}{\partial x_j}(gA_i)=\frac{\partial}{\partial x_i}(gA_j)}$

即要求 $\vec{A}\cdot(\nabla\times\vec{A})=0$ （三维） （移右边）

二、约束分类

约束方程 $f(\vec{r},\dot{\vec{r}},t)=0$ (≥0)

1、稳定与不稳定约束：$f(\vec{r},\dot{\vec{r}})=0$ (≥0)

2. 可解（单侧）与不可解（双侧）约束

$f\geq0$ $f=0$

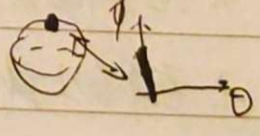
3、$\left\{\begin{array}{l}\text{几何约束}\\ \text{微分约束}\left\{\begin{array}{l}\text{可积约束}\\ \text{不可积约束}\end{array}\right.\end{array}\right.$ 完整约束

三、完整体系之运动学描述 $\left\{\begin{array}{ll}n\text{个粒子} & \vec{r}=(\vec{r}_1,\cdots,\vec{r}_n)\\ m\text{个约束} & f_\alpha(\vec{r},t)=0,\quad \alpha=1,\cdots m\end{array}\right.$

1、自由度 $s=3n-m$：完全描述体系位形所需独立变量之个数

2. 广义坐标 $q=(q_1,\cdots,q_s)$ → （若都正，就建立一一对应关系）

广义速度 $\dot{q}=(\dot{q}_1,\cdots,\dot{q}_s)$



（接上边）

$$\sum_i A_i dx_i = 0$$

$\vec{A} = \vec{A}(\vec{r})$ $\begin{cases} \text{积分曲线 } \vec{r} = \vec{r}(\tau)\,;\ \vec{r}(0) = \vec{r}_0,\ \frac{d\vec{r}}{d\tau} = g\vec{A}(\vec{r}) \\ \text{积分曲面 } f(\vec{r}) = 0\,;\ f(\vec{r}_0) = 0,\ \frac{d\vec{r}}{d\tau} = g\vec{A}(\vec{r}) \end{cases}$ 至少有局部解 存在唯一 在 g前不能直接确定

即 $f(x_1, \dots, x_n) = 0 \Rightarrow \sum_i \frac{\partial f}{\partial x_i} dx_i$ 有约束，一定能得到对速度的限制

$\sum \frac{\partial f}{\partial x_i} dx_i = 0$ 对速度的约束不一定能写成对坐标的约束.

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3. 变换方程 $\vec{r}_a = \vec{r}_a(q, t)$ → 注意与$\dot{q}$无关

$$\boxed{\dot{\vec{r}}_a = \frac{\partial \vec{r}_a}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_a}{\partial t} = \dot{\vec{r}}_a(q, \dot{q}, t)}$$

$$\frac{\partial \dot{\vec{r}}}{\partial q_k} = \frac{\partial^2 \vec{r}}{\partial q_k \partial q_i}\dot{q}_i + \frac{\partial \vec{r}}{\partial q_i}\overset{=0}{\boxed{\frac{\partial \dot{q}_i}{\partial q_k}}} + \frac{\partial^2 \vec{r}}{\partial q_k \partial t}$$

$$= \frac{\partial}{\partial q_i}\left(\frac{\partial}{\partial q_k}\right)\dot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}}{\partial q_k}\right) = \frac{d}{dt}\frac{\partial \vec{r}}{\partial q_k}$$

即：

$$\boxed{\begin{aligned} &\frac{\partial \dot{\vec{r}}}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_k}\right), \quad \frac{\partial \dot{\vec{r}}}{\partial t} = \frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial t}\right) \\ &\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_k} = \frac{\partial \vec{r}}{\partial q_i}\frac{\partial \dot{q}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}}{\partial q_i}\delta_{ik} = \frac{\partial \vec{r}}{\partial q_k} \end{aligned}}$$

对易关系

$$\boxed{\frac{d}{dt}\frac{\partial}{\partial q_k} = \frac{\partial}{\partial q_k}\frac{d}{dt}}$$

$$\ddot{\vec{r}} = \frac{d}{dt}(\dot{\vec{r}}) = \frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}}{\partial t}\right)$$

$$= \frac{\partial \vec{r}}{\partial q_k \partial q_i}\dot{q}_i\dot{q}_k + 2\frac{\partial^2 \vec{r}}{\partial q_i \partial t}\dot{q}_i + \frac{\partial \vec{r}}{\partial q_i}\ddot{q}_i + \frac{\partial^2 \vec{r}}{\partial t^2}$$

即 $\boxed{\frac{\partial \ddot{\vec{r}}}{\partial \ddot{q}_i} = \frac{\partial \vec{r}}{\partial q_i} = \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i}}$

对比：然而，$\frac{d}{dt}$ 与 $\frac{\partial}{\partial \dot{q}_k}$ 不可对易：$\boxed{\frac{\partial}{\partial \dot{q}_k}\frac{d}{dt} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_k} = \frac{\partial}{\partial q_k}}$

四、位形空间：以 $q_1 \cdots q_s$ 为直角坐标构建之 s 维空间。

一个点描述体系之位形

一条曲线可以描述系统位形之变换。

3维(母)空间 $(x_1, y_1, z_1, \cdots\cdots, x_n, y_n, z_n)$

此时 $f(\vec{r}, t)=0 = f(\vec{r}_1, \cdots, \vec{r}_n, t)$ 为 $(3n-1)$ 维超曲面。

故约束 $f_\alpha(\vec{r}, t)=0$，$(\alpha=1,\cdots,m)$

将空间限为 $S=3n-m$ 维位形曲面。

视 $\vec{r}=\vec{r}(q,t)$ 为参数方程。

对此位形曲面，有 S 个独立切矢量 $\boxed{\vec{\tau}_i = \frac{\partial \vec{r}}{\partial q_i}}$ $(i=1,\cdots,s)$

法向量共有m个 $\boxed{\vec{n}_\alpha = \frac{\partial f_\alpha}{\partial \vec{r}}}$，$(\alpha=1\cdots m)$

独立：m个法向量是独立的，或言线性无关的。(几何处成立)

$$\mathrm{rank}(\vec{n}_1, \cdots, \vec{n}_m) = m$$

五、(速度)相空间：以 $q_1,\cdots,q_s;\dot{q}_1,\cdots,\dot{q}_s$ 为直角坐标构建之 2s 维空间。

eg. $m\ddot{x} = -kx \Leftrightarrow \frac{d}{dt}\begin{pmatrix}x\\ \dot{x}\end{pmatrix} = \begin{pmatrix}\dot{x}\\ -\omega^2 x\end{pmatrix}\begin{matrix}\to v_x\\ \to v_{\dot{x}}\end{matrix}$

或能量守恒 $E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2$ 椭圆方程

且为顺时针转动。

相空间中，相点 $\leftrightarrow$ 状态.

在一相点确定后，演化过程将由动力学方程确定，.
表现为一条曲线：相轨迹.

六. 动能

1. 一般形式：$T=\frac{1}{2}m_a\dot{\vec{r}_a}^2=T(\vec{r}_1,\cdots,\vec{r}_n)=T(\vec{r})$

$$T=\frac{1}{2}m_a\left(\frac{\partial\vec{r}_a}{\partial q_i}\dot{q}_i+\frac{\partial\vec{r}_a}{\partial t}\right)\cdot\left(\frac{\partial\vec{r}_a}{\partial q_j}\dot{q}_j+\frac{\partial\vec{r}_a}{\partial t}\right)$$

$$\boxed{T=\frac{1}{2}m_a\frac{\partial\vec{r}_a}{\partial q_i}\cdot\frac{\partial\vec{r}_a}{\partial q_j}\dot{q}_i\dot{q}_j+m_a\frac{\partial\vec{r}_a}{\partial q_i}\cdot\frac{\partial\vec{r}_a}{\partial t}\dot{q}_i+\frac{1}{2}m_a\frac{\partial\vec{r}_a}{\partial t}\cdot\frac{\partial\vec{r}_a}{\partial t}}$$

$$\equiv\frac{1}{2}A_{ij}(q,t)\dot{q}_i\dot{q}_j+B_i(q,t)\dot{q}_i+C(q,t)$$

$$\boxed{T\equiv T_2+T_1+T_0=T(q,\dot{q},t)}$$

（用局部坐标描述时）几乎处处 A_{ij} 为对称正定矩阵.

2. Euler定理 $\dot{\vec{r}_a}\cdot\frac{\partial T}{\partial\dot{\vec{r}_a}}=\dot{\vec{r}_a}\cdot\vec{p}_a=2T$

$$\dot{q}_k\cdot\frac{\partial T}{\partial\dot{q}_k}=2T_2+T_1 \qquad \boxed{\left(\frac{\partial T}{\partial\dot{q}_k}=\frac{\partial T}{\partial\dot{\vec{r}_a}}\cdot\frac{\partial\dot{\vec{r}_a}}{\partial\dot{q}_k}=\vec{p}_a\cdot\frac{\partial\vec{r}_a}{\partial q_k}\right)}$$

特别：当变换方程不显含t. $\vec{r}=\vec{r}(q,\cancel{t})$

此时 $T_0=T_1=0$，$\dot{q}_k\cdot\frac{\partial T}{\partial\dot{q}_k}=2T_2$

$$\begin{pmatrix}p_1\\ \vdots\\ p_n\end{pmatrix}\leftarrow\vec{p}\cdot\underbrace{\frac{\partial\vec{r}}{\partial q_k}}_{\downarrow}$$

切矢量 $\begin{pmatrix}\partial\vec{r}_1/\partial q_k\\ \vdots\\ \partial\vec{r}_n/\partial q_k\end{pmatrix}$

例如，在球坐标系中，

$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\phi}^2\sin^2\theta)$$

$$\frac{\partial T}{\partial \dot{r}}=m\dot{r}=\vec{p}\cdot\hat{r}$$

$$\frac{\partial T}{\partial \dot{\theta}}=mr^2\dot{\theta}=\vec{p}\cdot r\hat{\theta}=\vec{p}\cdot(\hat{\phi}\times\vec{r})=\hat{\phi}\cdot(\vec{r}\times\vec{p})=\hat{\phi}\cdot\vec{l}$$

$$\frac{\partial T}{\partial \dot{\phi}}=mr^2\dot{\phi}\sin^2\theta=\vec{p}\cdot r\sin\theta\,\hat{\phi}=\vec{p}\cdot(\hat{z}\times\vec{r})=\hat{z}\cdot(\vec{r}\times\vec{p})=\hat{z}\cdot\vec{l}$$

$$\frac{\partial T}{\partial \dot{q}_k}=\vec{p}\cdot\frac{\partial \vec{r}}{\partial q_k}=\vec{p}\cdot h_k\hat{q}_k \quad (k不求和)$$

或：$\frac{\partial T}{\partial \dot{r}}=\frac{\partial T}{\partial \dot{\vec{r}}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{r}}=\vec{p}\cdot\hat{r}$

$$\frac{\partial T}{\partial \dot{\theta}}=\frac{\partial T}{\partial \dot{\vec{r}}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{\theta}}=\frac{\partial T}{\partial \dot{\vec{r}}}\cdot\frac{\partial \vec{r}}{\partial \theta}=\vec{p}\cdot r\hat{\theta}=\cdots$$

$$\frac{\partial T}{\partial \dot{\phi}}=\frac{\partial T}{\partial \dot{\vec{r}}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{\phi}}=\frac{\partial T}{\partial \dot{\vec{r}}}\cdot\frac{\partial \vec{r}}{\partial \phi}=\vec{p}\cdot r\sin\theta\,\hat{\phi}=\cdots$$

CH2. Lagrange 力学

CH2. Lagrange 力学.

§1 运动臆想.

一、匀速直线运动：$\int_{t_1}^{t_2} T\,dt = \int_{t_1}^{t_2} \frac{1}{2}m\dot{x}^2 dt$ 最小.

二、抛物运动. $\int_{t_1}^{t_2} T' dt = \int_{t_1}^{t_2} \frac{1}{2}m(\dot{x}+gt)^2\ dt$

$$= \int_{t_1}^{t_2} \frac{1}{2}m\dot{x}^2 dt + \int_{t_1}^{t_2} mgt\dot{x}\,dt + \underbrace{\int_{t_1}^{t_2} \frac{1}{2}mg^2t^2 dt}_{=const}$$

$$\int_{t_1}^{t_2} mgt\dot{x}\,dt = mgtx\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} mgx\,dt = C - \int_{t_1}^{t_2} mgx\,dt$$

$$= \int_{t_1}^{t_2} (T-V)\,dt = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - mgx\right) dt \text{ 最小}$$

T → 描述运动状态　　V → 描述相互作用

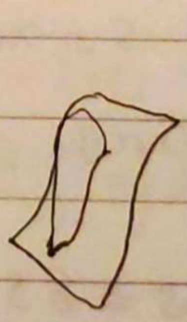
三. 遐想 1. 自由体系.

Lagrange 函数 $L \triangleq T-V = \sum_a \frac{1}{2}m_a v_a^2 + U(\vec{r},t)$

$= L(\vec{r}, \dot{\vec{r}}, t)$

作用量 $S = \int_{t_1}^{t_2} L(\vec{r}, \dot{\vec{r}}, t)\,dt$ 最小

四 遐想 2. 完整约束体系.



球面小人，只能感知到球面投影）方向上力，所定义"直线"、"三角形"各不同

$f_\alpha(\vec r,t)=0$ 理想约束力 $\vec N^{(\alpha)} = \begin{pmatrix}\vec N_1^{(\alpha)}\\ \vdots\\ \vec N_n^{(\alpha)}\end{pmatrix} \parallel \frac{\partial f_\alpha}{\partial \vec r} = \begin{pmatrix}\frac{\partial f_\alpha}{\partial r_1}\\ \vdots\\ \frac{\partial f_\alpha}{\partial r_n}\end{pmatrix}$

总约束力 $\vec N = \begin{pmatrix}\vec N_1\\ \vdots\\ \vec N_n\end{pmatrix} = \begin{pmatrix}\sum_{\alpha=1}^{7}\vec N_1^{(\alpha)}\\ \vdots\\ \sum_{\alpha=1}^{7}\vec N_n^{(\alpha)}\end{pmatrix}$ $\underline{\vec N\cdot\frac{\partial \vec r}{\partial q_k}=0}$

理想约束假设

主动力 $\begin{pmatrix}\vec F_1\\ \vdots\\ \vec F_n\end{pmatrix} = \begin{pmatrix}-\partial U/\partial \vec r_1\\ \vdots\\ -\partial U/\partial \vec r_n\end{pmatrix} \Longleftrightarrow \vec F_a = -\frac{\partial U}{\partial \vec r_a}$

将其~~在~~某个方向上投影， $\boxed{\vec F\cdot\frac{\partial \vec r}{\partial q_k} = -\frac{\partial U}{\partial \vec r}\cdot\frac{\partial \vec r}{\partial q_k} = -\frac{\partial U}{\partial q_k}}$

$U=U(\vec r,t)=U(\vec r(q,t),t)=U(q,t)$ (不一个函数)

引入广义坐标后，势能的改变完全由主动力的切向分量决定.

$L=L(\vec r,\dot{\vec r},t)=L(q,\dot q,t)$ (不一个函数)

五. 如何证明.

泛函 $S: x(t)\longmapsto S[x(t)]$

$$\Delta S[x(t)] = S[x(t)+\Delta x(t)] - S[x(t)] \approx \int_{t_1}^{t_2} G(t)\,\Delta x(t)\,dt = 0$$

$\Delta S \Rightarrow \delta S$ ， $\Delta x\Rightarrow \delta x$ $\quad G(t)\Rightarrow \frac{\delta L}{\delta x}$

此时 $S[x(t)] = \int L\,dt$

故极值路径 $\delta S[x(t)] = 0$

§2 泛函与变分

一. 泛函 $I: q(t) = (q_1(t), \cdots, q_n(t)) \mapsto I[q(t)]$,

$$I(q(t)) = \int_{t_1}^{t_2} f(q, \dot{q}, t)\,dt.$$

一组函数对应一个数

二. 变分:

路径变分 $\delta q_k(t) = \delta q_k$

速度变分（两条路径之变化率之差） $\delta \dot{q}_k \triangleq \dfrac{d\,\delta q_k}{dt}$

（$\boxed{\delta \frac{d}{dt} q_k = \frac{d}{dt} \delta q_k}$ 对坐标来说，变分微分可交换次序）

函数变分 $\delta f(q, \dot{q}, t) \triangleq f(q+\delta q, \dot{q}+\delta \dot{q}, t) - f(q, \dot{q}, t)$

泛函变分 $\delta I[q(t)] = I[q+\delta q] - I[q] = \delta \int_{t_1}^{t_2} f\,dt.$

（推致） $= \int_{t_1}^{t_2} \delta f\,dt$

$\boxed{\text{积分与变分可交换次序}}$

$$\delta f(q,\dot{q},t) = \frac{\partial f}{\partial q_k}\delta q_k + \frac{\partial f}{\partial \dot{q}_k}\delta \dot{q}_k$$ （等时变分, $\delta t \equiv 0$）

$$\delta f(g) = \frac{\delta f}{\delta g}\delta g$$

$$\delta f^n = n f^{n-1}\delta f$$

$$\delta(fg) = (\delta f)g + f(\delta g)$$

$$\boxed{\delta \frac{d}{dt} f(q,\dot{q},t) = \frac{d}{dt}\delta f(q,\dot{q},t)} \qquad (*)$$

$\boxed{\text{定理}}$ $0=\int_{t_1}^{t_2}[G_i\eta_i]\,dt$，$\eta_i$任意，则 $G_i=0$

(*)式之证明： $\frac{d}{dt}f = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial f}{\partial t}$.

则
$$\begin{aligned}\delta\frac{df}{dt} = {} & \frac{\partial^2 f}{\partial q_i \partial q_k}\dot{q}_i\delta q_k + \frac{\partial^2 f}{\partial q_i \partial \dot{q}_k}\dot{q}_i\delta \dot{q}_k + \frac{\partial f}{\partial q_k}\delta \dot{q}_k \\ & + \frac{\partial^2 f}{\partial \dot{q}_i \partial q_k}\ddot{q}_i\delta q_k + \frac{\partial^2 f}{\partial \dot{q}_i \partial \dot{q}_k}\ddot{q}_i\delta \dot{q}_k + \frac{\partial f}{\partial \dot{q}_i}\delta \ddot{q}_i \\ & + \frac{\partial^2 f}{\partial t \partial q_k}\delta q_k + \frac{\partial^2 f}{\partial t \partial \dot{q}_k}\delta \dot{q}_k\end{aligned}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\left(\frac{d}{dt}\frac{\partial f}{\partial q_k}\right)\delta q_k \qquad \left(\frac{d}{dt}\frac{\partial f}{\partial \dot{q}_k}\right)\delta \dot{q}_k$$

注意到 $\left(\frac{d}{dt}\frac{\partial f}{\partial q_k}\right)\delta q_k + \frac{\partial f}{\partial q_k}\delta \dot{q}_k = \frac{d}{dt}\left(\frac{\partial f}{\partial q_k}\delta q_k\right)$

$\left(\frac{d}{dt}\frac{\partial f}{\partial \dot{q}_k}\right)\delta \dot{q}_k + \frac{\partial f}{\partial \dot{q}_k}\delta \ddot{q}_k = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_k}\delta \dot{q}_k\right)$

故 $\delta\frac{df}{dt} = \frac{d}{dt}\left[\frac{\partial f}{\partial q_k}\delta q_k + \frac{\partial f}{\partial \dot{q}_k}\delta \dot{q}_k\right] = \frac{d}{dt}\delta f$. 证毕。

四. 极值路径（泛函之极值问题

$$\begin{cases} \delta I[q(t)] = \delta \int_{t_1}^{t_2} f(q, \dot{q}, t)\, dt = 0 \\ \delta q_k(t_1) = 0 = \delta q_k(t_2) \quad , \quad (k = 1, \cdots, n) \end{cases} \qquad (H)$$

$$0 = \delta I = \int_{t_1}^{t_2} \delta f\, dt = \int_{t_1}^{t_2} \left[\frac{\partial f}{\partial q_k} \delta q_k + \frac{\partial f}{\partial \dot{q}_k} \delta \dot{q}_k \right]$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial f}{\partial q_k} \delta q_k + \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_k} \delta q_k \right) - \left(\frac{d}{dt} \frac{\partial f}{\partial \dot{q}_k} \delta q_k \right) \right] dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial f}{\partial q_k} - \frac{d}{dt} \frac{\partial f}{\partial \dot{q}_k} \right] \delta q_k\, dt + \left. \frac{\partial f}{\partial \dot{q}_k} \delta q_k \right|_{t_1}^{t_2}$$

端点项为0

(H) 等价为：$\boxed{\dfrac{\delta f}{\delta q_k} \triangleq \dfrac{\partial f}{\partial q_k} - \dfrac{d}{dt} \dfrac{\partial f}{\partial \dot{q}_k} = 0}$, $k = 1, \cdots, n$

Euler-Lagrange 方程.

结论：1. $\boxed{\dfrac{\delta f}{\delta q_k} = 0 \Leftrightarrow \dfrac{\delta (cf)}{\delta q_k} = 0 \quad (c \neq 0)}$

→ 只是q,t之函数，不能包含q̇，因没有要求$\delta \dot{q}(t_i)$为0.

$\Leftrightarrow \boxed{\dfrac{\delta}{\delta q_k} \left[f + \dfrac{dF(q, t)}{dt} \right] = 0}$

2. $f(q,\dot{q},t)$ 的 Jacobi 积分

$$\boxed{h = h(q,\dot{q},t) = \dot{q}_k \frac{\partial f}{\partial \dot{q}_k} - f}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial f}{\partial t}$$

当取为极值路径时 $\left(\frac{d}{dt}\frac{\partial f}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial f}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial f}{\partial t} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_k}\dot{q}_k\right) + \frac{\partial f}{\partial t}$

$\boxed{\text{定理}}$ f在极值路径上 ① $\boxed{\frac{dh}{dt} = -\frac{\partial f}{\partial t}.}$

当f不显含t时, $h(q,\dot{q},t) = \text{const}$.

② $\boxed{\text{若}f\text{不显含某一个}q_k\text{，则}\ \frac{\partial f}{\partial \dot{q}_k} = \text{const}.}$

eg. 速降线 (Branchistochrone)

$$t = \int \frac{ds}{v} = \int_0^{x_0} \sqrt{\frac{1+y'(x)}{2gy}}\, dx = t[y(x)]$$

$f = \sqrt{\frac{1+y'(x)}{2gy}}$ ，不显含 x.

$$h \triangleq y' \frac{\partial f}{\partial y'} - f = \frac{-1}{\sqrt{2gy(1+y'^2)}} = \frac{-1}{\sqrt{2g}\,C}$$

即得 $C^2 = y + yy'^2 = (\sqrt{y})^2 + (\sqrt{y}\,y')^2$

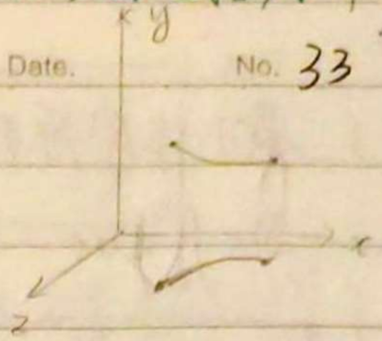
$$\Rightarrow \begin{cases} \sqrt{y} = C\sin\frac{\theta}{2} \\ \sqrt{y}\,y' = C\cos\frac{\theta}{2} \end{cases} \Rightarrow y = C^2\sin^2\frac{\theta}{2} \Rightarrow \sqrt{y}\frac{dy}{dx} = C\sin\frac{\theta}{2}C^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{d\theta}{dx}$$

$$\Rightarrow dx = C^2\sin^2\frac{\theta}{2}\,d\theta = \frac{C^2}{2}(1-\cos\theta)\,d\theta$$

$$\Rightarrow x = \frac{C^2}{2}(\theta - \sin\theta) \Rightarrow \begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{cases}$$

※当被积函数不显含t（有时是x）时，可利用雅可比积分守恒.

※在解 A ± B = 常数之方程时，可用三角式双曲函数代换求解



eg. 最小曲面.

$$S=\int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}\,dx$$

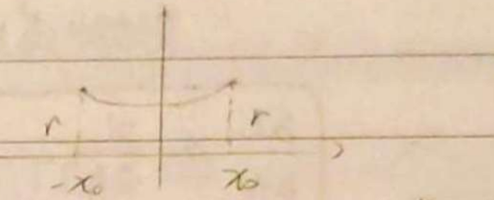
$f=y\sqrt{1+y'^2}$ 不显含x

$$h \triangleq y'\frac{\partial f}{\partial y'}-f=-\frac{y}{\sqrt{1+y'^2}}=a$$

$$\left(\frac{y}{a}\right)^2-(y')^2=1 \quad\Rightarrow\quad \begin{cases} y=a\cosh\theta \\ y'=\sinh\theta \end{cases}$$

$$\Rightarrow \frac{dy}{dx}=a\sinh\theta\cdot\frac{d\theta}{dx}=\sinh\theta \Rightarrow \frac{d\theta}{dx}=\frac{1}{a} \Rightarrow \theta=\frac{x+b}{a}$$

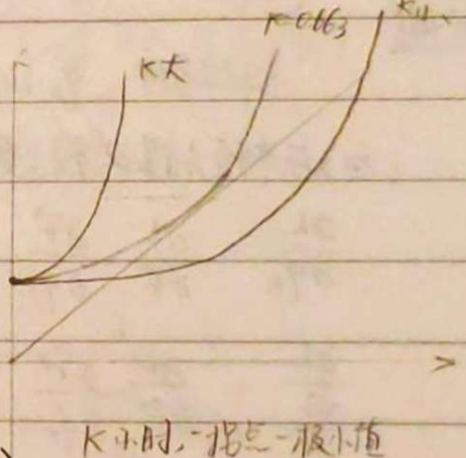
$$\Rightarrow \quad y=a\cosh\frac{x+b}{a}$$



当 b=0 时. $y=a\cosh\frac{x}{a}$

令 $u=\frac{r}{a}$, $k=\frac{x_0}{r}$

$$\Rightarrow \quad u=\cosh ku$$



当 k <0.663 时，存在两个解 a_1、a_2，不妨记 $a_1<a_2$.

k小时，一拐点一极小值

k大时，无解

$$S_{1,2}=\int_{-x_0}^{x_0} 2\pi a\cosh^2\frac{x}{a_{1,2}}\,dx$$

$$=2\pi a_{1,2}\left(x_0+a_{1,2}\sinh\frac{x_0}{a_{1,2}}\cosh\frac{x_0}{a_{1,2}}\right)$$

$$=2\pi a_{1,2}\left(x_0+r\sinh\frac{x_0}{a_{1,2}}\right)$$ 利用 $r=a_{1,2}\cosh\frac{x_0}{a_{1,2}}$

$$\frac{S_1-S_2}{2\pi a_1a_2}=\frac{x_0}{a_2}+\frac{r}{a_2}\sin\frac{x_0}{a_2}-\frac{x_0}{a_1}-\frac{r}{a_1}\sin\frac{x_0}{a_1}$$

$$=\sinh\left(\frac{x_0}{a_1}-\frac{x_0}{a_2}\right)-\left(\frac{x_0}{a_1}-\frac{x_0}{a_2}\right)$$

$$=\sinh p-p \quad . \quad p=\frac{x_0}{a_1}-\frac{x_0}{a_2}>0. \quad 故 \geq 0.$$

§3 最小作用原理

总动能

$$S=\int_{t_1}^{t_2} L\,dt \qquad L=T-U \qquad \boxed{L=L(q,\dot{q},t)=L(\vec{r}(q,t),\dot{\vec{r}}(q,\dot{q},t),t)}$$

一、两种表述：

条件极值.

$$\boxed{\begin{cases}\delta S=\delta\int_{t_1}^{t_2} L(\vec{r},\dot{\vec{r}},t)\,dt=0. \\ \delta\vec{r}(t_1)=0=\delta\vec{r}(t_2) \\ \left(\begin{cases}f(\vec{r},t)=0\\ f(\vec{r}+\delta\vec{r},t)=0\end{cases}\text{即}\right)\ \frac{\partial f_\alpha}{\partial\vec{r}}\cdot\delta\vec{r}=0\end{cases}}$$

作用量变分为0

端点相同

满足约束

在位形曲面上

极值问题

$$\boxed{\begin{cases}\delta S=\delta\int_{t_1}^{t_2} L(q,\dot{q},t)\,dt=0 \\ \delta q_k(t_1)=0=\delta q_k(t_2) \qquad k=1,\cdots,s\end{cases}}$$

作用量变分为0

相同端点

自动地满足约束

二、 $$\boxed{\frac{\delta L}{\delta q_k}\triangleq\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_k}=0} \qquad k=1,\cdots,s$$

与牛顿方程之关系

证明

初始条件 $q_k(0)$, $\dot{q}_k(0)$

（为了与牛顿方程比较，变成矢量方程）

$$\frac{\partial L}{\partial q_k}=\frac{\partial L}{\partial\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q_k}+\underline{\frac{\partial L}{\partial\dot{\vec{r}}}\cdot\frac{\partial\dot{\vec{r}}}{\partial q_k}}.$$

$$\frac{\partial L}{\partial\dot{q}_k}=\frac{\partial L}{\partial\dot{\vec{r}}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}_k}=\frac{\partial L}{\partial\dot{\vec{r}}}\frac{\partial\vec{r}}{\partial q_k}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_k}\right)=\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{r}}}\right)\cdot\frac{\partial\vec{r}}{\partial q_k}+\underline{\frac{\partial L}{\partial\dot{\vec{r}}}\cdot\frac{d}{dt}\frac{\partial\vec{r}}{\partial q_k}}$$

感觉是由于前两项恰好相消才写成了这样的形式，而不是直接就写。

故：

$$\boxed{\frac{\delta L}{\delta\vec{r}}\triangleq\frac{\partial L}{\partial\vec{r}}-\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{r}}}}$$

$$\frac{\delta U}{\delta \vec r} = \frac{\partial U}{\partial \vec r} - \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec r}} = -\vec F$$

$$\frac{\delta T}{\delta \vec r} = \frac{\partial T}{\partial \vec r} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\vec r}} = -\dot{\vec p}$$

$$\boxed{\frac{\delta L}{\delta q_k} = \frac{\delta L}{\delta \vec r}\cdot\frac{\partial \vec r}{\partial q_k} = \left(\frac{\partial L}{\partial \vec r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec r}}\right)\cdot\frac{\partial \vec r}{\partial q_k}}$$

又 $$\frac{\delta L}{\delta q_k} = \left[\left(\underbrace{\frac{\partial T}{\partial \vec r}}_{\parallel \atop 0} - \underbrace{\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec r}}}_{=-\dot{\vec p}}\right)\cdot\frac{\partial \vec r}{\partial q_k} + \left(\underbrace{-\frac{\partial U}{\partial \vec r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec r}}}_{=-\frac{\partial U}{\partial \vec r}=\vec F}\right)\cdot\frac{\partial \vec r}{\partial q_k}\right]$$

$$\vec p = \begin{pmatrix} \vec p_1 \\ \vdots \\ \vec p_n \end{pmatrix}$$

$$= (\vec F - \dot{\vec p})\cdot\frac{\partial \vec r}{\partial q_k} = 0$$ ，F是主动力.

1、拉格朗日方程是牛顿方程在位形曲面上的投影，

（投影过程中约束力在理想约束条件下消失）

2. 牛顿方程与拉朗日方程是"等价"的

确定体系位形变化时能给出相同的变化

但拉格朗日方程不包含约束力的信息（只有$s=3n-m$个方程）

牛顿方程包含约束力的信息（有$3n$个方程）.

3. 最小作用量原理中"最小"事实上指驻值路径.

δ算法：

有极好的性质 $$\boxed{\frac{\delta}{\delta q} = \frac{\partial \vec r}{\partial q}\cdot\frac{\delta}{\delta \vec r}}$$

$$\boxed{\frac{\delta}{\delta \vec r} = \frac{\partial q}{\partial \vec r}\cdot\frac{\delta}{\delta q}} \qquad \boxed{\delta \vec r = \frac{\partial \vec r}{\partial q_k}\delta q_k}$$

这并不显然！可称为"一阶变分的形式不变性" $\nearrow \frac{\partial r}{\partial q}$

证明：$$\frac{\delta}{\delta q} = \frac{\partial}{\partial q} - \frac{d}{dt}\frac{\partial}{\partial \dot q} = \frac{\partial}{\partial \vec r}\frac{\partial \vec r}{\partial q} + \frac{\partial}{\partial \dot{\vec r}}\frac{\partial \dot{\vec r}}{\partial q} - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\vec r}}\frac{\partial \dot{\vec r}}{\partial \dot q}\right)$$

$$= \frac{\partial}{\partial \vec r}\cdot\frac{\partial \vec r}{\partial q} + \frac{\partial}{\partial \dot{\vec r}}\frac{\partial \dot{\vec r}}{\partial q} - \left(\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\vec r}}\right)\cdot\frac{\partial \vec r}{\partial q}\right) - \frac{\partial}{\partial \dot{\vec r}}\cdot\frac{\partial \dot{\vec r}}{\partial q}$$

$$= \left(\frac{\partial}{\partial \vec r} - \frac{d}{dt}\frac{\partial}{\partial \dot{\vec r}}\right)\frac{\partial \vec r}{\partial q} = \frac{\delta}{\delta \vec r}\cdot\frac{\partial \vec r}{\partial q}.$$

求得自由度 s ⇒ 确定广义坐标 q ⇒ 写出 $T, U \sim q, \dot{q}$.

⇒ 确定拉格朗日函数 $L = L(q, \dot{q}, t)$

⇒ 代入拉格朗日方程 $\frac{\partial L}{\partial q_k} = 0$

eg. 自由粒子 $L = \frac{1}{2} m \dot{x}_i \dot{x}_i - U(x_1, x_2, x_3, t)$

$$= \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r}, t)$$

$\frac{\partial L}{\partial \vec{r}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}}$ 即 $-\frac{\partial U}{\partial \vec{r}} = m \ddot{\vec{r}}$ $\vec{F} = m\vec{a}$

eg. $L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2\theta) - U(r, \theta, \phi, t)$

$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 + m r \dot{\phi}^2 \sin^2\theta - \frac{\partial U}{\partial r} = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m \ddot{\vec{r}}$

即 $F_r = m r \dot{\theta}^2 + m r \dot{\phi}^2 \sin^2\theta - m \ddot{\vec{r}}$

ϕ 球坐标下 $-\frac{\partial U}{\partial \phi} = -\frac{\partial U}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial \phi} = \vec{F} \cdot r \hat{\theta}$ 别补

$\frac{\partial L}{\partial \theta} = m r^2 \dot{\phi}^2 \sin\theta \cos\theta - \frac{\partial U}{\partial \theta}$ 而 $-\frac{\partial U}{\partial \theta} = -\frac{\partial U}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial \theta} = \vec{F} \cdot r \hat{\theta}$

$= \hat{\phi} \cdot (\vec{r} \times \vec{F})$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} (\underbrace{m r^2 \dot{\theta}}_{\hat{\phi} \cdot \vec{L}})$

$\hat{\phi} \cdot \vec{\tau} = \hat{\phi} \cdot \frac{d\vec{L}}{dt} = \frac{d}{dt}(\hat{\phi} \cdot \vec{L}) - \frac{d\hat{\phi}}{dt} \cdot \vec{L}$

角动量定理在 ϕ 向投影

$\frac{\partial L}{\partial \phi} = -\frac{\partial U}{\partial \phi}$ $-\frac{\partial U}{\partial \phi} = -\frac{\partial U}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial \phi} = \vec{F} \cdot r \sin\theta \hat{\phi}$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt} (m r^2 \dot{\phi} \sin^2\theta)$ $= \hat{z} \cdot (\vec{r} \times \vec{F})$

$$\hat{z} \cdot \vec{\tau} = \hat{z} \cdot \frac{d\vec{L}}{dt}$$

角动量定理在z向投影.

三、术语

1. 广义力

广义主动力 $\boxed{Q_k \triangleq \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_k} \xrightarrow{\vec{F}=-\frac{\partial U}{\partial \vec{r}}} = -\frac{\partial U}{\partial q_k}}$

跟体系整体有关，"广义"指由其广义坐标决定，不单关心单个粒子。

广义约束力 $\boxed{Q_k' \triangleq \vec{N} \cdot \frac{\partial \vec{r}}{\partial q_k}}$ $\xrightarrow{\text{理想约束假设}} 0$

2. 与q_k共轭的广义动量 $\boxed{p_k \triangleq \frac{\partial L}{\partial \dot{q}_k}} = p_k(q, \dot{q}, t)$

①拉格朗日方程变为：$\boxed{\dot{p}_k = \frac{\partial L}{\partial q_k}}$

②若q_k为L的循环坐标（不显含于L），则$p_k = const.$

例：$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) - U(r)$

p_r 线动量

p_θ 角动量在$\hat{\phi}$的投影.

p_ϕ 角动量在$\hat{z}$的投影

3. $L(q, \dot{q}, t)$的 Jacobi 积分：

$$\boxed{h \triangleq \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = h(q, \dot{q}, t)}$$

→一次项.

①$L = T - U = T_2 + T_1 + (T_0 - U)$ 此时仅考虑 $U(\vec{r})$

$= L_2 + L_1 + L_0$

$\Rightarrow h = 2T_2 + T_1 - (T_2 + T_1 + T_0 - U)$ 利用欧拉定理.

$= T_2 - T_0 + U$ 一般不等于机械能.

$$\boxed{h = L_2 - L_0}$$

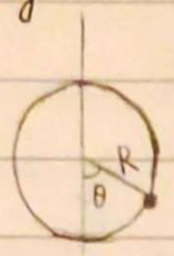
若 $\underline{\vec{r}=\vec{r}(q)}$，不显含 t，此时 $T=T_2$，有：

$$h = T + U = E \leftarrow \text{机械能}.$$

② $\boxed{\frac{dh}{dt} = -\frac{\partial L}{\partial t}}$，若 $\underline{L = L(q, \dot{q}, \cancel{t})}$ 不显含 t，

此时 h 将守恒.

eg.



$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$$

约束条件 $r = R$，$\phi = \omega$

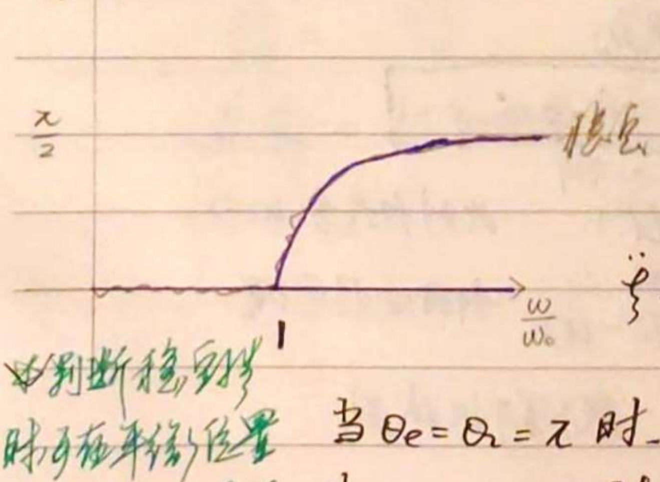
$$\Rightarrow T = \frac{1}{2} m(\dot{r}^2\theta^2 + R^2\omega^2\sin^2\theta)$$

故 $L = \frac{1}{2} mR^2\dot{\theta}^2 + \frac{1}{2} mR^2\omega^2\sin^2\theta + mgR\cos\theta$.

$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \theta}$：　$mR^2\ddot{\theta} = -mgR\sin\theta + mR^2\omega^2\sin\theta\cos\theta$

$$\Rightarrow \ddot{\theta} = -(\omega_0^2 - \omega^2\cos\theta)\sin\theta. \quad \omega_0^2 \triangleq \frac{g}{R}$$

1. 平衡位置 $\dot{\theta}_e = 0 = \ddot{\theta}_e$：$\theta_1 = 0$，$\theta_2 = \pi$，$\theta_3 = \arccos\frac{\omega_0^2}{\omega^2}$



2. 稳定性 记 $f(\theta) = -(\omega_0^2 - \omega^2\cos\theta)\sin\theta$

$$\theta = \theta_e + \xi$$

$$\ddot{\xi} = \overset{=0}{\cancel{f(\theta_e)}} + \xi f'(\theta_e)$$

$$\ddot{\xi} = -\Omega^2\xi, \quad \Omega^2 = \omega_0^2\cos\theta_e \ \omega^2\cos 2\theta_e$$

判断稳定性时可在平衡位置附近用 $\xi(\ll)$ 展开，代入其运动方程看系数正负.

当 $\theta_e = \theta_2 = \pi$ 时，$\Omega^2 = -\omega_0^2 - \omega_0^2 < 0$ 不稳定

当 $\theta_e = \theta_1 = 0$ 时，$\Omega^2 = \omega_0^2 - \omega^2$.

当 $\omega < \omega_0$ 稳定，当 $\omega > \omega_0$ 不稳定.

当 $\theta_e=\theta_3$，$\Omega^2=\frac{\omega^4-\omega_0^4}{\omega^2}>0 \Rightarrow \omega>\omega_0$. 稳定.

若 $\omega=\omega_0$，则 $\ddot{\theta}=-\omega_0^2(1-\cos\theta)\sin\theta \doteq -\frac{1}{2}\omega_0^2\theta^3$

非线性振动. 周期与振幅有关.

$h = L_2 - L_0 = \frac{1}{2}mR^2\dot{\theta}^2-\frac{1}{2}mR^2\omega^2\sin^2\theta-mgR\cos\theta$

四. 动力学含义

五. Lagrange 函数之一般性质.

1. 惯性系.

转动参考系中，考虑 $\vec{F}'=-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

$$U'=-\int\vec{F}'\cdot d\vec{r}=-m\int(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times d\vec{r})$$

$$=-m\int(\vec{\omega}\times\vec{r})\cdot d(\vec{\omega}\times\vec{r})=-\frac{1}{2}m(\vec{\omega}\times\vec{r})^2$$

$$\boxed{U'=-\frac{1}{2}m\omega^2R^2\sin^2\theta}$$

不影响 Lagrange 函数之形式、

一般求解时，选惯性系求解.

※ 写Lagrange 函数时在惯性系中写是关

2. 不确定性

① 可做一个标度变换

② 可做一个规范变换

$$\boxed{\begin{array}{l} L \Leftrightarrow cL \\ L \Leftrightarrow L + \frac{dF(q,t)}{dt} \end{array}}$$

$$S' = S + \text{const} \quad (\text{作用量})$$

③ 可加性、 $L = T_A + T_B - U_A - U_B - U_{A+B}$

当相距足够远时

则 $L = L_A(q_A, \dot{q}_A) + L_B(q_B, \dot{q}_B)$

例：两体问题

$$L = \frac{1}{2} m_1 \vec{r}_1^{\,2} + \frac{1}{2} m_2 \vec{r}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

取质心 $\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$ ， $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$

$$\Rightarrow L = \underbrace{\frac{1}{2} M \vec{R}^{\,2}}_{L_1} + \underbrace{\frac{1}{2} \mu \vec{r}_{12}^{\,2} - U(\vec{r}_{12})}_{L_2}$$

§4. 与速度有关之力. 前提: $L = T-U$.

（仍假设L有T-U之形式）.

$$\begin{cases} \vec{F}(r,\dot{\vec{r}},t)+\vec{N}-\dot{\vec{p}}=0 \\ \vec{N}\cdot\frac{\partial \vec{r}}{\partial q_k}=0 \end{cases}$$

$$\frac{\delta L}{\delta q_k} \triangleq \frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}=\frac{\delta L}{\delta \vec{r}}\cdot\frac{\partial \vec{r}}{\partial q_k}=\left(\frac{\partial L}{\partial \vec{r}}-\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}}\right)\cdot\frac{\partial \vec{r}}{\partial q_k}$$

$$\frac{\delta L}{\delta q_k}=\frac{\delta T}{\delta q_k}-\frac{\delta U}{\delta q_k}=\left[\underbrace{\left(\overset{=0}{\frac{\partial T}{\partial \vec{r}}}-\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{r}}}\right)}_{\overset{\parallel}{-\dot{\vec{p}}}}+\left(-\frac{\partial U}{\partial \vec{r}}+\frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{r}}}\right)\right]\frac{\partial \vec{r}}{\partial q_k}$$

一、势力 $\boxed{\vec{F}=-\frac{\partial U}{\partial \vec{r}}+\frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{r}}}=-\frac{\delta U}{\delta \vec{r}}}$

$$\boxed{Q_k=\vec{F}\cdot\frac{\partial \vec{r}}{\partial q_k}=-\frac{\partial U}{\partial q_k}+\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_k}=-\frac{\delta U}{\delta q_k}}$$

1、广义势能 $U=U(\vec{r},\dot{\vec{r}},t)=U''(q,\dot{q},t)$

2、不确定性 $U'=U-\frac{d\,G(q,t)}{dt}$ 给出同样之力.

eg. Lorentz 力 $\boxed{\vec{F}=e\vec{E}+e\vec{v}\times\vec{B}}$

$$\vec{E}=-\nabla\varphi-\partial_t\vec{A},\quad \vec{B}=\nabla\times\vec{A}$$

$$\vec{F}=-e\nabla\varphi-e\partial_t\vec{A}+e\vec{v}\times(\nabla\times\vec{A})$$

而 $[\vec{v}\times(\nabla\times\vec{A})]_i=\varepsilon_{ijk}\dot{x}_j(\nabla\times\vec{A})_k$

$$=\varepsilon_{ijk}\varepsilon_{mnk}\dot{x}_j\partial_m A_n$$

$$=\dot{x}_j\partial_i A_j-\dot{x}_j\partial_j A_i=\partial_i(\vec{v}\cdot\vec{A})-\dot{x}_j\partial_j A_i$$

故 $\vec{E} = -\frac{\partial\varphi}{\partial\vec{r}} - e\frac{\partial\vec{A}}{\partial t} + e\frac{\partial(\vec{v}\cdot\vec{A})}{\partial\vec{r}} - e\dot{x}_j\frac{\partial\vec{A}}{\partial x_j}$

$= -\frac{\partial}{\partial\vec{r}}(e\varphi - e\vec{v}\cdot\vec{A}) - e\frac{d}{dt}(\vec{A})$

注意到 φ 对 $\vec{v}$ 无关，$\frac{d}{dt}\frac{\partial}{\partial\vec{v}}(e\varphi - e\vec{v}\cdot\vec{A}) = -e\frac{d}{dt}\vec{A}$

（利用了 $\frac{\partial(\vec{F}\cdot\vec{r})}{\partial\vec{r}} = \vec{F}$）

(1) 故找到广义势能 $\boxed{U = e(\varphi - \vec{v}\cdot\vec{A})}$

(2) 其 Lagrange 方程 $L = T - V = \frac{1}{2}mv^2 + e\vec{v}\cdot\vec{A} - e\varphi$

直角坐标中，$p_k = \frac{\partial L}{\partial\dot{x}_k} = m\dot{x}_k + eA_k$

雅可比积分 $h = L_2 - L_0 = \frac{1}{2}mv^2 + e\varphi$.

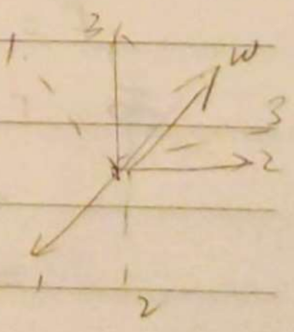
当电磁场随时间变化时 φ 不再直接与电势能（能量）关联.

(3) 规范变换. $\boxed{\varphi' = \varphi - \partial_t\psi(\vec{r},t)}\ \boxed{\vec{A}' = \vec{A} + \nabla\psi}$

$U' = U - e\frac{\partial\psi}{\partial t} - e\dot{x}_j\frac{\partial\psi}{\partial x_j} = U - \frac{d}{dt}(e\psi(\vec{r},t))$

$\boxed{L' = L + \frac{d}{dt}(e\psi(\vec{r},t))}$

例 $\vec{\omega} = \vec{\omega}(t)$. $\Rightarrow \vec{v} = \vec{v}' + \vec{\omega}\times\vec{r}$



$L = \frac{1}{2}mv^2 - U$

$= \frac{1}{2}mv'^2 + \frac{1}{2}m(\vec{\omega}\times\vec{r})^2 + m\vec{v}'\cdot(\vec{\omega}\times\vec{r}) - U$

(3) $\boxed{U' = -\frac{1}{2}m(\vec{\omega}\times\vec{r})^2 - m\vec{v}'\cdot(\vec{\omega}\times\vec{r})}$,

是转动惯性系中惯性力对应之广义势

※ $(\vec\omega\times\vec r)^2$ 写成 $[\omega^2r^2-(\vec\omega\cdot\vec r)^2]$

※ Lagrange 是前面动能在哪个系写，后面U势能也要在哪个系写

$$U'=-\frac{1}{2}m(\omega^2r^2-(\vec\omega\cdot\vec r)^2)-m\vec r\cdot(\vec v'\times\vec\omega)$$

$$-\frac{\partial U'}{\partial \vec r'}+\frac{d}{dt}\left(\frac{\partial U'}{\partial \dot{\vec r}'}\right)=m[\omega^2\vec r'-(\vec\omega\cdot\vec r')\vec\omega]+m\vec v'\times\vec\omega-m\frac{d(\vec\omega\times\vec r)}{dt}$$

$$=-m\vec\omega\times(\vec\omega\times\vec r')-2m\vec\omega\times\vec v'-m\dot{\vec\omega}\times\vec r'$$

這裡我將$\vec r$，$\dot{\vec r}$全加了撇。將$T=\frac{1}{2}m\vec v'^2$时，为了$\vec v'$函数，相当于已经在S'系中了，变量是$\vec v'$，$\vec r'$，t，也应对这些操作。

二、含有耗散力时

若引入耗散力$\vec F^D$，不能用上面广义势表示，其余力$\vec F^P$仍用广义势表示

$$0=(\vec F^P+\vec F^D+N-\dot{\vec p})\frac{\partial\vec r}{\partial q_k}$$

$$=\underbrace{(\vec F^P-\dot{\vec p})\cdot\frac{\partial\vec r}{\partial q_k}}_{\frac{\delta L}{\delta q_k}}+\underbrace{\vec F^D\cdot\frac{\partial\vec r}{\partial q_k}}_{D_k}+\vec N\cdot\frac{\partial\vec r}{\partial q_k}=0$$

即变为 $\boxed{\frac{\delta L}{\delta q_k}+D_k=\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial\dot q_k}+D_k=0}$

或：$\boxed{\frac{d}{dt}\frac{\partial L}{\partial\dot q_k}=\frac{\partial L}{\partial q_k}+D_k}$ $\boxed{D_k=\vec F^D\cdot\frac{\partial\vec r}{\partial q_k}}$ 广义耗散力.

此时仅Lagrange函数不足以描述动力学体系之性质，必须补充$\vec F^D$之性质.

eg. $\boxed{\vec F^D=-g(v)\hat v}$

1、定义耗散函数 $\boxed{\mathcal{F}\triangleq\int_0^v g(\xi)d\xi=\mathcal{F}(v)}$

故 $\vec F^D=-\frac{\partial\mathcal{F}}{\partial\vec v}$ $\left(=-\frac{\partial\mathcal{F}}{\partial v}\frac{\partial v}{\partial\vec v}\right)$

$\Rightarrow$ $\boxed{D_k=\vec F^D\cdot\frac{\partial\dot{\vec r}}{\partial\dot q_k}=-\frac{\partial\mathcal{F}}{\partial\vec v}\cdot\frac{\partial\vec v}{\partial\dot q_k}=-\frac{\partial\mathcal{F}}{\partial\dot q_k}}$

$$\mathcal{F} = \mathcal{F}(v(q,\dot q,t)) = \mathcal{F}(q,\dot q,t)$$

2. $$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q_k} = \frac{\partial L}{\partial q_k} - \frac{\partial \mathcal{F}}{\partial \dot q_k}}$$

3. Rayleigh 函数

$$\mathcal{F} = \frac{1}{2}\beta v^2 \Rightarrow \vec{F}^D = -\beta\vec{v}$$

$$\Rightarrow \vec{F}^D\cdot\vec{v} = -\beta v^2 = 2\mathcal{F}$$

eg. $L = \frac{1}{2}m(\dot x^2 - \omega_0^2 x^2)$, $\mathcal{F} = \frac{1}{2}\beta\dot x^2$

则 $m\ddot x = -m\omega_0^2 x - \beta\dot x$

$$\rightarrow \ddot x + 2\gamma\dot x + \omega_0^2 x = 0, \quad \gamma = \frac{\beta}{2m}$$

(注：若不限制 $L = T-U$ 之限制，此种耗散力情况仍可纳入到最小作用量原理之范畴. 若该具体设置. 适用将更加普遍)

(上例中，亦可定义 $L = e^{2\gamma t}\left[\frac{1}{2}m(\dot x^2 - \omega_0^2 x^2)\right]$

eg. $$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - U(\vec r,t)}$$ 相对论粒子.

$$p = \frac{\partial L}{\partial \vec v} = \frac{\partial L}{\partial v}\frac{\partial v}{\partial \vec v} = -mc^2\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(-\frac{1}{2}\frac{2v}{c^2}\right)\hat v = \boxed{\frac{m\vec v}{\sqrt{1-v^2/c^2}} = \vec p}$$

即相对论动量.

$$\frac{d}{dt}\frac{\partial L}{\partial \vec v} = \frac{\partial L}{\partial \vec r} \Rightarrow \dot p = -\frac{\partial U}{\partial \vec r} = \vec F$$

常用: $\frac{\partial}{\partial \vec v} = \frac{\partial}{\partial v}\cdot\frac{\partial v}{\partial \vec v}$, $\frac{\partial}{\partial v} = \frac{\partial}{\partial \vec v}\cdot\frac{\partial \vec v}{\partial v}$

$\frac{\partial}{\partial \vec r} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial \vec r}$, $\frac{\partial}{\partial r} = \frac{\partial}{\partial \vec r}\cdot\frac{\partial \vec r}{\partial r}$

补：虚功原理与 d'Alembert 原理

一、术语

1. 虚位移 δr 满足约束

$$\boxed{\delta \vec{r} = \frac{\partial \vec{r}}{\partial q_k}\delta q_k}$$ （等时变分）

2. 虚功

主动力 $\boxed{\delta A \triangleq \vec{F}\cdot\delta\vec{r} = \overset{\text{广义力}}{Q_k}\,\delta q_k}$ ✓

约束力 $\boxed{\delta A' \triangleq \vec{N}\cdot\delta\vec{r} = Q_k'\,\delta q_k.}$ ✓

惯性力 $\boxed{-\dot{\vec{p}}\cdot\delta\vec{r} = \left(-\frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{r}}}\right)\cdot\frac{\partial \vec{r}}{\partial q_k}\cdot\delta q_k = \left(\frac{\partial T}{\partial \vec{r}} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\vec{r}}}\right)\cdot\frac{\partial \vec{r}}{\partial q_k}\cdot\delta q_k = \frac{\delta T}{\delta \vec{r}}\cdot\frac{\partial \vec{r}}{\partial q_k}\delta \vec{q_k} = \delta T}$

3. 理想约束假设 $\delta A' = 0 = Q_k'\,\delta q_k \Rightarrow Q_k' = 0$ $(k=1,\dots,s)$

（约束力所做虚功总和为0。）

差了个符号，即P47

$\frac{\delta T}{\delta r}$ · δr × δT

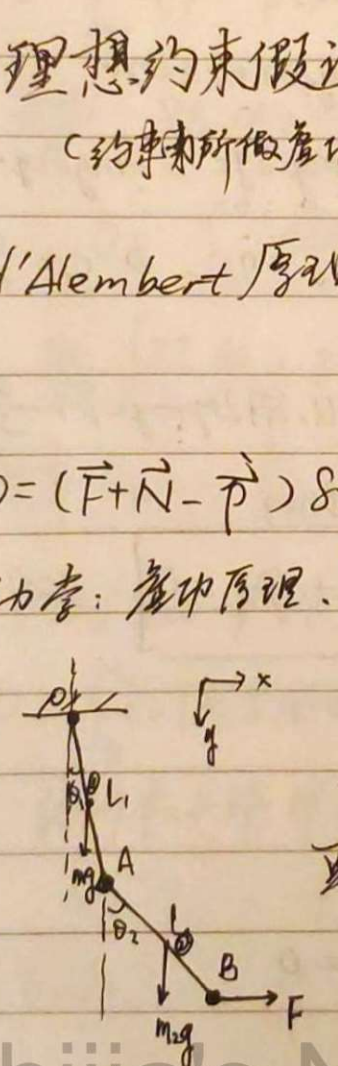
二、d'Alembert 原理 $\boxed{(\vec{F}-\dot{\vec{p}})\delta\vec{r} = 0}$

即 $\boxed{\left(Q_k + \frac{\delta T}{\delta q_k}\right)\delta q_k = 0}$

$$\left(0 = (\vec{F}+\vec{N}-\dot{\vec{p}})\delta\vec{r} = (\vec{F}+\vec{N}-\dot{\vec{p}})\frac{\partial \vec{r}}{\partial q_k}\delta q_k = 0.\right)$$

"静"力学：虚功原理. $\boxed{\vec{F}\cdot\delta\vec{r} = Q_k\,\delta q_k = 0}$

eg.



$$m_1\vec{g}\cdot\delta\vec{r_1} + m_2\vec{g}\cdot\delta\vec{r_2} + \vec{F}\cdot\delta\vec{r_B} = 0$$

$$= m_1 g\,\delta y_1 + m_2 g\,\delta y_2 + F\,\delta x_B = 0$$

~~再~~用独立二坐标表达出来.

$y_1 = \frac{1}{2}l_1\cos\theta_1$, $y_2 = l_1\cos\theta_1 + \frac{1}{2}l_2\cos\theta_2$

$x_B = l_1\sin\theta_1 + l_2\sin\theta_2$

四 或 Lagrange 方程，写出 L，用 $\frac{\partial L}{\partial q_k} = 0$.

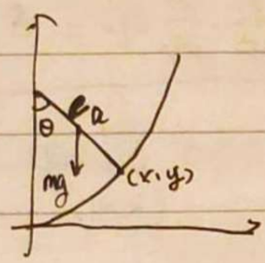
代入整理原式.

$$-\frac{1}{2}m_1 l_1 \sin\theta_1 \delta\theta_1 - m_2 g(l_1\sin\theta_1\delta\theta_1 + \frac{1}{2}l_2\sin\theta_2\delta\theta_2)$$
$$+ F(l_1\cos\theta_1\delta\theta_1 + l_2\cos\theta_2\delta\theta_2) = 0$$

$$\begin{cases} -(\frac{1}{2}m_1 + m_2)g l_1\sin\theta_1 + F l_1\cos\theta_1 = 0 \\ -\frac{1}{2}m_2 g l_2\sin\theta_2 + F l_2\cos\theta_2 = 0 \end{cases}$$

$$\Rightarrow \tan\theta_1 = \frac{F l_1}{(\frac{1}{2}m_1 + m_2)g} \qquad \tan\theta_2 = \frac{2F}{m_2 g}.$$

eg.



$x = a\sin\theta, \quad y = y(\theta)$

$x_c = \frac{1}{2}a\sin\theta, \quad y_c = y(\theta) + \frac{1}{2}a\cos\theta.$

$0 = m\vec{g}\cdot\delta\vec{r_c} = -mg\,\delta y_c = -mg\,\delta(y + \frac{1}{2}a\cos\theta)$

故 $y + \frac{1}{2}a\cos\theta = C$, 又 $y|_{\theta=0} = 0$. $\Rightarrow C = \frac{1}{2}a$

$\Rightarrow y = \frac{1}{2}a(1-\cos\theta)$

(回到 写势能 U, 用 Lagrange 方程 $\frac{\partial U}{\partial q} = 0$.

三. Lagrange 方程 (由 D'Alembert 原理推)

$$(Q_k + \frac{\delta T}{\delta q_k})\delta q_k = 0 \Rightarrow \boxed{\frac{\delta T}{\delta q_k} + Q_k = 0}. \quad k = (1, \cdots, s)$$

(静力学 $Q_k = 0$)

若 $\vec{F} = -\frac{\delta U}{\delta \vec{r}}$, 或者 $Q_k = -\frac{\delta U}{\delta q_k}$.

得到 $\frac{\delta L}{\delta q_k} = \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0$

(静力学 $\frac{\delta U}{\delta q_k} = 0$, 若 $\vec{F} = -\nabla U$, 则 $\frac{\partial U}{\partial q_k} = 0$ 平衡方程)

ssj注：$\frac{\delta T}{\delta \vec{r}}\delta\vec{r} \neq \delta T$，$\delta A \neq -\delta u$. 均相差一个全微分项，或言，它们只有在积分意义下才相等。

四、最小原理.

$$0=(\vec{F}-\dot{\vec{p}}) \Rightarrow 0=(\vec{F}-\dot{\vec{p}})\delta\vec{r} \Rightarrow 0=(\vec{F}-\dot{\vec{p}})\cdot\delta\vec{r}\,dt$$

$$\Rightarrow 0=\int_{t_1}^{t_2}(\vec{F}-\dot{\vec{p}})\delta\vec{r}\,dt$$

（注意到）$-\dot{\vec{p}}\cdot\delta\vec{r}=\frac{\delta T}{\delta q_k}\delta q_k=\frac{\partial T}{\partial q_k}\delta q_k-\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}\right)\delta q_k$

$$=\frac{\partial T}{\partial q_k}\delta q_k+\frac{\partial T}{\partial \dot{q}_k}\delta\dot{q}_k-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\delta q_k\right)$$

$$=\delta T(q,\dot{q},t)-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\delta q_k\right),$$

$$0=\int_{t_1}^{t_2}(\vec{F}-\dot{\vec{p}})\delta\vec{r}\,dt=\int_{t_1}^{t_2}(\delta A+\delta T)dt-\frac{\partial T}{\partial \dot{q}_k}\delta q_k\Big|_{t_1}^{t_2}$$

要求 $\delta q_k(t_1)=0=\delta q_k(t_2)$

$$\Rightarrow 0=\int_{t_1}^{t_2}(\delta A+\delta T)dt \quad \text{（一般情况）}.$$

因 $\int\delta T\,dt=\delta\int T\,dt$，但一般 $\int\delta A\,dt\neq\delta\int A\,dt$.

但当 $Q_K=-\frac{\delta u}{\delta q_k}$ 时，$\delta A=Q_k\delta q_k=-\frac{\delta u}{\delta q_k}\delta q_k=-\delta u$.

此时可用一个标量函数描述广功，有：　　也差$\frac{d}{dt}$一!

$$0=\int_{t_1}^{t_2}(\delta T+\delta A)dt=\int_{t_1}^{t_2}\delta(T-u)\,dt=\delta\int_{t_1}^{t_2}(T-u)\,dt,$$

得到最小作用量原理.

$$Q_k\delta q_k=+\vec{F}\cdot\frac{\partial\vec{r}}{\partial q_k}\delta q_k=+\vec{F}\cdot\delta\vec{r}=-\delta u$$

$$\delta A=Q_k\delta q_k=-\frac{\delta u}{\delta q_k}\delta q_k=\left(-\frac{\partial u}{\partial q_k}+\frac{d}{dt}\left(\frac{\partial u}{\partial \dot{q}_k}\right)\right)\delta q_k$$

$$=-\frac{\partial u}{\partial q_k}\delta q_k-\frac{\partial u}{\partial \dot{q}_k}\delta\dot{q}_k+\frac{d}{dt}\left(\frac{\partial u}{\partial \dot{q}_k}\delta q_k\right)$$

$$=-\delta u+\frac{d}{dt}\left(\frac{\partial u}{\partial \dot{q}_k}\delta q_k\right)$$

§5 Lagrange 乘子方法

一、如何求解约束 $f(\vec{r},t)$ 提供的约束力 $\vec{N}$？

$0=\vec{F}^{p}+\vec{N}-\dot{\vec{p}}$.　$\vec{F}^{p}=-\frac{\delta U}{\delta r}=-\frac{\partial U}{\partial r}+\frac{d}{dt}\frac{\partial U}{\partial \dot{r}}$ ✓

$-\dot{\vec{p}}=-\frac{d}{dt}\frac{\partial T}{\partial \dot{r}}=\frac{\delta T}{\delta r}$ ✓

$$\vec{N}\parallel\frac{\partial f}{\partial r}\Rightarrow\boxed{\vec{N}=\lambda\frac{\partial f}{\partial r}}$$

1、设想解除该约束

$\vec{r}=\vec{r}(q_1,\cdots,q_s,q_{s+1},t)$

$f(\vec{r},t)=f(\vec{r}(q,t),t)=f(q,t)=0$ （不再自然满足）

（当用s个独立坐标时，本应是自然满足约束方程的）

eg. $f=z-\frac{x^2+y^2}{a}=0$，$\vec{r}=x\hat{x}+y\hat{y}+\frac{x^2+y^2}{a}\hat{z}$

↙ 此时 $\frac{\partial \vec{r}}{\partial x},\frac{\partial \vec{r}}{\partial y}$ 均为曲面的切矢量

① 解除该约束，使 $q=(x,y,z)$，$\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$

$\frac{\partial \vec{r}}{\partial x}=\hat{x}$，$\frac{\partial \vec{r}}{\partial y}=\hat{y}$，$\frac{\partial \vec{r}}{\partial z}=\hat{z}$

在这三个方向上投影，可保留约束力信息了

② 使 $q=(s,\phi,z)$

$\frac{\partial \vec{r}}{\partial s}=\hat{s}$，$\frac{\partial \vec{r}}{\partial \phi}=s\hat{\phi}$，$\frac{\partial \vec{r}}{\partial z}=\hat{z}$

故此时，$\frac{\partial \vec{r}}{\partial q_k}$ 未必与位形曲面相切.

2. Newton方程在 $\frac{\partial \vec{r}}{\partial q_k}$ 上之"投影".

$$0=(\vec{F}^{P}+\vec{N}-\dot{\vec{p}})\frac{\partial \vec{r}}{\partial q_k}=\frac{\delta L}{\delta q_k}+\underline{N\cdot\frac{\partial \vec{r}}{\partial q_k}}$$

$$\boxed{Q_k'=\lambda\frac{\partial f}{\partial \vec{r}}\cdot\frac{\partial \vec{r}}{\partial q_k}=\lambda\frac{\partial f}{\partial q_k}}$$

$$\boxed{\vec{N}=\lambda_j\nabla f_j}$$

3. 方程.

$$\boxed{\begin{cases}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}=\lambda\frac{\partial f}{\partial q_k}\\ f(q,t)=0\end{cases}}\qquad (k=1,2,\cdots,s+1)$$

$f=f(q,t)$

约束力之虚功：$\delta A'=\vec{N}\cdot\delta\vec{r}=Q_k'\delta q_k=\lambda\frac{\partial f}{\partial q_k}\delta q_k=\lambda\delta f=0$

（故之虚功为0，但δq_k不独立，故Q_k'不一定为0）

☆偏导数做完之前不能代入约束方程，

会改变Lagrange方程之形式（对q_k、q_k之依赖关系）

※4. 方程也可写为 $\boxed{\tilde{L}=L+\lambda f}=T-(U-\lambda f)=\tilde{L}(q,\lambda,\dot{q},t)$

此时形式上，$\frac{\delta\tilde{L}}{\delta q_k}=0$ ，$\frac{\delta\tilde{L}}{\delta\lambda}=\frac{\partial\tilde{L}}{\partial\lambda}=f=0$.

把$-\lambda f$看做"约束力势能)

由于λf不依赖于速度，

$$\tilde{p}_k \triangleq \frac{\partial \tilde{L}}{\partial \dot{q}_k} = p_k \quad \tilde{p}_\lambda \triangleq \frac{\partial \tilde{L}}{\partial \dot{\lambda}} = 0$$

$$\tilde{h} = \tilde{p}_k \dot{q}_k + \tilde{p}_\lambda \dot{\lambda} - \tilde{L}$$

$$= p_k \dot{q}_k - L - \lambda f = h - \lambda f$$

若L和f都不显含q_k，$\tilde{p}_k$即p_k守恒

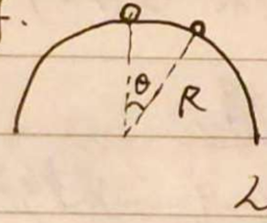
若L和f都不显含t，$\tilde{h} = h - \lambda f$在真实运动中$(f=0)$守恒

故仍有 $h = L_2 - L_0$ 守恒.

5. 当有m个约束时

$$\boxed{\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_k} - \dfrac{\partial L}{\partial q_k} = \sum\limits_{\alpha=1}^{m} \lambda_\alpha \dfrac{\partial f_\alpha}{\partial q_k} \\ f_\alpha = 0 \quad , \alpha = 1, \cdots, m \end{cases}}$$

eg.



$q = (r, \theta)$

设想没有约束，

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - mgr\cos\theta$$

$$f = r - R = 0$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} - \dfrac{\partial L}{\partial r} = \lambda \dfrac{\partial f}{\partial r} \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} - \dfrac{\partial L}{\partial \theta} = \lambda \dfrac{\partial f}{\partial \theta} \end{cases} \Rightarrow \begin{cases} m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda = Q_r' = N_n \text{ ①} \\ \dfrac{d}{dt}(mr^2\dot{\theta}) - mgr\sin\theta = 0 = Q_\theta' = \tau_\theta \text{ ②} \end{cases}$$

② $\Rightarrow mR^2\ddot{\theta} = mgR\sin\theta \Rightarrow R\dot{\theta}\dfrac{d\dot{\theta}}{dt} = g\sin\theta\dfrac{d\theta}{dt} \Rightarrow \frac{1}{2}R\dot{\theta}^2 + g\cos\theta = g$

将$R\dot{\theta}^2 = 2g(1-\cos\theta)$代入①：$\lambda = mg\cos\theta - mR\dot{\theta}^2 = mg(3\cos\theta - 2) = N_n$

ssj注: 当解出 $\lambda=0$ 时, 沈惠川 p124.

或: L不显含t, $h = L_2 - L_0$ 守恒得 $\frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta = mgR$.

或 $q=(x,y)$, $L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)-mgy$, $f=x^2+y^2-R^2=0$.

$$\begin{cases} m\ddot{x} - 0 = 2\lambda x & ① \\ m\ddot{y} + mg = 2\lambda y & ② \end{cases}$$

$f: x^2+y^2=R^2 \Rightarrow x\dot{x}+y\dot{y}=0 \Rightarrow x\ddot{x}+y\ddot{y}+\dot{y}^2+\dot{x}^2=0$

L, f 不显含时间t, $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+mgy=mgR$ (机械能守恒,

$x①+y②$: $m(x\ddot{x}+y\ddot{y})+mgy=2\lambda R^2$

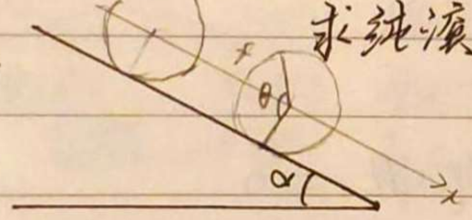
$\Rightarrow -m(\dot{x}^2+\dot{y}^2)+mgy=2\lambda R^2$

$\Rightarrow 2\lambda R^2 = mg(3y-2R)$

$\Rightarrow \lambda = mg\frac{3y-2R}{2R^2}$

$\Rightarrow \vec{N} = \lambda\frac{\partial f}{\partial x}\hat{x}+\lambda\frac{\partial f}{\partial y}\hat{y} = 2\lambda\vec{r} = mg(\frac{3y}{R}-2)\hat{r}$

eg.



求纯滚动约束提供之约束力.

$q=(x,\theta)$ $f=x-R\theta=0$

$L=\frac{1}{2}m\dot{x}^2+\frac{1}{4}mR^2\dot{\theta}^2+mgx\sin\alpha$

$$\begin{cases} m\ddot{x}-mg\sin\alpha = \lambda\frac{\partial f}{\partial x} = \lambda = Q_x' = N_x & ① \\ \frac{1}{2}mR^2\ddot{\theta}-0 = \lambda\frac{\partial f}{\partial \theta} = -\lambda R = Q_\theta' = \tau_\theta & ② \end{cases}$$

$② \Rightarrow \lambda = -\frac{1}{2}mR\ddot{\theta} = -\frac{1}{2}m\ddot{x}$ 代入① $\Rightarrow \ddot{x} = \frac{2}{3}g\sin\alpha$

$N_x = \lambda = -\frac{1}{3}mg\sin\alpha < 0$

$\tau_\theta = -\lambda R = \frac{1}{3}mgR\sin\alpha > 0.$

二. 最小作用原理

$$\left\{\begin{array}{l} \left.\begin{array}{l} \delta S = \delta\int L(q,\dot{q},t)\,dt = 0 \\ \delta q_k(t_1) = 0 = \delta q_k(t_2) \end{array}\right\} \Leftrightarrow \int_{t_1}^{t_2} \frac{\delta L}{\delta q_k}\,\delta q_k\,dt = 0 \\ \delta f = \frac{\partial f}{\partial q_k}\,\delta q_k = 0 \qquad ③ \end{array}\right.$$

$\Updownarrow$

对 $\forall\ \lambda(t)$，$\int_{t_1}^{t_2} \lambda \frac{\partial f}{\partial q_k}\,\delta q_k\,dt = 0$

故 $\Leftrightarrow \int_{t_1}^{t_2} \left[\frac{\delta L}{\delta q_k} + \lambda\frac{\partial f}{\partial q_k}\right]\delta q_k\,dt = 0$

必然有一个 $\frac{\partial f}{\partial q_k} \neq 0$，不妨假设 $\frac{\partial f}{\partial q_{s+1}} \neq 0$.

$\Rightarrow$ δq_{s+1} 可用 $\delta q_1, \cdots, \delta q_s$ 来表示

(又 $s+1$ 个变量仅一个约束③，故此假设下剩下的 s 个独立)

由 $\lambda = \lambda(t)$ 的任意性，必可找到个 λ, s.t.

$$\frac{\delta L}{\delta q_{s+1}} + \lambda\frac{\partial f}{\partial q_{s+1}} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \sum_{k=1}^{s} \left(\frac{\delta L}{\delta q_k} + \lambda\frac{\partial f}{\partial q_k}\right)\delta q_k\,dt = 0$$

故由 $\delta q_k\ (k=1,\cdots,s)$ 的独立性，也有：$\frac{\delta L}{\delta q_k} + \lambda\frac{\partial f}{\partial q_k} = 0 \quad (k=1,\cdots,s)$

综上. $\boxed{\frac{\delta L}{\delta q_k} + \lambda\frac{\partial f}{\partial q_k} = 0}$, $k = (1, \cdots, s+1)$

λ的物理意义不是很清晰

§6 对称与守恒

一. 何为守恒量 初次积分、运动常数

力学量 $\Gamma = \Gamma(q, \dot{q}, t)$.

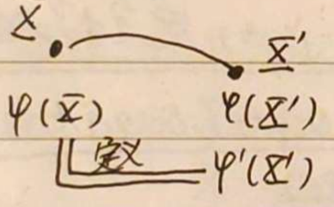
在真实运动中, $\frac{d\Gamma}{dt} = 0$ 即 $\Gamma(q, \dot{q}, t) = \Gamma(q^{(0)}, \dot{q}^{(0)}, t) = \Gamma_0$.

$$\begin{cases} q_k = f_k(q^{(0)}, \dot{q}^{(0)}, t) \\ \dot{q}_k = g_k(q^{(0)}, \dot{q}^{(0)}, t) \end{cases} \Rightarrow \begin{cases} q_k^{(0)} = f_k(q, p, -t) \\ p_k^{(0)} = g_k(q, p, -t) \end{cases}$$

eg
$$\begin{cases} x = x_0 + \dot{x}_0 t - \frac{1}{2} g t^2 \\ \dot{x} = \dot{x}_0 - g t \end{cases} \Rightarrow \begin{cases} x_0 = x - \dot{x} t - \frac{1}{2} g t^2 = \Gamma \\ \dot{x}_0 = \dot{x} + g t = \Gamma_1 \end{cases}$$

对一个自由度为 s 的体系, 最多能找到 2s 个独立的守恒量.
最多能找到 2s-1 个不显含 t 的独立守恒量.

二. 何为对称性(不变性)



主动把点 Σ 变为 Σ',

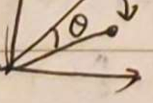
定义一个新的函数 $\varphi'(\Sigma') \doteq \varphi(\Sigma)$

新函数在该点的数值 $\varphi'(\Sigma')$ = 旧函数在该点的数值 $\varphi(\Sigma')$,

即当 $\boxed{\varphi(\Sigma') = \varphi(\Sigma)}$ 时, 称有不变性.

依赖参数 θ, 且 θ 连续的变换称为单参数变换

eg



主动变换
$$\begin{cases} X = x\cos\theta + y\sin\theta = X(x, y; \theta) \\ Y = -x\sin\theta + y\cos\theta = Y(x, y; \theta) \end{cases}$$

(绕点运是顺时针, 故反着)

三. 何为单参数点变换? (对应单参数变换群)

↘ 指位形空间之坐标变换.

$\boxed{q_k \mapsto Q_k = Q_k(q,t;\varepsilon)}$ 要求该变换可逆.

ε 要求连续. 且要求 $\exists\, \varepsilon_0$ s.t. $q_k \mapsto Q_k$ 为恒等变换

不妨设 $\varepsilon=0$ 时该变换为恒等变换 $\boxed{Q_k|_{\varepsilon=0} = q_k}$

$$\Rightarrow \boxed{\dot q_k \mapsto \dot Q_k = \frac{\partial Q_k}{\partial q_i}\dot q_i + \frac{\partial Q}{\partial t} = \dot Q_k(q,\dot q,t;\varepsilon)}$$

当 $\varepsilon \to 0$, 称为无穷小单参数变换.

$$\boxed{\begin{cases} q_k \mapsto Q_k = q_k + \varepsilon S_k(q,t) \\ \dot q_k \mapsto \dot Q_k = \dot q_k + \varepsilon \dot S_k(q,t) \end{cases}},\quad S_k = \frac{\partial Q_k}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

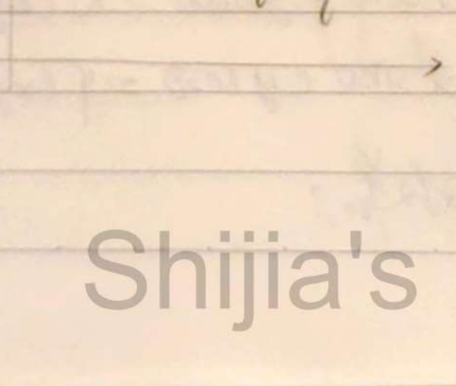
四. 何为动力学对称性?

对于之动力学性质可由 $L = L(q,\dot q,t)$ 完全描述.

定义 $\boxed{L_\varepsilon(q,\dot q,t) \triangleq L\underline{(Q,\dot Q,t)} = L(Q(q,t;\varepsilon),\dot Q(q,\dot q,t;\varepsilon),t)}$

若 $L_\varepsilon(q,\dot q,t) = L(q,\dot q,t) + \dfrac{dF_\varepsilon(q,t)}{dt}$

把真实路径变为真实路径.



$S'_\varepsilon = S' + F_\varepsilon\Big|_{t_1}^{t_2} = S' + \text{const}$

这儿指直接把$(Q,\dot Q,t)$代入$L(q,\dot q,t)$

如 $L = \frac{1}{2}m(\dot x^2+\dot y^2+\dot z^2)$

$L_\varepsilon = \frac{1}{2}m(\dot r^2+\dot\varphi^2+\dot z^2)$

$= \frac{1}{2}m((\dot x^2+\dot y^2+\dot z^2) + \cdots + \dot z^2)$

1、Lagrange 函数不变：$\boxed{L_\varepsilon(q,\dot{q},t)=L(q,\dot{q},t)}$

eg. $L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz$

作 $\begin{cases}\bar{X}=x\cos\theta-y\sin\theta\\ Y=y\sin\theta+x\cos\theta\\ Z=z\end{cases}$ $(S_x,S_y,S_z)=(-y,x,0)$ $\Gamma=-m\dot{x}y+m\dot{y}x=l_z$

则 $L_\varepsilon=\frac{1}{2}m(\dot{X}^2+\dot{Y}^2+\dot{Z}^2)-mgZ$

$\underline{\text{代入变换式}}$ $\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz=L$

作 $\begin{cases}\bar{X}=x+\varepsilon\\ y=y\\ z=z\end{cases}$ $(S_x,S_y,S_z)=(1,0,0)$ $\Gamma=m\dot{x}$

$\begin{cases}\bar{X}=x\\ Y=y+\varepsilon\\ Z=z\end{cases}$ $(S_x,S_y,S_z)=(0,1,0)$ $\Gamma=m\dot{y}$

而作 $\begin{cases}\bar{X}=x\\ Y=y\\ Z=z+\varepsilon\end{cases}$ 有 $L_\varepsilon=L-\varepsilon mg=L+\frac{d}{dt}(-\varepsilon mgt)$

$(S_x,S_y,S_z)=(0,0,1)$，$G=-mgt$

2、Lagrange 函数规范不变 $\Gamma=m\dot{z}+mgt$

$$\boxed{L_\varepsilon(q,\dot{q},t)=L(q,\dot{q},t)+\frac{dF_\varepsilon(q,t)}{dt}}$$

称1,2均为对称变换，可将真实轨道变为真实轨道

（但将真实轨道变为真实轨道的不一定是对称变换）.

eg. $L=\frac{1}{2}m(\dot{x}^2-\omega^2x^2)$ ，作变换 $\bar{x}=xe^{\varepsilon}$

$L_\varepsilon=e^{2\varepsilon}L$， 同样将真实轨道变为真实轨道.

（作用量相差 $e^{2\varepsilon}$倍）

但此变换不称为对称变换（没有对应的守恒量）

五. Noether 定理

若可逆(连续)单参数变换 $q_k\mapsto Q_k(q,\varepsilon;t)$ 为 $L(q,\dot{q},t)$ 的对称变换，即

$$\boxed{L_\varepsilon(q,\dot{q},t)\triangleq L(Q,\dot{Q},t)=L(q,\dot{q},t)+\frac{dF_\varepsilon(q,t)}{dt}}$$

则 $\boxed{\Gamma\triangleq p_kS_k-G}$ 守恒.

其中 $\boxed{p_k\triangleq\frac{\partial L}{\partial\dot{q}_k}，S_k=\left.\frac{\partial Q}{\partial\varepsilon}\right|_{\varepsilon=0}，G=\left.\frac{\partial F_\varepsilon}{\partial\varepsilon}\right|_{\varepsilon=0}}$

Proof. 1. 由 L_ε 定义 $\frac{\partial L_\varepsilon}{\partial\varepsilon}=\frac{\partial L}{\partial Q_k}\frac{\partial Q_k}{\partial\varepsilon}+\frac{\partial L}{\partial\dot{Q}_k}\frac{\partial\dot{Q}_k}{\partial\varepsilon}$

由于 $\varepsilon=0$ 是恒等变换，我们研究 $\varepsilon=0$ 附近的，即 $Q=q$ 附近的.

$$\left.\frac{\partial L_\varepsilon}{\partial\varepsilon}\right|_{\varepsilon=0}=\left.\frac{\partial L}{\partial Q_k}\right|_{Q=q}\left.\frac{\partial Q_k}{\partial\varepsilon}\right|_{\varepsilon=0}+\left.\frac{\partial L}{\partial\dot{Q}_k}\right|_{Q=q}\left.\frac{\partial\dot{Q}_k}{\partial\varepsilon}\right|_{\varepsilon=0}$$

$$=\frac{\partial L}{\partial q_k}S_k+\frac{\partial L}{\partial\dot{q}_k}\dot{S}_k$$

2. 对称变换(条件)

$$\left.\frac{\partial L_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{L_\varepsilon - L_{\varepsilon=0}}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{(L+\varepsilon \frac{dG}{dt}) - L}{\varepsilon} = \frac{dG}{dt}$$

1.2 ⇒ 故 $\frac{dG}{dt} = \frac{\partial L}{\partial q_k} s_k + \frac{\partial L}{\partial \dot{q}_k} \dot{s}_k$

☆ 想清在运算时针对的是"可能的运动"还是"真实的运动"，满足不同条件

3. 对真实运动， $\frac{\partial L}{\partial q_k} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \dot{p}_k$

$$\frac{dG}{dt} = \dot{p}_k s_k + p_k \dot{s}_k = \frac{d}{dt}(p_k s_k) \Rightarrow \frac{d}{dt}(p_k s_k - G) = 0$$

eg. $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$

$$\begin{cases} r \mapsto R = r \\ \theta \mapsto \Theta = \theta + \varepsilon \end{cases}$$

$s_r = 0$, $s_\theta = 1$, $L_\varepsilon = L$, $G = 0$

$\Gamma = p_r s_r + p_\theta s_\theta = m r^2 \dot{\theta}$

eg. $L = \frac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s, z + a\phi)$

$$\begin{cases} s \mapsto S = s & s_s = 0 \\ \phi \mapsto \Phi = \phi + \varepsilon & s_\phi = 1 \\ z \mapsto Z = z - a\varepsilon & s_z = -a \end{cases}$$

$\Rightarrow \Gamma = m s^2 \dot{\phi} - m\dot{z} a = l_z - a p_z$

三、Noether 定理之矢量表述：

$$\begin{tikzcd} q_k \arrow[r, mapsto] \arrow[d, leftrightarrow] & Q_k = Q_k(q,t;\varepsilon) \arrow[d, leftrightarrow] \\ \vec{r}_a = \vec{r}_a(q,t) & \vec{R}_a = \vec{r}_a(Q,t) \end{tikzcd}$$

若变换 $\boxed{\vec{r}_a \mapsto \vec{R}_a = \vec{R}_a(\vec{r}, t; \varepsilon)}$ 是某个体系 $L(\vec{r}, \dot{\vec{r}}, t)$ 之对称变换，即：

$$\boxed{L_\varepsilon(\vec{r}, \dot{\vec{r}}, t) \triangleq L(\vec{R}, \dot{\vec{R}}, t) = L(\vec{r}, \dot{\vec{r}}, t) + \frac{dF_\varepsilon(\vec{r}, t)}{dt}}$$

则 $\boxed{\Gamma = \frac{\partial L}{\partial \dot{\vec{r}}_a} \cdot \vec{\eta}_a - G}$ 守恒，$\vec{\eta}_a \triangleq \left.\frac{\partial \vec{R}_a}{\partial \varepsilon}\right|_{\varepsilon=0}$，$G \triangleq \left.\frac{\partial F_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}$

注意到 $p_k S_k = \frac{\partial L}{\partial \dot{q}_a} S_k = \frac{\partial L}{\partial \dot{\vec{r}}_a} \cdot \frac{\partial \dot{\vec{r}}_a}{\partial \dot{q}_k} S_k$

$$= \frac{\partial L}{\partial \dot{\vec{r}}_a} \frac{\partial \vec{r}_a}{\partial q_a} S_k,$$

而 $\left.\frac{\partial \vec{R}_a}{\partial \varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial \vec{r}_a}{\partial Q_k}\right|_{Q=q} \left.\frac{\partial Q_k}{\partial \varepsilon}\right|_{\varepsilon=0} = \boxed{\frac{\partial \vec{r}_a}{\partial q_k} S_k \triangleq \vec{\eta}_a}$

即 $p_k S_k = \frac{\partial L}{\partial \dot{\vec{r}}_a} \cdot \left.\frac{\partial \vec{R}_a}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\partial L}{\partial \dot{\vec{r}}_a} \cdot \vec{\eta}_a$

特别地，相互作用与速度无关 $L = T(\dot{\vec{r}}) - U(\vec{r}, \cancel{\dot{\vec{r}}}, t)$

则 $\Gamma = \vec{p}_k \cdot \vec{\eta}_a - G$ 守恒

七. 孤立体系（不采用求和约定）

↗ 与速度无关

$$L = T - U = \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^{\,2} - \frac{1}{2} \sum_{a \neq b} U_{ab}(r_{ab})$$

1、空间平移

$\vec{r}_a \longmapsto \vec{R}_a = \vec{r}_a + \varepsilon \hat{n}$，（每个粒子都沿相同方向平移相同）

$\vec{\eta}_a = \hat{n}$

$\dot{\vec{R}}_a = \dot{\vec{r}}_a$，$\vec{R}_{ab} = \vec{r}_{ab}$，$L$ 不变 $G = 0$

$$\Gamma = \sum_a \vec{p}_a \cdot \vec{\eta}_a = \sum_a \vec{p}_a \cdot \hat{n} = \hat{n} \cdot \vec{p}$$

在 $\hat{n}$ 任意取下，总动量 $\vec{p}$ 守恒

空间平移变换下体系总动量 $\vec{p} = \sum_a m_a \vec{v}_a$ 守恒、

2、空间转动

$\vec{r}_a \longmapsto \vec{R}_a = \vec{r}_a + \varepsilon \hat{n} \times \vec{r}_a$ 　$\vec{\eta}_a = \hat{n} \times \vec{r}_a$，$G = 0$

$$\Gamma = \sum_a \vec{p}_a \cdot (\hat{n} \times \vec{r}_a) = \hat{n} \cdot \sum_a (\vec{r}_a \times \vec{p}_a) = \hat{n} \cdot \vec{L}$$

总角动量 $\vec{L} \triangleq \sum_a (\vec{r}_a \times m_a \vec{v}_a)$ 守恒

3、速度变换

$\vec{r}_a \longmapsto \vec{R}_a = \vec{r}_a + \varepsilon \hat{n} t$，　$\dot{\vec{R}}_a = \dot{\vec{r}}_a + \varepsilon \hat{n}$，$\vec{R}_{ab} = \vec{r}_{ab}$

$$T_\varepsilon = \sum_a \frac{1}{2} m_a \vec{R}_a^{\,2} = \sum_a \frac{1}{2} m_a (\vec{r}_a + \varepsilon \hat{n} t)^2$$

$$= \sum_a \frac{1}{2} m_a (\vec{r}_a^{\,2}) + \varepsilon \hat{n} \cdot \sum_a m_a \dot{\vec{r}}_a + \varepsilon^2 \sum_a \frac{1}{2} m_a$$

故 $L_\varepsilon = L + \frac{d}{dt}\left[\varepsilon \hat{n} \cdot \sum_a m_a \vec{r}_a + \varepsilon^2 \left(\frac{1}{2} \sum_a m_a\right) t\right]$

是规范不变的，故存在守恒量

$\vec{\eta}_a = \hat{n} t$，　$G = \hat{n}\cdot\sum_a m_a \vec{r}_a$

$$\Gamma = \sum_a \vec{p}_a\cdot\vec{\eta}_a - G$$

$$= \sum_a m_a \dot{\vec{r}}_a\cdot\hat{n} t - \hat{n}\cdot\sum_a m_a \vec{r}_a$$

$$= \hat{n}\cdot\left[t\sum_a m_a \dot{\vec{r}}_a - \sum_a m_a \vec{r}_a\right]$$

$$= \hat{n}\cdot M\left[\dot{\vec{r}}_c t - \vec{r}_c\right] \xlongequal{\text{守恒}} \hat{n}\cdot M\vec{r}_{c0}$$

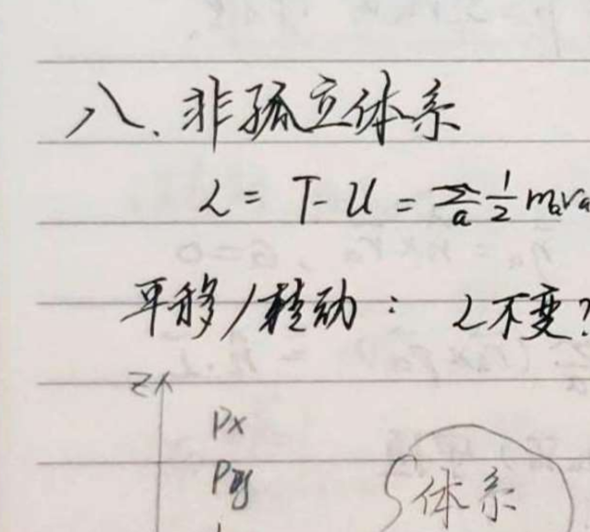
即　$\vec{r}_c = \vec{r}_{c0} + \dot{\vec{r}}_c t$

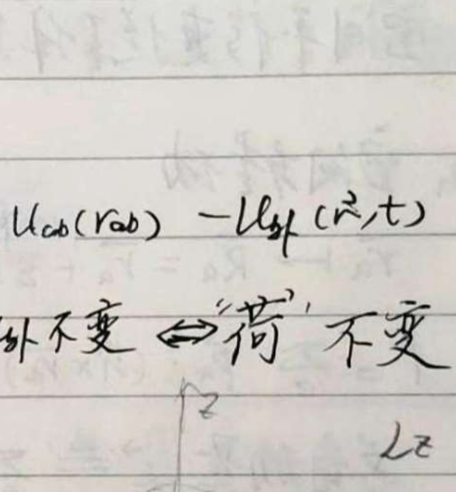
质心做匀速直线运动

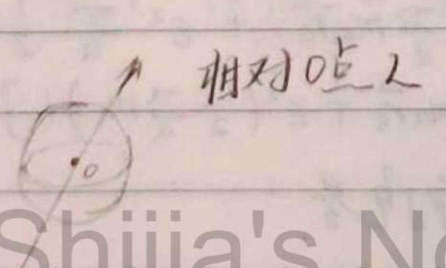
八、非孤立体系

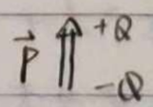
$$L = T - U = \sum_a \frac{1}{2} m_a v_a^2 - \frac{1}{2}\sum_{a\neq b} U_{ab}(r_{ab}) - U_{外}(\vec{r}, t)$$

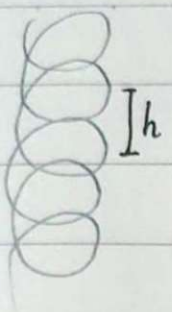
平移/转动：L不变？⇔ $U_{外}$不变 ⇔"荷"不变











$$\phi \mapsto \Phi = \phi + \varepsilon$$

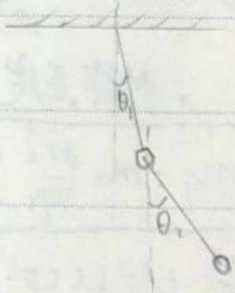
$$z \mapsto Z = z + \frac{\varepsilon h}{2\pi}$$

$$p_\phi + p_z \frac{h}{2\pi} = l_z + \frac{h}{2\pi} p_z$$

Date. No.

CH$_3$ 线性(微)振动

§1. 双摆:



一、$\mathcal{L} = \frac{1}{2}m[l^2\dot{\theta}_1^2 + l^2(\dot{\theta}_1+\dot{\theta}_2)^2] - mgl[\frac{1}{2}\theta_1^2 + \frac{1}{2}(\theta_1^2+\theta_2$

$$= ml^2[(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) - \omega_0^2(\theta_1^2 + \frac{1}{2}\theta_2^2)]$$

$$\omega_0 \triangleq \sqrt{\frac{g}{l}}$$

二、Lagrange Equ. $\begin{cases} 2\ddot{\theta}_1 + \ddot{\theta}_2 = -2\omega_0^2\theta_1 & ① \\ \ddot{\theta}_1 + \ddot{\theta}_2 = -\omega_0^2\theta_2 & ② \end{cases}$

三. 解耦: ①$+\alpha$②:

$$\frac{d^2}{dt^2}(\theta_1 + \frac{1+\alpha}{2+\alpha}\theta_2) = -\frac{2}{2+\alpha}\omega_0^2(\theta_1 + \frac{\alpha}{2}\theta_2)$$

令 $\frac{1+\alpha}{2+\alpha} = \frac{\alpha}{2} \Rightarrow \alpha = \pm\sqrt{2} \Rightarrow \frac{2}{2+\alpha} = 2\mp\sqrt{2}$

$$\Rightarrow \frac{d^2}{dt^2}(\theta_1 \pm \frac{\theta_2}{\sqrt{2}}) = -(2\mp\sqrt{2})\omega_0^2(\theta_1 \pm \frac{\theta_2}{\sqrt{2}})$$

$$\Rightarrow \begin{cases} \xi_1 = \theta_1 + \frac{\theta_2}{\sqrt{2}} = 2\lambda_1\cos(\omega_1 t+\varphi_1) & \omega_1 \triangleq \sqrt{2-\sqrt{2}}\,\omega_0 \\ \xi_2 = \theta_1 - \frac{\theta_2}{\sqrt{2}} = 2\lambda_2(\cos\omega_2 t+\varphi_2) & \omega_2 \triangleq \sqrt{2+\sqrt{2}}\,\omega_0 \end{cases}$$

$$\Rightarrow \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\frac{\xi_1+\xi_2}{2}\\ \frac{\xi_1-\xi_2}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}\lambda_1\cos(\omega_1 t+\varphi_1) + \begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}\lambda_2\cos(\omega_2 t+\varphi_2)$$

简正模 Normal Mode ω_1: $\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$ ω_2: $\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}$

ξ_1、ξ_2: 简正坐标。

§2 简谐近似

一、体系之描述　自由度 s，广义坐标 $q=(q_1,\cdots,q_n)$

1. 外部约束与外场稳定（条件1）

$\vec{r}=\vec{r}(q,t)$ ⇒ T 仅有二次项，$T=\frac{1}{2}m_{ij}\dot{q}_i\dot{q}_j$，$\boxed{m_{ij}=m_a\frac{\partial\vec{r}_a}{\partial q_i}\frac{\partial\vec{r}_a}{\partial q_j}=m_{ij}(q)}$

对称正定

$T>0$ 推得

$U=U(\vec{r},\dot{\vec{r}},t)$，再假设相互作用与速度无关，$U=U(\vec{r})$

故 $U=U(q)$．⇒ $\boxed{L=T-U=L(q,\dot{q})}$（不显含时间 t）

2. 体系有稳定平衡位置　$q^{(0)}=(q_1^{(0)},\cdots,q_s^{(0)})$

$$\left.\frac{\partial U}{\partial q_k}\right|_{q=q^{(0)}}=0 \qquad \boxed{K=K_{ij}\triangleq\left(\frac{\partial^2U}{\partial q_i\partial q_j}\right)_{q^{(0)}}}$$

对称性显然，要求 K 为正定，对称．

保证线性

二、平衡位置附近之 Lagrange 函数．

$$U(q)=U(q^{(0)})+\left.\frac{\partial U}{\partial q_i}\right|_{q^{(0)}}(q_i-q_i^{(0)})+\frac{1}{2}\left.\frac{\partial^2U}{\partial q_i\partial q_j}\right|_{q^{(0)}}(q_i-q_i^{(0)})(q_j-q_j^{(0)})+\cdots$$

令 $\xi=q-q^{(0)}$，则 $U\doteq\frac{1}{2}K_{ij}\xi_i\xi_j$．$T=\frac{1}{2}m_{ij}\dot{\xi}_i\dot{\xi}_j$

m_{ij} 只需保留常数项．$M_{ij}=m_{ij}(q^{(0)})$ 为常数

$$\boxed{L=\frac{1}{2}M_{ij}\dot{\xi}_i\dot{\xi}_j-\frac{1}{2}K_{ij}\xi_i\xi_j}.$$

广义速度之正定二次型　广义坐标之正定二次型

$$\boxed{\begin{cases}M_{ij}\triangleq\left.\dfrac{\partial^2T}{\partial\dot{\xi}_i\partial\dot{\xi}_j}\right|_{\xi=0}\\ K_{ij}\triangleq\left.\dfrac{\partial^2U}{\partial\xi_i\partial\xi_j}\right|_{\xi=0}\end{cases}}$$

三. Lagrange 方程 $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}=0$

$$\frac{\partial T}{\partial \dot\xi_k}=\frac{1}{2}M_{ij}\left(\frac{\partial \dot\xi_i}{\partial \dot\xi_k}\dot\xi_j+\dot\xi_i\frac{\partial \dot\xi_j}{\partial \dot\xi_k}\right)=\frac{1}{2}M_{kj}\dot\xi_j+\frac{1}{2}M_{ik}\dot\xi_i$$

$$\Rightarrow \frac{\partial T}{\partial \dot\xi_k}=M_{kj}\dot\xi_j \quad , \quad \frac{\partial U}{\partial \xi_k}=U_{kj}\xi_j$$

$$\Rightarrow \quad M_{kj}\ddot\xi_j+K_{kj}\xi_j=0$$ （线性近似）(对 ξ_k 的 Lagrange 方程)

记 $\xi=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_s\end{pmatrix}$

$$\boxed{L=\frac{1}{2}\dot\xi^T M\dot\xi-\frac{1}{2}\xi^T K\xi}$$

$$\boxed{M\ddot\xi+K\xi=0}$$

§3 简正坐标与简正模

$M\ddot\xi+K\xi=0$ M对称、正定，K对称

$\ddot\xi=-M^{-1}K\xi=-\Omega\xi$, $\Omega=M^{-1}K$

$O^T\Omega O=\Omega_d$ (对角化)，则令 $\xi=O\eta$

$\Rightarrow O\ddot\eta=-\Omega O\eta \Rightarrow \ddot\eta=-O^T\Omega O\eta=-\Omega_d\eta$

一、线性坐标变换 $\xi=A\eta$ (A可逆)

则 $MA\ddot\eta+\underbrace{KA}_{MA\Omega_d\,?}\eta=0$

正交相似对角化

M对称 ⇒ $\exists\, O_1$ 正交，$O_1^T M O_1=M_d=\mathrm{diag}\{m_1,\cdots m_s\}$

M正定 ⇒ $m_i>0$ ⇒ 可定义可逆 $D\triangleq\mathrm{diag}\left\{\frac{1}{\sqrt{m_1}},\cdots,\frac{1}{\sqrt{m_s}}\right\}=D$

$\Rightarrow D^T O_1^T M O_1 D=(O_1D)^T M(O_1D)=I$

即通过 O_1D 将 M 相合到 I

K对称 ⇒ $(O_1D)^TK(O_1D)$ 也对称，可对角化相似正交

⇒ $\exists O_2$ 正交，$O_2^TD^TO_1^TKO_1DO_2=(O_1DO_2)^TK(O_1DO_2)=\Omega_d$

Ω_d 是对角阵.（通过 O_1DO_2 将 K 相合到对角阵）

又 $(O_1DO_2)^TM(O_1DO_2)=I$，

故证明了，O_1DO_2 可将 M,K 同时对角化.

Theorem $\exists A=(A^{(1)},\cdots,A^{(s)})$, s.t.

$A^TMA=I$ 且 $A^TKA=\Omega_d=\mathrm{diag}\{\omega_1^2,\cdots,\omega_s^2\}$。

则作 $\xi=A\eta$ 后，$A^TMA\ddot{\eta}+A^TKA\eta=0$

$\Rightarrow \ddot{\eta}=-\Omega_d\eta$

1. 本征值 ω_α^2 为实数

K对称.
$$\begin{cases}O^TO=I\\ O^TKO=K_d\end{cases}\Rightarrow KO=OK_d\Rightarrow K(O^{(1)},\cdots,O^{(s)})=(KO^{(1)},\cdots,KO^{(s)})$$

$K_d=\mathrm{diag}\{k_1,\cdots,k_s\}$

$\Rightarrow(KO^{(1)},\cdots,KO^{(s)})=(k_1O^{(1)},\cdots,k_sO^{(s)})$

即 $KY=kY\Leftrightarrow(K-kI)Y=0$

$$\Rightarrow\begin{cases}\det(K-kI)=0\Rightarrow k_a\quad(a=1,\cdots,s)\\ (K-k_aI)Y=0\Rightarrow O^{(a)}=C_aY\end{cases}$$

$$\begin{cases} A^T M A = I \\ A^T K A = \Omega_d \end{cases} \Rightarrow KA = MA\Omega_d$$

$$\Rightarrow (KA^{(1)}, \cdots, KA^{(s)}) = (\omega_1^2 MA^{(1)}, \cdots, \omega_s^2 MA^{(s)})$$

$$\Rightarrow KA = \omega^2 MA$$

即解：$K\underline{X} = \omega^2 M\underline{X} \Leftrightarrow$

$$\begin{cases} \det(K-\omega^2 M) = 0 \Rightarrow \omega_a^2 \quad (共s个) \\ \xleftarrow{代入} \\ (K-\omega_a^2 M)\underline{X} = 0 \Rightarrow A^{(a)} = C_a\underline{X} \end{cases}$$

二、本征值与本征矢

久期方程 $\boxed{\det(K-\omega^2 M)=0} \Rightarrow \omega_a^2$

$\Rightarrow \boxed{(K-\omega_a^2 M)\underline{X}=0} \Rightarrow A^{(a)} = \underline{X}$,

1、这里本征值 ω_a^2 一定是实数、若K正定，则 $\omega_a^2 > 0$

2、本征矢 $\underline{X}$ 必可取为实数

3、如此得到A满足 $KA = MA\Omega_d$

—

$\underline{X} = \alpha + i\beta$ （α、β 均为实且不全为0）

$\underline{X}^+ K\underline{X} = (\alpha^T - i\beta^T)K(\alpha+i\beta) = (\alpha^T K\alpha + \beta^T K\beta) + i(\alpha^T K\beta - \beta^T K$

!

由M的正定性，此项必不为0

$\omega^2\underline{X}^+ M\underline{X} = \omega^2(\alpha^T M\alpha + \beta^T M\beta) + i\omega^2\underbrace{(\alpha^T M\beta - \beta^T M\alpha)}_{=0}$

由于 $\alpha^T M\beta$ 是数，$(\alpha^T M\beta)^T = (\beta^T M^T\alpha) = \beta^T M\alpha$

故 $\alpha^T K\alpha + \beta^T K\beta = \omega^2(\alpha^T M\alpha + \beta^T M\beta)$

$\Rightarrow \omega^2$ 是实数

ssj注：若ω有重根，例如$\omega_1=\omega_2=\omega_a$，对应有两振幅分别为 A_1，A_2，则应为 $A_1\lambda_1\cos(\omega_a t+\varphi_1)+A_2\lambda_2\cos(\omega_a t+\varphi_2)$

两个η ω相同，但λ，φ不一定相同.

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故可得 $\xi = A\eta = \sum_a A^{(a)}\eta_a$

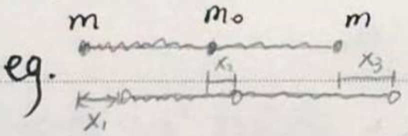
三、方程的解

$$\boxed{\ddot{\eta}+\Omega\eta=0\quad\begin{cases}\omega_a^2>0, & \eta_a(t)=\lambda_a\cos(\omega_a t+\varphi_a)\\ \omega_a^2=0, & \eta_a(t)=\lambda_a t+\varphi_a\\ \omega_a^2<0, & \eta_a(t)=c_a\cosh\Omega_a t+d_a\sinh\Omega_a t,\quad \Omega_a\triangleq\sqrt{-\omega_a^2}\end{cases}}$$

$$\boxed{M\ddot{\xi}+k\xi=0\xrightarrow{\xi=A\eta}\xi=A\eta=\sum_a A^{(a)}\eta_a(t)=A^{(1)}\eta_1+\cdots+A^{(s)}\eta_s}$$

四、简正模 ($\omega_a^2\geqslant 0$)

eg.



$$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m_0\dot{x}_2^2+\frac{1}{2}m\dot{x}_3^2$$

$$U=\frac{1}{2}k(x_2-x_1)^2+\frac{1}{2}k(x_3-x_2)^2$$

$$=\frac{1}{2}k(x_1^2+2x_2^2+x_3^2-2x_1x_2-2x_2x_3)$$

$$M=\begin{pmatrix}m&&\\&m_0&\\&&m\end{pmatrix}=m\begin{pmatrix}1&&\\&r&\\&&1\end{pmatrix},\ r\triangleq\frac{m_0}{m_1},\quad K=k\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}$$

$$\omega^2M-K=k\begin{pmatrix}\lambda-1&1&0\\1&r\lambda-2&1\\0&1&\lambda-1\end{pmatrix}\qquad \lambda\triangleq\frac{\omega^2}{\omega_0^2},\ \omega_0^2=\frac{k}{m}$$

$$0=\frac{1}{k}\det(\omega^2M-k)=\lambda(\lambda-1)[r\lambda-(r+2)]$$

$$\Rightarrow\ \lambda_1=1,\quad \lambda_2=\frac{r+2}{r},\quad \lambda_3=0$$

λ=1 时，$\omega=\omega_0$，$\begin{pmatrix}0&1&0\\1&r-2&1\\0&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\qquad A^{(1)}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$

当 $\lambda_2=\frac{r+2}{r}$ $\begin{pmatrix} \frac{2}{r} & 1 & 0 \\ 1 & r & 1 \\ 0 & 1 & \frac{2}{r} \end{pmatrix}$ $A^{(2)}=\begin{pmatrix} 1 \\ -\frac{2}{r} \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ -\frac{2m_0}{m} \\ 1 \end{pmatrix}$

$\omega_2=\omega_0\sqrt{1+\frac{2m}{m_0}}$

当 $\lambda_3=0$ 时，$\omega_3=0$，$A^{(3)}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}[\lambda_1'\cos(\omega_1 t+\varphi_1)]+\begin{pmatrix} 1 \\ -2\frac{m_0}{m} \\ 1 \end{pmatrix}[\lambda_2'\cos(\omega_2 t+\varphi_2)]+\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(b_3 t+$$

有时 $\omega=0$，称零模，通过取质心系可减少1个自由度（在上例中）

对 n质点体系，写全动量=0，角动量=0 减至 $3n-3-3$ 自由度.

$f(r)=f(|\vec{r}|)$ 在 $\vec{r}=\vec{R}$ 附近展开 $\vec{r}-\vec{R}\triangleq\vec{u}$

$$f(r)=f(|\vec{R}+\vec{u}|)=f(R)+\frac{\partial f}{\partial x_i}\Big|_{\vec{R}}u_i+\frac{1}{2!}\frac{\partial^2 f}{\partial x_i\partial x_j}\Big|_{\vec{R}}u_iu_j$$

$$\boxed{f(r)=f(R)+(\hat{R}\cdot\vec{u})f'(R)+\frac{1}{2}\left[(\hat{R}\cdot\vec{u})^2f''(R)+\frac{\vec{u}^2-(\hat{R}\cdot\vec{u})^2}{R}f'(R)\right]+}$$

$$\boxed{\frac{\partial f}{\partial x_i}=\frac{\partial r}{\partial x_i}\frac{\partial f}{\partial r}=\frac{x_i}{r}\frac{\partial f}{\partial r}}\rightarrow(\hat{R})_i$$

或：$\nabla f\cdot\vec{u}=f'\nabla R\cdot\vec{u}=f'(R)(\hat{R}\cdot\vec{u})$

$$\frac{\partial^2 f}{\partial x_i\partial x_j}=\frac{x_i}{r}\frac{\partial}{\partial x_j}\frac{\partial f}{\partial r}+\frac{\partial f}{\partial r}\left[\frac{1}{r}\frac{\partial x_i}{\partial x_j}+x_i\frac{\partial}{\partial x_j}\left(\frac{1}{r}\right)\right]$$

$$\boxed{\frac{\partial^2 f}{\partial x_i\partial x_j}=\frac{x_ix_j}{r^2}\frac{\partial^2 f}{\partial r^2}+\frac{r^2\delta_{ij}-x_ix_j}{r^3}\frac{\partial f}{\partial r}}$$

注 $\frac{\partial x_i}{\partial x_j}=\delta_{ij}$

内积 $(\underline{X},\underline{Y}) \triangleq \underline{X}^T M \underline{Y}$ M正定

模 $\|\underline{X}\| = \sqrt{(\underline{X},\underline{X})} = \sqrt{\underline{X}^T M \underline{X}}$

夹角 $\cos\theta = \dfrac{(\underline{X},\underline{Y})}{\|\underline{X}\|\|\underline{Y}\|}$

正交归一：$(A^{(i)}, A^{(j)}) = \delta_{ij} \Leftrightarrow A^T M A = I$

$\Rightarrow A^T K A = \Omega d.$

$$L = \frac{1}{2}\dot{\xi}^T M \dot{\xi} - \frac{1}{2}\xi^T K \xi$$

$$= \frac{1}{2}\dot{\eta}^T (A^T M A)\dot{\eta} - \frac{1}{2}\eta^T (A^T \underbrace{KA}_{\to MA\Omega d})\eta$$

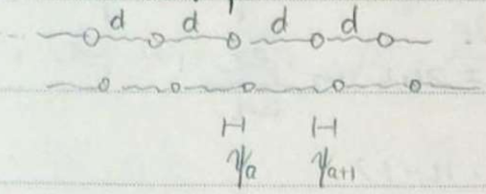
$$= \sum_{ij} \frac{1}{2}(A^T M A)_{ij}(\dot{\eta}_i\dot{\eta}_j - \omega_i^2 \eta_i \eta_j)$$

L 函数看起来是有耦合的，但 L 方程（动力学方程）解耦。

当 $A^T M A = I$（作了变化之后）

$$\boxed{L = \sum_i \frac{1}{2}(\dot{\eta}_i^2 - \omega_i^2 \eta_i^2)}$$

§4 一维链的振动



$$L=\frac{1}{2}m\sum_0^n\dot{\psi}_a^2-\frac{1}{2}k\sum_0^{n-1}(\psi_{a+1}-\psi_a)^2$$

$$=\left[\frac{1}{2}m\dot{\psi}_0^2-\frac{1}{2}k(\psi_1-\psi_0)^2\right]+\left[\frac{1}{2}m\dot{\psi}_n^2-\frac{1}{2}k(\psi_n-\psi_{n-1})^2\right]$$

$$+\sum_{a=1}^{n-1}\left[\frac{1}{2}m\dot{\psi}_a^2-\frac{1}{2}k(\psi_a-\psi_{a-1})^2-\frac{1}{2}k(\psi_{a+1}-\psi_a)^2\right]$$

L方程:
$$\begin{cases}\ddot{\psi}_0=-\omega_0^2(\psi_0-\psi_1)\\ \ddot{\psi}_a=-\omega_0^2(2\psi_a-\psi_{a-1}-\psi_{a+1})\\ \ddot{\psi}_n=-\omega_0^2(\psi_n-\psi_{n-1})\end{cases}\qquad \omega_0^2\triangleq\frac{k}{m}$$ 重了

$$\Rightarrow\ \ddot{\psi}=-K\psi\ ,\ K=\omega_0^2\begin{pmatrix}\ddots & & 0 & & \\ & 2 & -1 & & \\ 0 & -1 & \ddots & & \\ & & & -1 & 0\\ & & -1 & 2 & \\ & & 0 & & \ddots\end{pmatrix}$$

1.(1) 选择边界条件 $\psi_0=\psi_n=0$ → 本质上是作了傅里叶展开

(2) 猜 $\psi_a(t)=Ce^{i(kx_a-\omega t-\varphi)}$, $x_a\triangleq ad$

此时有 $\psi_{a\pm1}=e^{\pm ikd}\psi_a$

得 $\omega^2\psi_a=\omega_0^2(2-e^{-ikd}-e^{ikd})\psi_a$

对 $\forall t$ 的 ψ_a 恒成立, 故: $\omega^2=2\omega_0^2(1-\cos kd)$

$$\omega=2\omega_0\sin\frac{kd}{2}$$

(3) 继续猜 $\psi_a(t)=Ce^{-i(\omega t+\varphi)}\sin(kx_a)$ (*)

$$=\frac{C}{2i}\left[e^{i(kx_a-\omega t-\varphi)}-e^{-i(kx_a+\omega t+\varphi)}\right]$$

$\psi_0 = 0$ 自动满足, $\psi_n = 0 \Rightarrow kx_n = knd = \beta\pi$ $(\beta = 1, 2, \cdots)$

$$\Rightarrow k_\beta = \frac{\beta\pi}{nd} \Rightarrow \omega_\beta = 2\omega_0 \sin\frac{\beta\pi}{2n}$$

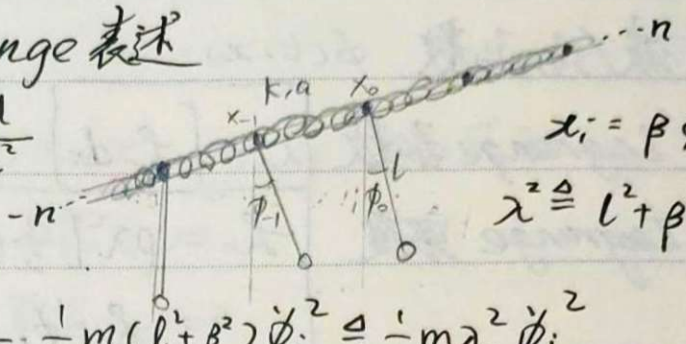
$$(\beta = 1, \cdots, n-1)$$

本征矢 $A_a^{(\beta)} = C\sin(k_\beta x_a) = C\sin(a\frac{\beta\pi}{n})$ (根据(*)式看出)

$$\lambda_\beta = \frac{2\pi}{k_\beta} \Rightarrow \frac{nd}{\lambda_\beta} = \frac{\beta}{2}$$

§5. 连续体系之 Lagrange 表述

一、def $\omega^2 = \frac{k\beta^2}{m\lambda^2}$, $\Omega^2 = \frac{gl}{\lambda^2}$



$x_i = \beta\phi_i$

$\lambda^2 \triangleq l^2 + \beta^2$

$$T_i = \frac{1}{2}m(l^2\dot\phi_i^2 + \dot x_i^2) = \frac{1}{2}m(l^2+\beta^2)\dot\phi_i^2 \triangleq \frac{1}{2}m\lambda^2\dot\phi_i^2$$

$$L = \frac{1}{2}m\lambda^2\sum_{-n}^{n}\dot\phi_i^2 - \frac{1}{2}k\beta^2\sum_{-n}^{n-1}(\phi_{i+1}-\phi_i)^2 - mgl\sum_{-n}^{n}(1-\cos\phi)$$

$$\Rightarrow L = m\lambda^2\left[\frac{1}{2}\sum_{-n}^{n}\dot\phi_i^2 - \omega^2\frac{1}{2}\sum_{-n}^{n-1}(\phi_{i+1}-\phi_i)^2 - \Omega^2\sum_{-n}^{n}(1-\cos\phi)\right]$$

L 方程：$\ddot\phi_i - \omega^2[(\phi_{i+1}-\phi_i)-(\phi_i-\phi_{i-1})] + \Omega^2\sin\phi_i = 0$, $n-1 \le i \le$

二. 连续极限：每摆替换成 s 个摆，并 $s \to +\infty$

① $m \mapsto \Delta m = \frac{m}{s}$, $a \mapsto \Delta x = \frac{a}{s}$, $k \mapsto sk$

$\Rightarrow$ 不变量：$\rho \triangleq \frac{\Delta m}{\Delta x} = \frac{m}{a}$, $Y \triangleq \frac{ak\beta^2}{\lambda}$

$\Rightarrow \omega^2 = \frac{k\beta^2}{m\lambda^2} = \frac{Y}{ma} \mapsto \frac{Y}{\Delta m\Delta x} = \frac{Y}{\rho(\Delta x)^2} = \frac{v^2}{(\Delta x)^2}$, $v \triangleq \sqrt{\frac{Y}{\rho}}$

② x：第 i 个摆在平衡时悬挂点之位置.

$\phi_i \mapsto \phi(x)$　或：$\phi_i(t) \mapsto \phi(t,x)$

则 $L = \sum \rho\Delta x\,\lambda^2\left\{\frac{1}{2}\dot\phi(x) - \frac{1}{2}v^2\left(\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}\right)^2 - \Omega^2[1-\cos\phi(x)]\right\}$

L 方程：$\ddot\phi(x) - v^2\frac{[\phi(x+\Delta x)-\phi(x)]-[\phi(x)-\phi(x-\Delta x)]}{(\Delta x)^2} + \Omega^2\sin\phi(x) = 0$

1、波/场函数 $\phi(t,x)$

2. Lagrange 函数 $\boxed{L=\int \mathcal{L}\,dx}$

Lagrange 密度 $$\boxed{\begin{aligned}\mathcal{L} &= \rho\lambda^2\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2-\frac{1}{2}v^2\left(\frac{\partial\phi}{\partial x}\right)^2-\Omega^2(1-\cos\phi)\right]\\ &= \mathcal{L}(\phi,\partial_t\phi,\partial_x\phi)\end{aligned}}$$

3. 运动方程 $$\boxed{\frac{\partial^2\phi}{\partial t^2}-v^2\frac{\partial^2\phi}{\partial x^2}+\Omega^2\sin\phi=0}$$

sine-Gordon 方程（sG方程）

eg. $g=0 \Rightarrow \Omega=0$. def $\psi=\lambda\phi$.

$$\mathcal{L}=\rho\left[\frac{1}{2}(\partial_t\psi)^2-\frac{1}{2}v^2(\partial_x\psi)^2\right],$$

$$\frac{\partial^2\psi}{\partial t^2}=v^2\frac{\partial^2\psi}{\partial x^2}$$

eg. $|\phi|<<1$ def $\psi=\lambda\phi$

$$\mathcal{L}=\rho\left[\frac{1}{2}(\partial_t\psi)^2-\frac{1}{2}v^2(\partial_x\psi)^2-\frac{1}{2}\Omega^2\psi^2\right]$$

$$\frac{\partial^2\psi}{\partial t^2}-v^2\frac{\partial^2\psi}{\partial x^2}+\Omega^2\psi=0$$ Klein-Gorden 方程

$q_k \mapsto \psi$, $\frac{\delta L}{\delta q_k}=\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}=0$

def: $$\frac{\delta\mathcal{L}}{\delta\psi}\triangleq\frac{\partial\mathcal{L}}{\partial\psi}-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}\right)$$

三. 符号

1. 时空坐标 $x = (x_0, x_1, x_2, x_3) = (ct, \vec{r})$

$$\begin{cases} \alpha, \beta, \gamma, \cdots & 取值\ 0,1,2,3 \\ i, j, k \cdots & 取值\ 1,2,3 \end{cases}$$

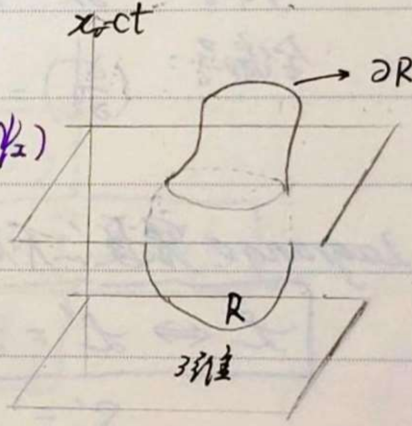
2. 场 $\psi_I(x) = \psi_I(t, \vec{r})$ $I = (1, 2, \cdots, N)$

（例如电磁势，$N=4$）

3. Lagrange密度 $\mathcal{L} = \mathcal{L}(\psi, \partial\psi, x)$

4. 作用量 $S = \int L\, dt = \int \mathcal{L}\, d^4x$ ，$(L = \int \mathcal{L}\, d^3x)$

四. 最小作用量原理



$\psi_I \mapsto \psi_I + \delta\psi_I$ $(\partial_\alpha \delta\psi_I = \delta \partial_\alpha \psi_I)$

$\partial_\alpha \psi_I \mapsto \partial_\alpha \psi_I + \partial_\alpha \delta\psi_I$

$$\begin{cases} \delta S = \delta \int_R \mathcal{L}(\psi, \partial\psi, x)\, d^4x = 0 \\ \delta\psi_I\ (x \in \partial R) = 0, \quad I = (1, \cdots, N) \end{cases}$$

$$\delta\mathcal{L} = \mathcal{L}(\psi + \delta\psi, \partial\psi + \delta\partial\psi, x) - \mathcal{L}(\psi, \partial\psi, x)$$

$$= \frac{\partial \mathcal{L}}{\partial \psi_I} \delta\psi_I + \frac{\partial \psi}{\partial(\partial_\alpha \psi)} \partial_\alpha(\delta\psi_I) \qquad \partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_I)}\delta\psi_I\right] - \left[\partial_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi_I)}\right] \delta\psi_I$$

$$= \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \psi_I} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)}\right]}_{\frac{\delta\mathcal{L}}{\delta\psi_I}} \delta\psi_I + \partial_\alpha \left[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \delta\psi_I}_{F_\alpha}\right]$$

$$0 = \delta S = \int_R \frac{\delta \mathcal{L}}{\delta \psi_I} \delta \psi_I \, d^4x + \underbrace{\int_R \partial_\alpha F_\alpha \, d^4x}_{\oint_{\partial R} F_\alpha dS_\alpha = 0}$$

1. Lagrange 方程

$$\boxed{\frac{\delta \mathcal{L}}{\delta \psi_I} \triangleq \frac{\partial \mathcal{L}}{\partial \psi_I} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)}}$$

$$\boxed{\frac{\delta \mathcal{L}}{\delta \psi_I} = \frac{\partial \mathcal{L}}{\partial \psi_I} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \psi_I)} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla \psi_I)} = 0}$$

2. ∂_α ↔ 全偏导

$f = f(x, y, z(x,y))$ 时

偏导：$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}$

全偏导：$\left(\frac{\partial f}{\partial x}\right) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z(x+\Delta x, y)) - f(x, y, z(x,y))}{\Delta x}$

3. Lagrange 密度的不确定性

$$\boxed{\mathcal{L} \Leftrightarrow \mathcal{L}' = \mathcal{L} + \partial_\alpha F_\alpha(\psi, x)}$$

$$S' = S + \text{const.}$$

（这里ψ对应以前的x，x对应以前的t.）

eg. $\mathcal{L} = \frac{i\hbar}{2}[\psi^*\partial_t\psi - (\partial_t\psi^*)\psi] - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - u\psi^*\psi$

因 ψ 具有实虚部，为二个场，不妨看成 ψ、ψ^* 两个场

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = \frac{i\hbar}{2}\partial_t\psi - u\psi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\psi^*)} = -\frac{i\hbar}{2}\psi \qquad \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\psi^*)} = -\frac{i\hbar}{2}\partial_t\psi$$

$$\frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)} = -\frac{\hbar^2}{2m}\nabla\psi \qquad \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)} = -\frac{\hbar^2}{2m}\nabla\cdot\nabla\psi$$

$$\Rightarrow \boxed{i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla + u\right)\psi}$$

schrödinger equ.

五、Maxwell 方程组

1、源 $\boxed{j_\alpha = (j_0, \vec{j}) = (\rho c, \rho\vec{v})}$

连续性方程：$\boxed{0 = \partial_t\rho + \nabla\cdot\vec{j} = \partial_\alpha j_\alpha}$

2、势 $\boxed{A_\alpha = (A_0, \vec{A}) = \left(-\frac{\varphi}{c}, \vec{A}\right)}$

场 $\vec{E} = -\nabla\varphi - \partial_t\vec{A}$

$$E_i = -\partial_i\varphi - \partial_t\vec{A_i}$$
$$= c(\partial_i A_0 - \partial_0 A_i)$$

$\vec{B} = \nabla\times\vec{A}$

$$B_i = \varepsilon_{imn}\partial_m A_n$$

如 $B_3 = \partial_1 A_2 - \partial_2 A_1$

3. Lagrange 密度.

$$\mathcal{L}(A,\partial A,x) = \frac{1}{2}\varepsilon_0(E^2-c^2B^2)-(\rho\varphi-\vec{j}\cdot\vec{A})$$

$$\boxed{\mathcal{L}(A,\partial A,x) = \frac{1}{2}\underbrace{\varepsilon_0(E^2-c^2B^2)}_{\partial A.}+\underbrace{j_\alpha}_{t,\vec{r}}\underbrace{A_\alpha}_{A}}$$

$$\frac{\delta\mathcal{L}}{\delta A_\beta}=\frac{\partial\mathcal{L}}{\partial A_\beta}-\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)}$$

$$=\frac{\partial\mathcal{L}}{\partial A_\beta}-\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t A_\beta)}-\nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla A_\beta)}=0$$

$$\frac{\partial\mathcal{L}}{\partial A_\beta}=j_\beta\ ,\quad \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)}=0\quad(\alpha=\beta)$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)}\overset{\Delta}{=}Y_{\alpha\beta}=-Y_{\beta\alpha}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_j A_0)}=\varepsilon_0E_i\frac{\partial E_i}{\partial(\partial_j A_0)}=\varepsilon_0E_i(\delta_{ij}c)=\varepsilon_0E_j$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_0 A_k)}=\varepsilon_0E_i\frac{\partial E_i}{\partial(\partial_0 A_k)}=\varepsilon_0E_i(-c\delta_{ik})=-C\varepsilon_0E_k$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_j A_k)}=-\varepsilon_0c^2B_i\frac{\partial B_i}{\partial(\partial_j A_k)}=-\varepsilon_0c^2B_i(\varepsilon_{imn}\delta_{jm}\delta_{kn})$$

$$=-C^2\varepsilon_0\varepsilon_{ijk}B_i$$

$\beta=0$ 时, $\frac{\partial\mathcal{L}}{\partial A_0}=\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_0)}$.

$$\rho c=j_0=\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i A_0)}=\partial_i(C\varepsilon_0E_i)$$

即 $\nabla\cdot E=\frac{\rho}{\varepsilon_0}$

当β=k时,

$$j_k = \frac{\partial \mathcal{L}}{\partial A_k} = \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 A_k)} + \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j A_k)}$$

$$= -\partial_0(c\varepsilon_0 E_k) - c^2 \varepsilon_0 \varepsilon_{kij} \partial_j B_i$$

$$= -(\varepsilon_0 \partial_t \vec{E})_k + c^2 \varepsilon_0 \varepsilon_{kji} \partial_j B_i$$

$$= -(\varepsilon_0 \partial_t \vec{E})_k + c^2 \varepsilon_0 (\nabla \times \vec{B})_k$$

即 $\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \partial_t \vec{E}$.

而由于 $\varphi, \vec{A}$ 之选取, 另外二式子自动成立.

$\psi_I \mapsto \Psi = \Psi(\psi, x; \varepsilon)$

$$\boxed{\mathcal{L}_\varepsilon(\psi, \partial\psi, x) \triangleq \mathcal{L}(\Psi, \partial\Psi, x) = \mathcal{L}(\psi, \partial\psi, x) + \partial_\alpha F_\alpha(\psi, x, \varepsilon)}$$

⇒ ∃ 守恒流 $\boxed{\Gamma_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_I)} \eta_I - G_\alpha}$

$$\eta_I \triangleq \frac{\partial \Psi_I}{\partial \varepsilon}\bigg|_{\varepsilon=0} \qquad G_\alpha = \frac{\partial F_\alpha}{\partial \varepsilon}\bigg|_{\varepsilon=0}.$$

守恒流即 $\boxed{\partial_\alpha \Gamma_\alpha = 0}$

□ □ □ □

Mo Tu We Th Fr Sa Su

Memo No.

Date 2019 / 11 / 16

CH4 中心力及散射

§1 一维运动 $L = T - U = \frac{1}{2}m\dot{x}^2 - U(x) \quad (m>0)$

一、运动方程及其解析解

$$m\ddot{x} = -\frac{\partial U}{\partial x} = F_x$$

$$E = \frac{1}{2}m\dot{x}^2 + U(x)$$

$$\Rightarrow \boxed{\int dt = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E-U(x)}} = t(x)} \Rightarrow x = x(t)$$

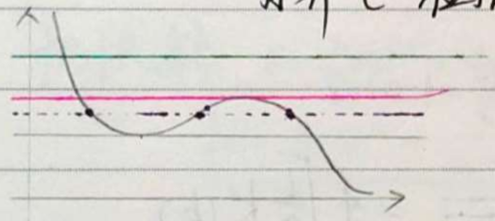
二、定性分析 $T \triangleq \frac{1}{2}m\dot{x}^2 = E - U(x) \geqslant 0$

1. 运动范围 $\boxed{U(x) \leq E}$ 与初始条件(位置)

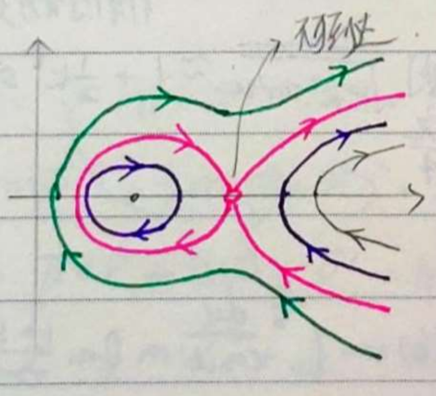
def: 转折点 $\dot{x}=0$ 即 $E = U(x)$

2. 运动分类: 无界

有界(一维情况下,很可能周期运动).



平衡: $U'(x) = 0 \begin{cases} U''(x) > 0 & 稳 \\ U''(x) < 0 & 不稳 \end{cases}$



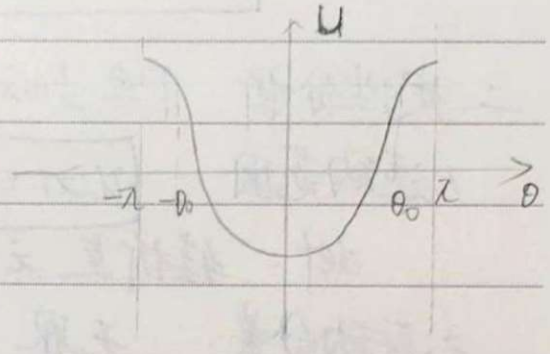
三. 有界运动的周期

$$\boxed{\tau=\sqrt{2m}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E-U(x)}}}\qquad U(1,2)=E$$

eg. 简谐振动

$$E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2=\frac{1}{2}m\omega^2A^2$$

$$\tau=2\sqrt{2m}\int_0^A\frac{dx}{\sqrt{\frac{1}{2}m\omega^2(A^2-x^2)}}=\frac{4}{\omega}\arcsin\frac{x}{A}\Big|_0^A=\frac{2\pi}{\omega}$$



eg. $E=\frac{1}{2}ml^2\dot{\theta}^2-mgl\cos\theta$

$=-mgl\cos\theta_0$

$$\tau=4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}$$

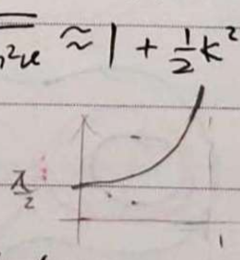
$$=2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}$$

def: $k\triangleq\sin\frac{\theta_0}{2}$, $\sin\frac{\theta}{2}=k\sin u$

则 $\tau=4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{1-k^2\sin^2u}}=4\sqrt{\frac{l}{g}}\underbrace{K(k)}_{\text{椭圆积分}}$

i) 当 $\theta_0\ll1$, $k\approx\frac{\theta_0}{2}\ll1$, 则 $\frac{1}{\sqrt{1-k^2\sin^2u}}\approx1+\frac{1}{2}k^2\sin^2u$.

则 $K(k)\approx\frac{\pi}{2}+\frac{1}{2}(\frac{\theta_0}{2})^2\cdot\frac{\pi}{4}$.



$\tau\approx2\pi\sqrt{\frac{l}{g}}[1+\frac{1}{16}\theta_0^2]$

ii) 当 $\theta_0=\pi$, $\Rightarrow k=1\Rightarrow K(k)=\int_0^{\frac{\pi}{2}}\frac{du}{\cos u}=\ln\frac{1+\sin u}{\cos u}\Big|_0^{\frac{\pi}{2}}\to\infty$

§2 中心力问题

$$\vec{F} = F(r)\hat{r} \begin{cases} F(r) > 0 & 斥力 \\ F(r) < 0 & 引力 \end{cases}$$

$$U = U(r) \quad \begin{cases} U = -\int \vec{F}\cdot d\vec{r} = -\int F dr \\ F = -\frac{\partial U}{\partial r} \end{cases}$$

$F=-\nabla U$ 别忘负号.

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - U(r)$$

一、运动常数

1. 相对于力心的角动量 $\vec{l} = \vec{r}\times m\vec{v} = 2m\frac{d\vec{S}}{dt}$

$$d\vec{S} = \frac{1}{2}\vec{r}\times d\vec{r}, \quad \frac{d\vec{S}}{dt} = \frac{1}{2}\vec{r}\times\vec{v}$$

coro. 粒子只能在同一个平面运动

$$\vec{r} = \vec{r}(r,\theta), \quad l = mr^2\dot{\theta} \Rightarrow \boxed{\dot{\theta} = \frac{l}{mr^2}}$$

2. 机械能 $E = \frac{1}{2}m\dot{\vec{r}}^2 + U(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r)$

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + U(r)}$$

二、径向运动

$E = \frac{1}{2}m\dot{r}^2 + V(r)$，有效势能 $V(r) = \overset{离心势能}{\frac{l^2}{2mr^2}} + U(r)$

$$\vec{v} = \vec{v}' + \vec{\omega}\times\vec{r}, \quad \vec{\omega} = \dot{\theta}\hat{z}, \quad \vec{v}' = \dot{r}\hat{r}$$

$$L = \frac{1}{2}m\vec{v}'^2 + \frac{1}{2}m(\vec{\omega}\times\vec{r})^2 + m\vec{v}'\cdot(\vec{\omega}\times\vec{r}) - U(r)$$

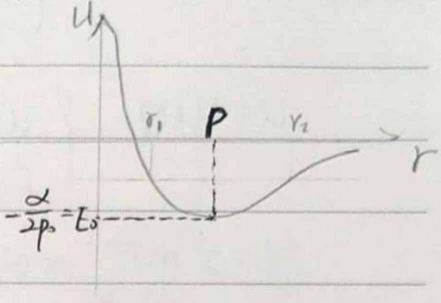
而 $\vec{\beta}\times\vec{r} + 2\vec{\omega}\times\vec{v}' = (r\ddot{\theta} + 2\dot{r}\dot{\theta})(\hat{z}\times\hat{r}) = \vec{a}_\theta = 0$.

$$\Rightarrow L = \frac{1}{2}m\vec{v}'^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r)$$

1. 定性分析.

eg. 平方反比引力: $U=-\frac{\alpha}{r}$, $\vec{F}=-\frac{\alpha}{r^2}\hat{r}$,

$$V(r)=-\frac{\alpha}{r}+\frac{l^2}{2mr^2}$$



def: $p\triangleq\frac{l^2}{m\alpha}$, ⇒ $V=\frac{\alpha}{2p}\left[\frac{p^2}{r^2}-2\frac{p}{r}\right]$

⇒ $V=\frac{\alpha}{2p}\left[\left(\frac{p}{r}-1\right)^2-1\right]$

i) 当$E=E_0$, $r=p$. 做圆周运动.

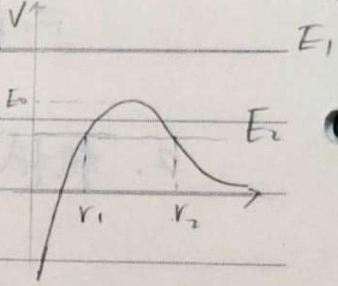
ii) 当 $E_0<E<0$, 有界

⇒ def: $\varepsilon\triangleq\sqrt{1+\frac{2pE}{\alpha}}=\sqrt{1+\frac{2El^2}{m\alpha^2}}$, $r_{1,2}=\frac{p}{1\pm\varepsilon}$

2. 拱点: $\dot{r}=0$, 拱线: 力心与拱点连线

轨道关于拱线一定是对称的.

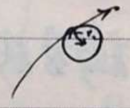
3. 进动角 $\Delta\theta$ 闭合条件: $\boxed{\Delta\theta=\frac{n_1}{n_2}2\pi}$

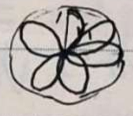


eg. $U=-\frac{\alpha}{r^3}$ ⇒ $V(r)=-\frac{\alpha}{r^3}+\frac{l^2}{2mr^2}$

$E_1>E_0$

$E_2<E_0$





三、圆周运动（以力心为圆心）及其稳定性.

$$E=\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\dot{\theta}^2+U(r)=\frac{1}{2}m\dot{r}^2+\underbrace{\frac{l^2}{2mr^2}+U(r)}_{\triangleq V(r)}$$

$$\left.\frac{\partial V}{\partial r}\right|_{r=R}=-F(R)-\frac{l^2}{mR^3}=0 \Rightarrow F(R)=-\frac{l^2}{mR^3}<0$$

即要求在平衡位置处中心力是引力

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=R}=-F'(R)-\frac{3F(R)}{R}>0$$

即 $\boxed{\frac{RF'(R)}{F(R)}>-3}$

当满足上述条件时，微扰 $r=R+\xi$

$$E=\frac{1}{2}m\dot{\xi}^2+\frac{1}{2}V''(R)\xi^2 \Rightarrow \boxed{\omega_r=\sqrt{\frac{V''(R)}{m}}}$$

eg. $\vec{F}=-\alpha r^n\hat{r}$ ，要求 $\alpha>0$

$$V''(R)=n\alpha R^{n-1}+3\alpha R^{n-1}>0 \Rightarrow \underset{\sim\sim\sim}{n>-3},\ \omega_r=\sqrt{\frac{(n+3)\alpha R^{n-1}}{m}}$$

① $\vec{F}=-\alpha\vec{r}\ (n=1) \Rightarrow \omega_r=2\sqrt{\frac{\alpha}{m}} \Rightarrow T_r=\pi\sqrt{\frac{m}{\alpha}}$

② $\vec{F}=-\frac{\alpha}{r^2}\hat{r}\ (n=-2) \Rightarrow \omega_r=\sqrt{\frac{\alpha}{mR^3}} \Rightarrow T_r=2\pi\sqrt{\frac{m}{\alpha}}R^{\frac{3}{2}}$

四. 轨道方程

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{L}{mr^2}\frac{dr}{d\theta} = -\frac{L}{m}\frac{du}{d\theta} \quad (u \triangleq \frac{1}{r})$$

$$\Rightarrow E = \frac{l^2}{2mr^4}\left(\frac{dr}{d\theta}\right)^2 + V(r)$$

$$\Rightarrow \quad d\theta\sqrt{E-V} = \pm\frac{l}{\sqrt{2m}r^2}dr$$

1. 积分形式 ⇒ $\boxed{\theta = \pm\frac{l}{\sqrt{2m}}\int\frac{dr}{r^2\sqrt{E-V(r)}}}$

进动角 $\Delta\theta = \frac{2l}{\sqrt{2m}}\int_{r_1}^{r_2}\frac{dr}{r^2\sqrt{E-V(r)}}$, $V(r_{1,2}) = E$

$$\boxed{E = \frac{l^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + U}$$

2. 一阶方程 $\boxed{\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{l^2}(E-U)}$

$$\Rightarrow 2u'u'' + 2uu' = -\frac{2m}{l^2}\frac{dU}{du}u'$$

$$\Rightarrow u'' + u + \frac{m}{l^2}\frac{dU}{du} = 0$$

→ ssj: 弄清楚这里 F(u) 的含义!

3. 二阶方程 (Binet 公式) $\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2}F(u)}$

指将 $r=\frac{1}{u}$ 代入 F(r)，而不是直接代 u.

作 θ → −θ 变换, 方程不变 $\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2}F(\frac{1}{u})$ → 代入.

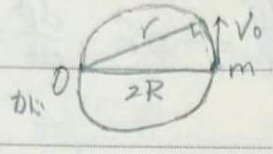
初始条件 $u(\theta=0) = u_0$ 不变

$$\frac{du}{d\theta}(\theta=0) = u_0' \longrightarrow \frac{du}{d\theta}(\theta=0) = -u_0'$$

不变, 当 $u_0' = 0$ 即 $\left.\frac{du}{d\theta}\right|_{\theta=0} = 0 \Leftrightarrow \left.\dot{r}\right|_{\theta=0} = 0$

即轨道一定是关于拱线对称的.

附注：一阶方程等价于能量守恒方程。因此下面第一个例子中已知 $r=r(\theta)$ 就不必再写成 $u=u(\theta)$ 代入1阶方程求势能，而是直接代入 $E=\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\dot{\theta}+U$ 即可

eg. 

$r=2R\cos\theta$

$U(r\to\infty)=0$

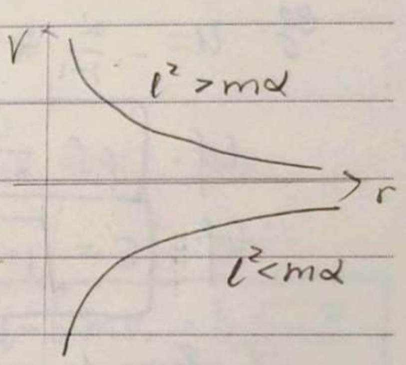
① $l=mr^2\dot{\theta}=2Rmv_0 \Rightarrow \dot{\theta}=\frac{2Rv_0}{r^2}$

② $\dot{r}=-2R\dot{\theta}\sin\theta \Rightarrow \boxed{v^2=\dot{r}^2+r^2\dot{\theta}^2}=4R^2\dot{\theta}^2=\frac{16R^4v_0^2}{r^4}$

③ $E=\frac{1}{2}mv^2+U(r)=\frac{8mR^4v_0^2}{r^4}+U(r)$

$E=0$, $U=-\frac{8mR^4v_0^2}{r^4}$, $F=-\frac{dU}{dr}$

为了保证能量在0处不发散



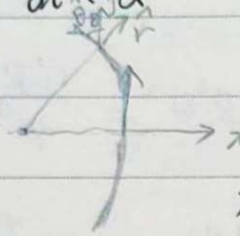
eg. $\vec{F}=-\frac{\alpha}{r^3}\hat{r}=-\alpha u^3\hat{r}$

$\Rightarrow \frac{d^2u}{d\theta^2}+\left(1-\frac{m\alpha}{l^2}\right)u=0.$

$V=\frac{l^2}{2mr^2}-\frac{\alpha}{2r^2}$

$$m\frac{d\vec{v}}{dt} = -\frac{\alpha}{r^2}\hat{r} = +\frac{\alpha}{r^2\dot{\theta}}\frac{d\hat{\theta}}{dt}$$ 利用：$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$，$\frac{d\hat{r}}{dt} = -\dot{\theta}\hat{\theta}$

$$\frac{d}{dt}\left(\frac{l}{\alpha}\vec{v} - \hat{\theta}\right) = 0$$ （选取如图为坐标）



$$\Rightarrow \frac{l}{\alpha}\vec{v} - \hat{\theta} = \vec{c} \triangleq \varepsilon\hat{y}$$

在θ处，将两边投影

$$1+\varepsilon\cos\theta = \frac{m^2r^2v_\theta^2}{\alpha m r}$$

$$\left(\frac{l}{\alpha}\vec{v} - \hat{\theta}\right)\cdot\hat{\theta} = \varepsilon\hat{y}\cdot\hat{\theta} = \varepsilon\cos\theta \Rightarrow r = \frac{\frac{l^2}{\alpha m}}{1+\varepsilon\cos\theta}$$

eg. $U = -\frac{\alpha}{r} = -\alpha u \Rightarrow \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2mE}{l^2} + \frac{2m\alpha}{l^2}u$

def: $\boxed{p \triangleq \frac{l^2}{m\alpha}}$ $\Rightarrow \left(\frac{du}{d\theta}\right)^2 + \left(u-\frac{1}{p}\right)^2 = \frac{1}{p^2}\left(1+\frac{2pE}{\alpha}\right)$

def: $\boxed{\varepsilon = \sqrt{1+\frac{2pE}{\alpha}} = \sqrt{1+\frac{2El^2}{m\alpha^2}}}$ $\Rightarrow \left(\frac{du}{d\theta}\right)^2 + \left(u-\frac{1}{p}\right)^2 = \left(\frac{\varepsilon}{p}\right)^2$

记 $\frac{\varepsilon}{p}\cos\phi = u - \frac{1}{p}$，$\frac{du}{d\theta} = \frac{\varepsilon}{p}\sin\phi = -\frac{\varepsilon}{p}\sin\phi\frac{d\phi}{d\theta}$

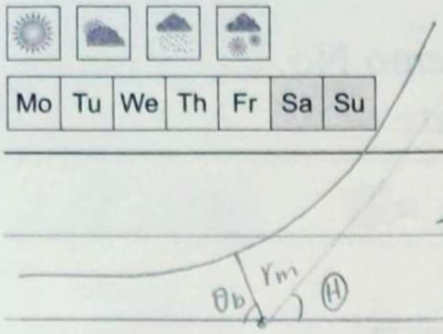
$$\Rightarrow \frac{d\phi}{d\theta} = -1, \quad \phi = -\theta + \theta_0$$

故 $\boxed{r = \frac{p}{1+\varepsilon\cos(\theta-\theta_0)}}$ 吸引力，以力心为内焦点的双曲线

$\boxed{r = \frac{p}{-1+\varepsilon\cos(\theta-\theta_0)}}$ ，排斥力，以力心为外焦点的双曲线

下面这个p和上面的p不太一样。

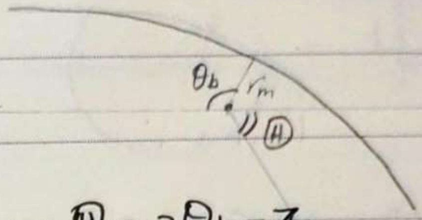
平方反比力



$\Theta = \pi - 2\theta_b$

$\cos\theta_b = \frac{1}{\varepsilon}$

$\Rightarrow \boxed{\sin\frac{\Theta}{2} = \frac{1}{\varepsilon}}$



$\Theta = 2\theta_b - \pi$

$\cos\theta_b = \frac{-1}{\varepsilon}$

$\boxed{\sin\frac{\Theta}{2} = \frac{1}{\varepsilon}}$

§3 散射 (Scattering)

一、假设：当 $r\to\infty$ 时 $F\to 0$ 且 $U\to 0$

入射粒子能量 $E=\frac{1}{2}mv_0^2$

碰撞参数（瞄准距离）b，则 $\boxed{l=b\cdot mv_0 = b\sqrt{2mE}}$

二、散射问题

1、正问题：$U=U(r)$ 已知，$\Rightarrow r=r(\theta, b, E)$

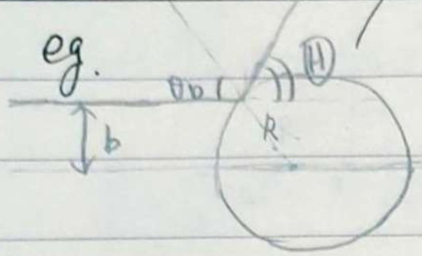
$\Rightarrow \Theta = \Theta(b, E)$

eg. $\varepsilon^2 = 1+\frac{2El^2}{m\alpha^2} = 1+\left(\frac{2bE}{\alpha}\right)^2 = \frac{1}{\sin^2\frac{\Theta}{2}} = 1+\cot^2\frac{\Theta}{2}$

$\Rightarrow \boxed{b=\frac{a}{2}\cot\frac{\Theta}{2} \quad , a \triangleq \frac{\alpha}{E}}$

$\boxed{b=\frac{\alpha}{2E}\cot\frac{\Theta}{2}}$

eg.



刚性

$$R\sin\theta_b = b$$

$$\Theta = \pi - 2\theta_b$$

$$\Rightarrow \boxed{b = R\cos\frac{\Theta}{2}}$$

中心排斥力

$$\Theta = \pi - 2\theta_b ,$$

$$\theta_b = \frac{l}{\sqrt{2m}}\int_{r_m}^{\infty}\frac{dr}{r^2\sqrt{E-V}} = \frac{l}{\sqrt{2m}}\int_{r_m}^{\infty}\frac{dr}{r^2\sqrt{E-U-\frac{l^2}{2mr^2}}}$$

$$\Rightarrow \boxed{\theta_b = \int_{r_m}^{\infty}\frac{b\,dr}{r^2\sqrt{1-\frac{U}{E}-\frac{b^2}{r^2}}}} , \quad 1-\frac{U(r_m)}{E}-\frac{b^2}{r_m^2}=0$$

eg. 上面刚性球，

$$U(r) = \begin{cases} 0 , & r>R \\ \infty , & r\leq R \end{cases} \qquad r_m = \begin{cases} R , & b<R \\ b , & b\geq R \end{cases}$$

$$\theta_b = \int_{r_m}^{\infty}\frac{b\,dr}{r^2\sqrt{1-\frac{b^2}{r^2}}} \overset{u=\frac{1}{r}}{=\!=\!=} \int_0^{r_m}\frac{b\,du}{\sqrt{1-b^2u^2}} = \arcsin(bu)\Big|_0^{\frac{1}{r_m}}$$

$$= \arcsin\frac{b}{r_m} = \begin{cases} \arcsin\frac{b}{R} , & b\leq R \\ \frac{\pi}{2} , & b>R \end{cases}$$

$$\Theta = \pi - 2\theta_b = \begin{cases} \pi - 2\arcsin\frac{b}{R} , & b\leq R \\ 0 , & b>R \end{cases}$$

2 逆问题. 已知 $\Theta = \Theta(b;E) \xrightarrow{?} U(r)$

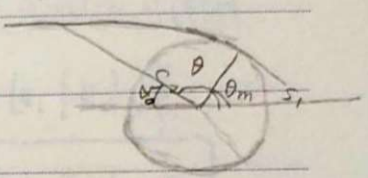
三 截面 (cross section)

问题：从多大面积 σ 内，入射粒子会：

① 与刚性球发生碰撞：$\sigma = \pi R^2$.

② 被地球俘获 $\sigma = \pi R^2\left(1+\frac{\alpha}{RE}\right)$ $\left(E = \frac{l^2}{2mr_m^2} - \frac{\alpha}{r_m}\right)$

$$r = \frac{p}{1+\varepsilon\cos(\theta-\theta_m)} = \frac{R(1+\varepsilon)}{1+\varepsilon\cos(\theta-\theta_m)}$$



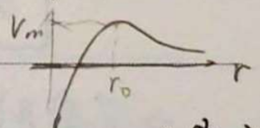
当 $\theta \to \pi$ 时 $r \to \infty$，$\cos\theta_m = \frac{1}{\varepsilon}$

再利用 $\varepsilon = \sqrt{1+\frac{2El^2}{m\alpha^2}}$

(或再利用 $r\sin\alpha_0 \to b$ 在 $\alpha \to 0$ 时.

③ 落向地心：i) $U = -\frac{\alpha}{r^3}$，$V = \frac{L^2}{2mr^2} - \frac{\alpha}{r^3} = \frac{b^2E}{r^2} - \frac{\alpha}{r^3}$

$V_{max} = \frac{\alpha}{2r_0^3}$ 在 $r_0 = \frac{3\alpha}{2b^2E}$ 取得.

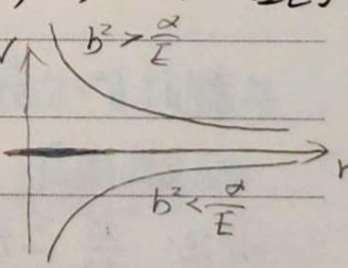


落向条件：$E > V_{max} \Rightarrow b^2 < 3\left(\frac{\alpha}{2E}\right)^{\frac{2}{3}}$，$\sigma = 3\pi\left(\frac{\alpha}{2E}\right)^{\frac{2}{3}}$

ii) $U = -\frac{\alpha}{r^2}$，$V = \frac{1}{r^2}(b^2E - \alpha)$

落向条件：$b^2 < \frac{\alpha}{E}$.

$\sigma = \frac{\pi\alpha}{E}$



入射流强度 $\boxed{J \triangleq \frac{N}{s \cdot t}}$

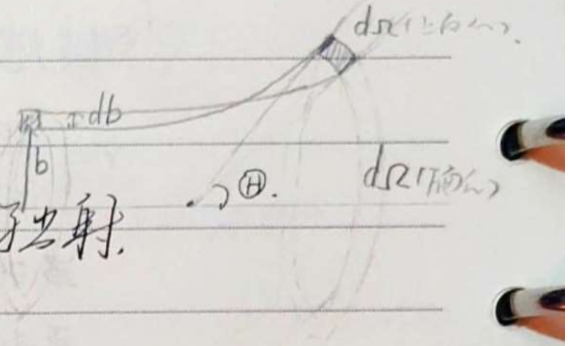
单位时间发生散射的粒子总数 $n \propto J$.

$\Rightarrow$ 总散射截面 $\boxed{\sigma \triangleq \frac{n}{J}}$

四. 微分散射截面.

$dn \propto J d\Omega \quad \Rightarrow \quad \boxed{\frac{d\sigma}{d\Omega} \triangleq \frac{1}{J} \frac{dn}{d\Omega}}$

均为从无穷远处测量,



单位时间有多少粒子入射,就有多少粒子出射.

$$\boxed{dn = J d\sigma = J \frac{d\sigma}{d\Omega} \cdot d\Omega.}$$

以下只考虑轴对称问题.

$$dn = |2\pi b \cdot db| J = J \frac{d\sigma}{d\Omega} |2\pi \sin\Theta \, d\Theta.|$$

考虑到 db, dΘ 正负.

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta} \left| \frac{db}{d\Theta} \right| = \frac{1}{2} \left| \frac{db^2}{d\cos\Theta} \right|}$$

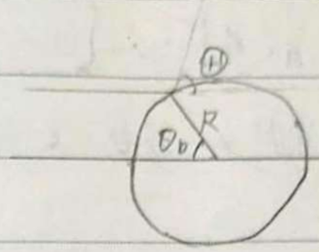
$\Rightarrow$ 总散射截面 $\boxed{\sigma = \int \frac{d\sigma}{d\Omega} \cdot d\Omega = 2\pi \int \frac{d\sigma}{d\Omega} \sin\Theta \, d\Theta}$

* 常见的 $b^2 = b^2(\Theta)$ 表达式中与Θ有关的不全是 $\cos\theta$,

也可直接微商 $2b \frac{db}{d\Theta}$ 又多出来个b 与外面的b 类似,

ssj 注: $\frac{d\sigma}{d\Omega}$ 的作为概率密度的意义

eg. 刚性球



① 弹性. $\sin\theta_b = \frac{b}{R} = \sin\frac{\pi-\Theta}{2}$

$\Rightarrow b = R\cos\frac{\Theta}{2}$

$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{4}R^2$

$\Rightarrow \sigma = \pi R^2$ 与前面结论一致.

eg, $U = -\frac{\alpha}{r^2}$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{l^2}(E-U) = \frac{1}{b^2} - \frac{\alpha}{b^2E}u^2$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + \Gamma^2u^2 = \frac{1}{b^2},\quad \Gamma^2 = 1+\frac{\alpha}{b^2E}$$

(类比谐振). $u = \frac{1}{r} = \frac{1}{b\Gamma}\cos\Gamma\theta \Rightarrow r = \frac{b\Gamma}{\cos\Gamma\theta}$

→ 最近点与最远点转过之角

当 $\theta_b = \frac{\pi}{2\Gamma}$ 时, $\Theta = \pi - 2\theta_b = \pi(1-\frac{1}{\Gamma})$

$$\Rightarrow b^2 = \frac{(\pi-\Theta)^2}{(2\pi-\Theta)\Theta}\frac{\alpha}{E}.$$

eg. $U = \frac{\alpha}{r}$. (Rutherford 散射)

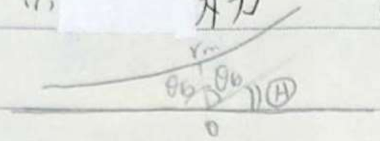
$$\left.\begin{aligned}&\sin\frac{\Theta}{2} = \frac{1}{\varepsilon},\\&\varepsilon^2 = 1+\frac{2El^2}{m\alpha^2} = 1+\left(\frac{2bE}{\alpha}\right)^2\end{aligned}\right\} \Rightarrow b = \frac{a}{2}\cot\frac{\Theta}{2}\quad \left(a=\frac{\alpha}{E}\right)$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{a^2}{16\sin^4\frac{\Theta}{2}}}$$ (积分所得 σ 发散 $\to +\infty$)

物理意义：散射发生之概率与动量改变量之四次方成反比

1. 中心力散射下之 $\boxed{\Theta = |(2\Theta_b-\pi)-2n\pi|}$，保证 $\Theta\in[0,\pi]$

(1) [redacted] 斥力 $\Theta = \pi - 2\Theta_b = \pi - 2\int_{r_m}^{\infty}\frac{b\,dr}{r^2\sqrt{1-\frac{U}{E}-\frac{b^2}{r^2}}}$

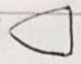


(2) [redacted] 引力.

假设转了k圈，考虑到 Θ 为入射、出射方向之夹角，

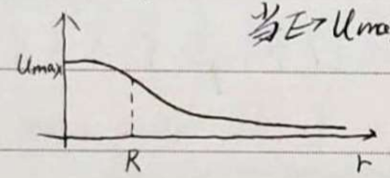
$\Theta = 2\Theta_b - \pi - 2k\pi$.

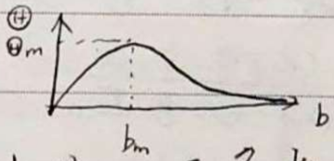
2. 积分范围

(1) 非中心力，例如  之刚性物体.

(2) 中心力也可能出现：

当 $E>U_{max}$，$b=0$ 时 $\Theta=0$，$b\to\infty$ 时 $\Theta=0$.





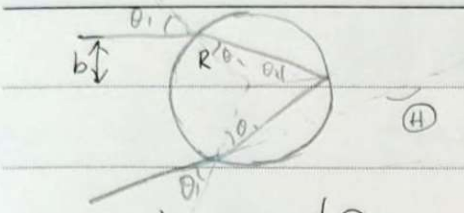
此时 $\Theta=\Theta(b)$ 不能找到唯一之反函数.

修正：$J\sum b_i 2\pi\cdot db_i = J\frac{d\sigma}{d\Omega}|2\pi\sin\Theta\, d\Theta|$

⇒

3. 彩虹(Rainbow)散射

$$\boxed{\frac{d\sigma}{d\Omega} = \sum_i \frac{b_i}{\sin\Theta}\left|\frac{db_i}{d\Theta}\right|}$$



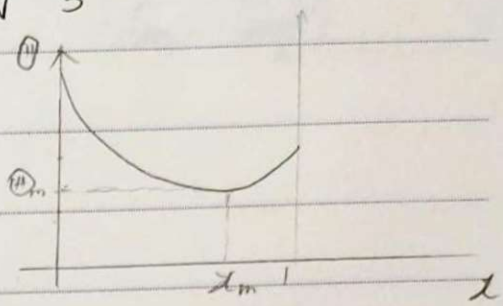
$$\Theta = \pi + 2\theta_1 - 4\theta_2$$

$$= \pi + 2\arcsin\frac{b}{R} - 4\arcsin\frac{b}{nR}$$

def: $x = \frac{b}{R} \Rightarrow \frac{d\Theta}{dx} = 0 \Rightarrow$ 极值点 $x_m = \sqrt{\frac{4-n^2}{3}}$

一般，$x_m = x(n) \approx 0.88$

$\Theta_m = \Theta_m(n) \approx 137.5°$



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Date / /

CH5. Hamilton 力学

§1. 相空间.

一、速度相空间 $(q,\dot q)$

"上半空间由左向右，下半空间由右向左"

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot q_k}}_{q,\dot q,\ddot q,t} = \underbrace{\frac{\partial L}{\partial q_k}}_{q,\dot q,t} \Rightarrow \frac{d}{dt}\underbrace{\begin{pmatrix} q \\ \dot q \end{pmatrix}}_{\xi} = \underbrace{\begin{pmatrix} \dot q \\ \omega(q,\dot q,t) \end{pmatrix}}_{\underline{X}}$$

→ 事实上是速度的定义.

→ 限制运动的部分，动力学方程.

↓ 一定关于 $\ddot q$ 是线性的

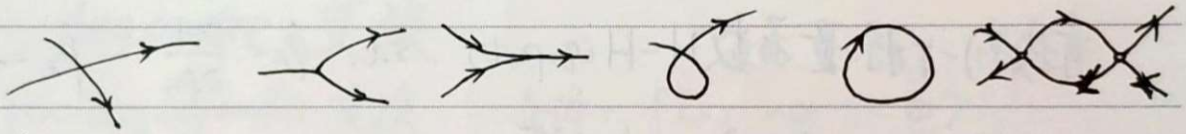
2s 个分量.

→ 相当速度场

$$\Rightarrow \dot\xi = \underline{X}(\xi,t), \quad \xi = \begin{pmatrix} q \\ \dot q \end{pmatrix}$$

$$\xi(t) \to \underline{X}(\xi,t) \to \dot\xi(t) \to \xi(t+\Delta t) = \xi(t) + \Delta t \cdot \dot\xi(t)$$

位置决定其运动(速度)



当 $\underline{X}=(\xi,$ ~~t~~$)$	×	×	×	×	✓	✓
当 $\underline{X}=(\xi,t)$				✓	✓	✓

二、(q,p) 相空间

$$p_k = \frac{\partial L}{\partial \dot q_k} = p_k(q,\dot q,t) \xrightarrow{反变换} \boxed{\dot q_k = \dot q_k(q,p,t)} : (q,p) \Rightarrow (q,\dot q)$$

即关键是是否存在这样的反变换，

导致就不确定性了的 $(q,\dot q)$

$$L = T - U = \frac{1}{2} A_{ij} \dot{q}_i \dot{q}_j + B_j \dot{q}_j + C$$

$$= \frac{1}{2} \dot{q}^T A \dot{q} + B^T \dot{q} + C$$ 在经典情况及考虑Lorentz力时,

$$\Rightarrow p = \frac{\partial L}{\partial \dot{q}} = A\dot{q} + B \Rightarrow \boxed{\dot{q} = A^{-1}(p - B)}$$

即使考虑相对论, 也可将(q,p)作为状态参量.

eg. $L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$

$$\Rightarrow \vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow c^2p^2 - p^2v^2 = m^2c^2v^2 \Rightarrow \vec{v} = \frac{c\vec{p}}{\sqrt{p^2 + mc^2}}$$

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} f(q,p,t) \\ g(q,p,t) \end{pmatrix} \Rightarrow \dot{\xi} = X(\xi, t), \quad \xi = \begin{pmatrix} q \\ p \end{pmatrix}$$

上面这个不再是定义.

可找到一个标量函数 $H = H(q,p,t)$ s.t. $\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}$

此时, q, p 运动力学地位对等.

$$\dot{q}_k = [q_k, H], \quad \dot{p}_k = [p_k, H] \Rightarrow \dot{\xi}_k = [\xi_k, H]$$

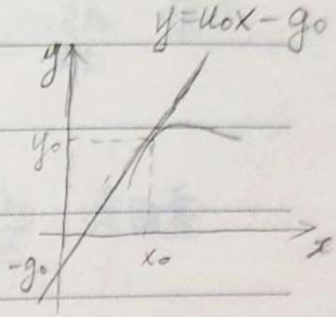
$$\begin{cases} Q \\ p \end{cases}$$

(通过正则变换) 使得每个 q_k 都不显含于 H ⇒ $H = H(p, t)$ 变成直线

q_k, p_k - - — - - ⇒ $H = H(t) \to 0$ 变成点

哈密顿-雅可比理论. 作用变量角变量理论.

函数曲线.



$$x_0 \rightarrow y_0=f(x_0) \rightarrow (x_0,y_0) \rightarrow \text{点} \longrightarrow \text{曲线}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow$$

$$u_0=f'(x_0) \rightarrow y_0=u_0x_0-g_0 \longrightarrow (u_0,g_0) \rightarrow \text{直线}$$

$$\begin{cases} u=f'(x)=u(x) \overset{?}{\Longrightarrow} x=x(u) . \\ g=ux-f(x)=g(x) \overset{?}{\Longrightarrow} g(u) \end{cases}$$

即要求 $u(x)$ 单调, 即 Hess 条件: $u'(x)=f''(x)>0\ (<0)$

↓ 允许在个别点上 $u'(x)=0$, 但以后不考虑.

即 $u'(x)=f''(x)\neq 0$

§2 Legendre 变换

一、$f(x)$ 的 Logendre 变换: 条件: $f''(x)>0\ (<0)$

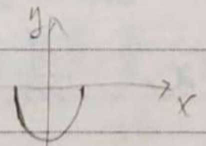
$\boxed{g(u)\triangleq u\cdot x-f(x)}$ 其中 $\boxed{u\triangleq f'(x)=u(x)} \Rightarrow x=x(u)$

从而 $u_0 \rightarrow g=g(u_0) \rightarrow$ 确定直线 $y=u_0x-g_0 \leftrightarrow$ 包络线 $y=f(x)$

eg. $f(x)=\frac{1}{2}ax^2 \quad (a>0)$

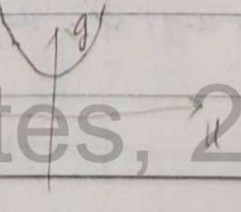
$u=ax, \quad g=\frac{1}{2}ax^2 \Rightarrow g(u)=\frac{u^2}{2a}$

eg. $f(x)=-b\sqrt{1-\frac{x^2}{a^2}}$



$$u=\frac{b\frac{x}{a^2}}{\sqrt{1-\frac{x^2}{a^2}}} \Rightarrow g=\frac{b}{\sqrt{1-\frac{x^2}{a^2}}}, \quad x=\frac{au}{\sqrt{u^2+\frac{b^2}{a^2}}}$$

$g(u)=\sqrt{b^2+a^2u^2}$



而 $\frac{dg}{du} = \frac{du}{du}x + u\frac{dx}{du} - \frac{df}{dx}\frac{dx}{du} = x$

变回去 $\frac{dg}{du} = x$，$\frac{d^2g}{du^2} = \frac{dx}{du}$ 满足Hess条件.

$xu - g(u) = f(x)$ （根据定义）

即新旧两个函数互为Legendre变换.

$f(x) + g(u) = ux$

$u = f'(x)$　　$x = g'(u)$

二. $f(x,y)$ 对x的 Legendre 变换：条件：对$\forall y$, $\frac{\partial^2 f}{\partial x^2} > 0\ (<0)$

$\boxed{g(u,y) \triangleq ux - f(x,y)}$

其中 $\boxed{u \triangleq \frac{\partial f}{\partial x} = u(x,y)}$ ⇒ $x = x(u,y)$

eg. $f(x,y) = \frac{1}{2}ax^2 - \frac{1}{2}by^2$　$(a,b>0)$

$u = ax$,　$g = \frac{1}{2}ax^2 + \frac{1}{2}by^2$

$\Rightarrow g(u,y) = \frac{u^2}{2a} + \frac{1}{2}by^2$

讨论：$\frac{\partial g}{\partial u} = \frac{\partial u}{\partial u}x + u\cancel{\frac{\partial x}{\partial u}} - \cancel{\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}} = x$

即仍有 $\frac{\partial g}{\partial u} = x$，$\frac{\partial f}{\partial x} = u$，

$$\frac{\partial g}{\partial y} = \overset{=0}{\cancel{\frac{\partial y}{\partial y}}}x + \underbrace{u\frac{\partial x}{\partial y}} - \left(\frac{\partial f}{\partial y} + \underbrace{\frac{\partial f}{\partial x}\frac{\partial x}{\partial y}}\right) = -\frac{\partial f}{\partial y}$$

但注意此时在 $g(u,y)$ 中，u,y 独立而 x,y 不独立.

$\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y} = 0$，$f(x,y) + g(u,y) = ux$

三. $f(x,y) = f(x_1,\cdots,x_n; y_1,\cdots,y_n)$ 对n个变量 $x=(x_1,\cdots,x_n)$ 的Legendre变换.

$$\boxed{g(u,y) \triangleq u_k x_k - f(x,y)}$$

其中 $u = \frac{\partial f}{\partial x} = u(x,y)$, 即 $u_k = \frac{\partial f}{\partial x_k} = u_k(x,y)$

找到 $x_k = x_k(u,y)$

即 $\frac{\partial u_i}{\partial x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ 方阵应满足 $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) > 0\ (<0)$

eg. $f = \frac{1}{2}ax^2 + \frac{1}{2}bx^2$ $(a,b>0)$

$u = \frac{\partial f}{\partial x} = ax$, $v = \frac{\partial f}{\partial y} = by$

$g = ux + vy - f = \frac{1}{2}ax^2 + \frac{1}{2}bx^2 = \frac{u^2}{2a} + \frac{v^2}{2b}$

$(u_0, v_0) \xrightarrow{知} g_0$, 确定平面 $z = u_0 x + v_0 y - g_0$

$L = L(q,\dot{q},t) \longrightarrow H(q,p,t)$

Legendre 变换的法则(信息)

(1) 新旧函数之和 = 新旧自变量乘积之和

$$\boxed{f(x,y) + g(u,y) = u_i x_i}$$

(2) 旧自变量 = 新函数对新自变量的偏导数

新自变量 = 旧函数对旧自变量的偏导数

$$\boxed{x_i = \frac{\partial g}{\partial u_i} \qquad u_i = \frac{\partial f}{\partial x_i}}$$

(3) 新旧函数对其同自变量(参数)的偏导数之和 = 0

$$\boxed{\frac{\partial f}{\partial y_k} + \frac{\partial g}{\partial y_k} = 0 \qquad \frac{\partial f}{\partial \lambda} + \frac{\partial g}{\partial \lambda} = 0}$$

§3. Hamilton 方程

$$(q,\dot{q}) \longleftrightarrow (q,p)$$

$$\Uparrow$$

$L(q,\dot{q},t)$　　$H(q,p,t)$　包含同样多之信息.

一. Hamilton 函数

$$H(q,p,t) = p_i\dot{q}_i - L(q,\dot{q},t) \quad . \quad \dot{q} = \dot{q}_i(q,p,t)$$

1. $\boxed{\text{数值上}, H(q,p,t) = h(q,\dot{q},t)}$ (后者是雅可比积分).

　(应有 $H(q,p,t) = h(q,\dot{q}(q,p,t),t)$)

2. $$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$
$$\|$$
$$\frac{dh}{dt} = \frac{dH}{dt}$$
$$\Rightarrow \boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

(或: $\frac{dH}{dt} = \frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k + \frac{\partial H}{\partial t} = -\dot{p}_k\dot{q}_k + \dot{q}_k\dot{p}_k + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$)

(但一般 $\frac{\partial h}{\partial t} \neq \frac{\partial H}{\partial t}$)

二. Hamilton 方程

$$\boxed{\begin{aligned}\dot{p}_k &= -\frac{\partial H}{\partial q_k}\\ \dot{q}_k &= \frac{\partial H}{\partial p_k}\end{aligned}} \qquad -\frac{\partial H}{\partial q_k} = \frac{\partial L}{\partial q_k} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \dot{p}_k$$

$(k = 1, \cdots, s)$

若H不显含t, 则 H为运动常数

若H 不显含q_k, 则 p_k 为运动常数, 称q_k为H之循环坐标

(同时自然地, 也为L之循环坐标)

eg. $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$

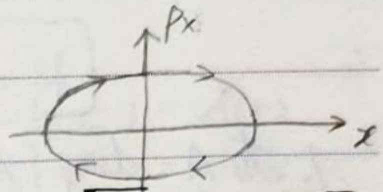
守恒

$p_x = m\dot{x}$.

$H = p_x\dot{x} - L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2x^2 = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 \quad (=E)$

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} & \text{(即将它又作了个反变换)} \\ \dot{p}_x = -m\omega^2 x \end{cases}$$

(若将 p_x, $\dot{p}_x$ 消去, $\ddot{x} = -\omega^2 x$)



1. 相轨迹: $\frac{d}{dt}\begin{pmatrix} x \\ p_x \end{pmatrix} = \begin{pmatrix} p_x/m \\ -m\omega^2 x \end{pmatrix} = \xi(x; p_x)$

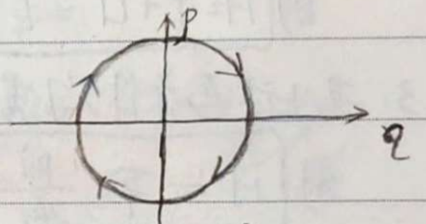
$a = \sqrt{\frac{2E}{m\omega^2}}$ $\quad b = \sqrt{2mE}$

$S_1 = \pi ab$

2. $q = \sqrt{m\omega}\, x$. $\Rightarrow L = \frac{\dot{q}^2}{2\omega} - \frac{1}{2}\omega q^2$

$p = \frac{p_x}{\sqrt{m\omega}}$ $\quad \Rightarrow H = \frac{1}{2}\omega(p^2 + q^2) \quad (=E)$

$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \omega\begin{pmatrix} p \\ -q \end{pmatrix}$



$R = \sqrt{\frac{2E}{\omega}}$, $\quad v = \omega\sqrt{p^2+q^2} = \omega R$

$S_2 = \pi R^2$

$S_1 = S_2$ (本质上是刘维尔定理)、

3. $L' = L + \frac{d}{dt}F(x,t) = L + \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial t}$

$$\underbrace{H' = L_2' - L_0'}\ = \frac{(p_x' - \frac{\partial F}{\partial x})^2}{2m} + \frac{1}{2}m\omega^2x^2 - \frac{\partial F}{\partial t}$$

($p_x' = m\dot{x} + \frac{\partial F}{\partial x}$, $h' = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2x^2 - \frac{\partial F}{\partial t}$.)

三. Hamilton 函数之一般形式:

设 $L = T - U = L_2 + L_1 + L_0 = \frac{1}{2}\dot{q}^T A \dot{q} + B^T \dot{q} + C$, A是正定对称阵.

$$\Rightarrow p = A\dot{q} + B \Rightarrow \dot{q} = A^{-1}(p - B)$$

则 $H = L_2 - L_0 = \frac{1}{2}(p-B)^T A^{-1} A A^{-1} (p-B) - C$

$$\boxed{H = \frac{1}{2}(p-B)^T A^{-1} (p-B) - C} \quad \boxed{H = L_2 - L_0}$$

1. 若 $A = \mathrm{diag}\{A_1, \cdots, A_s\}$

则 $\boxed{H = \sum_i \frac{(p_i - B_i)^2}{2A_i} - C}$

2. 若 $\vec{r} = \vec{r}(q, \cancel{t})$, 且 $U = U(q, \cancel{\dot{q}}, t)$

则 $\boxed{H = T + U = E}$ (不一定是常数)

3. 若上述两条件均满足, $B_i = 0$, $C = -U$

则 $\boxed{H = \sum_i \frac{p_i^2}{2A_i} + U}$

eg. $L = \frac{1}{2}mv^2 - e(\varphi - \vec{v}\cdot\vec{A}) = \frac{1}{2}m\dot{x}_i\dot{x}_i + eA_i\dot{x}_i - e\varphi$

$$H = \sum_i \frac{(p_i - eA_i)^2}{2m} + e\varphi \Rightarrow \begin{cases} \dot{x}_k = \frac{\partial H}{\partial p_k} = \frac{p_k - eA_k}{m} & ① \\ \dot{p}_k = -\frac{\partial H}{\partial x_k} = \frac{(p_i - eA_i)}{m} e\partial_k A_i - e\partial_k\varphi & ② \end{cases}$$

$$\left(\frac{\partial H}{\partial x_k} = \sum_i \frac{(p_i - eA_i)}{m} \cdot \frac{\partial(p_i - eA_i)}{\partial x_k} + e\frac{\partial\varphi}{\partial x_k} \right)$$

将①代入②. $m\ddot{x}_k + e\frac{dA_k}{dt} = e\dot{x}_i\frac{\partial A_i}{\partial x_k} - e\frac{\partial\varphi}{\partial x_k}$

$$\Rightarrow m\ddot{x}_k + e\frac{\partial A_k}{\partial x_i}\dot{x}_i + e\frac{\partial A_k}{\partial t} = e\frac{\partial A_i}{\partial x_k} - e\frac{\partial\varphi}{\partial x_k}$$

$$\Rightarrow m\ddot{x}_k = e\dot{x}_i(\partial_k A_i - \partial_i A_k) + e(-\partial_k\varphi - \partial_t A_k)$$

$$\Rightarrow \begin{aligned} &\parallel \\ &e\dot{x}_i \cdot \varepsilon_{kij}(\nabla\times\vec{A})_j \\ &e\dot{x}_i \varepsilon_{ijk}\dot{x}_i B_j \\ &e(\vec{v}\times\vec{B})_k \end{aligned} \quad \Rightarrow m\vec{a} = e(\vec{E} + \vec{v}\times\vec{B})$$

eg、 $L = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2] - U(r)$

$$A = \begin{pmatrix} m & & \\ & mr^2 & \\ & & mr^2\sin^2\theta \end{pmatrix} \Rightarrow H = \frac{p_r^2}{2m} + \frac{1}{2mr^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + U(r)$$

故 H、p_ϕ 总是守恒的.

$\parallel$ $L_z = mr^2\dot{\phi}\sin^2\theta$

Def: $H(r,\theta,p_r,p_\theta) = \frac{p_r^2}{2m} + \frac{1}{2mr^2}\left(p_\theta^2 + \frac{L_z^2}{\sin^2\theta}\right) + U(r)$

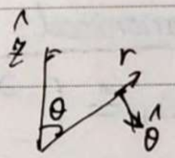
(In contrast) $L(r,\theta,p_r,p_\theta) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\theta^2 + \frac{L_z^2}{2mr^2\sin^2\theta} - U(r)$

没有改变 H 对 r、θ 之依赖关系，可以这样做

而改变了 L 对 r、θ 之依赖关系了，不可以这样做.

角动量 $\boxed{\vec{l} = \vec{r}\times\vec{p} = l_\theta\hat{\theta} + l_\phi\hat{\phi}}$ (对比 p24)

$\boxed{p_\theta = l_\phi}$ 、 $\boxed{p_\phi = \hat{z}\cdot\vec{l} = -l_\theta\sin\theta}$



$\boxed{p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = l^2}$ 是守恒的.

§4 相空间中的运动.

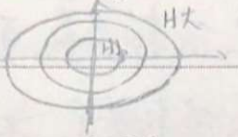
一、Hamilton 方程之动力学含义

对比：牛顿力学：$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{1}{m}\frac{\partial u}{\partial x} \end{pmatrix}$ $\begin{cases} x(t+\varepsilon) = x(t) + \varepsilon\dot{x}(t) \\ \dot{x}(t+\varepsilon) = \dot{x}(t) - \frac{\varepsilon}{m}\left(\frac{\partial u}{\partial x}\right)\Big|_t \end{cases}$

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} = X(q,p,t)$$ 给出了速度场

预言 $\begin{cases} q(t+\varepsilon) = q(t) + \varepsilon\cdot\left(\frac{\partial H}{\partial p}\right)_t \\ p(t+\varepsilon) = p(t) - \varepsilon\left(\frac{\partial H}{\partial q}\right)_t \end{cases}$

eg. $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$



正则变量 (q,p) 正则方程

canonical: in the simplest accepted term in mathematics.

$$\boxed{\Delta_H \triangleq \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix}}$$ 称 $H(q,p,t)$ 生成之 Hamilton 矢量场

二、ξ记号

$\alpha,\beta,\gamma,\cdots$ 取 $\underbrace{1,2,\cdots,\cdots,\cdots,2S}$

$i,j,k,\cdots$ 取 $\underbrace{1,2,\cdots,S}$

1. 正则变量 $\xi_\alpha = \begin{cases} \xi_k = q_k \\ \xi_{k+s} = p_k \end{cases}$ 即 $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2s} \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}$

同样地有：$\frac{\partial f}{\partial \xi} = \begin{pmatrix} \partial f/\partial\xi_1 \\ \vdots \\ \partial f/\partial\xi_{2s} \end{pmatrix} = \begin{pmatrix} \partial f/\partial q \\ \partial f/\partial p \end{pmatrix}$

有 $\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \frac{d}{dt}\xi = \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix} = \begin{pmatrix} 0_{s\times s} & I_{s\times s} \\ -I_{s\times s} & 0_{s\times s} \end{pmatrix}\begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix}$

2. 正则方程: $\boxed{\dot{\xi} = \Omega \frac{\partial H}{\partial \xi}}$, $\boxed{\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -\Omega^T \ , \Omega^{-1} = \Omega^T}$

3. $H(\xi, t)$ 生成的 Hamilton 矢量场 $\boxed{\Delta_H \triangleq \Omega \frac{\partial H}{\partial \xi} = \Delta_H(\xi, t)}$

因此, $H(\xi,t) \Rightarrow \Delta_H(\xi,t) \Rightarrow \dot{\xi} = \Delta_H(\xi,t) \Rightarrow$ 体系

三、Hamilton 体系

在2s维 ξ-相空间中, 由 $\dot{\xi} = \Omega \frac{\partial H}{\partial \xi} = \Delta_H(\xi,t)$ 决定其演化的体系称为由 $H(\xi,t)$ 生成的 Hamilton 体系.

1. H任意

eg. $s=1$

$H = C \Rightarrow \Delta_H = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$H = q \Rightarrow \Delta_H = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ 以单位速度向下运动的体系

$H = p \Rightarrow \Delta_H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 以单位速度向右运动的体系.

$H = \frac{1}{2}(p^2+q^2) \Rightarrow \Delta_H = \begin{pmatrix} p \\ -q \end{pmatrix}$ 以单位角速度绕原点顺时针转动.

不再在乎变量具体是什么, 只要偶数个就行.

但若只给定 $\underline{X}$, 不一定是 Hamilton 体系,

eg. $\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -p-2q \end{pmatrix} = \underline{X} \overset{?}{=} \begin{pmatrix} \partial H/\partial p \\ -\partial H/\partial q \end{pmatrix}$, 能描述 $\ddot{q} + \dot{q} + 2q = 0$

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存在 $H(\xi,t)$, s.t. $\bar{X}=\Omega\frac{\partial H}{\partial \xi}$, 即 $Y\triangleq\Omega\bar{X}=-\frac{\partial H}{\partial \xi}$

亦即"无旋",有 $\partial_\alpha Y_\beta=\partial_\beta Y_\alpha$

将$Y=\Omega\bar X$代入⇒ $\Omega_{\beta\gamma}\,\partial_\alpha\bar{X}_\gamma$ $\Omega_{\alpha\rho}\Omega_{\beta\sigma}\underbrace{\Omega_{\beta\sigma}\Omega_{\beta\gamma}}\bar{X}_\gamma$

$$(\Omega^T\Omega)_{\sigma\gamma}=\delta_{\sigma\gamma}$$

$\|$ ⇔ 而把微分算乘和去掉$\bar X$组合

$\Omega_{\alpha\gamma}\,\partial_\beta\bar{X}_\gamma$ $\underbrace{\Omega_{\alpha\rho}\Omega_{\beta\sigma}\Omega_{\alpha\gamma}}\partial_\beta\bar{X}_\gamma$

$$(\Omega^T\Omega)_{\rho\gamma}=\delta_{\rho\gamma}$$

$$\Rightarrow\ \Omega_{\alpha\rho}\,\partial_\alpha\bar{X}_\sigma=\Omega_{\beta\sigma}\,\partial_\beta\bar{X}_\rho$$

$$\Rightarrow\ \Omega_{\rho\alpha}\,\partial_\alpha\bar{X}_\sigma=\Omega_{\sigma\alpha}\,\partial_\alpha\bar{X}_\rho$$

Def: $D_\rho=\Omega_{\rho\alpha}\,\partial_\alpha$, $D_\sigma=\Omega_{\sigma\alpha}\,\partial_\alpha$

⇒ $\dot{\xi}=\bar{X}(\xi,t)$ 为 Hamilton 体系条件:

$$\boxed{D_\alpha\bar{X}_\beta=D_\beta\bar{X}_\alpha},\quad \boxed{D_\alpha\triangleq\Omega_{\alpha\rho}\,\partial_\rho}$$

事实上, $D_k=\Omega_{k\rho}\,\partial_\rho=\frac{\partial}{\partial p_k}$

$$D_{s+k}=\Omega_{s+k,\rho}\,\partial_\rho=-\frac{\partial}{\partial q_k}$$

ssj注: 在后面定义了Poisson括号后会发现,即

$$[\xi_\alpha,\bar{X}_\beta]_\xi=[\xi_\beta,\bar{X}_\alpha]_\xi$$

eg. $\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -\omega_0^2 q - 2\gamma p \end{pmatrix} = X(q,p)$ 不是H体系

作 $(q,p) \longmapsto (Q=q,\ P=pe^{2\gamma t})$

$\Rightarrow \frac{d}{dt}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} Pe^{-2\gamma t} \\ -\omega_0^2 Q e^{2\gamma t} \end{pmatrix}$ 是H体系.

$$H = \frac{1}{2}P^2 e^{-2\gamma t} + \frac{1}{2}\omega_0^2 Q^2 e^{2\gamma t}$$

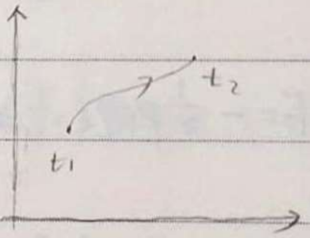
(因 $\dot{Q} = \frac{\partial H}{\partial P}$, $\frac{\partial^2 H}{\partial P^2} = e^{-2\gamma t} > 0$, 可做 Legrendre 变换)

$$L(Q,\dot{Q},t) = \frac{1}{2}(\dot{Q}^2 - Q^2)e^{2\gamma t}$$

同一个力学问题是否是 Hamilton 体系取决于描述所用之坐标，不是H体系之情况也有可能进行变换后变成H体系.

广义坐标与广义动量并没有唯一对应之关系，例如在L函数中，作 $L' = L + \frac{dF}{dt}$, $p_k' = p_k + \frac{\partial F}{\partial q_k}$.

四. 相空间的 Hamilton 原理



$$\boxed{\begin{cases} 0 = \delta\tilde{S} = \delta\int_{t_1}^{t_2} \tilde{L}(\xi,\dot{\xi},t)\,dt \\ \delta\xi(t_1) = 0 = \delta\xi(t_2) \end{cases}}$$

其中 $\tilde{L}(\xi,\dot{\xi},t) = p_i\dot{q}_i - H(q,p,t) = \hat{L}(q,p,\dot{q},\cancel{\dot{p}},t)$

(注意 H 未必满足 Hess 条件，是任意 q,p,t 之函数)

称是"相空间之 Lagrange 函数"

$$0=\int_{t_1}^{t_2}\left[\frac{\partial\tilde{L}}{\partial q_\alpha}\delta\xi_\alpha+\frac{\partial\tilde{L}}{\partial\dot{\xi}_\alpha}\delta\dot{q}_\alpha\right]dt$$

$$=\int_{t_1}^{t_2}\left[\frac{\partial\tilde{L}}{\partial q_\alpha}\delta\xi_\alpha+\frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{\xi}_\alpha}\delta\xi_\alpha\right)-\left(\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{\xi}_\alpha}\right)\delta\xi_\alpha\right]dt$$

$$=\int_{t_1}^{t_2}\left[\underbrace{\frac{\partial\tilde{L}}{\partial\xi_\alpha}-\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{\xi}_\alpha}}_{\triangleq\frac{\delta\tilde{L}}{\delta\xi_\alpha}}\right]\delta\xi_\alpha dt+\underbrace{\left(\frac{\partial\tilde{L}}{\partial\dot{\xi}_\alpha}\delta\xi_\alpha\right)\Bigg|_{t_1}^{t_2}}_{\left(\frac{\partial\tilde{L}}{\partial\dot{q}_k}\delta q_k+\frac{\partial\tilde{L}}{\partial\dot{p}_k}\delta p_k\right)\Big|_{t_1}^{t_2}}$$

(端点条件可稍放松些，只要 $\delta q_k(t_1)=0=\delta q_k(t_2)=0$ 即可)

$$\frac{\delta\tilde{L}}{\delta\xi_\alpha}=0\Rightarrow\begin{cases}\frac{\delta\tilde{L}}{\delta\xi_k}=\frac{\delta\tilde{L}}{\delta q_k}=\frac{\partial\tilde{L}}{\partial q_k}-\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{q}_k}=\left(-\frac{\partial H}{\partial q_k}\right)-\dot{p}_k=0\\ \frac{\delta\tilde{L}}{\delta\xi_{s+k}}=\frac{\delta\tilde{L}}{\delta p_k}=\frac{\partial\tilde{L}}{\partial p_k}-\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{p}_k}=\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)-0=0\end{cases}$$

规范变换 $\tilde{L}'(\xi,\dot{\xi},t)=\tilde{L}(\xi,\dot{\xi},t)+\frac{dF(\xi,t)}{dt}$

$$\tilde{S}'=\tilde{S}+G$$

eg. $F=-\frac{1}{2}p_iq_i\Rightarrow\tilde{L}'(\xi,\dot{\xi},t)=\frac{1}{2}(p_i\dot{q}_i-\dot{p}_iq_i)-H$

$$=\frac{1}{2}\dot{\xi}^T\Omega\xi-H$$

$\left((\dot{q},\dot{p})\begin{pmatrix}0&I\\-I&0\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix}=(\dot{q},\dot{p})\begin{pmatrix}q\\-p\end{pmatrix}\right)$

ssj 注：满足正则方程一定使作用量取极值

作用量取极值(变分为0)给出正则方程.

p.q (两组坐标之间)在使用正则方程前没有任何先验关系.

§4 Poisson 括号

在 2s 维空间中，坐标 ξ_α，力学量 $f(\xi, t) \rightarrow$ 参数

一、$f(\xi,t)$ 与 $g(\xi,t)$ 的 Poisson 括号：

$$\boxed{[f,g]_\xi \triangleq \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} = \left(\frac{\partial f}{\partial \xi_\alpha}\right)\Omega_{\alpha\beta}\frac{\partial g}{\partial \xi_\beta} = \left(\frac{\partial f}{\partial \xi}\right)^T \Omega \left(\frac{\partial g}{\partial \xi_\beta}\right)}$$

在选定的 ξ 坐标下定义的

$$\left(\left(\frac{\partial f}{\partial q}, \frac{\partial f}{\partial p}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial g}{\partial q} \\ \frac{\partial g}{\partial p} \end{pmatrix}\right)$$

1. 看做新定义的"乘法"：$f * g \triangleq [f,g]$

2. 看成由f生成的微分算子：$D_K g \triangleq [f,g]$

$$f \leftrightarrow \boxed{D_f = \frac{\partial f}{\partial \xi_\alpha}\Omega_{\alpha\beta}\frac{\partial}{\partial \xi_\beta}}$$

eg: $\boxed{f = q_K \text{ 时}, \ D_K = \frac{\partial}{\partial p_K}}$

eg: $\boxed{f = p_K \text{ 时}, \ D_{s+k} = -\frac{\partial}{\partial q_K}}$

二、数学性质

代数 { 1. 反对称：$\boxed{[f,g] = -[g,f]} \Rightarrow [f,f] = 0$

2. 双线性：$\boxed{\begin{aligned}[f, c_i g_i] &= c_i [f, g_i] \\ [c_i f_i, g] &= c_i [f_i, g]\end{aligned}}$

3. 雅可比恒等式：三个力学量 f, g, h（轮换：g,h,f；h,f,g）

$$\boxed{\begin{aligned}&[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0 \\ &[[f,g],h] + [[g,h],f] + [[h,f],g] = 0\end{aligned}}$$

1,2,3 合称 Lie 代数

$$[f,[g,h]] = \frac{\partial f}{\partial \xi_\alpha}\Omega_{\alpha\rho}\frac{\partial}{\partial \xi_\rho}\left(\frac{\partial g}{\partial \xi_\beta}\Omega_{\beta\gamma}\frac{\partial h}{\partial \xi_\gamma}\right)$$

$$= \Omega_{\alpha\rho}\Omega_{\beta\gamma}\frac{\partial f}{\partial \xi_\alpha}\underbrace{\frac{\partial}{\partial \xi_\rho}\Big(\frac{\partial g}{\partial \xi_\beta}}_{I_1}\frac{\partial h}{\partial \xi_\gamma}\Big).$$

ρ不变，

$$\begin{tikzcd} & f,\alpha \arrow[dr] & \\ h,\gamma \arrow[ur] & & g,\beta \arrow[ll] \end{tikzcd}$$

$$[g,[h,f]] = \Omega_{\rho\rho}\Omega_{\gamma\alpha}\frac{\partial g}{\partial \xi_\beta}\frac{\partial}{\partial \xi_\rho}\left(\frac{\partial h}{\partial \xi_\gamma}\frac{\partial f}{\partial \xi_\alpha}\right)$$

$$= \Omega_{\gamma\rho}\Omega_{\alpha\beta}\frac{\partial h}{\partial \xi_\gamma}\frac{\partial}{\partial \xi_\rho}\underbrace{\left(\frac{\partial f}{\partial \xi_\alpha}\frac{\partial g}{\partial \xi_\beta}\right)}_{I_2}$$

$$I_1 = \Omega_{\alpha\rho}\Omega_{\beta\gamma}\frac{\partial f}{\partial \xi_\alpha}\frac{\partial^2 g}{\partial \xi_\rho \partial \xi_\beta}\frac{\partial h}{\partial \xi_\gamma}$$

$$I_2 = \Omega_{\gamma\rho}\Omega_{\alpha\beta}\frac{\partial f}{\partial \xi_\alpha}\frac{\partial^2 g}{\partial \xi_\rho \partial \xi_\beta}\frac{\partial h}{\partial \xi_\gamma}$$

对I_2（$\rho \leftrightarrow \beta$ 对调）$\Rightarrow I_2 = \Omega_{\gamma\beta}\Omega_{\alpha\rho}\frac{\partial f}{\partial \xi_\alpha}\frac{\partial g}{\partial \xi_\beta \partial \xi_\rho}\frac{\partial h}{\partial \xi_\gamma} = -I_1$

4. Leibniz 法则：$\boxed{[f,gh] = [f,g]h + g[f,h]}$

（视为微分算符）

特例：若$[f,g]=0$，$[f,gh] = g[f,h]$

5. chain 法则：$\boxed{[f,g(h)] = \frac{\partial g}{\partial h}[f,h]}$

特例：$[f,g^n] = g^{n-1}[f,g]$

$\Rightarrow \boxed{[f,g(h_1,\dots,h_n)] = \frac{\partial g}{\partial h_k}[f,h_k]}$

$$\boxed{[f,g]_\xi = \frac{\partial f}{\partial \xi_\alpha}[\xi_\alpha, \xi_\beta]_\xi \frac{\partial g}{\partial \xi_\beta}}$$

只要告诉所有$[\xi_\alpha, \xi_\beta]_\xi$（状态变量间的Poisson括号），便可得到所有力学量的poisson括号

因此，定义 $[\xi_\alpha, \xi_\beta] = \Omega_{\alpha\beta}$（事实上即为前面的$\Omega_{\alpha\beta}$）:

6. 基本Poisson括号:

$$\boxed{[\xi_\alpha, \xi_\beta] = \Omega_{\alpha\beta}}$$

即：$[q_i, q_j] = 0$

$[p_i, p_j] = 0$

$[q_i, p_j] = \delta_{ij} = -[p_j, q_i]$

☆看清是$[q,p]$还是$[p,q]$

7. 对参数的偏导数:

$$\boxed{\partial_t[f,g] = [\partial_t f, g] + [f, \partial_t g]}$$

eg. $\xi(\vec r, \vec p) = \xi(x_1, x_2, x_3, p_1, p_2, p_3)$ $L_i = \varepsilon_{ijk} x_j p_k$

$\Rightarrow [L_i, x_j] = \varepsilon_{ijk} x_k$

$[L_i, p_j] = \varepsilon_{ijk} p_k$

$[L_i, L_j] = \varepsilon_{ijk} L_k$

$\cdot$	$\vec r$	$\vec p$	$\vec L$
$\vec r$	$\vec r^2$	$\vec r\cdot\vec p$	0
$\vec p$		$\vec p^2$	0
$\vec L$			$\vec r^2\vec p^2-(\vec r\cdot\vec p)^2$

所构造标量场只能是

$\vec f = f(r, p, \vec r, \vec p, t)$

所能构造的矢量场只能是?

$\vec A = f\vec r + g\vec p + h\vec L$

$\Rightarrow$ 更一般的，$[L_i, f] = 0$

$[L_i, A_j] = \varepsilon_{ijk} A_k$

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$$[f,g]_\xi \triangleq \frac{\partial f}{\partial \xi_\alpha}\Omega_{\alpha\beta}\frac{\partial g}{\partial \xi_\beta} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \triangleq D_f g$$

$$D_\alpha g = [\xi_\alpha, g] = \Omega_{\alpha\beta}\frac{\partial g}{\partial \xi_\beta}.$$

$$\boxed{[q_k, f] = \frac{\partial f}{\partial p_k}}, \quad \boxed{[p_k, f] = -\frac{\partial f}{\partial q_k}}$$

1. $[L_i, f] = \varepsilon_{ikl}[x_k p_l, f] = \varepsilon_{ikl}\left([x_k, f]p_l + x_k[p_l, f]\right)$

$$= \varepsilon_{ikl}\left(\frac{\partial f}{\partial p_k}p_l - x_k\frac{\partial f}{\partial x_l}\right)$$

2. $[L_i, x_j] = \varepsilon_{ijk}x_k \quad [L_i, p_j] = \varepsilon_{ijk}p_k$

$$[L_i, L_j] = \varepsilon_{jmn}[L_i, x_m p_n]$$

$$= \varepsilon_{ikl}\varepsilon_{jmn}(x_m\delta_{kn}p_l - x_k\delta_{lm}p_n)$$

$$= x_i p_j - x_j p_i = \varepsilon_{ijk}L_k$$

3.设由 $\vec{r}, \vec{p}, t$ 构成之标量 $f(r, p, \vec{r}\cdot\vec{p}, t)$

$$[L_i, f] = \frac{\partial f}{\partial r}\underbrace{[L_i, r]}_{} + \frac{\partial f}{\partial p}\underbrace{[L_i, p]}_{} + \frac{\partial f}{\partial(\vec{r}\cdot\vec{p})}\underbrace{[L_i, \vec{r}\cdot\vec{p}]}_{}$$

$$\frac{\partial r}{\partial x_j}[L_i, x_j] \qquad \frac{\partial p}{\partial p_j}[L_i, p_j] \qquad [L_i, x_j]p_j + x_j[L_i, p_j]$$

$$\frac{\partial f}{\partial r}\varepsilon_{ijk}\frac{x_j x_k}{r} + \frac{\partial f}{\partial p}\varepsilon_{ijk}\frac{p_j p_k}{p} + \frac{\partial f}{\partial(\vec{r}\cdot\vec{p})}\varepsilon_{ijk}(x_k p_j + x_j p_k) = 0.$$

ε_{ijk} 反对称，剩下部分总是对称。

故 $\boxed{[L_i, f] = 0}$

4. 仅由 $\vec{r}$、$\vec{p}$、$\vec{L}$ 构成的矢量 $\vec{A}=f\vec{r}+g\vec{p}+h\vec{L}$

（由于 $[L_i, f x_j] = f[L_i, x_j] = \varepsilon_{ijk} x_k f$ ）

$$\Rightarrow \boxed{[L_i, A_j] = \varepsilon_{ijk} A_k}$$

三. Poisson 括号应用于 Hamilton 体系. $\dot{\xi} = \Omega \frac{\partial H}{\partial \xi}$

对力学量 $\boxed{f(\xi,t)}$ $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi_\alpha}\dot{\xi}_\alpha = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi_\alpha}\Omega_{\alpha\beta}\frac{\partial H}{\partial \xi_\beta}$

1、力学量的运动方程. 器件并作用必须在定义动力学信息之后

$$\boxed{\frac{df}{dt} = [f,H] + \partial_t f = \frac{df}{dt}(\xi,t)}$$

① 正则方程 $\boxed{\dot{\xi}_\alpha = [\xi_\alpha, H]}$

② $[f,H] + \partial_t f = 0 \Rightarrow f$ 为运动常数；

若f不显含t，且 $[f,H]=0 \Rightarrow f$ 为运动常数.

eg. $H = \frac{p^2}{2m} + U(r)$，$H, L_i, \vec{L}$ 均为守恒量、

2、力学量的 Taylor 展开：设 $f = f(\xi,\cancel{t})$ 且 $H = H(\xi,\cancel{t})$

（此时 $\frac{df}{dt} = [f,H]$ 不显含t，$\frac{d^2f}{dt^2} = [[f,H],H]$，…）

$f(t) \triangleq f(\xi(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\left(\frac{d^nf}{dt^n}\right)$。

$$\boxed{f(t) = f_0 + [f,H]_0 t + \frac{t^2}{2!}[[f,H],H]_0 + \frac{t^3}{3!}[[[f,H],H],H] + \cdots}$$

Def: $[H,f] = D_H f$

$\Rightarrow f(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}(D_H^n f)$。

$$\Rightarrow \boxed{f(t+\tau) = \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!}(D_H^n f) = \exp(-\tau D_H) f}$$

eg. $H=\frac{p_x^2}{2m}+mgx$ 抛物运动

$x_0=x_0$, $[x,H]=[x,\frac{p_x^2}{2m}]=\frac{p_x}{m}$,

$[[x,H],H]=[\frac{p_x}{m},H]=[\frac{p_x}{m},mgx]=-g$

$x=x_0+\frac{p_x}{m}t-\frac{1}{2}gt^2$

eg. $H=\frac{1}{2}\omega(p^2+q^2)$

$D_H q=[H,q]=[\frac{1}{2}\omega p^2,q]=-\omega p$, $D_H p=[H,p]=[\frac{1}{2}\omega q^2,p]=\omega q$

$D_H^2 q=D_H(D_H q)=-\omega D_H p=-\omega^2 q$

$D_H^3 q=D_H(D_H^2 q)=\omega^3 p$

$\Rightarrow D_H^{2k}q=(-1)^k\omega^{2k}q$, $D_H^{2k+1}q=-(-1)^k\omega^{2k+1}p$

$\Rightarrow q(t)=\sum_0^{\infty}(-1)^k\frac{(\omega t)^{2k}}{(2k)!}q_0+\sum_0^{\infty}(-1)^k\frac{(\omega t)^{2k+1}}{(2k+1)!}p_0$

$= q_0\cos\omega t+p_0\sin\omega t$

或: $\frac{d}{dt}\begin{pmatrix}q\\p\end{pmatrix}=\begin{pmatrix}\omega p\\-\omega q\end{pmatrix}=\omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix}\hat{=}\Lambda\begin{pmatrix}q\\p\end{pmatrix}$

$\Rightarrow \begin{pmatrix}q(t)\\p(t)\end{pmatrix}=e^{\Lambda t}\begin{pmatrix}q_0\\p_0\end{pmatrix}$, $e^{\Lambda t}=\sum_{n=0}^{\infty}\frac{(\Lambda t)^n}{n!}$, $\Lambda^0=I$

3、对时间的全导数.

$$\boxed{\frac{d}{dt}[f,g]=[\frac{df}{dt},g]+[f,\frac{dg}{dt}]}$$

$$\frac{d}{dt}[f,g]=[[f,g],H]+\partial_t[f,g]$$

$$=-[[g,H],f]+[\partial_t f,g]-[[H,f],g]+[f,\partial_t g]$$

$$=[\partial_t f+[f,H],g]+[f,\partial_t g+[g,H]]$$

$$=[\frac{df}{dt},g]+[f,\frac{dg}{dt}]$$ 证毕.

若直接进行 $\frac{d}{dt}$ 操作时出现 $\dot{q}$，应理解为将 $\dot{q}$ 按运动方程代入 q,t，即得 $\frac{df}{dt}(q,t)$. (因未定义含 $\dot{q}$ 的 poisson 括号).

☆运动常数不是常数，每个初始条件下给出的都是不同的，不能乱提：

如 f 是运动常数，$[fg,H]\neq f[g,H]$ （后项 $g[f,H]$ 中虽 $\frac{df}{dt}=0$，但 $\frac{\partial f}{\partial t}$ 不一定为0，即 $[f,H]$ 不一定为0）

4. Poisson 定理：

若 f,g 为运动常数，则 $[f,g]$ 亦是

四. 判断体系是否是 Hamilton 体系.

（前面） $D_\alpha \bar{X}_\beta = D_\beta \bar{X}_\alpha$. 即 $[\xi_\alpha, \bar{X}_\beta]_\xi = [\xi_\beta, \bar{X}_\alpha]_\xi$

$\dot{\xi} = \bar{X}(\xi, t)$ 为 Hamilton 体系 $\Leftrightarrow$ $\boxed{[\dot{\xi}_\alpha, \xi_\beta]_\xi + [\xi_\alpha, \dot{\xi}_\beta]_\xi = 0}$

$\alpha, \beta = 1, \cdots, 2S$.（只需验证 $1 \le \alpha < \beta \le 2S$, C_{2S}^2 个）

eg. S=1 时, 验证 2 个. $[\dot{q}, p] + [q, \dot{p}] = 0$

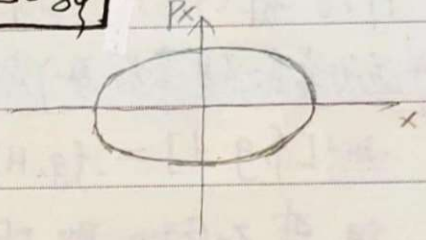
S=2 时. 验证 6 个

条件 $\Leftrightarrow$ $\boxed{\frac{d}{dt}[f, g] = [\frac{df}{dt}, g] + [f, \frac{dg}{dt}]}$ 对 ∀ f, g 成立.

（左证时取 $f = \xi_\alpha$, $g = \xi_\beta$. $[\xi_\alpha, \xi_\beta]_\xi = \Omega_{\alpha\beta}$, $\frac{d\Omega_{\alpha\beta}}{dt} = 0$ ）

$\Leftrightarrow$ $\boxed{\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t}}$ $\Leftrightarrow$ $\boxed{\dot{\xi} = \Omega \frac{\partial H}{\partial \xi}}$

eg 1. $H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2$



$\dot{x} = \frac{p_x}{m}$, $\dot{p}_x = -m\omega^2 x$

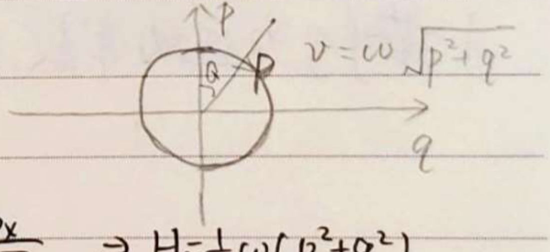
2. 变换 $q = \lambda x$, $p = \frac{p_x}{\lambda}$

$\Rightarrow$ $\dot{q} = \lambda^2 \frac{p}{m}$, $\dot{p} = -\frac{m\omega^2 q}{\lambda^2}$

（ $[\dot{q}, p]_{(q,p)} + [q, \dot{p}]_{(q,p)} = 0$, 是 Hamilton 体系）

$\frac{\partial H}{\partial p} = \lambda^2 \frac{p}{m}$, $\frac{\partial H}{\partial q} = \frac{m\omega^2 q}{\lambda^2}$

$\Rightarrow H = \frac{\lambda^2 p^2}{2m} + \frac{m\omega^2 q}{2\lambda^2}$



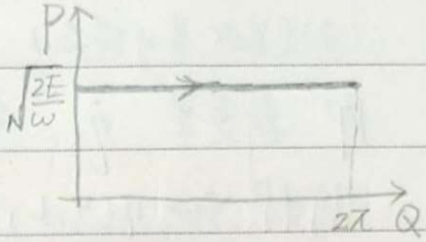
取 $\lambda = \sqrt{m\omega}$. $q = \sqrt{m\omega}\, x$, $p = \frac{p_x}{\sqrt{m\omega}}$ $\Rightarrow H = \frac{1}{2}\omega(p^2 + q^2)$

$\dot{q} = \omega p$, $\dot{p} = -\omega q$

3. $Q = \arctan\frac{q}{p}$ $P=\sqrt{p^2+q^2}$

$\dot{Q}=\omega$, $\dot{P}=0$ 是Hamilton体系，

新Hamilton函数：$K=\omega P$ （此时 $=\sqrt{2\omega E}$，不再是能量）



ssj注：每个H都是由ξ下各自组成的，要先获得正则方程之后再写对应的Hamilton函数。本例中虽然2中H和把变换代入1中H得到的结果一样但只是一个巧合。

4. 任一$H(p,q,t)$在上述3变换下满足~~辛~~条件能生成关于(Q,P)的场？

$$\left.\begin{aligned}\xi&=(q,p)\\ \eta&=(Q,P)\end{aligned}\right\}\Rightarrow[\dot{Q},P]_\eta+[Q,\dot{P}]_\eta=0$$

（在新η下验证是否是Hamilton体系）

在ξ中，可视Q，P为运动量，有：

$$\dot{Q}=[Q,H]_\xi=\frac{\partial H}{\partial P}[Q,P]_\xi+\cancel{\frac{\partial H}{\partial Q}[Q,Q]_\xi}$$

$$\dot{P}=[P,H]_\xi=-\frac{\partial H}{\partial Q}[Q,P]_\xi$$

而 $[Q,P]_\xi=[Q,\sqrt{p^2+q^2}]_\xi=(p[Q,p]_\xi+q[Q,q]_\xi)\frac{1}{\sqrt{p^2+q^2}}$

$$=\frac{1}{\sqrt{p^2+q^2}}\left(p\frac{\partial Q}{\partial q}-q\frac{\partial Q}{\partial p}\right)=\frac{1}{\sqrt{p^2+q^2}}=\frac{1}{P}$$

故 $[\dot{Q},P]_\eta+[Q,\dot{P}]_\eta$

$$=\left[\frac{1}{P}\frac{\partial H}{\partial P},P\right]_\eta+\left[Q,-\frac{1}{P}\frac{\partial H}{\partial Q}\right]_\eta$$

$$=\frac{\partial}{\partial Q}\left(\frac{1}{P}\frac{\partial H}{\partial P}\right)-\frac{\partial}{\partial P}\left(\frac{1}{P}\frac{\partial H}{\partial Q}\right)=\frac{1}{P^2}\frac{\partial H}{\partial Q}\overset{要求}{=}0$$

即要求 $H=H(\cancel{Q},P,t)=H(\sqrt{p^2+q^2},t)$

ssj注：以上推导中所有H均为ξ体系下的H，未出现η下的H。当原H可写成$H(\sqrt{p^2+q^2},t)$的形式，则新变量体系仍是Hamilton体系。

§6 正则变换（CT: Canonical Transformation）

一、①若由 $H(\xi,t)$ 生成的 Hamilton 体系，当以 $\eta=\eta(\xi,t)$ 作为状态参量时仍为 Hamilton 体系。

（变换显然是可逆的）

即：新变量 $\dot{\eta}_\alpha = \frac{\partial \eta_\alpha}{\partial \xi_\beta}\dot{\xi}_\beta + \partial_t \eta_\alpha = \Omega_{\alpha\beta}\frac{\partial K}{\partial \eta_\beta} = [\eta_\alpha, K]_\eta$

则称 $\xi \mapsto \eta(\xi,t)$ 对 $H(\xi,t)$ 是类正则的。

②若该变换对所有 $H(\xi,t)$ 都是类正则的，则称其为正则变换（CT）

eg. $\eta=\xi$

eg. $Q_k = p_k$，$P_k = -q_k$

$$\begin{cases} \dot{Q}_k = \dot{p}_k = -\frac{\partial H}{\partial q_k} = +\frac{\partial H}{\partial P_k} \\ \dot{P}_k = -\dot{q}_k = -\frac{\partial H}{\partial p_k} = -\frac{\partial H}{\partial Q_k} \end{cases}$$

只要找 $K(Q,P,t) \triangleq H(-P,Q,t)$

Def：$\boxed{\Lambda_{\alpha\beta} = [\xi_\alpha, \xi_\beta]_\eta = \frac{\partial \xi_\alpha}{\partial \eta_\rho}\Omega_{\rho\sigma}\frac{\partial \xi_\beta}{\partial \eta_\sigma} = \Lambda_{\alpha\beta}(\xi,t)}$ 2s维方阵

①上面 $\Lambda_{\alpha\beta}$ 仅由变换所决定，与要讨论的力学体系无关。

② $\Lambda_{\alpha\beta}$ 必为可逆、反对称。（因 $\frac{\partial \xi_\alpha}{\partial \eta_\rho}$、$\frac{\partial \xi_\beta}{\partial \eta_\sigma}$ 可视为 yacobi 阵可逆、$\Omega_{\rho\sigma}$ 可逆）

先假设对 $H(\xi,t)$ 类正则。则 η 是 Hamilton 体系，

$$\frac{d}{dt}[\xi_\alpha, \xi_\beta]_\eta = [\dot{\xi}_\alpha, \xi_\beta]_\eta + [\xi_\alpha, \dot{\xi}_\beta]_\eta$$

而 $\Lambda_{\alpha\beta}$ 是力学量，$\frac{d}{dt}\Lambda_{\alpha\beta} = [\Lambda_{\alpha\beta}, H]_\xi + \partial_t \Lambda_{\alpha\beta}$

$$\Rightarrow \boxed{[\dot{\xi}_\alpha, \xi_\beta]_\eta + [\xi_\alpha, \dot{\xi}_\beta]_\eta = [\Lambda_{\alpha\beta}, H]_\xi + \partial_t \Lambda_{\alpha\beta}} \quad (*)$$

某变换的 $\Lambda_{\alpha\beta}$（对特定的 H）是类正则变换的条件。

本页的思路：上页(*)式给出了$\Lambda_{\alpha\beta}$对特定H是正则的条件，我们尝试看$\Lambda_{\alpha\beta}$为正则变换时应满足什么条件。故对∀H，(*)应成立。先取了① H=C 证明$\Lambda_{\alpha\beta}$不显含t，再取② $H=\frac{1}{2}C_\gamma\xi_\gamma$证明了$\Lambda_{\alpha\beta}$是常数，再取③ $H=\frac{1}{2}C_{\rho\sigma}\xi_\rho\xi_\sigma$证明了Λ只能$=a\Omega$。三步是相承推进的。在③中还特殊了一次C=I证得$\Lambda\Omega=\Omega\Lambda$以代回。

I：① $H=\text{Const} \Rightarrow \dot{\xi}_\alpha=[\xi_\alpha,H]_\xi=0$，$[\Lambda_{\alpha\beta},H]_\xi=0$

$\Rightarrow \partial_t\Lambda_{\alpha\beta}=0 \Rightarrow \Lambda_{\alpha\beta}=\Lambda_{\alpha\beta}(\xi,\cancel{t})$

② $H=C_\gamma\xi_\gamma \Rightarrow \dot{\xi}_\alpha=C_\gamma[\xi_\alpha,\xi_\gamma]_\xi=\Omega_{\alpha\gamma}C_\gamma$，是常数

$\Rightarrow C_\gamma[\Lambda_{\alpha\beta},\xi_\gamma]_\xi=0 \Rightarrow \Lambda_{\alpha\beta}=\Lambda_{\alpha\beta}(\cancel{\xi},\cancel{t})=$常数

③ $H=\frac{1}{2}C_{\rho\sigma}\xi_\rho\xi_\sigma \quad (C_{\rho\sigma}=C_{\sigma\rho})$

$\Rightarrow \dot{\xi}_\alpha=\frac{1}{2}C_{\rho\sigma}(\Omega_{\alpha\rho}\xi_\sigma+\Omega_{\alpha\sigma}\xi_\rho)=\Omega_{\alpha\rho}C_{\rho\sigma}\xi_\sigma$

$\Rightarrow \Omega_{\alpha\rho}C_{\rho\sigma}[\xi_\sigma,\xi_\beta]_\eta+\Omega_{\beta\rho}C_{\rho\sigma}[\xi_\alpha,\xi_\sigma]_\eta$

即 $(\Omega C\Lambda)_{\alpha\beta}+(\Lambda C^T\Omega^T)_{\alpha\beta}=0$

$\Rightarrow \Omega C\Lambda=\Lambda C\Omega \Leftrightarrow C\Lambda\Omega=\Omega\Lambda C$

取 $C=I \Rightarrow \Lambda\Omega=\Omega\Lambda$ 代回

得到

$\Rightarrow C(\Omega\Lambda)=(\Omega\Lambda)C$ 再考虑阵$\Omega\Lambda$与任意对称阵C可交换，故$\Omega\Lambda$必为数量阵。

$\Rightarrow \Omega\Lambda=-aI \Rightarrow \Lambda=a\Omega$.

即若为CT $\Rightarrow \Lambda=a\Omega(a\neq0) \Leftrightarrow$

这样，$[f,g]_\eta=\frac{\partial f}{\partial\xi_\alpha}[\xi_\alpha,\xi_\beta]_\eta\frac{\partial g}{\partial\xi_\beta}=a[f,g]_\xi$

II：再证上述条件之必要性：

$\Lambda=a\Omega \Rightarrow [f,g]_\eta=a[f,g]_\xi$ 对∀f,g成立.

对∀H，$\frac{d}{dt}[f,g]_\xi=[\dot{f},g]_\xi+[f,\dot{g}]_\xi$

同乘a $\Rightarrow \frac{d}{dt}[f,g]_\eta=[\dot{f},g]_\eta+[f,\dot{g}]_\eta$

故以η仍为Hamilton体系.

二、(受限)正则变换的条件

令 $a=1$，称为受限正则变换：$\boxed{\Omega = \Lambda}$

1、Poisson 括号的不变性 $\boxed{[f,g]_\eta = [f,g]_\xi}$ $(\forall f,g)$

2、基本 Poisson 括号的不变性：$\boxed{[\xi_\alpha, \xi_\beta]_\eta = \Omega_{\alpha\beta}}$ 或 $\boxed{[\eta_\alpha, \eta_\beta]_\xi = \Omega_{\alpha\beta}}$

（2→1：$[f,g]_\xi = \frac{\partial f}{\partial \eta_\alpha}[\eta_\alpha, \eta_\beta]_\xi \frac{\partial g}{\partial \eta_\beta} = [f,g]_\eta$ ）

3、辛条件：$\boxed{M\Omega M^T = \Omega}$ 或 $\boxed{M^T\Omega M = \Omega}$

其中 $\boxed{M_{\alpha\beta} \triangleq \frac{\partial \eta_\alpha}{\partial \xi_\beta}}$ （称 M 是辛矩阵）（Ω 本身就是辛矩阵）

（$[\eta_\alpha, \eta_\beta] = \frac{\partial \eta_\alpha}{\partial \xi_\rho}\Omega_{\rho\sigma}\frac{\partial \eta_\beta}{\partial \xi_\sigma} = M_{\alpha\rho}\Omega_{\rho\sigma}M_{\beta\sigma}$，2,3 等价）

（$M\Omega M^T = \Omega \Rightarrow M\Omega M^T\Omega^T = \Omega\Omega^T \Rightarrow M\Omega M^T\Omega^T = I$

$\Rightarrow M^{-1} = \Omega M^T\Omega^T \Rightarrow M^{-1}M = \Omega M^T\Omega^T M = I$

$\Rightarrow M^T\Omega^T M = \Omega^T \Rightarrow M^T\Omega M = \Omega$ 证明了辛条件中二者等价）

4、可积条件：存在 $F(q,p,t)$ 使得

$$\boxed{p_k \delta q_k - P_i \delta Q_i = \delta F(q,p,t)}$$

（将其用旧变量表示：$p_k\delta q_k - P_i\frac{\partial Q_i}{\partial q_k}\delta q_k - P_i\frac{\partial Q_i}{\partial p_k}\delta p_k = \frac{\partial F}{\partial q_k}\delta q_k + \frac{\partial F}{\partial p_k}\delta p_k$ ）

即：
$$\boxed{\begin{cases} p_k - \frac{\partial Q_i}{\partial q_k}P_i = \frac{\partial F}{\partial q_k} \\ -\frac{\partial Q_i}{\partial p_k}P_i = \frac{\partial F}{\partial p_k} \end{cases}}$$

其中 F 称为（该正则变换）的生成函数

对生成函数的同一正则变换，$\boxed{F' = F + f(t)}$

证明4与3之等价性:

实践: $\frac{1}{2}(p_k\delta q_k - q_k\delta p_k) - \frac{1}{2}(P_k\delta Q_k - Q_i\delta P_i) = \delta(F - \frac{1}{2}q_k p_k + \frac{1}{2}Q_iP_i)$

注意到 $(\delta q, \delta p)\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} = p\delta q - q\delta p$

$$\Rightarrow \frac{1}{2}\Omega_{\alpha\gamma}\xi_\gamma\delta\xi_\alpha - \frac{1}{2}\Omega_{\rho\sigma}\eta_\sigma\underbrace{\delta\eta_\rho}_{\frac{\partial\eta_\rho}{\partial\xi_\alpha}\delta\xi_\alpha} = \delta G(\xi, t) = \frac{\partial G}{\partial\xi_\alpha}\delta\xi_\alpha$$

令 $\bar{X}_\alpha \triangleq \frac{1}{2}\Omega_{\alpha\gamma}\xi_\gamma - \frac{1}{2}\Omega_{\rho\sigma}\eta_\sigma\frac{\partial\eta_\rho}{\partial\xi_\alpha}$

$\Rightarrow \bar{X}_\alpha\delta\xi_\alpha = \delta G$ 即 $\bar{X}_\alpha = \frac{\partial G}{\partial\xi_\alpha}$

或 $\partial_\beta\bar{X}_\alpha - \partial_\alpha\bar{X}_\beta = 0$.

(kx)

$\Omega_{\rho\sigma}M_{\sigma\beta}M_{\rho\alpha} = (M^T\Omega M)_{\alpha\beta}$

$$\partial_\beta\bar{X}_\alpha = \frac{1}{2}(\Omega_{\alpha\gamma}\delta_{\beta\gamma} - \underbrace{\Omega_{\rho\sigma}\frac{\partial\eta_\sigma}{\partial\xi_\beta}\frac{\partial\eta_\rho}{\partial\xi_\alpha}} - \underbrace{\Omega_{\rho\sigma}\eta_\sigma\frac{\partial^2\eta_\rho}{\partial\xi_\beta\partial\xi_\alpha}})$$

$$\partial_\alpha\bar{X}_\beta = \frac{1}{2}(\Omega_{\beta\alpha} - \underbrace{\Omega_{\rho\sigma}\frac{\partial\eta_\sigma}{\partial\xi_\alpha}\frac{\partial\eta_\rho}{\partial\xi_\beta}} - \underbrace{\Omega_{\rho\sigma}\eta_\sigma\frac{\partial^2\eta_\rho}{\partial\xi_\alpha\partial\xi_\beta}})$$

$\Omega_{\rho\sigma}M_{\sigma\alpha}M_{\rho\beta} = (M^T\Omega^TM)_{\alpha\beta} = -(M^T\Omega M)_{\alpha\beta}$

即(kx)式即等价于说 $(\Omega - M^T\Omega M)_{\alpha\beta} = 0$

eg. $Q=\arctan\frac{q}{p}$ $P=\frac{1}{2}(p^2+q^2)$

① 证明 $[Q,P]_{(q,p)}=1$.

$[Q,P]$ (新的坐标) $= p[Q,p]+q[Q,q]$

$$= p\frac{\partial Q}{\partial q}-q\frac{\partial Q}{\partial p} = p\frac{p}{p^2+q^2}-q\frac{-q}{p^2+q^2} = 1$$

② 对称性:

$$p\delta q - P\delta Q = p\delta q - P\left(\frac{\partial Q}{\partial q}\delta q+\frac{\partial Q}{\partial p}\delta p\right)$$

$$= \frac{1}{2}(p\delta q+q\delta p) = \delta\left(\frac{1}{2}qp\right)$$

eg. $Q_i=Q_i(q,t)$ 可逆, 问 $P_i=P_i(q,p,t)$?

① (需验证 $[Q_i,Q_j]_q=0$, $[P_i,P_j]_q=0$, $[Q_i,P_j]_q=\delta_{ij}$)

② $\begin{cases} p_k-\frac{\partial Q_i}{\partial q_k}P_i=\frac{\partial F}{\partial q_k} \\ 0=\frac{\partial F}{\partial p_k} \end{cases}$ $\Rightarrow \frac{\partial q_k}{\partial Q_j}\frac{\partial Q_i}{\partial q_k}P_i = \left(p_k-\frac{\partial F}{\partial q_k}\right)\frac{\partial q_k}{\partial Q_j}$

$\Rightarrow F=F(q,t)$

$\Rightarrow P_j=\frac{\partial q_k}{\partial Q_j}\left(p_k-\frac{\partial F}{\partial q_k}\right)$

当 $F=0$ $\Rightarrow P_i=\frac{\partial q_k}{\partial Q_i}p_k \Rightarrow P_i$

当 $Q=\lambda q \Rightarrow P=\frac{p}{\lambda}$

当 $\vec{R}=\lambda\vec{r} \Rightarrow \vec{P}=\lambda^T\vec{p}$

令 $Q=q \Rightarrow P_i=p_i-\frac{\partial F}{\partial q_i}$ 即规范变换.

三. 数学性质

1. $|\det M| = 1$ （事实上，可证 $\boxed{\det M = 1}$ ）只能取+1.

eg. 对单自由度体系，

$$\det M = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = [Q,P]_{(q,p)} = 1$$

注：但对多自由度体系，不能只用 $\det M=1$ 证明是正则变换.

2. M 为辛矩阵，M^{-1} 也是（正则变换之逆变换仍是正则变换）

M_1，M_2 均为辛矩阵，M_1M_2 也是（二个连续之正则变换仍是正则变换）

$$\xi \xrightarrow{M_1=\frac{\partial\eta}{\partial\xi}} \eta(\xi,t) \xrightarrow{M_2=\frac{\partial\zeta}{\partial\eta}} \zeta(\eta,t)$$

$$M = M_2M_1$$

3. $$\boxed{\int d\eta_1 d\eta_2\cdots d\eta_{2s} = \int d\xi_1 d\xi_2\cdots d\xi_{2s}}$$

两相空间中对应"体"之"体积不变.

四. CT 的物理推论

$$\xi_\alpha(t+\tau) = \sum_0^\infty \frac{\tau^n}{n!}\frac{d^n\xi_\alpha(t)}{dt^n}$$ （所有出现之ξ对t之导数应以动力学方程代入）

定义新变换：$\eta_\alpha \triangleq \sum_{n=0}^\infty \frac{\tau^n}{n!}\frac{d^n\xi_\alpha}{dt^n} = \eta_\alpha(\xi, t; \tau)$ 视τ为参数.

$$[\eta_\alpha, \eta_\beta]_\xi = \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{\tau^{k+l}}{k!\,l!}\left[\frac{d^k\xi_\alpha}{dt^k}, \frac{d^l\xi_\beta}{dt^l}\right]_\xi$$

$$\overset{令k+l=n}{=\!=\!=} \sum_{n=0}^\infty \frac{\tau^n}{n!}\sum_{k+l=n}\frac{n!}{k!\,l!}\left[\frac{d^k\xi_\alpha}{dt^k}, \frac{d^l\xi_\beta}{dt^l}\right]_\xi$$

注意到 $\frac{d}{dt}[f,g] = [\frac{d}{dt}f, g] + [f, \frac{d}{dt}g]$

$$\boxed{\frac{d^n}{dt^n}[f,g] = \sum_{k+l=n}\frac{n!}{k!\,l!}\left[\frac{d^kf}{dt^k}, \frac{d^lg}{dt^l}\right]}$$

（定义新"乘法"运算 $f * g \triangleq [f, g]_\xi$，则可理解为：

$$\frac{d^n}{dt^n}fg = \sum_{k+l=n}\frac{n!}{k!\,l!}\frac{d^kf}{dt^k}\cdot\frac{d^lg}{dt^l}\ )$$

$$\Rightarrow [\eta_\alpha, \eta_\beta]_\xi = \sum_{n=0}^\infty \frac{\tau^n}{n!}\cdot\frac{d^n}{dt^n}\Omega_{\alpha\beta} = \Omega_{\alpha\beta}$$

1. Hamilton 体系的演化以被动角度看即为CT

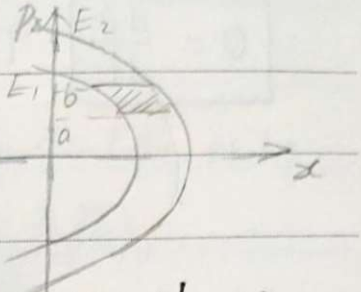
2. 正则区域（相空间中某一个区域中点按同样的正则方程进行（动力学上）的演化，区别只是初始条件不一样）.

正则区域的体积不随时间变换 (LV)

$$\boxed{\Gamma(t) \triangleq \int d\xi_1 d\xi_2 \cdots d\xi_{2s} = \Gamma(0)}$$

（Liouville 体积定理）

eg. $H=\frac{p_x^2}{2m}+mgx$



$$\begin{cases}\dot{x}=\frac{p_x}{m}\\ \dot{p}_x=-mg\end{cases}$$

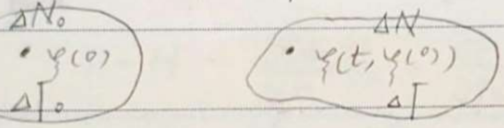
$$\Rightarrow\begin{cases}p_x=p_{x0}-mgt\\ x=\frac{E}{mg}-\frac{p_x^2}{2m^2g}\end{cases}\qquad \Gamma(0)=\int dx\int dp_x=\frac{(E_2-E_1)(b-a)}{mg}$$

$$\Delta p_x=\Delta p_{x0}=b-a,\quad \Gamma(t)=\frac{E_2-E_1}{mg}\int_{a-mgt}^{b-mgt}dp_x=\frac{(E_2-E_1)(b-a)}{mg}$$

$N\gg1$，定义：相点密度（数密度）$\boxed{n\hat{=}\frac{\Delta N}{\Delta\Gamma}=n(\xi,t)}$

归一化的密度分布函数（态密度）$\boxed{\rho\hat{=}\frac{n}{N}=\rho(\xi,t)}$，$\boxed{\int\rho d\xi=1}$

（边界变到边界，轨道不可能相交，跑不到外面去）



3. Liouville 定理：

$$\boxed{\rho(\xi^{(t)},t)=\rho(\xi^{(0)},0)}$$

随点向时间演化方向走，周围

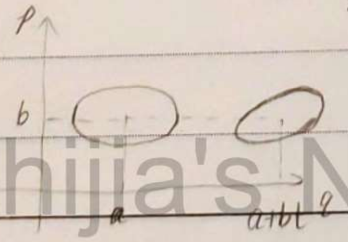
相空间中"流"不可压缩. "密度"不变.

eg. $\rho_0(q,p)=\rho(q,p,0)=\frac{1}{\pi\sigma_q\sigma_p}\exp\left[-\frac{(q-a)^2}{\sigma_q^2}-\frac{(p-b)^2}{\sigma_p^2}\right]$

$H=\frac{p^2}{2}\Rightarrow\dot{q}=p,\ \dot{p}=0\Rightarrow q=q_0+pt,\ p=p_0\Rightarrow q_0=q-pt,\ p_0=p$

则 $\rho(q,p,t)=\rho(q_0,p_0,0)$

$$=\frac{1}{\pi\sigma_p\sigma_q}\exp\left[-\frac{(q-pt-a)^2}{\sigma_q^2}-\frac{(p-b)^2}{\sigma_p^2}\right]$$



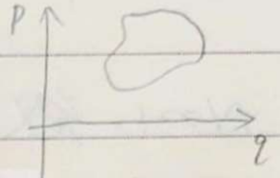
等值线变化

$$\boxed{0=\frac{d\rho}{dt}}=[\rho,H]_\xi+\partial_t\rho$$

$$=\frac{\partial\rho}{\partial\xi_\alpha}\Omega_{\alpha\beta}\frac{\partial H}{\partial\xi_\beta}+\partial_t\rho$$

$$=\frac{\partial}{\partial\xi_\alpha}\left(\rho\Omega_{\alpha\beta}\frac{\partial H}{\partial\xi_\beta}\right)-\overset{=0}{\rho\underbrace{\Omega_{\alpha\beta}}_{\text{反对称}}\underbrace{\frac{\partial^2H}{\partial\xi_\alpha\partial\xi_\beta}}_{\text{对称}}}+\partial_t\rho$$

Def：流密度矢量 $\boxed{J_\alpha\triangleq\rho\Omega_{\alpha\beta}\frac{\partial H}{\partial\xi_\beta}}$ $\boxed{J\triangleq\rho\Delta_H}$



$$\Rightarrow\boxed{\partial_\alpha J_\alpha+\partial_t\rho=0}$$

$$\Rightarrow\boxed{\frac{d}{dt}\int\rho\,d\Gamma=-\int\partial_\alpha J_\alpha\,d\Gamma=-\oint J_\alpha\,d\eta_\alpha}$$

在足够远，J_α足够快地$\to 0$，则总相点数目守恒、$\int_{\text{全空间}}\rho\,d\Gamma=\text{const.}$

五．新Hamilton函数 $K(\eta,t)=K(Q,P,t)$

$$\dot\xi_\alpha=[\xi_\alpha,H]_\xi$$

$$\dot\eta_\alpha=[\eta_\alpha,K]_\eta=[\eta_\alpha,K']_\eta$$

$$\Rightarrow\text{对}\forall\alpha,\ [\eta_\alpha,K'-K]_\eta=0\quad\Rightarrow K-K=f(\cancel{\eta},t)$$

1、任二个新的Hamilton函数最多只能差一个时间t的函数，

$$\boxed{K(Q,P,t)+f(t)=K'(Q,P,t)}$$

2. 变换不显含时间t时 → 在等号后面，视 $\eta_\alpha = (\xi, t)$

$$\dot{\eta}_\alpha = [\eta_\alpha, H]_\xi + \cancel{\partial_t \eta_\alpha} = [\eta_\alpha, H]_\xi = [\eta_\alpha, H]_\eta$$

⇒ 可取 $\boxed{K = H(\xi, t) = H(\xi(\eta), t)}$ （通过反变换） ← 不显含t变换

相空间的 Lagrange 函数
$$\begin{cases} \tilde{L}_H = p_k \dot{q}_k - H(\xi, t) \\ \tilde{L}_K = P_k \dot{Q}_k - K(\eta, t) \end{cases}$$

$$\Rightarrow \begin{cases} \delta \int_{t_1}^{t_2} \tilde{L}(\xi, \dot{\xi}, t)\, dt = 0 \\ \delta \xi_\alpha(t_1) = 0 = \delta \xi_\alpha(t_2) \end{cases}$$

$$\tilde{L} \triangleq \tilde{L}_H - \tilde{L}_K = (K - H) + (p_k \dot{q}_k - P_i \dot{Q}_i) \rightarrow \dot{Q}_i$$

$$= (K - H) + \left(p_k - \frac{\partial Q_i}{\partial q_k} P_i\right) \dot{q}_k + \left(- \frac{\partial Q_i}{\partial p_k} P_i\right) \dot{p}_k - \frac{\partial Q_i}{\partial t} P_i$$

$$= K - H - \frac{\partial Q_k}{\partial t} P_k + \frac{\partial F}{\partial q_k} \dot{q}_k + \frac{\partial F}{\partial p_k} \dot{p}_k + \frac{\partial F}{\partial t} - \frac{\partial F}{\partial t}$$

$$\boxed{\tilde{L} = \underbrace{K - H - \frac{\partial F}{\partial t} - \frac{\partial Q_k}{\partial t} P_k}_{\triangleq \tilde{L}'(\xi, \cancel{\dot{\xi}}, t)} + \frac{dF(\xi, t)}{dt}} \rightarrow \text{不显含 } \dot{\xi}$$

$$\Rightarrow \quad 0 = \delta \int_{t_1}^{t_2} \tilde{L}'(\xi, \dot{\xi}, t)\, dt = 0$$

Euler-Lagrange 方程: $\dfrac{\partial \tilde{L}'}{\partial \xi_\alpha} = \dfrac{d}{dt} \dfrac{\partial \tilde{L}'}{\partial \dot{\xi}_\alpha} = 0$

故 $\tilde{L}'$ 不显含 ξ_α. $\tilde{L}' = \tilde{L}'(\cancel{\xi}, \cancel{\dot{\xi}}, t)$

130 对注：点变换：广义坐标之间的变换，$Q=Q(q,t)$

3. $\boxed{K = H + \frac{\partial F(\xi,t)}{\partial t} + \frac{\partial Q_k}{\partial t}P_k = K(\eta,t)}$ （可以相关时间也一元函数）.

eg. $Q_i = Q_i(q,t)$ $\quad P_i = \frac{\partial q_k}{\partial Q_i}\left[p_k - \frac{\partial F(\xi,t)}{\partial q_k}\right]$

① $F=0$，点变换 $\quad K = H + \underbrace{\frac{\partial Q_k}{\partial t}}P_k$

先求导再做反变换

② 当 $Q_i = q_i$， $\quad K = H + \frac{\partial F(q,t)}{\partial t}$

新旧Hamilton函数之差只与变换的结构有关，与"物理"无关：$K-H = \frac{\partial F(\xi,t)}{\partial t} + \frac{\partial Q_k}{\partial t}P_k$.

$$\boxed{\tilde{L}_H - \tilde{L}_K = (p_k\dot{q}_k - H) - (P_k\dot{Q}_k - K) = \frac{dF(q,p,t)}{dt}}$$ (*)

例如，使左右两边dt前系数相等，有：$K-H = \frac{\partial F}{\partial t} + \frac{\partial Q_k}{\partial t}P_k$

使左右两边dq, dp前系数相等，有：$\begin{cases} \frac{\partial F}{\partial q_k} = p_k - \frac{\partial Q_i}{\partial q_k}P_i \\ \frac{\partial F}{\partial p_k} = - \frac{\partial Q_i}{\partial p_k}P_i \end{cases}$

(*) 包含了正则变换的所有信息.

$$(p_k\dot{q}_k - P_k\dot{Q}_k) + (K-H) = \frac{dF(q,p,t)}{dt}$$

§7、CT及其生成函数的分类

一、CT分类的含义

$$\begin{cases} Q_i = Q_i(q,p,t) & ① \\ P_i = P_i(q,p,t) & ② \end{cases}$$ 变换可逆

Thm：从可逆变换中，可取s个旧变量，s个新变量，使得组成的这2s个变量独立，即可描述这个体系。

（但不必是正则变量） s阶注：$\det\left(\frac{\partial \text{已知的新变量}}{\partial \text{未知的旧变量}}\right) \neq 0$

Type I (q,Q)： 可将 p_i 通过式①给出，$p_i = p_i(q,Q,t)$

再代入② $P_i = P_i(q,p,t)$ 得到 P_i

要满足Hess条件（能进行上述反变换条件）：

$$\det\left(\frac{\partial Q_i}{\partial p_j}\right) \neq 0$$

Type II：(q,P)： Hess条件：$\det\left(\frac{\partial P_i}{\partial p_j}\right) \neq 0$

Type III：(p,Q)： Hess条件：$\det\left(\frac{\partial Q_i}{\partial q_j}\right) \neq 0$

Type IV (p,P)： Hess条件：$\det\left(\frac{\partial P_i}{\partial q_j}\right) \neq 0$

eg. 既是第二类，也是第三类：

$Q_i = q_i$，$P_i = p_i$ 时

(q,P) ✓ (p,Q) ✓ (q,Q) × (p,P) ×

eg. 既是第一类，也是第四类

$Q_i = p_i$，$P_i = -q_i$ 时.

(q,Q) ✓ (p,P) ✓ (q,P) × (p,Q) ×

eg. 四类全是：$Q_i = \frac{q+p}{\sqrt{2}}$ $P = \frac{p-q}{\sqrt{2}}$ 全不是：$Q_1 = q_1$，$P_1 = p_1$

$Q_2 = p_2$，$P_2 = q_2$

Thm：属于第二类的正则变换也必属于第三类

属于第一类的正则变换也必属于第四类

二. Type I：(q,Q) 独立 $\det\left(\frac{\partial Q_i}{\partial p_j}\right)\neq 0$

此时 def：$F_1(q,Q,t) \triangleq F(q,p(q,Q,t),t)$

$$(K-H)dt + p_k dq_k - P_k dQ_k = dF_1(q,Q,t) = \frac{\partial F_1}{\partial q_k}dq_k + \frac{\partial F}{\partial Q_k}dQ_k + \frac{\partial F}{\partial t}$$

$$\Rightarrow \boxed{\begin{aligned} p_k &= \frac{\partial F_1}{\partial q_k} = p_k(q,Q,t) \\ P_k &= -\frac{\partial F_1}{\partial Q_k} = P_k(q,Q,t) \\ K &= H + \frac{\partial F_1(q,Q,t)}{\partial t} \end{aligned}}$$

"确定"了 F_1 → 仍可以相差 $f(t)$

代入

⇒ 得到新的Hamilton函数.

先取偏导数 再代反变换

已知一个 F_1，可由前二个式子唯一确定一个正则变换.

(又要求 $\det\left(\frac{\partial p_i}{\partial Q_j}\right)\neq 0 \neq \det\left(\frac{\partial^2 F_1}{\partial q_i \partial Q_j}\right)\neq 0$

↓

正则变换的条件　　对 F_1 提出的限制.

已知 F_1，通过代入前二个得到变换关系，再代到第三个并作反变换得到 K

eg. $F_1 = q_i Q_i$　$\frac{\partial^2 F_1}{\partial q \partial Q} = I$，是.

$\therefore p_k = Q_k,\ P_k = -q_k \Rightarrow Q_k = p_k,\ P_k = -q_k$

eg. $(q,p)\mapsto(Q,P)$　$[q,p]_{Q,P} = 1$ 故是正则变换

$$\left\{\begin{aligned} q &= \sqrt{2P}\sin Q \\ p &= \sqrt{2P}\cos Q \end{aligned}\right\}$$

$$\Downarrow$$

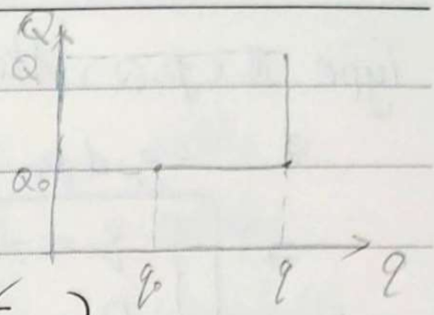
$$\left\{\begin{aligned} p &= \frac{q}{\tan Q} = \frac{\partial F_1}{\partial q} \quad ① \\ P &= \frac{q^2}{2\sin Q} = -\frac{\partial F_1}{\partial Q} \quad ② \end{aligned}\right.$$

法1 $F_1 = \int p dq - P dQ$

$$= \int_{q_0}^{q} \frac{q}{\tan Q_0} dq - \int_{Q_0}^{Q} \frac{q^2}{2\sin^2 Q} dQ$$

$$= \left(\cancel{\frac{q^2}{2\tan Q_0}} - \underbrace{\frac{q_0^2}{2\tan Q_0}}_{\text{常数，不要了}}\right) + \left(\frac{q^2}{2\tan Q} - \cancel{\frac{q^2}{2\tan Q_0}}\right)$$

$$= \frac{q^2}{2\tan Q}$$



法2 ① $\Rightarrow F_1 = \int \frac{q}{\tan Q} dq + f(Q, t) = \frac{q^2}{2\tan Q} + f$

代入②，$\frac{\partial f}{\partial Q} = 0 \Rightarrow f = f(t) \xRightarrow{\text{不妨设}} 0$

eg. $H = \frac{1}{2}\omega(p^2 + q^2) \Rightarrow K = H = \omega P$

$$\Rightarrow \begin{cases} \dot{Q} = \omega \\ \dot{P} = 0 \end{cases} \Rightarrow \begin{cases} Q = \omega t + \varphi \\ P = \text{const}. \end{cases}$$

$$\Rightarrow \begin{cases} q = \sqrt{2P_0} \sin(\omega t + \varphi) \\ p = \sqrt{2P_0} \cos(\omega t + \varphi) \end{cases}$$

Type II. (q, P) 独立，$\det\left(\frac{\partial P_i}{\partial p_j}\right) \neq 0$

约束式子：$p_k dq_k - P_k dQ_k + (K - H) dt = dF(q, p, t)$

变形为：$p_k dq_k + Q_k dP_k + (K - H) dt = dF_2(q, P, t)$

其中 def：$F_2 \triangleq F + Q_k P_k$，需满足 $\det\left(\frac{\partial^2 F_2}{\partial q_i \partial P_j}\right) \neq 0$

$$\Rightarrow \boxed{\begin{cases} p_k = \frac{\partial F_2}{\partial q_k} \\ Q_k = \frac{\partial F_2}{\partial P_k} \\ K = H + \frac{\partial F_2}{\partial t} \end{cases}}$$

Type III (p, Q) 独立 $\boxed{F_3 \triangleq F_1 - q_k p_k}$

$\Rightarrow \quad -q_k dp_k - P_k dQ_k + (K-H)dt = dF_3(p, Q, t)$

$$\Rightarrow \boxed{\begin{cases} q_k = -\frac{\partial F_3}{\partial p_k} \\ P_k = -\frac{\partial F_3}{\partial Q_k} \\ K = H + \frac{\partial F_3}{\partial t} \end{cases}}$$

Type IV (p, P) 独立 $F_4 \triangleq F - q_k p_k + Q_k P_k$

$\Rightarrow \quad -q_k dp_k + Q_k dP_k + (K-H)dt = dF_4(p, P, t)$

$$\Rightarrow \boxed{\begin{cases} q_k = -\frac{\partial F_4}{\partial p} \\ Q_k = \frac{\partial F_4}{\partial P_k} \\ K = H + \frac{\partial F_4}{\partial t} \end{cases}}$$

注意对 Type I，$\frac{\partial F_1}{\partial q_i \partial Q_j} = \frac{\partial p_i}{\partial Q_j}$ → 第二类变换

$$\quad = -\frac{\partial P_j}{\partial q_j}$$ → 第四类变换 ↔ 二者条件等价.

② $F_1(q, Q, t) = F_2(q, P, t) - Q_k P_k$

$$= -\left(\frac{\partial F_2}{\partial P_k} P_k - F_2\right)$$

F_1、F_2 互为"Legendre 变换"，但必须要求该变换同时属于第I、II类正则变换.

eg. $Q_i = Q_i(q,t)$ $\quad P_i = \frac{\partial q_k}{\partial Q_i} p_k$

是第二类但不是第一类

$$\Rightarrow \begin{cases} Q_i = Q_i(q,t) = \frac{\partial F_2}{\partial P_i} \\ p_i = \frac{\partial Q_k}{\partial q_i} P_k = \frac{\partial F_2}{\partial q_i} \end{cases} \Rightarrow F_2 = \int Q_k dP_k + f(q,t)$$

$$= Q_k(q,t) P_k + f$$

$$\xleftarrow{代入}$$

$$\Rightarrow \frac{\partial Q_k}{\partial q_i} P_k = \frac{\partial Q_k}{\partial q_i} P_k + \frac{\partial f}{\partial q_i} \Rightarrow f = f(t) \xrightarrow{不妨设} 0$$

故 $F_2 = Q_k P_k$ （应把新Q用旧q表示出来）

（若 $F_1 = F_2 - Q_k P_k = 0$，不满足$\left(\frac{\partial^2 F_1}{\partial q \partial Q}\right) \neq 0$，事实上也不能生成）.

例如，$(r,\theta) \mapsto (x,y)$，$(p_r, p_\theta) \mapsto (p_x, p_y)$

$$\Rightarrow F_2 = x p_x + y p_y = r\cos\theta\, p_x + r\sin\theta\, p_y$$

$$\Rightarrow p_r = \frac{\partial F_2}{\partial r} = p_x \cos\theta + p_y \sin\theta = \frac{x p_x + y p_y}{\sqrt{x^2+y^2}} = \hat{r} \cdot \vec{p}$$

$$p_\theta = \frac{\partial F_2}{\partial \theta} = -r p_x \sin\theta + r p_y \cos\theta = -y p_x + x p_y = l$$

$$x = \frac{\partial F_2}{\partial p_x} = r\cos\theta$$

$$y = \frac{\partial F_2}{\partial p_y} = r\sin\theta$$

证明：满足 $M\Omega M^T = \Omega$ 的 M 有 $\det M = 1$：

CT：$\eta = M\xi$ （不妨设为第二类的）

（这里"不妨设"是指通过类似的 $\begin{cases} Q_i = p_i \\ P_i = -q_i \end{cases}$ 将该变换再变为成第二类的）$\to \det = 1$

$$\begin{tikzcd} (q,p) \arrow[d, "可逆"] \\ (q,P) \arrow[d, "可逆"] \\ (Q,P) \end{tikzcd}$$

则总的雅可比 $\dfrac{\partial(Q,P)}{\partial(q,p)} = \dfrac{\partial(Q,P)}{\partial(q,P)} \Big/ \dfrac{\partial(q,p)}{\partial(q,P)}$

对此情况，q, P 独立，$\begin{cases} Q = Q(q,P,t) \\ P = P \end{cases}$ 故：$\dfrac{\partial(Q,P)}{\partial(q,P)} = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial P} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial P} \end{vmatrix} = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial P} \\ 0 & I \end{vmatrix} = \left|\frac{\partial Q}{\partial q}\right|$

$$\Rightarrow \frac{\partial(Q,P)}{\partial(q,p)} = \frac{\left(\frac{\partial Q}{\partial q}\right)_P}{\left(\frac{\partial p}{\partial P}\right)_q} = \frac{\det A}{\det B}$$

其中 $A_{ij} = \left(\dfrac{\partial Q_i}{\partial q_j}\right)_P$

$B_{ij} = \left(\dfrac{\partial p_i}{\partial P_j}\right)_q = \dfrac{\partial}{\partial P_i}\dfrac{\partial F_2}{\partial q_j} = \dfrac{\partial}{\partial q_i}\dfrac{\partial F_2}{\partial P_j} = \left(\dfrac{\partial Q_j}{\partial q_i}\right)_P$

故 $A = B^T$，$\dfrac{\det A}{\det B} = 1 \Rightarrow \dfrac{\partial(Q,P)}{\partial(q,p)} = 1$ □

eg. $Q = q\cos\omega t - p\sin\omega t$，$P = q\sin\omega t + p\cos\omega t$

$p = \dfrac{P - q\sin\omega t}{\cos\omega t} = \dfrac{\partial F_2}{\partial q} \Rightarrow F_2 = \int p\,dq + f(P,t) = \dfrac{2qP - q^2\sin\omega t}{2\cos\omega t} + f$

$Q = \dfrac{q - P\sin\omega t}{\cos\omega t} = \dfrac{\partial F_2}{\partial P}$ $\swarrow$ 代入 $= \dfrac{q}{\cos\omega t} + \dfrac{\partial f}{\partial P} \Rightarrow \dfrac{\partial f}{\partial P} = -P\tan\omega t$

$\Rightarrow f = \underset{\times 可丢时间函数}{P^2 \cdot \dfrac{\tan\omega t}{2}} \Rightarrow F_2 = \dfrac{2qP - (q^2 + P^2)\sin\omega t}{2\cos\omega t}$

$$\frac{\partial F_2}{\partial t} = -\frac{\omega}{2\cos^2\omega t}\left[q^2+P^2-2qP\sin\omega t\right]$$

$$= -\frac{\omega}{2\cos^2\omega t}\left[(q-P\sin\omega t)^2+P^2\cos^2\omega t\right]$$

$$= -\frac{1}{2}\omega(Q^2+P^2)$$

例如取 $H=\frac{1}{2}\omega(q^2+p^2)$， $H=\frac{1}{2}\omega(Q^2+P^2)$

$\Rightarrow K = H + \frac{\partial F_2}{\partial t} = 0$

$\Rightarrow Q = Q_0,\ P = P_0$ 不变

$$q = \underset{q_0}{\underset{\downarrow}{Q_0}}\cos\omega t + \underset{p_0}{\underset{\downarrow}{P_0}}\sin\omega t,\quad p = -\underset{-q_0}{\underset{\downarrow}{Q}}\sin\omega t + \underset{p_0}{\underset{\downarrow}{P}}\cos\omega t$$

§8 Hamilton - Jacobi 理论

$$H(q,p,t) \xrightarrow{F_2(q,P,t)} K = 0 = H + \frac{\partial F_2}{\partial t} \triangleq H + \frac{\partial S}{\partial t} = 0$$

一般记 $S(q,P,t) \triangleq F_2(q,P,t)$ （[illegible] $\frac{\partial S}{\partial q\partial p} \neq 0$ ）

$$H(q,p,t) + \frac{\partial S}{\partial t} = H(q,\frac{\partial S}{\partial q},t) + \frac{\partial S}{\partial t} = 0$$

一、HJ方程：

$$\boxed{-\frac{\partial S}{\partial t} = H(q,\frac{\partial S}{\partial q},t)}$$

S：Hamilton 主函数
$S = S(q,P,t)$

eg. $H = \frac{p^2}{2m} \rightarrow -\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2$

eg. $H = \frac{1}{2}\omega(p^2+q^2) \rightarrow -\frac{\partial S}{\partial t} = \frac{1}{2}\omega\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}\omega q^2$

一般，HJ方程是 S关于 s+1 个自变量 $q_1, \cdots q_s, t$ 的一阶（非线性）偏微分方程.

$$p_k=\frac{\partial S}{\partial q_k}=p_k(q,P,t)\ ;\ Q_k=\frac{\partial S}{\partial P_k}=Q_k(q,P,t)$$

(作反变换，给出s个守恒量 将P代入，给出另外s个守恒量

得到P随时间的演化。将q代入 作反变换，得到q随时间的变化

对任一条路径

$$\frac{dS}{dt}=\underbrace{\boxed{\frac{\partial S}{\partial q_k}}}_{=p_k}\dot q_k+\frac{\partial S}{\partial P_k}\dot P_k+\frac{\partial S}{\partial t}=-H$$

真实路径时，$\dot P_k=0$，$\frac{dS}{dt}=p_k\dot q_k-H \Rightarrow S=\int_0^t(p_k\dot q_k-H)dt$

S是不定限的作用量，是真实路径的作用量。

我们要完全积分解。由于有s+1个变量，有s+1个初始条件 $C_0, C_1, \cdots, C_s$

其中s个我们取为P，另外一个 C_0 是由S的不定引起的：

$$S=S_0+C_0$$

若将 C_0 取为某个P，则 $\frac{\partial S}{\partial P}=\frac{\partial S}{\partial C_0}=1$，$\frac{\partial S}{\partial q_i\partial C_0}=0$，不满足Hess条

一般地，偏微分方程的解中将依赖于一些任意函数，若我们给出的解只依赖于一些任意常数，则称是完全积分解。

eg. $H=p$. $-\frac{\partial S}{\partial t}=\frac{\partial S}{\partial q} \Rightarrow S=f(q-t)$ （通解）

取 $S=C(q-t)+C_0$，记 $P=C$，$\Rightarrow S=P(q-t)$

$\Rightarrow p=P$，$Q=q-t$

eg. $H=\frac{p^2}{2}$　$-\frac{\partial S}{\partial t}=\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 \Rightarrow S_a=\frac{(q-P_a)^2}{2t}$

$$\begin{cases} p=\frac{\partial S_a}{\partial q}=\frac{q-P_a}{t} \\ Q_a=\frac{\partial S_a}{\partial P_a}=\frac{P_a-q}{t} \end{cases}$$

守恒量：

$$\begin{cases} P_a=q-pt \\ Q_a=-p \end{cases}$$

也可取 $S_b=q\sqrt{2P_b}-P_b t \Rightarrow \begin{cases} p=\frac{\partial S_b}{\partial q}=2\sqrt{2P_b} \\ Q_b=\frac{\partial S_b}{\partial P_b}=\frac{q}{\sqrt{2P_b}}-t \end{cases}$

守恒量：

$$\begin{cases} P_b=\frac{p^2}{2} \\ Q_b=\frac{q}{p}-t \end{cases}$$

对同一个HJ方程，可有无穷多种完全积分解

不同的S各自给出2s个独立的守恒量（运动常数）.

二. 不显含t的HJ方程（$H=H(q,p,\cancel{t})$）

令主函数 $S=W(q)+T(t)$

（指一定存在至少一个完全积分解能写成这样的形式）

代入HJ方程，$-\frac{\partial T}{\partial t}=H(q,\frac{\partial W}{\partial q})\triangleq P_1$

$$\Rightarrow \boxed{S=W(q)-P_1 t}$$

$$\Rightarrow \boxed{H(q,\frac{\partial W}{\partial q})=P_1}$$

$W(q,P)$ Hamilton特征函数

事实上，是将H本身视为一个守恒量，P_1 即为H的数值.

（可形式说为，循环坐标t对应的守恒量）

eg. $H=\frac{1}{2}\omega(p^2+q^2) \Rightarrow \frac{1}{2}\omega\left(\frac{\partial W}{\partial q}\right)^2+\frac{1}{2}\omega q^2 = P$

→ 前面的正负号没有必要了

$$\Rightarrow S=\int\sqrt{\frac{2P}{\omega}-q^2}\,dq-Pt$$

$$\Rightarrow \begin{cases} p=\frac{\partial S}{\partial q}=\sqrt{\frac{2P}{\omega}-q^2} \Rightarrow p^2+q^2=\frac{2P}{\omega} & \text{：相空间轨道} \\ Q=\frac{\partial S}{\partial P}=\frac{1}{\omega}\int\frac{dq}{\sqrt{\frac{2P}{\omega}-q^2}}-t \ \hat{=}\ -t_0 \end{cases}$$

$\Rightarrow q=\sqrt{\frac{2P}{\omega}}\sin\omega(t-t_0)$ ：在相空间轨道上是匀速运动….

考虑如何再分出来一个q_s：

$W=\overline{W}(\bar{q})+W_s(q_s)$ ，$\bar{q}\doteq(q_1,\cdots,q_{s-1})$ $\bar{p}=(p_1,\cdots p_{s-1})$

① 当q_s为循环坐标，$\boxed{P_s\hat{=}p_s=\frac{\partial W_s}{\partial q_s}\Rightarrow W_s=P_sq_s}$

方程变为 $H(\bar{q},\frac{\partial\overline{W}}{\partial\bar{q}},P_s)=P_1$

② H能写为：$H=\overline{H}(\bar{q},\bar{p})+H_s(q_s,p_s)$

$$\Rightarrow \overline{H}(\bar{q},\frac{\partial\overline{W}}{\partial\bar{q}})+H_s(q_s,\frac{\partial W_s}{\partial q_s})=P_1$$

$$\Rightarrow \begin{cases}\boxed{H_s(q_s,\frac{\partial W_s}{\partial q_s})=P_s} \\ \boxed{H(\bar{q},\frac{\partial\overline{W}}{\partial\bar{q}})=P_1-P_s}\end{cases}$$ 变为常微分方程.

③ 若有 $f(\bar{q})H=\overline{H}(\bar{q},\bar{p})+H_s(q_s,p_s)$

$$\Rightarrow f(\bar{q})P_1=\overline{H}(\bar{q},\frac{\partial\overline{W}}{\partial\bar{q}})+H_s(q_s,\frac{\partial W_s}{\partial q_s})$$

($f(q_s)H=\overline{H}(\bar{q},\bar{p})+H_s(q_s,p_s)$ 也同理)

三 完全可分离体系

$W = \sum_{i=1}^{s} W_i(q_i, P)$ 或 $\boxed{S = \sum_{i=1}^{s} W_i(q_i, P) - P_1 t}$

此时 $\begin{cases} p_K = \dfrac{\partial W_K}{\partial q_K} \text{(不求和)} = p_K(q_K, P) & \text{Ia} \\ Q_K = \sum\limits_{i=1}^{s} \dfrac{\partial W_i}{\partial p_K}, (K \geq 2) = Q_K(q, P) & \text{Ib} \\ Q_1 = \sum\limits_{i=1}^{s} \dfrac{\partial W_i}{\partial P_1} - t = Q_1(q, P, t) \quad (\partial_t Q_1 = -1) & \text{Ic} \end{cases}$

2S个关系 $\begin{cases} \text{Ia} \Rightarrow P_K = P_K(q, p) & \text{IIa} \\ \text{IIa 代Ib} \Rightarrow Q_{K\geq 2} = Q_{K\geq 2}(q, p) & \text{IIb} \\ \text{IIa 代Ic} \Rightarrow Q_1 = Q_1(q, p, t) \quad (\partial_t Q_1 = -1) & \text{IIc} \end{cases}$

Ia 给出 (q_K, p_K) 面相轨迹之投影—仅与P有关

Ib 给出 位形空间中之轨道

而 Ia+Ib 或 IIa+Ib 给出相轨迹

Ic 或 IIc 给出在轨道中随时间之演化.

※注意观察 H 里面不显含啥坐标 (q_k)，则与其对应之 p_k 可令其 $\triangleq P_k$ 为一常数

eg. $B = B\hat{z}$，取 $\vec{A} = Bx\hat{y}$ （此时 $\nabla\times\vec{A} = (\partial_x A_y)\hat{z} = B\hat{z}$，朗道规范）

$$H = \sum \frac{(p_i - eA_i)^2}{2m} = \frac{p_x^2}{2m} + \frac{(p_y - eBx)^2}{2m} = E = T$$

$$S = W(x,y) - P_1 t = \bar{X}(x) + P_2 y - P_1 t$$

$$\Rightarrow\ P_1 = \frac{1}{2m}\left(\frac{\partial \bar{X}}{\partial x}\right)^2 + \frac{(P_2 - eBx)^2}{2m}$$

$$\Rightarrow\ S = \underbrace{\int\sqrt{2mP_1 - (P_2 - eBx)^2}\,dx}_{\bar{X}(x)} + P_2 y - P_1 t$$

$$p_x = \frac{\partial S}{\partial x} = \sqrt{2mP_1 - (P_2 - eBx)^2} \qquad \text{Def: } \omega \triangleq \frac{eB}{m}$$

$$p_y = \frac{\partial S}{\partial y} = P_2 \triangleq eBx_c = m\omega x_c$$

$$y_c \triangleq Q_2 = \frac{\partial S}{\partial P_2} = \frac{\partial \bar{X}}{\partial P_2} + y = \frac{\partial \bar{X}}{\partial (P_2 - eBx)} + y = -\frac{1}{eB}\frac{\partial \bar{X}}{\partial x} + y$$

$$= -\frac{\sqrt{2mP_1 - (P_2 - eBx)^2}}{eB} + y$$

$$-t_c \triangleq Q_1 = \frac{\partial S}{\partial P_1} = \int\frac{m\,dx}{\sqrt{2mP_1 - (P_2 - eBx)^2}} - t$$

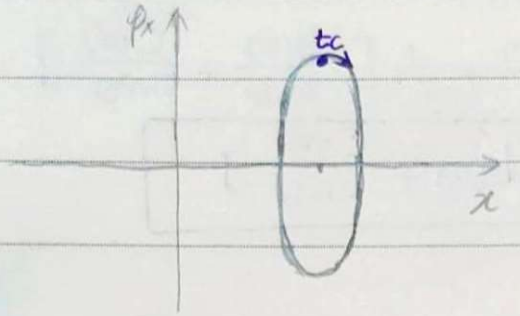
Def: $P_1 = \frac{m}{2}\omega^2 r^2$ （s.t. $2mP_1 = (m\omega r)^2$ ）

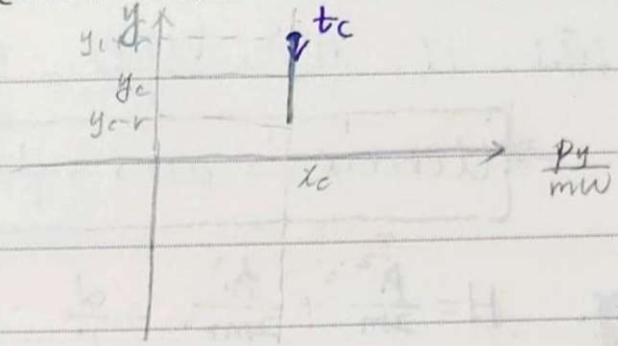
$$\Rightarrow \begin{cases} p_x = m\omega\sqrt{r^2 - (x - x_c)^2} \\ p_y = m\omega x_c \\ Q_2 = -\sqrt{r^2 - (x - x_c)^2} + y \\ Q_1 = \frac{1}{\omega}\int\frac{dx}{\sqrt{r^2 - (x - x_c)^2}} - t = \frac{1}{\omega}\arcsin\frac{x - x_c}{r} - t \end{cases}$$

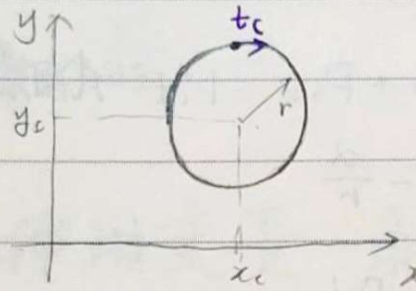
→ P_2相关

变形：
$$\begin{cases} \frac{(x-x_c)^2}{r^2} + \frac{p_x^2}{(m\omega r)^2} = 1 \\ p_y = m\omega x_c \\ (x-x_c)^2 + (y-y_c)^2 = r^2 \\ x-x_c = r\sin(\omega(t-t_c)) \end{cases}$$
→ P_1相关

(x, p_x)平面内相轨道投影







eg. $H = \frac{p_r^2}{2m} + \frac{1}{2mr^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + U(r,\theta,\phi)$

$W = R(r) + \Theta(\theta) + \Phi(\phi)$

$\Rightarrow \frac{1}{2m}\left(\frac{\partial R}{\partial r}\right)^2 + \frac{1}{2mr^2}\left[\left(\frac{\partial \theta}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial \Phi}{\partial \phi}\right)^2\right] = P_1 - U$

$\Rightarrow r^2\left(\frac{\partial R}{\partial r}\right)^2 + \left[\left(\frac{\partial \Theta}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial \Phi}{\partial \phi}\right)^2\right] = 2mr^2(P_1 - U)$

若 $r^2 U(r,\theta,\phi) = A(r) + V(\theta,\phi)$

$RHS = 2m(r^2 P_1 - A) - 2mV(\theta,\phi)$

$$\Rightarrow \begin{cases} r^2\left(\frac{\partial R}{\partial r}\right)^2 - 2m\left[r^2 P_1 - A(r)\right] \triangleq P_2 \\ \left(\frac{\partial \Theta}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial \Phi}{\partial \phi}\right)^2 + 2mV(\theta,\phi) = -P_2 \end{cases}$$

第二式改为: $\sin^2\theta\left(\frac{\partial\Theta}{\partial\theta}\right)^2+\left(\frac{\partial\Phi}{\partial\phi}\right)^2+2mV(\theta,\phi)\sin^2\theta=-P_2\sin^2\theta$

即要求 $V(\theta,\phi)\sin^2\theta=B(\theta)+C(\phi)$

$$\Rightarrow\begin{cases}\sin^2\theta\left(\frac{\partial\Theta}{\partial\theta}\right)^2+2mB(\theta)+P_2\sin^2\theta=P_3\\ \left(\frac{\partial\Phi}{\partial\phi}\right)^2+2mC(\phi)=-P_3\end{cases}$$

综上, $U=\frac{A(r)}{r^2}+\frac{V(\theta,\phi)}{r^2}=\frac{A(r)}{r^2}+\frac{1}{r^2}\left[\frac{B(\theta)}{\sin^2\theta}+\frac{C(\phi)}{\sin^2\theta}\right]$

$$\Rightarrow\boxed{U(r,\theta,\phi)=a(r)+\frac{1}{r^2}\left[b(\theta)+\frac{C(\phi)}{\sin^2\theta}\right]}$$

eg. $H=\frac{p_r^2}{2m}+\frac{p_\theta^2}{2mr^2}-\frac{\alpha}{r}$

$S=W(r,\theta)-P_1t=R(r)+P_2\theta-P_1t$ 代回方程,

$$P_1=\frac{1}{2m}\left(\frac{\partial R}{\partial r}\right)^2+\frac{P_2^2}{2mr^2}-\frac{\alpha}{r}$$

$$\Rightarrow S=\int\sqrt{2mP_1+\frac{2m\alpha}{r}-\frac{P_2^2}{r^2}}\,dr+P_2\theta-P_1t$$

给出轨道: $Q_2=\frac{\partial S}{\partial P_2}=-\int\frac{P_2\,dr}{r^2\sqrt{2mP_1+\frac{2m\alpha}{r}-\frac{P_2^2}{r^2}}}+\theta\triangleq\theta_0$

Def: $(P_2\triangleq p_\theta,\ P_1=E)$

$$\theta-\theta_0\overset{u=\frac{1}{r}}{=\!=}-\int\frac{du}{\sqrt{\left(\frac{m\alpha}{p_\theta^2}\right)^2\left(1+\frac{Ep_\theta^2}{m\alpha^2}\right)-\left(u-\frac{m\alpha}{p_\theta^2}\right)^2}}$$

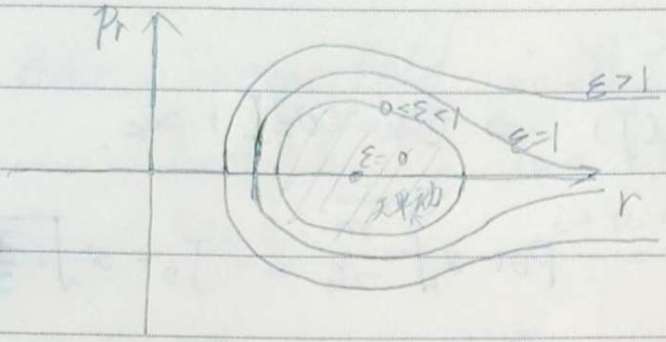
$$\triangleq-\int\frac{du}{\sqrt{\left(\frac{\varepsilon}{r_0}\right)^2-\left(u-\frac{1}{r_0}\right)^2}}=\arccos\frac{u-\frac{1}{r_0}}{\varepsilon/r_0}\Rightarrow r=\frac{r_0}{1+\varepsilon\cos(\theta-\theta_0)}$$

$$\varepsilon\triangleq\sqrt{1+\frac{Ep_\theta^2}{m\alpha^2}},\quad\frac{p_\theta^2}{m\alpha}=r_0$$

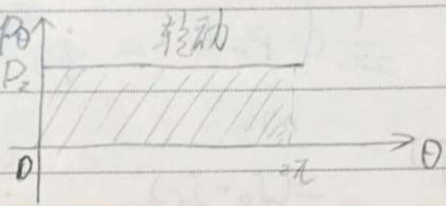
$$p_r = \frac{\partial S}{\partial r} = \sqrt{2mE + \frac{2m\alpha}{r} - \frac{p_\theta^2}{r^2}} = p_\theta\sqrt{\left(\frac{\varepsilon}{r_0}\right)^2 - \left(\frac{1}{r} - \frac{1}{r_0}\right)^2}$$

$$= p_\theta \sqrt{\left(\frac{1+\varepsilon}{r_0} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1-\varepsilon}{r_0}\right)}$$

给出 r-p_r 上之相轨道:



$$p_\theta = \frac{\partial S}{\partial \theta} = P_2$$



*§9 作用变量-角变量理论

对有界轨道 定义: $\boxed{J_k \triangleq \frac{1}{2\pi}\oint p_k dq_k}$ 作用变量 (s个)

天平动时是相轨道面积 (闭合)

转动时是θ取可能之一次 (大部分是2π)

不同相轨道"独立", 要求体系是可完全分离之.

$$S = \sum_i W_i(q_i, P) - P_1 t = W - P_1 t$$

将 $p_k = \frac{\partial S}{\partial q_k} = p_k(q_k, P)$ 代入.

$$J_k = J_k(P) \Rightarrow P_k = P_k(J)$$

定义 (第二类) 生成函数 $\boxed{\overline{W}(q, J) \triangleq W(q, P(J))}$ (不显含t)

与J共轭之广义坐标称:

角变量 $\boxed{\Theta_k = \frac{\partial W}{\partial J_k}}$

新 Hamilton 函数 $K = H = P_1(J)$

只出现 J 不出现 Θ，把所有 Θ 变为循环坐标。

（J 是守恒量）

$$\dot{\Theta}_k = \frac{\partial K}{\partial J_k} \triangleq \omega_k(J) \Rightarrow \Theta_k = \omega_k t + \alpha_k .$$

eg. $$J_r = \frac{1}{2\pi}\oint p_r dr = -p_\theta + \alpha\sqrt{-\frac{m}{2E}} = -J_\theta + \alpha\sqrt{-\frac{m}{2E}}$$

$$J_\theta = \frac{1}{2\pi}\oint p_\theta d\theta = p_\theta$$

$$H(J) = \frac{m\alpha^2}{2(J_\theta + J_r)^2} \quad (=E)$$

$$\Rightarrow \omega_r = \frac{\partial H}{\partial J_r} = \frac{m\alpha^2}{(J_r+J_\theta)^{\frac{3}{2}}} = \frac{m}{\alpha\left(-\frac{m}{2E}\right)^{\frac{3}{2}}} = \omega_\theta .$$

$w_r = w_\theta$ 故是周期性的，轨道是闭合的。

一般地，$\frac{w_r}{w_\theta}$ 是有理数，则轨道是闭合的

$\frac{w_r}{w_\theta}$ 是无理数 则轨道不是闭合的。

Bohr 模型：

① 定态假设 ② 量子化假设 $p_\theta = mvr = n\hbar$

③ 跃迁假设 $h\nu = E_{n_1} - E_{n_2}$

Bohr - Sommerfeld 之推广：

$J_r = k\hbar$, $J_\theta = l\hbar$ $\quad n = k - l = 1, 2, \cdots$

$k = 1, 2, \cdots$ $\quad l = 0, 1, 2, \cdots$

可适用于椭圆轨道

$\xi = (\vec{r}, \vec{p})$ $\quad H = H(\vec{r}, \vec{p})$

$$\Rightarrow -\frac{\partial S}{\partial t} = H(\vec{r}, \frac{\partial S}{\partial \vec{r}}) = H(\vec{r}, \nabla S)$$

$\partial_t \to -i\hbar \partial_t$, $\nabla \to -i\hbar \nabla$, $S \to \psi$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = H(\vec{r}, -i\hbar \nabla \psi)$$

eg. $H = \frac{p^2}{2m} + U(r)$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = (-\frac{\hbar^2}{2m}\nabla^2 + U)\psi$$

Mo Tu We Th Fr Sa Su

Memo No.
Date / /

CH6 刚体

§1 刚体运动学

一. 定义 $r_{ab} = |\vec{r}_{ab}| = |\vec{r}_a - \vec{r}_0| = C_{ab}$ $(a,b=1,\cdots,N)$

二. 自由度: 只要确定三个点的位置, 整个刚体位置便确定了

一般刚体 $s=6$, 线状刚体 $s=5$

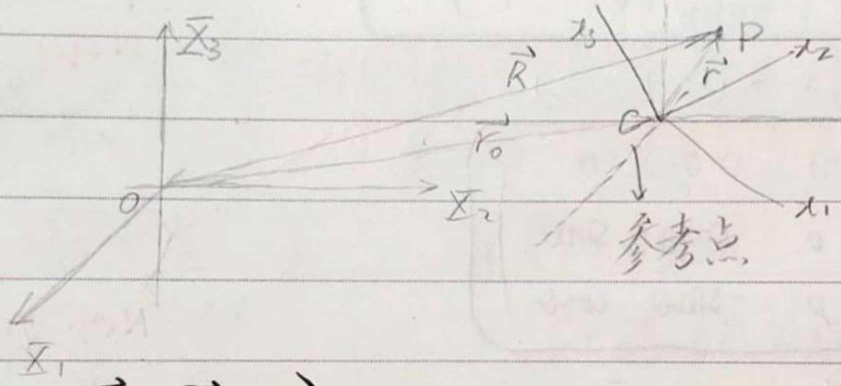
外部有约束时, 自由度进一步减小.

定点转动 $s=3$ 定轴转动 $s=1$

空间坐标系 $O\bar{X}_1\bar{X}_2\bar{X}_3$ — 惯性系

本体坐标系 $Cx_1x_2x_3$ — 固定于刚体上的坐标系.

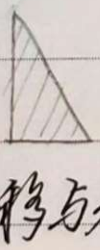
(相当于把刚体"延展", 只有运动学性质无动力学性质)

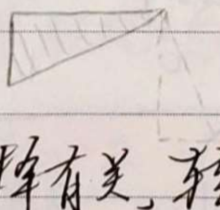


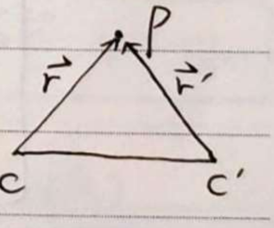
$\vec{R} = \vec{r}_0 + \vec{r}$

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{d\vec{r}_0}{dt} + \frac{d\vec{r}}{dt} = \vec{v}_0 + \vec{\omega}\times\vec{r}$$

刚体绕C, 即相对空间坐标系的角速度







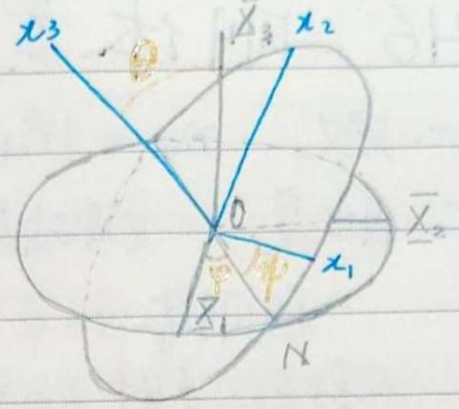
平移与参考点选择有关, 转动与参考点选择无关.

$$\vec{v} = \vec{v}_0 + \vec{\omega}\times\vec{r} = \vec{v}_0' + \vec{\omega}'\times\vec{r}' = \vec{v}_0 + \vec{\omega}\times(\vec{r}-\vec{r}') + \vec{\omega}'\times\vec{r}'$$

即 $(\vec{\omega}'-\vec{\omega})\times\vec{r}' = 0$ 对 $\forall\vec{r}'$ 成立

$\Rightarrow \boxed{\vec{\omega}' = \vec{\omega}}$

三、刚体角速度 $\vec{\omega}$



四、Euler角 (313型)

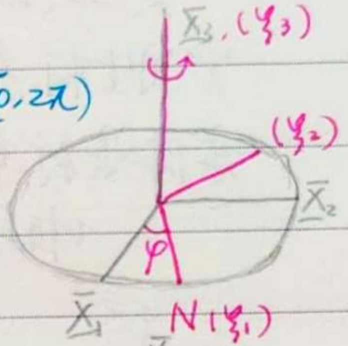
节线ON：$\bar{X}_1\bar{X}_2$与x_1x_2的交线

正方向选作 $\hat{\bar{X}}_3 \times \hat{x}_3$.

进动角φ，章动角θ，自转角ψ

看做连续三次的转动：

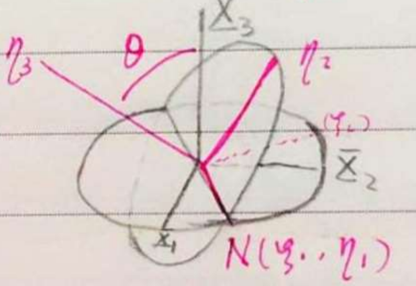
① 进动：绕$\bar{X}_3$转动角度φ，记$(\hat{\bar{X}}_3, \varphi)$，$\varphi \in [0, 2\pi)$



(对应之前的被动)

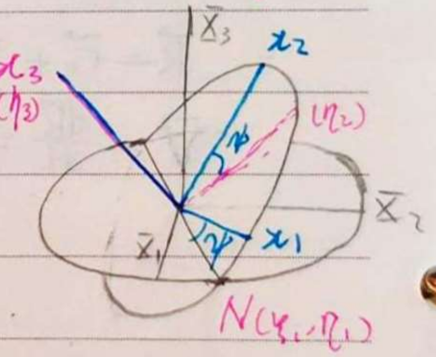
$$\boxed{\lambda_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

② 章动：$(\overrightarrow{ON}, \theta)$ $\theta \in [0, \pi)$



$$\boxed{\lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}}$$

③ 自转：$(\hat{x}_3, \psi)$ $\psi \in [0, 2\pi)$



$$\boxed{\lambda_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda_\psi \lambda_\theta \lambda_\varphi \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{pmatrix}}, \quad \lambda \triangleq \lambda_\psi \lambda_\theta \lambda_\varphi =$$

λ既有主动，也有被动的含义.

$d\vec{\Theta} = \hat{X}_3 d\varphi + \hat{ON} d\theta + \hat{x}_3 d\psi$

$\vec{\omega} = \dot{\varphi}\hat{X}_3 + \dot{\theta}\,\hat{ON} + \dot{\psi}\,\hat{x}_3$ 刚体相对空间坐标系の角速度

$$\boxed{\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta}$$

将$\hat{X}_3$投影. $\hat{X}_3 = \hat{x}_1 \overset{\lambda_{13}}{\sin\theta\sin\psi} + \hat{x}_2 \overset{\lambda_{23}}{\sin\theta\cos\psi} + \hat{x}_3 \overset{\lambda_{33}}{\cos\theta} = \lambda_{i3}\hat{x}_i$

将$\hat{ON}$投影. $\hat{ON} = \hat{x}_1\cos\psi + (-\hat{x}_2\sin\psi)$

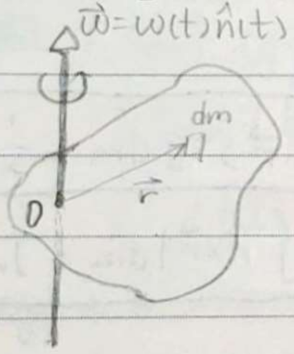
在本体系, $$\boxed{\begin{cases} \omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases}}$$

称 Euler 运动学方程

ssj注：同样地，反过来有：$\begin{cases} \dot{\varphi} = \omega_1 \csc\theta\sin\psi + \omega_2\csc\theta\cos\psi \\ \dot{\theta} = \omega_1\cos\psi - \omega_2\sin\psi \\ \dot{\psi} = -\omega_1\cot\theta\sin\psi - \omega_2\cot\theta\cos\psi + \omega_3 \end{cases}$

§2 定点转动刚体的角动量及动能

知 $\varphi, \theta, \psi, \omega_1, \omega_2, \omega_3$ 来求 $\dot{\varphi}, \dot{\theta}, \dot{\psi}$.



一. 角动量：对基点O の角动量. $\vec{r}_0 = \vec{0}$

$$\vec{L} = \int(\vec{r}\times\vec{v})dm$$
$$= \int[r^2\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r}]dm$$
$$= \int[r^2\overleftrightarrow{I} - \vec{r}\vec{r}]dm \cdot \vec{\omega}$$

Def: $\boxed{\overleftrightarrow{J} = \int(r^2\overleftrightarrow{I} - \vec{r}\vec{r})dm}$

$\boxed{\vec{L} = \overleftrightarrow{J}\cdot\vec{\omega}}$, $\overleftrightarrow{J}$ 称绕O点の惯量张量

(与参考点选择有关)

1. $J_{ij} \triangleq \hat{x}_i \cdot \overleftrightarrow{J} \cdot \hat{x}_j = \int(r^2\delta_{ij} - x_i x_j)dm$

$$\boxed{J = \begin{pmatrix} \int(x_2^2+x_3^2)dm & -\int x_1x_2\,dm & -\int x_1x_3\,dm \\ -\int x_1x_2\,dm & \int(x_3^2+x_1^2)dm & -\int x_2x_3\,dm \\ -\int x_1x_3\,dm & -\int x_2x_3\,dm & \int(x_1^2+x_2^2)dm \end{pmatrix}}$$

称 J_{11}, J_{22}, J_{33} 分别为绕 x_1, x_2, x_3 轴的转动惯量

称 $-J_{ij} = \int x_i x_j \, dm$ ，$i\neq j$ 为惯量积

（一般对于大部分坐标系，J 及其分量都是随时间变化的．当坐标系选为本体坐标系时，J 才不变．）

2．角动量分量 $L_i = \hat{x}_i\cdot\vec{L} = \hat{x}_i\cdot\overleftrightarrow{J}\cdot\vec{\omega} = (\hat{x}_i\cdot\overleftrightarrow{J})\cdot\hat{x}_j\hat{x}_j\cdot\vec{\omega}$

$$\boxed{L_i = J_{ij}\omega_j}$$ （$\vec{\omega}$ 与 $\vec{L}$ 一般不平行）

若 $\vec{L} \parallel \vec{\omega} = \omega\hat{n}$ ，则称 $\hat{n}$ 为主轴．

ssj注：当基点不为坐标原点时，$\vec{r}_0\neq\vec{0}$，一般运动角动量．

$$\vec{L} = M\vec{r}_0\times\vec{v}_0 + \vec{r}_0\times(\vec{\omega}\times M\vec{r}_c) + M\vec{r}_c\times\vec{v}_0 + \overleftrightarrow{J}\cdot\vec{\omega}$$

二．动能．

$$\boxed{T = \frac{1}{2}\int v^2 dm = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{J}\cdot\vec{\omega} = \frac{1}{2}J_{ij}\omega_i\omega_j}$$

$$= \frac{1}{2}\int(\vec{\omega}\times\vec{r})\cdot\vec{v}\,dm = \frac{1}{2}\int\vec{\omega}\cdot(\vec{r}\times\vec{v})dm$$

将 $\vec{\omega} = \omega(t)\hat{n}(t)$，

$$\boxed{T = \frac{1}{2}\omega^2(\hat{n}\cdot\overleftrightarrow{J}\cdot\hat{n}) = \frac{1}{2}\omega^2\int[r^2-(\hat{n}\cdot\vec{r})^2]dm = \frac{1}{2}J_n\omega^2}$$

（Def：绕 $\hat{n}$ 轴之转动惯量 $\boxed{J_n \triangleq \hat{n}\cdot\overleftrightarrow{J}\cdot\hat{n} = \int|\hat{n}\times\vec{r}|dm = J_n(t)}$ ）

（根据动能）．J 是对称正定之张量．

在基点O以 v_0 运动时，动能：

$$T = \frac{1}{2}\int(\vec{v}_0+\vec{\omega}\times\vec{r})^2 dm$$

$$\boxed{T = \frac{1}{2}Mv_0^2 + (\vec{v}_0\times\vec{\omega})\cdot M\vec{r}_c + \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{J}\cdot\vec{\omega}}$$

当基点O选为C（质心时），柯尼希：

$$\boxed{T = \frac{1}{2}Mv_c^2 + \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{J}\cdot\vec{\omega}}$$

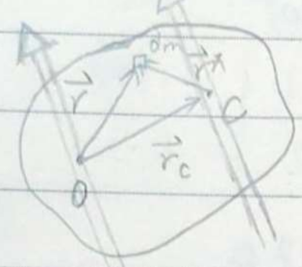
三、$\overleftrightarrow{J}$的基本性质

1. 叠加原理 $\boxed{\overleftrightarrow{J}_{A+B} = \overleftrightarrow{J}_A + \overleftrightarrow{J}_B}$

2. 正交轴定理 $\boxed{J_{11} + J_{22} \geqslant J_{33}}$ "=" 对 "$x_3 \equiv 0$"

3. 平行轴定理 $\boxed{\overleftrightarrow{J} = \overleftrightarrow{J}_c + \overleftrightarrow{J}^*}$, $\overleftrightarrow{J}_c = M(r_c^2 \overleftrightarrow{I} - \vec{r}_c \vec{r}_c)$

$$\overleftrightarrow{J}^* = \int (r^{*2} \overleftrightarrow{I} - \vec{r}^* \vec{r}^*) dm$$



若分别在O.C建立相互平行的坐标轴，有：

$$\boxed{J_{ij} = (J_c)_{ij} + (J^*)_{ij}}$$

即：

$$\boxed{J_{ij} = J^*_{ij} + M(r_c^2 \delta_{ij} - r_{ci} r_{cj})}$$

若在 $\hat{n}$ 转轴方向投影，

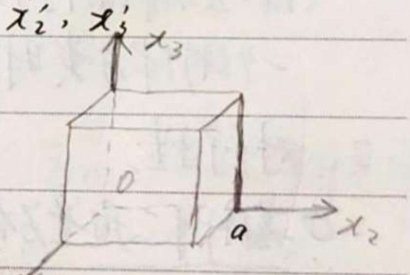
$$\boxed{J_n = J_n^* + M d_c^2}, \quad d_c = |\hat{n} \times \vec{r}_c|$$

*用来计算对非质心点的惯量张量.

四、主轴系(本体)

$$\boxed{\begin{cases} \det(b\overleftrightarrow{I} - \overleftrightarrow{J}) = 0 \\ (b\overleftrightarrow{I} - \overleftrightarrow{J})Y = 0 \end{cases}} \begin{aligned} &\Rightarrow b = \underbrace{J_1, J_2, J_3}_{} \to \text{主转动惯量} \\ &\Rightarrow Y = \hat{x}_1', \hat{x}_2', \hat{x}_3' \end{aligned}$$

eg. 匀质立方体.



$$J_{11} = \rho \int (x_2^2 + x_3^2) dx_1 dx_2 dx_3$$

$$= 2\rho \int_0^a dx_1 \int_0^a x_2^2 dx_2 \int_0^a dx_3 = \frac{2}{3} Ma^2 = J_{22} = J_{33}$$

$$J_{12} = -\rho \int x x_2 dx_1 dx_2 dx_3 = -\frac{1}{4} Ma^2 = J_{31} = J_{23}$$

$$J = \frac{1}{4} Ma^2 \begin{pmatrix} \alpha & -1 & -1 \\ -1 & \alpha & -1 \\ -1 & -1 & \alpha \end{pmatrix} = \frac{1}{4} Ma^2 \cdot A, \quad \alpha = \frac{8}{3}$$

$$0 = \det(bI - A) = (b - \alpha - 1)^2 (b - \alpha + 2) \Rightarrow b_1 = b_2 = \frac{11}{3}, \quad b_2 = \frac{2}{3}$$

$J_1=J_2=\frac{11}{12}Ma^2$, $J_3=\frac{1}{6}Ma^2$

$Y_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ 另外二个主轴在垂直于Y_1的平面内任取二相互垂直.

1. 主轴系 $L_1=J_1\omega_1$ $L_2=J_2\omega_2$ $L_3=J_3\omega_3$

$$T=\frac{1}{2}J_1\omega_1^2+\frac{1}{2}J_2\omega_2^2+\frac{1}{2}J_3\omega_3^2$$

$$J_n=J_1n_1^2+J_2n_2^2+J_3n_3^2$$

2、刚体分类:

① J_1、J_2、J_3 互不相同: 不对称陀螺 (top)

② $J_1=J_2\neq J_3$: 对称陀螺

③ $J_1=J_2=J_3$: 球形陀螺

对②对称陀螺, $J'=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}J_1 & & \\ & J_2(=J_1) & \\ & & J_3\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$

与第三个轴垂直的平面内的任一对正交方向均是主轴

(注: 立方体相对于中心是球形陀螺, 相对于一顶点是对称陀螺。将一个刚体分类时,与基点选择有关)

3、对称性.

①若刚体的质量分布具有对称平面,则该面的法向为主轴.

②若刚体的质量分布具有n次 ($n\geq2$) 对称轴,则对称轴为主轴之一. $\theta_n=\frac{2\pi}{n}$

具有n次 ($n\geq3$) 对称轴,则与对称轴垂直(相交)的任一轴皆为主轴.

ssj注①这里的"对称轴"是动力学意义上的对称轴,并不一定为几何对称轴。当是匀质时为二者相同.

③若刚体具有某阶的对称轴,则质心应位于该轴上.

eg.(对称轴为x_3) 记$\theta_n = \frac{2\pi}{n}$. $\cos\theta_n = C_n$, $\sin\theta_n = S_n$. $\lambda = \begin{pmatrix} C_n & S_n & 0 \\ -S_n & C_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$J'_{ij} = J_{ij} = \lambda_{ik}\lambda_{jl}J_{kl}$$

$$\begin{cases} J_{13} = \lambda_{1k}\lambda_{3l}J_{kl} = \lambda_{11}J_{13} + \lambda_{12}J_{23} \\ J_{23} = \lambda_{2k}\lambda_{3l}J_{kl} = \lambda_{21}J_{13} + \lambda_{22}J_{23} \end{cases}$$

整理: $\begin{cases} (1-C_n)J_{13} - S_nJ_{23} = 0 \\ S_nJ_{13} + (1-C_n)J_{23} = 0 \end{cases}$

$$\Delta = \begin{vmatrix} 1-C_n & -S_n \\ S_n & 1-C_n \end{vmatrix} = 2(1-C_n) > 0 \quad (n \geq 2,\ 0 < \theta_n \leq \pi)$$

故无非零解, 只可能 $J_{13} = J_{23} = 0$

则 J 具有形式: $J = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$, 则 x_3 是主轴. //

$$J_{11} = \lambda_{1k}\lambda_{1l}J_{kl} = \lambda_{11}^2J_{11} + \lambda_{12}^2J_{22} + 2\lambda_{11}\lambda_{12}J_{12}$$

$$J_{12} = \lambda_{1k}\lambda_{2l}J_{kl} = \lambda_{11}\lambda_{21}J_{11} + \lambda_{12}\lambda_{22}J_{22} + \lambda_{11}\lambda_{22}J_{12} + \lambda_{12}\lambda_{21}J_{21}$$

整理: $\begin{cases} S_n^2(J_{11}-J_{22}) - 2S_nC_nJ_{12} = 0 \\ S_nC_n(J_{11}-J_{22}) + 2S_n^2J_{12} = 0 \end{cases}$

$$\Delta = \begin{vmatrix} S_n^2 & -2S_nC_n \\ S_nC_n & 2S_n^2 \end{vmatrix} = 2S_n^2$$

当$n=2$, $\theta_n=\pi$, 有非零解.

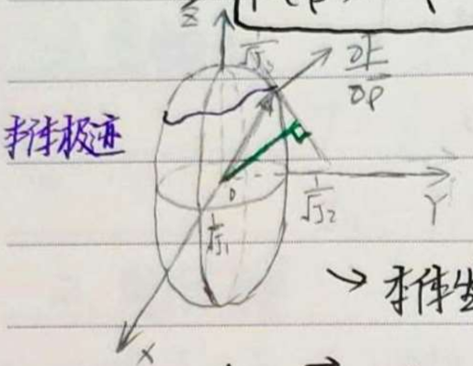
当$n \geq 3$时, $2S_n^2 > 0$, 无非零解, 故 $J_{12} = 0$, $J_{11} = J_{22}$

则 J 具有形式: $\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix}$, 是对称陀螺

五. 惯量椭球

1\. Def: $\boxed{\vec{\rho} = \frac{\hat{n}}{\sqrt{J_n}}}$

$$\Rightarrow \boxed{F(\vec{\rho}) = \vec{\rho}\cdot\overleftrightarrow{J}\cdot\vec{\rho} = J_1\rho_1^2 + J_2\rho_2^2 + J_3\rho_3^2 = 1}$$



① $\boxed{J_n = \frac{1}{\rho^2}}$

② 法向 $\boxed{\frac{\partial F}{\partial \vec{\rho}} = 2\overleftrightarrow{J}\cdot\vec{\rho}}$

→ 本体坐标系中，椭球不变

2\. 取 $\hat{n} = \frac{\vec{\omega}}{\omega} = \hat{n}(t)$,

$$\vec{\rho} = \frac{\vec{\omega}}{\omega\sqrt{J_n}} = \frac{\vec{\omega}}{\sqrt{2T}} \Rightarrow \boxed{\frac{\partial F}{\partial \vec{\rho}} = \sqrt{\frac{2}{T}}\,\vec{L}}$$

在本体坐标系中 $\vec{\rho}$ 箭头终点画出的曲线称本体极迹

在空间坐标系中，$\vec{\rho}$ 箭头终点画出的曲线称空间极迹

O到切平面的距离 $\boxed{\vec{\rho}\cdot\hat{L} = \vec{\rho}\cdot\frac{\vec{L}}{L} = \frac{\sqrt{2T}}{L}}$

§3 刚体动力学

一. 质点组动力学.

$$\vec{F}_a = \vec{F}_a^{外} + \sum_{b(\neq a)} \vec{f}_{ab} = m\ddot{\vec{r}}_a = \frac{d\vec{p}_a}{dt}$$

$$\vec{\tau}_a \hat{=} \vec{r}_a \times \vec{F}_a = \vec{r}_a \times \vec{F}_a^{外} + \sum_{b(\neq a)} \vec{r}_a \times \vec{f}_{ab} = \frac{d\vec{L}_a}{dt}$$

(利用 $\frac{d\vec{r}_a}{dt} \times \vec{p}_a = 0$)

$$P_a \hat{=} \vec{F}_a \cdot \vec{v}_a = \vec{F}_a^{外} \cdot \vec{v}_a + \sum_{b(\neq a)} \vec{f}_{ab} \cdot \vec{v}_a = \frac{dT_a}{dt}$$

(利用 $\frac{d\vec{p}_a}{dt} \cdot \vec{v}_a = m\frac{d(v^2)}{dt} = \frac{dT_a}{dt}$)

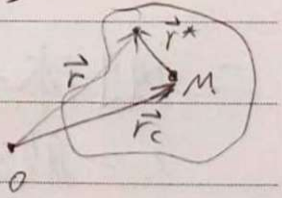
1. 动量定理.

$$\vec{F} = \vec{F}^{外} = \frac{d(\vec{P})}{dt} \qquad (\vec{P} = \sum_a m_a \vec{v}_a)$$

质心运动定理 $\begin{cases} \vec{F}^{外} = M\ddot{\vec{r}}_c, \ (\vec{r}_c = \sum \frac{m_a \vec{r}_a}{M}, \ M = M_c = \sum_a m_a) \\ \vec{\tau}_c = \vec{r}_c \times \vec{F}^{外} = \frac{d\vec{L}_c}{dt}, \ (\vec{L}_c = \vec{r}_c \times M\vec{v}_c) \\ \vec{F}_{外} \cdot \vec{r}_c = \frac{dT_c}{dt} = P_c \ (T_c = \frac{1}{2}Mv_c^2) \end{cases}$

将质心看做质点.

$$\begin{cases} \vec{r} = \vec{r}_c + \vec{r}^* \\ \vec{v} = \vec{v}_c + \vec{v}^* \end{cases}$$ (平移参考系).



$$\Rightarrow \vec{P} = \sum m_a \vec{v}_c + \sum m_a \vec{v}_c^* = \sum m_a \vec{v}_c = \vec{P}_c$$

Thm. $\boxed{\vec{P}^* = \sum_a m_a \vec{v}_a^* = 0}$

2、角动量定理

$$\vec{\tau} = \vec{\tau}_{外} = \frac{d\vec{L}}{dt} \quad , (\vec{L} = \sum_a \vec{r}_a \times m_a \vec{v}_a)$$

$$= \sum_a (\vec{r}_c + \vec{r}_a^*) \times m_a (\vec{v}_c + \vec{v}_a^*)$$

$$= \sum_a m_a \vec{r}_c \times \vec{v}_c + \sum_a \vec{r}_a^* \times m_a \vec{v}_a^*$$

$$\begin{array}{ccccc} \vec{\tau} & = & \vec{\tau}_c & + & \vec{\tau}^* \\ \| & & \| & & \\ \dot{\vec{L}} & = & \dot{\vec{L}}_c & + & \dot{\vec{L}}^* \end{array}$$

⇒ 质心系中之角动量定理.

$$\boxed{\vec{\tau}^* = \frac{d\vec{L}^*}{dt}}$$

3、动能定理:

$$P = \sum_a \vec{F}_a \cdot \vec{v}_a = \frac{dT}{dt} \quad , \quad T = \sum_a \frac{1}{2} m_a v_a^2$$

$$\begin{array}{ccccc} P & = & P_c & + & P^* \\ \| & & \| & & \\ \dot{T} & = & \dot{T}_c & + & \dot{T}^* \end{array}$$

$$\Rightarrow \quad \boxed{P^* = \sum_a \vec{F}_a \cdot \vec{v}_a^* = \frac{dT^*}{dt}}$$

注: 内力做功 $\vec{f}_{12} \cdot \vec{v}_1 + \vec{f}_{21} \cdot \vec{v}_2 = \vec{f}_{12} \cdot \vec{v}_{12} = \lambda \frac{d}{dt}(\frac{1}{2} v_{12}^2)$

只有在相对距离不变时内力做功为0，即刚体指的满足.

二、刚体动力学方程

1、一般运动

$$\boxed{\begin{array}{c} \vec{F}^{外} = M \ddot{\vec{r}}_c \\ \hline \vec{\tau}^* = \dfrac{d\vec{L}^*}{dt} \end{array}}$$

（三个）

（三个）

反映刚体自身取向之变化

$$P^* = \sum_a \vec{F_a} \cdot \vec{v_a^*} = \sum_a \vec{F_a} (\vec{\omega} \times \vec{r_a^*}) = \vec{\omega} \cdot \sum_a (\vec{r_a^*} \times \vec{F_a})$$

$$= \vec{\omega} \cdot \vec{\tau^*} = \vec{\omega} \cdot \frac{d}{dt} \left(\sum_a \vec{r_a^*} \times m_a \vec{v_a^*} \right) = \vec{\omega} \cdot \sum_a (\vec{r_a^*} \times m_a \dot{\vec{v_a^*}})$$

$$= \sum_a m_a (\vec{\omega} \times \vec{r_a^*}) \cdot \dot{\vec{v_a^*}} = \frac{dT^*}{dt}$$

即作为推论，

$$\boxed{P^* = \vec{\tau^*} \cdot \vec{\omega} = \frac{dT^*}{dt} = \sum_a \vec{F_a} \cdot \vec{v_a^*}}$$

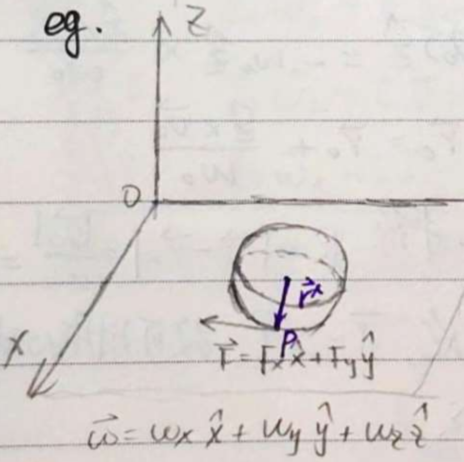
2\. 定点转动 $\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$ (三个)

主动力的力矩。定点外约束力的力矩=0.

3\. 定轴转动 $\boxed{\tau_n = \frac{dL_n}{dt}}$

4\. 平面平行运动 $\vec{F}^{外} = M\ddot{\vec{r}}_c$ (两个)，$\tau_n^* = \frac{dL_n^*}{dt}$

eg.



一般而言(包括这里)，指随质心平动的参考系。

$F_x = m\ddot{x}$ ① $F_y = m\ddot{y}$ ②

(在质心系) $\vec{L}^* = \overleftrightarrow{J} \cdot \vec{\omega} = \frac{2}{5} m a^2 \vec{\omega}$

$$\overleftrightarrow{J} = \frac{2}{5} m a^2 \overleftrightarrow{I}$$

$$\vec{\tau^*} = \vec{r^*} \times \vec{F} = (-a\hat{z}) \times (F_x \hat{x} + F_y \hat{y})$$

$$= aF_y \hat{x} - aF_x \hat{y}$$

$\Rightarrow J\dot{\omega}_x = aF_y$, $J\dot{\omega}_y = -aF_x$, $\dot{\omega}_z = 0$

$\Rightarrow F_x = -\frac{2}{5} m a \dot{\omega}_y$ ③ , $F_y = \frac{2}{5} m a \dot{\omega}_x$ ④ , $\dot{\omega}_z = 0$ ⑤

约束: $0 = \vec{v}_p = (\dot{x}\hat{x} + \dot{y}\hat{y}) + \vec{\omega} \times \vec{r^*} = (\dot{x} - a\omega_y)\hat{x} + (\dot{y} + a\omega_x)\hat{y}$

$\Rightarrow \ddot{x} = a\dot{\omega}_y$, $\ddot{y} = -a\dot{\omega}_x$, 五个方程

得到 $F_x = \ddot{x} = \dot{\omega}_y = 0$, $F_y = \ddot{y} = \dot{\omega}_x = 0$, $\dot{\omega}_z = 0$

eg. 桌子绕z以$\Omega\hat{z}$转动.

前五个方程不变.

约束方程(桌子上P点速度与球上P点速度相同).

$$(\dot{x} - a\omega_y)\hat{x} + (\dot{y} + a\omega_x)\hat{y} = \Omega\hat{z}\times(x\hat{x} + y\hat{y})$$

$\Rightarrow$ $\dot{x} + \Omega y = a\omega_y$ $\quad$ $\dot{y} - \Omega x = -a\omega_x$

$\Rightarrow$ $\ddot{x} + \Omega\dot{y} = a\dot{\omega}_y$ ⑥ $\quad$ $\ddot{y} - \Omega\dot{x} = -a\dot{\omega}_x$ ⑦

①.③.⑥ $\Rightarrow$ $\ddot{x} = -\frac{2}{5}(\ddot{x} + \Omega\dot{y})$ $\Rightarrow$ $\ddot{x} = -\frac{2}{7}\Omega\dot{y}$

②.④.⑦ $\Rightarrow$ $\ddot{y} = -\frac{2}{5}(\ddot{y} - \Omega\dot{x})$ $\quad$ $\ddot{y} = \frac{2}{7}\Omega\dot{x}$

即 $\frac{d\vec{v}}{dt} = \omega_0\hat{z}\times\vec{v}$, $\omega_0 = \frac{2}{7}\Omega$.

积分, $\vec{v} - \vec{v}_0 = \omega_0\hat{z}\times(\vec{r} - \vec{r}_0)$

又 $\vec{v}_0 = (\hat{z}\times\vec{v}_0)\times\hat{z} + \cancel{(\hat{z}\cdot\vec{v}_0)\hat{z}} = -\omega_0\hat{z}\times\frac{\hat{z}\times\vec{v}_0}{\omega_0}$

$\Rightarrow$ $\vec{v} = \omega_0\hat{z}\times(\vec{r} - \vec{r}_c)$, $\vec{r}_c = \vec{r}_0 + \frac{\hat{z}\times\vec{v}_0}{\omega_0}$

球心以r_c为圆心作圆周运动, 半径 $R = |\vec{r}_c - \vec{r}_0| = \frac{|\vec{v}_0|}{\omega_0} = \frac{7}{2}\frac{v_0}{\Omega}$

此二题之特殊之处是在质心系中讨论, $\overleftrightarrow{J} = \lambda\overleftrightarrow{I}$, 故可用质心中定理. 相对质心系之轴之质量分布是不变的.

三. 定点转动的 Euler 动力学方程.

1、 $\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{空间}$

$$\hat{x}_i\cdot\vec{\tau} = \hat{x}_i\cdot\left(\frac{d\vec{L}}{dt}\right)_{空间} + \overset{=0}{\left(\frac{d\hat{x}_i}{dt}\right)_{空间}}\cdot\vec{L} = \frac{d(\hat{x}_i\cdot\vec{L})}{dt} = \frac{d(L_i)}{dt}$$

据 $L_i = J_{ij}\Omega_j$，上式 $= \dot{J}_{ij}\Omega_j + J_{ij}\dot{\Omega}_j$，无法分离！不采用.

2、 $\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{空间} = \left(\frac{d\vec{L}}{dt}\right)_{转} + \vec{\omega}_{转}\times\vec{L}$ *

$$\overset{选择}{=} \left(\frac{d\vec{L}}{dt}\right)_{本体} + \vec{\omega}\times\vec{L}$$

此时 $\hat{x}_i\cdot\vec{\tau} = \frac{dL_i}{dt} + \varepsilon_{ijk}\omega_j L_k$

$$\xrightarrow[L_i=J_i\omega_i]{取主轴系} J_i\dot{\omega}_i - \varepsilon_{ijk}J_j\omega_j\omega_k$$

主轴系中分量

$\dot{J}_i = 0$　拆分了 ω 的求解与刚体位置之求解.

$$\Rightarrow \boxed{\begin{cases} \tau_1 = J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 \\ \tau_2 = J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 \\ \tau_3 = J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 \end{cases}}$$

Euler 动力学方程.
主轴系

若在本体系中写角动量定理.

$$\vec{\tau} + \vec{\tau}' = \left(\frac{d\vec{L}_{本体}}{dt}\right)_{本体} = 0$$

↓ 惯性力之力矩，$\vec{\tau}' = -\vec{\omega}\times\vec{L} - \dot{\vec{L}}$（大小等于这个）

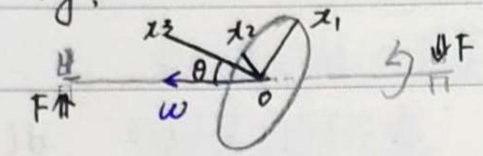
ssj 注：约定用 $\dot{\vec{L}}$ 表示本体坐标系中看到的角动量变化率，以 $\frac{d\vec{L}}{dt}$ 为空间坐标系中 $\vec{L}$ 变化率. 由：$\vec{\tau} = \frac{d\vec{L}}{dt}$，$\frac{d\vec{L}}{dt} = \dot{\vec{L}} + \vec{\omega}\times\vec{L}$ 结果：

$$\boxed{\vec{\tau} = \dot{\vec{L}} + \vec{\omega}\times\vec{L}}$$

即为本体系中的 Euler 动力学方程. 在选主轴系后，分量即简化为上面.

选本体系 x_1, x_2, x_3

e.g.



$\omega_1 = -\omega\sin\theta$, $\omega_2 = 0$, $\omega_3 = \omega\cos\theta$ （在本体系中投影$\vec{\omega}$）

又 x_1, x_2, x_3 是主轴，$J_3 = \frac{1}{2}mR^2$，$J_1 = J_2 = \frac{1}{4}mR^2$.

$\Rightarrow \tau_1 = \tau_3 = 0$. $\tau_2 = \frac{1}{4}mR^2\omega^2\sin\theta\cos\theta$.

Thm：动平衡之条件是转动轴沿着惯量主轴.

§4 欧拉陀螺（无外力矩情况）

$\vec{\tau} = 0 \quad \Rightarrow \vec{L}$ 守恒

$\vec{\tau}\cdot\vec{\omega} = 0 \quad \Rightarrow T$ 守恒

运动常数 $\begin{cases} T = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2 \\ L^2 = J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 \end{cases}$

角动量在本体系中之演化.

$$\begin{cases} L_1^2 + L_2^2 + L_3^2 = L^2 \\ \frac{L_1^2}{2J_1T} + \frac{L_2^2}{2J_2T} + \frac{L_3^2}{2J_3T} = 1 \end{cases}$$

设 $J_1 > J_2 > J_3$，并令

$L_{max} = \sqrt{2TJ_1}$ $\quad L_{mid} = \sqrt{2TJ_2}$, $L_{min} = \sqrt{2TJ_1}$

$$L_{min} \leq L \leq L_{max}$$

一、绕主轴转动的刚体的稳定性.

当绕 x_1 以 $\vec{\omega} = \omega_0 \hat{x}_1$ 转动时，施加微小扰动，角速度为：

$$\vec{\omega} = (\omega_0+\alpha_1)\hat{x}_1 + \alpha_2 \hat{x}_2 + \alpha_3 \hat{x}_3$$

① $J_1 > J_2 > J_3$

$$\Rightarrow \begin{cases} J_1\dot{\alpha}_1 = (J_2-J_3)\alpha_2\alpha_3 \approx 0 \\ J_2\dot{\alpha}_2 = (J_3-J_1)\alpha_3(\omega_0+\alpha_1) \approx (J_3-J_1)\omega_0\alpha_3 \\ J_3\dot{\alpha}_3 = (J_1-J_2)(\omega_0+\alpha_1)\alpha_2 \approx (J_1-J_2)\omega_0\alpha_2. \end{cases}$$

将②求导，两边乘 J_3 并代入③式，得：

$$J_2J_3\ddot{\alpha}_2 = (J_3-J_1)(J_1-J_2)\omega_0^2\alpha_2$$

$$\Rightarrow \ddot{\alpha}_2 = -\Omega_1^2\alpha_2 \quad , \quad \ddot{\alpha}_3 = -\Omega_1^2\alpha_3$$

其中 $\Omega_1 = \dfrac{(J_1-J_2)(J_1-J_3)}{J_2J_3} > 0$

同理，$\Omega_3 = \dfrac{(J_3-J_1)(J_3-J_2)}{J_1J_2} > 0$

绕 $\hat{x}_3$ 轴是稳定的.

$$\Omega_2 = \boxed{\dfrac{(J_2-J_1)(J_2-J_3)}{J_1J_3}} < 0$$

绕 $\hat{x}_2$ 轴是不稳定的.

Thm. 绕主转动惯量为最大或最小的主轴的转动是稳定的.
绕主转动惯量大小为中间的主轴的转动是不稳定的.

② 当 $J_1=J_2$ 时，考虑绕 x_1 轴之转动

$$\begin{cases} J_1\dot{\alpha}_1=(J_1-J_3)\alpha_2\alpha_3\approx 0 \\ J_1\dot{\alpha}_2=(J_3-J_1)\alpha_3(\omega_0+\alpha_1)\approx(J_3-J_1)\omega_0\alpha_3 \\ J_3\dot{\alpha}_3=0 \end{cases}$$

此时 α_3 是常数，α_1 近似为常数，α_2 随时间线性变化.

$$\alpha_2=C+Dt$$

Thm. 只有绕 x_3 轴之转动是稳定之.

绕 x_1 轴和 x_2 轴之轴之转动是不稳定之.

二、Poinsot（潘索）方法.

1. 运动模式：

守恒量 $$\begin{cases} T=\frac{1}{2}J_1\omega_1^2+\frac{1}{2}J_2\omega_2^2+\frac{1}{2}J_3\omega_3^2 \\ L^2=J_1^2\omega_1^2+J_2^2\omega_2^2+J_3^2\omega_3^2 \end{cases}$$

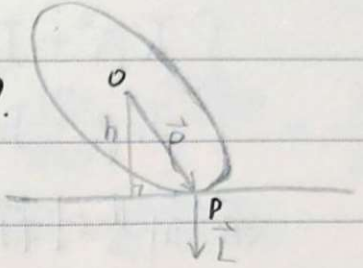
$$\xrightarrow{\vec{\rho}=\frac{\vec{\omega}}{\sqrt{2T}}}\boxed{\begin{cases} J_1\rho_1^2+J_2\rho_2^2+J_3\rho_3^2=1 \\ J_1^2\rho_1^2+J_2^2\rho_2^2+J_3^2\rho_3^2=\frac{L^2}{2T} \end{cases}}\Bigg\}\text{给出了本体极迹}$$

由于惯量椭球，选取之性质使其固连在刚体上，

我们可研究惯量椭球之"运动"

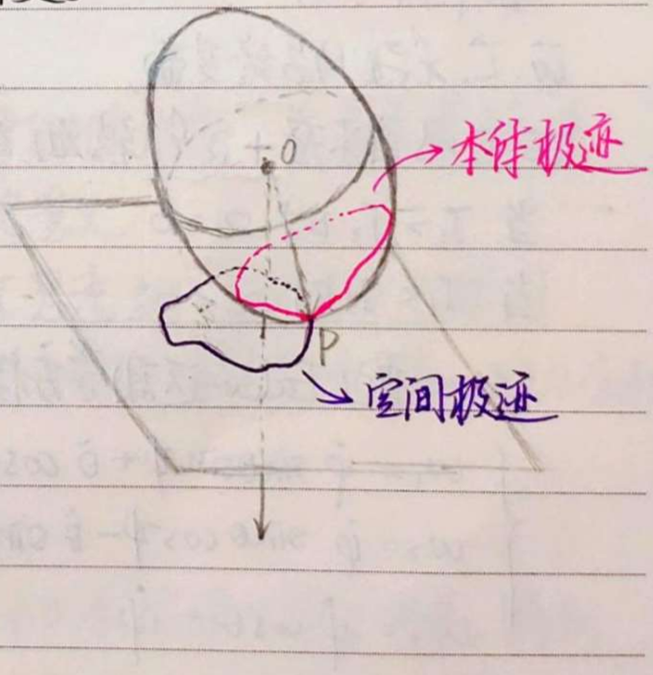
$$\frac{\partial F}{\partial \vec{\rho}} = 2\overleftrightarrow{J}\cdot\vec{\rho} = \sqrt{\frac{2}{T}}\,\vec{L}$$ 不变的法向(平行于$\vec{L}$)

$$h = \vec{\rho}\cdot\hat{L} = \frac{\vec{\rho}\cdot\vec{L}}{L} = \frac{\sqrt{2T}}{L}$$ 不变平面.



上面二式表示,在椭球运动过程中,角动量矢量方向(即为$\vec{\rho}$方向)所对应的$\vec{\rho}$终点所确定的平面在空间上位置(的法向)均不变。

我们可以想象与该平面重合的位置有一假想的实在平面,(这将有理于我们想象它的运动).由于$\vec{\rho}$是ω方向,OP即为瞬时转轴,故P点速度为0,即该椭球与假想实在平面间无滑滚动,且保持h不变。



2、对称陀螺（$J_1=J_2$）解的情况

$$\begin{cases} J_1\rho_1^2+J_2\rho_2^2+J_3\rho_3^2=1 \\ J_1^2\rho_1^2+J_2^2\rho_2^2+J_3^2\rho_3^2=\frac{L^2}{2T} \end{cases}$$

两个都是对称椭球，本体极迹为圆周。其Euler动力学方程:

$$\begin{cases} J_1\dot{\omega}_1=(J_1-J_3)\omega_2\omega_3 \\ J_1\dot{\omega}_2=(J_3-J_1)\omega_3\omega_1 \\ J_3\dot{\omega}_3=0 \end{cases} \Rightarrow \begin{cases} \dot{\omega}_1=+\Omega\omega_2 \\ \dot{\omega}_2=-\Omega\omega_1 \\ \omega_3=\text{const} \end{cases}, \boxed{\Omega=\frac{J_1-J_3}{J_1}\omega_3}$$

$$\Rightarrow \quad \omega_1=\omega_\perp\sin(\Omega t+\alpha) \quad , \omega_2=\omega_\perp\cos(\Omega t+\alpha)$$

即 ω_3、$\omega_\perp$. $L_3=J_3\omega_3$. $L_\perp=J_1\omega_\perp$ 均为运动常数.

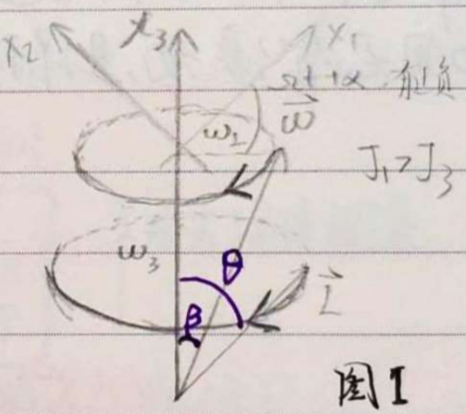
在本体系中，

$\vec{\omega}$. $\vec{L}$、x_3轴始终共面，

$\vec{\omega}$、$\vec{L}$以角速度$\underbrace{-\Omega\hat{x}_3}$转动.

当 $J_1>J_3$ 时 $\Omega>0$

当 $J_1<J_3$ 时 $\Omega<0$.



图I

再由 Euler 运动学方程，

$$\begin{cases} \omega_1=\dot{\varphi}\sin\theta\sin\psi+\dot{\theta}\cos\psi \\ \omega_2=\dot{\varphi}\sin\theta\cos\psi-\dot{\theta}\sin\psi \\ \omega_3=\dot{\varphi}\cos\theta+\dot{\psi} \end{cases}$$

将ω_1、ω_2、ω_3代入.

$$\Rightarrow \begin{cases} \dot{\varphi}=\frac{\omega_\perp}{\sin\theta}\cos(\Omega t+\alpha-\psi) \\ \dot{\theta}=\omega_\perp\sin(\Omega t+\alpha-\psi) \\ \dot{\psi}=\omega_3-\dot{\varphi}\cos\theta \end{cases}$$

看起来很复杂，但我们选取角动量方向 $(\hat{L})$ 为 $\underline{X}_3$ 正向，这样我们确定了（大致）空间坐标系之位置与欧拉角之实际意义。

$$\theta = \text{const} \Rightarrow \dot{\theta} = 0$$

$$\Rightarrow \varphi = \Omega t + \alpha$$

$$\Rightarrow \dot{\varphi} = \frac{\omega_1}{\sin\theta}$$

角动量与 $\hat{x}_3$ 轴夹角满足：

$$\boxed{\tan\theta = \frac{L_1}{L_3} = \frac{J_1\omega_1}{J_3\omega_3}}$$

角速度与 $\hat{x}_3$ 轴夹角满足：

$$\boxed{\tan\beta = \frac{\omega_1}{\omega_3}}$$

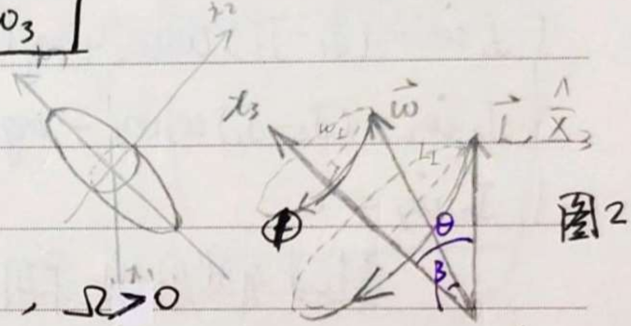


图2

① 当 $J_1 = J_2 > J_3$，$\theta > \beta$，$\Omega > 0$

② 当 $J_1 = J_2 < J_3$，$\theta < \beta$，$\Omega < 0$

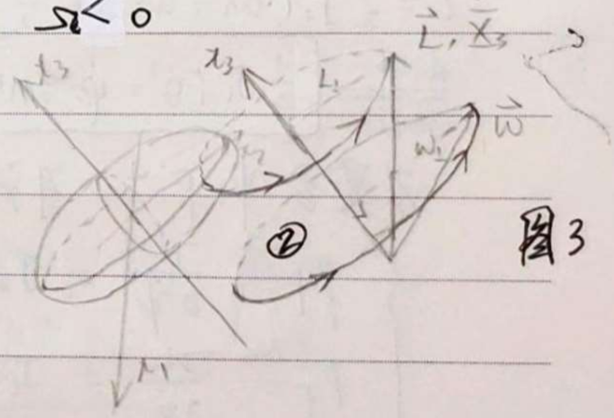
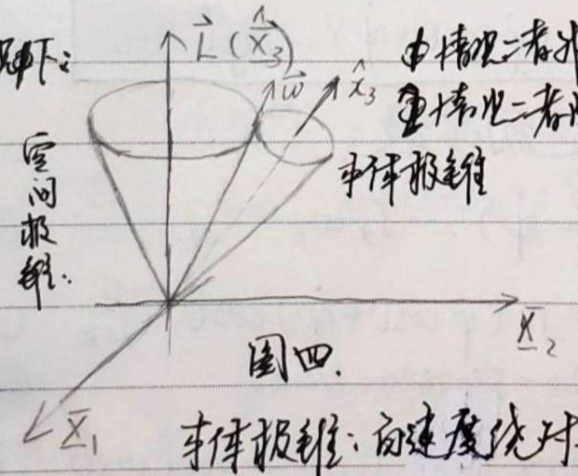


图3

情况①下：



①情况二者外切
②情况二者内切。

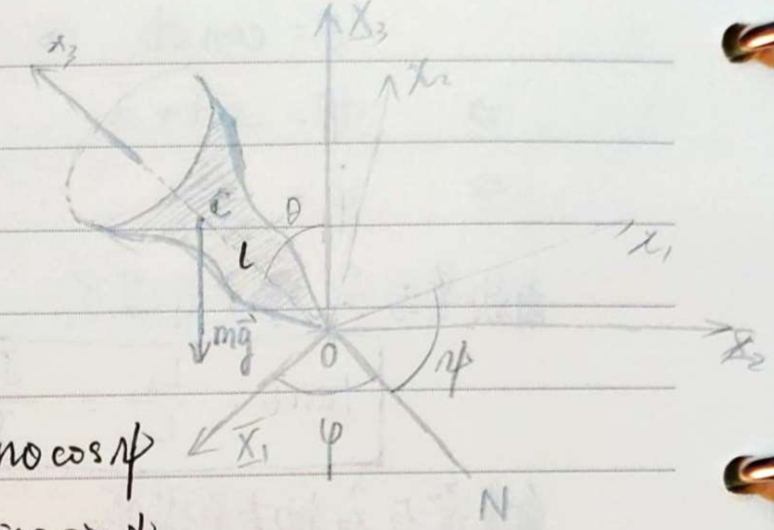
图四.

本体极锥：角速度绕对称轴旋转在刚体中描出之圆锥。
空间极锥：角速度绕角动量旋转描出之圆锥。

二者是无滑滚动之。（证明即：$\vec{\omega}$ 是沿刚体瞬时转轴之方向，即沿本体极锥滚动时瞬时转轴之方向，故为无滑接触）.

§5 Lagrange 陀螺

—— 定点转动的对称重陀螺.



$\vec{\tau} = l\hat{x}_3 \times (-mg\hat{X}_3)$

$= mgl\sin\theta\,\hat{ON}$

$= mgl\sin\theta(\hat{x}_1\cos\psi - \hat{x}_2\sin\psi)$

代入动力学方程:

$$\begin{cases} J_1\dot{\omega}_1 = (J_1 - J_3)\omega_2\omega_3 + mgl\sin\theta\cos\psi \\ J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 - mgl\sin\theta\sin\psi \\ J_3\dot{\omega}_3 = 0 \end{cases}$$

看起来有些麻烦. 采用 Lagrange 力学の方法.

$L = \frac{1}{2}J_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}J_3\omega_3^2 - mgl\cos\theta$

$\overset{\text{利Euler}}{\underset{\text{运动学方程}}{=\!=}} \boxed{\frac{1}{2}J_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}J_3(\dot{\varphi}\cos\theta + \dot{\psi})^2 - mgl\cos\theta}$

不显含 ψ. φ. t, 可给出三个运动常数:

$$\begin{cases} p_\psi = \frac{\partial L}{\partial\dot{\psi}} = J_3(\dot{\varphi}\cos\theta + \dot{\psi}) = J_3\omega_3 & ① \\ p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = J_1\dot{\varphi}\sin^2\theta + J_3(\dot{\varphi}\cos\theta + \dot{\psi})\cos\theta = L_z & ② \\ \qquad (L_z = J_1\dot{\varphi}\sin^2\theta + J_3\cos\theta\,\omega_3) & ②^* \\ E = T + U = \frac{1}{2}J_1(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}J_3\omega_3^2 + mgl\cos\theta & ③ \end{cases}$$

附注: p_ψ 为相对 $\hat{x}_3$ 的角动量, p_φ 为相对 $\hat{X}_3$ 的角动量.

将①②方程联立，解出进动与自转之部分：

$$\begin{cases} \dot{\varphi} = \dfrac{p_\varphi - p_\psi \cos\theta}{J_1 \sin^2\theta} \\ \dot{\psi} = \dfrac{p_\psi}{J_3} - \dfrac{p_\varphi - p_\psi \cos\theta}{J_1 \sin^2\theta} \cos\theta \end{cases}$$

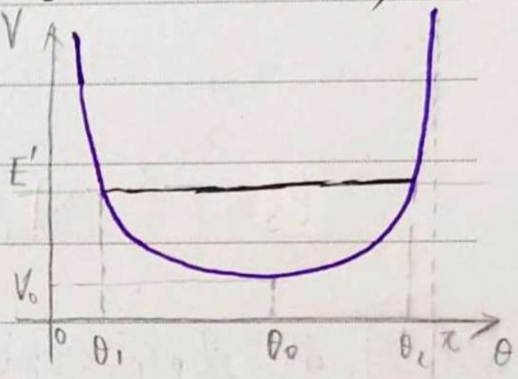
其中仅有θ是变量.

式③整理，处理章动θ之部分。

Def：$E' = E - \frac{1}{2} J_3 \omega_3^2 = \frac{1}{2} J_1 \dot{\theta}^2 + V(\theta)$

其中 $V(\theta) = \dfrac{(p_\varphi - p_\psi \cos\theta)^2}{2 J_1 \sin^2\theta} + mgl\cos\theta$ 有效势能

一般，$p_\varphi \neq p_\psi$，$V(\theta)$ 如右.



1. 规则进动：

、$E = V_0$，$\theta = \theta_0$，

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = 0 \Rightarrow (\cos\theta_0)\beta^2 + (-p_\psi \sin^2\theta_0)\beta + mgl J_1 \sin^4\theta_0 = 0,$$

$$\beta \triangleq p_\varphi - p_\psi \cos\theta_0$$

$$\Rightarrow \beta_\pm = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4mglJ_1\cos\theta_0}{p_\psi^2}}\right]$$

β之解之情况由根号里式子正负决定之讨论如下：

ssj注：或将p168的式回*代入，在不考虑 $\sin\theta_0 = 0$（直立或下垂）时有：

$$J_1 \dot{\varphi}^2 \cos\theta_0 - p_\psi \dot{\varphi} + mgh = 0$$

(1) 当 $\theta_0 < \frac{\pi}{2}$ 时，对 β 有解要求：

$$p_\psi^2 \geq 4mglJ_1\cos\theta_0 \Rightarrow \omega_3 \geq \frac{2}{J_3}\sqrt{mglJ_1\cos\theta_0} \triangleq \omega_0$$

Thm：仅当总 ω 在 $\hat{e}_3$ 之投影 $\omega_3 \geq \omega_0$ 时，才可能存在(固定倾角)之规则进动.

对 $\omega_3 > \omega_0$，β 有二解，由 $\dot{\varphi}_0 = \frac{\beta}{J_1\sin^2\theta_0}$，对每个 ω_3 有两个可能之进动角速度之值：

$$\dot{\varphi}_{0(+)} = \frac{\beta_+}{J_1\sin^2\theta_0}$$ 快进动

$$\dot{\varphi}_{0(-)} = \frac{\beta_-}{J_1\sin^2\theta_0}$$ 慢进动

(2) 当 $\theta_0 \geq \frac{\pi}{2}$，则 β 始终有两个解.

2. 对一般运动，陀螺对称轴在 $\theta_1 \leq \theta \leq \theta_2$ 范围内上下摆动(章动)，另一方面对称轴同时在绕着竖直轴转动(进动). 由于：

$$\boxed{\dot{\varphi} = \frac{p_\varphi - p_\psi\cos\theta}{J_1\sin^2\theta}}$$

进动方向是否改变取决于在 $\theta_1 \leq \theta \leq \theta_2$ 范围内上式右侧分子符号是否改变。