Solutions to some exercises in the book "J. E. Humphreys, An Introduction to Lie Algebras and Representation Theory"

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Contents

1 Definitions and First Examples

1. Let L be the real vector space $\mathbb{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) for $x, y \in L$, and verify that L is a Lie algebra. Write down the structure constants relative to the usual basis of $\mathbb{R}^3$.

Solution: Clearly, $[,]$ is bilinear and anti-commutative, it need only to check the Jacobi Identity:

$$\begin{aligned} [[x,y],z] &= (x \times y) \times z \\ &= (x.z)y - (y.z)x \\ &= (z.x)y - (y.x)z + (x.y)z - (z.y)x \\ &= [[z,y],x] + [[x,z],y] \end{aligned}$$

where $(.)$ is the inner product of $\mathbb{R}^3$.

Take the standard basis of $\mathbb{R}^3$: $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$. We can write down the structure equations of L:

$$\begin{aligned} [e_1, e_2] &= e_3 \\ [e_2, e_3] &= e_1 \\ [e_3, e_1] &= e_2 \end{aligned}$$

2. Verify that the following equations and those implied by $(L1)(L2)$ define a Lie algebra structure on a three dimensional space with basis (x, y, z) : $[xy] = z, [xz] = y, [yz] = 0$.

Solution:

$(L1)(L2)$ are satisfied, it is sufficient to show the Jacobi Identity hold for the basis:

$$\begin{aligned} [[x,y],z] &= [z,z] = 0 \\ [[y,z],x] &= [0,x] = 0 \\ [[x,z],y] &= [y,y] = 0 \end{aligned}$$

3. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis for $\mathfrak{sl}(2, F)$. Compute the matrices of $\mathrm{ad}x, \mathrm{ad}h, \mathrm{ad}y$ relative to this basis.

Solution:

By the structure equations of $\mathfrak{sl}(2, F)$:

$$\begin{aligned} \mathrm{ad}x(y) &= -\mathrm{ad}y(x) = h \\ \mathrm{ad}x(h) &= -\mathrm{ad}h(x) = -2x \\ \mathrm{ad}y(h) &= -\mathrm{ad}h(y) = 2y \end{aligned}$$

We can write down the matrices of $\mathrm{ad}x, \mathrm{ad}h, \mathrm{ad}y$ relative to this basis easily:

$$\mathrm{ad}x \sim \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathrm{ad}h \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \mathrm{ad}y \sim \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

4. Find a linear Lie algebra isomorphic to the nonabelian two dimensional algebra constructed in (1.4).

Solution:

Two dimensional Lie algebra constructed in (1.4) is given by basis (x, y) with commutation $[x, y] = x$.

In $\mathfrak{gl}(F^2)$, let

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

This is a isomorphism.

5. Verify the assertions made in (1.2) about $\mathfrak{t}(n, F), \mathfrak{d}(n, F), \mathfrak{n}(n, F)$, and compute the dimension of each algebra, by exhibiting bases.

Solution:

Assertions made in (1.2):

$$\mathfrak{t}(n, F) = \mathfrak{d}(n, F) + \mathfrak{n}(n, F) \quad \text{vector space direct sum} \tag{1}$$

$$[\mathfrak{d}(n, F), \mathfrak{n}(n, F)] = \mathfrak{n}(n, F) \tag{2}$$

$$[\mathfrak{t}(n, F), \mathfrak{t}(n, F)] = \mathfrak{n}(n, F) \tag{3}$$

Evidently, (1) holds and $[\mathfrak{d}(n, F), \mathfrak{n}(n, F)] \subseteq \mathfrak{n}(n, F)$. So we just need to show the converse conclusion is also true.

Let e_{ij} denotes the matrix with (i,j)-element is 1, and 0 otherwise.

$$\mathfrak{n}(n, F) = \text{span}_F\{e_{ij} | i < j\}$$

But we know

$$e_{ij} = e_{ii}e_{ij} - e_{ij}e_{ii} = [e_{ii}, e_{ij}] \subseteq [\mathfrak{d}(n, F), \mathfrak{n}(n, F)], \quad i < j$$

So (2) is correct.

(3) follows from (1) and (2):

$$\begin{aligned} [\mathfrak{t}(n, F), \mathfrak{t}(n, F)] &= [\mathfrak{d}(n, F) + \mathfrak{n}(n, F), \mathfrak{d}(n, F) + \mathfrak{n}(n, F)] \\ &\subseteq [\mathfrak{d}(n, F), \mathfrak{d}(n, F)] + [\mathfrak{n}(n, F), \mathfrak{n}(n, F)] + [\mathfrak{d}(n, F), \mathfrak{n}(n, F)] \\ &\subseteq \mathfrak{n}(n, F) \end{aligned}$$

Conversely, $\mathfrak{n}(n, F) = [\mathfrak{d}(n, F), \mathfrak{n}(n, F)] \subseteq [\mathfrak{t}(n, F), \mathfrak{t}(n, F)]$.

6. Let $x \in \mathfrak{gl}(n, F)$ have n distinct eigenvalues $a_1, \cdots, a_n$ in F. Prove that the eigenvalues of $\text{ad}x$ are precisely the n^2 scalars $a_i - a_j (1 \leqslant i, j \leqslant n)$, which of course need not be distinct.

Solution:

Let $v_i = \begin{pmatrix} v_{1i} \\ \vdots \\ v_{ni} \end{pmatrix} \in F^n, 1 \leqslant i \leqslant n$ are eigenvectors of x respect to eigenvalue a_i respectively. Then $v_i, 1 \leqslant i \leqslant n$ are linear independent. Let $A = (v_{ij})_{n \times n}$, E_{ij} denotes the $n \times n$ matrix with (i,j)-element is

1, and 0 otherwise, $e_{ij} = AE_{ij}A^{-1}$, Then we have A is a nonsingular matrix and

$$e_{ij}v_k = \delta_{jk}v_i$$

$$\mathrm{ad}x(e_{ij})v_k = x.e_{ij}.v_k - e_{ij}.x.v_k = (a_i - a_k)\delta_{jk}v_i = (a_i - a_j)e_{ij}.v_k$$

So

$$\mathrm{ad}x(e_{ij}) = (a_i - a_j)e_{ij}$$

i.e., $a_i - a_j$ are eigenvalues of $\mathrm{ad}x$, the eigenvectors are e_{ij} respectively. Hence $\mathrm{ad}x$ is diagonalizable. So we can conclude that the eigenvalues of $\mathrm{ad}x$ are precisely the n^2 scalars $a_i - a_j (1 \leqslant i, j \leqslant n)$.

7. Let $\mathfrak{s}(n, F)$ denote the **scalar matrices** (=scalar multiples of the identity) in $\mathfrak{gl}(n, F)$. If $char F$ is 0 or else a prime not dividing n, prove that $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) + \mathfrak{s}(n, F)$ (direct sum of vector spaces), with $[\mathfrak{s}(n, F), \mathfrak{gl}(n, F)] = 0$.

Solution:

$\forall A = (a_{ij}) \in \mathfrak{gl}(n, F)$, If $\mathrm{tr}(A) = 0$, $A \in \mathfrak{sl}(n, F)$; else $A = \frac{\mathrm{tr}(A)}{n}I + (A - \frac{\mathrm{tr}(A)}{n}I)$ with $\mathrm{tr}(A - \frac{\mathrm{tr}(A)}{n}I) = 0$, i.e., $A - \frac{\mathrm{tr}(A)}{n}I \in \mathfrak{sl}(n, F)$.($char F$ is 0 or else a prime not dividing n.) $\mathfrak{sl}(n, F) \cap \mathfrak{s}(n, F) = \{0\}$ is clearly. Hence $\mathfrak{gl}(n, F) = \mathfrak{sl}(n, F) + \mathfrak{s}(n, F)$ (direct sum of vector spaces).

For $aI \in \mathfrak{s}(n, F), \forall A \in \mathfrak{gl}(n, F)$, $[aI, A] = aA - aA = 0$, So

$$[\mathfrak{s}(n, F), \mathfrak{gl}(n, F)] = 0.$$

8. Verify the stated dimension of D_l.

Solution:

Suppose $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ $(m, n, p, q \in \mathfrak{gl}(n, F)$, $s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$. By $sx = -x^t s$, we have

$$m = -q^t, n = -n^t, p = -p^t, q = -m^t$$

We can enumerate a basis of D_l:

$$e_{i,j} - e_{l+j,l+i}, 1 \leqslant i, j \leqslant l; \quad e_{i,l+j} - e_{j,l+i}, e_{l+i,j} - e_{l+j,i}, 1 \leqslant i < j \leqslant l$$

where e_{ij} is the matrix having 1 in the (i, j) position and 0 elsewhere. Hence

$$\dim D_l = l^2 + \frac{1}{2}l(l-1) + \frac{1}{2}l(l-1) = 2l^2 - l$$

9. When $char F = 0$, show that each classical algebra $L = A_l, B_l, C_l$, or D_l is equal to $[LL]$. (This shows again that each algebra consists of trace 0 matrices.)

Solution:

$[L, L] \subseteq L$ is evident. It is sufficient to show $L \subseteq [L, L]$.

- A_1:

$$\begin{array}{rcl} e_{12} & = & \frac{1}{2}[h, e_{12}] \\ e_{21} & = & \frac{1}{2}[e_{21}, h] \\ h & = & [e_{12}, e_{21}] \end{array}$$

- $A_l (l \geqslant 2)$:

$$\begin{array}{rcll} e_{ij} & = & [e_{ik}, e_{kj}], & k \neq i, j, i \neq j \\ h_i & = & [e_{i,i+1}, e_{i+1,i}] & \end{array}$$

- $B_l (l \geqslant 2)$:

$$\begin{array}{rcl} e_{1,l+i+1} - e_{i+1,1} & = & [e_{1,j+1} - e_{l+j+1,1}, e_{j+1,l+i+1} - e_{i+1,l+j+1}] \\ e_{1,i+1} - e_{l+i+1,1} & = & [e_{1,l+j+1} - e_{j+1,1}, e_{l+j+1,i+1} - e_{l+i+1,j+1}] \\ e_{i+1,i+1} - e_{l+i+1,l+i+1} & = & [e_{i+1,1} - e_{1,l+i+1}, e_{1,i+1} - e_{l+i+1,1}] \\ e_{i+1,j+1} - e_{l+i+1,l+j+1} & = & [e_{i+1,1} - e_{1,l+i+1}, e_{1,j+1} - e_{l+j+1,1}] \\ e_{i+1,l+j+1} - e_{j+1,l+i+1} & = & [e_{i+1,i+1} - e_{l+i+1,l+i+1}, e_{i+1,l+j+1} - e_{j+1,l+i+1}] \\ e_{l+i+1,j+1} - e_{j+l+1,i+1} & = & [e_{l+i+1,l+i+1} - e_{i+1,i+1}, e_{l+i+1,j+1} - e_{j+l+1,i+1}] \end{array}$$

  where $1 \leqslant i \neq j \leqslant l$.

- $C_l (l \geqslant 3)$:

$$\begin{array}{rcll} e_{ii} - e_{l+i,l+i} & = & [e_{i,l+i}, e_{l+i,i}] & \\ e_{ij} - e_{l+j,l+i} & = & [e_{ii} - e_{l+i,l+i}, e_{ij} - e_{l+j,l+i}] & i \neq j \\ e_{i,l+j} + e_{j,l+i} & = & [e_{ii} - e_{l+i,l+i}, e_{i,l+j} + e_{j,l+i}] & \\ e_{l+i,j} + e_{l+j,i} & = & [e_{l+i,l+i} - e_{ii}, e_{l+i,j} + e_{l+j,i}] & \end{array}$$

- $D_l (l \geqslant 2)$:

$$\begin{array}{rcl} e_{ii} - e_{l+i,l+i} & = & \frac{1}{2}[e_{ij} - e_{l+j,l+i}, e_{ji} - e_{l+i,l+j}] \\ & & +\frac{1}{2}[e_{i,l+j} - e_{j,l+i}, e_{l+j,i} - e_{l+i,j}] \\ e_{ij} - e_{l+j,l+i} & = & [e_{ii} - e_{l+i,l+i}, e_{ij} - e_{l+j,l+i}] \\ e_{i,l+j} - e_{j,l+i} & = & [e_{ii} - e_{l+i,l+i}, e_{i,l+j} - e_{j,l+i}] \\ e_{l+i,j} - e_{l+j,i} & = & [e_{l+i,l+i} - e_{ii}, e_{l+i,j} - e_{l+j,i}] \end{array}$$

  where $i \neq j$.

10. For small values of l, isomorphisms occur among certain of the classical algebras. Show that A_1, B_1, C_1 are all isomorphic, while D_1 is the one dimensional Lie algebra. Show that B_2 is isomorphic to C_2, D_3 to A_3. What can you say about D_2?

Solution:

The isomorphism of A_1, B_1, C_1 is given as follows:

$$\begin{array}{ccccc} A_1 & \to & B_1 & \to & C_1 \\ e_{11} - e_{22} & \mapsto & 2(e_{22} - e_{33}) & \mapsto & e_{11} - e_{22} \\ e_{12} & \mapsto & 2(e_{13} - e_{21}) & \mapsto & e_{12} \\ e_{21} & \mapsto & 2(e_{12} - e_{31}) & \mapsto & e_{21} \end{array}$$

For B_2, C_2 we first calculate the eigenvectors for $h_1 = e_{22} - e_{44}, h_2 = e_{33} - e_{55}$ and $h'_1 = e_{11} - e_{33}, h'_2 = e_{22} - e_{44}$ respectively. We denote $\lambda = (\lambda(h_1), \lambda(h_2))$ for the eigenvalue of h_1, h_2, λ' is similar. See the following table:

B_2		C_2	
$\alpha = (1, 0)$	$e_{21} - e_{14}$	$\alpha' = (-1, 1)$	$e_{21} - e_{34}$
$-\alpha = (-1, 0)$	$e_{12} - e_{41}$	$-\alpha' = (1, -1)$	$e_{12} - e_{43}$
$\beta = (-1, 1)$	$e_{32} - e_{45}$	$\beta' = (2, 0)$	e_{13}
$-\beta = (1, -1)$	$e_{23} - e_{54}$	$-\beta' = (-2, 0)$	e_{31}
$\alpha + \beta = (0, 1)$	$e_{15} - e_{31}$	$\alpha' + \beta' = (1, 1)$	$e_{14} + e_{23}$
$-(\alpha + \beta) = (0, -1)$	$e_{13} - e_{51}$	$-(\alpha' + \beta') = (-1, -1)$	$e_{41} + e_{32}$
$2\alpha + \beta = (1, 1)$	$e_{25} - e_{34}$	$2\alpha' + \beta' = (0, 2)$	e_{24}
$-(2\alpha + \beta) = (-1, -1)$	$e_{43} - e_{52}$	$-(2\alpha' + \beta') = (0, -2)$	e_{42}

We make a linear transformation

$$\tilde{h_1}' = -\frac{1}{2}h_1' + \frac{1}{2}h_2', \tilde{h_2}' = \frac{1}{2}h_1 + \frac{1}{2}h_2$$

Then $\alpha(h_1) = \alpha'(\tilde{h_1}'), \alpha(h_2) = \alpha'(\tilde{h_2}'), \beta(h_1) = \beta'(\tilde{h_1}'), \beta(h_2) = \beta'(\tilde{h_2}')$. So the isomorphism of B_2, C_2 is given as follows:

$$\begin{array}{ccc}
B_2 & \to & C_2 \\
e_{22} - e_{44} & \mapsto & -\frac{1}{2}(e_{11} - e_{33}) + \frac{1}{2}(e_{22} - e_{44}) \\
e_{33} - e_{55} & \mapsto & \frac{1}{2}(e_{11} - e_{33}) + \frac{1}{2}(e_{22} - e_{44}) \\
e_{12} - e_{41} & \mapsto & \frac{\sqrt{2}}{2}(e_{12} - e_{43}) \\
e_{21} - e_{14} & \mapsto & \frac{\sqrt{2}}{2}(e_{21} - e_{34}) \\
e_{32} - e_{45} & \mapsto & e_{13} \\
e_{23} - e_{54} & \mapsto & e_{31} \\
e_{15} - e_{31} & \mapsto & \frac{\sqrt{2}}{2}(e_{14} + e_{23}) \\
e_{13} - e_{51} & \mapsto & \frac{\sqrt{2}}{2}(e_{32} + e_{41}) \\
e_{25} - e_{34} & \mapsto & e_{24} \\
e_{43} - e_{52} & \mapsto & e_{42}
\end{array}$$

For A_3 and D_3, we calculate the eigenvalues and eigenvectors for $h_1 = e_{11} - e_{22}, h_2 = e_{22} - e_{33}, h_3 = e_{33} - e_{44}$ and $h_1' = e_{11} - e_{44}, h_2' = e_{22} - e_{55}, h_3' = e_{33} - e_{66}$ respectively.

A_3		D_3	
$\alpha = (1, 1, -1)$	e_{13}	$\alpha' = (0, 1, 1)$	$e_{26} - e_{35}$
$-\alpha = (-1, -1, 1)$	e_{31}	$-\alpha' = (0, -1, -1)$	$e_{62} - e_{53}$
$\beta = (-1, 1, 1)$	e_{24}	$\beta' = (0, 1, -1)$	$e_{23} - e_{65}$
$-\beta = (1, -1, -1)$	e_{42}	$-\beta' = (0, -1, 1)$	$e_{32} - e_{56}$
$\gamma = (-1, 0, -1)$	e_{41}	$\gamma' = (1, -1, 0)$	$e_{12} - e_{54}$
$-\gamma = (1, 0, 1)$	e_{14}	$-\gamma' = (-1, 1, 0)$	$e_{21} - e_{45}$
$\alpha + \gamma = (0, 1, -2)$	e_{43}	$\alpha' + \gamma' = (1, 0, 1)$	$e_{16} - e_{34}$
$-(\alpha + \gamma) = (0, -1, 2)$	e_{34}	$-(\alpha' + \gamma') = (-1, 0, -1)$	$e_{61} - e_{43}$
$\beta + \gamma = (-2, 1, 0)$	e_{21}	$\beta' + \gamma' = (1, 0, -1)$	$e_{13} - e_{64}$
$-(\beta + \gamma) = (2, -1, 0)$	e_{12}	$-(\beta' + \gamma') = (-1, 0, 1)$	$e_{31} - e_{46}$
$\alpha + \beta + \gamma = (-1, 2, -1)$	e_{23}	$\alpha' + \beta' + \gamma' = (1, 1, 0)$	$e_{15} - e_{24}$
$-(\alpha + \beta + \gamma) = (1, -2, 1)$	e_{32}	$-(\alpha' + \beta' + \gamma') = (-1, -1, 0)$	$e_{51} - e_{42}$

We take a linear transformation

$$\tilde{h_1}' = -h_1' + h_3', \tilde{h_2}' = h_1' + h_2', \tilde{h_3}' = -h_1' - h_3'$$

then $\alpha(h_i) = \alpha'(\tilde{h_i}'), \beta(h_i) = \beta'(\tilde{h_i}'), \gamma(h_i) = \gamma'(\tilde{h_i}'), i = 1, 2, 3$; The isomorphism of A_3 and D_3 can be given as follows:

$$\begin{array}{ccc}
A_3 & \mapsto & D_3 \\
e_{11} - e_{22} & \mapsto & -(e_{11} - e_{44}) + e_{33} - e_{66} \\
e_{22} - e_{33} & \mapsto & (e_{11} - e_{44}) + (e_{22} - e_{55}) \\
e_{33} - e_{44} & \mapsto & -(e_{11} - e_{44}) - (e_{33} - e_{66}) \\
e_{13} & \mapsto & e_{26} - e_{35} \\
e_{31} & \mapsto & e_{62} - e_{53} \\
e_{24} & \mapsto & e_{23} - e_{65} \\
e_{42} & \mapsto & e_{32} - e_{56} \\
e_{41} & \mapsto & e_{12} - e_{54} \\
e_{14} & \mapsto & e_{21} - e_{45} \\
e_{43} & \mapsto & e_{16} - e_{34} \\
e_{34} & \mapsto & e_{61} - e_{43} \\
e_{21} & \mapsto & e_{13} - e_{64} \\
e_{12} & \mapsto & e_{31} - e_{46} \\
e_{23} & \mapsto & e_{15} - e_{24} \\
e_{32} & \mapsto & e_{51} - e_{42}
\end{array}$$

11. Verify that the commutator of two derivations of an F-algebra is again a derivation, whereas the ordinary product need not be.

Solution:

A is a F-algebra, $\delta, \delta' \in \mathrm{Der}(A), a, b \in A$

$$\begin{aligned}
[\delta, \delta'](ab) &= \delta\delta'(ab) - \delta'\delta(ab) \\
&= \delta\left(\delta'(a)b + a\delta'(b)\right) - \delta'\left(\delta(a)b + a\delta(b)\right) \\
&= \delta(\delta'(a))b + \delta'(a)\delta(b) + \delta(a)\delta'(b) + a\delta(\delta'(b)) \\
&\quad -\delta'(\delta(a))b - \delta(a)\delta'(b) - \delta'(a)\delta(b) - a\delta'(\delta(b)) \\
&= ([\delta, \delta'](a))b - a[\delta, \delta'](b) \\
\therefore [\delta, \delta'] &\in \mathrm{Der}(A)
\end{aligned}$$

12. Let L be a Lie algebra over an algebraically closed field and let $x \in L$. Prove that the subspace of L spanned by the eigenvectors of $\mathrm{ad}x$ is a subalgebra.

Solution:

Suppose y, z are eigenvectors of $\mathrm{ad}x$ respect to eigenvalues λ, μ, i.e., $[x, y] = \lambda y, [x, z] = \mu z$ then

$$\begin{aligned}
\mathrm{ad}x[y, z] &= [x, [y, z]] \\
&= [y, [x, z]] - [z, [x, y]] \\
&= (\lambda + \mu)[y, z]
\end{aligned}$$

So $[y, z]$ is also a eigenvector of $\mathrm{ad}x$. i.e., the subspace of L spanned by the eigenvectors of $\mathrm{ad}x$ is a subalgebra.

2 Ideals and Homomorphisms

1. Prove that the set of all inner derivations $\mathrm{ad}x, x \in L$, is an ideal of $\mathrm{Der}L$.

Solution:
$\forall \delta \in \mathrm{Der}L, x, y \in L$

$$\begin{aligned}
[\delta, \mathrm{ad}x](y) &= \delta([x,y]) - [x, \delta(y)] \\
&= [\delta(x), y] + [x, \delta(y)] - [x, \delta(y)] \\
&= \mathrm{ad}(\delta(x))(y) \\
\therefore [\delta, \mathrm{ad}x] &= \mathrm{ad}\delta(x) \text{is a inner derivations.}
\end{aligned}$$

2. Show that $\mathfrak{sl}(n, F)$ is precisely the derived algebra of $\mathfrak{gl}(n, F)$ (cf. Exercise 1.9).

Solution:
$\forall x, y \in \mathfrak{gl}(n, F), \mathrm{tr}[x, y] = \mathrm{tr}(xy) - \mathrm{tr}(yx) = 0$. We have

$$[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] \subseteq \mathfrak{sl}(n, F)$$

Conversely, by exercise 1.9,

$$\mathfrak{sl}(n, F) = [\mathfrak{sl}(n, F), \mathfrak{sl}(n, F)] \subseteq [\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)]$$

3. Prove that the center of $\mathfrak{gl}(n, F)$ equals $\mathfrak{s}(n, F)$ (the scalar matrices). Prove that $\mathfrak{sl}(n, F)$ has center 0, unless $char F$ divides n, in which case the center is $\mathfrak{s}(n, F)$.

Solution:
Clearly, we have $\mathfrak{s}(n, F) \subseteq Z(\mathfrak{gl}(n, F))$. Conversely, Let $a = \sum\limits_{i,j} a_{ij} e_{ij} \in Z(\mathfrak{gl}(n, F))$, then for each $e_{kl} \in \mathfrak{gl}(n, F)$,

$$\begin{aligned}
[a, e_{kl}] &= \sum_{i,j} a_{ij}[e_{ij}, e_{kl}] \\
&= \sum_{i,j} a_{ij}(\delta_{jk} e_{il} - \delta_{li} e_{kj}) \\
&= \sum_{i=1}^{n} a_{ik} e_{il} - \sum_{j=1}^{n} a_{lj} e_{kj} \\
&= (a_{kk} - a_{ll}) e_{kl} + \sum_{\substack{i=1 \\ i \neq k}}^{n} a_{ik} e_{il} - \sum_{\substack{j=1 \\ j \neq l}}^{n} a_{lj} e_{kj}
\end{aligned}$$

So

$$a_{kk} = a_{ll}, a_{ij} = 0, i \neq j$$

i.e.

$$a \in \mathfrak{s}(n, F)$$

For $\mathfrak{sl}(n,F)$, if $c \in Z(\mathfrak{sl}(n,F))$, $\forall x \in \mathfrak{sl}(n,F)$, $[x,c]=0$. But we know $\mathfrak{gl}(n,F)=\mathfrak{sl}(n,F)+\mathfrak{s}(n,F)$ and $\mathfrak{s}(n,F)$ is the center of $\mathfrak{gl}(n,F)$. Hence $c \in Z(\mathfrak{gl}(n,F))=\mathfrak{s}(n,F)$. We have

$$Z(\mathfrak{sl}(n,F))=\mathfrak{sl}(n,F)\cap\mathfrak{s}(n,F)$$

If $charF$ does not divide n, each $aI \in \mathfrak{s}(n,F)$ has trace $na \neq 0$, so $aI \notin \mathfrak{sl}(n,F)$. i.e., $Z(\mathfrak{sl}(n,F)) = \mathfrak{sl}(n,F)\cap\mathfrak{s}(n,F) = 0$. Else if $charF$ divides n, each $aI \in \mathfrak{s}(n,F)$ has trace $na = 0$, in this case $Z(\mathfrak{sl}(n,F)) = \mathfrak{sl}(n,F)\cap\mathfrak{s}(n,F)=\mathfrak{s}(n,F)$

4. Show that (up to isomorphism) there is a unique Lie algebra over F of dimension 3 whose derived algebra has dimension 1 and lies in $Z(L)$.

Solution:

Let L_0 be the 3-dimensional lie algebra over F with basis (x_0,y_0,z_0) and commutation:

$$[x_0,y_0]=z_0,[x_0,z_0]=[y_0,z_0]=0$$

.

Suppose L be any 3-dimensional lie algebra over F whose derived algebra has dimension 1 and lies in $Z(L)$. We can take a basis (x,y,z) of L such that $z \in [LL] \subseteq Z(L)$. By hypothesis, $[x,y]=\lambda z, [x,z]=[y,z]=0, \lambda \in F$. Then $L \to L_0, x \mapsto x_0, y \mapsto y_0, z \mapsto \lambda z_0$ is a isomorphism.

5. Suppose $\dim L = 3, L=[LL]$. Prove that L must be simple. [Observe first that any homomorphic image of L also equals its derived algebra.] Recover the simplicity of $\mathfrak{sl}(2,F)$, $charF \neq 2$.

Solution:

Let I is an ideal of L, then $[L/I, L/I]=[L,L]/I=L/I$.

Suppose L has a proper ideal $I \neq 0$, then I has dimension 1 or 2. If I has dimension 2, then L/I is a 1-dimensional algebra, $[L/I,L/I]=0 \neq L/I$. Else I has dimension 1, we can take a basis (x,y,z) of L such that z is a basis of I, so

$$[x,z]\in I,[y,z]\in I$$

Hence $[LL]$ is contained in the subspace of L spanned by $[x,y],z$. Its dimension is at most 2, this contradict with $[LL]=L$.

Now, we conclude that L has no proper nonzero ideal, i.e., L is a simple Lie algebra.

6. Prove that $\mathfrak{sl}(3,F)$ is simple, unless $charF = 3$ (cf. Exercise 3). [Use the standard basis $h_1,h_2,e_{ij}(i\neq j)$. If $I \neq 0$ is an ideal, then I is the direct sum of eigenspaces for $\mathrm{ad}h_1$ or $\mathrm{ad}h_2$; compare the eigenvalues of $\mathrm{ad}h_1, \mathrm{ad}h_2$ acting on the e_{ij}.]

Solution:

7. Prove that $\mathfrak{t}(n,F)$ and $\mathfrak{d}(n,F)$ are self-normalizing subalgebras of $\mathfrak{gl}(n,F)$, whereas $\mathfrak{n}(n,F)$ has normalizer $\mathfrak{t}(n,F)$.

Solution:

Let $a = \sum_{ij} a_{ij}e_{ij} \in \mathfrak{gl}(n,F), [a, \mathfrak{t}(n,F)] \subseteq \mathfrak{t}(n,F)$. But

$$\begin{aligned}[a, e_{kk}] &= \sum_{ij} a_{ij}\delta_{jk}e_{ik} - \sum_{ij} a_{ij}\delta_{ki}e_{kj} \\ &= \sum_{i} a_{ik}e_{ik} - \sum_{j} a_{kj}e_{kj} \\ &\subseteq \mathfrak{t}(n,F)\end{aligned}$$

It must be $a_{ik} = 0$ for $i > k$, and $a_{kj} = 0$ for $j < k$. Hence $a_{kl} = 0$ for all $k > l$. This implies $a \in \mathfrak{t}(n,F)$, i.e., $\mathfrak{t}(n,F)$ is the self-normalizing subalgebras of $\mathfrak{gl}(n,F)$.

Similarly for $\mathfrak{d}(n,F)$, let $a = \sum_{ij} a_{ij}e_{ij} \in \mathfrak{gl}(n,F), [a, \mathfrak{d}(n,F)] \subseteq \mathfrak{d}(n,F)$. But

$$\begin{aligned}[a, e_{kk}] &= \sum_{ij} a_{ij}\delta_{jk}e_{ik} - \sum_{ij} a_{ij}\delta_{ki}e_{kj} \\ &= \sum_{i} a_{ik}e_{ik} - \sum_{j} a_{kj}e_{kj} \\ &\subseteq \mathfrak{d}(n,F)\end{aligned}$$

It must be $a_{ik} = 0$ for $i \neq k$, and $a_{kj} = 0$ for $j \neq k$. Hence $a_{kl} = 0$ for all $k \neq l$. This implies $a \in \mathfrak{d}(n,F)$, i.e., $\mathfrak{d}(n,F)$ is the self-normalizing subalgebras of $\mathfrak{gl}(n,F)$.

8. Prove that in each classical linear Lie algebra (1.2), the set of diagonal matrices is a self-normalizing subalgebra, when $char F = 0$.

Solution:

9. Prove Proposition 2.2.

Solution:

10. Let σ be the automorphism of $\mathfrak{sl}(2,F)$ defined in (2.3). Verify that $\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$.

Solution:

$$\begin{aligned}\exp \mathrm{ad} x(x) &= x \\ \exp \mathrm{ad} x(h) &= h - 2x \\ \exp \mathrm{ad} x(y) &= y + h - x \\ \exp \mathrm{ad}(-y)(x) &= x + h - y \\ \exp \mathrm{ad}(-y)(h) &= h - 2y \\ \exp \mathrm{ad}(-y)(y) &= y\end{aligned}$$

$$\begin{aligned}\sigma(x) &= \exp \mathrm{ad}x \exp \mathrm{ad}(-y)(x)\\ &= \exp \mathrm{ad}x(x+h-y)\\ &= x+h-2x-y-h+x\\ &= -y\end{aligned}$$

$$\begin{aligned}\sigma(y) &= \exp \mathrm{ad}x \exp \mathrm{ad}(-y)(y+h-x)\\ &= \exp \mathrm{ad}x(y+h-2y-x-h+y)\\ &= \exp \mathrm{ad}x(-x)\\ &= -x\end{aligned}$$

$$\begin{aligned}\sigma(h) &= \exp \mathrm{ad}x \exp \mathrm{ad}(-y)(h-2x)\\ &= \exp \mathrm{ad}x(h-2y-2(x+h-y))\\ &= \exp \mathrm{ad}x(-h-2x)\\ &= -h+2x-2x=-h\end{aligned}$$

11. If $L = \mathfrak{sl}(n, F)$, $g \in GL(n, F)$, prove that the map of L to itself defined by $x \mapsto -gx^tg^{-1}$ (x^t= transpose of x) belongs to $\mathrm{Aut}L$. When $n = 2$, g = identity matrix, prove that this automorphism is inner.

Solution:

$g \in GL(n, F)$ and $\mathrm{tr}(-gx^tg^{-1}) = -\mathrm{tr}(x)$, i.e, $\mathrm{tr}(x) = 0$ if and only if $\mathrm{tr}(-gx^tg^{-1}) = 0$. so the map $x \mapsto -gx^tg^{-1}$ is a linear space automorphism of $\mathfrak{sl}(n, F)$. We just verify it is a homomorphism of lie algebras:

$$\begin{aligned}[-gx^tg^{-1}, -gy^tg^{-1}] &= gx^ty^tg^{-1} - gy^tx^tg^{-1}\\ &= -g((xy)^t - (yx)^t)g^{-1}\\ &= -g[x, y]^tg^{-1}\end{aligned}$$

When $n = 2$, g=identity matrix, the automorphism $\sigma : x \mapsto -x^t$, i.e.

$$\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$$

So $\sigma = \exp \mathrm{ad}x \exp \mathrm{ad}(-y) \exp \mathrm{ad}x$ is an inner automorphism. (**Warning:** An inner automorphism is not exactly of form $\exp \mathrm{ad}x$ with $\mathrm{ad}x$ is nilpotent. It can be the composition of elements with this form.)

12. Let L be an orthogonal Lie algebra (type B_l or D_l). If g is an **orthogonal** matrix, in the sense that g is invertible and $g^tsg = s$, prove that $x \mapsto gxg^{-1}$ defines an automorphism of L.

Solution:

$x \in B_l$ or D_l, $sx = -x^ts$. Hence

$$\begin{aligned}sgxg^{-1} &= (g^{-1})^tsxg^{-1}\\ &= -(g^{-1})^tx^tsg^{-1}\\ &= -(g^{-1})^tx^tg^ts\\ &= -(gxg^{-1})^ts\end{aligned}$$

So the map $x \mapsto gxg^{-1}$ is a linear automorphism of B_l or C_l. We just verify it is a homomorphism of lie algebras:

$$[gxg^{-1}, gyg^{-1}] = gxyg^{-1} - gyxg^{-1} = g[x,y]g^{-1}$$

3 Solvable and Nilpotent Lie Algebras

1. Let I be an ideal of L. Then each member of the derived series or descending central series of I is also an ideal of L.

Solution:

For derived series, we need to show: if I is an ideal of L, $[II]$ is an ideal of L. Let $x, \in L, y, z \in I$

$$[x,[y,z]] = [[z,x],y] + [[x,y],z] \in [II]$$

So $[II]$ is an ideal of L.

For descending central series, we need to show: if I, J are ideals of L, and $J \subseteq I$, $[IJ]$ is an ideal of L. Let $x \in L, y \in I, z \in J$

$$[x,[y,z]] = [[z,x],y] + [[x,y],z] \in [IJ]$$

Because $[z,x] \in J$ and $[x,y] \in I$.

2. Prove that L is solvable if and only if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and such that each quotient L_i/L_{i+1} is abelian.

Solution:

If L is solvable, its derived series : $L = L^{(0)} \supset L^{(1)} \supset \cdots \supset L^{(k)} = 0$, $L^{(m+1)} = [L^{(m)}, L^{(m)}]$, so $L^{(m)}$ is an ideal of $L^{(m)}$ and $L^{(m)}/L^{(m+1)}$ is abelian.

Conversely, if there exists a chain of subalgebras $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and such that each quotient L_i/L_{i+1} is abelian.

Claim: if I is an ideal of L and L/I is abelian, then $I \subseteq [LL]$. This is clearly. Because L/I is abelian, $\forall x, y \in L, [x,y] \in I$, i.e., $[L,L] \subseteq I$.

By the above claim, we can deduced by induction that $L^{(m)} \subseteq L_m$. In fact $L^{(1)} \subseteq L_1$ is true. If $L^{(m)} \subseteq L_m$, $L^{(m+1)} = [L^{(m)}, L^{(m)}] \subseteq [L_m, L_m] \subseteq L_{m+1}$.

By the hypothesis, $L^{(k)} = 0$, so L is solvable.

3. Let $charF = 2$. Prove that $\mathfrak{sl}(2,F)$ is nilpotent.

Solution:

Let (x,h,y) is the standard basis for $\mathfrak{sl}(2,F)$.

$$[hx] = 2x = 0, [xy] = h, [hy] = -2y = 0$$

Hence

$$[\mathfrak{sl}(2,F), \mathfrak{sl}(2,F)] = Fh$$

$$[\mathfrak{sl}(2,F),[\mathfrak{sl}(2,F),\mathfrak{sl}(2,F)]] = [\mathfrak{sl}(2,F),Fh] = 0$$

i.e., $\mathfrak{sl}(2,F)$ is nilpotent.

4. Prove that L is solvable (resp. nilpotent) if and only if ad L is solvable (resp. nilpotent).

Solution: $\mathrm{ad}: L \to \mathrm{ad}L$ is a homomorphism, by proposition 2.2, $\mathrm{ad}L \cong L/Z(L)$ because of $ker(\mathrm{ad}) = Z(L)$. $[Z(L),Z(L)] = 0$, so $Z(L)$ is a solvable ideal.

By proposition 3.1, L is solvable if and only if $\mathrm{ad}L$ is solvable.

5. Prove that the nonabelian two dimensional algebra constructed in (1.4) is solvable but not nilpotent. Do the same for the algebra in Exercise 1.2.

Solution:

The nonabelian two dimensional algebra L is given by basis (x,y) and commutations $[x,y] = x$. We can deduce that: $L^{(1)} = Fx, L^{(2)} = 0$. So L is solvable. But $L^1 = Fx, L^2 = Fx, \cdots, L^k = Fx, \cdots$, so L is not nilpotent.

The algebra in Exercise 1.2 L is given by basis (x,y,z) and commutations $[x,y] = z, [x,z] = y, [y,z] = 0$. We can deduce that: $L^{(1)} = Fy + Fz, L^{(2)} = 0$. So L is solvable. But $L^1 = Fy + Fz, L^2 = Fy + Fz, \cdots, L^k = Fy + Fz, \cdots$, so L is not nilpotent.

6. Prove that the sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. Therefore, L possesses a unique maximal nilpotent ideal. Determine this ideal for each algebra in Exercise 5.

Solution:

Let I, J are nilpotent ideals of L, $I^m = 0, J^n = 0$.

$$[I+J, I+J] \subseteq [II] + [JJ] + [IJ] \subseteq [II] + [JJ] + I \cap J$$

We can deduce by induction that

$$(I+J)^k \subseteq I^k + J^k + I \cap J$$

If we let $k > \max(m,n)$, then $I^k = 0, J^k = 0$, $(I+J)^k \subseteq I \cap J$.

$$(I+J)^{k+l} = \underbrace{[I+J, \cdots, [I+J}_{k}, I \cap J] \cdots] \subseteq I^l \cap J + I \cap J^l$$

If we let $k > \max(m,n)$, then $(I+J)^{k+l} = 0$, $I+J$ is a nilpotent ideal of L.

7. Let L be nilpotent, K a proper subalgebra of L. Prove that $N_L(K)$ includes K properly.

Solution:

Let $L^0 = L, L^1 = [LL], L^2 = [L, L^1], \cdots, L^n = 0$ be the descending central series of L. K is a proper subalgebra of L. Hence there exists a k, such that $L^{k+1} \subseteq K$, but $L^k \not\subseteq K$.

Let $x \in L^k, x \notin K$, $[x, K] \subseteq L^{k+1} \subseteq K$, so $x \in N_L(K)$, but $x \notin K$. i.e., $N_L(K)$ includes K properly.

In exercise 3.5, the 2 dimensional algebra has a maximal nilpotent ideal Fx; the 2 dimensional algebra has a maximal nilpotent ideal $Fy + Fz$.

8. Let L be nilpotent. Prove that L has an ideal of codimension 1.

Solution:

L is nilpotent, $[LL] \neq L$. We have a natural homomorphism $\pi : L \to L/[LL]$. $L/[LL]$ is a nonzero abelian algebra, so it has a subspace $\bar{I}$ of codimension 1. $\bar{I}$ must be a ideal of $L/[LL]$ as $L/[LL]$ being abelian. So $\pi^{-1}(\bar{I})$ is a ideal of L with codimension 1.

9. Prove that every nilpotent Lie algebra L has an outer derivation (see (1.3)), as follows: Write $L = K + Fx$ for some ideal K of codimension one (Exercise 8). Then $C_L(K) \neq 0$ (why?). Choose n so that $C_L(K) \subset L^n$,$C_L(K) \not\subset L^{n+1}$, and let $z \in C_L(K) - L^{n+1}$. Then the linear map δ sending K to 0, x to z, is an outer derivation.

Solution:

L is nilpotent, there exists k such that $L^k = 0, L^{k-1} \neq 0$. So $[L^{k-1}, K] \subseteq [L^{k-1}, L] = 0$, i.e., $0 \neq L^{k-1} \subseteq C_L(K)$. Then we have n satisfying $C_L(K) \subseteq L^n, C_L(K) \nsubseteq L^{n+1}$. Let $z \in C_L(K) \backslash L^{n+1}$. We make a linear map δ send K to 0,x to z.

For all $k_1 + \lambda_1 x, k_2 + \lambda_2 x \in L, [k_1 + \lambda_1 x, k_2 + \lambda_2 x] \in K$, so $\delta([k_1 + \lambda_1 x, k_2 + \lambda_2 x]) = 0$.

In the other hand,

$$\begin{aligned}
& [\delta(k_1 + \lambda_1 x), k_2 + \lambda_2 x] + [k_1 + \lambda_1 x, \delta(k_2 + \lambda_2 x)] \\
= \; & [\lambda_1 z, k_2 + \lambda_2 x] + [k_1 + \lambda_1 x, \lambda_2 z] \\
= \; & \lambda_1 \lambda_2 [z, x] + \lambda_1 \lambda_2 [x, z] \\
= \; & 0
\end{aligned}$$

We conclude that δ is a derivation. If δ is a inner derivation, $\delta = \text{ad} y$, then $[y, K] = \delta(K) = 0$, so $y \in C_L(K) \subseteq L^n$. Then we have $[y, x] \subseteq L^{n+1}$. But $[y, x] = \delta(x) = z \notin L^{n+1}$. This is a contradiction. So δ is a outer derivation.

10. Let L be a Lie algebra, K an ideal of L such that L/K is nilpotent and such that $\text{ad}x|_K$ is nilpotent for all $x \in L$. Prove that L is nilpotent.

Solution:

L/K is nilpotent, for all $x \in L$, $\text{ad}\bar{x}$ is a nilpotent endomorphism in $End(L/K)$. i.e., there exists a n such that $(\text{ad}x)^n(y) \in K, \forall y \in L$.

In the other hand, $\text{ad}x|_K$ is nilpotent, so we have a m such that $(\text{ad}x)^m((\text{ad}x)^n(y)) = 0$, i.e., $(\text{ad}x)^{m+n}(y) = 0$. So $\text{ad}x$ is a nilpotent endomorphism in $\mathfrak{gl}(L)$. By Engel's Theorem, L is nilpotent.

4 Theorems of Lie and Cartan

1. Let $L = \mathfrak{sl}(V)$. Use Lie's Theorem to prove that $\mathrm{Rad}L = Z(L)$; conclude that L is semisimple (cf. Exercise 2.3). [Observe that $\mathrm{Rad}L$ lies in each maximal solvable subalgebra B of L. Select a basis of V so that $B = L \cap \mathfrak{t}(n, F)$, and notice that the transpose of B is also a maximal solvable subalgebra of L. Conclude that $\mathrm{Rad}L \subset L \cap \mathfrak{d}(n, F)$, then that $\mathrm{Rad}L = Z(L)$.]

   Solution:

2. Show that the proof of Theorem 4.1 still goes through in prime characteristic, provided $\dim V$ is less than $char F$.

   Solution:

3. This exercise illustrates the failure of Lie's Theorem when F is allowed to have prime characteristic p. Consider the $p \times p$ matrices:

$$x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad y = \mathrm{diag}(0, 1, 2, 3, \cdots, p-1)$$

   Check that $[x, y] = x$, hence that x and y span a two dimensional solvable subalgebra L of $\mathfrak{gl}(p, F)$. Verify that x, y have no common eigenvector.

   Solution:

4. When $p = 2$, Exercise 3.3 show that a solvable Lie algebra of endomorphisms over a field of prime characteristic p need not have derived algebra consisting of nilpotent endomorphisms (cf. Corollary C of Theorem 4.1). For arbitrary p, construct a counterexample to Corollary C as follows: Start with $L \subset \mathfrak{gl}(p, F)$ as in Exercise 3. Form the vector space direct sum $M = L + F^p$, and make M a Lie algebra by decreeing that F^p is abelian, while L has its usual product and acts on F^p in the given way. Verify that M is solvable, but that its derived algebra $(= Fx + F^p)$ fails to be nilpotent.

   Solution:

5. If $x, y \in \mathrm{End}V$ commute, prove that $(x + y)_s = x_s + y_s$, and $(x + y)_n = x_n + y_n$. Show by example that this can fail if x, y fail to commute. [Show first that x, y semisimple (resp. nilpotent) implies x+y semisimple (resp. nilpotent).]

   Solution:

6. Check formula (*) at the end of (4.2).

Solution:

7. Prove the converse of Theorem 4.3.

Solution:

8. Note that it suffices to check the hypothesis of Theorem 4.3 (or its corollary) for x, y ranging over a basis of L. For the example given in Exercise 1.2, verify solvability by using Cartan's Criterion.

Solution:

5 Killing Form

1. Prove that if L is nilpotent, the Killing form of L is identically zero.

Solution: L is nilpotent. There is a k such that $L^{2k+1} = 0$, So

$$\underbrace{\mathrm{ad}x\mathrm{ad}y \cdots \mathrm{ad}x\mathrm{ad}y}_{k}(x) \in L^{2k+1} = 0, \forall x, y, z \in L$$

Hence we have $\mathrm{ad}x\mathrm{ad}y$ is a nilpotent endomorphism of L.

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}x\mathrm{ad}y) = 0$$

2. Prove that L is solvable if and only if $[LL]$ lies in the radical of the Killing form.

Solution:

"$\Leftarrow$": $[LL]$ lies in the radical of the Killing form, then $\forall x \in [LL], y \in L$, $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}x\mathrm{ad}y) = 0$. By corollary 4.3, L is solvable.

"$\Rightarrow$": L is solvable. By Lie theorem, L has a basis $x_1, \cdots, x_n$ such that any $x \in L$, $\mathrm{ad}x$ is a upper triangular matrix relative to $(x_1, \cdots, x_n)$. Hence

$$\mathrm{ad}[x, y] = \mathrm{ad}x\mathrm{ad}y - \mathrm{ad}y\mathrm{ad}x$$

is a strictly upper triangular matrix. We have $\mathrm{ad}[xy]\mathrm{ad}y$ is a strictly upper triangular matrix for all $x, y, z \in L$. Therefore, $\mathrm{tr}(\mathrm{ad}[xy]\mathrm{ad}y) = 0$ and then $[LL] \subseteq \mathrm{Rad}(L)$.

3. Let L be the two dimensional nonabelian Lie algebra (1.4), which is solvable. Prove that L has nontrivial Killing form.

Solution:
L be the two dimensional nonabelian Lie algebra (1.4). (x, y) is a basis of L and $[x, y] = x$. We can write down the matrix of $\mathrm{ad}x, \mathrm{ad}y$ relative to the basis (x, y) as follows:

$$\mathrm{ad}x \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{ad}y \sim \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

So $\kappa(y, y) = \mathrm{tr}(\mathrm{ad}y\mathrm{ad}y) = 1$, κ is nontrivial.

4. Let L be the three dimensional solvable Lie algebra of Exercise 1.2. Compute the radical of its Killing form.

Solution:
Let (x, y, z) be the basis of L, $[xy] = z, [xz] = y, [yz] = 0$. The matrices of $\mathrm{ad}x, \mathrm{ad}y, \mathrm{ad}z$ are

$$\mathrm{ad}x \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{ad}y \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}z \sim \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We compute the matrix of Killing Form κ relative to the basis (x, y, z):

$$\kappa = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $ax + by + cz$ is any element in the radical of κ.

$$(a, b, c) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So $a = 0, b, c$ can be any number in F. We conclude that the radical of the Killing form is $Fy + Fz$.

5. Let $L = \mathfrak{sl}(2, F)$. Compute the basis of L dual to the standard basis, relative to the Killing form.

Solution:
The matrix of the Killing form relative to the basis (x, h, y) is

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

The basis of L dual to the standard basis is $(\frac{1}{4}y, \frac{1}{8}h, \frac{1}{4}x)$

6. Let $char F = p \neq 0$. Prove that L is semisimple if its Killing form is nondegenerate. Show by example that the converse fails. [Look at $\mathfrak{sl}(3, F)$ modulo its center, when $char F = 3$.]

Solution:

If $\mathrm{Rad}(L) \neq 0$, the last nonzero term I in its derived series is a abelian subalgebra of L, and by exercise 3.1, I is a ideal of L. In another words, L has a nonzero abelian ideal. It is suffice to prove any abelian ideal of L is zero.

Let S be the radical of the Killing form, which is nondegenerate. So $S = 0$. To prove that L is semisimple, it will suffice to prove that every abelian ideal I of L is included in S. Suppose $x \in I, y \in L$. Then $\mathrm{ad}x\mathrm{ad}y$ maps $L \to L \to I$, and $(\mathrm{ad}x\mathrm{ad}y)^2$ maps L into $[II] = 0$. This means that $\mathrm{ad}x\mathrm{ad}y$ is nilpotent, hence that $0 = Tr(\mathrm{ad}x\mathrm{ad}y) = \kappa(x, y)$, so $I \subseteq S = 0$.

7. Relative to the standard basis of $\mathfrak{sl}(3, F)$, compute the determinant of κ. Which primes divide it?

Solution:

We write down the matrix of $\mathrm{ad}x$ relative to basis $(e_{11} - e_{22}, e_{22} - e_{33}, e_{12}, e_{13}, e_{21}, e_{23}, e_{31}, e_{33}$ when x runs over this basis.

$$\mathrm{ad}(e_{11} - e_{22}) = \mathrm{diag}(0, 0, 2, 1, -2, -1, -1, 1)$$
$$\mathrm{ad}(e_{22} - e_{33}) = \mathrm{diag}(0, 0, -1, 1, 1, 2, -1, -2)$$

$$\mathrm{ad}e_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \mathrm{ad}e_{21} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{ad}e_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathrm{ad}e_{31} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{ad}e_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathrm{ad}e_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix of the Killing form relative to this basis is

$$\kappa = \begin{pmatrix} 12 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}$$

Its determinant is $\det(\kappa) = 2^8 3^9$, so prime 2 and 3 divide the determinant of κ

8. Let $L = L_1 \oplus \cdots \oplus L_t$ be the decomposition of a semisimple Lie algebra L into its simple ideals. Show that the semisimple and nilpotent parts of $x \in L$ are the sums of the semisimple and nilpotent parts in the various L_i of the components of x.

Solution:

Let $x \in L$, $x = x_1 + \cdots + x_t$ with $x_i \in L_i$, and $x_i = u_i + v_i$ is the Jordan decomposition of x_i in L_i, u_i is semisimple and v_i is nilpotent.

Because $\mathrm{ad}_L u_i|_{L_i} = \mathrm{ad}_{L_i} u_i$ is a semisimple endomorphism of L_i. In the other hand, $\mathrm{ad}_L u_i|_{L_j} = 0, j \neq i$. Hence $\mathrm{ad}_L u_i$ is a semisimple endomorphism of L. We know $[u_i, u_j] =$ for all $i \neq j$. Let $u = u_1 + \cdots + u_t$, then $\mathrm{ad}_L u$ is a semisimple endomorphism of L.

Similarly, let $v = v_1 + \cdots + v_t$, $\mathrm{ad} v$ is a nilpotent endomorphism of L.

Furthermore, $[u, v] = [u_1, v_1] + \cdots + [u_t, v_t] = 0$, so $x = u + v$ is the Jordan decomposition of x.

6 Complete Reducibility of Representations

1. Using the standard basis for $L = \mathfrak{sl}(2, F)$, write down the Casimir element of the adjoint representation of L (cf. Exercise 5.5). Do the same thing for the usual (3-dimensional) representation of $\mathfrak{sl}(3, F)$, first computing dual bases relative to the trace form.

Solution:

For the adjoint representation of $L = \mathfrak{sl}(2, F)$, The matrix of β respect to basis (x, h, y) is

$$\beta \sim \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

we can deduce the dual basis of (x, h, y) is $(\frac{1}{4}y, \frac{1}{8}h, \frac{1}{4}x)$. So the Casimir element of this representation is

$$c_{\mathrm{ad}} = \frac{1}{4}\mathrm{ad}x\mathrm{ad}y + \frac{1}{8}\mathrm{ad}h\mathrm{ad}h + \frac{1}{4}\mathrm{ad}y\mathrm{ad}x$$

For the usual representation of $L = \mathfrak{sl}(3, F)$, The matrix of β respect to basis $(e_{11}-e_{22}, e_{22}-e_{33}, e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32})$ is

$$\beta \sim \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We can deduce the dual basis is

$$\frac{2}{3}e_{11} - \frac{1}{3}e_{22} - \frac{1}{3}e_{33}, \frac{1}{3}e_{11} + \frac{1}{3}e_{22} - \frac{2}{3}e_{33}, e_{21}, e_{31}, e_{32}, e_{12}, e_{13}, e_{23}$$

So

$$c_\varphi = \sum_x xx' = \begin{pmatrix} \frac{8}{3} & 0 & 0 \\ 0 & \frac{8}{3} & 0 \\ 0 & 0 & \frac{8}{3} \end{pmatrix}$$

2. Let V be an L-module. Prove that V is a direct sum of irreducible submodules if and only if each L-submodule of V possesses a complement.

Solution:

"$\Rightarrow$" Let $V = V_1 \oplus \cdots \oplus V_n$ and V_i is irreducible submodule of V. Let W is any submodule of V.

Let W' be the maximal submodule of V which trivial intersection with W.(Such a module exists because V has finite dimensional.) Then $W \cap W' = 0$. If $W + W'$ is a proper submodule of V, then there is a V_i such that $V_i \not\subset W + W'$. But $V_i \cap (W + W')$ is a submodule of V_i and we know it is not the V_i, so $V_i \cap (W + W') = 0$. So we can make a module $W' + V_i$ which trivially intersection with W, and proper including W', which contradict with W' is maximal. So $V = W \oplus W'$.

"$\Leftarrow$" V is a finite dimensional module of L. Let U be the maximal submodule of V such that it is a direct sum of irreducible submodule. Such a U exists because a irreducible submodule of V is a direct sum of itself. If $U \neq V$, then there is a submodule W such that $V = U \oplus W$. Let W_1 is a irreducible submodule of W. Then $U \oplus W_1$ is a submodule of V and it is a direct sum of irreducible submodules. This contradicts to the choice of U.

3. If L is solvable, every irreducible representation of L is one dimensional.

Solution:

Let V is a irreducible representation of L, $\varphi : L \to \mathfrak{gl}(V)$ is a representation. Then $\varphi(L)$ is a solvable subalgebra of $\mathfrak{gl}(V)$. By Lie Theorem, there is a $0 \neq v \in V$, $\phi(L).v \subseteq Fv$. So Fv is a submodule of V. Hence $V = Fv$ has dimension 1 as V is irreducible.

4. Use Weyl's Theorem to give another proof that for L semisimple, $\mathrm{ad}L = \mathrm{Der}L$ (Theorem 5.3). [If $\delta \in \mathrm{Der}L$, make the direct sum $F+L$ into an L-module via the rule $x.(a, y) = (0, a\delta(x)+[xy])$. Then consider a complement to the submodule L.]

Solution:

Let $\delta \in \mathrm{Der}L$, make $F + L$ into an L-module by

$$x.(a, y) = (0, a\delta(x) + [x, y])$$

We can check the above formula defines a module as follows:

$$\begin{aligned}
[x, z].(a, y) &= (0, a\delta([x, z]) + [[x, z], y]) \\
&= (0, a[\delta(x), z] + a[x, \delta(z)] + [x, [z, y]] + [z, [y, x]] \\
x.z.(a, y) &= x.(0, a\delta(z) + [z, y]) = (0, [x, a\delta(z) + [z, y]]) \\
z.x.(a, y) &= z.(0, a\delta(x) + [x, y]) = (0, [z, a\delta(x) + [x, y]]) \\
\therefore [x, z].(a, y) &= x.z.(a, y) - z.x.(a, y)
\end{aligned}$$

Clearly, L is a submodule of $F + L$. By Weyl's Theorem, it has a complement of dimension 1. Let $(a_0, x_0), a_0 \neq 0$ be its basis. Then L acts on it trivially. Hence

$$0 = x.(a_0, x_0) = (0, a_0\delta(x) + [x, x_0])$$

i.e.

$$\delta(x) = [\frac{1}{a_0}x_0, x] = \mathrm{ad}\frac{1}{a_0}x_0(x)$$

So $\delta \in \mathrm{Int}L$.

5. A Lie algebra L for which $\mathrm{Rad}L = Z(L)$ is called reductive. (Examples: L abelian, L semisimple, $L = \mathfrak{gl}(n, F)$.)
   1. If L is reductive, then L is a completely reducible $\mathrm{ad}L$-module. [If $\mathrm{ad}L \neq 0$, use Weyl's Theorem.] In particular, L is the direct sum of $Z(L)$ and $[LL]$, with $[LL]$ semisimple.
   2. If L is a classical linear Lie algebra (1.2), then L is semisimple. [Cf. Exercise 1.9.]
   3. If L is a completely reducible $\mathrm{ad}L$-module, then L is reductive.
   4. If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphisms are completely reducible.

Solution:

(1)L is reductive, $\mathrm{ad}L \cong L/Z(L) \cong L/\mathrm{Rad}(L)$, so If $\mathrm{ad}L \neq 0$, $\mathrm{ad}L$ is a semisimple Lie algebra. By weyl theorem, L is a completely reducible $\mathrm{ad}L$-module. If $\mathrm{ad}L = 0$, L is abelian, each 1-dimensional subspace of L is a irreducible $\mathrm{ad}L$-module. So L is a completely reducible $\mathrm{ad}L$-module.

We know $L/Z(L)$ is semisimple, so $[LL]/Z(L) \cong [L/Z(L), L/Z(L)] \cong L/Z(L)$. i.e., for all $x \in L$, there exists $y, z \in L$, such that $x + Z(L) = [y, z] + Z(L)$, so we have $x = [y, z] + c$ with $c \in Z(L)$. We conclude that

$$L = Z(L) + [LL]$$

On the other hand, $Z(L)$ is a $\mathrm{ad}L$-submodule of L and L is a completely reducible $\mathrm{ad}L$-module. So $Z(L)$ has a component M in L.

$$L = M \oplus Z(L)$$

where M is a ideal of L.

$$[LL] \subset [M \oplus Z(L), M \oplus Z(L)] \subset [M, M] \subset M$$

We conclude that

$$L = [LL] \oplus Z(L)$$

Hence $[LL] \cong L/Z(L)$ is semisimple.

(2) If L is a classical linear Lie algebra, by exercise 4.1, $\mathrm{Rad}L = Z(L)$. And by exercise 1.9, $Z(L) = 0$, so $L = [LL]$ is semisimple.

(3)L is a completely reducible $\mathrm{ad}L$-module. Clearly $Z(L)$ is a submodule. So

$$L = Z(L) \oplus M$$

where M is a direct sum of some simple ideal of L. So M is semisimple. $L/Z(L) \cong M$ is semisimple. We conclude that $\mathrm{Rad}(L/Z(L)) = \mathrm{Rad}L/Z(L) = 0$. Hence $\mathrm{Rad}L \subseteq Z(L)$.

On the other hand, $Z(L) \subseteq \mathrm{Rad}L$ is clearly.

We conclude that $\mathrm{Rad}L = Z(L)$, L is reductive.

(4)

6. Let L be a simple Lie algebra. Let $\beta(x, y)$ and $\gamma(x, y)$ be two symmetric associative bilinear forms on L. If β, γ are nondegenerate, prove that β and γ are proportional. [Use Schur's Lemma.]

Solution:

L is a irreducible L-module by ad, and L^* is a L-module, We can define a linear map $\phi : L \to L^*, x \mapsto \beta_x$, where $\beta_x \in L^*$ defined by $\beta_x(y) = \beta(x, y)$. Then we have

$$\begin{aligned} \phi(\mathrm{ad}(x).y)(z) &= \beta_{[x,y]}(z) = \beta([x, y], z) \\ &= -\beta([y, x], z) = -\beta(y, [x, z]) \\ &= -\beta_y(\mathrm{ad}(x).z) = (\mathrm{ad}(x).\beta_y)(z) \\ \therefore \phi(\mathrm{ad}(x).y) &= \mathrm{ad}(x).\phi(y) \end{aligned}$$

From the above formula we know ϕ is a module homomorphism of L-module.

Similarly, we can define a linear map $\psi : L^* \to L, f \to x_f$, where x_f defined by $f(z) = \gamma(x_f, z)$ for all $z \in L$. This x_f exists because γ is non-degenerate. Then we have

$$\begin{aligned} \gamma(x_{\mathrm{ad}(x).f}, z) &= (\mathrm{ad}(x).f)(z) = -f([x, z]) \\ \gamma(\mathrm{ad}(x).x_f, z) &= -\gamma([x_f, x], z) = -\gamma(x_f, [x, z]) = -f([x, z]) \\ \psi(\mathrm{ad}(x).f) &= x_{\mathrm{ad}(x).f} \\ \mathrm{ad}(x).\psi(f) &= [x, x_f] \\ \therefore \psi(\mathrm{ad}(x).f) &= \mathrm{ad}(x).\psi(f) \end{aligned}$$

Hence ψ is also a homomorphism of L-modules. So $\psi \circ \phi$ is a homomorphism from L to L, i.e, $\psi \circ \phi$ is a endomorphism of L which commutative with all $\mathrm{ad}x, x \in L$, and L is a irreducible L-module. By Schur's lemma we have

$$\psi \circ \phi = \lambda I$$

So

$$x_{\beta_x} = \psi(\beta_x) = \lambda x$$

$$\gamma(\lambda x, y) = \gamma(x_{\beta x}, y) = \beta_x(y) = \beta(x, y)$$

i.e.,

$$\beta(x, y) = \lambda\gamma(x, y), \forall x, y \in L$$

7. It will be seen later on that $\mathfrak{sl}(n, F)$ is actually simple. Assuming this and using Exercise 6, prove that the Killing form κ on $\mathfrak{sl}(n, F)$ is related to the ordinary trace form by $\kappa(x, y) = 2nTr(xy)$.

Solution:

Clearly $Tr(xy)$ is a nonzero symmetric associative bilinear form on $\mathfrak{sl}(n, F)$, its radical is a ideal of $\mathfrak{sl}(n, F)$, but $\mathfrak{sl}(n, F)$ is a simple Lie algebra, So $Tr(xy)$ is nondegenerate. By exercise 6.6, $\kappa(x, y) = \lambda Tr(xy)$. We can only compute it for $x = y = e_{11} - e_{22}$. In this case, $Tr(xy) = 2$.

$e_{ii} - e_{i+1,i+1}, 1 \leqslant i \leqslant n - 1$ and $e_{ij}, i, j \neq 1, 2$ are eigenvectors for $\mathrm{ad}(e_{11} - e_{22})$ with eigenvalue 0.

e_{12} is the eigenvector for $\mathrm{ad}(e_{11} - e_{22})$ with eigenvalue 2. e_{21} is the eigenvector for $\mathrm{ad}(e_{11} - e_{22})$ with eigenvalue -2. $e_{1k}, k \neq 1, 2$ and $e_{k2}, k \neq 1, 2$ are eigenvectors for $\mathrm{ad}(e_{11} - e_{22})$ with eigenvalue 1. $e_{k1}, k \neq 1, 2$ and $e_{2k}, k \neq 1, 2$ are eigenvectors for $\mathrm{ad}(e_{11} - e_{22})$ with eigenvalue -1.

So the matrix of $\mathrm{ad}(e_{11} - e_{22})$ relative to the standard basis of $\mathfrak{sl}(n, F)$ is a diagonal matrix

$$\mathrm{diag}(\underbrace{0, \cdots, 0}_{n-1}, 2, -2, \underbrace{1, \cdots, 1}_{2n-4}, \underbrace{-1, \cdots, -1}_{2n-4}, 0, \cdots, 0)$$

Hence $\kappa(x, y) = Tr(\text{ad}x\text{ad}y) = 4 + 4 + 2(2n - 4) = 4n = 2nTr(xy)$.

8. If L is a Lie algebra, then L acts (via ad) on $(L \otimes L)^*$, which may be identified with the space of all bilinear forms β on L. Prove that β is associative if and only if $L.\beta = 0$.

Solution:

By definition,

$$\begin{aligned} z.\beta(x \otimes y) &= -\beta(z.(x \otimes y)) \\ &= -\beta(z.x \otimes y + x \otimes z.y) \\ &= \beta([x, z] \otimes y) - \beta(x \otimes [z, y]) \end{aligned}$$

Hence

$$L.\beta = 0 \Leftrightarrow \beta([x, z] \otimes y) = \beta(x \otimes [z, y]), \forall x, y, z \in L \Leftrightarrow \beta \text{is associative.}$$

9. Let L' be a semisimple subalgebra of a semisimple Lie algebra L. If $x \in L'$, its Jordan decomposition in L' is also its Jordan decomposition in L.

Solution:

The map $\phi : L' \to \mathfrak{gl}(L), x \mapsto \text{ad}_L x$ make L be a L' module. Let $x \in L'$ and $x = x_s + x_n$ is its Jordan decomposition in L'. By Corollary 6.4, $\text{ad}_L x = \text{ad}_L x_s + \text{ad}_L x_n$ is the Jordan decomposition of $\text{ad}_L x$. So $x = x_s + x_n$ is the Jordan decomposition of x in L as the uniqueness of the Jordan decomposition.

7 Representations of $\mathfrak{sl}(2, F)$

In these exercises, $L = \mathfrak{sl}(2, F)$.

1. Use Lie's Theorem to prove the existence of a maximal vector in an arbitrary finite dimensional L-module. [Look at the subalgebra B spanned by h and x.]

Solution:

Let V be an arbitrary finite dimensional L-module. $\phi : L \to \mathfrak{gl}(V)$ is a representation. Let B be the subalgebra of L spanned by h and x. Then $\phi(B)$ is a solvable subalgebra of $\mathfrak{gl}(V)$. And $\phi(x)$ is a nilpotent endomorphism of V. By Lie's theorem, there is a common eigenvector v for B. So $h.v = \lambda v, x.v = 0$, v is a maximal vector.

2. $M = \mathfrak{sl}(3, F)$ contains a copy of L in its upper left-hand 2×2position. Write M as direct sum of irreducible L-submodules (M viewed as L- module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.

Solution:

Let $h = e_{11} - e_{22}, x = e_{12}, y = e_{21}$, M is a L-module.

First, we know $\text{ad}h.e_{12} = 2e_{12}, \text{ad}x.e_{12} = 0$. So e_{12} is a maximal vector with highest weight 2. It can generate a irreducible module isomorphic to $V(2)$. Let $v_0 = e_{12}$, $v_1 = [e_{21}, e_{12}] = -(e_{11} - e_{22}), v_2 = [e_{21}, -(e_{11} - e_{22})] = -e_{21}$. So $V(2) \cong \text{span}\{e_{12}, e_{11} - e_{22}, e_{21}\}$.

$\text{ad}h.e_{13} = e_{13}, \text{ad}x.e_{13} = 0$. So e_{13} is a maximal vector with weight 1. It can generate a irreducible module isomorphic to $V(1)$. $[e_{21}, e_{13}] = e_{23}$. We have $V(1) \cong \text{span}\{e_{13}, e_{23}\}$.

$\text{ad}h.e_{32} = e_{32}, \text{ad}x.e_{32} = 0$. So e_{32} is a maximal vector with weight 1. It can generate a irreducible module isomorphic to $V(1)$. $[e_{21}, e_{32}] = -e_{31}$. We have another $V(1) \cong \text{span}\{e_{31}, e_{32}\}$.

At last, we have a 1-dimensional irreducible submodule of $V(0) \cong \text{span}\{e_{22} - e_{33}\}$. And

$$M = V(0) \oplus V(1) \oplus V(1) \oplus V(2)$$

3. Verify that formulas $(a) - (c)$ of Lemma 7.2 do define an irreducible representation of L. [To show that they define a representation, it suffices to show that the matrices corresponding to x, y, h satisfy the same structural equations as x, y, h.]

Solution:

$$\begin{aligned}
[h,x].v_i &= 2x.v_i = 2(\lambda - i + 1)v_{i-1} \\
h.x.v_i - x.h.v_i &= (\lambda - i + 1)h.v_{i-1} - (\lambda - 2i)x.v_i \\
&= (\lambda - i + 1)(\lambda - 2i + 2)v_{i-1} - (\lambda - 2i)(\lambda - i + 1)v_{i-1} \\
&= 2(\lambda - i + 1)v_{i-1} \\
[h,y].v_i &= -2y.v_i = -2(i+1)v_{i+1} \\
h.y.v_i - y.h.v_i &= (i+1)h.v_{i-1} - (\lambda - 2i)y.v_i \\
&= (i+1)(\lambda - 2i - 2)v_{i+1} - (\lambda - 2i)(i+1)v_{i+1} \\
&= -2(i+1)v_{i+1} \\
[x,y].v_i &= hv_i = (\lambda - 2i)v_i \\
x.y.v_i &= y.x.v_i = (i+1)x.v_{i+1} - (\lambda - i + 1)y.v_{i-1} \\
&= (i+1)(\lambda - i)v_i - (\lambda - i + 1)iv_i \\
&= (\lambda - 2i)v_i
\end{aligned}$$

4. The irreducible representation of L of highest weight m can also be realized "naturally", as follows. Let X, Y be a basis for the two dimensional vector space F^2, on which L acts as usual. Let $\mathscr{R} = F[X, Y]$ be the polynomial algebra in two variables, and extend the action of L to $\mathscr{R}$ by the derivation rule: $z.fg = (z.f)g + f(z.g)$, for $z \in L, f, g \in \mathscr{R}$. Show that this extension is well defined and that $\mathscr{R}$ becomes an L-module. Then show that the subspace of homogeneous polynomials of degree m, with basis $X^m, X^{m-1}Y, \cdots, XY^{m-1}, Y^m$, is invariant under L and irreducible of highest weight m.

Solution:

First we show that the extension is well defined.

If $X^mX^n = X^sX^t$, $m+n = s+t$. Suppose $s < m$.

$$\begin{aligned} z.(X^mX^n) &= (z.X^m)X^n + X^m(z.X^n) \\ &= (z.(X^sX^{m-s}))X^n + X^m(z.X^n) \\ &= (z.X^s)X^t + X^s((z.X^{m-s})X^n + X^{m-s}(z.X^n)) \\ &= (z.X^s)X^t + X^s(z.X^t) \\ &= z.(X^sX^t) \end{aligned}$$

The other cases can be check similarly. So $\mathscr{R}$ is a L-module. And we have:

$$\begin{array}{lll} x.X = 0 & h.X = X & y.X = Y \\ x.Y = X & h.Y = -Y & y.Y = 0 \end{array}$$

By the definition of extension,

$$\begin{array}{lll} x.X^k = 0 & h.X^k = kX^k & y.X^k = kX^{k-1}Y \\ x.Y^k = kXY^{k-1} & h.Y^k = -kY^k & y.Y^k = 0 \end{array}$$

Hence the subspace of homogeneous polynomials of degree m is a L-module. Let

$$v_i = \binom{m}{i} X^{m-i}Y^i, i = 1, \cdots, m$$

Then

$$\begin{aligned} h.v_i &= \binom{m}{i}((h.X^{m-i})Y^i + X^{m-i}(h.Y^i)) \\ &= \binom{m}{i}((m-i)X^{m-i}Y^i - iX^{m-i}Y^i) \\ &= (m-2i)v_i \\ y.v_i &= \binom{m}{i}(y.X^{m-i})Y^i \\ &= \binom{m}{i}(m-i)X^{m-i-1}Y^{i+1} \\ &= (i+1)v_{i+1} \\ x.v_i &= \binom{m}{i}X^{m-i}(x.Y^i) \\ &= \binom{m}{i}iX^{m-i+1}Y^{i-1} \\ &= (m-i+1)v_{i-1} \end{aligned}$$

So the subspace of homogeneous polynomials of degree m isomorphic to $V(m)$. So it is a irreducible L-module of highest weight m.

5. Suppose $charF = p > 0$, $L = \mathfrak{sl}(2, F)$. Prove that the representation $V(m)$ of L constructed as in Exercise 3 or 4 is irreducible so long as the highest weight m is strictly less than p, but reducible when $m = p$.

Solution:

Let U be a submodule of $V(m), U \neq 0$. $0 \neq v \in U$, $v = \sum_{i=0}^{m} \lambda_i v_i$. Let $k = \max\{i | \lambda_i \neq 0\}$, then

$$x^k.v = \lambda_k(m-k+1)(m-k+2)\cdots m v_0 \in U$$

If $m < p$, $\lambda_k(m-k+1)(m-k+2)\cdots m \neq 0$, hence $v_0 \in U$. So $y^k.v_0 = k!v_k \in U$,$v_k \in U$. We conclude that $U = V$.

If $m = p$, $U = \text{span}_F\{v_0, \cdots, v_{m-1}\}$ is a submodule of V. So V is reducible.

6. Decompose the tensor product of the two L-modules $V(3), V(7)$ into the sum of irreducible submodules : $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.

Solution:

Let u_0, u_1, u_2, u_3 is the standard basis of $V(3)$ and $v_0, \cdots, v_7$ is the standard basis of $V(7)$, $M = V(3) \otimes V(7)$ has a basis $u_i \otimes v_j, i = 0, \cdots, 3, j = 0, \cdots, 7$.

$$h.(u_i \otimes v_j) = (h.u_i) \otimes v_j + u_i \otimes (h.v_j) = (10 - 2(i+j))u_i \otimes v_j$$

Hence

$$M_{10-2k} = \text{span}_F\{u_i \otimes v_j, i + j = k\}$$

In M_{10}, $x.(u_0 \otimes v_0) = 0$, so $u_0 \otimes v_0$ is a maximal vector with weight 10. It generates a irreducible submodule of M isomorphic to $V(10)$.

In M_8, $x.(7u_1 \otimes v_0 - 3u_0 \otimes v_1) = 0$, so $7u_1 \otimes v_0 - 3u_0 \otimes v_1$ is a maximal vector with weight 8. It generates a irreducible submodule of M isomorphic to $V(8)$.

In $M(6)$, $x.(7u_2 \otimes v_0 - 2u_1 \otimes v_1 + u_0 \otimes v_2) = 0$, so $7u_2 \otimes v_0 - 2u_1 \otimes v_1 + u_0 \otimes v_2$ is a maximal vector with weight 6. It generates a irreducible submodule of M isomorphic to $V(6)$.

In $M(4)$, $x.(105u_3 \otimes v_0 - 15u_2 \otimes v_1 + 5u_1 \otimes v_2 - 3u_0 \otimes v_3) = 0$, so $105u_3 \otimes v_0 - 15u_2 \otimes v_1 + 5u_1 \otimes v_2 - 3u_0 \otimes v_3$ is a maximal vector with weight 4. It generates a irreducible submodule of M isomorphic to $V(4)$.

Hence $V(4) \oplus V(6) \oplus V(8) \oplus V(10) \subseteq M$, but $\dim M = 4 * 8 = 32, \dim V(4) + \dim V(6) + \dim V(8) + \dim V(10) = 5 + 7 + 9 + 11 = 32$. So $M = V(4) \oplus V(6) \oplus V(8) \oplus V(10)$.

In general, for $M = V(m) \otimes V(n)$. We suppose $m \leqslant n$. $u_i, i = 0, \cdots, m$ is the basis of $V(m)$ and $v_j, j = 1, \cdots, n$ is the basis of $V(n)$.

$$h.(u_i \otimes v_j) = (m + n - 2(i+j))u_i \otimes v_j$$

Hence

$$M_{m+n-2k} = \text{span}\{u_i \otimes v_j, i + j = k\}$$

For $k = 0, \cdots, m$, suppose $w = \sum_{i=0}^{k} \lambda_i u_i \otimes v_{k-i} \in M_{m+n-2k}$ is a maximal vector. Then

$$\begin{aligned} x.w &= \sum_{i=0}^{k} \lambda_i((x.u_i) \otimes v_{k-i} + u_i \otimes (x.v_{k-i})) \\ &= \sum_{i=1}^{k} \lambda_i(m-i+1)u_{i-1} \otimes v_{k-i} + \sum_{i=0}^{k-1} \lambda_i(n-k+i+1)u_i \otimes v_{k-i-1} \\ &= \sum_{i=1}^{k}(\lambda_i(m-i+1) + \lambda_{i-1}(n-k+i))u_{i-1} \otimes v_{k-i} \\ &= 0 \end{aligned}$$

Therefore

$$\lambda_i(m-i+1) + \lambda_{i-1}(n-k+i)$$

We conclude that

$$\lambda_i = (-1)^i \frac{(m-k+i)!(m-i)!}{(m-k)!m!} = (-1)^i \binom{m-k+i}{i} / \binom{m}{i} \lambda_0$$

Let $\lambda_0 = 1$, then $w = \sum_{i=0}^{k} \lambda_i u_i \otimes v_{k-i}$ is a maximal vector with weight $m+n-2k$. It generates a irreducible submodule of M isomorphic to $V(m+n-2k)$.

So $\bigoplus_{k=0}^{m} V(m+n-2k) \subseteq V(m) \otimes V(n)$. Compare the dimensional of two sides.

$$\begin{aligned} & \dim \left(\bigoplus_{k=0}^{m} V(m+n-2k) \right) \\ =& \sum_{k=0}^{m}(m+n-2k+1) \\ =& (m+1)(m+n+1) - m(m+1) \\ =& (m+1)(n+1) \\ =& \dim V(m) \otimes V(n) \end{aligned}$$

So we have

$$V(m) \otimes V(n) = V(n-m) \oplus V(n-m+2) \oplus \cdots \oplus V(m+n)$$

7. In this exercise we construct certain infinite dimensional L-modules. Let $\lambda \in F$ be an arbitrary scalar. Let $Z(\lambda)$ be a vector space over F with countably infinite basis $(v_0, v_1, v_2, \cdots)$.

   1. Prove that formulas (a)-(c) of Lemma 7.2 define an L-module structure on $Z(\lambda)$, and that every nonzero L-submodule of $Z(\lambda)$ contains at least one maximal vector.
   2. Suppose $\lambda + 1 = i$ is a nonnegative integer. Prove that v_i is a maximal vector (e.g., $\lambda = -1, i = 0$). This induces an L-module homomorphism $Z(\mu) \overset{\phi}{\to} Z(\lambda)$, $\mu = \lambda - 2i$, sending v_0 to v_i. Show that ϕ is a monomorphism, and that $\mathrm{Im}\phi, Z(\lambda)/\mathrm{Im}\phi$ are both irreducible L- modules (but $Z(\lambda)$ fails to be completely reducible when $i > 0$).
   3. Suppose $\lambda + 1$ is not a nonnegative integer. Prove that $Z(\lambda)$ is irreducible.

Solution:

(1) We can check $Z(\lambda)$ is a L-module as we do in exercise 2.4.3.

Let $U \subseteq Z(\lambda)$ be an arbitrary nonzero submodule of $Z(\lambda)$, $0 \neq v = \sum_{k=1}^{n} a_{i_k} v_{i_k} \in U$ with all $a_{i_k} \neq 0$. We have

$$h.v = \sum_{k=1}^{n} a_{i_k}(\lambda - 2i_k)v_{i_k} \in U$$

This implies all $v_{i_k} \in U$. So

$$U = \bigoplus_{j\in J} \mathbb{C}v_j, J \subseteq \mathbb{N}$$

Let $k = \min J$, then $v_k \in U$, and $v_{k-1} \notin U$, so $x.v_k = (\lambda - k + 1)v_{k-1} = 0$. We conclude that v_k is a maximal vector in U.

(2) If $\lambda + 1 = i$ is a nonnegative integer. $x.v_i = (\lambda - i + 1)v_{i-1} = 0$, so v_i is a maximal vector.

Next we show $v_0 \to v_i$ induces a homomorphism $\phi : Z(\mu) \to Z(\lambda), v_k \mapsto \binom{k+i}{i} v_{k+i}$.

$$\begin{aligned}
\phi(h.v_k) &= (\mu - 2k)\phi(v_k) = (\mu - 2k)\binom{k+i}{i} v_{k+i} \\
h.\phi(v_k) &= \binom{k+i}{i} h.v_{k+i} = (\lambda - 2k - 2i)\binom{k+i}{i} v_{k+i} \\
&= (\mu - 2k)\binom{k+i}{i} v_{k+i} \\
\phi(x.v_k) &= (\mu - k + 1)\phi(x_{k-1}) = (\mu - k + 1)\binom{k+i-1}{i} v_{k+i-1} \\
&= -(k+i)\binom{k+i-1}{i} v_{k+i-1} = -k\binom{k+i}{i} v_{k+i-1} \\
x.\phi(v_k) &= \binom{k+i}{i} x.v_{k+i} = (\lambda - k - i + 1)\binom{k+i}{i} v_{k+i-1} \\
&= -k\binom{k+i}{i} v_{k+i-1}
\end{aligned}$$

$$\begin{aligned}
\phi(y.v_k) &= (k+1)\phi(v_{k+1}) = (k+1)\binom{k+i+1}{i} v_{k+i+1} \\
&= (k+i+1)\binom{k+i}{i} v_{k+i+1} \\
y.\phi(v_k) &= \binom{k+i}{i} y.v_{k+i} = (k+i+1)\binom{k+i}{i} v_{k+i+1}
\end{aligned}$$

Clear ϕ is a monomorphism. $\mathrm{Im}\phi \cong Z(\mu)$ is a submodule of $Z(\lambda)$ and by (1) it has a maximal vector of form v_s. But

$$x.v_s = (\mu - s + 1) = -(i+s)v_{s-1} = 0$$

From $i + s > 0$, we have $v_{s-1} = 0$. So v_0 is the unique maximal vector in $Z(\mu)$ and $Z(\mu)$ is irreducible.

$Z(\lambda)/\mathrm{Im}\phi \cong V(i-1)$ is a irreducible module.

Next we show $Z(\lambda)$ is not completely reducible. If $Z(\lambda)$ is completely reducible, U is a proper nonzero submodule of $Z(\lambda)$, then $Z(\lambda) = U \oplus W$. By (1) U has a maximal vector $v_s, s \geqslant 1$, W has a maximal vector $v_t, t \geqslant 1$.

$$x.v_s = (\lambda - s + 1)v_{s-1} = 0, x.v_t = (\lambda - t + 1)v_{t-1} = 0$$

Hence $s = t = \lambda + 1$. This contradict with $U \cap W = 0$.

(3) If $Z(\lambda)$ reducible, it has a proper nonzero submodule U. By (1) U has a maximal vector v_k with $k > 0$.

$$x.v_k = (\lambda - k + 1)v_{k-1} = 0$$

Hence $\lambda + 1 = k$ is a positive integer. We get a contradiction.

8 Root Space Decomposition

1. If L is a classical linear Lie algebra of type A_l, B_l, C_l, or D_l (see (1.2)), prove that the set of all diagonal matrices in L is a maximal toral subalgebra, of dimension l (Cf. Exercise 2.8.)

Solution:

The set of all diagonal matrices in L is a toral subalgebra. It is enough to show it is maximal.

Let $\mathfrak{h}$ be the maximal toral subalgebra contains all diagonal matrices in L. We know that $\mathfrak{h}$ is abelian.

If $a = (a_{ij}) \in \mathfrak{h}$, we claim there is a matrix of form $h = \operatorname{diag}(a_1, \cdots, a_n)$ with $a_i \neq a_j, i \neq j$ in $\mathfrak{h}$. Hence $ah = ha$ implies a is a diagonal matrix.

$$\begin{aligned}
A_l: &\qquad h = \operatorname{diag}(1, \cdots, l, -\frac{l(l+1)}{2}) \\
B_l: &\qquad h = \operatorname{diag}(0, 1, \cdots, l, -1, \cdots, -l) \\
C_l: &\qquad h = \operatorname{diag}(1, \cdots, l, -1, \cdots, -l) \\
D_l: &\qquad h = \operatorname{diag}(1, \cdots, l, -1, \cdots, -l)
\end{aligned}$$

2. For each algebra in Exercise 1, determine the roots and root spaces. How are the various h_α expressed in terms of the basis for H given in (1.2)?

Solution:

3. If L is of classical type, compute explicitly the restriction of the Killing form to the maximal toral subalgebra described in Exercise 1.

Solution:

4. If $L = \mathfrak{sl}(2, F)$, prove that each maximal toral subalgebra is one dimensional.

Solution:

$\mathfrak{h}$ is a maximal toral subalgebra of L, $L = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} L_\alpha$, $\dim L_\alpha = 1$. $\alpha \in \Phi$ then $-\alpha \in \Phi$. This implies $\operatorname{Card}(\Phi)$ is even and nonzero. So $\dim \mathfrak{h} = 1$.

5. If L is semisimple, H a maximal toral subalgebra, prove that H is self-normalizing (i.e., $H = N_L(H)$).

Solution:
L is semisimple and H is a maximal toral subalgebra. $L = H \oplus \sum_{\alpha\in\Phi} L_\alpha$. $x \in N_L(H), x = h_0 + \sum_{\alpha\in\Phi} x_\alpha, x_\alpha \in L_\alpha$. Let $h \in H$ such that $\alpha(h) \neq 0, \forall \alpha \in \Phi$.

$$[h, x] = \sum_{\alpha\in\Phi} \alpha(h) x_\alpha \in H$$

Hence $x_\alpha = 0, \forall \alpha \in \Phi$. $x = h_0 \in H$. i.e, $N_L(H) = H$.

6. Compute the basis of $\mathfrak{sl}(n, F)$ which is dual (via the Killing form) to the standard basis. (Cf. Exercise 5.5.)

Solution:

7. Let L be semisimple, H a maximal toral subalgebra. If $h \in H$, prove that $C_L(h)$ is reductive (in the sense of Exercise 6.5). Prove that H contains elements h for which $C_L(h) = H$; for which h in $\mathfrak{sl}(n, F)$ is this true ?

Solution:
L is semisimple. We have a decomposition $L = H \dot{+} \sum_{\alpha\in\Phi} L_\alpha$.

$$\begin{aligned} & x = h_0 + \sum_{\alpha\in\Phi} x_\alpha \in C_L(h) \\ \Leftrightarrow \quad & [h, x] = \sum_{\alpha\in\Phi} \alpha(h) x_\alpha = 0 \\ \Leftrightarrow \quad & \alpha(h) = 0 \quad \text{or} \quad x_\alpha = 0 \end{aligned}$$

Hence

$$C_L(h) = H \dot{+} \sum_{\substack{\alpha\in\Phi \\ \alpha(h)=0}} L_\alpha$$

Denote $\Phi_h = \{\alpha \in \Phi | \alpha(h) = 0\}$. Now we claim that

$$Z(C_L(h)) = \{h' \in H | \alpha(h') = 0, \forall \alpha \in \Phi_h\}$$

Let $x = h_0 + \sum_{\alpha\in\Phi_h} x_\alpha \in Z(C_L(h))$. We can find a $h' \in H$ such that $\alpha(h') \neq 0, \forall \alpha \in \Phi_h$. Then $[h, x] = \sum_{\alpha\in\Phi_h} \alpha(h') x_\alpha = 0$. It implies $x_\alpha = 0$. We have $x = h_0 \in H$. Next we take $0 \neq x_\alpha \in L_\alpha, \forall \alpha \in \Phi_h$, then $[x, x_\alpha] = \alpha(h_0) x_\alpha = 0$. Hence $\alpha(x) = \alpha(h_0) = 0, \forall \alpha \in \Phi_h$.
Next we show $Z(C_L(h)) = \mathrm{Rad}(C_L(h))$. Clearly $Z(C_L(h))$ is a solvable ideal of $C_L(h)$, it is enough to show it is a maximal solvable ideal.
If $x = h_0 + \sum_{\alpha\in\Phi_h} x_\alpha \in \mathrm{Rad}(C_L(h)) \backslash Z(C_L(h))$. We have a $h' \in H$ such that $\alpha(h') \neq 0$ and $\alpha(h') \neq \beta(h'), \forall \alpha \neq \beta \in \Phi_h$. Then $[h', x] = \sum_{\alpha\in\Phi_h} \alpha(h') x_\alpha \in \mathrm{Rad}(C_L(h))$. Hence $h_0, x_\alpha \in \mathrm{Rad}(C_L(h)), \alpha \in \Phi_h$.
If there is a $\alpha \in \Phi_h$ such that $x_\alpha \neq= 0$, then $h_\alpha = [x_\alpha, y_\alpha] \in \mathrm{Rad}(C_L(h))$, $2y_\alpha = -[h_\alpha, y_\alpha] \in \mathrm{Rad}(C_L(h))$. Hence $\mathfrak{sl}(2, F) \cong S_\alpha \subseteq \mathrm{Rad}(C_L(h))$ which contradict with the solvability of $\mathrm{Rad}(C_L(h))$.

Now we get $x = h_0 \in \mathrm{Rad}(C_L(h)) \backslash Z(C_L(h))$. So there is a $\alpha \in \Phi_h$ such that $\alpha(h_0) \neq 0$. Then $[h_0, x_\alpha] = \alpha(h_0)x_\alpha \in \mathrm{Rad}(C_L(h)), [h_0, y_\alpha] = -\alpha(h_0)y_\alpha \in \mathrm{Rad}(C_L(h))$. We also have $S_\alpha \subseteq \mathrm{Rad}(C_L(h))$ which contradict with the solvability of $\mathrm{Rad}(C_L(h))$.

All of the above show that $Z(C_L(h)) = \mathrm{Rad}(C_L(h))$. i.e., $C_L(h)$ is reductive. We know there is a $h \in H$, $\alpha(h) \neq 0, \forall \alpha \in \Phi$. In this case, $C_L(h) = H$.

8. For $\mathfrak{sl}(n, F)$ (and other classical algebras), calculate explicitly the root strings and Cartan integers. In particular, prove that all Cartan integers $2(\alpha, \beta)/(\beta, \beta), \alpha \neq \pm\beta$, for $\mathfrak{sl}(n, F)$ are $0, \pm 1$.

Solution:

9. Prove that every three dimensional semisimple Lie algebra has the same root system as $\mathfrak{sl}(2, F)$, hence is isomorphic to $\mathfrak{sl}(2, F)$.

Solution:

Let L be a three dimensional semisimple Lie algebra. Then L has a maximal toral subalgebra H.

$$L = H \dot{+} \sum_{\phi \in \Phi} L_\alpha$$

Since $\alpha \in \Phi$ implies $-\alpha \in \Phi$ and $\dim L_\alpha = 1, \forall \alpha \in \Phi$. Hence $\sum\limits_{\alpha \in \Phi} L_\alpha$ has even dimensional. But L is semisimple. We have $\dim H = 1$, $\Phi = \{\alpha, -\alpha\}$.

Hence there is a subalgebra $\mathfrak{sl}(2, F) \subseteq S_\alpha \subseteq L$ with $\dim S_\alpha = \dim L = 3$. So we have

$$L \cong \mathfrak{sl}(2, F)$$

10. Prove that no four, five or seven dimensional semisimple Lie algebras exist.

Solution:

Let L is a semisimple Lie algebra with a maximal toral subalgebra H. $L = H \dot{+} \sum \alpha \in \Phi L_\alpha$. Since $\alpha \in \Phi$ implies $-\alpha \in \Phi$, $\sum\limits_{\alpha \in \Phi} L_\alpha$ has dimensional $2k$ with $k \geqslant 1$. And $\Phi = \{\pm\alpha 1, \cdots, \pm\alpha_k\}$ which span a space of dimension at most k.

$$\therefore \dim H = \dim L - \dim(\sum_{\alpha \in \Phi} L_\alpha) = \dim L - 2k$$

In the other hands, Φ span H^*.

$$\dim H = \dim H^* \leqslant k$$

We conclude

$$\frac{\dim L}{3} \leqslant k < \frac{\dim L}{2} \qquad (*)$$

If $\dim L = 4$, we can not find a integer k satisfying $(*)$.

If $\dim L = 5$, $k = 2$. Then $\dim H = 1$,i.e, Φ spans a 1-dimensional space. $\alpha_2 = m\alpha_1$ with $m = \pm 1$. We get a contradiction.

If $\dim L = 7$. $k = 3$. Then $\dim H = 1$. We can deduce a contradiction as the case $\dim L = 5$.

Hence, there is no four,five or seven dimensional semisimple Lie algebra.

11. If $(\alpha, \beta) > 0$, and $\alpha \neq \pm\beta$, prove that $\alpha - \beta \in \Phi(\alpha, \beta \in \Phi)$. Is the converse true?

Solution:

We have $\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Phi$ since $\alpha, \beta \in \Phi$.

Let the β string through α is $\alpha - r\beta, \cdots, \alpha, \cdots, \alpha + q\beta$. We have $r > 0$ since $(\alpha, \beta) > 0$. Hence $\alpha - \beta$ appeals in the string. It is a root.

9 Axiomatics

Unless otherwise specified, Φ denotes a root system in E, with Weyl group $\mathcal{W}$.

1. Let E' be a subspace of E. If a reflection σ_α leaves E' invariant, prove that either $\alpha \in E'$ or else $E' \subset P_\alpha$.

Solution:

If $E' \nsubseteq P_\alpha$. Let $\lambda \in E' \backslash P_\alpha$, $\sigma_\alpha(\lambda) = \lambda - < \lambda, \alpha > \alpha \in E'$. Since $\lambda \notin P_\alpha$, $< \lambda, \alpha > \neq 0$. Hence $\alpha \in E'$.

2. Prove that $\Phi^\vee$ is a root system in E, whose Weyl group is naturally isomorphic to $\mathcal{W}$; show also that $< \alpha^\vee, \beta^\vee > = < \beta, \alpha >$, and draw a picture of $\Phi^\vee$ in the cases A_1, A_2, B_2, G_2.

Solution:

$(R1)$ and $(R2)$ are clearly. For $(R4)$, we have

$$< \beta^\vee, \alpha^\vee > = \frac{2(\beta^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)} = \frac{2(\beta, \alpha)}{(\beta, \beta)} = < \alpha, \beta >$$

Then we check $(R3)$:

$$\begin{aligned} \sigma_{\alpha^\vee}(\beta^\vee) &= \beta^\vee - < \beta^\vee, \alpha^\vee > \alpha^\vee \\ &= \frac{2\beta}{(\beta, \beta)} - < \alpha, \beta > \frac{2\alpha}{(\alpha, \alpha)} \\ &= \frac{2}{(\beta, \beta)} (\beta - < \beta, \alpha > \alpha) \\ &= \frac{2\sigma_\alpha(\beta)}{(\sigma_\alpha(\beta), \sigma_\alpha(\beta))} \\ &= (\sigma_\alpha(\beta))^\vee \end{aligned}$$

$\sigma_{\alpha^\vee}$ and σ_α both leaves P_α pointwise fixed and send α to $-\alpha$. So they are the same linear transformation of E. The Weyl group is naturally isomorphic.

3. In Table 1, show that the order of $\sigma_\alpha\sigma_\beta$ in $\mathcal{W}$ is (respectively) 2,3,4,6 when $\theta = \pi/2$, $\pi/3$ (or $2\pi/3$), $\pi/4$ (or $3\pi/4$), $\pi/6$(or $5\pi/6$). [Note that $\sigma_\alpha\sigma_\beta$ = rotation through 2θ.]

Solution: Let $\alpha, \beta \in \Phi$, then $\sigma_\alpha\sigma_\beta$ fixes pointwise $P_\alpha \cap P_\beta$. So $\sigma_\alpha\sigma_\beta$ determined by their restriction on the subspace spanned by α, β. We take a standard orthogonal basis of it.

$$\begin{aligned}
\alpha_1 &= \frac{\alpha}{\|\alpha\|} \\
\beta' &= \beta - \frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \frac{1}{2} < \beta, \alpha > \alpha \\
\|\beta'\|^2 &= (\beta,\beta) - \frac{(\beta,\alpha)^2}{(\alpha,\alpha)} = \|\beta\|^2(1-\cos^2\theta) = \|\beta\|^2\sin^2\theta \\
\beta_1 &= \frac{1}{\|\beta\|\sin\theta}\left(\beta - \frac{1}{2} < \beta, \alpha > \alpha\right)
\end{aligned}$$

$\{\alpha_1, \beta_1\}$ is a standard orthogonal basis of the subspace spanned by α, β. And we have

$$\begin{aligned}
\sigma_\alpha\sigma_\beta(\alpha_1) &= \frac{1}{\|\alpha\|}\sigma_\alpha\sigma_\beta(\alpha) \\
&= \frac{1}{\|\alpha\|}\sigma_\alpha(\alpha - < \alpha, \beta > \beta) \\
&= \frac{1}{\|\alpha\|}(-\alpha - < \alpha, \beta > \beta + < \alpha, \beta >< \beta, \alpha > \alpha) \\
&= \frac{1}{\|\alpha\|}(-\alpha - < \alpha, \beta > \beta' + \frac{1}{2} < \alpha, \beta >< \beta, \alpha > \alpha) \\
&= \frac{1}{\|\alpha\|}(\alpha\cos 2\theta - 2\beta_1\|\alpha\|\cos\theta\sin\theta) \\
&= \alpha_1\cos 2\theta - \beta_1\sin 2\theta
\end{aligned}$$

$$\begin{aligned}
\sigma_\alpha\sigma_\beta(\beta_1) &= \frac{1}{\|\beta\|\sin\theta}\sigma_\alpha\sigma_\beta(\beta - \frac{1}{2} < \beta, \alpha > \alpha) \\
&= \frac{1}{\|\beta\|\sin\theta}\sigma_\alpha(-\beta - \frac{1}{2} < \beta, \alpha > \alpha + \frac{1}{2} < \beta, \alpha >< \alpha, \beta > \beta) \\
&= \frac{1}{\|\beta\|\sin\theta}\sigma_\alpha(\beta\cos 2\theta - \frac{1}{2} < \beta, \alpha > \alpha) \\
&= \frac{1}{\|\beta\|\sin\theta}(\beta\cos 2\theta - < \beta, \alpha > \alpha\cos 2\theta + \frac{1}{2} < \beta, \alpha > \alpha) \\
&= \frac{1}{\|\beta\|\sin\theta}(\beta'\cos 2\theta + \frac{1}{2}\cos 2\theta < \beta, \alpha > \alpha - \\
&\quad < \beta, \alpha > \alpha\cos 2\theta + \frac{1}{2} < \beta, \alpha > \alpha) \\
&= \frac{1}{\|\beta\|\sin\theta}(\beta'\cos 2\theta + \alpha < \beta, \alpha > \sin^2\theta) \\
&= \beta_1\cos 2\theta + \alpha_1\sin 2\theta
\end{aligned}$$

Hence we have $\sigma_\alpha\sigma_\beta$ is a rotation through 2θ. So the order of $\sigma_\alpha\sigma_\beta$ in $\mathcal{W}$ is respectively 2,3,4,6.

4. Prove that the respective Weyl groups of $A_1 \times A_1, A_2, B_2, G_2$ are dihedral of order 4,6,8,12. If Φ is any root system of rank 2, prove that its Weyl group must be one of these.

Solution:

5. Show by example that $\alpha - \beta$ may be a root even when $(\alpha, \beta) \leqslant 0$(cf. Lemma 9.4).

Solution:
In G_2, $\alpha, \beta + \alpha \in \Phi, < \beta + \alpha, \alpha >= -3 + 2 = -1 < 0$, but $\beta + \alpha - \alpha = \beta \in \Phi$.

6. Prove that $\mathcal{W}$ is a normal subgroup of $Aut\Phi$(=group of all isomorphisms of Φ onto itself).

Solution:
If $\alpha \in \Phi, \phi \in Aut\Phi$,

$$\begin{aligned} \phi\sigma_\alpha\phi^{-1}(\phi(\beta)) &= \phi\sigma_\alpha(\beta) \\ &= \phi(\beta) - <\beta, \alpha> \phi(\alpha) \\ &= \phi(\beta) - <\phi(\beta), \phi(\alpha)> \phi(\alpha) \\ &= \sigma_{\phi(\alpha)}(\phi(\beta)) \end{aligned}$$

Hence $\phi\sigma_\alpha\phi^{-1} \in \mathcal{W}$. However, any elements of $\mathcal{W}$ can be written as $\sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ with $\alpha_i \in \Phi$. So

$$\phi\sigma_{\alpha_1} \cdots \sigma_{\alpha_t}\phi^{-1} = \phi\sigma_{\alpha_1}\phi^{-1} \cdots \phi\sigma_{\alpha_t}\phi^{-1} = \sigma_{\phi(\alpha_1)} \cdots \sigma_{\phi(\alpha_t)} \in \mathcal{W}$$

i.e., $\mathcal{W}$ is a normal subgroup of $Aut\Phi$.

7. Let $\alpha, \beta \in \Phi$ span a subspace E' of E. Prove that $E' \cap \Phi$ is a root system in E'. Prove similarly that $\Phi \cap (Z\alpha + Z\beta)$ is a root system in E' (must this coincide with $E' \cap \Phi$?). More generally, let Φ' be a nonempty subset of Φ such that $\Phi' = -\Phi'$, and such that $\alpha, \beta \in \Phi', \alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Phi'$. Prove that Φ' is a root system in the subspace of E it spans. [Use Table 1].

Solution:

8. Compute root strings in G_2 to verify the relation $r - q =< \beta, \alpha >$.

Solution:

9. Let Φ be a set of vectors in a euclidean space E, satisfying only (R1), (R3), (R4). Prove that the only possible multiples of $\alpha \in \Phi$ which can be in Φ are $\pm 1/2\alpha, \pm\alpha, \pm 2\alpha$. Verify that $\{\alpha \in \Phi | 2\alpha \notin \Phi\}$ is a root system.

Solution:

10. Let $\alpha, \beta \in \Phi$. Let the α-string through β be $\beta - r\alpha, \cdots, \beta + q\alpha$, and let the β-string through α be $\alpha - r'\beta, \cdots, \alpha + q'\beta$. Prove that $\frac{q(r+1)}{(\beta,\beta)} = \frac{q'(r'+1)}{(\alpha,\alpha)}$.

Solution:

$$\text{ad}e_{-\alpha}\text{ad}e_{\alpha}e_{\beta} = \frac{1}{2}(r+1)q(\alpha,\alpha)e_{\beta}$$

$$\text{ad}e_{-\beta}\text{ad}e_{\beta}e_{\alpha} = \frac{1}{2}(r'+1)q'(\beta,\beta)e_{\alpha}$$

$$\begin{aligned}(\text{ad}e_{-\alpha}\text{ad}e_{\alpha}e_{\beta}, e_{-\beta}) &= ([e_{\alpha}, e_{\beta}], [e_{-\alpha}, e_{-\beta}]) \\ (\text{ad}e_{-\beta}\text{ad}e_{\beta}e_{\alpha}, e_{-\alpha}) &= ([e_{\beta}, e_{\alpha}], [e_{-\beta}, e_{-\alpha}])\end{aligned}$$

Hence,

$$\frac{1}{2}(r+1)q(\alpha,\alpha) = (\text{ad}e_{-\alpha}\text{ad}e_{\alpha}e_{\beta}, e_{-\beta}) = (\text{ad}e_{-\beta}\text{ad}e_{\beta}e_{\alpha}, e_{-\alpha}) = \frac{1}{2}(r'+1)q'(\beta,\beta)$$

i.e.

$$\frac{q(r+1)}{(\beta,\beta)} = \frac{q'(r'+1)}{(\alpha,\alpha)}$$

11. Let c be a positive real number. If Φ possesses any roots of squared length c, prove that the set of all such roots is a root system in the subspace of E it spans. Describe the possibilities occurring in Figure 1.

Solution:

10 Simple Roots and Weyl Group

1. Let $\Phi^\vee$ be the dual system of Φ, $\Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\}$. Prove that $\Delta^\vee$ is a base of $\Phi^\vee$.[Compare Weyl chambers of Φ and $\Phi^\vee$.]

Solution: Since $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, the hyper planes $P_{\alpha^\vee}$ and P_α in E which orthogonal to $\alpha^\vee$ and α, respectively, are coincide. We conclude that γ is a regular element with respect to the root system Φ if and only if γ is a regular element with respect to the root system $\Phi^\vee$.

Let γ be a regular element with respect to the root system Φ (so a regular element for $\Phi^\vee$). For any $\alpha \in \Phi$,

$$(\gamma,\alpha)(\gamma,\alpha^\vee) = \frac{2}{(\alpha,\alpha)}(\gamma,\alpha)^2 > 0,$$

i.e., (γ,α) and $(\gamma,\alpha^\vee)$ either positive simultaneously or negative simultaneously. Hence, $\alpha \in \Phi^+(\gamma)$ if and only if $\alpha^\vee \in {\Phi^\vee}^+(\gamma)$.

Now let γ be a regular element with respect to Φ such that $\Delta = \Delta(\gamma)$. From Theorem 10.3, we deduce that every $\alpha \in \Phi^+(\gamma)$ can be written as

$$\alpha = \sigma(\beta),$$

where $\beta \in \Delta$ and $\sigma \in W$, the Weyl group of Φ. Then

$$\alpha^\vee = \sigma(\beta)^\vee = \frac{2\sigma(\beta)}{(\sigma(\beta),\sigma(\beta))} = \frac{2}{(\beta,\beta)}\sigma(\beta) = \sigma(\beta^\vee).$$

Let $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$. We claim that, for any $\sigma \in W$, $\sigma(\alpha_i^\vee)$ is a linear combination of $\alpha_1^\vee, \cdots, \alpha_\ell^\vee$ with integer coefficients. Note that we may write $\sigma = \sigma_{\alpha_{i_1}} \cdots \sigma_{\alpha_{i_t}}$. It is enough to show $\sigma_{\alpha_j}(\alpha_i^\vee)$ is a linear combination of $\alpha_1^\vee, \cdots, \alpha_\ell^\vee$ with integer coefficients.

$$\begin{aligned}
\sigma_{\alpha_j}(\alpha_i^\vee) &= \alpha_i^\vee - \langle \alpha_i^\vee, \alpha_j \rangle \alpha_j \\
&= \alpha_i^\vee - \frac{2(\alpha_i^\vee, \alpha_j)}{(\alpha_j, \alpha_j)} \alpha_j \\
&= \alpha_i^\vee - \frac{4(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} \alpha_j \\
&= \alpha_i^\vee - \frac{4(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \alpha_j^\vee \\
&= \alpha_i^\vee - \langle \alpha_j, \alpha_i \rangle \alpha_j^\vee.
\end{aligned}$$

Note that $\langle \alpha_j, \alpha_i \rangle$ is an integer, we proved the claim.

Hence, for $\alpha \in \Phi$,

$$\alpha^\vee = \sum_{i=1}^{\ell} k_i \alpha_i^\vee, k_i \in \mathbb{Z}.$$

Next we show either all k_i's are nonnegative or all k_i's are nonpositive.

In fact, we may write $\alpha = \sum\limits_{i=1}^{\ell} k_i' \alpha_i$, where either all k_i's are nonnegative integers or all k_i's are nonpositive integers. Then

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)} \alpha = \frac{2}{(\alpha, \alpha)} \sum_{i=1}^{\ell} k_i' \alpha_i = \sum_{i=1}^{\ell} \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} k_i' \alpha_i^\vee.$$

Note that $\{\alpha_1^\vee, \cdots, \alpha_\ell^\vee\}$ is linear independent, we obtain that

$$k_i = \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} k_i'.$$

Since $\frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} > 0$ for all $i = 1, \cdots, \ell$ and $\alpha \in \Phi$, k_i and k_i' have the same sign. Hence, either all k_i's are nonnegative or all k_i's are nonpositive.

2. If Δ is a base of Φ, prove that the set $(Z\alpha + Z\beta) \cap \Phi(\alpha \neq \beta \text{in} \Delta)$ is a root system of rank 2 in the subspace of E spanned by α, β (cf. Exercise 9.7). Generalize to an arbitrary subset of Δ.

Solution:

3. Prove that each root system of rank 2 is isomorphic to one of those listed in (9.3).

Solution:

4. Verify the Corollary of Lemma 10.2A directly for G_2.

Solution:

5. If $\sigma \in \mathcal{W}$ can be written as a product of t simple reflections, prove that t has the same parity as $l(\sigma)$.

Solution:

6. Define a function $sn : \mathcal{W} \to \{\pm 1\}$ by $sn(\sigma) = (-1)^{l(\sigma)}$. Prove that sn is a homomorphism (cf. the case A_2, where $\mathcal{W}$ is isomorphic to the symmetric group $\mathcal{S}_3$).

Solution:

7. Prove that the intersection of "positive" open half-spaces associated with any basis $\gamma_1, \cdots, \gamma_l$ of E is nonvoid. [If δ_i is the projection of γ_i on the orthogonal complement of the subspace spanned by all basis vectors except γ_i, consider $\gamma = \sum r_i \delta_i$ when all $r_i > 0$.]

Solution:

8. Let Δ be a base of Φ, $\alpha \neq \beta$ simple roots, $\Phi_{\alpha\beta}$ the rank 2 root system in $E_{\alpha\beta} = R\alpha + R\beta$(see Exercise 2 above). The Weyl group $\mathcal{W}_{\alpha\beta}$ of $\Phi_{\alpha\beta}$ is generated by the restrictions τ_α, τ_β to $E_{\alpha\beta}$ of $\sigma_\alpha, \sigma_\beta$, and $\mathcal{W}_{\alpha\beta}$ may be viewed as a subgroup of $\mathcal{W}$. Prove that the "length" of an element of $\mathcal{W}_{\alpha\beta}$(relative to τ_α, τ_β) coincides with the length of the corresponding element of $\mathcal{W}$.

Solution:

9. Prove that there is a unique element σ in $\mathcal{W}$ sending Φ^+ to Φ^- (relative to Δ). Prove that any reduced expression for σ must involve all $\sigma_\alpha(\alpha \in \Delta)$. Discuss $l(\sigma)$.

Solution:

10. Given $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ in Φ, let $\lambda = \sum_{i=1}^{l} k_i\alpha_i (k_i \in Z, \text{all} k_i \geqslant 0 \text{or all} k_i \leqslant 0)$. Prove that either λ is a multiple (possibly 0) of a root, or else there exists $\sigma \in \mathcal{W}$ such that $\sigma\lambda = \sum_{i=1}^{l} k_i'\alpha_i$, with some $k_i' > 0$ and some $k_i' < 0$. [Sketch of proof: If λ is not a multiple of any root, then the hyperplane P_λ orthogonal to λ is not included in $\bigcup_{\alpha\in\Phi} P_\alpha$. Take $\mu \in P_\lambda - \bigcup_{\alpha\in\Phi} P_\alpha$. Then find $\sigma \in \mathcal{W}$ for which all $(\alpha, \sigma\mu) > 0$. It follows that $0 = (\lambda, \mu) = (\sigma\lambda, \sigma\mu) = \sum k_i(\alpha_i, \sigma\mu)$.]

Solution:

11. Let Φ be irreducible. Prove that $\Phi^\vee$ is also irreducible. If Φ has all roots of equal length, so does $\Phi^\vee$ (and then $\Phi^\vee$ is isomorphic to Φ). On the other hand, if Φ has two root lengths, then so does $\Phi^\vee$; but if α is long, then $\alpha^\vee$ is short (and vice versa). Use this fact to prove that Φ has a unique maximal short root (relative to the partial order $\prec$ defined by Δ).

Solution:

12. Let $\lambda \in \mathfrak{C}(\Delta)$. If $\sigma\lambda = \lambda$ for some $\sigma \in \mathcal{W}$, then $\sigma = 1$.

Solution:

13. The only reflections in $\mathcal{W}$ are those of the form $\sigma_\alpha (\alpha \in \Phi)$. [A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in $\mathcal{W}$.]

Solution:

14. Prove that each point of E is $\mathcal{W}$-conjugate to a point in the closure of the fundamental Weyl chamber relative to a base Δ. [Enlarge the partial order on E by defining $\mu \prec \lambda$ iff $\lambda - \mu$ is a nonnegative R-linear combination of simple roots. If $\mu \in E$, choose $\sigma \in \mathcal{W}$ for which $\lambda = \sigma\mu$ is maximal in this partial order.]

11 Classification

1. Verify the Cartan matrices (Table 1).
2. Calculate the determinants of the Cartan matrices (using induction on l for types $A_l - D_l$), which are as follows:

$$A_l : l+1; B_l : 2; C_l : 2; D_l : 4; E_6 : 3; E_7 : 2; E_8, F_4 \text{and} G_2 : 1$$

3. Use the algorithm of (11.1) to write down all roots for G_2. Do the same for $C_3 : \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.
4. Prove that the Weyl group of a root system Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.
5. Prove that each irreducible root system is isomorphic to its dual, except that B_l, C_l are dual to each other.
6. Prove that an inclusion of one Dynkin diagram in another (e.g., E_6 in E_7 or E_7 in E_8) induces an inclusion of the corresponding root systems.

12 Construction of Root Systems and Automorphisms

1. Verify the details of the constructions in (12.1).
2. Verify Table 2.

Type	Long	Short
A_l	$\alpha_1 + \alpha_2 + \cdots + \alpha_l$	
B_l	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_l$	$\alpha_1 + \alpha_2 + \cdots + \alpha_l$
C_l	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$
D_l	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$	
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$3\alpha_1 + 2\alpha_2$	$2\alpha_1 + \alpha_2$

3. Let $\Phi \subset E$ satisfy (R1), (R3), (R4), but not (R2), cf. Exercise 9.9. Suppose moreover that Φ is irreducible, in the sense of §11. Prove that Φ is the union of root systems of type B_n, C_n in $E(n = \dim E)$, where the long roots of B_n are also the short roots of C_n. (This is called the non-reduced root system of type BC_n in the literature.)

4. Prove that the long roots in G_2 form a root system in E of type A_2.

5. In constructing C_l, would it be correct to characterize Φ as the set of all vectors in I of squared length 2 or 4? Explain.

6. Prove that the map $\alpha \mapsto -\alpha$ is an automorphism of Φ. Try to decide for which irreducible Φ this belongs to the Weyl group.

7. Describe $Aut\Phi$ when Φ is not irreducible.

13 Abstract Theory of Weights

1. Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ be the decomposition of Φ into its irreducible components, with $\Delta = \Delta_1 \cup \cdots \cup \Delta_t$. Prove that Λ decomposes into a direct sum $\Lambda_1 \oplus \cdots \oplus \Lambda_t$; what about Λ^+?

2. Show by example (e.g., for A_2) that $\lambda \notin \Lambda^+, \alpha \in \Delta, \lambda - \alpha \in \Lambda^+$ is possible.

3. Verify some of the data in Table 1, e.g., for F_4.

4. Using Table 1, show that the fundamental group of A_l is cyclic of order $l+1$, while that of D_l is isomorphic to $Z/4Z$ (l odd), or $Z/2Z \times Z/2Z$ (l even). (It is easy to remember which is which, since $A_3 = D_3$.)

5. If Λ' is any subgroup of Λ which includes Λ_r, prove that Λ' is $\mathcal{W}$-invariant. Therefore, we obtain a homomorphism $\phi : Aut\Phi/\mathcal{W} \to Aut(\Lambda/\Lambda_r)$. Prove that ϕ is injective, then deduce that $-1 \in \mathcal{W}$ if and only if $\Lambda_r \supset 2\Lambda$ (cf. Exercise 12.6). Show that $-1 \in \mathcal{W}$ for precisely the irreducible root systems $A_1, B_l, C_l, D_l(l\text{even}), E_7, E_8, F_4, G_2$.

6. Prove that the roots in Φ which are dominant weights are precisely the highest long root and (if two root lengths occur) the highest short root (cf. (10.4) and Exercise 10.11), when Φ is irreducible.

7. If $\varepsilon_1, \cdots, \varepsilon_l$ is an obtuse basis of the euclidean space E (i.e., all $(\varepsilon_i, \varepsilon_j) \leqslant 0$ for $i \neq j$), prove that the dual basis is acute (i.e., all $(\varepsilon_i^*, \varepsilon_j^*) \geqslant 0$ for $i \neq j$). [Reduce to the case $l = 2$.]

8. Let Φ be irreducible. Without using the data in Table 1, prove that each λ_i is of the form $\sum\limits_j q_{ij}\alpha_j$, where all q_{ij} are positive rational numbers. [Deduce from Exercise 7 that all q_{ij} are nonnegative. From $(\lambda_i, \lambda_j) > 0$. Then show that if $q_{ij} > 0$ and $(\alpha_j, \alpha_k) < 0$, then $q_{ik} > 0$.]

9. Let $\lambda \in \Lambda^+$. Prove that $\sigma(\lambda + \delta) - \delta$ is dominant only for $\sigma = l$.

10. If $\lambda \in \Lambda^+$, prove that the set Π consisting of all dominant weights $\mu \prec \lambda$ and their $\mathcal{W}$-conjugates is saturated, as asserted in (13.4).

11. Prove that each subset of Λ is contained in a unique smallest saturated set, which is finite if the subset in question is finite.

12. For the root system of type A_2, write down the effect of each element of the Weyl group on each of λ_1, λ_2. Using this data, determine which weights belong to the saturated set having highest weight $\lambda_1 + 3\lambda_2$. Do the same for type G_2 and highest weight $\lambda_1 + 2\lambda_2$.

13. Call $\lambda \in \Lambda^+$ **minimal** if $\mu \in \Lambda^+, \mu \prec \lambda$ implies that $\mu = \lambda$. Show that each coset of Λ_r in Λ contains precisely one minimal Λ. Prove that λ is minimal if and only if the $\mathcal{W}$-orbit of λ is saturated (with highest weight λ), if and only if $\lambda \in \Lambda^+$ and $< \lambda, \alpha >= 0, 1, -1$ for all roots α. Determine (using Table 1) the nonzero minimal λ for each irreducible Φ, as follows:

A_l:$\lambda_1, \cdots, \lambda_l$
B_l:λ_l
C_l:λ_1
D_l:$\lambda_1, \lambda_{l-1}, \lambda_l$
E_6:λ_1, λ_6
E_7:λ_7

14 Isomorphism Theorem

1. Generalize Theorem 14.2 to the case: L semisimple.
2. Let $L = \mathfrak{sl}(2, F)$. If H, H' are any two maximal toral subalgebras of L, prove that there exists an automorphism of L mapping H onto H'.
3. Prove that the subspace M of $L \oplus L'$ introduced in the proof of Theorem 14.2 will actually equal D, if x and x' are chosen carefully.
4. Let σ be as in Proposition 14.3. Is it necessarily true that $\sigma(x_\alpha) = -y_\alpha$ for nonsimple α, where $[x_\alpha y_\alpha] = h_\alpha$?
5. Consider the simple algebra $\mathfrak{sl}(3, F)$ of type A_2. Show that the subgroup of $IntL$ generated by the automorphisms τ_α in (14.3) is strictly larger than the Weyl group (here $\mathfrak{S}_3$). [View $IntL$ as a matrix group and compute τ_α^2 explicitly.]
6. Use Theorem 14.2 to construct a subgroup $\Gamma(L)$ of $AutL$ isomorphic to the group of all graph automorphisms (12.2) of Φ.
7. For each classical algebra (1.2), show how to choose elements $h_\alpha \in H$ corresponding to a base of Φ (cf. Exercise 8.2).

15 Cartan Subalgebras

1. A semisimple element of $\mathfrak{sl}(n, F)$ is regular if and only if its eigenvalues are all distinct (i.e., if and only if its minimal and characteristic polynomials coincide).
2. Let L be semisimple ($charF = 0$). Deduce from Exercise 8.7 that the only solvable Engel subalgebras of L are the CSA's.
3. Let L be semisimple ($charF = 0$), $x \in L$ semisimple. Prove that x is regular if and only if x lies in exactly one CSA.
4. Let H be a CSA of a Lie algebra L. Prove that H is maximal nilpotent, i.e., not properly included in any nilpotent subalgebra of L. Show that the converse is false.
5. Show how to carry out the proof of Lemma A of (15.2) if the field F is only required to be of cardinality exceeding $\dim L$.
6. Let L be semisimple ($charF = 0$), L' a semisimple subalgebra. Prove that each CSA of L' lies in some CSA of L. [Cf. Exercise 6.9.]

16 Conjugacy Theorems

1. Prove that $\mathcal{E}(L)$ has order one if and only if L is nilpotent.

Solution: $\Rightarrow$ Suppose $\mathcal{E}(L) = 1$. Let $x \in L$. Then

$$L = \bigoplus_a L_a(\mathrm{ad}x).$$

If there is $a \neq 0$ such that $L_a(\mathrm{ad}x) \neq 0$, we take $0 \neq y \in L_a(\mathrm{ad}x)$. Then y is strongly ad-nilpotent. Since $\mathcal{E}(L) = 1$, we conclude that $\exp(\mathrm{ad}y) = 1$. It follows that $\mathrm{ad}y = 0$, i.e., $y \in L_0(\mathrm{ad}x)$. This yields a contradiction. Hence, $L = L_0(\mathrm{ad}x)$ and hence $\mathrm{ad}x$ is nilpotent. Therefore, L is nilpotent.

$\Leftarrow$ Suppose L is nilpotent. Let $x \in L$ be strongly ad-nilpotent. Then there exists $y \in L$ such that $x \in L_a(\mathrm{ad}y)$ for some $a \neq 0$, i.e., $(\mathrm{ad}y - a)^n x = 0$ for some n. Since $\mathrm{ad}y$ is nilpotent and $a \neq 0$, we obtain that $(\mathrm{ad}y - a)$ is invertible, and hence $x = 0$. Hence, $\mathcal{E}(L) = \{\exp(\mathrm{ad}x) | x \text{ is strongly ad-nilpotent.}\} = 1$.

2. Let L be semisimple, H a CSA, Δ a base of Φ. Prove that any subalgebra of L consisting of nilpotent elements, and maximal with respect to this property, is conjugate under $\mathcal{E}(L)$ to $N(\Delta)$, the derived algebra of $B(\Delta)$.

3. Let Ψ be a set of roots which is **closed** ($\alpha, \beta \in \Psi, \alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Psi$) and satisfies $\Psi \cup -\Psi = \emptyset$. Prove that Ψ is included in the set of positive roots relative to some base of Φ. [Use Exercise 2.] (This exercise belongs to the theory of root systems, but is easier to do using Lie algebras.)

4. How does the proof of Theorem 16.4 simplify in case $L = \mathfrak{sl}(2, F)$?

5. Let L be semisimple. If a semisimple element of L is regular, then it lies in only finitely many Borel subalgebras. (The converse is also true, but harder to prove, and suggests a notion of "regular" for elements of L which are not necessarily semisimple.)

6. Let L be semisimple, $L = H + \sum^{\cdot} L_\alpha$. A subalgebra P of L is called **parabolic** if P includes some Borel subalgebra. (In that case P is self-normalizing, by Lemma 15.2B.) Fix a base $\Delta \subset \Phi$, and set $B = B(\Delta)$. For each subset $\Delta' \subset \Delta$, define $P(\Delta')$ to be the subalgebra of L generated by all $L_\alpha (\alpha \in \Delta \text{or} - \alpha \in \Delta')$, along with H.

   1. $P(\Delta')$ is a parabolic subalgebra of L (called **standard** relative to Δ.
   2. Each parabolic subalgebra of L including $B(\Delta)$ has the form $P(\Delta')$ for some $\Delta' \subset \Delta$. [Use the Corollary of Lemma 10.2A and Proposition 8.4(d).]
   3. Prove that every parabolic subalgebra of L is conjugate under $\mathcal{E}(L)$ to one of the $P(\Delta')$.

4. Let $L = \mathfrak{sl}(2, F)$, with standard basis (x, h, y). For $c \in F$, write $x(c) = \exp \mathrm{ad}(cx), y(c) = \exp \mathrm{ad}(cy)$. Define inner automorphisms $w(c) = x(c)y(-c^{-1})x(c), h(c) = w(c)w(1)^{-1} (= w(c)w(-1))$, for $c \neq 0$. Compute the matrices of $w(c), h(c)$ relative to the given basis of L, and deduce that all diagonal automorphisms (16.5) of L are inner. Conclude in this case that $AutL = IntL = \mathcal{E}(L)$.

5. Let L be semisimple. Prove that the intersection of two Borel subalgebras B, B' of L always includes a CSA of L. [The proof is not easy; here is one possible outline:

   1. Let N, N' be the respective ideals of nilpotent elements in B, B'. Relative to the Killing form of L, $N = B^\perp, N' = B'^\perp$, where $\perp$ denotes orthogonal complement.
   2. Therefore $B = N^\perp = (N + (N \cap N'))^\perp = (N + (B \cap N'))^\perp = N^\perp \cap (B^\perp + N'^\perp) = B \cap (N + B') = N + (B \cap B')$.
   3. Note that $A = B \cap B'$ contains the semisimple and nilpotent parts of its elements.
   4. Let T be a maximal toral subalgebra of A, and find a T-stable complement A' to $A \cap N$. Then A' consists of semisimple elements. Since B/N is abelian, $[TA'] = 0$, forcing $A' = T$.
   5. Combine (b),(d) to obtain $B = N + T$; thus T is a maximal toral subalgebra of L.]

17 Universal Enveloping Algebras

1. Prove that if $\dim L < \infty$, then $\mathfrak{U}(L)$ has no zero divisors. [Hint: Use the fact that the associated graded algebra $\mathfrak{G}$ is isomorphic to a polynomial algebra.]

2. Let L be the two dimensional nonabeiian Lie algebra (1.4), with $[xy] = x$. Prove directly that $i : L \to \mathfrak{U}(L)$ is injective (i.e., that $J \cap L = 0$).

3. If $\in L$, extend $\mathrm{ad}x$ to an endomorphism of $\mathfrak{U}(L)$ by defining $\mathrm{ad}x(y) = xy - yx(y \in \mathfrak{U}(L))$. If $\dim L < \infty$, prove that each element of $\mathfrak{U}(L)$ lies in a finite dimensional L-submodule. [If $x, x_1, \cdots, x_m \in L$, verify that $\mathrm{ad}x(x_l, \cdots, x_n) = \sum_{i=1}^{m} x_1 x_2 \cdots \mathrm{ad}x(x_i) \cdots x_m$.]

4. If L is a free Lie algebra on a set X, prove that $\mathfrak{U}(L)$ is isomorphic to the tensor algebra on a vector space having X as basis.

5. Describe the free Lie algebra on a set $X = \{x\}$.

6. How is the PBW Theorem used in the construction of free Lie algebras?

18 Generators and Relations

1. Using the representation of L_0 on V (Proposition 18.2), prove that the algebras X, Y described in Theorem 18.2 are (respectively) free Lie algebras on the sets of x_i, y_i.

2. When $rank\Phi = 1$, the relations $(S_{ij}^+), (S_{ij}^-)$ are vacuous, so $L_0 = L \cong \mathfrak{sl}(2, F)$. By suitably modifying the basis of V in (18.2), show that V is isomorphic to the module $Z(0)$ constructed in Exercise 7.7.

3. Prove that the ideal K of L_0 in (18.3) lies in every ideal of L_0 having finite codimension (i.e., L is the largest finite dimensional quotient of L_0).

4. Prove that each inclusion of Dynkin diagrams (e.g., $E_6 \subset E_7 \subset E_8$) induces a natural inclusion of the corresponding semisimple Lie algebras.

19 The Simple Algebras

1. If L is a Lie algebra for which $[LL]$ is semisimple, then L is reductive.

2. Supply details for the argument outlined in (19.2).

3. Verify the assertions made about $\mathfrak{C}_0$ in (19.3).

4. Verify that $\delta(x), x \in \mathfrak{sl}(3, F)$, as defined in (19.3), is a derivation of $\mathfrak{C}$.

5. Show that the Cayley algebra $\mathfrak{C}$ satisfies the "alternative laws":$x^2y = x(xy), yx^2 = (yx)x$. Prove that, in any algebra $\mathfrak{U}$ satisfying the alternative laws, an endomorphism of the following form is actually a derivation: $[\lambda_a, \lambda_b] + [\lambda_a, \rho_b] + [\rho_a, \rho_b](a, b\mathfrak{U}, \lambda_a$ =left multiplication in $\mathfrak{U}$ by a, ρ_b= right multiplication in $\mathfrak{U}$ by b, bracket denoting the usual commutator of endomorphisms).

Show that the Cayley algebra $\mathfrak{C}$ satisfies the "alternative laws":$x^2y = x(xy), yx^2 = (yx)x$. Prove that, in any algebra $\mathfrak{U}$ satisfying the alternative laws, an endomorphism of the following form is actually a derivation: $[\lambda_a, \lambda_b] + [\lambda_a, \rho_b] + [\rho_a, \rho_b](a, b\mathfrak{U}, \lambda_a$ =left multiplication in $\mathfrak{U}$ by a, ρ_b= right multiplication in $\mathfrak{U}$ by b, bracket denoting the usual commutator of endomorphisms).

Solution: Let $x = a + b, y = c$ in $x^2 y = x(xy)$ and $yx^2 = (yx)x$, we obtain

$$(ab)c + (ba)c = a(bc) + b(ac) \qquad \text{and} \qquad c(ab) + c(ba) = (ca)b + (cb)a$$

Let $D = [\lambda_a, \lambda_b] + [\lambda_a, \rho_b] + [\rho_a, \rho_b]$. Note that

$$D(x) = a(bx) - b(ax) + a(xb) - (ax)b + (xb)a - (xa)b.$$

Then

$$\begin{aligned} D(x)y &= (a(bx))y - (b(ax))y + (a(xb))y - ((ax)b)y + ((xb)a)y - ((xa)b)y \\ &= (a(bx))y - b((ax)y) + a((xb)y) - (ax)(by) + (xb)(ay) - ((xa)b)y \\ xD(y) &= x(a(by)) - x(b(ay)) + x(a(yb)) - x((ay)b) + x((yb)a) - x((ya)b) \\ &= x(a(by)) - (xb)(ay) + (xa)(yb) - (x(ay))b + (x(yb))a - x((ya)b) \end{aligned}$$

And

$$\begin{aligned} -((xa)b)y + (xa)(yb) &= ((xa)y)b - (xa)(by) \\ x(a(by)) - (ax)(by) &= -a(x(by)) + (xa)(by) \end{aligned}$$

Thus,

$$\begin{aligned} &D(x)y + xD(y) \\ &= a((xb)y) - a(x(by)) + (x(yb))a - b((ax)y) - (x(ay))b + ((xa)y)b + (a(bx))y - x((ya)b) \end{aligned}$$

Note,

$$D(xy) = a(b(xy)) - b(a(xy)) + a((xy)b) - (a(xy))b + ((xy)b)a - ((xy)a)b,$$

Since,

$$\begin{aligned} &a(b(xy)) + a((xy)b) - a((xb)y) + a(x(by)) \\ &\quad = a(b(xy) + (xy)b - (xb)y + x(by)) \\ &\quad = a((bx)y + (xy)b - (xb)y + (xb)y) \\ &\quad = a((bx)y + (xy)b) \\ &- (a(xy))b - ((xy)a)b + (x(ay))b - ((xa)y)b \\ &\quad = -(a(xy) + (xy)a - x(ay) + (xa)y)b \\ &\quad = -(a(xy) + x(ya) - (xa)y + (xa)y)b \\ &\quad = -(a(xy) + x(ya))b \end{aligned}$$

Thus

$$\begin{aligned} D(xy) - D(x)y - xD(y) =& a((bx)y + (xy)b) + ((xy)b - x(yb))a \\ &+ b((ax)y - a(xy)) - (a(xy) + x(ya))b \\ &- (a(bx))y + x((ya)b) \\ =& a((bx)y + (xy)b) + ((bx)y - b(xy))a \\ &+ b((ya)x - y(ax)) - (a(xy) + x(ya))b \\ &- (a(bx))y + x((ya)b) \\ =& a((bx)y) - (a(bx))y + ((bx)y)a \\ &- (x(ya))b + x((ya)b) + b((ya)x) \\ &+ a((xy)b) - (a(xy))b - (b(xy))a - b(y(ax)) \\ =& (bx)(ya) - ((bx)y)a + ((bx)y)a \\ &- (bx)(ya) + b(x(ya)) + b((ya)x) \\ &- b((xy)a) + (b(xy))a - (b(xy))a - b(y(ax)) \\ =& b(x(ya)) + b((ya)x) - b((xy)a) - b(y(ax)) \\ =& b(x(ya) - (xy)a) + b((ya)x - y(ax)) \end{aligned}$$

6. Fill in details of the argument at the conclusion of (19.3).

20 Weights and Maximal Vectors

1. If V is an arbitrary L-module, then the sum of its weight spaces is direct.

2. (a) If V is an irreducible L-module having at least one (nonzero) weighs space, prove that V is the direct sum of its weight spaces.

   (b) Let V be an irreducible L-module. Then V has a (nonzero) weight space if and only if $\mathfrak{U}(H).v$ is finite dimensional for all $v \in V$, or if and only if $\mathfrak{U}.v$ is finite dimensional for all $v \in V$ (where $\mathfrak{U}$=subalgebra with 1 generated by an arbitrary $h \in H$ in $\mathfrak{U}(H)$).

   (c) Let $L = \mathfrak{sl}(2, F)$, with standard basis (x, y, h). Show that $1 - x$ is not invertible in $\mathfrak{U}(L)$, hence lies in a maximal left ideal I of $\mathfrak{U}(L)$. Set $V = \mathfrak{U}(L)/I$, so V is an irreducible L-module. Prove that the images of $1, h, h^2, \cdots$ are all linearly independent in V (so $\dim V = \infty$), using the fact that

   $$(x-1)^r h^s \equiv \begin{cases} 0 \pmod I, & r > s \\ (-2)^r r! \cdot 1 \pmod I & r = s \end{cases}$$

   Conclude that V has no (nonzero) weight space.

3. Describe weights and maximal vectors for the natural representations of the linear Lie algebras of types $A_l - D_l$ described in (l.2).

4. Let $L = \mathfrak{sl}(2, F), \lambda \in H^*$. Prove that the module $Z(\lambda)$ for $\lambda = \lambda(h)$ constructed in Exercise 7.7 is isomorphic to the module $Z(\lambda)$ constructed in (20.3). Deduce that $\dim V(\lambda) < \infty$ if and only if $\lambda(h)$ is a nonnegative integer.

5. If $\mu \in H^*$, define $\mathcal{P}(\mu)$ to be the number of distinct sets of nonnegative integers $k_\alpha (\alpha \succ 0)$ for which $\mu = \sum_{\alpha \succ 0} k_\alpha \alpha$. Prove that $\dim Z(\lambda)_\mu = \mathcal{P}(\lambda - \mu)$, by describing a basis for $Z(\lambda)_\mu$.

6. Prove that the left ideal $I(\lambda)$ introduced in (20.3) is already generated by the elements $x_\alpha, h_\alpha, -\lambda(h_\alpha).1$ for α simple.

7. Prove, without using the induced module construction in (20.3), that $I(\lambda) \cap \mathfrak{U}(N^-) = 0$, in particular that $I(\lambda)$ is properly contained in $\mathfrak{U}(L)$. [Show that the analogous left ideal $I'(\lambda)$ in $\mathfrak{U}(B)$ is proper, white $I(\lambda) = \mathfrak{U}(N^-) I'(\lambda)$ by PBW.]

8. For each positive integer d, prove that the number of distinct irreducible L-modules $V(\lambda)$ of dimension $\leqslant d$ is finite. Conclude that the number of nonisomorphic L-modules of dimension $\leqslant d$ is finite. [If $\dim V(\lambda) < \infty$, view $V(\lambda)$ as S_α-module for each $\alpha \succ 0$; notice that $\lambda(h_\alpha) \in Z$, and that $V(\lambda)$ includes an S_α-submodule of dimension $\lambda(h_\alpha) + 1$.]

9. Verify the following description of the unique maximal submodule $Y(\lambda)$ of $Z(\lambda)$ (20.3) : If $v \in Z(\lambda)_\mu, \lambda - \mu = \sum_{\alpha \succ 0} c_\alpha \alpha (c_\alpha \in Z^+)$, observe that $\prod_{\alpha \succ 0} x_\alpha^{c_\alpha}.v$ has weight λ (the positive roots in any fixed order), hence is a scalar multiple of the maximal vector v^+. If this multiple is 0 for every possible choice of the c_α (cf. Exercise 5), prove that $v \in Y(\lambda)$. Conversely, prove that $Y(\lambda)$ is the span of all such weight vectors v for weights $\mu \neq \lambda$.

10. A maximal vector w^+ of weight μ in $Z(\lambda)$ induces an L-module homomorphism $\phi : Z(\mu) \to Z(\lambda)$, with $\text{Im}\phi$ the submodule generated by w^+. Prove that ϕ is injective.

11. Let V be an arbitrary finite dimensional L-module, $\lambda \in H^*$. Construct in the L-module $W = Z(\lambda) \otimes V$ a chain of submodules $W = W_1 \supset W_2 \supset \cdots \supset W_{n+1} = 0 (n = \dim V)$ so that W_i / W_{i+1} is isomorphic to $Z(\lambda + \lambda_l)$, where the weights of V in suitable order (multiplicities counted) are $\lambda_1, \cdots, \lambda_n$.

21 Finite Dimensional Modules

1. The reader can check that we have not yet used the simple transitivity of $\mathcal{W}$ on bases of Φ (Theorem 10.3(e)), only the transitivity. Use representation theory to obtain a new proof, as follows: There exists a finite dimensional irreducible module $V(\lambda)$ for which all $< \lambda, \alpha >$ $(\alpha \in \Delta)$ are distinct and positive. If $\sigma \in \mathcal{W}$ permutes Δ, then $\sigma\lambda = \lambda$, forcing $\sigma = 1$.

2. Draw the weight diagram for the case B_2, $\lambda = \lambda_1 + \lambda_2$ (notation of Chapter 3).

3. Let $\lambda \in \Lambda^+$. Prove that 0 occurs as a weight of $V(\lambda)$ if and only if λ is a sum of roots.

4. Recall the module $Z(\lambda)$ constructed in (20.3). Use Lemma 21.2 to find certain maximal vectors in $Z(\lambda)$, when $\lambda \in \Lambda$: the coset of $y_i^{m_i+1}, m_i =< \lambda, \alpha_i >$, is a maximal vector provided m_i is nonnegative (Cf. Exercise 7.7.)

5. Let V be a faithful finite dimensional L-module, $\Lambda(V)$ the subgroup of Λ generated by the weights of V. Then $\Lambda(V) \supset \Lambda_r$. Show that every subgroup of Λ including Λ_r is of this type.

6. If $V = V(\lambda), \lambda \in \Lambda^+$, prove that V^* is isomorphic (as L-module) to $V(-\sigma\lambda)$, where $\sigma \in \mathcal{W}$ is the unique element of $\mathcal{W}$ sending Δ to $-\Delta$ (Exercise 10.9, cf. Exercise 13.5).

7. Let $V = V(\lambda), W = V(\mu)$, with $\lambda, \mu \in \Lambda^+$. Prove that $\Pi(V \otimes W) = \{\nu + \nu' | \nu \in \Pi(\lambda), \nu' \in \Pi(\mu)\}$ and that $\dim(V \otimes W)_{\nu+\nu'}$ equals

$$\sum_{\pi+\pi'=\nu+\nu'} \dim V_\pi \cdot \dim W_{\pi'}$$

   In particular, $\lambda + \mu$ occurs with multiplicity one, so $V(\lambda + \mu)$ occurs exactly once as a direct summand of $V \otimes W$.

8. Let $\lambda_1, \cdots, \lambda_l$ be the fundamental dominant weights for the root system Φ of L (13.1). Show how to construct an arbitrary $V(\lambda), \lambda \in \Lambda^+$, as a direct summand in a suitable tensor product of modules $V(\lambda_1), \cdots, V(\lambda_l)$ (repetitions allowed).

9. Prove Lemma 21.4 and deduce Lemma 21.2 from it.

10. Let $L = \mathfrak{sl}(l+1, F)$, with CSA $H = \mathfrak{d}(l+1, F) \cap L$. Let $\mu_1, \cdots, \mu_{l+1}$ be the coordinate functions on H, relative to the standard basis of $\mathfrak{gl}(l+1, F)$. Then $\sum \mu_i = 0$, and $\mu_1, \cdots, \mu_l$ form a basis of H^*, while the set of $\alpha_i = \mu_i - \mu_{i+1} (1 \leqslant i \leqslant l)$ is a base Δ for the root system Φ. Verify that $\mathcal{W}$ acts on H^* by permuting the μ_i; in particular, the reflection with respect to α_i interchanges μ_i, μ_{i+1} and leaves the other μ_j fixed. Then show that the fundamental dominant weights relative to Δ are given by $\lambda_k = \mu_1 + \cdots + \mu_k (1 \leqslant k \leqslant l)$.

11. Let $V = F^{l+1}, L = \mathfrak{sl}(V)$. Fix the CSA H and the base $\Delta = (\alpha_1, \cdots, \alpha_l)$ of Φ as in Exercise 10. The purpose of this exercise is to construct irreducible L-modules $V_k (1 \leqslant k \leqslant l)$ of highest weight λ_k.

    1. For $k = 1, V_1 = V$ is irreducible of highest weight λ_1.
    2. In the k-fold tensor product $V \otimes \cdots \otimes V, k \geqslant 2$, define V_k to be the subspace of **skew-symmetric tensors**: If $(v_1, \cdots, v_{l+1})$ is the canonical basis of V, V_k has basis consisting of the $\binom{l+1}{k}$ vectors

    $$[v_{i_1}, \cdots, v_{i_k}] = \sum_{\pi \in \mathcal{S}_k} sn(\pi) v_{\pi(i_1)} \otimes \cdots \otimes v_{\pi(i_k)} \qquad (*)$$

       where $i_1 < i_2 < \cdots < i_k$. Show that (*) is of weight $\mu_{i_1} + \cdots + \mu_{i_k}$.
    3. Prove that L leaves the subspace V_k invariant and that all the weights $\mu_{i_1} + \cdots + \mu_{i_k} (i_l < \cdots < i_k)$ are distinct and conjugate under $\mathcal{W}$. Conclude that V_k is irreducible, of highest weight λ_k. (Cf. Exercise 13.13.)

22 Multiplicity Formula

1. Let $\lambda \in \Lambda^+$. Prove, without using Freudenthal's formula, that $m_\lambda(\lambda - k\alpha) = 1$ for $\alpha \in \Delta$ and $0 \leqslant k \leqslant< \lambda, \alpha >$.
2. Prove that c_L is in the center of $\mathfrak{U}(L)$ (cf. (23.2)). [Imitate the calculation in (6.2), with ϕ omitted.] Show also that c_L is independent of the basis chosen for L.
3. In Example 1 (22.4), determine the $\mathcal{W}$-orbits of weights, thereby verifying directly that $\mathcal{W}$-conjugate weights have the same multiplicity (cf. Theorem 21.2). [Cf. Exercise 13.12.]
4. Verify the multiplicities shown in Figure 1 of (21.3).
5. Use Freudenthal's formula and the data for A_2 in Example 1 (22.4) to compute multiplicities for $V(\lambda), \lambda = 2\lambda_1 + 2\lambda_2$. Verify in particular that $\dim V(\lambda) = 27$ and that the weight 0 occurs with multiplicity 3. Draw the weight diagram.
6. For L of type G_2, use Table 2 of (22.4) to determine all weights and their multiplicities for $V(\lambda), \lambda = \lambda_1 + 2\lambda_2$. Compute $\dim V(\lambda) = 286$. [Cf. Exercise 13.12.]
7. Let $L = \mathfrak{sl}(2, F)$, and identify $m\lambda_1$ with the integer m. Use Propositions A and B of (22.5), along with Theorem 7.2, to derive the **Clebsch-Gordan formula**: If $n \leqslant m$, then $V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n), n+1$ summands in all. (Cf. Exercise 7.6.)
8. Prove the uniqueness part of Proposition 22.5A.

23 Characters

1. In the Example in (23.3), verify that the trace polynomial is given correctly.
2. For the algebras of type A_2, B_2, G_2, compute explicit generators for $\mathfrak{B}(H)^{\mathcal{W}}$ in terms of the fundamental dominant weights λ_1, λ_2. Show how some of these lift to $\mathcal{E}$, using the algorithm of this section. (Notice too that in each case $\mathfrak{B}(H)^{\mathcal{W}}$ is a polynomial algebra with $l = 2$ generators.)
3. Show that Proposition 23.2 remains valid when λ is an arbitrary linear function on H, provided only that $< \lambda, \alpha >$ is an integer.
4. From the formula $(*)\chi_\lambda(z) = \lambda(\xi(z))$ of (23.3), compute directly the value of the universal Casimir element c_L (22.1) on $V(\lambda), \lambda \in \Lambda^+ : (\lambda+\delta, \lambda+\delta) - (\delta, \delta)$. [Recall how t_α and h_α, resp. z_α and y_α, are related. Rewrite c_L in the ordering of a PBW basis, and use the fact derived in (22.3) that $(\mu, \mu) = \sum_i \mu(h_i)\mu(k_i)$ for any weight μ.]
5. Prove that any polynomial in n variables over F ($char F = 0$) is a linear combination of powers of linear polynomials. [Use induction on n. Expand $(T_1 + aT_2)^k$ and then use a Vandermonde determinant argument to show that kth powers of linear polynomials span a space of correct dimension when $n = 2$.
6. If $\lambda \in \Lambda^+$ prove that all μ linked to λ satisfy $\mu \prec \lambda$, hence that all such μ occur as weights of $Z(\lambda)$.
7. Let $\mathcal{D} = [\mathfrak{U}(L), \mathfrak{U}(L)]$ be the subspace of $\mathfrak{U}(L)$ spanned by all $xy - yx (x, y \in \mathfrak{U}(L))$. Prove that $\mathfrak{U}(L)$ is the direct sum of the subspaces $\mathcal{D}$ and $\mathcal{E}$ (thereby allowing one to extend x_λ to all of $\mathfrak{U}(L)$ by requiring it to be 0 on $\mathcal{D}$). [Recall from Exercise 17.3 that $\mathfrak{U}(L)$ is the sum of finite dimensional L-modules, hence is completely reducible because L is semisimple. Show that $\mathcal{E}$ is the sum of all trivial L-submodules of $\mathfrak{U}(L)$, while $\mathcal{D}$ coincides with the space of all $\operatorname{ad} x(y), x \in L, y \in \mathfrak{U}(L)$, the latter being complementary to $\mathcal{E}$.]
8. Prove that the weight lattice Λ is Zariski dense in H^* (see Appendix), H^* being identified with affine l-space. Use this to give another proof that Corollary' in (23.2) extends to all $\lambda, \mu H^*$.
9. Every F-algebra homomorphism $\chi : \mathcal{E} \to F$ is of the form ξ_λ for some $\lambda \in H^*$. [View χ as a homomorphism $\mathcal{C}(H)^{\mathcal{W}} \to F$ and show that its kernel generates a proper ideal in $\mathcal{C}(H)$.]
10. Prove that the map $\psi : \mathcal{E} \to \mathcal{C}(H)^{\mathcal{W}}$ is independent of the choice of Δ.

24 Formulas of Weyl, Kostant, and Steinberg

1. Give a direct proof of Weyl's character formula (24.3) for type A_1.
2. Use Weyl's dimension formula to show that a faithful irreducible finite dimensional L-module of smallest possible dimension has highest weight λ_i for some $1 \leqslant i \leqslant l$.
3. Use Kostant's formula to check some of the multiplicities listed in Example 1 (22.4), and compare ch_λ there with the expression given by Weyl's formula.
4. Compare Steinberg's formula for the special case A_1 with the Clebsch-Gordan formula (Exercise 22.7).
5. Using Steinberg's formula, decompose the G_2-module $V(\lambda_1) \otimes V(\lambda_2)$ into its irreducible constituents. Check that the dimensions add up correctly to the product $\dim V(\lambda_1) \cdot \dim V(\lambda_2)$, using Weyl's formula.
6. Let $L = \mathfrak{sl}(3, F)$. Abbreviate $\lambda = m_1\lambda_1 + m_2\lambda_2$ by (m_1, m_2). Use Steinberg's formula to verify that $V(1,0) \otimes V(0,1) \cong V(0,0) \oplus V(1,1)$.
7. Verify the degree formulas in (24.3) ; derive such a formula for type C_3. How can the integers $c_i^{(\alpha)}$ be found in general?
8. Let $\lambda \in \Lambda$. If there exists $\sigma \neq 1$ in $\mathcal{W}$ fixing λ, prove that $\sum\limits_{\sigma(\lambda)=\lambda} sn(\sigma)\varepsilon_{\sigma(\lambda)} = 0$. [Use the fact that λ lies in the closure but not the interior of some Weyl chamber to find a reflection fixing λ, and deduce that the group fixing λ has even order.]
9. The purpose of this exercise is to obtain another decomposition of a tensor product, based on explicit knowledge of the weights of one module involved. Begin, as in (24.4), with the equation (2) $ch_{\lambda'} * \omega(\lambda''+\delta) = \sum\limits_{\lambda\in\Lambda^+} n(\lambda)\omega(\lambda+\delta)$. Replace $ch_{\lambda'}$ on the left side by $\sum\limits_{\lambda\in\Lambda} m_{\lambda'}(\lambda)\varepsilon_\lambda$, and combine to get: $\sum\limits_{\sigma\in\mathcal{W}} sn(\sigma) \sum\limits_{\lambda} m_{\lambda'}(\lambda)\varepsilon_{\sigma(\lambda+\lambda''+\delta)}$, using the fact that $\mathcal{W}$ permutes weight spaces of $V(\lambda')$. Next show that the right side of (2) can be expressed as $\sum\limits_{\sigma\in\mathcal{W}} sn(\sigma) \sum\limits_{\lambda\in\Lambda^+} n(\lambda)\varepsilon_{\sigma(\lambda+\delta)}$. Define $t(\mu)$ to be 0 if some element $\sigma \neq 1$ of $\mathcal{W}$ fixes μ, and to be $sn(\sigma)$ if nothing but 1 fixes μ and if $\sigma(\mu)$ is dominant. Then deduce from Exercise 8 that:

$$ch_{\lambda'} * ch_{\lambda''} = \sum_{\lambda\in\Pi(\lambda')} m_{\lambda'}(\lambda)t(\lambda+\lambda''+\delta)ch_{\{\lambda+\lambda''+\delta\}-\delta}$$

   where the braces denote the unique dominant weight to which the indicated weight is conjugate.
10. Rework Exercises 5, 6, using the approach of Exercise 9.
11. With notation as in Exercise 6, verify that $V(1,1) \otimes V(1,2) \cong V(2,3) \oplus V(3,1) \oplus V(0,4) \oplus V(1,2) \oplus V(1,2) \oplus V(2,0) \oplus V(0,1)$.
12. Deduce from Steinberg's formula that the only possible $\lambda \in \Lambda^+$ for which $V(\lambda)$ can occur as a summand of $V(\lambda') \otimes V(\lambda'')$ are those of the form $\mu + \lambda''$, where $\mu \in \Pi(\lambda')$. In case all such $\mu + \lambda''$ are dominant, deduce from Exercise 9 that $V(\mu + \lambda'')$ does occur in the tensor product, with multiplicity $m_{\lambda'}(\mu)$. Using these facts, decompose $V(1,3) \otimes V(4,4)$ for type A_2 (cf. Example 1 of (22.4)).
13. Fix a sum π of positive roots, and show that for all sufficiently large n, $m_{n\delta}(n\delta - \pi) = p(-\pi)$.

25 Chevalley basis of L

1. Prove Proposition 25.1(c) by inspecting root systems of rank 2. [Note that one of α, β may be assumed simple.]
2. How can the bases for the classical algebras exhibited in (1.2) be modified so as to obtain Chevalley bases? [Cf. Exercise 14.7.]

3. Use the proof of Proposition 25.2 to give a new proof of Exercise 9.10.

4. If only one root length occurs in each component of Φ (i.e., Φ has irreducible components of types A, D, E), prove that all $c_{\alpha\beta} = \pm 1$ in Theorem 25.2 (when $\alpha, \beta, \alpha+\beta \in \Phi$).

5. Prove that different choices of Chevalley basis for L lead to isomorphic Lie algebras $L(Z)$ over Z. ("Isomorphism over Z" is defined just as for a field.)

6. For the algebra of type B_2, let the positive roots be denoted $\alpha, \beta, \alpha+\beta, 2\beta+\alpha$. Check that the following equations are those resulting from a Chevalley basis (in particular, the signs $\pm$ are consistent):

$$\begin{array}{llllll}
[h_\beta, x_\beta] & = & 2x_\beta & [x_\beta, x_\alpha] & = & x_{\alpha+\beta} \\
[h_\beta, x_\alpha] & = & -2x_\alpha & [x_\beta, x_{\alpha+\beta}] & = & 2x_{2\beta+\alpha} \\
[h_\beta, x_{\alpha+\beta} & = & 0 & [x_\beta, x_{-\alpha-\beta}] & = & -2x_{-\alpha} \\
[h_\beta, x_{2\beta+\alpha}] & = & 2x_{2\beta+\alpha} & [x_\beta, x_{-2\beta-\alpha}] & = & -x_{-\alpha-\beta} \\
[h_\alpha, x_\beta] & = & -x_\beta & [x_\alpha, x_{-\alpha-\beta}] & = & x_{-\beta} \\
[h_\alpha, x_\alpha] & = & 2x_\alpha & [x_{\alpha+\beta}, x_{-2\beta-\alpha}] & = & x_{-\beta} \\
[h_\alpha, x_{\alpha+\beta} & = & x_{\alpha+\beta} & [h_\alpha, x_{2\beta+\alpha}] & = & 0
\end{array}$$

7. Let $F = C$. Fix a Chevalley basis of L, and let L' be the R-subspace of L spanned by the elements $\sqrt{-1}h_i(1 \leqslant i \leqslant l), x_\alpha - x_{-\alpha}$, and $\sqrt{-1}(x_\alpha + x_{-\alpha})(\alpha \in \Phi^+)$. Prove that these elements form a basis of L over C (so $L \cong L' \otimes_R C$) and that L' is closed under the bracket (so L' is a Lie algebra over R). Show that the Killing form κ' of L' is just the restriction to L' of κ, and that κ' is negative definite. (L' is a "compact real form" of L, associated with a compact Lie group).

8. Let $L = \mathfrak{sl}(l+1, F)$, and let K be any field of characteristic p. If $p \nmid l+1$, then $L(K)$ is simple. If $p = 2, l = 1$, then $L(K)$ is solvable. If $l > 1, p \mid l+1$, then $\mathrm{Rad} L(K) = Z(L(K))$ consists of the scalar matrices.

9. Prove that for L of type A_l, the resulting Chevalley group $G(K)$ of adjoint type is isomorphic to $PSL(l+1, K) = SL(l+1, K)$ modulo scalars (the scalars being the $l+1$st roots of unity in K).

10. Let L be of type G_2, K a field of characteristic 3. Prove that $L(K)$ has a 7-dimensional ideal M(cf. the short roots). Describe the representation of $L(K)$ on $L(K)/M$.

11. The Chevalley group $G(K)$ acts on $L(K)$ as a group of Lie algebra automorphisms.

12. Is the basis of G_2 exhibited in (19.3) a Chevalley basis?

26 Kostant's Theorem

1. Let $L = \mathfrak{sl}(2, F)$. Let $(v_0, v_1, \cdots, v_m)$ be the basis constructed in (7.2) for the irreducible L-module $V(m)$ of highest weight m. Prove that the Z-span of this basis is invariant under $\mathfrak{U}(L)_Z$. Let $(w_0, w_1, \cdots, w_m)$ be the basis of $V(m)$ used in (22.2). Show that the Z-span of the w_i is not invariant under $\mathfrak{U}(L)_Z$.

2. Let $\lambda \in \Lambda^+ \subset H^*$ be a dominant integral linear function, and recall the module $Z(\lambda)$ of (20.3), with irreducible quotient $V(\lambda) = Z(\lambda)/Y(\lambda)$. Show that the multiplicity of a weight μ of $V(\lambda)$ can be effectively computed as follows, thanks to Kostant's Theorem: If v^+ is a maximal vector of $Z(\lambda)$, then the various $f_A.v^+$ for which $\sum a_i\alpha_j = \lambda - \mu$ form an F-basis of the weight space for μ in $Z(\lambda)$. (Cf. Lemma D of (24.1).) In turn if $\sum a_i\alpha_i = \sum c_i\alpha_i$, then $e_C f_A.v^+$ is an integral multiple $n_{CA}v^+$. This yields a $d \times d$ integral matrix (n_{CA}) (d= multiplicity of μ in $Z(\lambda)$), whose $rank = m_\lambda(\mu)$. (Cf. Exercise 20.9). Moreover, this integral matrix is computable once the Chevalley basis structure constants are known. Carry out a calculation of this kind for type A_2, taking $\lambda - \mu$ small.

27 Admissible Lattices

1. If M is an admissible lattice in V, then $M \cap V_\mu$ is a lattice in V_μ for each weight μ of V.
2. Prove that each admissible lattice in L which includes $L(Z)$ and is closed under the bracket has the form L_V. [Imitate the proof of Proposition 27.2; cf. Exercise 21.5.]
3. If M (resp. N) is an admissible lattice in V (resp. W), then $M \otimes N$ is an admissible lattice in $V \otimes W$ (cf. Lemma 26.3A). Use this fact, and the identification (as L-modules) of $V^* \otimes V$ with $\mathrm{End} V$ (6.1), to prove that L_V is stable under all $(\mathrm{ad} x_\alpha)^m/m!$ in Proposition 27.2 (without using Lemma 27.2).

   In the following exercises, $L = \mathfrak{sl}(2, F)$, and weights are identified with integers.
4. Let $V = V(\lambda), \lambda \in \Lambda^+$. Prove that $L_V = L(Z)$ when λ is odd, while $L_V = Z\left(\frac{h}{2}\right) + Zx + Zy$ when λ is even.
5. If $char K > 2$, prove that $L(K) \to L_V(K)$ is an isomorphism for any choice of V.
6. Let $V = V(\lambda), \lambda \in \Lambda^+$. Prove that $G_V(K) \cong SL(2, K)$ when $\Lambda(V) = \Lambda, PSL(2, K)$ when $\Lambda(V) = \Lambda_r$.
7. If $0 \leqslant \lambda < char K, V = V(\lambda)$, prove that $V(K)$ is irreducible as $L(K)$- module.
8. Fix $\lambda \in \Lambda^+$. Then a minimal admissible lattice $M_{\min}$ in $V(\lambda)$ has a Z-basis $(v_0, \cdots, v_\lambda)$ for which the formulas in Lemma 7.2 are valid:

$$\begin{aligned} h.v_i &= (h - 2i)v_i, \\ y.v_i &= (i+1)v_{i+1}, \quad (v_{\lambda+1} = 0) \\ x.v_i &= (\lambda - i + 1)v_{i-1}, \quad (v_{-1} = 0). \end{aligned}$$

   Show that the corresponding maximal admissible lattice $M_{\max}$ has a Z-basis $(w_0, \cdots, w_\lambda)$ with $w_0 = v_0$ and action given by:

$$\begin{aligned} h.w_i &= (\lambda - 2i)w_i \\ y.w_i &= (\lambda - i)w_{i+1} \\ x.w_i &= iw_{i-1} \end{aligned}$$

   Deduce that $v_i = \binom{\lambda}{i} w_i$. Therefore, $[M_{\max} : M_{\min}] = \prod_{i=0}^{\lambda} \binom{\lambda}{i}$.
9. Keep the notation of Exercise 8. Let M be any admissible lattice, $M_{\max} \supset M \supset M_{\min}$. Then M has a Z-basis $(z_0, \cdots, z_\lambda)$ with $z_i = a_i w_i (a_i \in Z), a_0 = a_\lambda = 1$. Define integers b_i, c_i by: $x.z_i = b_i z_{i-1} (b_0 = 1), y.z_i = c_i z_{i+1} (c_\lambda = 1)$. Show that $c_i = \pm b_{\lambda - i}$ and that $\prod b_i = \lambda!$.
10. Keep the notation of Exercise 8. Let M be a subgroup of $M_{\max}$ containing $M_{\min}$, with a Z-basis $(w_0, a_1 w_1, \cdots, a_\lambda w_\lambda)$. Find necessary and sufficient conditions on the a_i for M to be an admissible lattice. Work out the possibilities when $\lambda = 4$.