Exercise 6 for 2022∼ 2023 USTC Course

'Introduction to Quantum Information'

Jun-Hao Wei, Shu-Ming Hu and Kai Chen

Hefei National Research Center for Physical Sciences
at the Microscale and School of Physical Sciences,
University of Science and Technology of China, Hefei 230026, China

1. Let ρ be an arbitrary 2-qubit density operator. Suppose we perform a projective measurement of the second qubit in the computational basis. Let ρ' be the density matrix which would be assigned to the system after the measurement by an observer who did not learn the measurement result. Prove that the reduced density matrix for the first qubit is not affected by the measurement, i.e.,

$$\mathrm{tr}_2(\rho') = \mathrm{tr}_2(\rho).$$

Answer:

Let $P_0 = |0\rangle\langle 0|_2$, $P_1 = |1\rangle\langle 1|_2$ be the projectors of the second qubit, then

$$\rho' = P_0\rho P_0 + P_1\rho P_1.$$

Note that a 2-qubit density operator can be written as

$$\rho = \frac{1}{4}(I\otimes I + \vec{r}\cdot\vec{\sigma}\otimes I + I\otimes\vec{s}\cdot\vec{\sigma} + \sum_{i,j} t_{ij}\sigma_i\otimes\sigma_j).$$

Thus,

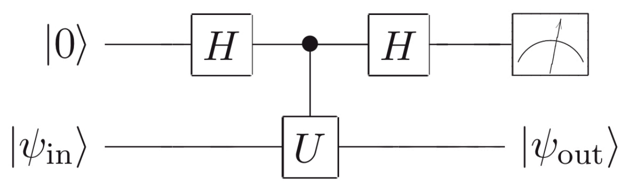
$$\rho' = \frac{1}{4}(I\otimes I + \vec{r}\cdot\vec{\sigma}\otimes I + I\otimes[P_0(\vec{s}\cdot\vec{\sigma})P_0 + P_1(\vec{s}\cdot\vec{\sigma})P_1] + \sum_{i,j} t_{ij}\sigma_i\otimes[P_0\sigma_j P_0 + P_1\sigma_j P_1]).$$

For instance, the contribution of the last term to the reduced density matrix is

$$\sigma_i\otimes\mathrm{tr}[P_0\sigma_j P_0 + P_1\sigma_j P_1] = \sigma_i\otimes\mathrm{tr}[\sigma_j P_0 + \sigma_j P_1] = \sigma_i\otimes\mathrm{tr}(\sigma_j),$$

where we used cyclic property of trace, $P_0^2 = P_0, P_1^2 = P_1$ and $P_0 + P_1 = I$. Similarly for the rest. Therefore, $\mathrm{tr}_2(\rho') = \mathrm{tr}_2(\rho) = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$.

2. Suppose we have a single qubit operator U with eigenvalues ± 1, so that U is both Hermitian and unitary, so it can be regarded both as an observable and a quantum gate. Suppose we wish to measure the observable U. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving a post-measurement state which is the corresponding eigenvector. Show that the following circuit implements a measurement of U.



Answer:
The state after the second Hadamard is

$$\frac{1}{\sqrt{2}} |0\rangle \left(\frac{|\psi_{\mathrm{in}}\rangle + U |\psi_{\mathrm{in}}\rangle}{\sqrt{2}} \right) + \frac{1}{\sqrt{2}} |1\rangle \left(\frac{|\psi_{\mathrm{in}}\rangle - U |\psi_{\mathrm{in}}\rangle}{\sqrt{2}} \right).$$

If the first qubit is projected to $|0\rangle$, the second qubit becomes $|\psi_{\mathrm{out}}\rangle = (|\psi_{\mathrm{in}}\rangle + U |\psi_{\mathrm{in}}\rangle)/\sqrt{2} \propto P_+ |\psi_{\mathrm{in}}\rangle$, where $P_+ = (I + U)/2$ is the projector to the positive eigenspace of U. If the first qubit is projected to $|1\rangle$, the second qubit becomes $|\psi_{\mathrm{out}}\rangle = (|\psi_{\mathrm{in}}\rangle + U |\psi_{\mathrm{in}}\rangle)/\sqrt{2} \propto P_- |\psi_{\mathrm{in}}\rangle$, where $P_- = (I - U)/2$ is the projector to the negative eigenspace of U. Therefore, the measurement results of the first qubit indicates one of the two eigenvalues, and leaves $|\psi_{\mathrm{out}}\rangle$ the corresponding eigenvector.

3. Please construct the quantum SWAP gate to swap two qubits using the C-NOT gate.

Answer:
The circuit swapping two qubits is as following:

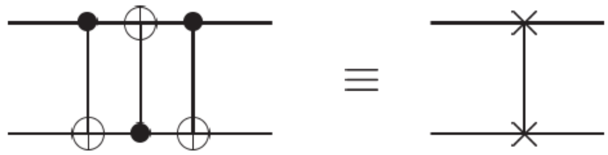


Figure 1.7. Circuit swapping two qubits, and an equivalent schematic symbol notation for this common and useful circuit.

To see that this circuit accomplishes the swap operation, note that the sequence of gates has the following sequence of effects on a computational basis state $|a, b\rangle$,

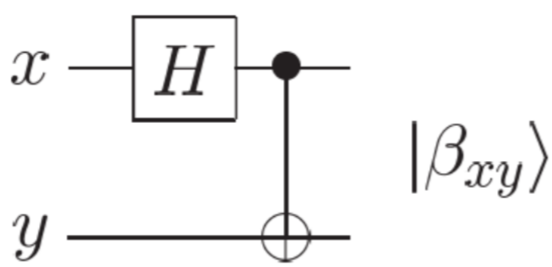
$$\begin{aligned} |a, b\rangle &\longrightarrow |a, a \oplus b\rangle \\ &\longrightarrow |a \oplus (a \oplus b), a \oplus b\rangle = |b, a \oplus b\rangle \\ &\longrightarrow |b, (a \oplus b) \oplus b\rangle = |b, a\rangle\ , \end{aligned}$$

where all additions are done modulo 2.

4. Please design a quantum circuit which converts the state $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ into four Bell states.

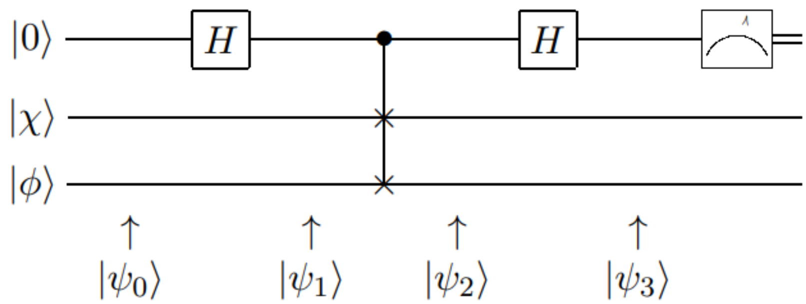
Answer:

The quantum circuit to create Bell state is as following:



The proof is omitted.

5. Consider the following three-qubit quantum circuit, in which $|\chi\rangle$ and $|\phi\rangle$ are arbitrary qubit states:



1.) Give the intermediate states of the circuit, $|\psi_0\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$.

2.) If the measurement result is zero, what is the state of the bottom two qubits?

3.) If $\langle\chi|\phi\rangle = \alpha$, with what probability is the measurement result zero?

Answer:

1.)

$$
\begin{aligned}
|\psi_0\rangle &= |0\rangle|\chi\rangle|\phi\rangle, \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\chi\rangle|\phi\rangle, \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}|0\rangle|\chi\rangle|\phi\rangle + \frac{1}{\sqrt{2}}|1\rangle|\phi\rangle|\chi\rangle, \\
|\psi_3\rangle &= \frac{1}{2}|0\rangle(|\chi\rangle|\phi\rangle + |\phi\rangle|\chi\rangle) + \frac{1}{2}|1\rangle(|\chi\rangle|\phi\rangle - |\phi\rangle|\chi\rangle).
\end{aligned}
$$

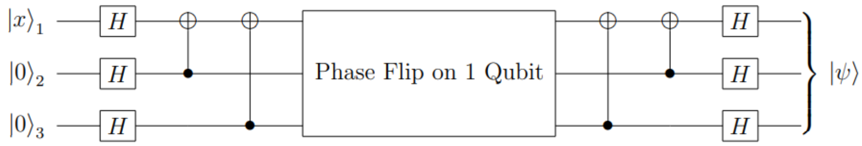
2.)

$$
\frac{|\chi\rangle|\phi\rangle + |\phi\rangle|\chi\rangle}{\sqrt{2(1 + |\langle\chi|\phi\rangle|^2)}}
$$

3.) The measurement is going to result in 0 with probability

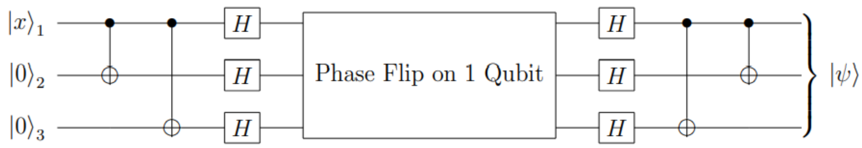
$$
\frac{1}{4}(\langle\chi|\langle\phi| + \langle\phi|\langle\chi|)(|\chi\rangle|\phi\rangle + |\phi\rangle|\chi\rangle) = \frac{1 + |\alpha|^2}{2}.
$$

6. Verify that the following circuit is the appropriate encoder/decoder circuit for the 3 qubit phase flip code. In other words, exhibit a measurement on the two ancillae in the circuit's output $|\psi\rangle$ that will detect whether a phase flip error occurred on one of the three qubits.

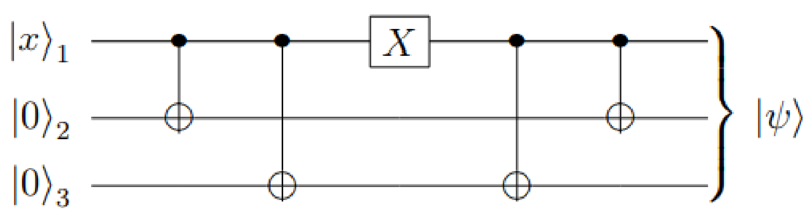


Answer:

The phase flip coder/decoder circuit is equivalent to



Thus, if the noise is Z_1, then the facts $H^2 = I$ and $HZH = X$ simplify the circuit to



So, we can have

Phase Flip on	Circuit Output $|\psi\rangle$
No qubits	$|x\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$
3rd qubit	$|x\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3$
2nd qubit	$|x\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3$
1st qubit	$|x \oplus 1\rangle_1 \otimes |1\rangle_2 \otimes |1\rangle_3$

we see an error detection procedure is to measure the two ancillae and an error correction procedure is to apply a bit flip to the first qubit if the ancillae are in the state $|1\rangle_2 \otimes |1\rangle_3$ and then, in all cases where a phase flip has occurred, throw out both ancillae and replace them with fresh ancillae initialized in the state $|0\rangle_2 \otimes |0\rangle_3$.

7. Please draw the quantum circuit of Deutsch algorithm, and analysis how it

works.

Answer: Please read page 58-61 of lecture "QIP2022chapt_6_Kai Chen.pdf" for reference.

8. Please write down the DiVincenzo criterion that quantum computer implementation must satisfy.

   Answer:

   (1) **Scalability:** A scalable physical system with well characterized parts, usually qubits.

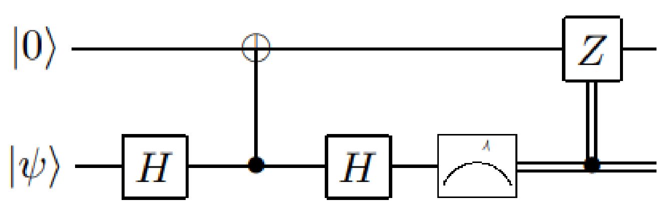
   (2) **Initialization:** The ability to initialize the system in a simple fiducial state.

   (3) **Control:** The ability to control the state of the computer using sequences of elementary universal gates.

   (4) **Stability:** Decoherence times much longer than gate times, together with the ability to suppress decoherence through error correction and fault-tolerant computation.

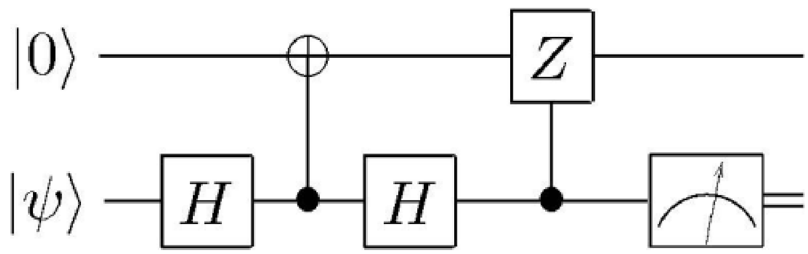
   (5) **Measurement:**The ability to read out the state of the computer in a convenient product basis.

9. Let $|\psi\rangle = a|0\rangle + b|1\rangle$ and consider the following circuit. What is the output state of the top qubit?

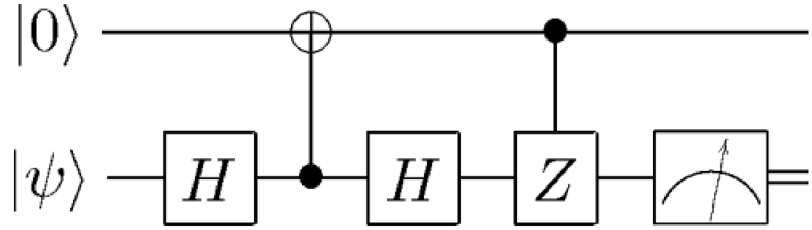


   Answer:

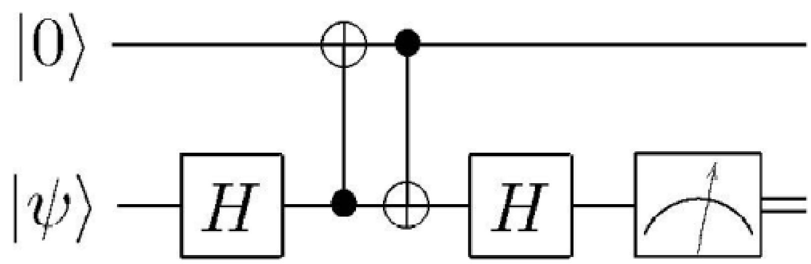
   Classical control operation after measurement is equivalent to quantum control operation before measurement, hence the circuit is equivalent to



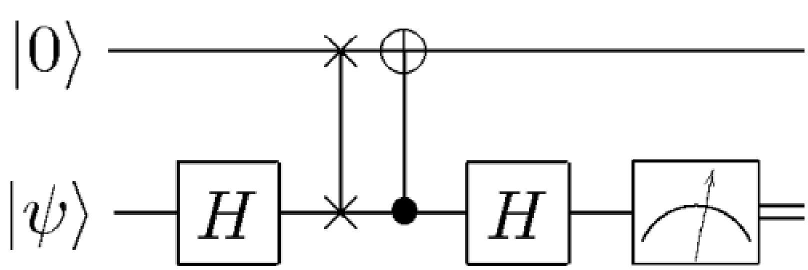
Control-Z operation is symmetric between control bit and target bit, hence the circuit is equivalent to



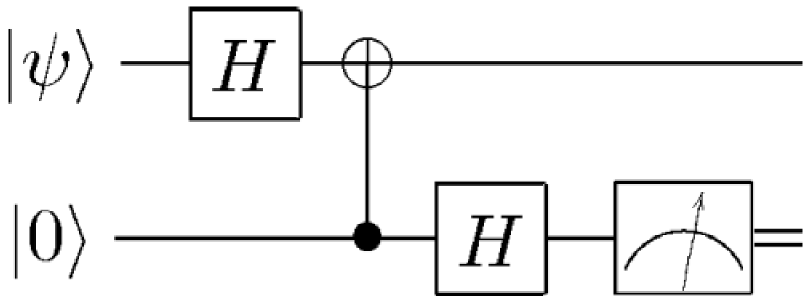
Commuting control-Z through Hadamard we get



Three control-nots with alternating control and target qubit gives swap gates, therefore the circuit is equivalent to



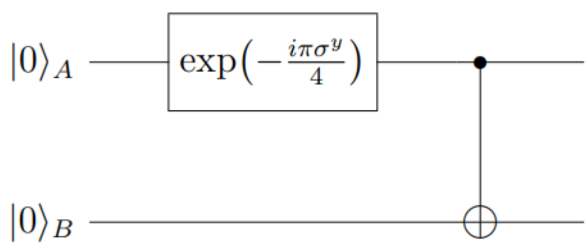
which is equal to



The control-not operation does not change the state as it has $|0\rangle$ as control qubit, therefore it is easy to see that the output qubit on the top line is

$$|\phi\rangle = H|\psi\rangle = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle.$$

10. Evaluate the output of the following quantum circuit.



Answer:

The circuit's input state is

$$|\psi_{in}\rangle_{AB} = |00\rangle.$$

After the first gate, the state is

$$\begin{aligned}
|\psi_1\rangle_{AB} &= [exp(\frac{-i\pi\sigma_A^y}{4}) \otimes I_B]|\psi_{in}\rangle_{AB} \\
&= [(\cos(\frac{\pi}{4})I_A - i\sin(\frac{\pi}{4})\sigma_A^y) \otimes I_B]|00\rangle \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle).
\end{aligned}$$

After the second and final gate, the state is

$$\begin{aligned}
|\psi_{out}\rangle_{AB} &= U_{CNOT}|\psi_1\rangle_{AB} \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).
\end{aligned}$$