Shared-Variable Concurrency

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Parallel Composition (or Concurrency Composition)

Syntax:

$$(comm)\ c ::= \ldots \mid c_0 \parallel c_1 \mid \ldots$$

Note we allow nested parallel composition, e.g., $(c_0\,;(c_1 \parallel c_2)) \parallel c_3$.

Operational Semantics:

$$\frac{(c_0, \sigma) \longrightarrow (c_0', \sigma')}{(c_0 \parallel c_1, \sigma) \longrightarrow (c_0' \parallel c_1, \sigma')} \qquad \frac{(c_1, \sigma) \longrightarrow (c_1', \sigma')}{(c_0 \parallel c_1, \sigma) \longrightarrow (c_0 \parallel c_1', \sigma')}$$

$$\frac{}{(\mathbf{Skip} \parallel \mathbf{Skip}, \sigma) \longrightarrow (\mathbf{Skip}, \sigma)} \qquad \frac{(c_i, \sigma) \longrightarrow (\mathbf{abort}, \sigma'), \quad i \in \{0, 1\}}{(c_0 \parallel c_1, \sigma) \longrightarrow (\mathbf{abort}, \sigma')}$$

We have to use small-step semantics (instead of big-step semantics) to model concurrency.

Interference

Example:

$$\begin{array}{lcl} y := x + 1\,; & & y := x + 1\,; \\ x := y + 1 & \| & x := x + 1 \end{array}$$

Suppose initially $\sigma\, x = \sigma\, y = 0$. What are the possible results?

$(1) y = 1, x = 2;\ (2) y = 1, x = 3;\ (3) y = 3, x = 3;\ (4) y = 2, x = 3$

Two commands c_0 and c_1 are said to *interfere* if:

$$(fv(c_0) \cap fa(c_1)) \cup (fv(c_1) \cap fa(c_0)) \neq \emptyset$$

If c_0 and c_1 interfere, we say there are *race conditions* (or *races*) in $c_0 \parallel c_1$.

When c_0 and c_1 do not interfere, nor terminate by failure, the concurrent composition $c_0 \parallel c_1$ is determinate.

Another Example

A *benign race*:

$$k := -1;$$

$$(\textbf{newvar } i := 0 \textbf{ in while } i \leq n \wedge k = -1 \textbf{ do}$$
$$\quad \textbf{if } f(i) \geq 0 \textbf{ then } k := i \textbf{ else } i := i + 2$$
$$\| \textbf{ newvar } i := 1 \textbf{ in while } i \leq n \wedge k = -1 \textbf{ do}$$
$$\quad \textbf{if } f(i) \geq 0 \textbf{ then } k := i \textbf{ else } i := i + 2)$$

A problematic version:

$$k := -1;$$

$$(\textbf{newvar } i := 0 \textbf{ in while } i \leq n \wedge k = -1 \textbf{ do}$$
$$\quad \textbf{if } f(i) \geq 0 \textbf{ then print}(i)\,;\textbf{print}(f(i)) \textbf{ else } i := i + 2$$
$$\| \textbf{ newvar } i := 1 \textbf{ in while } i \leq n \wedge k = -1 \textbf{ do}$$
$$\quad \textbf{if } f(i) \geq 0 \textbf{ then print}(i)\,;\textbf{print}(f(i)) \textbf{ else } i := i + 2)$$

Conditional Critical Regions

We could use a critical region to achieve mutual exclusive access of shared variables.

Syntax:

$$(\textit{comm}) \quad c \ ::= \ \textbf{await}\ b\ \textbf{then}\ \hat{c}$$

where $\hat{c}$ is a sequential command (a command with no **await** and parallel composition).

Semantics:

$$\frac{[\![b]\!]_{\textit{boolexp}}\,\sigma = \textbf{true} \qquad (\hat{c}, \sigma) \longrightarrow^* (\textbf{Skip}, \sigma')}{(\textbf{await}\ b\ \textbf{then}\ \hat{c}, \sigma) \longrightarrow (\textbf{Skip}, \sigma')}$$

$$\frac{[\![b]\!]_{\textit{boolexp}}\,\sigma = \textbf{false}}{(\textbf{await}\ b\ \textbf{then}\ \hat{c}, \sigma) \longrightarrow (\textbf{Skip}\,;\textbf{await}\ b\ \textbf{then}\ \hat{c}, \sigma)}$$

The second rule gives us a “busy-waiting” semantics. If we eliminate that rule, the thread will be blocked when the condition does not hold.

Achieving Mutual Exclusion

$$
\begin{array}{l}
k := -1; \\[6pt]
(\textbf{newvar } i := 0 \textbf{ in while } i \le n \wedge k = -1 \textbf{ do} \\
\quad (\textbf{if } f(i) \ge 0 \textbf{ then } (\textbf{await } busy = 0 \textbf{ then } busy := 1)\,; \\
\qquad\qquad\qquad\qquad \textbf{print}(i)\,; \textbf{print}(f(i))\,; busy := 0 \\
\qquad\qquad\quad \textbf{else } i := i + 2) \\[6pt]
\parallel \textbf{newvar } i := 1 \textbf{ in while } i \le n \wedge k = -1 \textbf{ do} \\
\quad (\textbf{if } f(i) \ge 0 \textbf{ then } (\textbf{await } busy = 0 \textbf{ then } busy := 1)\,; \\
\qquad\qquad\qquad\qquad \textbf{print}(i)\,; \textbf{print}(f(i))\,; busy := 0 \\
\qquad\qquad\quad \textbf{else } i := i + 2))
\end{array}
$$

A syntactic sugar:

$$\textbf{atomic}\{c\} \stackrel{\text{def}}{=} \textbf{await true then } c$$

We may also use the short-hand notation $\langle c \rangle$.

Semantics:

$$\frac{(c,\ \sigma) \longrightarrow^* (\textbf{Skip}, \sigma')}{(\textbf{atomic}\{c\},\ \sigma) \longrightarrow (\textbf{Skip}, \sigma')}$$

It gives the programmer control over the size of atomic actions.

Reynolds uses **crit** c instead of $\textbf{atomic}\{c\}$.

Deadlock

$$
\begin{array}{lcl}
\textbf{await } busy0 = 0 & & \textbf{await } busy1 = 0 \\
\quad \textbf{then } busy0 := 1; & & \quad \textbf{then } busy1 := 1; \\
\textbf{await } busy1 = 0 & & \textbf{await } busy0 = 0 \\
\quad \textbf{then } busy1 := 1; & \parallel & \quad \textbf{then } busy0 := 1; \\
\ldots & & \ldots \\
busy0 := 0; & & busy0 := 0; \\
busy1 := 0; & & busy1 := 0;
\end{array}
$$

Fairness

$$k := -1;$$

$$\begin{array}{l}(\textbf{newvar } i := 0 \textbf{ in while } k = -1 \textbf{ do} \\ \quad \textbf{if } f(i) \geq 0 \textbf{ then } k := i \textbf{ else } i := i + 2 \\ \| \textbf{ newvar } i := 1 \textbf{ in while } k = -1 \textbf{ do} \\ \quad \textbf{if } f(i) \geq 0 \textbf{ then } k := i \textbf{ else } i := i + 2)\end{array}$$

Suppose $f(i) < 0$ for all even number i. Then there's an infinite execution in the form of:

$$\ldots \longrightarrow (c_1 \parallel c', \sigma_1) \longrightarrow (c_2 \parallel c', \sigma_2) \longrightarrow \ldots \longrightarrow (c_n \parallel c', \sigma_n) \longrightarrow \ldots$$

An execution of concurrent processes is *unfair* if it does not terminate but, after some finite number of steps, there is an unterminated process that never makes a transition.

Fairness — More Examples

A fair execution of the following program would always terminate:

$$\textbf{newvar}\ y := 0\ \textbf{in}\ (x := 0\textbf{;}((\textbf{while}\ y = 0\ \textbf{do}\ x := x + 1) \parallel y := 1))$$

Stronger fairness is needed to rule out infinite execution of the following program:

$$\begin{array}{l} \textbf{newvar}\ y := 0\ \textbf{in} \\ \quad (x := 0\textbf{;} \\ \quad\ ((\textbf{while}\ y = 0\ \textbf{do}\ x := 1 - x) \parallel (\textbf{await}\ x = 1\ \textbf{then}\ y := 1)) \\ \quad) \end{array}$$

Trace Semantics

Can we give a denotational semantics to concurrent programs? The domain-based approach is complex. Here we use *transition traces* to model the execution of programs.

Execution of (c_0, σ_0) in a concurrent setting:

$(c_0, \sigma_0) \longrightarrow (c_1, \sigma'_0), (c_1, \sigma_1) \longrightarrow (c_2, \sigma'_1), \ldots, (c_{n-1}, \sigma_{n-1}) \longrightarrow (\mathbf{Skip}, \sigma'_{n-1})$

The gap between σ'_i and σ_{i+1} reflects the intervention of the environment (other threads).

It could be infinite if (c_0, σ_0) does not terminate:

$$(c_0, \sigma_0) \longrightarrow (c_1, \sigma_1), (c_1, \sigma'_1) \longrightarrow (c_2, \sigma_2), \ldots$$

We omit the commands to get a transition trace:

$$(\sigma_0, \sigma'_0), (\sigma_1, \sigma'_1), \ldots, (\sigma_{n-1}, \sigma'_{n-1})$$

or

$$(\sigma_0, \sigma'_0), (\sigma_1, \sigma'_1), \ldots$$

Interference-Free Traces

A trace $(\sigma_0, \sigma'_0), (\sigma_1, \sigma'_1), \ldots, (\sigma_{n-1}, \sigma'_{n-1})$ (or $(\sigma_0, \sigma'_0), (\sigma_1, \sigma'_1), \ldots)$ is said to be *Interference-Free* iff $\forall i.\, \sigma'_i = \sigma_{i+1}$.

Operations over Traces

We use τ to represent individual transition traces, and $\mathcal{T}$ for a set of traces.

$$
\begin{array}{lcl}
\epsilon & & \text{empty trace} \\
\tau_1 {+\!\!+} \tau_2 & \stackrel{\text{def}}{=} & \text{concatenation of } \tau_1 \text{ and } \tau_2 \\
 & & \quad \tau_1 \text{ if } \tau_1 \text{ is infinite.} \\
\mathcal{T}_1\,;\mathcal{T}_2 & \stackrel{\text{def}}{=} & \{\tau_1 {+\!\!+} \tau_2 \mid \tau_1 \in \mathcal{T}_1 \text{ and } \tau_2 \in \mathcal{T}_2\} \\
\mathcal{T}^0 & \stackrel{\text{def}}{=} & \{\epsilon\} \\
\mathcal{T}^{n+1} & \stackrel{\text{def}}{=} & \mathcal{T}\,;\mathcal{T}^n \\
\mathcal{T}^* & \stackrel{\text{def}}{=} & \bigcup\limits_{n=0}^{\infty} \mathcal{T}^n \\
\mathcal{T}^\omega & \stackrel{\text{def}}{=} & \{\tau_0 {+\!\!+} \tau_1 {+\!\!+} \ldots \mid \tau_i \in \mathcal{T}\}
\end{array}
$$

Note the difference between $\mathcal{T}^*$ and $\mathcal{T}^\omega$.

Trace Semantics — First Try

$$
\begin{aligned}
\mathcal{T}[\![x := e]\!] &= \left\{(\sigma, \sigma') \mid \sigma' = \sigma\{x \rightsquigarrow [\![e]\!]_{intexp}\,\sigma\}\right\} \\
\mathcal{T}[\![\mathbf{Skip}]\!] &= \{(\sigma, \sigma) \mid \sigma \in \Sigma\} \\
\mathcal{T}[\![c_0\,;c_1]\!] &= \mathcal{T}[\![c_0]\!]\,;\mathcal{T}[\![c_1]\!] \\
\mathcal{T}[\![\mathbf{if}\ b\ \mathbf{then}\ c_1\ \mathbf{else}\ c_2]\!] &= (\mathcal{B}[\![b]\!]\,;\mathcal{T}[\![c_1]\!]) \cup (\mathcal{B}[\![\neg b]\!]\,;\mathcal{T}[\![c_2]\!]) \\
&\quad \text{where } \mathcal{B}[\![b]\!] = \{(\sigma, \sigma) \mid [\![b]\!]_{boolexp}\,\sigma = \mathbf{true}\} \\
\mathcal{T}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!] &= ((\mathcal{B}[\![b]\!]\,;\mathcal{T}[\![c]\!])^*\,;\mathcal{B}[\![\neg b]\!]) \cup (\mathcal{B}[\![b]\!]\,;\mathcal{T}[\![c]\!])^\omega
\end{aligned}
$$

How to give semantics to **newvar**$x := e$ **in** c?

Definition: *local-global*$(x, e, \tau, \hat{\tau})$ iff the following are true (suppose $\tau = (\sigma_0, \sigma'_0), (\sigma_1, \sigma'_1), \ldots$ and $\hat{\tau} = (\hat{\sigma_0}, \hat{\sigma'_0}), (\hat{\sigma_1}, \hat{\sigma'_1}), \ldots$):

- they have the same length;
- for all $x' \neq x$, $\sigma_i\, x' = \hat{\sigma}_i\, x'$ and $\sigma'_i\, x' = \hat{\sigma}'_i\, x'$;
- for all i, $\sigma_{i+1}\, x = \sigma'_i\, x$;
- for all i, $\hat{\sigma}_i\, x = \hat{\sigma}'_i\, x$;
- $\sigma_0\, x = [\![e]\!]_{intexp}\, \hat{\sigma}_0$.

$$\mathcal{T}[\![\mathbf{newvar}\, x := e\ \mathbf{in}\ c]\!] = \{\hat{\tau} \mid \tau \in \mathcal{T}[\![c]\!] \text{ and } \textit{local-global}(x, e, \tau, \hat{\tau})\}$$

Fair Interleaving

We view a trace τ as a function mapping indices to the corresponding transitions.

Definition: *fair-merge*(τ_1, τ_2, τ) iff there exist functions $f \in \mathrm{dom}(\tau_1) \to \mathrm{dom}(\tau)$ and $g \in \mathrm{dom}(\tau_2) \to \mathrm{dom}(\tau)$ such that the following are true:

- f and g are *monotone injections*:

$$i < j \Longrightarrow (f\, i < f\, j) \wedge (g\, i < g\, j)$$

- $\mathrm{ran}(f) \cap \mathrm{ran}(g) = \emptyset$ and $\mathrm{ran}(f) \cup \mathrm{ran}(g) = \mathrm{dom}(\tau)$;
- $\forall i.\, \tau_1(i) = \tau(f\, i) \wedge \tau_2(i) = \tau(g\, i)$

Then $\mathcal{T}_{fair}[\![c_1 \parallel c_2]\!] =$
$\{\tau \mid \exists \tau_1 \in \mathcal{T}_{fair}[\![c_1]\!], \tau_2 \in \mathcal{T}_{fair}[\![c_2]\!].\ \textit{fair-merge}(\tau_1, \tau_2, \tau)\}$

Definition: *unfair-merge*(τ_1, τ_2, τ) if one of the following are true:

- *fair-merge*(τ_1, τ_2, τ)
- τ_1 is infinite and there exist τ_2' and τ_2'' such that $\tau_2 = \tau_2' {+\!\!+} \tau_2''$ and *fair-merge*(τ_1, τ_2', τ)
- τ_2 is infinite, and there exist τ_1' and τ_1'' such that $\tau_1 = \tau_1' {+\!\!+} \tau_1''$ and *fair-merge*(τ_1', τ_2, τ)

$$\begin{aligned}
&\mathcal{T}_{unfair}[\![c_1 \parallel c_2]\!] \\
&\quad = \{\tau \mid \exists \tau_1 \in \mathcal{T}_{unfair}[\![c_1]\!], \tau_2 \in \mathcal{T}_{unfair}[\![c_2]\!].\ \textit{unfair-merge}(\tau_1, \tau_2, \tau)\}
\end{aligned}$$

Trace Semantics for **await**

$$
\begin{aligned}
\mathcal{T}[\![\textbf{await } b \textbf{ then } c]\!] \;=\;& \\
& (\mathcal{B}[\![\neg b]\!]\,;\mathcal{T}[\![\textbf{Skip}]\!])^*\,; \\
& \quad \{(\sigma,\sigma') \mid [\![b]\!]_{boolexp}\,\sigma = \textbf{true} \\
& \qquad\qquad \text{and there exist } \sigma'_0,\sigma_1,\sigma'_1,\ldots,\sigma_n \text{ such that} \\
& \qquad\qquad\quad (\sigma,\sigma'_0),(\sigma_1,\sigma'_1),\ldots,(\sigma_n,\sigma') \in \mathcal{T}[\![c]\!] \\
& \qquad\qquad\qquad \text{and it is Interference-Free.}\} \\
& \cup(\mathcal{B}[\![\neg b]\!]\,;\mathcal{T}[\![\textbf{Skip}]\!])^\omega
\end{aligned}
$$

The semantics is equivalent to the following:

$$
\begin{array}{l}
\mathcal{T}[\![c]\!] \stackrel{\text{def}}{=} \\
\quad \{(\sigma_0, \sigma'_0), \ldots, (\sigma_n, \sigma'_n) \mid \\
\quad\quad \text{there exist } c_0, \ldots, c_n \text{ such that } c_0 = c, \\
\quad\quad\quad \forall i \in [0, n-1].\, (c_i, \sigma_i) \longrightarrow (c_{i+1}, \sigma'_i), \\
\quad\quad\quad \text{and } (c_n, \sigma_n) \longrightarrow (\textbf{Skip}, \sigma'_n)\} \\
\quad \cup \{(\sigma_0, \sigma'_0), (\sigma_1, \sigma'_1), \ldots \mid \\
\quad\quad\quad \text{there exist } c_0, c_1, \ldots \text{ such that } c_0 = c, \\
\quad\quad\quad\quad \text{and for all } i, (c_i, \sigma_i) \longrightarrow (c_{i+1}, \sigma'_i)\}
\end{array}
$$

Problem with This Semantics

The trace semantics we just defined is not abstract enough.
It distinguishes the following programs (which should be viewed equivalent):

$$x := x+1$$

$$x := x+1\,\mathbf{;}\,\mathbf{Skip}$$

$$\mathbf{Skip}\,\mathbf{;}\,x := x+1$$

Also consider the following two programs:

$$x := x+1\,\mathbf{;}\,x := x+1$$

$$(x := x+1\,\mathbf{;}\,x := x+1)\ \mathbf{choice}\ x := x+2$$

Stuttering and Mumbling

$$\frac{}{\tau \prec \tau} \qquad \frac{}{\tau \prec (\sigma, \sigma), \tau} \qquad \frac{}{(\sigma, \sigma'), (\sigma', \sigma''), \tau \prec (\sigma, \sigma''), \tau}$$

$$\frac{\tau \prec \tau' \quad \tau' \prec \tau''}{\tau \prec \tau''} \qquad \frac{\tau \prec \tau'}{(\sigma, \sigma'), \tau \prec (\sigma, \sigma'), \tau'}$$

$$\mathcal{T}^{\dagger} \stackrel{\text{def}}{=} \{\tau \mid \tau \in \mathcal{T} \text{ or } \exists \tau' \in \mathcal{T}.\, \tau' \prec \tau\}$$

$$\mathcal{T}^{*}[\![c]\!] \stackrel{\text{def}}{=} (\mathcal{T}[\![c]\!])^{\dagger}$$

Stuttering and Mumbling (cont'd)

The new semantics $\mathcal{T}^*[\![c]\!]$ is equivalent to the following:

$$
\begin{array}{l}
\mathcal{T}[\![c]\!] \stackrel{\text{def}}{=} \\
\quad \{(\sigma_0, \sigma'_0), \ldots, (\sigma_n, \sigma'_n) \mid \\
\quad\quad \text{there exist } c_0, \ldots, c_n \text{ such that } c_0 = c, \\
\quad\quad\quad \forall i \in [0, n-1].\, (c_i, \sigma_i) \longrightarrow^* (c_{i+1}, \sigma'_i), \\
\quad\quad\quad \text{and } (c_n, \sigma_n) \longrightarrow^* (\textbf{Skip}, \sigma'_n)\} \\
\quad \cup \{(\sigma_0, \sigma'_0), (\sigma_1, \sigma'_1), \ldots \mid \\
\quad\quad\quad \text{there exist } c_0, c_1, \ldots \text{ such that } c_0 = c, \\
\quad\quad\quad\quad \forall i.\, (c_i, \sigma_i) \longrightarrow^* (c_{i+1}, \sigma'_i), \\
\quad\quad\quad\quad \text{and for infinitely many } i \geq 0, (c_i, \sigma_i) \longrightarrow^+ (c_{i+1}, \sigma'_i)\}
\end{array}
$$