An Introduction to Modern Astrophysics notes

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Peculiar Motions and the Local Standard of Rest

The velocity components of stars in the solar neighborhood are traditionally labeled

$$\Pi \equiv \frac{dR}{dt}, \Theta \equiv R\frac{d\theta}{dt}, Z \equiv \frac{dz}{dt}$$

Defining the **dynamical local standard of rest** (dynamical LSR) to be a point that is instantaneously centered on the Sun and moving in a perfectly circular orbit along the solar circle about the Galactic center.

$$\Pi_{LSR} = 0, \Theta_{LSR} = \Theta_0, Z_{LSR} = 0$$

An alternative definition for the LSR known as the **kinematic local standard of rest** (kinematic LSR) is based on the average motions of stars in the solar neighborhood.

$$< \Pi >= 0, < \Theta >< \Theta_0, < Z >= 0$$

The velocity of a star relative to the dynamical LSR is known as the star's **peculiar velocity** and is given by

$$u = \Pi - \Pi_{LSR} = \Pi, v = \Theta - \Theta_{LSR} = \Theta - \Theta_0, w = Z - Z_{LSR} = Z$$

The average of u, v and w for all stars in the solar neighborhood, excluding the Sun, is

$$< u >= 0，\ < v >= C\sigma_u^2 = C < u^2 >< 0，\ < w >= 0$$

Why $< \Theta >< \Theta_0 (< v >< 0)$?

- The stars inside the Sun's orbit are in the **apogalacticon** of their elliptical orbits when they pass through the Sun's neighborhood, so the tangential velocity Θ is smaller.
- The stars outside the Sun's orbit are in the **perigalacticon** of their elliptical orbits when they pass through the Sun's neighborhood, so the tangential velocity Θ is larger.
- There are more stars inside the Sun's orbit than beyond it.

Why should σ_u correlate with < v > ?

- larger σ_u, wider range of elliptical orbits included, more negative < v >
- smaller σ_u, fewer stars with orbits noncircular, < v > ~ 0

Differential Galactic Rotation and Oort's Constants

The relative radial and transverse velocities of a star (at point S) to the Sun (at point O) are, respectively,

$$v_r = \Theta cos\alpha - \Theta_0 sinl$$
$$v_t = \Theta sin\alpha - \Theta_0 cosl$$

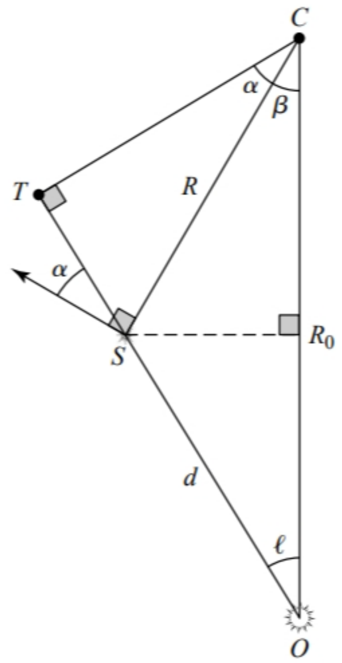

FIGURE 22 The geometry of analyzing differential rotation in the Galactic plane. The Sun is at point O, the center of the Galaxy is located at C, and the star is at S, located a distance d from the Sun. ℓ is the Galactic longitude of the star at S, and α and β are auxiliary angles. The directions of motion reflect the clockwise rotation of the Galaxy as viewed from the NGP.

where Θ_0 is the orbital velocity of the Sun in the idealized case of perfectly circular motion (actually the orbital velocity of the LSR) and α is defined in the figure. Defining the *angular-velocity curve* to be

$$\Omega(R) \equiv \frac{\Theta(R)}{R},$$

the relative radial and transverse velocities become

$$v_r = \Omega R \cos\alpha - \Omega_0 R_0 \sin\ell,$$

$$v_t = \Omega R \sin\alpha - \Omega_0 R_0 \cos\ell.$$

Now, by referring to the geometry of Fig. 22 and considering the right triangle ΔOTC, we find

$$R\cos\alpha = R_0 \sin\ell,$$

$$R\sin\alpha = R_0 \cos\ell - d.$$

Substituting these relations into the previous expressions, we have

$$v_r = (\Omega - \Omega_0)\, R_0 \sin\ell, \qquad (37)$$

$$v_t = (\Omega - \Omega_0)\, R_0 \cos\ell - \Omega d. \qquad (38)$$

Equations (37) and (38) are valid as long as the assumption of circular motion is justified.

Oort derived a set of approximate equations for v_r and v_t that are valid only in the region near the Sun. Defining the **Oort constants**

$$A \equiv -\frac{1}{2}(\frac{d\Theta}{dR}|_{R_0} - \frac{\Theta_0}{R_0})$$
$$B \equiv -\frac{1}{2}(\frac{d\Theta}{dR}|_{R_0} + \frac{\Theta_0}{R_0})$$

Using the Taylor expansion of Ω(R), the difference between Ω and Ω_0, and the approximate value of Ω is

$$\Omega - \Omega_0 \approx \frac{d\Omega}{dR}|_{R_0}(R-R_0) = (\frac{1}{R_0}\frac{d\Theta}{dR}|_{R_0} - \frac{\Theta_0}{R_0^2})(R-R_0) = (-2A/R_0)(-dcosl)$$
$$\Omega \approx \Omega_0 = A - B$$

Inserting equations above into Eqs. (37) and (38) results in

$$v_r \approx Adsin2l$$
$$v_t \approx Adcos2l + Bd$$

Some Equations of Cosmology

I'm going to skip the derivation of these equations here, which takes general relativity to understand them completely, but the reader should at least know how to derive these equations from Newtonian physics.

- Friedmann equation, Hubble parameter and density parameter:

$$H^2(t) \equiv (\frac{1}{R(t)}\frac{dR(t)}{dt})^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{R^2(t)} = \frac{8\pi G}{3}\rho_c(t)$$
$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho(t)}{3H^2(t)} = 1 + \frac{kc^2}{R^2(t)H^2(t)}$$
$$where\ \rho(t) = \rho_m(t) + \rho_r(t) + \rho_\Lambda, \rho_\Lambda = \frac{\Lambda c^2}{8\pi G} = const$$
$$\Omega(t) = \Omega_m(t) + \Omega_r(t) + \Omega_\Lambda(t), \Omega_\Lambda(t) = \frac{\Lambda c^2}{3H^2(t)}$$

- Fluid equation:

$$\frac{d(R^3(t)\rho(t))}{dt} = -\frac{P(t)}{c^2}\frac{d(R^3(t))}{dt}$$
$$where\ P(t) = P_m(t) + P_r(t) + P_\Lambda$$

- Equation of state:

$$P(t) = w\rho(t)c^2$$
$$where\ w_m = 0, w_r = \frac{1}{3}, w_\Lambda = -1$$

- Acceleration equation:

$$\frac{d^2R(t)}{dt^2} = -\frac{4\pi G}{3}(\rho(t) + \frac{3P(t)}{c^2})R(t)$$

- Deceleration parameter:

$$q(t) \equiv -\frac{R(t)}{(dR(t)/dt)^2}\frac{d^2R(t)}{dt^2} = -\frac{1}{R(t)H^2(t)}\frac{d^2R(t)}{dt^2} = \frac{1}{2}(1+3w_i)\Omega_i(t)$$

- Cosmological redshift:

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{R(t_0)}{R(t)} = \frac{1}{R(t)}$$

- Density and Hubble parameter as a function of redshift z:

$$\rho(z) = \rho_0(1+z)^{3(1+w)}$$
$$H^2(z) = H_0^2[\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0} + (1-\Omega_0)(1+z)^2]$$

where the subscript 0 represents the present value.

- The constant k determines the ultimate fate of the universe:
  - If $k > 0$, the total energy of the shell is negative, and the universe is *bounded*, or **closed**. In this case, the expansion will someday halt and reverse itself.
  - If $k < 0$, the total energy of the shell is positive, and the universe is *unbounded*, or **open**. In this case, the expansion will continue forever.
  - If $k = 0$, the total energy of the shell is zero, and the universe is **flat**, neither open nor closed. In this case, the expansion will continue to slow down, coming to a halt only as $t \to \infty$ and the universe is infinitely dispersed.

Transition from the Radiation Era to the Matter Era to the Λ Era

The behavior of the scale factor R(t) for a flat universe can be found by setting k = 0 in the Friedmann equation.

Radiation era when t << $t_{r,m}$: $R(t) \propto t^{1/2}$

The transition from the radiation era to the matter era occurred when the scale factor satisfied

$$\rho_r = \rho_m \rightarrow R_{r,m} \approx 3.05 \times 10^{-4}, z_{r,m} \approx 3270, t_{r,m} \approx 5.52 \times 10^4 yr$$

Matter era when $t_{r,m}$ << t << t_H: $R(t) \propto t^{2/3}$

The acceleration of the universe changed sign (from negative to positive) when the scale factor was

$$R_{accel} \approx 0.57, z_{accel} \approx 0.76, t_{accel} \approx 7.08 Gyr$$

The transition from the matter era to the Λ era occurred when the scale factor satisfied

$$\rho_m = \rho_\Lambda \rightarrow R_{m,\Lambda} \approx 0.72, z_{m,\Lambda} \approx 0.39, t_{m,\Lambda} \approx 9.55 Gyr$$

Λ **era** when t >> t_H: $R(t) \propto e^{H_0 t \sqrt{\Omega_{\Lambda,0}}}$

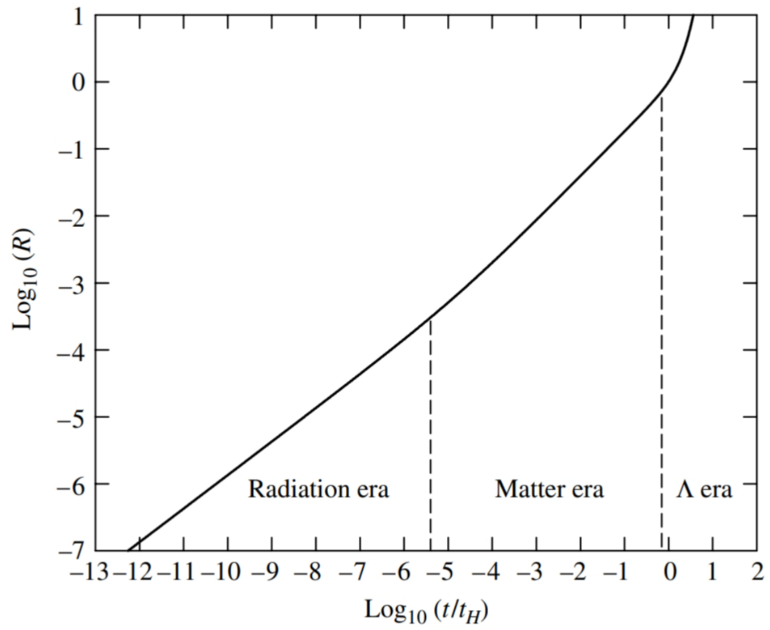

FIGURE 19 A logarithmic graph of the scale factor R as a function of time. During the radiation era, $R \propto t^{1/2}$; during the matter era, $R \propto t^{2/3}$; and during the Λ era, R grows exponentially.

Distances to the Most Remote Objects in the Universe

The **Robertson–Walker metric** determines the spacetime interval between two events in an isotropic, homogeneous universes.

$$ds^2 = c^2dt^2 - R^2(t)(\frac{d\varpi^2}{1-k\varpi^2} + \varpi^2 d\theta^2 + \varpi^2 sin^2\theta d\phi^2)$$

With ds = 0 for a light ray, and dθ = dφ = 0 for a radial path traveled from the point of the light's emission at **comoving coordinate** ϖ_e to its arrival at Earth at ϖ = 0, taking the negative square root (so ϖ decreases with increasing time) gives

$$\frac{-cdt}{R(t)} = \frac{d\varpi}{\sqrt{1-k\varpi^2}}$$

$$\therefore \int_{t_e}^{t_0} \frac{cdt}{R(t)} = \int_0^{\varpi_e} \frac{d\varpi}{\sqrt{1-k\varpi^2}} = \begin{cases} \varpi_e \cdots for\ \Omega_0 = 1, then\ k = 0 \\ sin^{-1}\varpi_e \cdots for\ \Omega_0 > 1, then\ k > 0 \\ sinh^{-1}\varpi_e \cdots for\ \Omega_0 < 1, then\ k < 0 \end{cases}$$

Defining two dimensionless integrals

$$I(z) \equiv H_0 \int_{t_e}^{t_0} \frac{dt}{R(t)} = H_0 \int_0^z \frac{dz'}{H(z')} = z - \frac{1}{2}(1+q_0)z^2 + (\frac{1}{6} + \frac{2}{3}q_0 + \frac{1}{2}q_0^2 + \frac{1}{6}(1-\Omega_0))z^3 - \cdots$$

$$where\ \frac{dz}{dt} = -\frac{1}{R^2(t)}\frac{dR(t)}{dt} = -\frac{H(t)}{R(t)}$$

$$S(z) \equiv \begin{cases} I(z) \cdots for\ \Omega_0 = 1, then\ k = 0 \\ \frac{1}{\sqrt{\Omega_0 - 1}} sin(I(z)\sqrt{\Omega_0 - 1}) \cdots for\ \Omega_0 > 1, then\ k > 0 \\ \frac{1}{\sqrt{1-\Omega_0}} sinh(I(z)\sqrt{1-\Omega_0}) \cdots for\ \Omega_0 < 1, then\ k < 0 \end{cases} \approx z - \frac{1}{2}(1+q_0)z^2 \cdots for\ z << 1$$

Therefore the comoving coordinate as a function of the redshift is $\varpi(z) = \frac{c}{H_0}S(z)$. Now we are ready for the concept of four distances at time t_0.

$$Coordinate\ distance: r_0(z) = \varpi(z) = \frac{c}{H_0}S(z)$$

$$Proper\ distance: d_{p,0}(z) = \int_{t_e}^{t_0}\frac{cdt}{R(t)} = \frac{c}{H_0}I(z)$$

$$Luminosity\ distance: d_{L,0}(z) = r_0(z)(1+z) = \frac{c}{H_0}S(z)(1+z)$$

$$Angular\ diameter\ distance: d_{A,0}(z) = \frac{r_0(z)}{1+z} = \frac{c}{H_0}\frac{S(z)}{1+z}$$

Multiplying by the scale factor R(t) then converts these to the distances at some other time t.

The proper distance to the farthest observable point at time t is called the particle horizon

$$d_h(t) = R(t)\int_0^t \frac{cdt'}{R(t')} = \begin{cases} 2ct \cdots during\ \ radiation\ \ era \\ 3ct \cdots during\ \ matter\ \ era \\ \rightarrow 19.3Gpc \cdots during\ \ \Lambda\ \ era \end{cases}$$