COROLLARY. *For every* $(n-1)$*-tuple* $a_1 \cdots a_{n-1}$ *in* G_n *there exists a unique* $b \in G_n$ *such that* $(ba_1 \cdots a_{n-1}) = e$.

THEOREM II. *If* G_n *is an abstract algebra with one operation and if there exists a group* G *defined on the same set as* G_n *and whose binary operation is derived from the operation in* G_n *and such that each n-product in* G_n *can be rewritten as an iteration of the binary operation in* G*, then it is necessary and sufficient that* G_n *be a hypergroup.*

Proof. We know from Theorem I that G_n is a hypersemigroup with identity. For any $a \in G_n$ there exists a unique $(n-1)$-tuple $a^{-1}e \cdots e$ such that $(a^{-1}e \cdots ea) = (aa^{-1}e \cdots e) = e$, since G is a group. On the other hand, we define a binary operation $(*)$ in the same way as in Theorem I. This results in a semigroup with identity G. We see from the corollary to Lemma II that if $ae \cdots e$ is an $(n-1)$-tuple in G_n, there exists a unique a^{-1} such that $(a^{-1}ae \cdots e) = a^{-1} * a = e$, so that the pair $(G_n, *)$ is a group. The rest of the proof follows as in Theorem I.

I wish to thank the referee for a very valuable suggestion.

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CLASSROOM NOTES

THE SCARCITY OF CROSS PRODUCTS ON EUCLIDEAN SPACES

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When first introduced to the dot product on $\mathbf{R}^n$ (n arbitrary) and the cross product on $\mathbf{R}^3$, students are not unlikely to inquire about the definition of a cross product for $\mathbf{R}^n$, $n \neq 3$. This note points out that by proving the elementary propositions and theorem below, students may answer this question themselves. Namely, they may show that if reasonable demands are made of "cross products" (that they satisfy the axioms for cross products in Apostol's recent Calculus text [2]), then only on $\mathbf{R}^1$, $\mathbf{R}^3$ and $\mathbf{R}^7$ can cross products exist—the one on $\mathbf{R}^1$ being trivial and the ones on $\mathbf{R}^7$ somewhat pathological. We indicate the elementary proofs, for whose rediscovery students will need knowledge of the fact that orthonormal (o.n.) sets in $\mathbf{R}^n$ have cardinalities $\leqq n$, with equality if and only if the o.n. set is a basis; orthogonal projection is required a couple of times, but projection is onto a subspace for which an orthonormal basis has already been constructed, so that the projection can be written out explicitly, componentwise.

The axioms [2, p. 275] are

$$(5.28)\qquad \mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$$

$$(5.29)\qquad \mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$$

$$(5.30)\qquad \lambda(\mathbf{a}\times\mathbf{b}) = (\lambda\mathbf{a})\times\mathbf{b}$$

$$(5.31)\qquad \mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$$

$$(5.32)\qquad |\mathbf{a}\times\mathbf{b}| = [|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2]^{1/2}$$

for all scalars λ and all vectors **a, b, c.** One can easily verify that in the presence of the other four axioms, (5.32) is equivalent to the statement that the cross product of two orthogonal unit vectors is also a unit vector, so that in spite of its complicated appearance the axiom is natural.

Our principal tools are the identities

$$(1)\qquad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = -[\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})]$$

and

$$(2)\qquad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 2(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c};$$

the first of these, which expresses the fact that the linear transformation $\mathbf{b}\rightarrow(\mathbf{a}\times\mathbf{b})$ is skew-symmetric, is easily derived from (5.31) by setting $\mathbf{a}+\mathbf{c}$ where **a** occurs, using linearity and symmetry, and canceling what one already knows about. The second is derived by two applications of the "polarization" trick: writing (5.32) as

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2$$

and setting $\mathbf{b}+\mathbf{d}$ where **b** is, using bilinearity and symmetry to get an expansion of both sides and canceling what one already knows about, one gets

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d}).$$

This can be written, using (1) and the linearity of the dot product, as

$$-[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]\cdot\mathbf{d} = [(\mathbf{a}\cdot\mathbf{a})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{a}]\cdot\mathbf{d}$$

and since $\mathbf{d}\in\mathbf{R}^n$ is arbitrary, this gives

$$(3)\qquad \mathbf{a}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b},$$

a useful special case of (2). Now setting $\mathbf{a}+\mathbf{c}$ wherever **a** is in (3), expanding by linearity, canceling what one already knows about and using the skew-symmetry of the cross product (i.e. (5.28)) leads directly to (2). A useful special case, incidentally, is

(4) For mutually orthogonal **a, b, c,** $\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

which one sees by observing that the right side of (2) is zero whenever **a, b** and **c** are mutually orthogonal. The identities (1), (2), (3) and (4) hold for any cross product satisfying the axioms, on any $\mathbf{R}^n$.

Given such a cross product on $\mathbf{R}^n$, we make two definitions:

(I) A vector subspace A of $\mathbf{R}^n$ is *closed under* $\times$ if whenever $\mathbf{a}_1 \in A$ and $\mathbf{a}_2 \in A$, then also $\mathbf{a}_1 \times \mathbf{a}_2 \in A$.

(II) A subspace B of $\mathbf{R}^n$ is *stable under* $A\times$, where $A \subseteq \mathbf{R}^n$, if whenever $\mathbf{a} \in A$ and $\mathbf{b} \in B$, then $\mathbf{a} \times \mathbf{b} \in B$.

So A is closed under $\times$ iff stable under $A\times$. We now have

PROPOSITION 1. *Let $A \subseteq \mathbf{R}^n$, B be a subspace of $\mathbf{R}^n$. If B is stable under $A\times$, then so is its orthogonal complement $B^\perp = \{\mathbf{c} \mid \mathbf{c} \in \mathbf{R}^n,\ \mathbf{b}\cdot\mathbf{c} = 0$ for all $\mathbf{b} \in B\}$.*

Proof. If $\mathbf{c} \in B^\perp$, then for any $\mathbf{a} \in A$ and $\mathbf{b} \in B$

$$\mathbf{b}\cdot(\mathbf{a} \times \mathbf{c}) = -[(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}] = 0 \quad \text{since } \mathbf{a} \times \mathbf{b} \in B.$$

Thus for any $\mathbf{b} \in B$, $\mathbf{b} \perp (\mathbf{a} \times \mathbf{c})$, and so $\mathbf{a} \times \mathbf{c} \in B^\perp$. ∎

PROPOSITION 2. *Let A be a subspace of $\mathbf{R}^n$ which is closed under $\times$ and possesses an orthonormal basis $\{\mathbf{f}_1, \cdots, \mathbf{f}_k\}$. Let $\mathbf{b} \in A^\perp$. Then the vectors $\{\mathbf{b}, \mathbf{f}_1 \times \mathbf{b}, \cdots, \mathbf{f}_k \times \mathbf{b}\}$ lie in $A^\perp$ and are mutually orthogonal and of the same length as $\mathbf{b}$. In particular if $\mathbf{b}$ is a unit vector they form an orthonormal set of $k+1$ elements in $A^\perp$.*

Proof. That $\{\mathbf{b}, \mathbf{f}_1 \times \mathbf{b}, \cdots\} \subseteq A^\perp$ follows from Proposition 1. The orthogonality relations and lengths follow from $\mathbf{b}\cdot(\mathbf{f}_i \times \mathbf{b}) = 0$ and

$$\begin{aligned}(\mathbf{f}_i \times \mathbf{b})\cdot(\mathbf{f}_j \times \mathbf{b}) &= (\mathbf{b} \times \mathbf{f}_i)\cdot(\mathbf{b} \times \mathbf{f}_j) \\ &= -[\mathbf{b} \times (\mathbf{b} \times \mathbf{f}_i)\cdot\mathbf{f}_j] \quad \text{by (1)} \\ &= -[(\mathbf{b}\cdot\mathbf{f}_i)\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\mathbf{f}_i]\cdot\mathbf{f}_j = (\mathbf{f}_i\cdot\mathbf{f}_j)(\mathbf{b}\cdot\mathbf{b}). \ \blacksquare\end{aligned}$$

It is easy to verify that the only cross product on $\mathbf{R}^1$ is identically zero, and that there is no cross product on $\mathbf{R}^2$: it couldn't be zero, and there aren't enough dimensions for it to be nonzero. Suppose we have a cross product on $\mathbf{R}^n$, then, $n \geqq 3$. If $\mathbf{e}_1$ and $\mathbf{e}_2$ are an orthonormal pair of elements of $\mathbf{R}^n$, then $\mathbf{e}_1 \times \mathbf{e}_2$ is a unit (by (5.32)) vector normal to both $\mathbf{e}_1$ and $\mathbf{e}_2$; set $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. Let H be the linear subspace of $\mathbf{R}^n$ spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Regardless of what n may be,

PROPOSITION 3. *H is closed under $\times$.*

Proof. Bilinearity of $\times$ insures that it suffices to check this on basis vectors. By identity (3)

$$\begin{aligned}\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_2 \times (\mathbf{e}_1 \times \mathbf{e}_2) &= -[\mathbf{e}_2 \times (\mathbf{e}_2 \times \mathbf{e}_1)] \\ &= -[(\mathbf{e}_2\cdot\mathbf{e}_1)\mathbf{e}_2 - (\mathbf{e}_2\cdot\mathbf{e}_2)\mathbf{e}_1] = \mathbf{e}_1\end{aligned}$$

and similarly $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$. ∎

Thus the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ has the same multiplication table as the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for $\mathbf{R}^3$ with the usual cross product.

One can resolve any vector $\mathbf{a} \in \mathbf{R}^n$ into

$$\mathbf{a} = \sum_{i=1}^{3} (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i + \left[\mathbf{a} - \sum_{i=1}^{3} (\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i\right]$$

with the first component in H and the second in $H^\perp$. Thus either $H=\mathbf{R}^n$, in which case $n=3$, or else it is possible to find a unit vector $\mathbf{m}\in H^\perp$. Suppose the latter. Then by Proposition 2, the set $\{\mathbf{m}, \mathbf{e}_1\times\mathbf{m}, \mathbf{e}_2\times\mathbf{m}, \mathbf{e}_3\times\mathbf{m}\}$ is an orthonormal set in $H^\perp$, and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{m}, \mathbf{e}_1\times\mathbf{m}, \mathbf{e}_2\times\mathbf{m}, \mathbf{e}_3\times\mathbf{m}\}$ is an orthonormal set in $\mathbf{R}^n$, so $n\geqq 7$. Let C denote the subspace of $\mathbf{R}^n$ spanned by this last set.

PROPOSITION 4. *C is closed under $\times$.*

Proof. One need only check the five types of products which can occur:

(i) $\mathbf{e}_i \times \mathbf{e}_j$. Belongs to $H \subseteq C$, by Proposition 3.

(ii) $\mathbf{e}_i \times \mathbf{m}$. Belongs to C by definition.

(iii) $$\begin{aligned}\mathbf{e}_i \times (\mathbf{e}_j \times \mathbf{m}) &= 2(\mathbf{e}_i\cdot\mathbf{m})\mathbf{e}_j - (\mathbf{e}_j\cdot\mathbf{m})\mathbf{e}_i - (\mathbf{e}_i\cdot\mathbf{e}_j)\mathbf{m} + \mathbf{m}\times(\mathbf{e}_i\times\mathbf{e}_j)\\ &= -(\mathbf{e}_i\times\mathbf{e}_j)\times\mathbf{m} - (\mathbf{e}_i\cdot\mathbf{e}_j)\mathbf{m} \in C.\end{aligned}$$

(iv) $\mathbf{m}\times(\mathbf{e}_i\times\mathbf{m}) = -\mathbf{m}\times(\mathbf{m}\times\mathbf{e}_i) = (\mathbf{m}\cdot\mathbf{m})\mathbf{e}_i - (\mathbf{m}\cdot\mathbf{e}_i)\mathbf{m} = \mathbf{e}_i \in C.$

(v) $$\begin{aligned}(\mathbf{e}_i\times\mathbf{m})\times(\mathbf{e}_j\times\mathbf{m}) &= (\mathbf{m}\times\mathbf{e}_i)\times(\mathbf{m}\times\mathbf{e}_j)\\ &= \mathbf{e}_j\times[(\mathbf{m}\times\mathbf{e}_i)\times\mathbf{m}] \quad \text{by (4)}\\ &= \mathbf{e}_j\times[\mathbf{m}\times(\mathbf{e}_i\times\mathbf{m})]\\ &= \mathbf{e}_j\times\mathbf{e}_i \quad \text{by (iv) above.}\end{aligned}$$ ∎

One can resolve any vector $\mathbf{a}\in\mathbf{R}^n$ into

$$\begin{aligned}\mathbf{a} = &\left[\sum_{i=1}^{3}(\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i + (\mathbf{a}\cdot\mathbf{m})\mathbf{m} + \sum_{i=1}^{3}(\mathbf{a}\cdot(\mathbf{e}_i\times\mathbf{m}))(\mathbf{e}_i\times\mathbf{m})\right]\\ &+\left[\mathbf{a} - \sum_{i=1}^{3}(\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i - (\mathbf{a}\cdot\mathbf{m})\mathbf{m} - \sum_{i=1}^{3}(\mathbf{a}\cdot(\mathbf{e}_i\times\mathbf{m}))(\mathbf{e}_i\times\mathbf{m})\right]\end{aligned}$$

with the first component in C and the second in $C^\perp$. Thus either $C=\mathbf{R}^n$, in which case $n=7$, or it is possible to find a unit vector $\mathbf{n}\in C^\perp$.

THEOREM. *If there exists a cross product on $\mathbf{R}^n$ which satisfies Apostol's axioms, then $n=1$, 3 or 7. Conversely, there exist cross products on these three spaces.*

Proof. We have seen that for $n\leqq 3$, cross products exist precisely when $n=1$ or 3, the cross product on $\mathbf{R}^1$ being identically zero and the two cross products on $\mathbf{R}^3$ being the classical ones, and that if $\mathbf{R}^n$ has a cross product for $n>3$, then $n\geqq 7$. If $n>7$, then one can find a unit vector $\mathbf{n}\in C^\perp$, where C is constructed as above. We shall show, however, that the existence of such a vector would lead to a contradiction. Indeed, should such an $\mathbf{n}$ exist, then $\mathbf{m}\times\mathbf{n}$ is also a unit vector in $C^\perp$; set $\mathbf{p}=\mathbf{m}\times\mathbf{n}$. Just as in Proposition 3, we have $\mathbf{n}\times\mathbf{p}=\mathbf{m}$ and $\mathbf{p}\times\mathbf{m}=\mathbf{n}$. Let us compute some products: we find that, for $i\neq j$,

$$
\begin{aligned}
(\mathbf{e}_i \times \mathbf{m}) \times (\mathbf{e}_j \times \mathbf{n}) \\
&= (\mathbf{e}_i \times \mathbf{m}) \times (\mathbf{e}_j \times (\mathbf{p} \times \mathbf{m})) \quad \text{by (4)} \\
&= (\mathbf{e}_i \times \mathbf{m}) \times (\mathbf{m} \times (\mathbf{e}_j \times \mathbf{p})) \quad \text{by (4)} \\
&= (\mathbf{e}_j \times \mathbf{p}) \times ((\mathbf{e}_i \times \mathbf{m}) \times \mathbf{m}) \\
&= (\mathbf{e}_j \times \mathbf{p}) \times (\mathbf{m} \times (\mathbf{m} \times \mathbf{e}_i)) = (\mathbf{e}_j \times \mathbf{p}) \times [(\mathbf{m}\cdot\mathbf{e}_i)\mathbf{m} - (\mathbf{m}\cdot\mathbf{m})\mathbf{e}_i] \\
&= -(\mathbf{e}_j \times \mathbf{p}) \times \mathbf{e}_i = \mathbf{e}_i \times (\mathbf{e}_j \times \mathbf{p}) = \mathbf{p} \times (\mathbf{e}_i \times \mathbf{e}_j).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{e}_i \times \mathbf{m}) &= \\
(\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{e}_i \times (\mathbf{n} \times \mathbf{p})) &= \\
-(\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{e}_i \times (\mathbf{p} \times \mathbf{n})) &= \quad \text{by (4)} \\
-(\mathbf{e}_j \times \mathbf{n}) \times (\mathbf{n} \times (\mathbf{e}_i \times \mathbf{p})) &= \quad \text{by (4)} \\
-(\mathbf{e}_i \times \mathbf{p}) \times ((\mathbf{e}_j \times \mathbf{n}) \times \mathbf{n}) &= \cdots \text{by (3)} \\
&= -(\mathbf{e}_i \times \mathbf{p}) \times (-\mathbf{e}_j) \\
&= \mathbf{p} \times (\mathbf{e}_i \times \mathbf{e}_j).
\end{aligned}
$$

(The computations indicated by the ellipsis are straightforward.) But putting these together gives $(\mathbf{e}_i\times\mathbf{m})\times(\mathbf{e}_j\times\mathbf{n})=(\mathbf{e}_j\times\mathbf{n})\times(\mathbf{e}_i\times\mathbf{m})$; this contradicts the skew-symmetry of $\times$ unless this particular product is zero. However, $\mathbf{p}\times(\mathbf{e}_i\times\mathbf{e}_j)$ is the product of two perpendicular unit vectors and thus has unit norm. This contradiction shows that the case $n>7$ cannot arise.

It remains to exhibit a cross product on $\mathbf{R}^7$ which satisfies the axioms. Such a product can be constructed from the familiar one on $\mathbf{R}^3$ as follows: $\mathbf{R}^7$ corresponds 1-1 with the set of all triples $(\mathbf{a}, \lambda, \mathbf{b})$ where $\mathbf{a}$ and $\mathbf{b}$ are in $\mathbf{R}^3$ and $\lambda\in\mathbf{R}$ under the correspondence

$$a_1\mathbf{i}_1 + \cdots + a_7\mathbf{i}_7 \rightarrow (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},\ a_4,\ a_5\mathbf{i} + a_6\mathbf{j} + a_7\mathbf{k}).$$

It is easy to see that this correspondence turns addition and scalar multiplication in $\mathbf{R}^7$ into component-by-component addition and scalar multiplication of triples; defining the dot product of triples by

$$(\mathbf{a}_1, \lambda_1, \mathbf{b}_1)\cdot(\mathbf{a}_2, \lambda_2, \mathbf{b}_2) = (\mathbf{a}_1\cdot\mathbf{a}_2) + \lambda_1\lambda_2 + (\mathbf{b}_1\cdot\mathbf{b}_2)$$

we see easily that the dot product of two triples equals the dot product of the two elements of $\mathbf{R}^7$ to which they correspond. We actually define the cross product on these triples: if one recalls what Propositions 3 and 4 told one the multiplication table of C should be, and if one thinks of $(\mathbf{a}, \lambda, \mathbf{b})$ as representing a vector whose component in H is $\mathbf{a}$, whose component along $\mathbf{m}$ is λ, and whose component in the space spanned by $\{\mathbf{e}_1\times\mathbf{m}, \mathbf{e}_2\times\mathbf{m}, \mathbf{e}_3\times\mathbf{m}\}$ is $\mathbf{b}\times\mathbf{m}$, then one is led to the definition

$$(\mathbf{a}_1, \lambda_1, \mathbf{b}_1) \times (\mathbf{a}_2, \lambda_2, \mathbf{b}_2) = ([\lambda_1\mathbf{b}_2 - \lambda_2\mathbf{b}_1 + (\mathbf{a}_1 \times \mathbf{a}_2) - (\mathbf{b}_1 \times \mathbf{b}_2)], [-(\mathbf{a}_1\cdot\mathbf{b}_2) + (\mathbf{b}_1\cdot\mathbf{a}_2)], [\lambda_2\mathbf{a}_1 - \lambda_1\mathbf{a}_2 - (\mathbf{a}_1 \times \mathbf{b}_2) - (\mathbf{b}_1 \times \mathbf{a}_2)]).$$

(Note that the cross products inside the brackets in the triple are products in $\mathbf{R}^3$ and thus well defined.) Knowing that the cross product in $\mathbf{R}^3$ satisfies the axioms, one easily verifies that this cross product on $\mathbf{R}^7$ (via identification of $\mathbf{R}^7$ with the triples) satisfies the axioms also. ∎

Remark: The pathological behavior of the cross product on $\mathbf{R}^7$ begins with its lack of anything near uniqueness: in $\mathbf{R}^3$ there are only two choices for the cross product of a given orthonormal pair of vectors, while in $\mathbf{R}^7$ our construction can be modified to yield cross products which choose any of the unit vectors in the five-dimensional space orthogonal to the given pair. The cross products on $\mathbf{R}^7$ also fail to satisfy the Lie identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = 0;$$

indeed

$$\mathbf{m} \times (\mathbf{e}_1 \times \mathbf{e}_2) + \mathbf{e}_2 \times (\mathbf{m} \times \mathbf{e}_1) + (\mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{m})) = -3(\mathbf{e}_3 \times \mathbf{m}).$$

Any number of other standard identities also fail in $\mathbf{R}^7$.

Postscript. The discussion above has concentrated on the geometric problem of defining a cross product on a real inner product space, and has attacked this problem via reasoning processes close to those of three-dimensional vector geometry. The theorem it produces is actually equivalent to a classical theorem of Hurwitz which states that a bilinear "multiplication" operation (denoted by juxtaposition) with the property $|\mathbf{xy}| = |\mathbf{x}|\,|\mathbf{y}|$ can be introduced on a finite-dimensional real inner product space if and only if the space has dimension 1, 2, 4 or 8, and that the operation is then "essentially" the multiplication of the reals, complexes, Hamilton quaternions, or Cayley numbers, and actually is one of those multiplications if an identity is present. Indeed, assuming Hurwitz's theorem and being given an $\mathbf{R}^n$ with a cross product as before, write $\mathbf{R}^{n+1}$ as (isometric to) $\mathbf{R}\mathbf{e}\oplus\mathbf{R}^n$ where $\mathbf{e}\in\mathbf{R}^{n+1}$, and define a multiplication on $\mathbf{R}^{n+1}$ by

$$(\lambda\mathbf{e} + \mathbf{a})(\mu\mathbf{e} + \mathbf{b}) = (\lambda\mu - \mathbf{a}\cdot\mathbf{b})\mathbf{e} + (\mu\mathbf{a} + \lambda\mathbf{b} + \mathbf{a} \times \mathbf{b});$$

then it is easy to verify, using the cross product axioms, that this bilinear multiplication has the property $|\lambda\mathbf{e}+\mathbf{a}|\,|\mu\mathbf{e}+\mathbf{b}| = |(\lambda\mathbf{e}+\mathbf{a})(\mu\mathbf{e}+\mathbf{b})|$ and hence $n+1=2$, 4 or 8, $n=1$, 3, or 7. On the other hand, Hurwitz's theorem follows quite readily from ours: given a real inner product space A with a multiplication satisfying $|\mathbf{xy}| = |\mathbf{x}|\,|\mathbf{y}|$ and a left-and-right identity $\mathbf{e}$, one finds that $(\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y}) = (\mathbf{xy}\cdot\mathbf{xy})$ leads, after the usual couple of polarizations, to

$$2(\mathbf{z}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{w}) = (\mathbf{zy}\cdot\mathbf{xw}) + (\mathbf{zw}\cdot\mathbf{xy}), \tag{5}$$

of which we shall need only the cases

$$(e\cdot x)(y\cdot y) = (y\cdot xy), \qquad (x\cdot x)(e\cdot w) = (x\cdot xw) \tag{6, 6'}$$

and

$$2(z\cdot e)(y\cdot e) = (zy\cdot e) + (z\cdot y). \tag{7}$$

Now let V denote the subspace $\mathbf{R}e^{\perp}$ of A, and define an operation $\times$ on V by $a\times b = ab + (a\cdot b)e$.

Since $(a\times b)\cdot e = (ab\cdot e) + (a\cdot b) = 2(a\cdot e)(b\cdot e) = 0$ by (7), $\times$ maps V bilinearly into itself, i.e. satisfies (5.29) and (5.30), and the version of (5.30) with λ applied to b. It follows easily from (6) and (6′) that $(a\times b)\cdot b = 0 = a\cdot(a\times b)$, so (5.31) holds; moreover, it is not difficult to show from these two versions of (5.31) that (5.28) also holds. Finally, (5.32) follows quickly from (7). Thus by our theorem (which, it is easy to check, does not rest on finite-dimensionality hypotheses) V is 7, 3, 1 or possibly 0-dimensional, A is 8, 4, 2 or 1-dimensional, and quite easily seen to be isomorphic to the Cayley numbers, quaternions, complexes or reals. An elegant algebraic treatment of Hurwitz's theorem is to be found in [**3**].

Hurwitz's classical theorem suggests generalizations in various directions. We have noted in passing that (in the presence of an identity) finite dimensionality is a useless hypothesis; Wright [**4**] has shown that a nonassociative real normed division algebra with $|xy| = |x|\,|y|$ must actually have an inner-product norm, thus be one of our four standard algebras. A "nonassociative Gelfand-Mazur theorem," with merely $|xy| \leqq |x|\,|y|$, seems still to be lacking. In the situation of the classical Hurwitz theorem, the unit sphere of $\mathbb{R}^n$ ($n = 1, 2, 4$ or 8) comes equipped with a continuous product operation, with identity; a long series of profound topological results, culminating in [**1**], has shown that these values of n are the only ones for which such a continuous product on the unit sphere can exist.

The author thanks the referee for his suggestion that material be appended which would explicitly point out the relation of the preceding material to the Hurwitz theorem and to the recent topological results.

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