

参考书 ① 彭家贵、陈卿，微分几何

② 沈一兵，整体微分几何

③ Michael. Spivak. A comprehensive introduction to Diff. Geometry. Vol II Chap 1-3. Vol III

④ do Carmo. Diff. Geo. of curves and surfaces

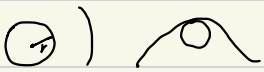
⑤ B. O'Neil Elementary diff. geo.

⑥ 高斯. 曲面一般研究. 1827 和 1825. 商务社

微分几何：直观几何 → 抽象的几何（内蕴/外蕴）

空间的弯曲

·曲线的形状？

曲率 curvature? 圆  局部量

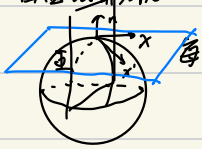
事实上，平面曲线的形状由其每点处的曲率唯一确定

空间曲线的形状由其每点处的曲率和挠率唯一确定

找一个最接近的平面

1847 Serret-Frenet Formula
Serret 1851
Frenet 1847 thesis
1852 abstract published

·曲面的形状？



每个切方向 x，用曲线形状衡量弯曲 K_x

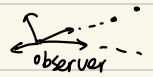
Euler 1760：若 K_x 不对所有方向 x 全相对，

则存在一个方向 X_1 使 K_{X_1} 达到最大值，和另一个方向 X_2 使 K_{X_2} 达最大值，

且 X_1, X_2 相互垂直，且任一 K_x 均可由 K_{X_1}, K_{X_2} 线性表出

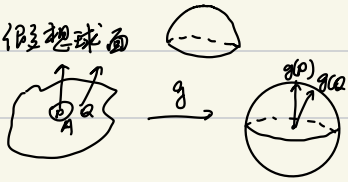
Meusnier 1776 $K = K_x \cdot \frac{1}{\cos\theta}$

3维

Gauss 1827 天文观测 

只能看方向，不知道距离

假想球面

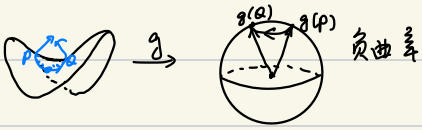


$$K(p) = \lim_{A \to p} \frac{\text{Area}(g(A))}{\text{Area}(A)}$$

平面 $K \equiv 0$

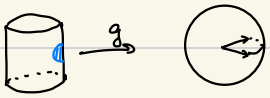
球面 半径为1 $K \equiv 1$

R $K \equiv \frac{1}{R^2}$



负曲率

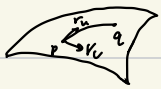
$K(p) = K_{x_1} \cdot K_{x_2}$



$K \equiv 0$ 弯曲的曲线 在此定义下成平坦的了

高斯绝妙定理 (Theorema Egregium)

$$4(EG-F^2)K(p) = E(E_v G_v - 2F_u G_v + G_u^2) + F(E_u G_v - E_v G_u - 2E_v F_v + 4F_u F_v - 2F_u G_u) + G(E_u G_u - 2E_u F_v + E_v^2) - 2(EG-F^2)(E_{vv} - 2F_{uv} + G_{uu})$$



不依赖于向量，只依赖于曲线上自身的曲线长度！ ⇝ 内蕴几何

∴ 球冠不能拍成平面 (必须破坏曲线长度)，内蕴弯曲，而圆柱面可以，外蕴弯曲

∴ 如果生活在二维空间中，通过了解曲线长度，仍然可以了解空间是不是弯曲的

但如果生活在一维空间中，测量出来相当于直线，一维内蕴几何是平凡的

Integral curvature

$$\int_{\triangle ABC} K dV = \angle A + \angle B + \angle C - \pi$$



测地三角形

Bonnet 1848 引入"测地曲率"概念 曲率 ⟷ 拓扑 整体微分几何

Gauß - Bonnet Formula

Riemann 几何，不通过背景空间，从自身定义曲线长度



高维情况，二维嵌面 高斯曲率 (必须由切向量计算，法向量不适用)

(I) 欧氏空间

三维欧氏空间 $\mathbb{E}^3$ 点、线、面 欧氏空间的基本假设

齐次性 homogeneous 各向同性 isotropic

向量既有大小又有方向的量 不区分起点

1) 加法 a) 交换律 b) 结合律 2) 数乘 a) 结合律 $\lambda(\mu\vec{a})=(\lambda\mu)\vec{a}$ b) 分配律 3) 内积 定义 $\langle\vec{a},\vec{b}\rangle=\frac{1}{2}(|\vec{a}+\vec{b}|^2-|\vec{a}|^2-|\vec{b}|^2)$

4) 外积 $(\mathbb{E}^3,\langle,\rangle)$ $\vec{a}\wedge\vec{b}$ a) 反交换律 b) 分配律 $\vec{a}\wedge(\lambda\vec{b}+\mu\vec{c})=\lambda\vec{a}\wedge\vec{b}+\mu\vec{a}\wedge\vec{c}$ 注记. 不满足结合律. 若取 e_1,e_2,e_2+e_3. 其中 e_1,e_2,e_3 标准

5) 混合积 $(\vec{v}_1,\vec{v}_2,\vec{v}_3):=\langle\vec{v}_1,\vec{v}_2\wedge\vec{v}_3\rangle$ 平行六面体体积 (可正可负)

线数结构：抽象的(三维)向量空间 $\longrightarrow$ 一组基 v_1,v_2,v_3，内积 $\langle,\rangle$ 双线性 对称 正定

例. 3维数组空间 内积-外积

通过 $v_1\longleftrightarrow(1,0,0)$ $v_2\longleftrightarrow(0,1,0)$ $v_3\longleftrightarrow(0,0,1)$ 可以将 $\mathbb{E}^3$ 对应于 $\mathbb{R}^3$

$(\mathbb{E}^3,\langle,\rangle)$ 标准正交基 $\{e_1,e_2,e_3\}$

$(\mathbb{R}^3,\langle,\rangle)$ 标准正交基 $\{(1,0,0),(0,1,0),(0,0,1)\}$

通过与 $(\mathbb{R}^3,\langle,\rangle)$ 对应，$(\vec{v}_1,\vec{v}_2,\vec{v}_3)=\begin{vmatrix}v_1^1&v_1^2&v_1^3\\v_2^1&v_2^2&v_2^3\\v_3^1&v_3^2&v_3^3\end{vmatrix}$

性质(练习)

(1) Lagrange 恒等式 $\langle v_1\wedge v_2,v_3\wedge v_4\rangle=\langle v_1,v_3\rangle\langle v_2,v_4\rangle-\langle v_1,v_4\rangle\langle v_2,v_3\rangle$

(2) $v_1\wedge(v_2\wedge v_3)=\langle v_1,v_3\rangle v_2-\langle v_1,v_2\rangle v_3$

$\mathbb{E}^3$ 坐标

标架 $\{O,e_1,e_2,e_3\}$ 正交标架

$\forall P\in\mathbb{E}^3$, $P=\vec{OP}=\sum_{i=1}^{3}a_ie_i=(a_1,a_2,a_3)$

选另外一个标架 $\{O',e_1',e_2',e_3'\}$，$P=(b_1,b_2,b_3)$，问 (a_1,a_2,a_3) 和 (b_1,b_2,b_3) 关系？

$\sum a_ie_i=\vec{OP}=\vec{O'P}+\vec{OO'}=\sum b_ie_i+\sum c_ie_i$，$\begin{pmatrix}e_1'\\e_2'\\e_3'\end{pmatrix}=T\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$ T 正交

$\Rightarrow(a_1,a_2,a_3)=(c_1,c_2,c_3)+(b_1,b_2,b_3)T$

$\mathbb{E}^3$ 欧氏变换（不动点变换）

定理：设 $T:\mathbb{E}^3\to\mathbb{E}^3$ 欧氏变换，则存在 $T\in O(3)$ 及 $P\in\mathbb{E}^3$ s.t. $T(x)=xT+P$. $\forall x\in\mathbb{E}^3$

证：(1) 保距 $\Rightarrow$ 保内积

$$d^2(P,Q)=d^2(T(P),T(Q))$$

$\langle Q-P, Q-P\rangle$ $\qquad$ $d^2(T(P),T(0))+d^2(T(Q),T(0))-2\langle T(Q),T(P)\rangle$

$\langle Q,Q\rangle+\langle P,P\rangle-2\langle Q,P\rangle$

$d^2(Q,0)+d^2(P,0)-2\langle Q,P\rangle$ $\qquad\Rightarrow\langle Q,P\rangle=\langle T(P),T(Q)\rangle$

(2) 线性 $\quad T$ 保内积 $\Rightarrow\{w_i=T(e_i)\}$ 标准正交

$\forall p.\ T(p)=\sum_{i=1}^3\langle T(p),w_i\rangle w_i=\sum_{i=1}^3\langle T(p),T(e_i)\rangle w_i=\sum_{i=1}^3\langle p,e_i\rangle w_i$

$\sum\langle p,e_i\rangle e_i$

不好设 $T(0)=0\Rightarrow\exists T$ s.t. $T(x)=xT$

$\forall p\in\mathbb{E}^3$, $\langle p,p\rangle=\langle T(p),T(p)\rangle\Rightarrow TT^t=Id$

pp^t $\qquad pT\cdot T^tp^t$

□

$\mathbb{E}^3$ 向量值函数

$t\mapsto v(t)=(v_1(t),v_2(t),v_3(t))\quad t\in\mathbb{R}$

定义 $\dfrac{dv(t)}{dt}$

性质（练习）：(1) $\dfrac{d}{dt}(\lambda\vec v)=\lambda\dfrac{d\vec v}{dt}$ $\quad$ (2) $\dfrac{d}{dt}\langle\vec v,\vec w\rangle=\langle\dfrac{d\vec v}{dt},\vec w\rangle+\langle\vec v,\dfrac{d\vec w}{dt}\rangle$

(3) $\dfrac{d}{dt}(\vec v\wedge\vec w)=\dfrac{d\vec v}{dt}\wedge\vec w+\vec v\wedge\dfrac{d\vec w}{dt}$ $\quad$ (4) $\dfrac{d}{dt}(\vec u,\vec v,\vec w)=(\dfrac{d\vec u}{dt},\vec v,\vec w)+(\vec u,\dfrac{d\vec v}{dt},\vec w)+(\vec u,\vec v,\dfrac{d\vec w}{dt})$

定理 向量值函数 $\vec\alpha(t)$，连续可微，则 $\vec\alpha(t)$ 的长度是常数 $\Leftrightarrow\langle\vec\alpha(t),\dfrac{d\vec\alpha(t)}{dt}\rangle=0$ □

性质：$\vec\alpha(t)$ 连续可微，处处非零，则

(1) $\vec\alpha(t)$ 之方向不变 $\Leftrightarrow\dfrac{d}{dt}\vec\alpha(t)\wedge\vec\alpha(t)\equiv0$ $\quad$ （当 $\vec\alpha(t)=f(t)\hat e(t)$）

(2) 若 $\vec\alpha(t)$ 与一个固定方向垂直，则 $(\vec\alpha(t),\dfrac{d}{dt}\vec\alpha(t),\dfrac{d^2}{dt^2}\vec\alpha(t))=0$

反之，若 $(\vec\alpha(t),\vec\alpha'(t),\vec\alpha''(t))=0$ 且处处有 $\vec\alpha(t)\wedge\vec\alpha'(t)\neq0$，则 $\vec\alpha(t)$ 必与一固定方向垂直

（考虑 $b(t)=\vec\alpha'(t)\wedge\vec\alpha(t)$）

向量场

$\mathbb{E}^3 = \mathbb{R}^3$

定义 $\mathbb{E}^3$ 上一个切向量 v_p 包含两个组成部分：向量 v、点 p

两个切向量 $v_p = v_q \Leftrightarrow v = w,\ p = q$

设 $p \in \mathbb{E}^3$，定义 $T_p(\mathbb{E}^3) = \{p$ 点处 $\mathbb{E}^3$ 上所有切向量$\}$，称为 $\mathbb{E}^3$ 在 p 点上切空间

定义 $\mathbb{E}^3$ 上一个向量场是一个将每个点 $p \in \mathbb{E}^3$ 赋一个切向量 $v_p \in T_p\mathbb{E}^3$ 上函数

例 u_1 向量场. $\forall p$ $u_1(p) = (1,0,0)_p$

u_2 $u_2(p) = (0,1,0)_p$

u_3 $u_3(p) = (0,0,1)_p$

$\{u_1, u_2, u_3\}$ 称为 $\mathbb{E}^3 = \mathbb{R}^3$ 上自然标架场

性质 $\mathbb{E}^3 = \mathbb{R}^3$ 上任一个向量场总可表为 $V = x^1 u_1 + x^2 u_2 + x^3 u_3$，其中 x^1, x^2, x^3 为三个唯一确定实函数

称 V 光滑向量场. 如果 x^1, x^2, x^3 均为光滑函数

$\mathbb{E}^3$ 上的微分形式

定义 $\mathbb{E}^3 = \mathbb{R}^3$ 上的线性 1-形式 ϕ 是 $\mathbb{E}^3$ 上所有切向量集合上的一个实值函数，使得 对 $\forall a, b \in \mathbb{R}$

和 $\forall p \in \mathbb{E}^3$，$v_p, w_p \in T_p\mathbb{E}^3$，有 $\phi(av_p + bw_p) = a\phi(v_p) + b\phi(w_p)$

运算 给定 1-形式 ϕ 和 ψ. $f: \mathbb{R}^3 \to \mathbb{R}$

$(\phi + \psi)(v_p) := \phi(v_p) + \psi(v_p)$ $\forall v_p \in T_p\mathbb{R}^3,\ \forall p \in \mathbb{R}^3$

$(f\phi)(v_p) := f(p)\,\phi(v_p)$

给定 1-形式 ϕ 和一个向量场 V，可定义一个实值函数 $\phi(V): \mathbb{R}^3 \to \mathbb{R}$，$p \mapsto \phi(V(p))$

称 1-形式 ϕ 为光滑 1-形式，如果对任意上光滑向量场 V，函数 $\phi(V)$ 均光滑

性质. 给定 1-形式 ϕ, ψ，向量场 V, W，函数 $f, g: \mathbb{R}^3 \to \mathbb{R}$

有：(i) $\phi(fV + gW) = f\phi(V) + g\cdot\phi(W)$

(ii) $(f\phi + g\psi)(V) = f\cdot\phi(V) + g\psi(W)$

给定一个 $f:\mathbb{R}^3\to\mathbb{R}$ 光滑函数，定义如下 1-形式 $df:\bigcup_{p\in\mathbb{E}^3}T_p\mathbb{E}^3\to\mathbb{R}$：

$$df(v_p):=\frac{d}{dt}\Big|_{t=0}f(r_{v_p}(t))\in\mathbb{R}\quad .\ \forall v_p\in T_p\mathbb{E}^3,\ \forall p\in\mathbb{E}^3$$

其中，$\mathbb{R}^3$ 上一条参数化直线 $r:(a,b)\to\mathbb{R}^3$：

$p=(p_1,p_2,p_3)$，$v_p=(v^1,v^2,v^3)$，$r_{v_p}(t)=(p_1+tv^1,p_2+tv^2,p_3+tv^3)$

$\therefore df(v_p)=\langle \operatorname{grad} f(p),v_p\rangle$ 是线性的

坐标函数 $x^1,x^2,x^3:\mathbb{R}^3\to\mathbb{R}$ 光滑函数 $\Rightarrow$ 三个 1-形式 dx^1,dx^2,dx^3

$\forall p\in\mathbb{R}^3$，$dx^1(U_1(p))=\frac{d}{dt}\big|_{t=0}x^1(p_1+t,p_2,p_3)=\frac{d}{dt}\big|_{t=0}(p_1+t)=1$

$dx^1(U_2(p))=\frac{d}{dt}\big|_{t=0}p_1=0$

$dx^1(U_3(p))=0$

一般地 $dx^i(U_j)=\delta^i_j$

$\forall$ 向量场 $V=v^1U_1+v^2U_2+v^3U_3$，$dx^i(V)=v^i$ $\Rightarrow dx^1,dx^2,dx^3$ 是三个光滑 1-形式

性质 设 ϕ 是 $\mathbb{R}^3$ 上 1-形式，则 ϕ 可表为 $\phi=f_1dx^1+f_2dx^2+f_3dx^3=\sum_{i=1}^3f_i\,dx^i$，其中 $f_i:=\phi(U_i)$，$i=1,2,3$

证：左、右均为 1-形式

$$\forall p\in\mathbb{R}^3,\ \forall v_p\in T_p\mathbb{R}^3,\quad \phi(v_p)=\phi\Big(\sum_{i=1}^3v^iU_i(p)\Big)=\sum_{i=1}^3v^i\phi(U_i(p))=\sum_{i=1}^3v^i\phi(U_i)(p)=\sum_{i=1}^3f_i(p)\,dx^i(v_p)$$

练习：对 $\mathbb{R}^3$ 上任意 C^∞ 参数化曲线 $r(t)$，满足 $r(0)=p$，$r'(0)=v_p$，均有 $\frac{d}{dt}\big|_{t=0}f(r(t))=df(v_p)$

$f:\mathbb{R}^3\to\mathbb{R}$ 光滑，$df=\sum_{i=1}^3df(U_i)\,dx^i=\sum_{i=1}^3\frac{\partial f}{\partial x^i}dx^i$

性质 $\forall f,g:\mathbb{R}^3\to\mathbb{R}$，$\forall h:\mathbb{R}\to\mathbb{R}$ 光滑，有

(i) $d(f+g)=df+dg$

(ii) $d(fg)=f\cdot dg+g\cdot df$

(iii) $d(h(f))=h'(f)\,df$

$\Omega_0=\{\text{光滑 0-form}\}=\{f:\mathbb{R}^3\to\mathbb{R},C^\infty\}$

$\Omega_1=\{\text{光滑 1-form}\}$

$\Omega_2=\{\cdots\ 2\text{-form}\}$

$\Omega_3=\{\cdots\ 3\text{-form}\}$

定义 $\mathbb{R}^3$上的一个2-形式 η 是 $\mathbb{R}^3$ 上所有的向量对 (v_p, w_p) 集合上的实值函数，使得在任意点 p 处，

η 是双线性的，且反称，即

$\eta(v_p, w_p) = -\eta(w_p, v_p)$

$\eta(av_p + bu_p, w_p) = a\eta(v_p, w_p) + b\eta(u_p, w_p)$

若对任意 C^∞ 向量场 V, W，实值函数 $\eta(V,W): \mathbb{R}^3 \to \mathbb{R}$，$p \to \eta(V,W)(p) = \eta(V_p, W_p) \in \mathbb{R}$ 是 C^∞ 的，则称 η 为光滑2-形式

例. $\eta(v_p, w_p) := dx^1(v_p)\cdot dx^2(w_p) - dx^1(w_p)\cdot dx^2(v_p)$

$= dx^1 \otimes dx^2(v_p, w_p) - dx^2 \otimes dx^1(v_p, w_p)$

$= dx^1 \wedge dx^2(v_p, w_p)$

$\eta := dx^1 \wedge dx^2 = dx^1 \otimes dx^2 - dx^2 \otimes dx^1$

$dx^2 \wedge dx^1 = -dx^1 \wedge dx^2$

$dx^1 \wedge dx^1 = 0$

$\rightsquigarrow$ 三个2-形式 $dx^1\wedge dx^2$, $dx^1 \wedge dx^3$, $dx^2 \wedge dx^3$

性质：设 η 为 $\mathbb{R}^3$ 上一个2-form，则 η 可表为 $\eta = \eta(U_1, U_2)\, dx^1 \wedge dx^2 + \eta(U_1, U_3)\, dx^1 \wedge dx^3 + \eta(U_2, U_3)\, dx^2 \wedge dx^3$

证：左、右均为2-form，直接计算检查即可

定义 $\mathbb{R}^3$ 上的一个3-形式 ψ 是 $\mathbb{R}^3$ 上所有向量三元组 (v_p^1, v_p^2, v_p^3) 集合上的实值函数，使得在任意 p 点处，ψ 关于每个分量线性且alternative，即 $\psi(v_p^1, v_p^2, v_p^3) = \mathrm{sgn}(\sigma)\, \phi(v_p^{\sigma(1)}, v_p^{\sigma(2)}, v_p^{\sigma(3)})$

V, W, U 向量场，$\psi(V, W, U)(p) = \psi(V(p), W(p), U(p))$ 光滑性

例 $\psi(v_p, w_p, u_p) := (v_p, w_p, u_p)$

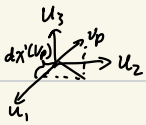
$\psi(v_p, w_p, u_p) = \sum_\sigma \mathrm{sgn}(\sigma)\, dx^{\sigma(1)}(v_p)\, dx^{\sigma(2)}(w_p)\, dx^{\sigma(3)}(u_p)$

$= dx^1 \wedge dx^2 \wedge dx^3$

性质 设 ψ 为 $\mathbb{R}^3$ 上任一3-形式，则 ψ 可表为 $\psi = \psi(U_1, U_2, U_3)\, dx^1 \wedge dx^2 \wedge dx^3$

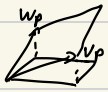
由这种反称性，4-形式往上只有0

v_p 有向长度 dx^i, $i=1,2,3$ $dx^i(v_p)=v_p^i$



$v_p \wedge w_p$ 有向面积 $dx^i \wedge dx^j$, $i<j$

$$dx^1\wedge dx^2(v_p,w_p)=v_p^1 w_p^2 - v_p^2 w_p^1 = \det\begin{pmatrix} v_p^1 & v_p^2 \\ w_p^1 & w_p^2 \end{pmatrix}$$



$v_p \wedge w_p \wedge u_p$ 有向体积

外微分运算

$$\Omega_0 \xrightarrow{d} \Omega_1 \xrightarrow{d} \Omega_2 \xrightarrow{d} \Omega_3$$
$$f \mapsto df$$

定义 $d:\Omega_1\to\Omega_2$ $\forall \phi\in\Omega_1$, 有 $\phi=\sum_i \phi(U_i)dx^i$, 定义 $d\phi = \sum_i d(\phi(U_i))\wedge dx^i$

$d:\Omega_2\to\Omega_3$ $\forall \eta\in\Omega_2$, 有 $\eta=\sum_{i<j}\eta(U_i,U_j)dx^i\wedge dx^j$, 定义 $d\eta=\sum_{i<j} d(\eta(U_i,U_j))\wedge dx^i\wedge dx^j$

性质: $d\circ d=0$ (练习)

$$df=\sum_i \frac{\partial f}{\partial x^i}dx^i$$

$$d(df)=\sum_i d\left(\frac{\partial f}{\partial x^i}\right)\wedge dx^i=\sum_i\sum_j \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)dx^j\wedge dx^i=\sum_{i<j}\left(\frac{\partial f}{\partial x^j\partial x^i}-\frac{\partial f}{\partial x^i\partial x^j}\right)dx^i\wedge dx^j=0$$

$f\in\Omega_0=C^\infty(\mathbb{R}^3)$ $\operatorname{grad}\{\mathbb{R}^3\text{上}C^\infty\text{向量场}\}=\mathfrak{X}(\mathbb{R}^3)$

检查

$$\begin{array}{ccccccc} C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\text{rot}} & \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\ \updownarrow \text{Id} & & \updownarrow u_i\leftrightarrow dx^i & & \updownarrow \begin{matrix} u_1\leftrightarrow dx^2\wedge dx^3 \\ u_2\leftrightarrow -dx^1\wedge dx^3 \\ u_3\leftrightarrow dx^1\wedge dx^2 \end{matrix} & & \updownarrow \\ \Omega_0(\mathbb{R}^3) & \xrightarrow{d} & \Omega_1(\mathbb{R}^3) & \xrightarrow{d} & \Omega_2(\mathbb{R}^3) & \longrightarrow & \Omega_3(\mathbb{R}^3) \longrightarrow \end{array}$$

(II) 曲线论局部理论

1. 研究什么样的曲线？

参数化曲线 映射 $r:(a,b)\to\mathbb{E}^3\cong\mathbb{R}^3$

$t\mapsto r(t)=(x(t),y(t),z(t))$

定义 $r:(a,b)\to\mathbb{E}^3$ 称为一条正则曲线，如果下述条件成立：

(1) 光滑性. 每个分量 $x(t),y(t),z(t)$ 均为光滑函数

(2) $|r'(t)|>0,\ \forall t\in(a,b)$ (浸入局部 1-1)

弧长 $s:(a,b)\to(0,L)$, $t\mapsto\int_0^t|r'|$ $\Rightarrow \frac{ds}{dt}=|r'(t)|>0$ $s:(a,b)\to(0,L)$ 可逆

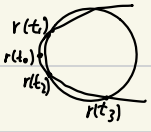
记它的逆函数 $t=t(s):(0,L)\to(a,b)$

正则曲线 $r(t)$

弧长参数化 $c(s)=r(t(s))$ $c:(0,L)\to\mathbb{E}^3$ t 的向量 $\dot c(s)=r'(t(s))\cdot\frac{dt}{ds}=\frac{r'(t(s))}{|r'(t(s))|}$, $|\dot c(s)|\equiv1$

2. 平面曲线 $r:(a,b)\to\mathbb{E}^2\cong\mathbb{R}^2$

如何量化它的弯曲程度？



$\Leftarrow t_1,t_2,t_3\to t_0$

实现 ① 直线不弯曲

② 圆曲线 R越大，弯曲程度越小 $\rightsquigarrow$ 密切圆 osculating circle

不妨 $t_1<t_2<t_3$, $r(t_1),r(t_2),r(t_3)$ 三点不共线，存在一个圆，圆心 $C(t_1,t_2,t_3)$

考虑函数 $t\mapsto\langle r(t)-C(t_1,t_2,t_3),r(t)-C(t_1,t_2,t_3)\rangle$，在 t_1,t_2,t_3 处取值相等

$\Rightarrow$ 存在 $\eta_1\in(t_1,t_2)$, $\eta_2\in(t_2,t_3)$ s.t. $0=\langle r'(t),r(t)-C(t_1,t_2,t_3)\rangle\big|_{t=\eta_1,\eta_2}$

考虑函数 $t\mapsto\langle r'(t),r(t)-C(t_1,t_2,t_3)\rangle$

$\Rightarrow$ 存在 $\eta\in(\eta_1,\eta_2)$ s.t. (*) $0=\langle r''(\eta),r(\eta)-C(t_1,t_2,t_3)\rangle+\langle r'(\eta),r'(\eta)\rangle$

总结：若有 $C(t_1,t_2,t_3)\to C$ as $t_1,t_2,t_3\to t_0$.

则 $\begin{cases}0=\langle r'(t_0),r(t_0)-C\rangle\\ 0=\langle r''(t_0),r(t_0)-C\rangle+\langle r'(t_0),r'(t_0)\rangle\end{cases}$ $\Rightarrow$ 告诉 C 位置

希望 $r'(t_0),r''(t_0)$ 不共线，由于 $|r'(t_0)|>0$，否则不共线.

改写为弧长参数. $\begin{cases} 0=\langle r'(s), r(s)-C\rangle \\ -1=\langle r''(s), r(s)-C\rangle \end{cases}$ $\langle r'(s), r'(s)\rangle \equiv 1 \Rightarrow \langle r(s), r''(s)\rangle \equiv 0$

若 $r''(s)\neq 0$, $r(s_0)-C=a r''(s_0)$, $-1=a|r''(s_0)|^2 \Rightarrow a=-\frac{1}{|r''(s_0)|^2} \Rightarrow$ 半径 $|r(s_0)-C|=\frac{1}{|r''(s_0)|}$

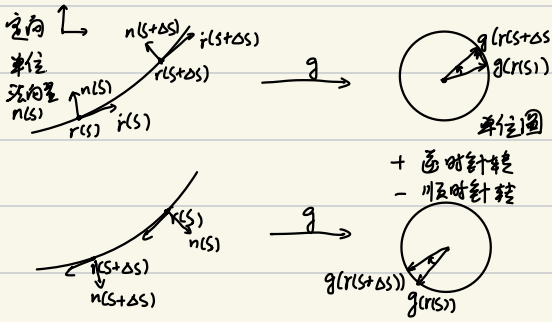
定理: 设 $r:(0,L)\to \mathbb{R}^3$ 是一条弧长参数化曲线

(1) 如果 $r''(s_0)\neq 0$, 那么当 s_1, s_2, s_3 充分靠近 s_0 时, $r(s_i)$ $(i=1,2,3)$ 三点不共线, 且当 $s_i\to s$ 时, 三点确定的圆收敛到一个以 $\frac{1}{|r''(s)|}$ 为半径, 过点 $r(s_0)$ 的圆, 圆心在与 $r''(s_0)$ 垂直的直线上

(2) 如果 $r''(s)=0$, 即使 $r(s_i)$ $(i=1,2,3)$ 不共线, 其确定的圆也不收敛

证: 练习. □

$\rightsquigarrow$ 定义了弧长参数曲线 $r=r(s)$ 的曲率 $K(s)=|r''(s)|$



$$r''(s)=\lim_{\Delta s\to 0}\frac{|r'(s+\Delta s)-r'(s)|}{\Delta s}=\lim_{\Delta s\to 0}\frac{|g(r(s+\Delta s))-g(r(s))|}{\Delta s}$$

$$=\lim_{\Delta s\to 0}\frac{g(r)\text{在}[s,s+\Delta s]\text{的长度}}{\Delta s}$$

Frenet 标架

$\{r(s); t(s), n(s)\}$ s 弧长, $t(s)=r'(s)$ (tangent vector) 回顾 $\langle r''(s), r'(s)\rangle=0 \Rightarrow r''(s)=k(s)n(s)$, $k(s)\in\mathbb{R}$

$t(s), n(s)$ 定向约定

当 $\begin{cases} t'(s)=r''(s)=k(s)n(s) \\ n'(s)=-k(s)t(s) \end{cases}$

($\langle n(s), n(s)\rangle\equiv 1 \Rightarrow \langle n'(s), n(s)\rangle=0 \Rightarrow n'(s)$ 与 $t(s)$ 共线,

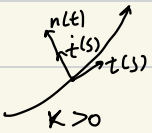
$\langle n(s), t(s)\rangle\equiv 0 \Rightarrow \underbrace{\langle n'(s), t(s)\rangle}_{\text{记为 } b(s)}+\underbrace{\langle n(s), t'(s)\rangle}_{k(s)}\equiv 0 \Rightarrow b(s)+k(s)=0$)

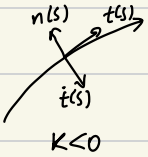
Frenet 标架运动方程

$$\begin{cases} \dot{t}(s) \\ \dot{n}(s) \end{cases} = \begin{pmatrix} 0 & K(s) \\ -K(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \end{pmatrix} \Bigg\}$$ 其中 $|K(s)| = |\ddot{r}(s)|$，称 $K(s)$ 为带符号曲率

参数增长方向反向，则 K 变号

$r = r(s),\ s \in [0, L]$ $K(s)$





令 $c = c(s) = r(L-s)$，$s \in [0, L]$

$\dot{c}(s) = -\dot{r}(L-s)$，$\ddot{c}(s) = \ddot{r}(L-s)$

$\Rightarrow c = c(s)$ 在 $r(L-s)$ 点处的曲率为 $-K(s)$

带符号曲率 $K(s) = \dot{r}(s), \ddot{r}(s)$ 张成矩形的带符号面积 $= \det(\dot{r}(s), \ddot{r}(s)) = \det\begin{pmatrix} \dot{x}(s) & \dot{y}(s) \\ \dot{x}(s) & \dot{y}(s) \end{pmatrix}$

给定正则曲线 $r: (a,b) \to \mathbb{R}^3$，问该曲线在点 $r(t)$ 处带符号曲率？

$$K(r(t)) = K(c(s(t))) = \det(\dot{c}(s), \ddot{c}(s)) = \det\left(r'(t)\frac{dt}{ds},\ r''(t)\left(\frac{dt}{ds}\right)^2 + r'(t)\frac{d^2t}{ds^2}\right)$$

$$K(t) = \frac{\det(r'(t), r''(t))}{|r'(t)|^3} = \frac{\det\begin{pmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{pmatrix}}{\left(|x'(t)|^2 + |y'(t)|^2\right)^{\frac{3}{2}}}$$

$K(s)$ 合理性

① $K(s) \equiv 0 \Leftrightarrow \dot{r}(s) \equiv 0 \Leftrightarrow r(s) \equiv \text{const} \equiv v_0 \Rightarrow r$ 直线

② $K(s) \equiv a \in \mathbb{R}$，$a \neq 0$

$$\begin{cases} \ddot{r}(s) = \dot{t}(s) = a \cdot n(s) \\ \dot{n}(s) = -a \cdot t(s) \end{cases}$$

考查 $\frac{d}{ds}\left(r(s) + \frac{1}{a}n(s)\right) = t(s) + \frac{1}{a}\dot{n}(s) = 0$

$\Rightarrow r(s) + \frac{1}{a}n(s) \equiv v_0$

$\Rightarrow |r(s) - v_0| = \left|-\frac{1}{a}n(s)\right| = \frac{1}{|a|}$

定理（平面曲线基本定理）设 $K: [a,b] \to \mathbb{R}$ C^∞ 函数，则存在弧长参数正则曲线 $r: [a,b] \to \mathbb{R}^2$，其在 s 处的带符号曲率为 $K(s)$，$\forall$ 另存在两条这样的曲线 $r, \bar{r}$，则存在一个刚体运动（平移，旋转）A s.t.

$\bar{r} = A \circ r$

记：若存在这样的曲线 $r=r(s)$，则

$$\begin{cases}\frac{dr}{ds}=t(s)\\ \frac{dt}{ds}=K(s)n(s)\\ \frac{dn}{ds}=-K(s)t(s)\end{cases}\qquad t(s)=\begin{pmatrix}t_1(s)\\ t_2(s)\end{pmatrix}\Downarrow\quad n(s)=\begin{pmatrix}-t_2(s)\\ t_1(s)\end{pmatrix}$$

$$\Rightarrow\begin{cases}\frac{dr}{ds}=\begin{pmatrix}t_1(s)\\ t_2(s)\end{pmatrix}\\ \frac{dt}{ds}=K(s)\begin{pmatrix}-t_2(s)\\ t_1(s)\end{pmatrix}\end{cases}\qquad \text{初值 } r(0)=r_0,\ t(0)=t_0=\begin{pmatrix}t_1(0)\\ t_2(0)\end{pmatrix}$$

Picard-Lindelof $\Rightarrow$ 存 $r(s)=\begin{pmatrix}r_1(s)\\ r_2(s)\end{pmatrix}$，$t(s)=\begin{pmatrix}t_1(s)\\ t_2(s)\end{pmatrix}$

$\frac{d}{ds}\langle t(s),t(s)\rangle=2\langle t(s),\dot t(s)\rangle=2\langle t(s),K(s)\begin{pmatrix}-t_2(s)\\ t_1(s)\end{pmatrix}\rangle=0\Rightarrow r=r(s)$ 是一条弧长参数曲线

$r,\bar r:[a,b]\to\mathbb{R}^2$ 刚体运动 $Ar=\overset{\text{旋转}}{B}r+C$ 使 $\begin{cases}Ar(0)=\bar r(0)\\ Bt(0)=\bar t(0)\end{cases}$

$r(0)$ $\bar r(0)$

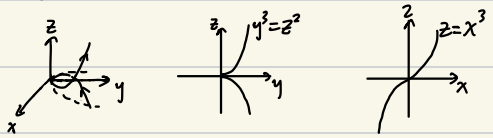
$$\begin{cases}\frac{d}{ds}(A(r(s)))=B\frac{dr}{ds}=B(t(s))\\ \frac{d}{ds}(B(t(s)))=B\frac{dt}{ds}=BK(s)\begin{pmatrix}-t_2(s)\\ t_1(s)\end{pmatrix}=K(s)\begin{pmatrix}-(Bt)_2(s)\\ (Bt)_1(s)\end{pmatrix}\end{cases}$$

$$K(s)\underbrace{B\begin{pmatrix}0&-1\\1&0\end{pmatrix}}_{\parallel}\begin{pmatrix}t_1(s)\\ t_2(s)\end{pmatrix}=K(s)\underbrace{\begin{pmatrix}0&-1\\1&0\end{pmatrix}B}_{\parallel}\begin{pmatrix}t_1(s)\\ t_2(s)\end{pmatrix}$$

□

3. 空间曲线　设 $r=r(t)\in\mathbb{E}^3$，$t\in[a,b]$

例. $r(t)=(t,t^2,t^3)$　$t\in\mathbb{R}$



在 s_0，取 s_1,s_2,s_3 s.t. 三点不共线 $\Rightarrow$ 定一个圆，圆心 $C(s_1,s_2,s_3)$

若 $C(s_1,s_2,s_3)\to C$ as $s_i\to s_0\Rightarrow\begin{cases}\langle i(s_0),r(s_0)-C\rangle=0\\ \langle\dot i(s_0),r(s_0)-C\rangle=-1\end{cases}$ (*)　求不出 C！$\rightsquigarrow$ 应该先定平面！

密切平面

$r(s_1),r(s_2),r(s_3)$ 不共线，确定一个平面，记其法向量为 $\vec a(s_1,s_2,s_3)$（方向暂不确定）

取该平面上一点 $p(s_1,s_2,s_3)$，则 $\langle r(s_i)-p(s_1,s_2,s_3),\vec a(s_1,s_2,s_3)\rangle=0$，$i=1,2,3$

考查函数 $s\mapsto\langle r(s)-p(s_1,s_2,s_3),\vec a(s_1,s_2,s_3)\rangle$

中值定理，$\exists\xi_1,\xi_2$，$\langle\dot r(s),\vec a(s_1,s_2,s_3)\rangle\big|_{\xi_1,\xi_2}=0$

中值 $\Rightarrow\exists\eta\in(\xi_1,\xi_2)$ s.t. $\langle\vec r(\eta),\vec a(s_1,s_2,s_3)\rangle=0$

若平面收敛，即 $\vec{\alpha}(s_1,s_2,s_3)\longrightarrow\vec{a}$ as $s_i\to s_0$，则

$$\begin{cases}\langle\dot{r}(s),\vec{\alpha}\rangle=0\\ \langle\ddot{r}(s),\vec{\alpha}\rangle=0\end{cases}$$

若 $\ddot{r}(s)\neq 0$，则密切平面为 $\dot{r}(s)$，$\ddot{r}(s)$ 确定的平面

定理：$r:(0,L)\to\mathbb{R}^3$，弧长参数化的曲线 s.t. $\ddot{r}(s_0)\neq 0$

则(1)当 s_1,s_2,s_3 充分靠近 s_0 时，$r(s_1),r(s_2),r(s_3)$ 不共线

(2)当 $s_i\to s_0$ 时，由三点 $r(s_i)$ 确定的平面收敛到过 $r(s_0)$，由 $\dot{r}(s),\ddot{r}(s_0)$ 张成的平面

由(*)和此定理，空间曲线的曲率 $k(s)=|\ddot{r}(s)|$（若 $\ddot{r}(s)\neq 0$）

若 $\ddot{r}(s)=0$，定义 $k(s)=0$

$\ddot{r}(s)\neq 0$，切向量

$$\begin{cases}t(s)=\dot{r}(s)\\ n(s)=\dfrac{\ddot{r}(s)}{\|\ddot{r}(s)\|}\ \text{主法向量 principal normal}\\ b(s)=t(s)\wedge n(s)\ \text{副法向量 binormal}\end{cases}$$

$\{r(s);t(s),n(s),b(s)\}$ Frenet 标架

决定密切平面变化的：$\dot{b}(s)=\dfrac{d}{ds}b(s)$

$\langle b(s),b(s)\rangle\equiv 1$，求导 $\langle\dot{b}(s),b(s)\rangle\equiv 0$

$\Rightarrow\exists f(s),g(s)$，s.t. $\dot{b}(s)=f(s)t(s)+g(s)n(s)$

$$\left.\begin{matrix}\langle b(s),t(s)\rangle\equiv 0\\ \langle b(s),n(s)\rangle\equiv 0\end{matrix}\right\}\text{求导},\ \langle\dot{b}(s),t(s)\rangle=-\langle b(s),\dot{t}(s)\rangle=-\langle b(s),k(s)n(s)\rangle\equiv 0$$

$\Rightarrow\dot{b}(s)=g(s)n(s)$

引入记号 $\tau(s)=-g(s)$ torsion / second curvature 挠率

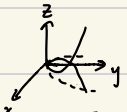
$$\left.\begin{matrix}\langle n,n\rangle\equiv 1\\ \langle n,t\rangle\equiv 0\\ \langle n,b\rangle\equiv 0\end{matrix}\right\}\Rightarrow\begin{matrix}\langle\dot{n}(s),n(s)\rangle\equiv 0\\ \langle\dot{n}(s),t(s)\rangle=-\langle n(s),\dot{t}(s)\rangle=-k(s)\\ \langle\dot{n}(s),b(s)\rangle=-\langle n(s),\dot{b}(s)\rangle=\tau(s)\end{matrix}$$

1851 1847/1852

$$\begin{cases} \frac{dt}{ds}=k(s)n(s) \\ \frac{dn}{ds}=-k(s)t(s)+\tau(s)b(s) \\ \frac{db}{ds}=-\tau(s)n(s) \end{cases}$$

Serre-Frenet公式

$$\frac{d}{ds}\begin{pmatrix} t \\ n \\ b \end{pmatrix}=\begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} t \\ n \\ b \end{pmatrix}$$



$\dot{r}(0)=(1,0,0)$ $\ddot{r}(0)=(0,2,0)$

n-b平面 法平面

t-b平面为xz平面，在0附近，曲线在它一侧 从切平面 ragnified plane "包络出一个曲面"

曲率 $k(s)=|\dot{t}(s)|=|\dot{r}(s)\wedge\ddot{r}(s)|$

$\tau(s)=-\langle n(s),\dot{b}(s)\rangle=-\langle \frac{\ddot{r}(s)}{k(s)}, t(s)\wedge\dot{n}(s)\rangle=-\langle \frac{\ddot{r}(s)}{k(s)}, t(s)\wedge\left(\frac{\dot{r}(s)}{k(s)}\right)^{\cdot}\rangle$

$$\boxed{\tau(s)=\frac{(\dot{r}(s),\ddot{r}(s),\dddot{r}(s))}{|\dot{r}(s)\wedge\ddot{r}(s)|^2}}=\frac{(r',r'',r''')}{|r''|^2}$$

练习. $k(t)=\frac{|r'(t)\wedge r''(t)|}{|r'(t)|^3}$

$\tau(t)=\frac{(r',r'',r''')}{|r'\wedge r''|^2}$

定理：空间正则曲线 $r=r(s)$，设其曲率 $k>0$，则 r 落在某个平面上 $\Leftrightarrow \tau\equiv 0$

证：($\Rightarrow$) 取平面上一点 r_0，记平面法向量为 a

$\langle r(s)-r_0, a\rangle\equiv 0 \Rightarrow \langle t(s), a\rangle\equiv 0 \Rightarrow \langle k(s)n(s), a\rangle=0 \Rightarrow \langle n(s), a\rangle=0$

$\Rightarrow \langle -k(s)t(s)+\tau(s)b(s), a\rangle\equiv 0 \Rightarrow \tau(s)\langle b(s), a\rangle\equiv 0 \Rightarrow \tau(s)=0$

($\Leftarrow$) $\tau\equiv 0 \Rightarrow \dot{b}(s)=-\tau(s)n(s)=0 \Rightarrow b(s)\equiv b$

而 $\frac{d}{ds}\langle r(s), b\rangle=\langle t(s), b\rangle=0 \Rightarrow \langle r(s), b\rangle\equiv c=\langle r_0, b\rangle$

$\Rightarrow \langle r(s)-r_0, b\rangle=0 \Rightarrow$ 在平面内 □

$\tau(s)$ 的符号

$\tau(s)=\frac{(\dot{r},\ddot{r},\dddot{r})}{|\dot{r}\wedge\ddot{r}|^2}$

$r=r(s)$，$s\in[0,L]$，$\bar{r}=\bar{r}(s)=r(L-s)$，则 $\dot{r}$-$\ddot{r}$ 反号，$\dddot{r}$ 不变 $\therefore \tau(s)$ 不变

$\therefore$ 给定曲线，$\tau(s)$ 与前进方向无关

$r = r(s) \in \mathbb{E}^3$, $s \in \mathbb{R}$

考虑 $T: \mathbb{E}^3 \to \mathbb{E}^3$ 欧氏变换

$\tilde{r}(s) = T \circ r(s)$

$\frac{d}{ds}\tilde{r}(s) = T\left(\frac{dr}{ds}\right) = T \circ t(s) \Rightarrow \left|\frac{d}{ds}\tilde{r}(s)\right| = 1 \Rightarrow s$ 是 $\tilde{r}$ 的弧长参数

问题：$\tilde{\kappa}(s) \neq \kappa(s)$　$\tilde{\kappa}(s) = |\ddot{\tilde{r}}(s)| = |T \circ \ddot{r}(s)| = |\ddot{r}(s)| = \kappa(s)$

$\tilde{\tau}(s) \neq \tau(s)$

注意 $\tilde{t}(s) = T \circ t(s)$, $\tilde{n}(s) = T \circ n(s)$

副法向量 $\tilde{b}(s) = \tilde{t}(s) \wedge \tilde{n}(s) = (T \circ t(s)) \wedge (T \circ n(s)) = (\det T) T(t(s) \wedge n(s)) = \pm T \circ b(s)$

$\Rightarrow$ 若 $\det T = 1$（刚体）则 $\tilde{\tau} = \tau$，若 $\det T = -1$（反向刚体），则 $\tilde{\tau} = -\tau$

$r = r(s) \in \mathbb{E}^3$

考查 $s = 0$ 处，Taylor expansion，$\dot{r}(0) \neq 0$

$r(s) = r(0) + \underbrace{\dot{r}(0)}_{t(0)} s + \underbrace{\ddot{r}(0)}_{\kappa(0)n(0)} \cdot \frac{s^2}{2} + \underbrace{\dddot{r}(0)}_{(\kappa(s)n(s))'|_{s=0} = \dot{\kappa}(0)n(0) + \kappa(0)(-\kappa(0)t(0) + \tau(0)b(0))} \cdot \frac{s^3}{6} + \varepsilon(s) \longrightarrow$ 高阶小量

$= \dot{\kappa}(0)n(0) - \kappa(0)^2 t(0) + \kappa(0)\tau(0)b(0)$

Frenet 标架 $(r(0); t(0), n(0), b(0))$

$$r(s) = r(0) + \left(s - \frac{\kappa(0)^2}{6}s^3 + o(s^3)\right) t(0) + \left(\frac{\kappa(0)}{2}s^2 + \frac{\dot{\kappa}(0)}{6}s^3 + o(s^3)\right) n(0) + \left(\frac{\kappa(0)\tau(0)}{6}s^3 + o(s^3)\right) b(0)$$

坐标表示 $r(s) = \left(s - \frac{\kappa(0)^2}{6}s^3 + o(s^3),\ \frac{\kappa(0)}{2}s^2 + \frac{\dot{\kappa}(0)}{6}s^3 + o(s^3),\ \frac{\kappa(0)\tau(0)}{6}s^3 + o(s^3)\right)$

圆柱螺旋线 $r(t) = (a\cos t, a\sin t, bt)$　$a, b \in \mathbb{R}$　$\kappa(t) \equiv \frac{|a|}{a^2+b^2}$　$\tau(t) \equiv \frac{b}{a^2+b^2}$

由 κ, τ，可反解出 a, b

$\rightsquigarrow$ 圆柱螺旋线是否代表了全部常曲率、常挠率曲线？

空间曲线基本定理：设 $\kappa, \tau: [a,b] \to \mathbb{R}$ 两个 C^∞ 函数，$\kappa > 0$ on $[a,b]$，则

(i) 存在一条弧长参数正则曲线 $r: [a,b] \to \mathbb{R}^3$ 以 κ, τ 为曲率、挠率函数

(ii) 若存在两条这样的曲线 $r_1, r_2: [a,b] \to \mathbb{R}^3$，则存在 $\mathbb{E}^3$ 中一个刚体运动 T s.t. $r_1 = T \circ r_2$

证：(i) $\begin{cases} \frac{dr}{ds}=t(s) \\ \frac{dt}{ds}=\kappa(s)n(s) \\ \frac{dn}{ds}=-\kappa(s)t(s)+\tau(s)b(s) \\ \frac{db}{ds}=-\tau(s)n(s) \end{cases}$ 给定初值 $r(0), t(0), n(0), b(0)$，则解存在唯一

$\begin{cases} \text{是否弧长参数？} \\ \text{是否 } t(s), n(s), b(s) \text{ 处处标准正交？} \\ \text{是否处处右手系定向？} \end{cases}$

$v_1(s), v_2(s), v_3(s)$

$$\frac{d}{ds}\langle v_i(s), v_j(s)\rangle = \left\langle \sum_{k=1}^{3} a_{ik}(s)v_k(s), v_j(s)\right\rangle + \left\langle v_i(s), \sum_{k=1}^{3} a_{jk}(s)v_k(s)\right\rangle$$

$$= \sum_{k=1}^{3} a_{ik}(s)\langle v_k(s), v_j(s)\rangle + \sum_{k=1}^{3} a_{jk}\langle v_j, v_k\rangle$$

$$\frac{d}{dt}\begin{pmatrix} \langle v_1,v_2\rangle \\ \langle v_1,v_3\rangle \\ \langle v_2,v_3\rangle \\ \langle v_1,v_1\rangle \\ \langle v_2,v_2\rangle \\ \langle v_3,v_3\rangle \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

初值 $\langle v_i(0), v_j(0)\rangle = \delta_{ij}$

希望 $\langle v_i(s), v_j(s)\rangle \equiv \delta_{ij}$

而 (δ_{ij}) 代入显然满足方程，由解的唯一性，t, n, b 处处标准正交，$\left|\frac{dr}{ds}\right| \equiv 1 \Rightarrow$ 弧长参数

$s \mapsto (t(s), n(s), b(s)) \in \{\pm 1\}$，$(t(0), n(0), b(0)) = 1$，连续性 $\Rightarrow (t(s), n(s), b(s)) \equiv 1$

(ii) 与平面曲线类似 □

问题（整体问题）

闭曲线 $r: [a,b] \to \mathbb{E}^3$ $r(a)=r(b)$，$r'(a)=r'(b)$，$\int_0^l \kappa(s)\,ds = ?$ （$\kappa \sim$ 转角变化率）

凸曲线 in 曲率？ Frenchel 1929 Fary 1949 / Milnor 1950

(III) 曲面的局部理论

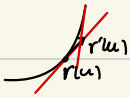
正则曲面(片)

参数曲面 $D\subset\mathbb{R}^2\to\mathbb{E}^3$, $(u,v)\mapsto r(u,v)$ 即 $(x(u,v),y(u,v),z(u,v))$

使得 (1) 每个分量 x,y,z 是 C^∞ 的

(2) $r_u(u,v)=\left(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u},\frac{\partial z}{\partial u}\right)$, $r_v(u,v)=\left(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v},\frac{\partial z}{\partial v}\right)$ 满足 $r_u\wedge r_v\neq 0$

例. $r:(a,b)\to\mathbb{E}^3$ 是正则曲线, 考察 $r(u,v)=r(u)+vr'(u)$, $D=(a,b)\times(-\varepsilon,\varepsilon)$



$r_u=r'(u)+vr''(u)$ $r_v=r'(u)$ $\Rightarrow r_u\wedge r_v=vr''(u)\wedge r'(u)$

$D_1=(a,b)\times(0,\varepsilon)$, $D_2=(a,b)\times(-\varepsilon,0)$ 若 $r''(u)\wedge r'(u)\neq 0$, 则 $r:D_1\to\mathbb{E}^3$, $D_2\to\mathbb{E}^3$ 两张正则曲面片

P点处切平面: = 正则曲面上过P点所有(参数化)光滑曲线在P点处切向量的全体

或者, 曲面 M, $p\in M$, 切平面 $T_pM=\{M$ 上过 p 点的所有光滑参数化曲面$\}/\sim$ $\sim$ 为切向量相对

Claim: T_pM 是一个2维的向量空间

$p=r(u_0,v_0)$, 考虑光滑曲线 $u\mapsto r(u,v_0)$, 在 p 处切向量 $\frac{d}{du}\big|_{u=u_0}r(u,v_0)=r_u(u_0,v_0)\in T_pM$

$v\mapsto r(u_0,v)$, 在 p 处切向量 $\frac{d}{dv}\big|_{v=v_0}r(u_0,v)=r_v(u_0,v_0)\in T_pM$

称 $r_u(u,v)$, $r_v(u,v)$ 为 M 在 p 点处的两个坐标切向量

$\forall a\in\mathbb{R}$, 曲线 $s\mapsto r(u_0+as,v_0)$ ($s\in(-\varepsilon,\varepsilon)$) $\frac{d}{ds}\big|_{s=0}r(u_0+as,v_0)=ar_u(u_0,v_0)\in T_pM$

类似地, $\forall b\in\mathbb{R}$, $br_v(u_0,v_0)\in T_pM$

一般地, $t\mapsto r(u(t),v(t))\in\mathbb{E}^3$ 在 p 处切向量 $\frac{d}{dt}\big|_{t=0}r(u(t),v(t))=r_u(u,v)u'(0)+r_v(u,v)v'(0)$ (隐函数定理局部光滑)

$\forall a,b\in\mathbb{R}$, $s\mapsto r(u_0+as,v_0+bs)$ 在 p 处切向量 $ar_u(u,v)+br_v(u,v)\in T_pM$

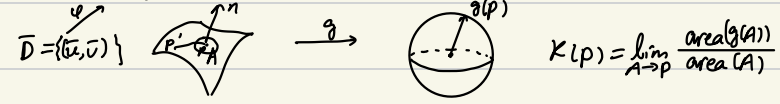
$\therefore T_pM=$ $r_u(u,v)$, $r_v(u,v)$ 张成的平面 $=\{ar_u(u,v)+br_v(u,v)\mid a,b\in\mathbb{R}\}$, Claim 得证

若曲面弯曲, 不可能在一个邻域内处处标准正交 (Gauss 绝妙定理)

局部上, 正则曲线总可写成 $r(u,v)=(u,v,f(u,v))$ (或其轮换形式)

$r_u\wedge r_v\neq 0$, 不妨设 $\frac{\partial(x,y)}{\partial(u,v)}\neq 0$, 局部 $u=u(x,y)$, $v=v(x,y)$, $r(u,v)=(u,v,z(u(x,y),v(x,y)))$

参数变换 $r:D=\{(u,v)\}\to\mathbb{E}^3$



$K(p)=\lim_{A\to p}\frac{\text{area}(g(A))}{\text{area}(A)}$

长度 $t\mapsto r(t)\in M\subset\mathbb{E}^3$ ($t\in(0,a)$)

$L(t)=\int_0^t|r'(t)|\,dt$

光滑曲线 $t\mapsto r(t)$ 在一点 t_0 处总可以写成 $t\mapsto r(u(t),v(t))$, $t\in(t_0-\varepsilon,t_0+\varepsilon)$

$$|r'(t)|^2 = \langle r'(t), r'(t)\rangle = \langle r_u\frac{du}{dt} + r_v\frac{dv}{dt},\ r_u\frac{du}{dt} + r_v\frac{dv}{dt}\rangle$$

$$= \underbrace{\langle r_u, r_u\rangle}_{\| \\ E}\left(\frac{du}{dt}\right)^2 + 2\underbrace{\langle r_u, r_v\rangle}_{\| \\ F}\frac{du}{dt}\frac{dv}{dt} + \underbrace{\langle r_v, r_v\rangle}_{\| \\ G}\left(\frac{dv}{dt}\right)^2$$

第一基本形式

坐标函数 $u: M \to \mathbb{R}$, $r(u,v) \mapsto u$　　$v: M \to \mathbb{R}$, $r(u,v) \mapsto v$

$du(V) = \frac{d}{dt}\big|_{t=0} u(r(t))$, $r'(0) = V$　　$dv \cdots$

$du(r_u) = \frac{d}{dt}\big|_{t=0} u(r(t,0)) = 1$　　类似地, $dv(r_u) = 0$

$du(r_v) = \frac{d}{dt}\big|_{t=0} u(r(0,t)) = 0$　　$dv(r_v) = 1$

$I := E\,du\otimes du + F\,du\otimes dv + F\,dv\otimes du + G\,dv\otimes dv$

在 p 处, $I|_p : T_pM \times T_pM \to \mathbb{R}$ 双线性映射, 对称

Claim: $\forall V, W \in T_pM$, $I(V,W) = \langle V, W\rangle$ ($\Rightarrow$ I 的取值与参数化无关)

证: 只需验证 $I(r_u, r_u) = E\,du(r_u)dv(r_u) + F\,du(r_u)dv(r_u) + F\,dv(r_u)dv(r_u) + G\,dv(r_u)dv(r_u) = E\cdot 1 = E = \langle r_u, r_u\rangle$

$I(r_u, r_v) = \cdots$

$I(r_v, r_v) = \cdots$

□

若 $V = \lambda_1 r_u + \lambda_2 r_v$, $W = \mu_1 r_u + \mu_2 r_v$, 则 $I(V,W) = (\lambda_1, \lambda_2)\begin{pmatrix} E & F \\ F & G\end{pmatrix}\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}$

Rmk. $\begin{pmatrix} E & F \\ F & G\end{pmatrix}$ 为定矩阵, $\det = EG - F^2 = \langle r_u, r_u\rangle\langle r_v, r_v\rangle - \langle r_u, r_v\rangle^2 = \langle r_u, r_u\rangle\langle r_v, r_v\rangle(1-\cos^2\theta) > 0$

面积 $\iint_D |r_u \wedge r_v|\,du\,dv = \iint_D \sqrt{EG-F^2}\,du\,dv$ (Lagrange 恒等式 $|r_u \wedge r_v|^2 = \langle r_u, r_u\rangle\langle r_v, r_v\rangle - \langle r_u, r_v\rangle^2 = EG - F^2$)

$g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$　$x^1 = u$, $x^2 = v$ $\Rightarrow$ $I = \sum_{i,j=1}^{2} g_{ij}\,dx^i \otimes dx^j$

参数变换 $\begin{cases}\overline{x^1} = \overline{x^1}(x^1, x^2) \\ \overline{x^2} = \overline{x^2}(x^1, x^2)\end{cases}$

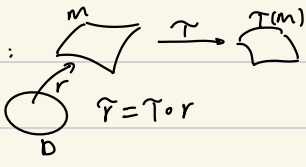
$$\overline{I} = \sum_{i,j=1}^{2} \overline{g_{ij}}\,\overline{dx^i}\otimes\overline{dx^j},\quad \overline{g_{ij}} = \left\langle \sum_k r_{x^k}\frac{dx^k}{d\overline{x^i}},\ \sum_\ell r_{x^\ell}\frac{dx^\ell}{d\overline{x^j}}\right\rangle = \sum_{k,\ell} g_{k\ell}\frac{dx^k}{d\overline{x^i}}\frac{dx^\ell}{d\overline{x^j}}$$

$$= \sum_{i,j,k,\ell} g_{k\ell}\frac{dx^k}{d\overline{x^i}}\frac{dx^\ell}{d\overline{x^j}}\,d\overline{x^i}\otimes d\overline{x^j}$$

$$= \sum_{i,j,k,\ell,s,\eta} g_{k\ell}\frac{\partial x^k}{\partial\overline{x^i}}\frac{\partial x^\ell}{\partial\overline{x^j}}\frac{\partial\overline{x^i}}{\partial x^\eta}\frac{\partial\overline{x^j}}{\partial x^s}\,dx^\eta\otimes dx^s$$

I 在参数变换下不变

I 在刚体变换下不变:



$\tilde{r} = T\circ r$

$T^*\tilde{I}(V,W) = \tilde{I}(T(V), T(W)) = I(V,W)$, $\forall V, W \in T_pM$

$\therefore T^*(\tilde{I}) = I$

例. (1) $\mathbb{R}^2 \subset \mathbb{R}^3$ $r(u,v)=(u,v,1)$ $r_u=(1,0,0)$, $r_v=(0,1,0)$ $\Rightarrow E=1, F=0, G=1 \Rightarrow I=du\otimes du+dv\otimes dv$ M_1

(2) 柱面 $(u,v)\in(a,b)\times\mathbb{R}$ $r(u,v)=(x(u),y(u),v)$ $r_u=(x'(u),y'(u),0)$, $r_v=(0,0,1)$

$\Rightarrow E=(x')^2+(y')^2$, $F=0$, $G=1$ $\Rightarrow I=(x'(u)^2+y'(u)^2)\,du\otimes du+dv\otimes dv$

若 $u\mapsto(x(u),y(u))$ 为弧长参数，则 $I=du\otimes du+dv\otimes dv$ M_2

$f: M_1\to M_2$

$(u,v,1)\mapsto(x(u),y(v),v)$

$t\mapsto(u(t),v(t),1)$ t 方向向量模长 $\sqrt{u'(t)^2+v'(t)^2}$

$t\mapsto f(u(t),v(t),1)=(x(u(t)),y(u(t)),v(t))$ t 方向向量模长 $\sqrt{(x'(u)\frac{du}{dt})^2+(y'(u)\frac{du}{dt})^2+v'(t)^2}=\sqrt{u'(t)^2+v'(t)^2}$

不能直接换

一般地，$r(u,v)$, $r'(u,v)$ 第一基本形式相同 $\Rightarrow |r_u|=|r'_u|$, $|r_v|=|r'_v|$, r_u, r_v 夹角与 r'_u, r'_v 夹角相同

$\Rightarrow \left|\frac{\partial}{\partial t}r(u(t),v(t))\right|=\sqrt{(\frac{\partial u}{\partial t})^2|r_u|^2+(\frac{\partial v}{\partial t})^2|r_v|^2+2\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\langle r_u,r_v\rangle}$

$=\sqrt{(\frac{\partial u}{\partial t})^2|r'_u|^2+(\frac{\partial v}{\partial t})^2|r'_v|^2+2\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\langle r'_u,r'_v\rangle}=\left|\frac{\partial}{\partial t}r'(u(t),v(t))\right|$

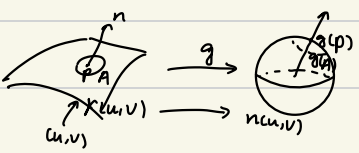
$\bar{D}=\{(\theta,\varphi): -\frac{\pi}{2}<\theta<\frac{\pi}{2}, 0<\varphi<\pi\}$ $r:\bar{D}\to\mathbb{E}^3$ $\Rightarrow I=a^2(d\theta\otimes d\theta+\cos^2\theta\, d\varphi\otimes d\varphi)$

$(\theta,\varphi)\mapsto(a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, a\sin\theta)$ 球面

$(u,v)\in\mathbb{R}^2$ $r(u,v)=\left(\frac{2a^2u}{a^2+u^2+v^2}, \frac{2a^2v}{a^2+u^2+v^2}, a\cdot\frac{u^2+v^2-a^2}{u^2+v^2+a^2}\right)$, $I=\frac{4}{(1+\frac{1}{a^2}(u^2+v^2))^2}(du\otimes du+dv\otimes dv)$

Gauss Curvature

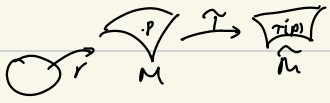
$$K(p)=\lim_{A\to p}\frac{\text{area}(g(A))}{\text{area}(A)}=\lim_{A\to p}\frac{\iint|n_u\wedge n_v|\,du\,dv}{\iint|r_u\wedge r_v|\,du\,dv}$$



定义. $K(p)=\frac{n_u\wedge n_v}{r_u\wedge r_v}$, i.e. $n_u\wedge n_v=k(p)\,r_u\wedge r_v$ ($r_u\wedge r_v\neq 0$, 良定)

容许参数变换: $(u,v)=\sigma(\bar{u},\bar{v})$ $\bar{r}(\bar{u},\bar{v})=r\circ\sigma$ $\left.\begin{array}{l}\bar{r}_{\bar{u}}\wedge\bar{r}_{\bar{v}}=\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}r_u\wedge r_v\\ \bar{n}_{\bar{u}}\wedge\bar{n}_{\bar{v}}=\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}n_u\wedge n_v\end{array}\right\}\Rightarrow K$ 不依赖参数化

$\tilde{r}_u = T \circ r_u$, $\tilde{r}_v = T \circ r_v$



$\Rightarrow \tilde{r}_u \wedge \tilde{r}_v = \det(T)\, T(r_u \wedge r_v)$

$\tilde{n} = \frac{\tilde{r}_u \wedge \tilde{r}_v}{|\tilde{r}_u \wedge \tilde{r}_v|} = \frac{\det(T) T(r_u \wedge r_v)}{|r_u \wedge r_v|} = \det(T)\, T \circ n$

$\tilde{K}(T(p)) = K(p)$

$\tilde{n}_u \wedge \tilde{n}_v (T(p)) = [\det(T) T \circ n_u] \wedge [\det(T) T \circ n_v] = (T \circ n_u) \wedge (T \circ n_v) = \det(T)\, T(n_u \wedge n_v) = K(p) \det(T) T(r_u \wedge r_v)$

($K(p)\, r_u \wedge r_v$; $\tilde{r}_u \wedge \tilde{r}_v$)

$\therefore K$ 在欧氏变换下不变

Gauss 1827 与 Euler 工作产生联系

显式公式 $r(u,v) = (u, v, f(u,v))$ $(u,v) \in D \subset \mathbb{R}^2$

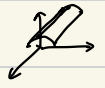
$r_u = (1, 0, f_u)$

$r_v = (0, 1, f_v)$ $\Rightarrow r_u \wedge r_v = (-f_u, -f_v, 1)$

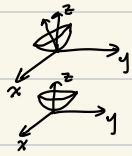
$\Rightarrow n = \frac{r_u \wedge r_v}{|r_u \wedge r_v|} = \frac{1}{\sqrt{f_u^2 + f_v^2 + 1}} (-f_u, -f_v, 1) \Rightarrow n_u, n_v \Rightarrow \cdots$

$$K(u,v,f(u,v)) = \frac{f_{uu} f_{vv} - f_{uv}^2}{(f_u^2 + f_v^2 + 1)^2}$$

可展性 developable $\Leftrightarrow f_{uu} f_{vv} - f_{uv}^2 = 0$

例.  $r(u,v) = (u, v, \sqrt{1-v^2})$

不妨设 $f(0,0) = 0$, $\langle n(0,0), (0,0,1) \rangle = \frac{1}{\sqrt{1 + f_u^2(0,0) + f_v^2(0,0)}}$



通过一个欧氏变换使曲面变为 $\tilde{r} = T \circ r$

$D \xrightarrow{r} r(D) \subset E^3 \xrightarrow{T} T \circ r(D)$

$(u,v) \longmapsto (u,v,f(u,v)) \longmapsto (u, v, h(u,v)) = (u, v, \langle \tilde{r}(u,v) - \tilde{r}(0,0), (0,0,1) \rangle)$, $h_u(0,0) = h_v(0,0) = 0$

$K(0,0) = (h_{uu} h_{vv} - h_{uv}^2)(0,0)$

$D \xrightarrow{R_\theta} D \longrightarrow \tilde{r}(D)$

$(\bar{u}, \bar{v}) \mapsto (u,v) \mapsto (u, v, h(u,v))$

$(\bar{u}, \bar{v}) \longmapsto (u, v, \underbrace{h(\bar{u}\cos\theta - \bar{v}\sin\theta, \bar{u}\sin\theta + \bar{v}\cos\theta)}_{f_\theta(\bar{u}, \bar{\theta})})$

检查 $(f_\theta)_{\bar{u}}(0,0) = h_u \cos\theta + h_v \sin\theta \big|_{(0,0)} = 0$ $\quad (f_\theta)_{\bar{v}}(0,0) = 0$

$(f_\theta)_{\bar{u}\bar{v}}\big|_{(0,0)} = \cos 2\theta\, h_{uv} + \frac{\sin 2\theta}{2}(h_{vv} - h_{uu})$

总结一下，$\exists\, r(u,v) = (u, v, f(u,v))$ 使得 $\begin{cases} f_u(0,0) = 0 \\ f_v(0,0) = 0 \\ f_{uu}(0,0) = 0 \end{cases}$ 且 $K(0,0) = f_{uu}(0,0) f_{vv}(0,0) = \boxed{\frac{\partial f}{\partial u^2}(0,0)} \boxed{\frac{\partial f}{\partial v^2}(0,0)}$

(x-z 平面；y-z 平面)

平面曲线 $t \mapsto (x(t), y(t))$ 带符号曲率 $K(t) = \frac{\det \begin{pmatrix} x' & y' \\ x'' & y'' \end{pmatrix}}{((x')^2 + (y')^2)^{\frac{3}{2}}}$ $\quad u \mapsto (u, f(u,0))$, $K(0) = \frac{\partial f}{\partial u^2}(0,0)$

$\therefore K(0,0)$ = 这两个方向上曲率乘积

$v=e_1\cos\theta+e_2\sin\theta$　　v,n 平面截得曲线为 $r(t)=(t, f(t\cos\theta, t\sin\theta))$

验证：$r'(0)=(1, \frac{\partial f}{\partial u}\cos\theta+\frac{\partial f}{\partial v}\sin\theta)\Big|_{t=0}=(1,0)$

则 $K_n(v)=\frac{d^2}{dt^2}f(t\cos\theta, t\sin\theta)\Big|_{t=0}=\frac{d}{dt}(\frac{\partial f}{\partial u}\cos\theta+\frac{\partial f}{\partial v}\sin\theta)\Big|_{t=0}$

$=(\frac{\partial^2 f}{\partial u^2}\cos^2\theta+2\frac{\partial^2 f}{\partial u\partial v}\cos\theta\sin\theta+\frac{\partial^2 f}{\partial v^2}\sin^2\theta)\Big|_{t=0}$

$=\cos^2\theta\frac{\partial^2 f}{\partial u^2}(0,0)+\sin^2\theta\frac{\partial^2 f}{\partial v^2}(0,0)$

也即所有法曲率 K_n 取值均在 $\frac{\partial^2 f}{\partial u^2}(0,0)=K_n(e_1)$ 和 $\frac{\partial^2 f}{\partial v^2}(0,0)=K_n(e_2)$ 之间

换言之，$K_n(e_1)$，$K_n(e_2)$ 给出法曲率的最大、最小值

法曲率　设 $M: r=r(u,v)$ 为一光滑曲面，$p=r(0,0)$，设 $v\in T_pM$，n 为 p 点处法向量，M 在 p 点处沿 v 方向的法曲率 $K_n(v)$ 定义为 $v-n$ 平面截 M 所得平面曲线（对弧参数化 $\overset{t\mapsto r(t)}{r(0)=p}$ 使得 $r'(0)=v$）

在定向 $v-n$ 下的带符号曲率.

注记：$K_n(v)=K_n(\lambda v)$，$\forall\lambda\neq 0$

设 s 为 $r=r(s)=r(u(s),v(s))$ 的弧长参数

$\dot r(0)=r_u\frac{du}{ds}+r_v\frac{dv}{ds}\Big|_{s=0}$

$\ddot r(0)=r_{uu}(\frac{du}{ds})^2+r_{uv}\frac{du}{ds}\frac{dv}{ds}+r_{vu}\frac{dv}{ds}\frac{du}{ds}+r_{vv}(\frac{dv}{ds})^2+r_u\frac{d^2u}{ds^2}+r_v\frac{d^2v}{ds^2}$.

考虑 $\langle\ddot r(s), n(s)\rangle_{s=0}=K_n(\dot r(0))$

$=\langle r_{uu}, n\rangle(\frac{du}{ds})^2+\langle r_{uv}, n\rangle\frac{du}{ds}\frac{dv}{ds}+\langle r_{vu}, n\rangle\frac{dv}{ds}\frac{du}{ds}+\langle r_{vv}, n\rangle(\frac{dv}{ds})^2\Big|_{s=0}$

↑ 曲面决定之量　$\frac{du}{ds}(0)$，$\frac{dv}{ds}(0)$ 切向量

$\Rightarrow M$　$r=r(u,v)$　$p=r(0,0)$　$0\neq w\in T_pM$，不妨设 $w=\lambda r_u+\mu r_v$

$K_n(w)=K_n(\frac{w}{|w|})=\langle r_{uu}, n\rangle\frac{\lambda^2}{|w|^2}+2\langle r_{uv}, n\rangle\frac{\lambda}{|w|}\cdot\frac{\mu}{|w|}+\langle r_{vv}, n\rangle\frac{\mu^2}{|w|^2}$

T_pM　$I(\cdot,\cdot)=E\,du\otimes du+F\,du\otimes dv+F\,dv\otimes du+G\,dv\otimes dv$

$\forall v, w\in T_pM$，$I(v,w)=\langle v, w\rangle$

定义　$L:=\langle r_{uu}, n\rangle$

$M:=\langle r_{uv}, n\rangle=\langle r_{vu}, n\rangle$

$N:=\langle r_{vv}, n\rangle$

第二基本形式 $I = L\,du\otimes du + M\,du\otimes dv + M\,dv\otimes du + N\,dv\otimes dv$

$$\mathrm{II}\left(\frac{w}{|w|},\frac{w}{|w|}\right) = L\left(\frac{\lambda}{|w|}\right)^2 + M\frac{\lambda}{|w|}\frac{\mu}{|w|} + M\frac{\mu}{|w|}\frac{\lambda}{|w|} + N\left(\frac{\mu}{|w|}\right)^2$$

$$K_n\left(\frac{w}{|w|}\right) = \mathrm{II}\left(\frac{w}{|w|},\frac{w}{|w|}\right) = \frac{1}{|w|^2}\mathrm{II}(w,w) = \frac{\mathrm{II}(w,w)}{I(w,w)}$$

$$\Rightarrow K_n(w) = \frac{\mathrm{II}(w,w)}{I(w,w)}$$

第二基本形式在参数变换下是否不变？在欧氏变换下是否不变？

L, M, V 涉及二阶导，不好算

注意到 $\langle r_u, n\rangle \equiv 0$，求导 $\langle r_{uu}, n\rangle + \langle r_u, n_u\rangle \equiv 0$，...

$$\therefore L = \langle r_{uu}, n\rangle = -\langle r_u, n_u\rangle$$

$$M = \langle r_{uv}, n\rangle = -\langle r_u, n_v\rangle = -\langle r_v, n_u\rangle$$

$$V = \langle r_{vv}, n\rangle = -\langle r_v, n_v\rangle$$

$u = u(\bar u, \bar v)$, $v = v(\bar u, \bar v)$

$$\mathrm{II} = \sum_{i,j=1}^{2} -\langle r_{ui}, n_{uj}\rangle\, du^i\otimes du^j$$

$$\bar{\mathrm{II}} = \sum_{i,j=1}^{2} -\langle r_{\bar u i}, n_{\bar u j}\rangle\, d\bar u^i\otimes d\bar u^j$$

$$r_{\bar u i} = \sum_{k=1}^{2} r_{u^k}\frac{\partial u^k}{\partial \bar u^i},\quad n_{\bar u i} = \sum_{k=1}^{2} n_{u^k}\frac{\partial u^k}{\partial \bar u^i}\quad d\bar u^i = \sum_{l=1}^{2}\frac{\partial \bar u^i}{\partial u^l}du^l$$

$\Rightarrow$ 在同定向参数变换下不变，在反定向参数变换下变号

$$n = \frac{r_u\wedge r_v}{|r_u\wedge r_v|}\qquad T(n) = \frac{T(r_u\wedge r_v)}{|T(r_u)\wedge T(r_v)|} = \frac{\det T\,(T(r_u)\wedge T(r_v))}{|T(r_u)\wedge T(r_v)|} = \frac{\mathrm{sgn}(\det T)\,(T(r_u)\wedge T(r_v))}{|T(r_u)\wedge T(r_v)|}$$

$\Rightarrow$ 在刚体变换下不变，在反向刚体变换下变号

但 $K(p)$ 为两个法曲率之乘积，在参数变换和欧氏变换下都不变 $\rightsquigarrow$ 内蕴

$$K(p) = \left(\max_{w\in T_pM\setminus\{0\}} K_n(w)\right)\left(\min_{w\in T_pM\setminus\{0\}} K_n(w)\right)$$

$I(V,W) = \langle V, W\rangle$

取 $\mathcal{W}: T_pM\to T_pM$ linear s.t. $\mathrm{II}(V,W) = \langle V, \mathcal{W}(W)\rangle$

$V = \lambda r_u + \mu r_v$, $W = \xi r_u + \eta r_v$

$$\begin{aligned}\mathrm{II}(V,W) &= L\lambda\xi + M(\lambda\eta+\mu\xi) + N\mu\eta = (\lambda,\mu)\begin{pmatrix} L & M\\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\\ &= -\langle r_u, n_u\rangle\lambda\xi - \langle r_u, n_v\rangle\lambda\eta - \langle r_v, n_u\rangle\mu\xi - \langle r_v, n_v\rangle\mu\eta\\ &= \langle \lambda r_u+\mu r_v,\ -\xi n_u - \eta n_v\rangle = \langle V, \mathcal{W}(W)\rangle\end{aligned}$$

T_pM (1830-1910)

定义 $W(\xi r_u+\eta r_v)=-\xi n_u-\eta n_v$ Weingarten 变换

线性变换 $W: T_pM \to T_pM$

$r_u \mapsto -n_u$

$r_v \mapsto -n_v$

引 $\mathrm{II}(V,W)=\langle V, W(W)\rangle$

$\mathrm{II}(W,V)=\langle W, W(V)\rangle$

$\Rightarrow \langle V, W(W)\rangle=\langle W(V), W\rangle \cdot \forall V, W$

W 的 eigenvalue $\lambda_1\neq\lambda_2$ (自伴算子可对角化, $\lambda_1=\lambda_2$ 时任取两个正交特征向量即可

$\langle W(v_1), v_2\rangle=\lambda_1\langle v_1, v_2\rangle$

$\langle v_1, W(v_2)\rangle=\lambda_2\langle v_1, v_2\rangle$

设 $e_1, e_2 \in T_pM$ 为 W 的两个特征向量, 标准正交, $W=a^1e_1+a^2e_2$

$$\max_{W\in T_pM\setminus\{0\}} \frac{\mathrm{II}(W,W)}{\mathrm{I}(W,W)}=\max_{W\in T_pM\setminus\{0\}} \frac{\langle W, W(W)\rangle}{\langle W, W\rangle}=\max_{a_1,a_2\in\mathbb{R}} \frac{\lambda_1(a^1)^2+\lambda_2(a^2)^2}{(a^1)^2+(a^2)^2}=\max\{\lambda_1,\lambda_2\}$$

$$\min_{W\in T_pM\setminus\{0\}} \frac{\mathrm{II}(W,W)}{\mathrm{I}(W,W)}=\min\{\lambda_1,\lambda_2\}$$

$$\mathrm{II}(W,W)=(\xi\ \eta)\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

$$=(\xi\ \eta)\begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1}\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

$$\Rightarrow W(W)=\begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1}\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

$$\Rightarrow K(P)=\lambda_{\max}(W)\lambda_{\min}(W)=\det(W)$$

$$=\frac{\det\begin{pmatrix}L & M\\ M & N\end{pmatrix}}{\det\begin{pmatrix}E & F\\ F & G\end{pmatrix}}=\frac{LN-M^2}{EG-F^2}$$

W 即为 Gauss 映射的 differential, 特征值均变号 $\therefore K(P)=\det(dg)$

第二基本形式 II

例. 平面 $r(u,v)=(u,v,\underset{\text{常数}}{c})$

$r_{uu}=r_{uv}=r_{vv}=0 \Leftarrow \left\{\begin{array}{l} r_u=(1,0,0)\\ r_v=(0,1,0)\end{array}\right\} \Rightarrow n=(0,0,1)$

$\Rightarrow L=\langle r_{uu}, n\rangle=0,\ M=0,\ N=0$

$\Rightarrow \mathrm{II}=0$

例. 柱面 $r(u,v)=(x(u), y(u), v)$ 假定 $x_u^2+y_u^2=1$

$r_u=(x_u, y_u, 0)\quad r_v=(0,0,1)$

$\mathrm{I}=du\otimes du+dv\otimes dv$

$n=(y_u, -x_u, 0)$

$r_{uu}=(x_{uu}, y_{uu}, 0) \Rightarrow L=x_{uu}y_u-x_uy_{uu}$

$r_{uv}=r_{vv}=0 \Rightarrow M=N=0$

$$\text{II} = (x_{uu}y_u - x_u y_{uu})\,du\otimes du$$

$$= -\frac{(x_u y_{uu} - x_{uu}y_u)}{(x_u^2+y_u^2)^{3/2}}\,du\otimes du$$

$$= -K(u)\,du\otimes du$$

$$K_n(r_u) = \frac{\text{II}(r_u, r_u)}{\text{I}(r_u, r_u)} = -K(u)$$





$$K_n(r_v) = 0$$

$w = \cos\theta\, r_u + \sin\theta\, r_v \quad K_n(w) = \frac{\text{II}(w,w)}{\text{I}(w,w)} = -K(u)\,du(w)\,du(w) = -K(u)\cos^2\theta$

例. 球面 $a>0$ $r(\theta,\varphi) = (a\cos\theta\cos\varphi,\ a\cos\theta\sin\varphi,\ a\sin\theta)$ $-\frac{\pi}{2}<\theta<\frac{\pi}{2},\ 0<\varphi<2\pi$

$$\text{I}(\theta,\varphi) = a^2(d\theta\otimes d\theta + \cos^2\theta\, d\varphi\otimes d\varphi)$$

$$\text{II}(\theta,\varphi) = a(d\theta\otimes d\theta + \cos^2\theta\, d\varphi\otimes d\varphi)$$

$$\frac{\text{II}(w,w)}{\text{I}(w,w)} \equiv \frac{1}{a} \quad (\text{球面对称})$$

例. 环面 $r(u,v) = ((R+r\cos u)\cos v,\ (R+r\cos u)\sin v,\ r\sin u)$ $0<u<2\pi,\ 0<v<2\pi$



$$\text{I} = r^2\,du\otimes du + (R+r\cos u)^2\,dv\otimes dv$$

$$\text{II} = r\,du\otimes du + (R+r\cos u)\cos u\,dv\otimes dv$$

$$K(p) = \frac{LN-M^2}{EG-F^2} = \frac{\cos u}{r(R+r\cos u)},\ 0<u<2\pi$$

内圈负曲率，外圈正曲率，最高最低两个圈零曲率

内圈曲率积分 + 外圈曲率积分 = 0

高斯绝妙定理 (Theorem Egregium)

定理：已知曲面 $r = r(u,v)$ 在 p 点处 in 高斯曲率 $K(p)$ 满足

$$4(EG-F^2)^2K(p) = E(E_vG_v - 2F_uG_v + (G_u)^2)$$
$$+ F(E_uG_v - E_vG_u - 2E_vF_v + 4F_uF_v - 2F_uG_u)$$
$$+ G(E_uG_u - 2E_uF_v + (E_v)^2)$$
$$- 2(EG-F^2)(E_{vv} - 2F_{uv} + G_{uu})$$

要点：$K(p)$ 只依赖于 E, F, G

记忆：$K(p) = \frac{LN-M^2}{EG-F^2}$

$K(p)(EG-F^2)^2 = (LN-M^2)(EG-F^2)$

$L=\langle r_{uu}, n\rangle=\left\langle r_{uu}, \frac{r_u\wedge r_v}{|r_u\wedge r_v|}\right\rangle=\frac{\cdots}{\sqrt{EG-F^2}}$ ($\because LN-M^2$ 分母中含出 $EG-F^2$ $\therefore$ 要乘 $EG-F^2$)

$K(p)=\langle r_{uu}, r_u\wedge r_v\rangle\langle r_{vv}, r_u\wedge r_v\rangle-\langle r_{uv}, r_u\wedge r_v\rangle^2$

$=\det\begin{pmatrix} r_{uu}\\ r_u\\ r_v\end{pmatrix}\det(r_u\ r_v\ n)-\det\begin{pmatrix} r_{uv}\\ r_u\\ r_v\end{pmatrix}\det(r_u\ r_v\ n)$

$=\det\begin{pmatrix} \langle r_{uu}, r_{vv}\rangle & \langle r_{uu}, r_u\rangle & \langle r_{uu}, r_v\rangle\\ \langle r_u, r_{vv}\rangle & E & F\\ \langle r_v, r_{vv}\rangle & F & G\end{pmatrix}-\det\begin{pmatrix} \langle r_{uv}, r_{uv}\rangle & \langle r_{uv}, r_u\rangle & \langle r_{uv}, r_v\rangle\\ \langle r_u, r_{uv}\rangle & E & F\\ \langle r_v, r_{uv}\rangle & F & G\end{pmatrix}$

$\langle r_{uu}, r_u\rangle=\frac{1}{2}\frac{\partial}{\partial u}\langle r_u, r_u\rangle=\frac{1}{2}E_u$ ...

$\langle r_{uu}, r_v\rangle=\frac{\partial}{\partial u}\langle r_u, r_v\rangle-\langle r_u, r_{uv}\rangle=F_u-\frac{1}{2}E_v$ ...

$$\left.\begin{aligned}\langle r_{uu}, r_{vv}\rangle&=\frac{\partial}{\partial v}\langle r_{uu}, r_v\rangle-\langle r_{uuv}, r_v\rangle\\ \langle r_{uv}, r_{uv}\rangle&=\frac{\partial}{\partial u}\langle r_{uv}, r_v\rangle-\langle r_{uuv}, r_v\rangle\end{aligned}\right\}\Rightarrow \langle r_{uu}, r_{vv}\rangle-\langle r_{uv}, r_{uv}\rangle=\frac{\partial}{\partial v}\langle r_{uu}, r_v\rangle-\frac{\partial}{\partial u}\langle r_{uv}, r_v\rangle$$
$$=(F_u-\tfrac{1}{2}E_v)_v-\tfrac{1}{2}G_{uu}$$

行列式按行展开可知，只要知道这个差即可求出 $K(p)$

□

给定 I（不需要曲面的信息，内积诱导），即可定义 $K(p)$，抽象曲面 内蕴几何

球面 Gauss 曲率不为 0，由 Gauss 绝妙定理，不可能在局部参数化使得 $I=du\otimes du+dv\otimes dv$

弯曲的曲面不可能 才距变换到平面

平均曲率

$H=\frac{1}{2}(\lambda_{max}(W)+\lambda_{min}(W))$

$=\frac{1}{2}\mathrm{tr}(W)=\frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} E & F\\ F & G\end{pmatrix}^{-1}\begin{pmatrix} L & M\\ M & N\end{pmatrix}\right)=\frac{1}{2}\frac{1}{EG-F^2}\mathrm{tr}\begin{pmatrix} GL-FM & *\\ * & -FM+EN\end{pmatrix}=\frac{1}{2}\frac{LG-2MF+NE}{EG-F^2}$

$H=\frac{1}{2}$（最大法曲率 + 最小法曲率），这两个曲率称为主曲率，主曲率对应的切向量方向称为主方向，

从而 H 为两个主曲率的平均值

$r: D\to E^3$ 正则曲面 $D\subset\mathbb{R}^2$ 是区域，∂D 分段光滑闭曲线

光滑映射 $\alpha:(-\varepsilon,\varepsilon)_t\times D\to E^3$ 若满足 $\alpha(0,u,v)=r(u,v)$，则称 α 为 $r=r(u,v)$ 的一个变分 (variation)

对充分小的 $t\in(-\varepsilon,\varepsilon)$，$r_t=r_t(u,v)=\alpha(t,u,v)$ 仍为正则曲面片

考虑 $t\mapsto\alpha(t,u,v)$，$W(u_0,v_0):=\frac{\partial}{\partial t}\big|_{t=0}\alpha(t,u_0,v_0)$，$W$ 称为变分向量场 (variation vector field)

考虑 $\alpha:(-\varepsilon,\varepsilon)\times D\to E^3$，$\alpha(t,u,v)=r(u,v)+t\varphi(u,v)n(u,v)$

变分向量场 $W(u,v)=\frac{\partial}{\partial t}\big|_{t=0}\alpha(t,u,v)=\varphi(u,v)n(u,v)$ i.e. $W=\varphi n$

性质：设 r, α 如上所述，考查正则曲面 $r_t, t\in(-\varepsilon,\varepsilon)$ 之面积 $A(t)=\mathrm{Area}(r_t(D))=\iint_D\sqrt{E_tG_t-F_t^2}\,du\,dv$

我们有 $A'(0)=\frac{d}{dt}\big|_{t=0}A(t)=-\iint_D 2\varphi H\underbrace{|r_u\wedge r_v|}_{d\sigma}\,du\,dv$

注：任一对向量场，可在局部重新参数化使变分向量场与曲面垂直，从而 H 与面积变分仍有关系

$H\not\equiv 0\Rightarrow\exists(u_0,v_0)$ s.t. $H(u_0,v_0)\neq 0$ 且邻域或 ∂，$H|_\partial\neq 0$ 且不变号

取 $\varphi|_{M|\partial}\equiv 0$，在 ∂ 上与 H 同号 $\Rightarrow A'(0)<0$

$\rightsquigarrow$ Minimal surface 极小曲面：$H\equiv 0$

证：$E_t=\langle\gamma_t u,\gamma_t u\rangle=\langle r_u+t\varphi_u n+t\varphi n_u,\ r_u+t\varphi_u n+\varphi n_u\rangle=E-2t\varphi L+t^2(\varphi_u^2+\varphi^2\langle n_u,n_u\rangle)$

$F_t=F-2t\varphi M+t^2(\cdots)$

$G_t=G-2t\varphi+t^2(\cdots)$

$\Rightarrow E_tG_t-F_t^2=(E-2t\varphi L)(G-2t\varphi N)-(F-2t\varphi M)^2+t^2(\cdots)$

$=EG-F^2-2t\varphi(LG+EN-2MF)+t^2(\cdots)$

$=(EG-F^2)(1-4t\varphi H)+t^2(\cdots)$

$\frac{d}{dt}\big|_{t=0}A_t=\iint_D\frac{d}{dt}\big|_{t=0}\sqrt{E_tG_t-F_t^2}\,du\,dv=\iint_D\frac{\frac{d}{dt}(E_tG_t-F_t^2)}{2\sqrt{EG-F^2}}du\,dv=-2\iint_D\varphi H\sqrt{EG-F^2}\,du\,dv$ □

例. 二次曲面 $z=\frac{1}{2}(ax^2+by^2)$，$a,b\in\mathbb{R}$　　$r(x,y)=(x,y,\frac{1}{2}(ax^2+by^2))$

$I=(1+a^2x^2)\,dx\otimes dx+abxy\,dx\otimes dy+(1+b^2y^2)\,dy\otimes dy$

$II=\frac{1}{\sqrt{1+a^2x^2+b^2y^2}}(a\,dx\otimes dx+b\,dy\otimes dy)$

在一点 p 处，取单位切向量 $V=\lambda r_x+\mu r_y$

$\kappa_n(V)=II(V,V)=\frac{1}{\sqrt{1+a^2x^2+b^2y^2}}(a\,dx(V)dx(V)+b\,dy(V)dy(V))=\frac{a\lambda^2+b\mu^2}{\sqrt{1+a^2x^2+b^2y^2}}$

$ab>0$，过一点 p 所有曲线弯曲方向相同

$ab<0$，　…

$ab=0$，　…

$\mathrm{II} = L\,du\otimes du + M\,du\otimes dv + M\,dv\otimes du + N\,dv\otimes dv$

$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ 为高度函数 h 的 Hessian.

$h(u,v) = \langle r(u,v) - r(0,0),\ n(0,0)\rangle \qquad h_u(0,0) = h_v(0,0) = 0$

$\mathrm{Hess}\,h = \begin{pmatrix} h_{uu} & h_{uv} \\ h_{vu} & h_{vv} \end{pmatrix} = \begin{pmatrix} \langle r_{uu}, n\rangle & \langle r_{uv}, n\rangle \\ \langle r_{vu}, n\rangle & \langle r_{vv}, n\rangle \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

$r(u,v) = (u, v, h(u,v)) \quad p = r(0,0) \quad h(0,0) = 0 \quad h_u(0,0) = h_v(0,0) = 0 \quad h_{uv}(0,0) = 0$

$r_u(0,0) = (1,0,0) \quad r_v(0,0) = (0,1,0)$

$h(u,v) = h(0,0) + \frac{1}{2}\left(h_{uu}(0,0)u^2 + h_{vv}(0,0)v^2\right) + o(u^2+v^2)$

$= \frac{1}{2}(k_1u^2 + k_2v^2) + o(u^2+v^2)$, k_1, k_2 主曲率 $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ 主方向

二次近似曲面

$(u, v, \frac{1}{2}(k_1u^2 + k_2v^2))$ 称为曲面在 p 点处的密切抛物面 (osculating paraboloid)

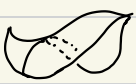
(1) 椭圆点 (elliptic point) k_1, k_2 同号, 即 $\det \begin{pmatrix} L & M \\ M & N \end{pmatrix} > 0$

此时密切抛物面为椭圆抛物面



(2) 双曲点 (hyperbolic point) k_1, k_2 异号, 即 $\det \begin{pmatrix} L & M \\ M & N \end{pmatrix} < 0$

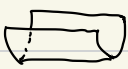
∴ 双曲抛物面



马鞍上曲面在密切抛物面两侧

(3) 抛物点 (parabolic point) k_1, k_2 有一个为 0, 即 $\det \begin{pmatrix} L & M \\ M & N \end{pmatrix} = 0$ 但 L, M, N 不全为 0

∴ 抛物柱面



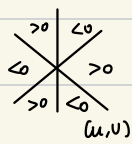
马鞍上曲面不一定在密切抛物面两侧

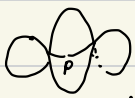
(4) 平面 $k_1 = k_2 = 0$, 即 $L = M = N = 0$



但曲面可能很复杂, 如 $(u, v, u^3 - 3uv^2)$

$u^3 - 3uv^2 = u(u+\sqrt{3}v)(u-\sqrt{3}v)$





monkey saddle 猴鞍面

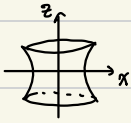
曲面论形状

给定 Gauss 曲率函数和平均曲率函数，是否唯一确定一个曲面？

$x(u,v)=(\sinh v\sin u,\ -\sinh v\cos u,\ u)\quad u\in(0,\pi),\ v\in(-\infty,\infty)$

v-线 $(\sinh v\sin u_0,\ -\sinh v\cos u_0,\ u_0)$ 直线 $\hookrightarrow$ 直纹面 (ruled surface)

u-线 $(c\sin u,\ -c\cos u,\ u)$ 螺旋线



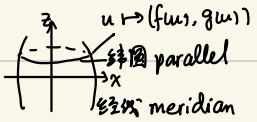
旋转曲面

悬链面 (catenoid)

$y(u,v)=(\cosh v\cos u,\ \cosh v\sin u,\ v)\quad u\in(0,\pi),\ v\in(-\infty,\infty)$

$M_t:\ (u,v)\longrightarrow \cos t\,x(u,v)+\sin t\,y(u,v)$

保持第一、第二基本形式，特别地，保持 Gauss 曲率，$H_t\equiv 0$，但迥然曲面完全不同



$u\mapsto(f(u),g(u))$

$\Rightarrow r(u,v)=(f(u)\cos v,\ f(u)\sin v,\ g(u))$

$r_u=(f'(u)\cos v,\ f'(u)\sin v,\ g'(u))$

$r_v=(-f(u)\sin v,\ f(u)\cos v,\ 0)$

$E=(f'(u))^2+(g'(u))^2\quad F=\langle r_u,r_v\rangle=0\quad G=f(u)^2$

$r_u\wedge r_v=(-f(u)g'(u)\cos v,\ -f(u)g'(u)\sin v,\ f(u)f'(u))$

$n=\frac{1}{\sqrt{(f')^2+(g')^2}}(-g'(u)\cos u,\ -g'(u)\sin v,\ f'(u))$

$r_{uu}=(f''(u)\cos v,\ f''(u)\sin v,\ g''(u))$

$r_{uv}=(-f'(u)\sin v,\ f'(u)\cos v,\ 0)$

$r_{vv}=(-f(u)\cos v,\ -f(u)\sin v,\ 0)$

$\Rightarrow L=\frac{f'g''-f''g'}{\sqrt{(f')^2+(g')^2}},\ M=0,\ N=\frac{fg'}{\sqrt{(f')^2+(g')^2}}$

Weigarten 变换在 $\{r_u,r_v\}$ 基下的矩阵表示

$$\begin{pmatrix}E&0\\0&G\end{pmatrix}^{-1}\begin{pmatrix}L&0\\0&N\end{pmatrix}=\frac{1}{\sqrt{(f')^2+(g')^2}}\begin{pmatrix}\frac{f'g''-f''g'}{\sqrt{(f')^2+(g')^2}}&0\\0&\frac{g'}{f}\end{pmatrix}$$

两个主曲率 $k_1=\frac{f'g''-f''g'}{((f')^2+(g')^2)^{\frac{3}{2}}}$，$k_2=\frac{g'}{f\sqrt{(f')^2+(g')^2}}$

假设 $u\mapsto(f(u),g(u))$ 为弧长参数曲线 $((f')^2+(g')^2\equiv 1)$

则 $k_1=f'g''-f''g'$，$k_2=\frac{g'}{f}$

$\Rightarrow K=\frac{(f'g''-f''g')g'}{f}=\frac{f'g'g''-f''(g')^2}{f}=\frac{f'(-f'f'')-f''(1-(f')^2)}{f}=-\frac{f''}{f}$

$H=\frac{1}{2}\big(\underbrace{f'g''+f''g'}_{=-\frac{f''}{g'}}+\frac{g'}{f}\big)$

$K=-\frac{f''}{f} \Leftrightarrow f''+Kf=0$

常Gauss曲率曲面+弧长参数：形如 $f''+Kf=0 \quad (f')^2+(g')^2=1$

$K=0$，$f''=0$

$K=c^2>0$，$f(u)=A\cos cu+B\sin cu$

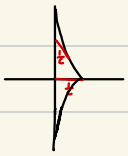
$K=-c^2<0$，$f(u)=A\sinh cu+B\cosh cu$（伪球面 pseudosphere）

$$\begin{cases} f(u)=\frac{1}{c}e^{cu} & f'(u)=e^{cu} \\ g(u)=\pm\int_0^u\sqrt{1-e^{2ct}}\,dt & u\in(-\infty,0) \end{cases}$$

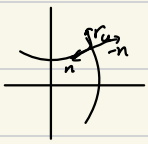
$(f'(u),g'(u))=(e^{cu},\pm\sqrt{1-e^{2cu}})$

$\cos\theta=\left\langle\frac{(f',g')}{\sqrt{(f')^2+(g')^2}},(1,0)\right\rangle=\frac{f'}{\sqrt{(f')^2+(g')^2}}=e^{cu}$

$\Rightarrow \frac{f(u)}{\cos\theta}=\frac{1}{c}$



曳物线 fractrix



r_u，r_v 主方向in几何解释：n 落在 $x-z$ 平面上 $\Rightarrow n_u$ 落在 $x-z$ 平面上

$\langle n_u,n\rangle=0 \Rightarrow n_u$ 必与 r_u 共线 $-n_u=W(r_u)=k_1r_u$

$r(u,v)=(f(u)\cos v,f(u)\sin v,g(u))$

$r_u(u,v)=(f'\cos v,f'\sin v,g')|_{v=0}$

$=(f',0,g')$



n_v，r_v 落在纬圆所在平面内

$\Rightarrow -n_v$ 与 r_v 共线

也即 $W(r_v)=-n_v=k_2r_v$

$$n=\frac{(-g'\cos v,-g'\sin v,f')}{\sqrt{(f')^2+(g')^2}},\quad k_2=\frac{\text{分量}}{|r_v|}=\frac{\text{分量}}{f(u)}=\frac{\sqrt{1-\langle -n,(0,0,1)\rangle^2}}{f(u)}=\frac{g'}{f\sqrt{(f')^2+(g')^2}}$$

$H=\frac{1}{2}\left(\frac{g'}{f}-\frac{f''}{g'}\right) \quad (f')^2+(g')^2=1$

$H\equiv 0 \Leftrightarrow \frac{g'}{f}=\frac{f''}{g'} \Leftrightarrow ff''=(g')^2=1-(f')^2 \Leftrightarrow ff''+(f')^2\equiv 1 \Leftrightarrow (ff')'=1 \Leftrightarrow ff'=u+A$

$\left(\frac{1}{2}f^2\right)'$

$\Rightarrow \frac{1}{2}f^2=\frac{1}{2}u^2+Au+\frac{B}{2}$，$f^2=u^2+2Au+B=(u+A)^2+B-A^2>0$

$f=\sqrt{u^2+2Au+B} \quad B>A^2$

$f'=\frac{u-A}{\sqrt{u^2+2Au+B}} \Rightarrow g'=\pm\sqrt{\frac{B-A^2}{u^2+2Au+B}}$

通过参数变换可设 $f(u)=\sqrt{u^2+a^2} \Rightarrow g'(u)=\pm\frac{a}{\sqrt{u^2+a^2}}$

$\Rightarrow g(u)=\pm\int_0^u\frac{a}{\sqrt{t^2+a^2}}dt=\pm a\left(\operatorname{arcsinh}\frac{u}{a}\right)$

$\Rightarrow (f(u),g(u))=\left(\sqrt{u^2+a^2},\pm a\operatorname{arcsinh}\frac{u}{a}\right)=\left(a\cosh\frac{z}{a},z\right) \Rightarrow$ Catenoid

$\operatorname{arcsinh}\frac{u}{a}=\pm\frac{z}{a}$

$\sinh\frac{z}{a}=\frac{u}{a}$

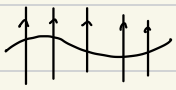
Ruled surface

$r(u,v) = a(u) + \underset{\in \mathbb{R}}{v} b(u)$

$r_{vv} = 0 \Rightarrow N = 0 \Rightarrow K = -\frac{M^2}{EG-F^2}$

① $b(u) \equiv b_0$

$r(u,v) = a(u) + v b_0$ 广义柱面



② $a(u) \equiv a_0$

$r(u,v) = a_0 + v b(u)$ 广义锥面



③ $b(u) \equiv a'(u)$

$r(u,v) = a(u) + v a'(u)$ 切线面

$\Big\} K \equiv 0$

④ 单叶双曲面 (Elliptic hyperboloid of one sheet)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



$r(u,v) = (a(\cos u - v\sin u),\ b(\sin u + v\cos u),\ cv)$

$= (a\cos u, b\sin u, 0) + v(-a\sin u, b\cos u, c)$

or $r(u,v) = (a(\cos u + v\sin u),\ b(\sin u - v\cos u),\ cv)$

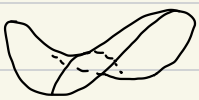
$= (a\cos u, b\sin u, 0) + v(a\sin u, -b\cos u, c)$

$a = b$ 时 旋转面：



⑤ 双曲抛物面 hyperbolic paraboloid

$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$



$r(u,v) = (a(u+v),\ bv,\ u^2 + 2uv)$

$= (au, 0, u^2) + v(a, b, 2u)$

or $r(u,v) = (a(u+v),\ -bv,\ u^2 + 2uv)$

$= (au, 0, u^2) + v(a, -b, 2u)$

Gauss 曲率 $\equiv 0$ 的直纹面 (可展曲面)

性质：一个直纹面 $r(u,v) = a(u) + vb(u)$ 的 Gauss 曲率为 $0 \Leftrightarrow (a', b, b') = 0$

证: $r_v = b(u)$, $r_{vv}=0 \Rightarrow N=0 \Rightarrow K = \frac{-M^2}{EG-F^2}$

$K=0 \Leftrightarrow M = \langle r_{uv}, n\rangle = 0 \Leftrightarrow \langle b'(u), n\rangle = 0$

$\Leftrightarrow b'(u) = \lambda r_u + \mu r_v = \lambda(a'(u)+vb'(u)) + \mu b(u) \Leftrightarrow (a', b, b') = 0$ □

Gauss curvature $\equiv 0 \Leftrightarrow (a', b, b') = \langle a', b\wedge b'\rangle \equiv 0$

情形1 $b\wedge b'(u)\equiv 0$ 此时 $b(u)$ 方向不变, $r(u,v) = a(u)+vb(u) = a(u)+v|b(u)|\frac{b(u)}{|b(u)|}$ ($\frac{b(u)}{|b(u)|}$ 为 c_0)

$\Rightarrow$ 曲面是广义柱面

情形2 $\exists u$ s.t. $b\wedge b'(u)\neq 0 \Leftrightarrow b(u)$ 与 $b'(u)$ 线性无关

$(a', b, b') = 0 \Rightarrow \exists \lambda(u), \mu(u)$ s.t. $a'(u) = \lambda(u)\, b(u) + \mu(u) b'(u)$

令 $\tilde{a}(u) = a(u) - \mu(u)\, b(u)$

则 $\tilde{a}'(u) = a'(u) - \mu'(u) b(u) - \mu(u) b'(u) = (\lambda(u) - \mu'(u))\, b(u)$

$r(u,v) = \tilde{a}(u) + (\mu(u)+v)\, b(u)$

若 $\tilde{a}'(u)\neq 0 \Rightarrow r(u,v) = \tilde{a}(u) + \frac{\mu(u)+v}{\lambda(u)-\mu'(u)}\tilde{a}'(u)$ 切线面 (参数变换 $\tilde{v} = \frac{v+\mu(u)}{\lambda(u)-\mu'(u)}$)

考察直纹面 $r(u,v) = a(u)+vb(u)$ 假设 $b'(u)\neq 0$

设 $H\equiv 0$. 则该曲面必为正螺面 (helicoid, $r_{u,v} = (u\cos v, u\sin v, bv)$)

证: Claim: 我们总可以假设 $1 = |b| = |b'(u)|$, $\langle a', b'\rangle = 0$

前两个要求容易, $\sigma(u) = a(u) + \lambda(u) b(u)$

希望 $\langle \sigma'(u), b'(u)\rangle = 0 \Leftrightarrow \langle a'(u) + \lambda'(u) b(u) + \lambda(u) b'(u), b'(u)\rangle = 0$ 注意 $1 = \langle b, b\rangle$

$\Leftrightarrow \langle a'(u) + \lambda(u) b'(u), b'(u)\rangle = 0$

$\Leftrightarrow -\langle a'(u), b'(u)\rangle = \lambda(u)$

所以, 取 $\sigma(u) = a(u) - \langle a'(u), b'(u)\rangle b(u)$ 即可;

$r_u = a'(u) + vb'(u)$ $\quad r_v = b(u)$

$r_{uu} = a''(u) + vb''(u)$ $\quad r_{uv} = b'(u)$ $\quad r_{vv} = 0$

$\Rightarrow$ $F = \langle r_u, r_v\rangle = \langle a', b\rangle$

$G = \langle r_v, r_v\rangle = 1$

$L = \langle r_{uu}, n\rangle = \langle a'' + vb'', \frac{r_u\wedge r_v}{|r_u\wedge r_v|}\rangle = \frac{1}{|r_u\wedge r_v|}\langle a''+vb'', (a'+vb')\wedge b\rangle$

$M = \langle r_{uv}, n\rangle = \frac{1}{|r_u\wedge r_v|}\langle b', (a'+vb')\wedge b\rangle$

$H = 0 \Leftrightarrow LG - 2MF = 0$

$\Leftrightarrow L - 2\langle a', b\rangle M = 0$

$$\Leftrightarrow \det\begin{pmatrix} a''+vb'' \\ a'+vb' \\ b \end{pmatrix} - 2\langle a', b\rangle \det\begin{pmatrix} b' \\ a'+vb' \\ b \end{pmatrix} = 0$$

$$\Leftrightarrow 0 = -2\langle a', b\rangle \det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix} + \det\begin{pmatrix} a'' \\ a' \\ b \end{pmatrix} + v\left(\det\begin{pmatrix} a'' \\ b' \\ b \end{pmatrix} + \det\begin{pmatrix} b'' \\ a' \\ b \end{pmatrix}\right) + v^2 \det\begin{pmatrix} b'' \\ b' \\ b \end{pmatrix}$$

$$\Leftarrow \begin{cases} -2\langle a', b\rangle \det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix} + \det\begin{pmatrix} a'' \\ a' \\ b \end{pmatrix} = 0 & ① \\ \det\begin{pmatrix} a'' \\ b' \\ b \end{pmatrix} + \det\begin{pmatrix} b'' \\ a' \\ b \end{pmatrix} = 0 & ② \\ \det\begin{pmatrix} b'' \\ b' \\ b \end{pmatrix} = 0 & ③ \end{cases}$$

③ $\Rightarrow b'' = \langle b'', b'\rangle b' + \langle b'', b\rangle b = \langle b'', b\rangle b = -b$

($\langle b', b\rangle \equiv 0$ 求导 $\langle b'', b\rangle = -\langle b', b'\rangle = -1$，$|b'| \equiv 1$)

总结 $b'' = -b$ 空间曲线 $u \mapsto b(u)$ 曲率为 1，$|b''| \equiv 1$

Frenet 运动方程 $(b')' = k(u)n(u) = n(u) \Rightarrow -b$ 是 $u \mapsto b(u)$ 主法向

$\Rightarrow$ 副法向 $b' \wedge (-b)$

Frenet $(b' \wedge (-b))' = -\tau(u)(-b) \Rightarrow \tau(u) \equiv 0$

$b'' \wedge (-b) + b' \wedge (-b') = 0$

结论：$u \mapsto b(u)$，$k(u) \equiv 1$，$\tau(u) \equiv 0 \Rightarrow u \mapsto b(u)$ 为单位圆

不妨设 $b(u) = (\cos u, \sin u, 0)$

② $\Rightarrow \det\begin{pmatrix} a'' \\ b' \\ b \end{pmatrix} = 0$

a'' 落在 b', b 张成的平面内（xy 平面）

$a(u) = (\alpha(u), \beta(u), \gamma(u))$

$\Rightarrow \gamma''(u) = 0 \Rightarrow \gamma(u) = \lambda(u + \frac{c}{\lambda})$

另外 $a''(u) = \langle a'', b'\rangle b' + \langle a'', b\rangle b$

① $\Rightarrow -2\langle a', b\rangle \det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix} + \langle a'', b'\rangle \det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix} = 0$ 注意 $\langle a', b'\rangle \equiv 0 \Rightarrow \langle a'', b'\rangle + \langle a', b''\rangle = 0$

$\Rightarrow \langle a', b\rangle \det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix} = 0$ $\det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix} = \det\begin{pmatrix} -\sin u & \cos u & 0 \\ \alpha'(u) & \beta'(u) & \lambda \\ \cos u & \sin u & 0 \end{pmatrix} = \lambda$ $\Rightarrow \langle a'', b'\rangle = -\langle a', b''\rangle = \langle a', b\rangle$

$\Rightarrow \lambda\langle a', b\rangle = 0$

若 $\lambda = 0$，$a(u) = (\alpha(u), \beta(u), 0)$ 平面

若 $\lambda \neq 0$，即 $\langle a', b\rangle = 0 \Leftrightarrow \langle (\alpha', \beta', \lambda), (\cos u, \sin u, 0)\rangle = 0$

$\langle a', b'\rangle = 0 \longrightarrow \begin{cases} \alpha'\cos u + \beta'\sin u = 0 \\ -\alpha'\sin u + \beta'\cos u = 0 \end{cases} \Leftrightarrow \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \alpha' = \beta' = 0 \Rightarrow \alpha, \beta$ 常数 □

$\begin{matrix} E & F & G \\ L & M & N \end{matrix}$ 给定，能否唯一确定曲面？

$(EG-F^2>0, E>0, G>0)$

① 如果 $L=M=N=0$，曲面是否必为平面（$\exists$ 一部分）？

② 如果 $\frac{L}{E}=\frac{M}{F}=\frac{N}{G}=k\in\mathbb{R}$，是否必为球面或平面（$\exists$ 一部分）

① 对：
$$\left.\begin{aligned} 0=L=\langle r_{uu},n\rangle &= -\langle r_u,n_u\rangle \\ 0=M=\langle r_{uv},n\rangle &= -\langle r_u,n_v\rangle=-\langle r_v,n_u\rangle \\ 0=N=\langle r_{vv},n\rangle &= -\langle r_v,n_v\rangle \end{aligned}\right\}\Rightarrow n_u=n_v=0\Rightarrow \langle r,n\rangle \text{常数}\Rightarrow \text{平面}(\exists\text{一部分})$$

给定 E,F,G s.t. $\det\begin{pmatrix} E & F \\ F & G\end{pmatrix}>0$，$L=M=N=0$，问存在性？

不一定，E,F,G 代入高斯绝妙定理须算出 0.

Q：这一条件充分吗？

② 假设存在 $r: D\to E^3$

给定一点 p 和 $w\in T_pM$，$w=\lambda r_u+\mu r_v$

$$K_n(w)=\frac{\mathrm{II}(w,w)}{\mathrm{I}(w,w)}=\frac{\lambda^2L+2\lambda\mu M+\mu^2N}{\lambda^2E+2\lambda\mu F+\mu^2G},\ \forall w$$

$$\frac{L}{E}=\frac{M}{F}=\frac{N}{G}=k\Leftrightarrow K_n(w)=k,\ \forall w\in T_pM,\ \forall p\in M \quad \text{全脐点曲面}$$

（若点 p 处 $\exists$ 任一切方向主法曲率均相等，则称 p 为 M 的一个脐点 (umbilic) $\begin{cases} k=0 & \text{平点} \\ k\neq 0 & \text{圆点}\end{cases}$）

性质：设 $r: D\to E^3$ 是（连通）正则曲面片，若为全脐点曲面，则必为球面或平面 $\exists$ 一部分

证：脐点处 W：$\begin{matrix} T_pM & \to & T_pM \\ v & \mapsto & k(p)v \end{matrix}$ 则 $\begin{cases} W(r_u)=-n_u=kr_u \\ W(r_v)=-n_v=kr_v \end{cases}$ (*)

目标：证明 $k(p)$ 是常值函数

对 (*) 求导，$\begin{cases} -n_{uv}=k_vr_u+kr_{uv} \\ -n_{vu}=k_ur_v+kr_{vu}\end{cases}$

相减得 $0=k_vr_u-k_ur_v\xrightarrow{\text{正则性}}k_u=k_v=0\Rightarrow k$ 常值函数

$\begin{cases} -n_u=kr_u \\ -n_v=kr_v \end{cases}$ $k\in\mathbb{R}$　　$n=-kr+c_0$　　$k=0$ 平面（$\exists$ 一部分）

$k\neq 0$ $\left|r-\frac{1}{k}c_0\right|=\frac{1}{|k|}$ 球面（$\exists$ 一部分）

□

修正一下，给定 E, F, G, L, M, N s.t.

① $\begin{pmatrix} E & F \\ F & G \end{pmatrix} > 0$ 处处成立 ② $\frac{L}{E} = \frac{M}{F} = \frac{N}{G} = k$ ③ $\mathcal{G}(E,F,G) = k^2$

问是否存在这样的曲面？

曲面的结构方程：Gauss 方程

Codazzi 方程

记号（指标） $r = r(u,v) \xrightarrow{u^1 = u,\ u^2 = v} r(u^1, u^2)$

$r_u = r_{u^1} = r_1$

$r_v = r_{u^2} = r_2$ $\quad r_{\alpha\beta} := \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta}$ $\alpha, \beta = 1, 2$

(r_{uu}, r_{uv}, r_{vv})

$n = \frac{r_1 \wedge r_2}{|r_1 \wedge r_2|}$

$g_{\alpha\beta} = \langle r_\alpha, r_\beta \rangle$ $\quad \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ $\quad \mathrm{I} = g_{\alpha\beta}\, du^\alpha \otimes du^\beta$ （Einstein 加和约定）

$b_{\alpha\beta} = \langle r_{\alpha\beta}, n \rangle$ $\quad \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ $\quad \mathrm{II} = b_{\alpha\beta}\, du^\alpha \otimes du^\beta$

$(g_{\alpha\beta}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} > 0$，$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix} \xrightarrow{\text{记号}} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$

$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$1 = \delta_1^1 = g^{11}g_{11} + g^{12}g_{21}$

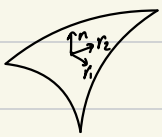
$0 = \delta_2^1 = g^{21}g_{11} + g^{22}g_{21}$ $\quad g^{\alpha\gamma} g_{\gamma\beta} = \delta_\beta^\alpha$

$g = \det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$

$b = \det \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

运动方程

$$\begin{cases} \frac{\partial r_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^\gamma r_\gamma + C_{\alpha\beta} n \\ \frac{\partial n}{\partial u^\alpha} = D_\alpha^\beta r_\beta + A_\alpha n \end{cases} \quad (\alpha, \beta = 1, 2)$$



自然标架场

有解的必要条件：mixed differential equal

$$\boxed{\frac{\partial^2 r_\alpha}{\partial u^\gamma \partial u^\beta} = \frac{\partial^2 r_\alpha}{\partial u^\beta \partial u^\gamma}\ (A) \qquad \frac{\partial^2 n}{\partial u^\gamma \partial u^\beta} = \frac{\partial^2 n}{\partial u^\beta \partial u^\gamma}\ (B)}$$

Gauss 方程、Codazzi 方程、Gauss 绝妙定理 就来自这个非常的性质

由 $\langle n, n \rangle \equiv 1$ 得 $A_\alpha = 0$

$\langle n_\alpha, r_\gamma \rangle = D_\alpha^\beta \langle r_\beta, r_\gamma \rangle = D_\alpha^\beta g_{\beta\gamma}$

$-\langle n, r_{\alpha\gamma} \rangle \Rightarrow -b_{\alpha\gamma} = D_\alpha^\beta g_{\beta\gamma} \Rightarrow -b_{\alpha\gamma} g^{\gamma\xi} = D_\alpha^\beta g_{\beta\gamma} g^{\gamma\xi} = D_\alpha^\beta \delta_\beta^\xi = D_\alpha^\xi$

$\Rightarrow D_\alpha^\xi = -b_{\alpha\gamma} g^{\gamma\xi} \xrightarrow{\text{记号}} -b_\alpha^\xi$

b_α^ξ 为 $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$ 在 (α, β) 位置元素

(0,2) 2张量对应两个下指标，(1,1) 张量时是一个上指标一个下指标

$$C_{\alpha\beta} = \left\langle \frac{\partial r_\alpha}{\partial u^\beta}, n \right\rangle = \langle r_{\alpha\beta}, n\rangle = b_{\alpha\beta}$$

$$\langle r_{\alpha\beta}, r_\xi \rangle = \left\langle \Gamma_{\alpha\beta}^\gamma r_\gamma, r_\xi \right\rangle = \Gamma_{\alpha\beta}^\gamma \langle r_\gamma, r_\xi \rangle = \Gamma_{\alpha\beta}^\gamma g_{\gamma\xi}$$

$$(\cdot)\begin{cases} g_{\alpha\xi,\beta} = \dfrac{\partial g_{\alpha\xi}}{\partial u^\beta} = \langle r_{\alpha\beta}, r_\xi\rangle + \langle r_\alpha, r_{\xi\beta}\rangle & (1) \\ g_{\beta\alpha,\xi} = \dfrac{\partial g_{\beta\alpha}}{\partial u^\xi} = \langle r_{\beta\xi}, r_\alpha\rangle + \langle r_\beta, r_{\alpha\xi}\rangle & (2) \\ g_{\xi\beta,\alpha} = \dfrac{\partial g_{\xi\beta}}{\partial u^\alpha} = \langle r_{\xi\alpha}, r_\beta\rangle + \langle r_\xi, r_{\beta\alpha}\rangle & (3) \end{cases}$$

$$\Rightarrow \langle r_{\beta\xi}, r_\alpha\rangle = \frac{1}{2}(g_{\alpha\xi,\beta} + g_{\beta\alpha,\xi} - g_{\xi\beta,\alpha})$$

$$\Gamma_{\xi\beta}^\gamma g_{\gamma\alpha} g^{\alpha\eta} = \frac{1}{2} g^{\alpha\eta}(g_{\alpha\xi,\beta} + g_{\beta\alpha,\xi} - g_{\xi\beta,\alpha})$$

$$\Rightarrow \boxed{\Gamma_{\beta\xi}^\eta = \frac{1}{2} g^{\eta\alpha}(g_{\alpha\xi,\beta} + g_{\beta\alpha,\xi} - g_{\xi\beta,\alpha})}$$

由 E, F, G 决定

Christoffel symbol 1869 →

1883–1888 意大利, Ricci → 协变微分 coordinate diff → Absolute diff. calculus ↓ 求导不依赖坐标卡

平行移动 张量分析 → Levi-Civita

由于 r 光滑，易知 $\Gamma_{\alpha\beta}^\gamma$、$C_{\alpha\beta}$ 关于 α,β 对称

Christoffel 第二类符号：$\Gamma_{\xi\beta\alpha} = \frac{1}{2}(g_{\xi\alpha,\beta} + g_{\beta\xi,\alpha} - g_{\beta\alpha,\xi})$

∴运动方程简化为

$$\begin{cases} \dfrac{\partial r_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^\gamma r_\gamma + b_{\alpha\beta} n \\ \dfrac{\partial n}{\partial u^\alpha} = -b_\alpha^\beta r_\beta \end{cases}$$

$$(A) \Rightarrow \frac{\partial}{\partial u^\gamma}\left(\Gamma_{\alpha\beta}^\xi r_\xi + b_{\alpha\beta} n\right) = \frac{\partial}{\partial u^\beta}\left(\Gamma_{\alpha\gamma}^\xi r_\xi + b_{\alpha\gamma} n\right)$$

$$\text{LHS} = \frac{\partial \Gamma_{\alpha\beta}^\xi}{\partial u^\gamma} r_\xi + \Gamma_{\alpha\beta}^\xi \frac{\partial r_\xi}{\partial u^\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} n + b_{\alpha\beta}\frac{\partial n}{\partial u^\gamma}$$

$$= \frac{\partial \Gamma_{\alpha\beta}^\xi}{\partial u^\gamma} r_\xi + \Gamma_{\alpha\beta}^\xi\left(\Gamma_{\xi\gamma}^\eta r_\eta + b_{\xi\gamma} n\right) + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} n + b_{\alpha\beta}\left(-b_\gamma^\eta r_\eta\right)$$

$$= \left(\frac{\partial \Gamma_{\alpha\beta}^\xi}{\partial u^\gamma} + \Gamma_{\alpha\beta}^\eta \Gamma_{\eta\gamma}^\xi - b_{\alpha\beta} b_\gamma^\xi\right) r_\xi + \left(\Gamma_{\alpha\beta}^\xi b_{\xi\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma}\right) n$$

$$\text{RHS} = \left(\frac{\partial \Gamma_{\alpha\gamma}^\xi}{\partial u^\beta} + \Gamma_{\alpha\gamma}^\eta \Gamma_{\eta\beta}^\xi - b_{\alpha\gamma} b_\beta^\xi\right) r_\xi + \left(\Gamma_{\alpha\gamma}^\xi b_{\xi\beta} + \frac{\partial b_{\alpha\gamma}}{\partial u^\beta}\right) n$$

$$\Rightarrow \begin{cases} \dfrac{\partial \Gamma_{\alpha\beta}^\xi}{\partial u^\gamma} - \dfrac{\partial \Gamma_{\alpha\gamma}^\xi}{\partial u^\beta} + \Gamma_{\alpha\beta}^\eta \Gamma_{\eta\gamma}^\xi - \Gamma_{\alpha\gamma}^\eta \Gamma_{\eta\beta}^\xi - b_{\alpha\beta} b_\gamma^\xi + b_{\alpha\gamma} b_\beta^\xi = 0 & \text{(Gauss)} \\ \Gamma_{\alpha\beta}^\xi b_{\xi\gamma} - \Gamma_{\alpha\gamma}^\xi b_{\xi\beta} + \dfrac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \dfrac{\partial b_{\alpha\gamma}}{\partial u^\beta} = 0 & \text{(Codazzi)} \end{cases}$$

(Codazzi – Mainardi, K.M. Peterson) 1868–1869, 1856, 分别, 1853

练习：(B) ⇔ Codazzi 方程

Gauss 方程 $\Leftrightarrow g_{\delta\xi}\left(\frac{\partial\Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}}-\frac{\partial\Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}}+\Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma}-\Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta}\right)=\left(b_{\alpha\beta}b^{\xi}_{\gamma}+b_{\alpha\gamma}b^{\xi}_{\beta}\right)g_{\delta\xi}=b_{\alpha\beta}b_{\gamma\delta}-b_{\alpha\gamma}b_{\beta\delta}\triangleq R_{\delta\alpha\beta\gamma}$

对称性：$R_{\delta\alpha\beta\gamma}=-R_{\delta\alpha\gamma\beta}=-R_{\alpha\delta\beta\gamma}=R_{\beta\gamma\delta\alpha}$

左边也有相同的对称性.

从而当 $\delta=\alpha$ 或 $\beta=\gamma$ 时左右均为 0，只有 R_{1212} 这一个独立方程：

$R_{1212}=b_{21}b_{21}-b_{22}b_{11}=LN-M^2$

$\Rightarrow \frac{R_{1212}}{g_{11}g_{22}-g_{12}^2}=\frac{LN-M^2}{EG-F^2}=K$

$E=g_{11}\quad F=g_{12}=g_{21}\quad G=g_{12}$

$g^{11}=\frac{E}{EG-F^2}\quad g^{12}=g^{21}=\frac{-F}{EG-F^2}\quad g^{22}=\frac{E}{EG-F^2}$

$$R_{\delta\alpha\beta\gamma}=-g_{\delta\xi}\frac{\partial\Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}}+g_{\delta\xi}\frac{\partial\Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}}-g_{\delta\xi}\Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma}+g_{\delta\xi}\Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta}$$

$$=-\frac{\partial\Gamma_{\delta\alpha\beta}}{\partial u^{\gamma}}+\frac{\partial g_{\delta\xi}}{\partial u^{\gamma}}\Gamma^{\xi}_{\alpha\beta}+\frac{\partial\Gamma_{\delta\alpha\gamma}}{\partial u^{\beta}}-\frac{\partial g_{\delta\xi}}{\partial u^{\beta}}\Gamma^{\xi}_{\alpha\gamma}-\Gamma^{\eta}_{\alpha\beta}\Gamma_{\delta\eta\gamma}+\Gamma^{\eta}_{\alpha\gamma}\Gamma_{\delta\eta\beta}$$

$$=-\frac{\partial\Gamma_{\delta\alpha\beta}}{\partial u^{\gamma}}+\frac{\partial\Gamma_{\delta\alpha\gamma}}{\partial u^{\beta}}-\Gamma^{\eta}_{\alpha\beta}\left(\Gamma_{\delta\eta\gamma}-\frac{\partial g_{\delta\xi}}{\partial u^{\gamma}}\right)+\Gamma^{\eta}_{\alpha\gamma}\left(\Gamma_{\delta\eta\beta}-\frac{\partial g_{\delta\xi}}{\partial u^{\beta}}\right)$$

$$=-\frac{1}{2}\frac{\partial}{\partial u^{\gamma}}\left(\frac{\partial g_{\delta\xi}}{\partial u^{\beta}}+\frac{\partial g_{\beta\delta}}{\partial u^{\alpha}}-\frac{\partial g_{\alpha\beta}}{\partial u^{\delta}}\right)+\frac{1}{2}\frac{1}{\partial u^{\beta}}\left(\frac{\partial g_{\alpha\delta}}{\partial u^{\gamma}}+\frac{\partial g_{\gamma\delta}}{\partial u^{\alpha}}-\frac{\partial g_{\alpha\gamma}}{\partial u^{\delta}}\right)$$

$$-\Gamma^{\eta}_{\alpha\beta}\left[\left(\frac{1}{2}\right)\left(\frac{\partial g_{\delta\eta}}{\partial u^{\alpha}}+\frac{\partial g_{\eta\delta}}{\partial u^{\gamma}}-\frac{\partial g_{\gamma\alpha}}{\partial u^{\delta}}\right)-\frac{\partial g_{\eta\delta}}{\partial u^{\gamma}}\right]+\Gamma^{\eta}_{\alpha\gamma}\left[\left(\frac{1}{2}\right)\cdot\left(\frac{\partial g_{\eta\delta}}{\partial u^{\beta}}+\frac{\partial g_{\beta\delta}}{\partial u^{\eta}}-\frac{\partial g_{\beta\eta}}{\partial u^{\delta}}\right)-\frac{\partial g_{\beta\delta}}{\partial u^{\eta}}\right]$$

$\Leftrightarrow$

$$\boxed{R_{\delta\alpha\beta\gamma}=\frac{1}{2}\left(-\frac{\partial^2 g_{\delta\beta}}{\partial u^{\gamma}\partial u^{\alpha}}+\frac{\partial^2 g_{\alpha\beta}}{\partial u^{\gamma}\partial u^{\delta}}+\frac{\partial^2 g_{\gamma\delta}}{\partial u^{\beta}\partial u^{\alpha}}-\frac{\partial^2 g_{\alpha\gamma}}{\partial u^{\beta}\partial u^{\delta}}\right)+\Gamma^{\eta}_{\alpha\beta}\Gamma_{\eta\delta\gamma}-\Gamma^{\eta}_{\alpha\gamma}\Gamma_{\eta\delta\beta}}$$

另一方面 $\Gamma^{\eta}_{\alpha\beta}=g^{\eta\xi}\Gamma_{\xi\alpha\beta}$

也即 $R_{\delta\alpha\beta\gamma}=\frac{1}{2}\left(-\frac{\partial^2 g_{\delta\beta}}{\partial u^{\gamma}\partial u^{\alpha}}+\frac{\partial^2 g_{\alpha\beta}}{\partial u^{\gamma}\partial u^{\delta}}+\frac{\partial^2 g_{\gamma\delta}}{\partial u^{\beta}\partial u^{\alpha}}-\frac{\partial^2 g_{\alpha\gamma}}{\partial u^{\beta}\partial u^{\delta}}\right)+g^{\eta\xi}\left(\Gamma_{\xi\alpha\beta}\Gamma_{\eta\beta\gamma}-\Gamma_{\xi\alpha\gamma}\Gamma_{\eta\delta\beta}\right)$

从而 $R_{1212}=\frac{1}{2}\left(-E_{vv}+2F_{uv}-G_{uu}\right)$

$$+g^{11}\left(\Gamma_{121}\Gamma_{112}-\Gamma_{122}\Gamma_{111}\right)$$

$$+g^{12}\left(\Gamma_{221}\Gamma_{112}-\Gamma_{222}\Gamma_{111}+\Gamma_{121}\Gamma_{212}-\Gamma_{122}\Gamma_{211}\right)$$

$$+g^{22}\left(\Gamma_{221}\Gamma_{212}-\Gamma_{222}\Gamma_{211}\right)$$

$\Rightarrow (EG-F^2)R_{1212}=\frac{1}{2}\left(-E_{vv}+2F_{uv}-G_{uu}\right)(EG-F^2)$

$$+G\left(\Gamma_{121}\Gamma_{112}-\Gamma_{122}\Gamma_{111}\right)$$

$$-F\left(\Gamma_{221}\Gamma_{112}-\Gamma_{222}\Gamma_{111}+\Gamma_{121}\Gamma_{212}-\Gamma_{122}\Gamma_{211}\right)$$

$$+E\left(\Gamma_{221}\Gamma_{212}-\Gamma_{222}\Gamma_{211}\right)$$

练习：验证这即为 Gauss 绝妙定理

Codazzi方程关于β,γ反称

$\therefore$ 简化为 $\begin{cases} \frac{\partial b_{11}}{\partial u^2}-\frac{\partial b_{12}}{\partial u^1}+\Gamma_{11}^{\beta}b_{\beta 2}-\Gamma_{12}^{\beta}b_{\beta 1}=0 \\ \frac{\partial b_{21}}{\partial u^2}-\frac{\partial b_{22}}{\partial u^1}+\Gamma_{21}^{\beta}b_{\beta 2}-\Gamma_{22}^{\beta}b_{\beta 1}=0 \end{cases}$

问题：给定 E,F,G,L,M,N s.t. $\begin{pmatrix} E & F \\ F & G \end{pmatrix}>0$

问是否存在正则曲面(片)有相应的第一、第二基本形式？

必要条件 (1) $R_{1212}=LN-M^2$

(2) $\frac{\partial b_{\alpha\beta}}{\partial u^\gamma}-\frac{\partial b_{\alpha\gamma}}{\partial u^\beta}+\Gamma_{\alpha\beta}^{\delta}b_{\delta\gamma}-\Gamma_{\alpha\gamma}^{\delta}b_{\delta\beta}=0$

若条件给为 (1) $\det(E,F,G)=k^2$ k const.

(2) $\frac{L}{E}=\frac{M}{F}=\frac{N}{G}=k \Leftrightarrow b_{\alpha\beta}=kg_{\alpha\beta}$

则必要条件 (2) 中 LHS $=k(g_{\alpha\beta,\gamma}-g_{\alpha\gamma,\beta}+\Gamma_{\alpha\beta}^{\delta}b_{\delta\gamma}-\Gamma_{\alpha\gamma}^{\delta}g_{\delta\beta})$

$=k[\cancel{g_{\alpha\beta,\gamma}}-\cancel{g_{\alpha\gamma,\beta}}+\frac{1}{2}(g_{\gamma\beta,\alpha}+\cancel{g_{\alpha\gamma,\beta}}-\cancel{g_{\alpha\beta,\gamma}})-\frac{1}{2}(\cancel{g_{\beta\gamma,\alpha}}+\cancel{g_{\alpha\beta,\gamma}}-\cancel{g_{\alpha\gamma,\beta}})]$

$=0$

定理：设 $D\times V\subset\mathbb{R}^2\times\mathbb{R}^n$ 为开区域，$f_i:D\times V\to\mathbb{R}^n$ 为 C^∞ 函数，$i=1,2$

考查方程 $\frac{\partial F}{\partial u^j}(u^1,u^2)=f_j(u^1,u^2,F(u^1,u^2))$ $(F:D\to V,\ j=1,2)$ $(*)$

(1) (唯一性) $\forall x\in V$，若 $F_1,F_2:D\to V$ 都是 $(*)$ 的解且 $F_1(0,0)=F_2(0,0)=x$，则 $F_1=F_2$ on D

(2) (存在性) 若 $\frac{\partial}{\partial u^i}(f_j(u^1,u^2,F(u^1,u^2)))=\frac{\partial}{\partial u^j}(f_i(u^1,u^2,F(u^1,u^2)))$，

则存在 $(0,0)$ 的一个小邻域 W 和 C^∞ 函数 $F:W\to V$ 满足方程

考虑方程 ($r:D\xrightarrow{C^\infty}E^3\cong\mathbb{R}^3$，$g,b$ 满足这节推出的条件)

$$\begin{cases} \frac{\partial r}{\partial u^\alpha}=r_\alpha,\ \alpha=1,2 \\ \frac{\partial r_\alpha}{\partial u^\beta}=\Gamma_{\alpha\beta}^{\gamma}r_\gamma+b_{\alpha\beta}n,\ \alpha,\beta=1,2 \\ \frac{\partial n}{\partial u^\alpha}=-b_\alpha^\beta r_\beta,\ \alpha=1,2 \end{cases} \qquad \frac{\partial}{\partial u^\alpha}\begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix}=\underbrace{\begin{pmatrix} & \text{linear eq.} & \\ & & \end{pmatrix}}_{7\times 7}\begin{pmatrix} r \\ r_1 \\ r_2 \\ n \end{pmatrix}$$

存在性一步(相差刚体变换)可以仿二维情况证明.

存在性定理：给定 $E,F,G,L,M,N: D \to \mathbb{R}$ C^∞，$\begin{pmatrix} E & F \\ F & G \end{pmatrix} > 0$

若 (1) $R_{1212} = LN - M^2$ (2) $\frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} + \Gamma_{\alpha\beta}^\xi b_{\xi\gamma} - \Gamma_{\alpha\gamma}^\xi b_{\xi\beta} = 0$

则存在唯一的曲面片 $r: D \to \mathbb{E}^3$ s.t. $\langle r_\alpha, r_\beta \rangle = g_{\alpha\beta}$，$\langle r_{\alpha\beta}, \frac{r_1 \wedge r_2}{|r_1 \wedge r_2|} \rangle = b_{\alpha\beta}$

证明：上面 in - 所线性 PDE 是有解 in

简要验证 $\begin{cases} \langle r_\alpha, r_\beta \rangle = g_{\alpha\beta} \\ \langle r_\alpha, n \rangle = 0 \\ \langle n, n \rangle = 1 \end{cases}$ $(\Rightarrow \langle r_{\alpha\beta}, n \rangle = b_{\alpha\beta})$

$\frac{\partial}{\partial u^\gamma} \begin{pmatrix} \langle r_\alpha, r_\beta \rangle \\ \langle r_\alpha, n \rangle \\ \langle n, n \rangle \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} \langle r_\alpha, r_\beta \rangle \\ \langle r_\alpha, n \rangle \\ \langle n, n \rangle \end{pmatrix}$，希望解是 $\begin{pmatrix} g_{\alpha\beta} \\ 0 \\ 1 \end{pmatrix}$

$\frac{\partial}{\partial u^\gamma} \langle r_\alpha, r_\beta \rangle = \langle \Gamma_{\alpha\gamma}^\xi r_\xi + b_{\alpha\gamma} n, r_\beta \rangle + \langle r_\alpha, \Gamma_{\beta\gamma}^\xi r_\xi + b_{\beta\gamma} n \rangle$

$= \Gamma_{\alpha\gamma}^\xi \langle r_\xi, r_\beta \rangle + b_{\alpha\gamma} \langle n, r_\beta \rangle + \Gamma_{\beta\gamma}^\xi \langle r_\alpha, r_\xi \rangle + b_{\beta\gamma} \langle r_\alpha, n \rangle$

$\frac{\partial}{\partial u^\gamma} \langle r_\alpha, n \rangle = \langle \Gamma_{\alpha\gamma}^\xi r_\xi + b_{\alpha\gamma} n, n \rangle + \langle r_\alpha, -b_\alpha^\beta r_\beta \rangle$

$\frac{\partial}{\partial u^\gamma} \langle n, n \rangle = 2 \langle n, \frac{\partial n}{\partial u^\gamma} \rangle = 2 \langle n, -b_\gamma^\beta r_\beta \rangle$

其中 $\langle r_\alpha^\circ, r_\beta^\circ \rangle = g_{\alpha\beta}(0,0)$，$\langle r_\alpha^\circ, n^\circ \rangle = 0$，$\langle n^\circ, n^\circ \rangle = 1$，$r_1^\circ, r_2^\circ, n^\circ$ 成为手系

可验证 $\begin{pmatrix} g_{\alpha\beta} \\ 0 \\ 1 \end{pmatrix}$ 满足方程，从而由解 in 的唯一性得证 □

活动标架

$\{r, r_1, r_2, n\}$ Schmidt 正交化

曲面的活动标架运动方程

$$(i) \begin{cases} \frac{\partial r}{\partial u^\alpha} = r_\alpha \\ \frac{\partial r_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^\gamma r_\gamma + b_{\alpha\beta} n, \quad \alpha, \beta = 1,2 \\ \frac{\partial n}{\partial u^\alpha} = -b_\alpha^\gamma r_\gamma \end{cases}$$

曲线 Frenet 标架运动方程

$$(ii) \begin{cases} \frac{dr}{ds} = t(s) \\ \frac{dt}{ds} = k(s) n(s) \\ \frac{dn}{ds} = -kt + \tau b \\ \frac{db}{ds} = -\tau n \end{cases}$$

微分形式

设 $r(u,v) = (x(u,v), y(u,v), z(u,v))$

则 $dr := (dx, dy, dz) = (x_u du + x_v dv, y_u du + y_v dv, z_u du + z_v dv)$

$= (x_u du, y_u du, z_u du) + (x_v dv, y_v dv, z_v dv)$

$= (x_u, y_u, z_u) du + (x_v, y_v, z_v) dv$

$= r_u du + r_v dv$

$= (du)\cdot r_u + (dv)\cdot r_v$

$dr_\alpha = \frac{\partial r_\alpha}{\partial u^\beta} du^\beta = (\Gamma_{\alpha\beta}^\gamma r_\gamma + b_{\alpha\beta} n)\, du^\beta$

$= (\Gamma_{\alpha\beta}^\gamma du^\beta) r_\gamma + (b_{\alpha\beta} du^\beta) n$

$dn = \frac{\partial n}{\partial u^\alpha} du^\alpha = (-b_\alpha^\gamma r_\gamma)\, du^\alpha = (-b_\alpha^\gamma du^\alpha) r_\gamma$

以上述两个方程组(i)和(ii)可写作

(i)
$$\begin{cases} dr = (du^1) r_1 + (du^2) r_2 \\ dr_1 = (\Gamma_{1\beta}^1 du^\beta) r_1 + (\Gamma_{1\beta}^2 du^\beta) r_2 + (b_{1\beta} du^\beta) n \\ dr_2 = (\Gamma_{2\beta}^1 du^\beta) r_1 + (\Gamma_{2\beta}^2 du^\beta) r_2 + (b_{2\beta} du^\beta) n \\ dn = (-b_\alpha^1 du^\alpha) r_1 + (-b_\alpha^2 du^\alpha) r_2 \end{cases}$$

$$\begin{pmatrix} dr_1 \\ dr_2 \\ dn \end{pmatrix} = \begin{pmatrix} \Gamma_{1\beta}^1 du^\beta & \Gamma_{1\beta}^2 du^\beta & b_{1\beta} du^\beta \\ \Gamma_{2\beta}^1 du^\beta & \Gamma_{2\beta}^2 du^\beta & b_{2\beta} du^\beta \\ -b_\beta^1 du^\beta & -b_\beta^2 du^\beta & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ n \end{pmatrix}$$

↑ 复杂

和

(ii')
$$\begin{cases} dc = (ds)\cdot t \\ dt = (\kappa ds) n \\ dn = (-\kappa ds) t + (\tau ds) b \\ db = (-\tau ds) n \end{cases}$$

$$\begin{pmatrix} dt \\ dn \\ db \end{pmatrix} = \begin{pmatrix} 0 & \kappa ds & 0 \\ -\kappa ds & 0 & \tau ds \\ 0 & -\tau ds & \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

↑ 反对称，且形式简单

实际上，Frenet 标架处处为单位正交标架这个性质正是其运动方程矩阵反称的原因：

设 $c(s)$ 是 $\mathbb{E}^3$ 中一条弧长参数曲线，每点处取一个正定向的单位正交标架 $\{c; e_1(s), e_2(s), e_3(s)\}$，

且设 $e_i(s)$, $i=1,2,3$ 光滑地依赖于参数 s. 特别地，曲线的切向量可有如下表示：

$\frac{dc}{ds} = a^i e_i = a^1 e_1 + a^2 e_2 + a^3 e_3$

我们取 $e_1(s) = \frac{dc}{ds}$，也即 $a^1 = 1$, $a^2 = a^3 = 0$ （将最初取定的标架作适当正交变换）

我们还有 $\frac{de_i(s)}{ds} = q_i^j(s) e_j(s)$, $i=1,2,3$

$\langle e_i(s), e_j(s) \rangle = \delta_{ij}$. 求导，$0 = \langle \frac{de_i}{ds}, e_j \rangle + \langle e_i, \frac{de_j}{ds} \rangle = \langle q_i^k e_k, e_j \rangle + \langle e_i, q_j^k e_k \rangle = q_i^j + q_j^i$

故 (q_i^j) 反称. 所以，标架的单位正交性保证了运动方程矩阵形如：

$$\begin{pmatrix} 0 & q_1^2 & q_1^3 \\ -q_1^2 & 0 & q_2^3 \\ -q_1^3 & -q_2^3 & 0 \end{pmatrix}$$

能否把矩阵变得更简单？

我们取一个新标架 $\{c;\tilde{e}_1,\tilde{e}_2,\tilde{e}_3\}$ 使得

$\tilde{e}_1(s)=e_1(s)$

$\begin{pmatrix}\tilde{e}_2\\ \tilde{e}_3\end{pmatrix}(s)=\begin{pmatrix}\cos\theta(s) & \sin\theta(s)\\ -\sin\theta(s) & \cos\theta(s)\end{pmatrix}\begin{pmatrix}e_2\\ e_3\end{pmatrix}(s)$ （转一下）

显见 $\{c;\tilde{e}_1,\tilde{e}_2,\tilde{e}_3\}$ 仍为单位正交标架

可设 $\frac{d\tilde{e}_1}{ds}=\tilde{q}_1^2\tilde{e}_2+\tilde{q}_1^3\tilde{e}_3$

$\Rightarrow\begin{pmatrix}q_1^2\\ q_1^3\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\tilde{q}_1^2\\ \tilde{q}_1^3\end{pmatrix}$

$\Leftarrow\begin{pmatrix}\tilde{q}_1^2\\ \tilde{q}_1^3\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}q_1^2\\ q_1^3\end{pmatrix}$

从而 $(q_1^2)^2+(q_1^3)^2=(\tilde{q}_1^2)^2+(\tilde{q}_1^3)^2$ 不依赖于标架选取（确定 $e_1=\frac{dc}{ds}$ 后）

观察到 $\sqrt{(q_1^2)^2+(q_1^3)^2}=\left|\frac{d\tilde{e}_1}{ds}\right|=k(s)$ 是曲线 c 的曲率

当 $k>0$，即 $(q_1^2)^2+(q_1^3)^2>0$ 时，可选取 θ 使得

$\tilde{q}_1^3=-\sin\theta q_1^2+\cos\theta q_1^3=0$ 且 $\tilde{q}_1^2=\cos\theta q_1^2+\sin\theta q_1^3>0$

从而在这样一个标架下，有

$$\begin{cases}\frac{d\tilde{e}_1}{ds}=\tilde{q}_1^2\tilde{e}_2\\ \frac{d\tilde{e}_2}{ds}=-\tilde{q}_1^2\tilde{e}_1+\tilde{q}_2^3\tilde{e}_3\\ \frac{d\tilde{e}_3}{ds}=-\tilde{q}_2^3\tilde{e}_2\end{cases}$$

其中 $\tilde{q}_1^2=k(s)>0$，故上述标架就是曲线 c 的 Frenet 标架

⇝ 希望在曲面上也能这么做

活动标架 (Moving frames)

Cotton、Darboux、Frenet、Semet、*Élie Cartan

刚体运动，连参数时间 t → 随参数变化的标架

把活动标架从欧氏合同变换群推广到任意李群

定义（活动标架场）以参数曲面 $r=r(u,v)$ 上的点为原点的 $\mathbb{E}^3$ 上的标架 $\{r(u,v);x_1(u,v),x_2(u,v),x_3(u,v)\}$（其中 x_1,x_2,x_3 为处处线性无关的光滑向量场，即 $(x_1,x_2,x_3)\neq0$）称为曲面上的活动标架场

一般我们要求 $(x_1,x_2,x_3)>0$，如果 $\{x_1,x_2,x_3\}$ 为单位正交标架，则称 $\{r(u,v);x_1,x_2,x_3\}$ 为曲面上的正交活动标架

在 $\mathbb{E}^3$ 中一个曲面上，我们可在各点的切平面上取单位正交向量 e_1, e_2，且 e_1, e_2 为光滑向量场。(如设 $r=r(u,v)$ 为曲面 M 的一个参数表示，则可通过对 $\{r_u, r_v\}$ 施行 Schmidt 正交化，取 $\begin{cases} e_1 = \frac{r_u}{|r_u|} \\ e_2 = \frac{r_v - \langle r_v, e_1\rangle e_1}{|r_v - \langle r_v, e_1\rangle e_1|} \end{cases}$)

取 $e_3 := e_1 \wedge e_2$，则 e_3 为曲面的单位法向量场。这样我们就得到了 M 的一个正交活动标架

正交活动标架运动方程

$\{r; e_1, e_2, e_3\}$

$\quad\quad\quad

第一基本形式和第二基本形式

考查在仿射活动标架下，第一、第二基本形式的表达式

$$I(V,W)=\langle V,W\rangle=\langle\langle V,e_1\rangle e_1+\langle V,e_2\rangle e_2,\ \langle W,e_1\rangle e_1+\langle W,e_2\rangle e_2\rangle$$

$$=\langle V,e_1\rangle\langle W,e_1\rangle+\langle V,e_2\rangle\langle W,e_2\rangle$$

$$=\omega^1(V)\omega^1(W)+\omega^2(V)\omega^2(W),\ \forall V,W$$

$$\Rightarrow \boxed{I=\omega^1\otimes\omega^1+\omega^2\otimes\omega^2}$$

考查 Weigarten 变换，记它在基 $\{e_1,e_2\}$ 下的矩阵为：

$$\mathcal{W}\begin{pmatrix}e_1\\e_2\end{pmatrix}=\underbrace{\begin{pmatrix}h_1^1 & h_1^2\\ h_2^1 & h_2^2\end{pmatrix}}_{B}\begin{pmatrix}e_1\\e_2\end{pmatrix}$$

而 $\begin{pmatrix}-n_1\\-n_2\end{pmatrix}=\mathcal{W}\begin{pmatrix}r_1\\r_2\end{pmatrix}=\mathcal{W}\left(A\begin{pmatrix}e_1\\e_2\end{pmatrix}\right)=A\begin{pmatrix}\mathcal{W}(e_1)\\ \mathcal{W}(e_2)\end{pmatrix}=AB\begin{pmatrix}e_1\\e_2\end{pmatrix}$

注意到 $de_3=dn=(n_1\ n_2)\begin{pmatrix}du^1\\du^2\end{pmatrix}=\omega_3^1e_1+\omega_3^2e_2=-(\omega_1^3\ \omega_2^3)\begin{pmatrix}e_1\\e_2\end{pmatrix}$ (※)

且 (**) 即 $\begin{pmatrix}\omega^1\\\omega^2\end{pmatrix}=\begin{pmatrix}a_1^1 & a_2^1\\ a_1^2 & a_2^2\end{pmatrix}\begin{pmatrix}du^1\\du^2\end{pmatrix}=A^T\begin{pmatrix}du^1\\du^2\end{pmatrix}$

而 (※) $\Rightarrow(du^1\ du^2)\begin{pmatrix}n_1\\n_2\end{pmatrix}=-(\omega_1^3\ \omega_2^3)\begin{pmatrix}e_1\\e_2\end{pmatrix}$

应用 $(du^1\ du^2)A=(\omega^1,\omega^2)\Rightarrow(du^1\ du^2)=(\omega^1\ \omega^2)A^{-1}\Rightarrow(du^1\ du^2)\begin{pmatrix}\mathcal{W}(r_1)\\ \mathcal{W}(r_2)\end{pmatrix}=(\omega^1\ \omega^2)A^{-1}AB\begin{pmatrix}e_1\\e_2\end{pmatrix}$

$$=(\omega^1\ \omega^2)B\begin{pmatrix}e_1\\e_2\end{pmatrix}$$

还告诉我们 $\begin{pmatrix}\omega_1^3\\\omega_2^3\end{pmatrix}=B^T\begin{pmatrix}\omega^1\\\omega^2\end{pmatrix}=\begin{pmatrix}h_1^1 & h_2^1\\ h_1^2 & h_2^2\end{pmatrix}\begin{pmatrix}\omega^1\\\omega^2\end{pmatrix}$

即 $\begin{cases}\omega_1^3=h_1^1\omega^1+h_2^1\omega^2\\ \omega_2^3=h_1^2\omega^1+h_2^2\omega^2\end{cases}$

考查第二基本形式

易见 $\langle\mathcal{W}(V),W\rangle=(\langle V,e_1\rangle\ \langle V,e_2\rangle)\begin{pmatrix}h_1^1 & h_1^2\\ h_2^1 & h_2^2\end{pmatrix}\begin{pmatrix}\langle W,e_1\rangle\\ \langle W,e_2\rangle\end{pmatrix}$

$\langle V,\mathcal{W}(W)\rangle=(\langle V,e_1\rangle\ \langle V,e_2\rangle)\begin{pmatrix}h_1^1 & h_2^1\\ h_1^2 & h_2^2\end{pmatrix}\begin{pmatrix}\langle W,e_1\rangle\\ \langle W,e_2\rangle\end{pmatrix}$

故 $\mathcal{W}$ 自共轭 $\Rightarrow B=B^T$ 为对称阵

又注意到 $II(V,W)=(\omega^1(V)\ \omega^2(V))B\begin{pmatrix}\omega^1(W)\\ \omega^2(W)\end{pmatrix}=(\omega^1(V)\ \omega^2(V))\begin{pmatrix}\omega_1^3(W)\\ \omega_2^3(W)\end{pmatrix}=\omega^1(V)\omega_1^3(W)+\omega^2(V)\omega_2^3(W)$

有 $\boxed{II=\omega^1\otimes\omega_1^3+\omega^2\otimes\omega_2^3}$

回忆：在自然标架下讨论 I, II 时，说明了 I 与曲面参数选取无关，II 与固定向的参数选取无关.

在正交活动标架下，我们有：

命题：曲面的第一基本形式与正交标架的选取无关；曲面的第二基本形式与固定向的正交活动标架的选取无关.

若 e_1, e_2 取为对应主曲率 k_1, k_2 的两个主方向 $(e_1 \wedge e_2 = n)$，则 $\omega_1^3 = k_1\omega^1$，$\omega_2^3 = k_2\omega^2$

$\Rightarrow$ 曲面的第二基本形式 $\mathrm{II} = k_1\omega^1\otimes\omega^1 + k_2\omega^2\otimes\omega^2$

$$0 = d(dr) = d(\omega^\alpha e_\alpha) = d\omega^\alpha e_\alpha - \omega^\alpha \wedge de_\alpha = d\omega^\alpha e_\alpha - \omega^\beta \wedge \omega_\beta^j e_j$$

$$= (d\omega^1 - \omega^2\wedge\omega_2^1)e_1 + (d\omega^2 - \omega^1\wedge\omega_1^2)e_2 - (\omega^1\wedge\omega_1^3 + \omega^2\wedge\omega_2^3)e_3$$

$$\Rightarrow \begin{cases} d\omega^1 = \omega^2\wedge\omega_2^1 \\ d\omega^2 = \omega^1\wedge\omega_1^2 \\ \omega^1\wedge\omega_1^3 + \omega^2\wedge\omega_2^3 = 0 \end{cases}$$

$$\begin{pmatrix}\omega_1^3 \\ \omega_2^3\end{pmatrix} = B\begin{pmatrix}\omega^1 \\ \omega^2\end{pmatrix} = \begin{pmatrix}h_1^1 & h_2^1 \\ h_1^2 & h_2^2\end{pmatrix}\begin{pmatrix}\omega^1 \\ \omega^2\end{pmatrix}$$

$$0 = \omega^1\wedge(h_1^1\omega^1 + h_2^1\omega^2) + \omega^2\wedge(h_1^2\omega^1 + h_2^2\omega^2)$$

$$= (h_2^1 - h_1^2)\omega^1\wedge\omega^2 \Leftrightarrow h_1^2 = h_2^1 \Leftrightarrow B = B^T$$

$$\star\ \begin{cases} d\omega^1 = \omega^2\wedge\omega_2^1 \\ d\omega^2 = \omega^1\wedge\omega_1^2 \end{cases} \qquad \omega_1^2 = -\omega_2^1 \quad \omega_1^2 = a\omega^1 + b\omega^2$$

$$\Rightarrow d\omega^1 = -a\omega^2\wedge\omega^1 = a\omega^1\wedge\omega^2$$

$$d\omega^2 = b\omega^1\wedge\omega^2$$

若有两个 1-形式 $\omega_1^2, \overline{\omega_1^2}$ 同时满足 $\star$，则 $\overline{\omega_1^2} = \omega_1^2$

$$0 = d(de_\alpha) = d(\omega_\alpha^j e_j) = d\omega_\alpha^j e_j - \omega_\alpha^j\wedge de_j = d\omega_\alpha^j e_j - \omega_\alpha^j\wedge\omega_j^k e_k = (d\omega_\alpha^k - \omega_\alpha^j\wedge\omega_j^k)e_k$$

$$\Rightarrow \begin{array}{l} d\omega_\alpha^1 - \omega_\alpha^j\wedge\omega_j^1 \Rightarrow \alpha=1 \text{ 平凡 } \alpha=2\ d\omega_2^1 - \omega_2^3\wedge\omega_3^1 \Rightarrow d\omega_1^2 = \omega_1^3\wedge\omega_3^2 \\ d\omega_\alpha^2 - \omega_\alpha^j\wedge\omega_j^2 \Rightarrow \alpha=2 \text{ 平凡 } \alpha=1\ d\omega_1^2 - \omega_1^3\wedge\omega_3^2 = 0 \\ d\omega_\alpha^3 - \omega_\alpha^j\wedge\omega_j^3 \Rightarrow \begin{cases} d\omega_1^3 = \omega_1^2\wedge\omega_2^3 \\ d\omega_2^3 = \omega_2^1\wedge\omega_1^3 \end{cases} \end{array} \Longrightarrow$$

结构方程 $d\omega_1^2 = \omega_1^3\wedge\omega_3^2$ (G)

$d\omega_1^3 = \omega_1^2\wedge\omega_2^3$

$d\omega_2^3 = \omega_2^1\wedge\omega_1^3$ (C)

$$d\omega_1^2 = \omega_1^3\wedge\omega_3^2 = -\omega_1^3\wedge\omega_2^3 = -\det B\ \omega^1\wedge\omega^2 = -K\ \omega^1\wedge\omega^2$$

$$\Rightarrow K = -\frac{d\omega_1^2}{\omega^1\wedge\omega^2}$$ Gauss 绝妙定理

$r=r(u,v)$ $F=0$ i.e. $I=E du\otimes du+G dv\otimes dv$ $\langle r_u, r_v\rangle = F=0$

取 $e_1=\frac{r_u}{|r_u|}$, $e_2=\frac{r_v}{|r_v|}$ $dr=(r_u\ r_v)\begin{pmatrix}du\\dv\end{pmatrix}=(e_1\ e_2)\begin{pmatrix}w^1\\w^2\end{pmatrix}\Rightarrow w^1=\sqrt{E}du,\ w^2=\sqrt{G}dv$

计算 $dw^1=d(\sqrt{E}du)=(\sqrt{E})_u du+(\sqrt{E})_v dv)\wedge du$

$=-(\sqrt{E})_v du\wedge dv=-\frac{(\sqrt{E})_v}{\sqrt{EG}}w^1\wedge w^2$

$dw^2=\frac{(\sqrt{G})_u}{\sqrt{EG}}w^1\wedge w^2$

$\Rightarrow w_1^2=-\frac{(\sqrt{E})_v}{\sqrt{EG}}w^1+\frac{(\sqrt{G})_u}{\sqrt{EG}}w^2=-\frac{(\sqrt{E})_v}{\sqrt{G}}du+\frac{(\sqrt{G})_u}{\sqrt{E}}dv$

继续求导 $dw_1^2=d(-\frac{(\sqrt{E})_v}{\sqrt{G}})\wedge du+d(\frac{(\sqrt{G})_u}{\sqrt{E}})\wedge dv$

$=(\frac{(\sqrt{E})_v}{\sqrt{G}})_v du\wedge dv+(\frac{(\sqrt{G})_u}{\sqrt{E}})_u du\wedge dv$

$\begin{pmatrix}w_1^3\\w_2^3\end{pmatrix}=B\begin{pmatrix}\sqrt{E}&\\&\sqrt{G}\end{pmatrix}\begin{pmatrix}du\\dv\end{pmatrix}\Rightarrow w_1^3\wedge w_3^2=-\det(B\begin{pmatrix}\sqrt{E}&0\\0&\sqrt{G}\end{pmatrix})du\wedge dv$

$dw_1^2=w_1^3\wedge w_3^2\Rightarrow(\frac{(\sqrt{E})_v}{\sqrt{G}})_v+(\frac{(\sqrt{G})_u}{\sqrt{E}})_u=-\frac{LN-M^2}{\sqrt{EG}}$

进一步假设 $M=0$ $\begin{pmatrix}w_1^3\\w_2^3\end{pmatrix}=\begin{pmatrix}\frac{L}{\sqrt{E}}&\\&\frac{N}{\sqrt{G}}\end{pmatrix}\begin{pmatrix}du\\dv\end{pmatrix}$

(C) $\Rightarrow d(\frac{L}{\sqrt{E}}du)=-\frac{(\sqrt{E})_v}{\sqrt{G}}\cdot\frac{N}{\sqrt{G}}du\wedge dv$

$\Rightarrow\frac{L_v\sqrt{E}-L(\sqrt{E})_v}{E}=\frac{(\sqrt{E})_v N}{G}$

$\Rightarrow L_v=\frac{1}{2}(\frac{L}{E}+\frac{N}{G})2(\sqrt{E})_v\sqrt{E}\Rightarrow L_v=HE_v$

同理，$N_u=HG_u$

常主曲率曲面的分类

$k_1=k_2$ 全脐点曲面，必是球面或平面(in一部分)

$k_1\neq k_2$ 常值函数 (claim：必是圆柱面in一部分)

记：e_1,e_2 两个主方向，$e_3=n$. $W\begin{pmatrix}e_1\\e_2\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_2\end{pmatrix}\begin{pmatrix}e_1\\e_2\end{pmatrix}$

$\Rightarrow\begin{pmatrix}w_1^3\\w_2^3\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_2\end{pmatrix}\begin{pmatrix}w^1\\w^2\end{pmatrix}$

$dw_1^3=d(k_1w^1)=k_1dw^1=k_1w^2\wedge w_2^1\Rightarrow(k_1-k_2)w^2\wedge w_2^1=0$

$w_1^3\wedge w_2^3=k_2w_1^2\wedge w^2$

类似地 $dw_2^3=d(k_2w^2)\Rightarrow(k_2-k_1)w^1\wedge w_1^2=0$

$\Rightarrow w_1^2\wedge w^2=0,\ w_1^2\wedge w^1=0\Rightarrow w_1^2=0$

$\Rightarrow dw_1^2=0\Rightarrow K=0$

不妨设 $k_1 \neq 0,\ k_2 = 0 \Rightarrow \omega_2^3 = k_2 \cdot \omega^2 = 0$

$de_1 = \omega_1^2 e_2 + \omega_1^3 e_3 = k_1 \omega^1 e_3$

$de_2 = \omega_2^1 e_1 + \omega_2^3 e_3 = 0 \longrightarrow e_2$ 为常向量

$de_3 = \omega_3^1 e_1 + \omega_3^2 e_2 = -k_1 \omega^1 e_1$

当 e_2 垂直 im 平面 $\Rightarrow$ 曲面于一平面曲线，每一点 $im\,t$ 切向量即为 e_1

$\{e_1, e_3\}$ $\begin{cases} de_1 = k_1 \omega^1 e_3 \\ de_3 = -k_1 \omega^1 e_1 \end{cases}$

$\forall\, V = e_1,\ \omega^1(V) = \langle V, e_1 \rangle = 1 \Rightarrow \omega^1 = ds$

$\begin{cases} \frac{de_1}{ds} = k_1 e_3 \\ \frac{de_3}{ds} = -k_1 e_1 \end{cases} \Rightarrow$ 常曲率，只有（半径相同 im）圆

设另一个正交活动标架 $\{r;\ \bar{e}_1, \bar{e}_2, \bar{e}_3 = n\}$ $(\bar{e}_1, \bar{e}_2, \bar{e}_3) > 0$

有 $\begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix} = R_\theta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ $R_\theta^T R_\theta = R_\theta R_\theta^T = I_d$

因为 $(\omega^1\ \omega^2)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = (\bar{\omega}^1\ \bar{\omega}^2)\begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix} = dr = (\bar{\omega}^1\ \bar{\omega}^2) R_\theta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$

$\Rightarrow \boxed{\begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{pmatrix} = R_\theta \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}}$

$I = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$

$\bar{I} = \bar{\omega}^1 \otimes \bar{\omega}^1 + \bar{\omega}^2 \otimes \bar{\omega}^2 = \sum_{\alpha=1}^{2} \bar{\omega}^\alpha \otimes \bar{\omega}^\alpha = \sum_{\alpha,\beta,\gamma=1}^{2} (R_\theta)_\beta^\alpha \omega^\beta \otimes (R_\theta)_\gamma^\alpha \omega^\gamma$

$= \sum_{\beta,\gamma=1}^{2} \Big(\underbrace{\sum_{\alpha=1}^{2} (R_\theta)_\beta^\alpha (R_\theta)_\gamma^\alpha}_{\sum_{\alpha=1}^{2} (R_\theta)_\beta^\alpha (R_\theta^T)_\alpha^\gamma = \delta_\beta^\gamma} \Big) \omega^\beta \otimes \omega^\gamma = \sum_{\alpha=1}^{2} \omega^\alpha \otimes \omega^\alpha = I$

$\bar{\omega}_1^2 = \langle d\bar{e}_1, \bar{e}_2 \rangle \overset{\det R_\theta > 0}{=\!=} \langle d(\cos\theta e_1 + \sin\theta e_2),\ -\sin\theta e_1 + \cos\theta e_2 \rangle$

$= \langle -\sin\theta\, d\theta\, e_1 + \cos\theta\, de_1 + \cos\theta\, d\theta\, e_2 + \sin\theta\, de_2,\ -\sin\theta e_1 + \cos\theta e_2 \rangle$

$= \sin^2\theta\, d\theta + \cos^2\theta\, \omega_1^2 + \cos^2\theta\, d\theta - \sin^2\theta\, \omega_2^1$

$\Rightarrow \bar{\omega}_1^2 = \omega_1^2 + d\theta$，当 $\det R_\theta = 1$

反过来 $\bar{\omega}_1^2 = -\omega_1^2 - d\theta$，当 $\det R_\theta = -1$

$\mathbb{E}^3$ $\{x, e_1, e_2, e_3\}$ $(e_1, e_2, e_3) > 0$

3个参数 2 1

$\begin{cases} dx = \omega^i e_i & \omega^i(X) = \langle X, e_i \rangle \\ de_i = \omega_i^j e_j & \omega_i^j = -\omega_j^i \end{cases}$

6个一形式 $\omega^1, \omega^2, \omega^3, \omega_1^2, \omega_1^3, \omega_2^3$

结构方程 $0 = d(dx) = d\omega^i e_i - \omega^i \wedge \omega_i^j e_j \Rightarrow d\omega^i = \omega^j \wedge \omega_j^i$

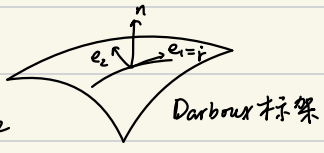
$0 = d(de_i) = d\omega_i^j e_j - \omega_i^j \wedge \omega_j^k e_k \Rightarrow d\omega_i^j = \omega_i^k \wedge \omega_k^j$

限制到曲面上就没有 ω^3

ω_1^2：联络1-形式

找曲面上两点之间的最短线

$r = r(u^1, u^2)$ $r(s) = r(u^1(s), u^2(s))$ $\exists a^1, a^2$, $e_2 = a^1 r_1 + a^2 r_2$



过一方向曲面上的曲线

$s \in [0, l]$, $f: [0, l] \to \mathbb{R}$ 光滑, $f(0) = f(l) = 0$

$r_f^\lambda(s) = r((u^1(s), u^2(s)) + \lambda f(s)(a^1(s), a^2(s)))$

$r_f^\lambda|_{\lambda=0} = r$

$\frac{\partial r_f^\lambda}{\partial \lambda}|_{\lambda=0} = r_1(s) f(s) a^1(s) + r_2(s) f(s) a^2(s) = f(s) e_2(s)$

若 $r = r(s)$ 是最短线，则

$$\frac{dL(r_f^\lambda|_{[0,l]})}{d\lambda}\Big|_{\lambda=0} = \frac{d}{d\lambda}\Big|_{\lambda=0} \int_0^l \langle \frac{\partial r_f^\lambda}{\partial s}, \frac{\partial r_f^\lambda}{\partial s} \rangle^{\frac{1}{2}} ds$$

$$= \int_0^l \frac{1}{2} \langle \frac{\partial r_f^\lambda}{\partial s}, \frac{\partial r_f^\lambda}{\partial s} \rangle^{-\frac{1}{2}} 2 \langle \frac{\partial}{\partial \lambda} \frac{\partial r_f^\lambda}{\partial s}, \frac{\partial r_f^\lambda}{\partial s} \rangle|_{\lambda=0} ds$$

$$= \int_0^l \langle \frac{\partial}{\partial \lambda} \frac{\partial r_f^\lambda}{\partial s}, \frac{\partial r_f^\lambda}{\partial s} \rangle|_{\lambda=0} ds = \int_0^l \langle \frac{\partial}{\partial s} \frac{\partial r_f^\lambda}{\partial \lambda}|_{\lambda=0}, \frac{\partial r_f^\lambda}{\partial s}|_{\lambda=0} \rangle ds$$

$$= \int_0^l \langle \frac{\partial}{\partial s}(f(s) \cdot e_2(s)), e_1(s) \rangle ds$$

$$= \int_0^l \left(\frac{d}{ds} \langle f(s) \cdot e_2(s), e_1(s) \rangle - \langle f(s) e_2(s), \frac{de_1}{ds} \rangle \right) ds$$

$$= -\int_0^l f(s) \langle \frac{de_1}{ds}, e_2 \rangle(s) ds$$

$\forall$ 光滑函数 $f: [0, l] \to \mathbb{R}$ s.t. $f(0) = f(l) = 0$, $\frac{dL}{ds} \geq 0$

$\Rightarrow \langle \frac{de_1}{ds}, e_2 \rangle(s) \equiv 0$

而 $\langle \frac{de_1}{ds}, e_2\rangle = 0 \Rightarrow \frac{de_1}{ds}$ 与 $t\Sigma$ 平面垂直

$\Leftrightarrow \langle d e_1 \dot{r}(s)), e_2\rangle = \langle (\omega_1^2 e_2 + \omega_1^3 e_3)(\dot{r}(s)), e_2\rangle = \omega_1^2(\dot{r}(s)) = 0$

$\rightsquigarrow$ 测地线 geodesic

曲面 $r = r(u,v)$ 上一条弧长参数曲线 $\gamma = r(s) = r(u(s), v(s))$,

主法向量 $\Leftarrow \langle \frac{de_1}{ds}, e_2\rangle = 0$, 其中 $e_1 = \dot{r}(s)$, $e_2 = -e_1 \wedge n$.

若 M 上一条曲线 $\gamma = r(s)$, $\forall s$, $\dot{r}(s)$ 与 $n(s)$ 张成 M 平面线性, 得 $r(s)$ 则是测地线.

$\frac{de_1}{ds} = \frac{d}{ds}(\frac{dr}{ds}) = \frac{d^2r}{ds^2}$ 曲率向量 空间曲率曲率 $K = |\frac{d^2r}{ds^2}|$

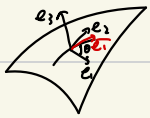
$\langle \frac{d^2r}{ds^2}, n\rangle = K_n(\dot{r}(s))$ 法曲率

$\frac{d^2r}{ds^2} - \langle \frac{d^2r}{ds^2}, n\rangle n = K_g e_2$. K_g geodesic curvature 测地曲率

$K^2 = K_n^2 + K_g^2$

曲面上一条弧长参数曲线为测地线 $\Leftrightarrow$ 测地曲率 $= 0$

飞点活动标架 $\{r; e_1, e_2, e_3 = n\}$



$\bar{e}_1 = \dot{r}(s)$
$\bar{e}_2 = e_3 \wedge \bar{e}_1$ $\begin{pmatrix}\bar{e}_1\\ \bar{e}_2\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}$

$K_g = \overline{\omega_1^2}(\bar{e}_1) = (\omega_1^2 + d\theta)(\cos\theta e_1 + \sin\theta e_2)$

$= \cos\theta\, \omega_1^2(e_1) + \sin\theta\, \omega_1^2(e_2) + \underbrace{d\theta(\bar{e}_1)}_{d\theta(\dot{r}(s))}$

$\Rightarrow \boxed{K_g = \frac{d\theta}{ds} + \omega_1^2(e_1)\cos\theta + \omega_1^2(e_2)\sin\theta}$

$r = r(u,v)$ $F = 0$

可以取 $e_1 = \frac{r_u}{\sqrt{E}}$, $e_2 = \frac{r_v}{\sqrt{G}}$, $e_3 = n$

$K_g(r(s)) = \frac{d\theta}{ds} + (k_g)_u \cos\theta + (k_g)_v \sin\theta$ (Liouville 公式)

$\omega^1 = \sqrt{E}\,du$ $\omega^2 = \sqrt{G}\,dv$

$\left.\begin{aligned} d\omega^1 &= (\sqrt{E})_v\, dv \wedge du = -\frac{(\sqrt{E})_v}{\sqrt{EG}}\,\omega^1 \wedge \omega^2 \\ d\omega^2 &= (\sqrt{G})_u\, du \wedge dv = \frac{(\sqrt{G})_u}{\sqrt{EG}}\,\omega^1 \wedge \omega^2 \end{aligned}\right\} \Rightarrow \omega_1^2 = -\frac{(\sqrt{E})_v}{\sqrt{EG}}\omega^1 + \frac{(\sqrt{G})_u}{\sqrt{EG}}\omega^2$

$$k_g(r(s)) = \frac{d\theta}{ds} + \frac{(\sqrt{E})_v}{\sqrt{EG}}\cos\theta + \frac{(\sqrt{G})_u}{\sqrt{EG}}\sin\theta$$

($r(s)$ 的单位切向量 $= r_u\frac{du}{ds} + r_v\frac{dv}{ds}$)

$$\frac{\langle r_u, r(s)'\rangle}{\sqrt{E}} = \cos\theta \qquad \frac{1}{\sqrt{G}}\frac{(\sqrt{E})_v}{\sqrt{E}} = \frac{1}{\sqrt{G}}(\ln\sqrt{E})_v = \frac{1}{2\sqrt{G}}(\ln E)_v$$

$$\cos\theta = \frac{E\frac{du}{ds}}{\sqrt{E}} = \sqrt{E}\frac{du}{ds}$$

$\frac{d^2r}{ds^2}$ 的切向部分消失

$$\frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{d}{ds}\left(r_\alpha \frac{du^\alpha}{ds}\right)$$

$$= r_\alpha \frac{d^2u^\alpha}{ds^2} + \frac{\partial r_\alpha}{\partial u^\beta}\frac{du^\beta}{ds}\frac{du^\alpha}{ds} = r_\alpha \frac{d^2u^\alpha}{ds^2} + \Gamma_{\alpha\beta}^\gamma r_\gamma \frac{du^\beta}{ds}\frac{du^\alpha}{ds} + b_{\alpha\beta}n\frac{du^\beta}{ds}\frac{du^\alpha}{ds}$$

$$= \left(\frac{d^2u^\gamma}{ds^2} + \Gamma_{\alpha\beta}^\gamma \frac{du^\alpha}{ds}\frac{du^\beta}{ds}\right) r_\gamma + b_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}n \qquad \left\langle \frac{d^2r}{ds^2}, n\right\rangle = k_n(r(s)) = b_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} = \mathrm{II}(r(s), r(s))$$

测地线 $\Leftrightarrow \frac{d^2u^\gamma}{ds^2} + \Gamma_{\alpha\beta}^\gamma \frac{du^\alpha}{ds}\frac{du^\beta}{ds} = 0$, $\gamma = 1, 2$

$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2}g^{\gamma\eta}(g_{\eta\beta,\alpha} + g_{\alpha\eta,\beta} - g_{\alpha\beta,\eta})$ (∴ 完全由第一基本形式决定)

性质. $\forall p \in M$, $\forall v \in T_pM$, $\exists \varepsilon > 0$, 存在一条测地线 $r = r(s) = r(u^1(s), u^2(s))$ $(s \in (-\varepsilon, \varepsilon))$,

满足 $r(0) = p$. $r'(0) = v$.

注: 只能在局部存在, 比如  扣掉北极点, 则一些测地线(大圆)无法延拓过北极点.

(和曲面的完备性有关)

记: $r(0) = p = (u_0^1, u_0^2) \Leftrightarrow u^1(0) = u_0^1$, $u^2(0) = u_0^2$

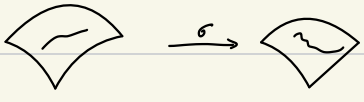
$r'(0) = v = v^1 r_1 + v^2 r_2 \Leftrightarrow \frac{du^1}{ds}(0) = v^1$, $\frac{du^2}{ds}(0) = v^2$

结合上面测地线满足的方程即证 □

才距变换

$M, \widetilde{M}$ 正则曲面片, $\sigma: M \to \widetilde{M}$ 1:1的 σ, σ^{-1} C^∞

称 σ 为一个才距变换 (isometry), 若 σ 保持所有曲线的长度



$I(r(s), r(s)) = \widetilde{I}(\sigma\circ r(s), \sigma\circ r(s))$ $\forall r(s)$

σ 的切映射 (differential) $(d\sigma)_p = (\sigma_*)_p : T_pM \to T_{\sigma(p)}\widetilde{M}$

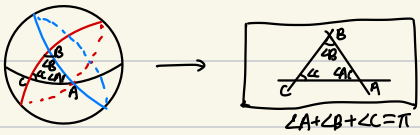
σ 才距 $\Leftrightarrow$ $\boxed{I(v, w) = \widetilde{I}(\sigma_*(v), \sigma_*(w))}$ $\sigma^*\widetilde{I} = I$

$\langle v, w\rangle$ $\quad$ $\langle \sigma_*(v), \sigma_*(w)\rangle$

∴测地线、高斯曲率在才距变换下均不变

内蕴几何 从 $I = g_{\alpha\beta} du^\alpha \otimes du^\beta$ 出发，讨论其几何.

才距变换下不变∴几何



不变量 Gauß curvature

$\frac{2\theta}{2\pi} 4\pi R^2 = 4\theta R^2$

$4(\angle A)R^2 + 4(\angle B)R^2 + 4(\angle C)R^2 = 4\pi R^2 + 4\,\mathrm{Area}(\triangle ABC) \Rightarrow \angle A+\angle B+\angle C = \pi + \frac{\mathrm{Area}(\triangle ABC)}{R^2}$

$\int_{\triangle ABC} K\,dS = \frac{\mathrm{Area}(\triangle ABC)}{R^2}$

$\boxed{\int_{\triangle ABC} K\,dS = \angle A+\angle B+\angle C-\pi}$ Gauß 可能通过此式以及∠仅依赖第一基本形式猜出绝妙定理

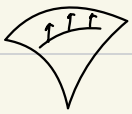
求导沿曲线∴切向量场

$V(s) \in T_pM$, $\frac{DV(s)}{ds} := \frac{dV}{ds} - \langle \frac{dV}{ds}, n\rangle n$ 测地线 $\Leftrightarrow \frac{De(s)}{ds} = 0$

定义（Levi-Civita平行性）沿着曲线 $r = r(s)$ ∴光滑切向量场 $V(s)$ 称为平行∴，若 $\frac{DV}{ds} = 0$

$\frac{dV}{ds} = \frac{d}{ds}(V^\alpha r_\alpha) = \frac{dV^\alpha}{ds} r_\alpha + V^\alpha \frac{dr_\alpha}{ds}$

$= \frac{dV^\alpha}{ds} r_\alpha + V^\alpha (\Gamma_{\alpha\beta}^\gamma r_\gamma + b_{\alpha\beta} n) \frac{du^\beta}{ds}$



$0 = \frac{DV}{ds} \Leftrightarrow \frac{dV^\alpha}{ds} + V^\gamma \Gamma_{\gamma\beta}^\alpha \frac{du^\beta}{ds} = 0$, $\alpha = 1,2$

$\begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$ $\begin{cases} V^1(0) = V_0^1 \\ V^2(0) = V_0^2 \end{cases}$ 一阶线性ODEs ∴存全局存在

$r = r(s)$, $V(s)$ $\{e_1, e_2, n\}$

$\frac{DV}{ds} = \frac{dV}{ds} - \langle \frac{dV}{ds}, n\rangle n = \langle \frac{dV}{ds}, e_1\rangle e_1 + \langle \frac{dV}{ds}, e_2\rangle e_2$

性质：(1) $V(t), W(t)$ 沿曲线 $r = r(t)$ ∴切向量场

$\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$

(2) $\frac{D(fV)}{dt} = \frac{df}{dt} V + f \frac{DV}{dt}$

(3) $\frac{d\langle V,W\rangle}{dt} = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$ ($\Rightarrow$ Levi-Civita 平行性保持长度和角度)

平行性 $\Leftrightarrow$ 求导

沿曲线的切向量场 $V(t)$

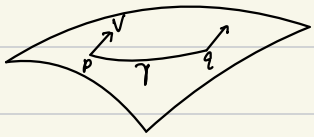
$\frac{dV(t)}{dt} = \lim_{\Delta t\to 0}\frac{V(t+\Delta t)-V(t)}{\Delta t}$ 实际上利用了欧氏平行性将切向量挪到同一点，从而求导

Connection：不同点处的切平面之间

平行：

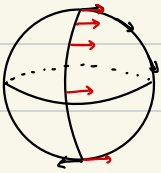
$P_{\gamma,p,q}: T_pM \longrightarrow T_qM$ 同构映射

$V \longmapsto P_{\gamma,p,q}(V)$



保内积 线性映射：$P_{\gamma,p,q}(\lambda V_1+\mu V_2) = \lambda P_{\gamma,p,q}(V_1) + \mu P_{\gamma,p,q}(V_2)$

Levi-Civita 平行性依赖于路径的选取



原因：曲面是弯曲的

M_1, M_2 两个曲面(片)沿着曲线 $C(t)$ 相切 (i.e. $T_{C(t)}M_1 = T_{C(t)}M_2, \forall t$)，两个观察：

① 设 $V_0 \in T_{C(t_0)}M_1 = T_{C(t_0)}M_2$

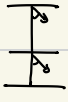
则 $V(t)$ 为 V_0 在 M_1 上沿 $C(t)$ 平行移动所得切向量场 $\Leftrightarrow$ V_0 在 M_2 上沿 $C(t)$ 平行移动所得切向量场

(从而可能可以用简单曲面代替复杂曲面)

② $\sigma: M_1 \longrightarrow M_2$ 才能变换，$V(t)$ 在 M_1 上沿 $C(t)$ 平行，则 $\sigma_* V(t)$ 在 M_2 上沿 $\sigma\circ C(t)$ 平行

($V(t) = V^\alpha(t) r_\alpha(t) \Rightarrow \frac{dV^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma}(r(t))\frac{du^\gamma}{dt}V^\beta(t) = 0$，$\alpha = 1,2$

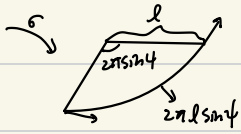
$\sigma_* V(t) = V^\alpha(t)\sigma_* r_\alpha(t)$ 由于第一基本形式不变，仍保持上式)

上图中沿经线的平行移动与平面  一样

沿纬圈 in 移动



纬圆半径 $R\cos\varphi = l\sin\varphi$ $\Rightarrow$ 纬圆周长 $2\pi l\sin\varphi$



$\therefore$ (除了沿东西以外) 不是平行移动

协变微分 $de_1(v) = \frac{de_1}{ds}\big|_{s=0}$

$De_1 = \langle de_1, e_1\rangle e_1 + \langle de_1, e_2\rangle e_2 = \omega_1^2 e_2$ 联络 1-形式

测地线是否为最短线？

极坐标 $I = dx\otimes dx + dy\otimes dy$

$= (\cos\theta\, d\rho - \rho\sin\theta\, d\theta)\otimes(\cos\theta\, d\rho - \rho\sin\theta\, d\theta) + \cdots$

$= d\rho\otimes d\rho + \rho^2 d\theta\otimes d\theta$

$\rightsquigarrow$ 对一般曲面，也试图找正交坐标

曲面上

Gauss lemma  指数映射 $\exp_p: T_pM \to M$, $v \mapsto \exp_p(v)$

(ρ,θ) 测地圆 测地线 $r = r(s)$, $r(0)=p$, $r'(0)=\frac{v}{|v|}$

$r: (\rho,\theta)\mapsto r(\rho,\theta) = \exp_p(\rho(\cos\theta,\sin\theta))$

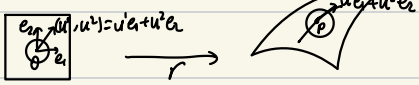
$\exp_p(v) := r(|v|)$

这是一个正则参数化！

测地线 in 局部最短性

法坐标系 (normal coordinate)

找一个"好" in 正则参数化



$r(u^1,u^2) = \exp_p(u^1 e_1 + u^2 e_2)$ e_1, e_2 两个标准正交切向量 $\in T_pM$

$= r_{u^1e_1+u^2e_2}\left(\sqrt{(u^1)^2+(u^2)^2}\right)$

$\exists \varepsilon > 0$ s.t. $\exp_p$ is well-defined on $B(0,\varepsilon)$

① r C^∞

② $r = r(u^1,u^2)$ $r_1 \wedge r_2 = e_1\wedge e_2 \neq 0$:

At $p\in M$, $r_1(p):=\frac{\partial r}{\partial u^1}(p)$ u^1-线 in 切向量

$$=\frac{\partial}{\partial u^1}\Big|_{u^1=0} r(u^1,0)=\frac{\partial}{\partial u^1}\Big|_{u^1=0} r_{u^1 e_1}(u^1) \longrightarrow \text{a geodesic, with tangent vector at } p \text{ is } \frac{u^1 e_1}{u^1}=e_1$$

$$=e_1$$

$$r_2(p)=\frac{\partial}{\partial u^2}\Big|_{u^2=0} r(0,u^2)=e_2$$

$\Rightarrow \exists \varepsilon'>0$ s.t. $r: B(0,\varepsilon)\to M$ 正则参数化

Remark: $\exp_p: B(0,\varepsilon)\subset T_pM \to M$

$(\exp_p)_*|_0: T_0(T_pM)\to T_pM$

$e_1 \mapsto e_1$

$e_2 \mapsto e_2$ 恒等映射

① $\langle r_\alpha, r_\beta\rangle = g_{\alpha\beta}$ $g_{\alpha\beta}(p)=\delta_{\alpha\beta}$

Claim: $g_{\alpha\beta,\gamma}(p)=\frac{\partial}{\partial u^\gamma} g_{\alpha\beta}|_p=0$ (但不能使 $g_{\alpha\beta,\gamma\delta}(p)=0$, 否则 $K(p)=0$)

proof: $g_{\alpha\beta,\gamma}=\frac{\partial}{\partial u^\gamma}\langle r_\alpha, r_\beta\rangle=\langle r_{\alpha\gamma}, r_\beta\rangle+\langle r_\alpha, r_{\beta\gamma}\rangle$

$$=\langle \Gamma^\eta_{\alpha\gamma} r_\eta + b_{\alpha\gamma} n^{\to 0}, r_\beta\rangle + \langle r_\alpha, \Gamma^\eta_{\beta\gamma} r_\eta\rangle$$

$$=\Gamma^\eta_{\alpha\gamma}(p)\, g_{\eta\beta}(p)+\Gamma^\eta_{\beta\gamma}(p)\, g_{\alpha\eta}(p)=\Gamma^\beta_{\alpha\gamma}(p)+\Gamma^\alpha_{\beta\gamma}(p)$$

可证: $\Gamma^\alpha_{\beta\gamma}(p)=0$, $\forall \alpha,\beta,\gamma=1,2$

$r=r(u^1,u^2)$ 坐标为 $(u^1,u^2)=(\rho\cos\theta,\rho\sin\theta)$, θ 为某给定角度

是一条测地线 in 坐标, i.e. $\rho\mapsto r(\underbrace{\rho\cos\theta}_{u^1},\underbrace{\rho\sin\theta}_{u^2})$ 是一条测地线

测地线方程 $\frac{d^2u^\alpha}{d\rho^2}+\Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{d\rho}\frac{du^\gamma}{d\rho}=0$, $\alpha=1,2$

$\Rightarrow \Gamma^\alpha_{11}(\rho\cos\theta,\rho\sin\theta)\cos^2\theta+2\Gamma^\alpha_{12}(\rho\cos\theta,\rho\sin\theta)\sin\theta\cos\theta+\Gamma^\alpha_{22}(\rho\cos\theta,\rho\sin\theta)\sin^2\theta=0$

取 $\theta=0\Rightarrow \Gamma^\alpha_{11}(\rho,0)=0$

取 $\theta=\frac{\pi}{2}\Rightarrow \Gamma^\alpha_{22}(0,\rho)=0$

令 $\rho\to 0\Rightarrow \Gamma^\alpha_{12}(p)=0$ □

测地极坐标系

参数变换 $\begin{cases}u^1=\rho\cos\theta\\ u^2=\rho\sin\theta\end{cases}$

$r=r(u^1,u^2)=\bar r(\rho,\theta)$

$\det\begin{pmatrix}\frac{\partial u^1}{\partial\rho} & \frac{\partial u^1}{\partial\theta}\\ \frac{\partial u^2}{\partial\rho} & \frac{\partial u^2}{\partial\theta}\end{pmatrix}=\rho\neq 0$ 去掉原点后为容许 in 参数变换

坐标曲线 ρ-线 从 p 点发出测地线

θ-线 测地圆(以 p 为心)

定理(Gauß lemma) 从 $p\in M$ 发出的测地线与以 p 为心的测地圆彼此正交

(即在测地极坐标下 $F\equiv 0$, $I=d\rho\otimes d\rho+G\,d\theta\otimes d\theta$)

证: ① $\lim_{\rho\to 0}F(\rho,\theta)=0$

$r_\theta=\frac{\partial r}{\partial u^1}\frac{\partial u^1}{\partial\theta}+\frac{\partial r}{\partial u^2}\frac{\partial u^2}{\partial\theta}=-r_1\rho\sin\theta+r_2\rho\cos\theta$

$\lim_{\rho\to 0}r_\theta=0\Rightarrow\lim_{\rho\to 0}F(\rho,\theta)=0$

② $\frac{\partial}{\partial\rho}F(\rho,\theta)\equiv 0$

$\frac{\partial}{\partial\rho}F(\rho,\theta)=\frac{\partial}{\partial\rho}\langle r_\rho,r_\theta\rangle=\langle r_{\rho\rho},r_\theta\rangle+\underbrace{\langle r_\rho,r_{\theta\rho}\rangle}_{\frac{1}{2}\frac{\partial}{\partial\theta}\overbrace{\langle r_\rho,r_\rho\rangle}^{0}}=\langle r_{\rho\rho},r_\theta\rangle=\langle\Gamma_{11}^{\alpha}r_\alpha,r_\theta\rangle$

ρ-线是测地线, 坐标 $(u^1=\rho,u^2=\theta)$

$\frac{d^2u^\alpha}{d\rho^2}+\Gamma_{\beta\gamma}^{\alpha}\frac{du^\beta}{d\rho}\frac{du^\gamma}{d\rho}=0$, $\alpha=1,2$

$0+\Gamma_{\beta\gamma}^{\alpha}\frac{du^\beta}{d\rho}\frac{du^\gamma}{d\rho}=0$, $\alpha=1,2$

$\Leftrightarrow\Gamma_{11}^{\alpha}(\rho,\theta)=0$, $\alpha=1,2$ □

定理: $\forall p\in M$, $\exists p$ 点的一个小邻域 U 使得 $\forall q\in U$, 包含在 U 内的联接 p,q 的测地线是曲面上

联接 p,q 的所有曲线中(分段)长度最短的

证: U 上应可建立以 p 为心的测地极坐标系

$\|r=r(\rho,\theta)$

$\{\exp_p(w)\mid w\in T_pM,\ w\in B(0,\varepsilon)\}$

令 $r=r(t)$, $t\in[0,t_0]$, $r(0)=p$, $r(t_0)=q$ 是一条联接 p,q 的曲线

(1) $C:r=r(t)$ 完全落入 U 内, 可表 $r(t)=r(\rho(t),\theta(t))$

$r'(t)=r_\rho\frac{d\rho}{dt}+r_\theta\frac{d\theta}{dt}$

$|r'(t)|=\sqrt{\left(\frac{d\rho}{dt}\right)^2+G\langle r_\theta,r_\theta\rangle\left(\frac{d\theta}{dt}\right)^2}$

$L(C)=\int_0^{t_0}|r'(t)|\,dt\geq\int_0^{t_0}\left|\frac{d\rho}{dt}\right|dt\geq\left|\int_0^{t_0}\frac{d\rho}{dt}\,dt\right|=|\rho(t_0)-\rho(0)|=\rho(t_0)$

(2) $C:r=r(t)$ 不完全落入 U 中

$t_1=\inf\{t\in[0,t_0]\mid r(t)\notin U\}$

$L(C)\geq L(r|_{[0,t_1]})\geq\rho(t_0)$ 令 $t_1\to t_1\Rightarrow L(C)\geq\varepsilon>\rho(q)$ □

$\gamma=\gamma(\rho,\theta)$ 测地极坐标系

$I=d\rho\otimes d\rho+G(\rho,\theta)\,d\theta\otimes d\theta$

$e_1=\gamma_\rho,\ e_2=\frac{\gamma_\theta}{\sqrt{G}}$

$\Rightarrow \begin{matrix}\omega^1=d\rho\\ \omega^2=\sqrt{G}\,d\theta\end{matrix} \xrightarrow{d} \begin{matrix}d\omega^1=0\\ d\omega^2=(\sqrt{G})_\rho\,d\rho\wedge d\theta=\frac{(\sqrt{G})_\rho}{\sqrt{G}}\omega^1\wedge\omega^2\end{matrix} \Rightarrow \omega_1^2=\frac{(\sqrt{G})_\rho}{\sqrt{G}}\omega^2=(\sqrt{G})_\rho\,d\theta$

$\Rightarrow K=\frac{-d\omega_1^2}{\omega^1\wedge\omega^2}=\frac{-(\sqrt{G})_{\rho\rho}\,d\rho\wedge d\theta}{d\rho\wedge\sqrt{G}\,d\theta}$

$$\boxed{K=-\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}}$$

$K\equiv C$. 即 $-\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}=C \Leftrightarrow (\sqrt{G})_{\rho\rho}+C\sqrt{G}=0$

比如 $K\equiv C\equiv 0 \Rightarrow (\sqrt{G})_{\rho\rho}=0 \Rightarrow (\sqrt{G})_\rho$ 只依赖于 θ（$=f(\theta)$）

性质 (1) $\lim_{\rho\to 0}\sqrt{G}=0$

(2) $\lim_{\rho\to 0}(\sqrt{G})_\rho=1$

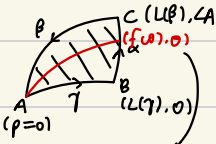
证：$G=\langle\gamma_\theta,\gamma_\theta\rangle=\langle\gamma_\alpha\frac{\partial u^\alpha}{\partial\theta},\gamma_\beta\frac{\partial u^\beta}{\partial\theta}\rangle=g_{\alpha\beta}(\rho,\theta)\frac{\partial u^\alpha}{\partial\rho}\frac{\partial u^\beta}{\partial\theta}$

$(\sqrt{G})_\rho=\frac{1}{2}\frac{G_\rho}{\sqrt{G}}=\frac{1}{2}\cdot\frac{g_{\alpha\beta,\rho}\frac{\partial u^\alpha}{\partial\theta}\frac{\partial u^\beta}{\partial\theta}+g_{\alpha\beta}\frac{\partial}{\partial\rho}\left(\frac{\partial u^\alpha}{\partial\theta}\frac{\partial u^\beta}{\partial\theta}\right)}{\sqrt{G}} \xrightarrow[\text{洛}]{\rho\to 0} 1$ □

测地三角形内角和

定理：假设测地三角形 ΔABC 包含在点 A 的附近在测地极坐标邻域内，则 $\int_{\Delta ABC}K\,dvol=\angle A+\angle B+\angle C-\pi$

证：$dvol=\sqrt{EG-F^2}\,d\rho\wedge d\theta=\sqrt{G}\,d\rho\wedge d\theta$　　$K\,dvol=-(\sqrt{G})_{\rho\rho}\,d\rho\wedge d\theta$



不妨将图形转一下，使 γ 对应 $\theta=0$，假定 θ 对应作 $f(\theta)$

$$\begin{aligned}LHS&=\int-(\sqrt{G})_{\rho\rho}\,d\rho\wedge d\theta=\int_0^{\angle A}d\theta\int_0^{f(\theta)}-(\sqrt{G})_{\rho\rho}\,d\rho\\&=\int_0^{\angle A}d\theta\left((-\sqrt{G})_\rho(f(\theta),\theta)+\underbrace{(\sqrt{G})_\rho(0,\theta)}_{\|\ 1}\right)\\&=\angle A-\int_0^{\angle A}(\sqrt{G})_\rho(f(\theta),\theta)\,d\theta\\&=\angle A-\int_0^{\angle A}(\sqrt{G})_\rho(f(\theta(s)),\theta(s))\frac{d\theta}{ds}ds\end{aligned}$$

$\alpha(s)=(f(\theta(s)),\theta(s))$　$\alpha'(s)=\underbrace{\langle\alpha'(s),\gamma_\rho\rangle}_{\cos\varphi}\gamma_\rho+\underbrace{\langle\alpha'(s),\frac{\gamma_\theta}{\sqrt{G}}\rangle}_{\sin\varphi}\frac{\gamma_\theta}{\sqrt{G}}$

$\cos\varphi=\langle\gamma_\rho\frac{d\rho}{ds}+\gamma_\theta\frac{d\theta}{ds},\gamma_\rho\rangle=\frac{d\rho}{ds}$　　$\sin\varphi=\langle\gamma_\rho\frac{d\rho}{ds}+\gamma_\theta\frac{d\theta}{ds},\frac{\gamma_\theta}{\sqrt{G}}\rangle=\sqrt{G}\frac{d\theta}{ds}$

$\frac{d\cos\varphi}{ds}=-\sin\varphi\frac{d\varphi}{ds}\Rightarrow\frac{d^2\rho}{ds^2}=-\sqrt{G}\frac{d\theta}{ds}\frac{d\varphi}{ds}$

$\|$

$\frac{1}{2}G_\rho\left(\frac{d\theta}{ds}\right)^2$

$\therefore LHS = \angle A + \int_0^{L(\alpha)} \frac{d\varphi}{ds} ds = \angle A + \varphi(L(\alpha)) - \varphi(0)$

$= \angle A + \angle C - (\pi - \angle B)$

只需验证 Δ: $g = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}$ $g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{G} \end{pmatrix}$

$\Gamma_{11}^1 = \frac{1}{2} g^{11}(0+0-0) = 0$, $\Gamma_{12}^1 = \frac{1}{2} g^{11}(0+0-0) = 0$, $\Gamma_{22}^1 = \frac{1}{2} g^{11}(0+0-G_\rho) = -\frac{1}{2} G_\rho$

$\therefore$ 由测地线方程: $\frac{d^2\rho}{ds^2} + \Gamma_{\alpha\beta}^1 \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 0$

$\Rightarrow \frac{d^2\rho}{ds^2} - \frac{1}{2} G_\rho \left(\frac{d\theta}{ds}\right)^2 = 0$

alternative proof of (1):

$$\frac{d\cos\varphi}{ds} = \frac{d}{ds} \langle \alpha'(s), r_\rho \rangle$$

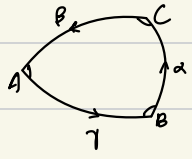
$$= \underbrace{\langle \alpha''(s), r_\rho \rangle}_{k_g \langle e_2^{\perp}, r_\rho \rangle = 0} + \langle \alpha'(s), r_{\rho\rho} \frac{d\rho}{ds} + r_{\rho\theta} \frac{d\theta}{ds} \rangle$$

$$= \langle r_\rho \frac{d\rho}{ds} + r_\theta \frac{d\theta}{ds}, r_{\rho\rho} \frac{d\rho}{ds} + r_{\rho\theta} \frac{d\theta}{ds} \rangle$$

$$= \underbrace{\langle r_\rho, r_{\rho\rho} \rangle}_{0} \left(\frac{d\rho}{ds}\right)^2 + \underbrace{\langle r_\theta, r_{\rho\theta} \rangle}_{\frac{1}{2} G_\rho} \left(\frac{d\theta}{ds}\right)^2 + \left(\underbrace{\langle r_\theta, r_{\rho\rho} \rangle}_{\frac{\partial}{\partial \rho} \langle r_\theta, r_\rho \rangle - \langle r_{\theta\rho}, r_\rho \rangle = 0} + \underbrace{\langle r_\rho, r_{\rho\theta} \rangle}_{0} \right) \frac{d\rho}{ds} \frac{d\theta}{ds}$$

□

定理 (Gauss-Bonnet I): 假设三角形 $\triangle ABC$ 包含在以 A 为原点 in 测地极坐标邻域内, 假设 α 在



以 A 为原点in测地极坐标为 $(f(\theta), \theta)$ (θ 对应的 $\in -\inf(\theta)$), 则

$$\int_{\triangle ABC} K \, dvol + \int_\alpha k_g \, ds = \angle A + \angle B + \angle C - \pi$$

证: 上个定理证明中 $\langle \alpha''(s), r_\rho \rangle = k_g \langle -\sin\varphi \, r_\rho + \cos\varphi \frac{r_\theta}{\sqrt{G}}, r_\rho \rangle = -k_g \sin\varphi$

$$\frac{d\varphi}{ds} = -\Gamma(G)_\rho \frac{d\theta}{ds} + k_g$$

□

推论 (Gauss-Bonnet II): 假设 n 边形 (记所围区域为 $\square_{A_1 \cdots A_n}$, 边界 $\partial \square_{A_1 \cdots A_n}$) 落在以其内部某点 O 为原点in测地极坐标邻域内, 且每条边 $\overset{\frown}{A_i A_{i+1}}$ 可坐标表示为 $(f_i(\theta), \theta)$,

则 $\int_{\square_{A_1 \cdots A_n}} K \, dvol + \int_{\partial \square_{A_1 \cdots A_n}} k_g \, ds = \sum_{i=1}^n \angle A_i - n\pi + 2\pi = 2\pi - \sum_{i=1}^n (\pi - \angle A_i)$

□

定理 (Gauß-Bonnet III) 设 C 是曲面 M 上一条分段光滑简单闭曲线，所围区域落入一个测地极坐标邻域中，则有 $\int_\Omega K\,dvol + \oint_C k_g\,ds + \sum(\pi - \angle A_i) = 2\pi$ □

定理 (Gauß-Bonnet IV) 设 C 是曲面 M 上一条分段光滑简单闭曲线，所围区域 Ω 与 $\mathbb{R}^2$ 上一个单连通区域分数分同胚，则有 $\int_\Omega K\,dvol + \oint_C k_g\,ds + \sum(\pi - \angle A_i) = 2\pi$ □

$K = \frac{-d\omega_{12}}{\omega_1 \wedge \omega_2}$

假定 C 光滑简单闭曲线，所围区域 Ω 与 $\mathbb{R}^2$ 上一个单连通区域分数分同胚。

$\int_\Omega K\,dvol = \int_\Omega -d\omega_{12} \overset{Stokes}{=\!=} \oint_C -\omega_{12} = -\oint_C \frac{\omega_{12}}{ds}\,ds$

$\int_\Omega K\,dvol = -\oint_C k_g\,ds + \oint_C \frac{d\varphi}{ds}\,ds$

$\int_\Omega K\,dvol + \oint_C k_g\,ds = 2\pi$

$\Rightarrow \oint_C d\varphi = 2\pi$

定理 (Hopf) 设 C 为一条光滑简单平面闭曲线，则 $\frac{1}{2\pi}\oint_C k(s)\,ds = \frac{\pm 1}{\text{定向}} = \frac{1}{2\pi}\int_C d\varphi$ □

定理：设 C 是曲面 M 上一条分段光滑简单闭曲线，所围区域 Ω 与 $\mathbb{R}^2$ 上一个单连通区域分数分同胚，

C 参数表示 $r = r(s)$ 弧长参数，$V: [0, L(C)] \to \mathbb{R}^3$ 为沿着 C 平行的切向量场，

$V(s) = \cos\varphi(s)\,e_1(s) + \sin\varphi(s)\,e_2(s)$，则 $\varphi(L(C)) - \varphi(0) = \int_\Omega K\,dvol$

证：$0 = \frac{DV}{ds} = \frac{D(\cos\varphi(s)e_1 + \sin\varphi(s)e_2)}{ds}$

$= \frac{d\varphi}{ds}(-\sin\varphi e_1 + \cos\varphi e_2) + \frac{\omega_{12}e_2}{ds}\cos\varphi + \frac{-\omega_{12}e_1}{ds}\sin\varphi$

$= (\frac{d\varphi}{ds} + \frac{\omega_{12}}{ds})(e_2\cos\varphi - e_1\sin\varphi)$

$\Leftrightarrow \frac{d\varphi}{ds} + \frac{\omega_{12}}{ds} = 0$ □

$\mathbb{R}^3$ 中的二维流形

闭曲面三角剖分 - 欧拉公式

定理 (Gauß-Bonnet) M 是一个紧致、无边、可定向的光滑曲面，Gauß 曲率为 K，则

$\int_M K\,dvol = 2\pi(V - E + F) = 2\pi\chi(M)$

注：经小扰动，局部曲率改变，但整体的弯曲程度不变

微分形式in积分 M: 正则曲面片

定义: 设 ϕ 是 m 上 1 形式, 设 $\alpha:[a,b]\to M$ 是 M 上in一条曲线, 则 ϕ 沿 α in积分定义为

$$\int_\alpha \phi := \int_{[a,b]} \alpha^*(\phi) = \int_a^b \alpha^*\phi\left(\frac{\partial}{\partial t}\right) dt$$

这里 $\alpha^*\phi\left(\frac{\partial}{\partial t}\right) = \phi\left(\alpha_*\left(\frac{\partial}{\partial t}\right)\right) = \phi(\alpha'(t))$

$$\Rightarrow \int_\alpha \phi = \int_a^b \phi(\alpha'(t))\,dt$$

定理: 设 f 是 M 上in光滑函数. 设 $\alpha:[a,b]\to M$ 是 M 上in一条曲线, 满足 $\alpha(a)=p$, $\alpha(b)=q$, 则有

$$\int_\alpha df \left(= \int_a^b df(\alpha'(t))\,dt = \int_a^b \frac{d}{dt} f(\alpha(t))\,dt\right) = f(q) - f(p)$$ □

定义. 设 η 是 M 上一个 2 形式, 设 $r: R\to M$ 是 M 上区域, $R=[a,b]\times[c,d]$, 则 η 沿 r in积分定义为

$$\int_r \eta := \iint_R r^*\eta = \int_a^b\int_c^d r^*\eta\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) du\wedge dv = \int_a^b\int_c^d \eta\big(\underbrace{r_*\left(\tfrac{\partial}{\partial u}\right)}_{r_u}, \underbrace{r_*\left(\tfrac{\partial}{\partial v}\right)}_{r_v}\big)\,du\wedge dv$$

Stokes' thm ϕ 1-形式 $\iint_D d\phi = \int_{\partial D}\phi$ □

角度函数

$V, W\in T_pM$. φ 称为 V 到 W in有向角, 若 $\frac{W}{|W|} = \cos\varphi\frac{V}{|V|} + \sin\varphi\frac{(V)^\perp}{|V|}$, 其中 $(V)^\perp = n\wedge\frac{V}{|V|}$

定理: 设 $c: I\to M$ 是 M in一条曲线, V, W 为 M 上沿 c in处处非零in向量场,

则存在一个光滑函数 $\varphi: I\to\mathbb{R}$ 使得 $\forall t\in I$. $\varphi(t)$ 是从 $V(t)$ 到 $W(t)$ in一个有向角

证: 不妨设 V, W 处处为单位切向量, 则 $W = \langle V, W\rangle V + \langle W, V^\perp\rangle V^\perp$, $V^\perp = n\wedge V$

记沿 c in两个函数 $f := \langle W, V\rangle$, $g := \langle W, V^\perp\rangle$, 有 $f^2+g^2=1$

存一点 t_0 处, $\varphi(t_0)$ 为从 $V(t_0)$ 到 $W(t_0)$ in有向角, 则构造 $\varphi(t) := \varphi(t_0) + \int_{t_0}^t fg' - f'g\,du$

Claim: $\cos\varphi(t) = f(t)$, $\sin\varphi(t) = g(t)$

Claim in证明: 考查 $L(t) := (f(t)-\cos\varphi(t))^2 + (g(t)-\sin\varphi(t))^2$, 有 $L(t_0)=0$, $L'(t)=0$ □

整体 Gauss-Bonnet 定理: M 紧致无边可定向in光滑曲面, 有

$$\iint_M K\,dV = 2\pi\chi(M) = 2\pi(2-2g)$$ ✓亏格

几何 拓扑

M 上是否存在处处非零的光滑切向量场？

假设存在这样的切向量场 V. 记 $e_1 = \frac{V}{|V|}$. M 可定向 $\Rightarrow$ 单位法向量场 $n := e_3 \Rightarrow$ 正交活动标架

$\iint_M K dV = \iint_M K \omega^1 \wedge \omega^2 = -\iint_M d\omega_1^2 \overset{\text{Stokes}}{=\!=} 0 \quad \Rightarrow \chi(M) = 0$ 环面

Poincare-Hopf index formula

M 紧致，无边，可定向，V 光滑切向量场

p 孤立奇点，设 Ω 是 p 的一个小邻域：Ω 中不包含其他奇点，Ω 包含在一个坐标邻域之内，

有向边界 $C = \partial\Omega$

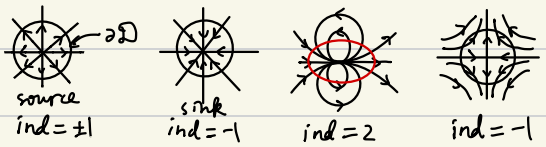
Ω 上总存在处处非零的光滑切向量场 X

比较 V 和 X：

令 $\alpha: [a,b] \to C = \partial\Omega$ 为切向边界 C 的一个参数化，取沿曲线 C 的有向角函数 $\varphi = L_\alpha(X, V)$ (从 X 到 V 的有向角)

定义(指标). 向量场 V 在 p 点处的指标(index)定义为 $ind(V,p) := \frac{\varphi(b) - \varphi(a)}{2\pi}$

例. 平面 $X \Rightarrow$



定理(Poincare-Hopf)：M 紧致无边可定向，V 是 M 上光滑切向量场，只有孤立奇点 $p_1, \cdots, p_K$，

则 $\chi(M) = \frac{1}{2\pi}\int_M K dV = \sum_{i=1}^K ind(V, p_i)$

例.  double torus $g = 2$

记：p_i，Ω_i (两两不相交)，$C_i = \partial\Omega_i$，处处非零切向量场 X_i，$M := M \setminus \bigsqcup_{i=1}^K \Omega_i$

(i) V 在 M 上处处非零，可取 $\overline{e_1} = \frac{V}{|V|}$，$\overline{e_2} = n \wedge \overline{e_1}$，$\overline{e_3} = n$ $\overline{\omega^1}, \overline{\omega^2}, \overline{\omega_1^2}$

(ii) 在 Ω_i 上 $e_1 = \frac{X_i}{|X_i|}$，$e_2 = n \wedge e_1$，$e_3 = n$. $\omega^1, \omega^2, \omega_1^2$

考虑 $\iint_M K dV = \int_{\overline{M}} K dV + \sum_{i=1}^K \iint_{\Omega_i} K dV$

$\iint_M K dV = \int_M d\overline{\omega_1^2} \overset{\text{Stokes}}{=\!=} -\int_{\partial M} \overline{\omega_1^2} = \sum_{i=1}^K \int_{\partial\Omega_i} \overline{\omega_1^2}$

设 P_i 是沿 $C_i=\partial\mathcal{D}_i$ 平行的切向量场，记 $\overline{\varphi_i}:=\angle(V,P_i)$ 从 V 到 P_i 的有向角

由 $\int_{\partial\mathcal{D}_i}\overline{\omega_i^2}=\int_{\partial\mathcal{D}_i}d\overline{\varphi_i}=\overline{\varphi_i}(L(C_i))-\overline{\varphi_i}(0)$

$\varphi_i:=\angle(X_i,P_i)$ $\varphi_i(L(C_i))-\varphi_i(0)=\int_{\partial\mathcal{D}_i}d\varphi_i=-\int_{\partial\mathcal{D}_i}\omega_i^2=-\iint_{\mathcal{D}_i}K\,dV$

$$\Rightarrow\iint_M K\,dV=\iint_{M\setminus\bigcup\mathcal{D}_i}K\,dV+\sum_{i=1}^{k}\iint_{\mathcal{D}_i}K\,dV$$
$$=-\sum_{i=1}^{k}(\overline{\varphi_i}(L(C_i))-\overline{\varphi_i}(0))+\sum_{i=1}^{k}(\varphi_i(L(C_i))-\varphi_i(0))$$
$$=\sum_{i=1}^{k}((-\overline{\varphi_i}+\varphi_i)(L(C_i))-(-\overline{\varphi_i}+\varphi_i)(0))$$
$$=\sum_{i=1}^{k}(\angle(X_i,V)(L(C_i))-\angle(X_i,V)(0))$$

□

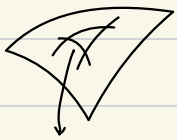
Jacobi's remarkable curve theorem

定理 (Jacobi 1842) 设 C 是空间中的光滑闭曲线，设其主法向量处处非零，若其主法向量的终点在单位球面上形成的曲线（主法标线）是一条简单闭曲线，则它将单位球面分成面积相等的两部分

历史：Gauß 1827 $\iint_{\triangle ABC}K\,dV=\angle A+\angle B+\angle C-\pi$

Bonnet 1848 $\oint_{\partial\triangle ABC}k_g\,ds+\iint_{\triangle ABC}K\,dV=\angle A+\angle B+\angle C-\pi$

Jacobi 1836



测地线 $\Rightarrow$ 主法向量与曲面的法向量共线

定理 (Jacobi) 考虑由三条空间曲线给出的三角形 $\triangle ABC$，若三条曲线的主法向量处处非零，且相交点处主法向量吻合，则由主法向量确定的单位球面上的三角形 $\triangle XYZ$ 有：

$$\text{area}(\triangle ABC)=\angle A+\angle B+\angle C-\pi$$

Thomas Clausen 1842 认为 Jacobi 的证法有误，Jacobi 在回复中给出定理的推广

记：C：$r=r(s)$ 弧长参数化 $\tau(s)\neq 0$ Frenet 标架 Frenet 运动方程

主法标线 $s\mapsto e_2(s)$ 设 ρ 为主法标线的弧长参数

$$\rho(s_0)=\int_0^{s_0}\left|\frac{de_2}{ds}\right|ds=\int_0^{s_0}\sqrt{k^2(s)+\tau^2(s)}\,ds,\quad \frac{d\rho}{ds}=\sqrt{k^2+\tau^2}$$

由 Gauß-Bonnet 定理，只需证 k_g 沿主法标线积分为 0

$$k_g = \left\langle \frac{d^2 e_2(s(\rho))}{d\rho^2}, -\frac{de_2(s(\rho))}{d\rho} \wedge e_2 \right\rangle = \left\langle \frac{d^2 e_2}{d\rho^2}, (\kappa e_3 + \tau e_1)\frac{ds}{d\rho} \right\rangle$$

$$\frac{de_2}{d\rho} = \frac{de_2}{ds}\cdot\frac{ds}{d\rho} = (-\kappa e_1 + \tau e_3)\frac{ds}{d\rho} \Rightarrow e_2 \wedge \frac{de_2}{d\rho} = (\kappa e_3 + \tau e_1)\frac{ds}{d\rho}$$

$$\frac{d^2 e_2}{d\rho^2} = \frac{d}{ds}\left(\frac{de_2}{ds}\frac{ds}{d\rho}\right)\frac{ds}{d\rho} = \frac{d}{ds}\left(\frac{de_2}{ds}\right)\left(\frac{ds}{d\rho}\right)^2 + \frac{de_2}{ds}\frac{d}{ds}\left(\frac{ds}{d\rho}\right)\frac{ds}{d\rho}$$

第二项代入内积为0

$$(-\kappa' e_1 + \tau' e_3 - (\kappa^2+\tau^2)e_2)\left(\frac{ds}{d\rho}\right)^2 + \frac{de_2}{ds}\frac{d}{ds}\left(\frac{ds}{d\rho}\right)\frac{ds}{d\rho}$$

$$\Rightarrow k_g = \left\langle (-\kappa' e_1 + \tau' e_3 - (\kappa^2+\tau^2)e_2)\left(\frac{ds}{d\rho}\right)^2, (\kappa e_3 + \tau e_1)\frac{ds}{d\rho} \right\rangle$$

$$= \frac{-\kappa'\tau + \tau\kappa}{\kappa^2+\tau^2}\frac{ds}{d\rho} = \frac{d}{ds}\left(\arctan\frac{\tau}{\kappa}\right)\frac{ds}{d\rho} = \frac{d}{d\rho}\left(\arctan\frac{\tau}{\kappa}\right)$$

□

(III) 整体微分几何

球面的刚性：Liebmann 定理

定理(Liebmann 1899) 设 M 是 $\mathbb{E}^3$ 中一个紧致、无边、连通且具有常Gauss曲率的曲面，则 M 是一个球面

下面的证明来自 S. S. Chern：Some new characterization of the Euclidean sphere

引理1 M 为 $\mathbb{E}^3$ 中紧致曲面，则存在一点 $p_0\in M$ 使 $K(p_0)>0$

证：记曲面的位置向量为 $r(p)=(x(p),y(p),z(p))$

考虑 M 上的一个函数 $f: M\to\mathbb{R}$，$p\mapsto \langle r(p),r(p)\rangle$

f 光滑 $\Rightarrow \exists p_0\in M$ s.t. f 在 p_0 达极大值

对任意过 p 点的 M 上弧长参数曲线 $r=r(s)$ $(r(0)=p_0)$

$0=\frac{d}{ds}\big|_{s_0} f(r(s))=\frac{d}{ds}\big|_{s=0}\langle r(s),r(s)\rangle=2\langle \dot r(0),r(0)\rangle$ 由 $r(s)$ 任意性，$r(0)=\lambda n(p_0)$，$\lambda\neq 0$

$0\geq \frac{d^2}{ds^2}\big|_{s_0} f(r(s))=2\underbrace{\langle \ddot r(0),r(0)\rangle}_{\lambda n(p_0)}+2\underbrace{\langle \dot r(0),\dot r(0)\rangle}_{1}$

法曲率

$\Rightarrow \lambda k_n(\dot r(0))\leq -1$ $(\forall \dot r(0))$ $\Rightarrow K(p)>0$ □

引理2 (Hilbert Grundlagen der Geometrie 1909)

设 p_0 是 $\mathbb{E}^3$ 中光滑曲面 M 上一点，k_1,k_2 主曲率函数，满足

(1) k_1 在 p_0 达局部极大值

(2) k_2 在 p_0 达局部极小值

(3) $k_1(p_0)>k_2(p_0)$

则 Gauss 曲率 $K(p_0)\leq 0$

证：(3) $\Rightarrow p$ 不是脐点 $\Rightarrow$ 在 p_0 小邻域上 k_1,k_2 光滑，曲率线网 $r=r(u,v)$，特别地，$I=E\,du\otimes du+G\,dv\otimes dv$

正交活动标架 $e_1=\frac{r_u}{\sqrt{E}}$，$e_2=\frac{r_v}{\sqrt{G}}$，$e_3=e_1\wedge e_2$，$\omega^1=\sqrt{E}\,du$，$\omega^2=\sqrt{G}\,dv$，$\omega_1^3=k_1\omega^1$，$\omega_2^3=k_2\omega^2$

求导，$\omega_1^2\wedge\omega_2^3=d\omega_1^3=dk_1\wedge\omega^1+k_1\,d\omega^1=dk_1\wedge\omega^1+k_1\omega^2\wedge\omega_2^1$

$\omega_2^1\wedge\omega_1^3=d\omega_2^3=dk_2\wedge\omega^2+k_2\,d\omega^2=dk_2\wedge\omega^2+k_2\omega^1\wedge\omega_1^2$

$\Rightarrow 0=dk_1\wedge\omega^1+(k_1-k_2)\omega_1^2\wedge\omega^2=\left[-\frac{(k_1)_v}{\sqrt{G}}\omega^1+(k_1-k_2)\omega_1^2\right]\wedge\omega^2$

$0=dk_2\wedge\omega^2+(k_1-k_2)\omega_1^2\wedge\omega^1=\left[-\frac{(k_2)_u}{\sqrt{E}}\omega^2+(k_1-k_2)\omega_1^2\right]\wedge\omega^1$

于是 $(k_1-k_2)\omega_1^2=a\omega^1+b\omega^2$，于是 $(k_1-k_2)\omega_1^2=\frac{(k_1)_v}{\sqrt{G}}\omega^1+\frac{(k_2)_u}{\sqrt{E}}\omega^2$

求导，$d(k_1-k_2)\wedge\omega_1^2+(k_1-k_2)\,d\omega_1^2=d\left(\frac{(k_1)_v}{\sqrt{G}}\omega^1+\frac{(k_2)_u}{\sqrt{E}}\omega^2\right)$ 在 p_0 处取值

$$(k_1(p_0)-k_2(p_0))(-K(p_0)\omega^1\wedge\omega^2(p_0)) = d\left(\frac{(k_1)_v}{\sqrt{G}}\right)\omega^1 + \cancel{\frac{(k_1)_v}{\sqrt{G}}d\omega^1} + d\left(\frac{(k_2)_u}{\sqrt{E}}\right)\omega^2 + \cancel{\frac{(k_2)_u}{\sqrt{E}}d\omega^2} \quad \text{at } p_0$$

$$-(k_1(p_0)-k_2(p_0))K(p_0)\omega^1\wedge\omega^2 = \left(\frac{(k_1)_v}{\sqrt{G}}\right)_v\frac{1}{\sqrt{G}}\omega^2\wedge\omega^1 + \left(\frac{(k_2)_u}{\sqrt{E}}\right)_u\frac{1}{\sqrt{E}}\omega^1\wedge\omega^2$$

$$= \left(-\frac{(k_1)_{vv}}{G} + \frac{(k_2)_{uu}}{E}\right)\omega^1\wedge\omega^2$$

$$\Rightarrow K(p_0) = -\frac{1}{k_1(p_0)-k_2(p_0)}\left(\frac{(k_2)_{uu}}{E} - \frac{(k_1)_{vv}}{G}\right) \le 0$$ □

Liebmann 定理 in 证明：常 Gauß 曲率 + 引理 1 $\Rightarrow$ Gauß 曲率 $\equiv C > 0$

k_1, k_2 主曲率函数，$k_1 \ge k_2$，可知 k_1, k_2 为连续函数

设 $p_0 \in M$ 为主曲率函数 k_1 的极大值点 $\Rightarrow k_2 = \frac{C}{k_1}$ 在 p_0 点取极小值

· 若 $k_1(p_0) > k_2(p_0)$，则由 Hilbert 引理 2 知 $K(p_0) \le 0$ 与 $K \equiv C > 0$ 矛盾

· 若 $k_1(p_0) = k_2(p_0)$，即 p_0 是脐点 $k_2(p_0) \le k_2(p) \le k_1(p) \le k_1(p_0) \Rightarrow k_1(p) \equiv k_2(p)$，$\forall p \in M$

即 M 是全脐点曲面 $\Rightarrow M$ 是球面（半径 $\frac{1}{\sqrt{C}}$）in 一部分

① M 连通，M 闭子集
② M 曲面，M 开子集
$\Rightarrow M$ 是整个球面（半径 $\frac{1}{\sqrt{C}}$） □

事实上我们证明了

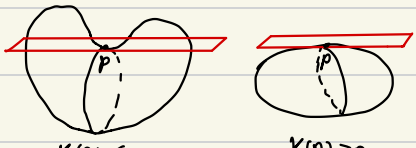
定理：设 M 是 $\mathbb{E}^3$ 中 in 紧致（=无边）、连通、Gauß 曲率为正 in 光滑曲面，若存在一个连续函数 f 使得 M in 两个曲率函数 k_1, k_2（$k_1 \ge k_2$）满足 $k_2 = f(k_1)$，则 M 是一个球面 □

推论：M 是 $\mathbb{E}^3$ 中 in 一个紧致、连通、Gauß 曲率为正 in 光滑曲面（卵形面），若 M 有常平均曲率，则 M 是球面 □

Gauß 曲率为正 $\xrightarrow{\text{Gauß-Bonnet}}$ 亏格 $g=0$，M 同胚于球面 → 弱化为 → M 同胚于球面 Hopf 1950s

↓ 弱化为

紧致 Alexandorff 1956



$K(p)<0$ 曲面在切平面两侧　　$K(p)>0$ 曲面在切平面同侧

定义（凸性）$\mathbb{E}^3$ 中 in 光滑曲面称为凸曲面，如果 M 位于每点处切平面 in 同侧，即 $\forall p_0 \in M$，函数

$f: M \to \mathbb{R}$，$f(p) := \langle r(p) - r(p_0), n(p_0)\rangle$ 不变号

定理(Hadamard) 设 M 为 $\mathbb{E}^3$ 中紧致连通曲面，若 Gauss 曲率 K 处处为正，则 M 是凸曲面

注意到在上面第一张图中，Gauss 映射不是双射.

$\rightsquigarrow$ 引理：M 为 $\mathbb{E}^3$ 中紧致连通曲面，$K>0$，则 Gauss 映射 $g: M \to S^2$ 是双射

证：① g 是满射，即 $\forall e \in S^2$，$\exists p_0 \in M$ 使 $g(p_0) = n(p_0) = e$

考查函数 $f: M \to \mathbb{R}$，$f(p) := \langle n(p), e \rangle$，则 $f \leq 1$

设 p_0 为 f 在 M 上的最大值点，$df(p_0) = 0$

取含 p_0 的曲面片，$r = r(u,v)$

$$\begin{cases} \frac{\partial f}{\partial u}(p_0) = 0 \\ \frac{\partial f}{\partial v}(p_0) = 0 \end{cases} \Rightarrow \begin{cases} \langle n_u, e \rangle (p_0) = 0 \\ \langle n_v, e \rangle (p_0) = 0 \end{cases} \xrightarrow{K>0,\ n_u, n_v \text{不平行}} e = \pm n(p_0)$$

$$n_u = -\frac{L}{E} r_u - \frac{M}{G} r_v$$
$$n_v = -\frac{M}{E} r_u - \frac{N}{G} r_v$$

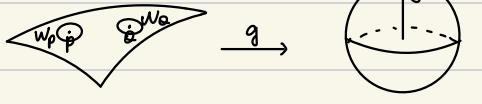
再由极大性，$0 \geq f_{uu}(p_0) = \langle (-\frac{L}{E} r_u - \frac{M}{G} r_v)_u, e \rangle (p_0) = \langle -\frac{L}{E} r_{uu} - \frac{M}{G} r_{vu}, e \rangle (p_0)$

若 $e = -n(p_0)$，则 $\langle r_{uu}, e \rangle (p_0) = -L$，$\langle r_{vu}, e \rangle = -M$，$\frac{L^2}{E} + \frac{M^2}{G} \leq 0 \Rightarrow L = M = 0 \Rightarrow K(p_0) = 0$，矛盾

故 $e = n(p_0)$

② g 是单射. 反证：设存在两点 $P, Q \in M$ s.t. $g(P) = g(Q) = e \in S^2$

$K = \frac{n_u \wedge n_v}{r_u \wedge r_v} = \det(dg) > 0$



由反函数定理知 g：$W_P \to g(W_P)$，$W_Q \to g(W_Q)$ 是微分同胚

不妨设 $W_P \cap W_Q = \emptyset$ 且 $g(W_P) \subset g(W_Q)$，g 在 $M \setminus W_P$ 上仍是满射

$4\pi \xrightarrow{\text{Gauss-Bonnet}} \int_M K dV > \int_{M \setminus W_P} K dV \geq \int_{S^2} d\sigma = 4\pi$，矛盾 □

定理证明：反证，若 M 不是凸的，即 $\exists p_0 \in M$ s.t. $f(p) = \langle r(p) - r(p_0), n(p_0) \rangle$ 变号，

即 $\{p \in M : f(p) > 0\}$，$\{p \in M : f(p) < 0\}$ 非空.

设 Q_1, Q_2 为 f 在 M 上的最大、最小值点

三个不同的点 Q_1, Q_2, p_0

由极值原理，$n(p_0)$ 与 Q_1, Q_2 处的切平面均垂直 $\Rightarrow$ 与 g 为单射矛盾 □

注："$K>0$" 可弱化为 "$K \geq 0$"

S.S Chern, R K. Lashof: On the total curvature of immersed manifolds I & II

嵌入曲面：积分公式

M：$\mathbb{E}^3$中嵌入曲面 $\phi=(r,n,dn)$ 是 M 上整体定义的微分1-形式

$r=(r^1,r^2,r^3)$
$n=(n^1,n^2,n^3)$
$dn=(dn^1,dn^2,dn^3)$

$$\phi=\begin{vmatrix} r^1 & r^2 & r^3 \\ n^1 & n^2 & n^3 \\ dn^1 & dn^2 & dn^3 \end{vmatrix}=(r^2n^3-r^3n^2)dn^1-(r^1n^3-r^3n^1)dn^2+(r^1n^2-r^2n^1)dn^3$$

$$d\phi=d(r^2n^3-r^3n^2)\wedge dn^1-d(r^1n^3-r^3n^1)\wedge dn^2+d(r^1n^2-r^2n^1)\wedge dn^3=(dr,n,dn)+(r,dn,dn)+(r,dn,d^2n)$$

$$d\phi(r,n,dn)=\begin{vmatrix} dr^1 & dr^2 & dr^3 \\ n^1 & n^2 & n^3 \\ dn^1 & dn^2 & dn^3 \end{vmatrix}+\begin{vmatrix} r^1 & r^2 & r^3 \\ dn^1 & dn^2 & dn^3 \\ dn^1 & dn^2 & dn^3 \end{vmatrix}=(dr,n,dn)+(r,dn,dn)$$

限制到一个曲面片，取正交活动标架 $e_1,e_2,e_3=n$

$$(dr,n,dn)=(\omega^1e_1+\omega^2e_2,\ e_3,\ \omega_3^1e_1+\omega_3^2e_2)$$

$$=\langle \omega^1e_1+\omega^2e_2,\ e_3\wedge(\omega_3^1e_1+\omega_3^2e_2)\rangle$$

$$=\langle \omega^1e_1+\omega^2e_2,\ \omega_3^1e_2+\omega_3^2(-e_1)\rangle$$

$$=-\omega^1\wedge\omega_3^2+\omega^2\wedge\omega_3^1$$

$$=\omega_1^3\wedge\omega^2+\omega^1\wedge\omega_2^3 \qquad \begin{pmatrix}\omega_1^3\\ \omega_2^3\end{pmatrix}=\begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$

$$=h_{11}\omega^1\wedge\omega^2+h_{22}\omega^1\wedge\omega^2$$

$$=2H\,\omega^1\wedge\omega^2$$

$$(r,dn,dn)=(r,\omega_1^1e_1+\omega_2^2e_2,\ \omega_3^1e_1+\omega_3^2e_2)$$

$$=\langle r,\ (\omega_3^1e_1+\omega_3^2e_2)\wedge(\omega_3^1e_1+\omega_3^2e_2)\rangle$$

$$=\langle r,\ \omega_3^1\wedge\omega_3^2n+\omega_3^2\wedge\omega_3^1(-n)\rangle$$

$$=2\langle r,n\rangle\omega_3^1\wedge\omega_3^2=2\langle r,n\rangle\omega_1^3\wedge\omega_2^3$$

$$=2\langle r,n\rangle\det\begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}\omega^1\wedge\omega^2$$

$$=2\langle r,n\rangle K\,\omega^1\wedge\omega^2$$

$$0=\int_M d\phi=\int_M(2H+2\langle r,n\rangle K)\,dV$$

$$\Rightarrow\int_M H\,dV=\int_M-\langle r,n\rangle K\,dV \quad (\text{Minkowski 公式})$$

$\varphi: M\to\mathbb{R}$，$p\mapsto-\langle r(p),n(p)\rangle$ 称为 M 的支撑函数

M：$\mathbb{E}^3$中嵌入曲面，则 $\int_M H\,dV=\int_M\varphi K\,dV$

练习：

① 选 $\phi=(r,n,dr)\Rightarrow\int_M dV=\int_M \varphi H\,dV$

② 选 $\phi=(r,dr,\overset{常向量}{a})\Rightarrow\int_M\langle n,a\rangle dV=0$

③ 选 $\phi=(dr,n,a)\Rightarrow\int_M H\langle n,a\rangle dV=0$

④ 选 $\phi=(n,dn,a)\Rightarrow\int_M K\langle n,a\rangle dV=0$

由 a 的任意性，$\int_M n\,dV=0$，$\int_M Hn\,dV=0$，$\int_M Kn\,dV=0$

φ 支撑函数，M 是单射凸曲面，不妨将原点置于 M 内部，从而可以通过选取单位法向量场使得 $\varphi>0$ (<0)

$H^2=\left(\frac{1}{2}(k_1+k_2)\right)^2=\left(\frac{1}{2}(k_1-k_2)\right)^2+k_1k_2=K+\frac{1}{4}(k_1-k_2)^2$

$\Rightarrow H^2\geq K$ 且 "=" 成立 $\Leftrightarrow k_1=k_2$

应用1 Liebmann定理

证：已知 M 是单射，连通，$K\equiv C>0$ 的曲面

取法向量场使得 $\varphi>0$，这同时保证 $H>0$

由 Minkowski 的公式得

$$\int_M K\varphi\,dV\overset{第一公式}{=\!=\!=}\int_M H\,dV\geq\int_M\sqrt{K}\,dV=\sqrt{K}\int_M dV\overset{第二公式}{=\!=\!=}\sqrt{K}\int_M H\varphi\,dV\geq\sqrt{K}\int_M\sqrt{K}\varphi\,dV=\int_M K\varphi\,dV$$

$\Rightarrow$ 对号取到 $\Rightarrow k_1=k_2$ on M，M 全脐点曲面 $\Rightarrow M$ 是球面 □

应用2 推论：设 M 是 $\mathbb{E}^3$ 中单射连通凸曲面，若平均曲率为常数，则 M 是球面

证：$\int_M K\varphi\,dV\overset{第一公式}{=\!=\!=}\int_M H\,dV=H\int_M dV\overset{第二公式}{=\!=\!=}H\int_M H\varphi\,dV=\int_M H^2\varphi\,dV$ □

卵形面的支撑函数 $\varphi(p)=-\langle r(p),n(p)\rangle$ "$\varphi\Rightarrow r$"

$g:M\to S^2$ 1:1 $\quad S^2\xrightarrow{g^{-1}}M\xrightarrow{r}\mathbb{E}^3\quad\varphi:S^2\to\mathbb{R}$

选 S^2 上正交活动标架场 $\{e_1,e_2,\overset{n}{e_3}\}$，五个线性1-形式 $\theta^1,\theta^2,\theta_1^3,\theta_2^3,\theta_3^3$

$d\varphi = \varphi_1\theta^1 + \varphi_2\theta^2$

$\| $

$d(-\langle r, n\rangle) = -\langle dr, n\rangle - \langle r, dn\rangle$

在 S^2 上，$dn = d^{\mathbb{R}^3}n = \theta^1 e_1 + \theta^2 e_2 \Rightarrow -\langle r, dn\rangle = -\langle r, e_1\rangle\theta^1 - \langle r, e_2\rangle\theta^2$

$\forall p \in S^2$，$\forall v \in T_pS^2$，找 S^2 上曲线 $c(t)$ $\begin{cases} c(0) = p \\ c'(0) = v \end{cases}$

$d(r\circ g^{-1})(v) = \frac{d}{dt}\big|_{t=0} r\circ g^{-1}(c(t)) = dr((g^{-1})_*(v)) = v \quad \Rightarrow \langle dr, n\rangle = 0$

$(g^{-1})_* \in T_{g^{-1}(p)}M$

联立得 $\varphi_1 = -\langle r, e_1\rangle$，$\varphi_2 = -\langle r, e_2\rangle$

$\Rightarrow r = \langle r, e_1\rangle e_1 + \langle r, e_2\rangle e_2 + \langle r, e_3\rangle e_3$

$= -\varphi_1 e_1 - \varphi_2 e_2 - \varphi e_3$

定理（Cohn-Vossen 1927）$\mathbb{E}^3$ 中两个充分接近的卵形面是合同的，即它们只差 $\mathbb{E}^3$ 中的一个合同变换

证：积分公式 G. Herglotz 1943

$M, \overline{M}$ 卵形面，$f: M \to \overline{M}$ 等距变换。

$\{r; e_1, e_2, e_3 = e_1\wedge e_2\}$（右手系），$\{\overline{r}; \overline{e_1}, \overline{e_2}, \overline{n}\}$（不一定右手系）其中 $\overline{e_1} = f_* e_1$，$\overline{e_2} = f_* e_2$。调整 $r, \overline{r}, n$ 使 $\varphi, \overline{\varphi} > 0$

只需考虑 $n = \overline{e_1}\wedge\overline{e_2}$（即右手系）的情形（否则考虑 $-n$ 与 f 的复合）。此时要证 M、$\overline{M}$ 相差刚体变换

运动方程 $\begin{cases} dr = \omega^1 e_1 + \omega^2 e_2 \\ de_1 = \omega_1^2 e_2 + \omega_1^3 e_3 \\ de_2 = \omega_2^1 e_1 + \omega_2^3 e_3 \\ de_3 = \omega_3^1 e_1 + \omega_3^2 e_2 \end{cases}$，$\begin{cases} d\overline{r} = \overline{\omega^1}\,\overline{e_1} + \overline{\omega^2}\,\overline{e_2} \\ d\overline{e_1} = \overline{\omega_1^2}\,\overline{e_2} + \overline{\omega_1^3}\,\overline{e_3} \\ d\overline{e_2} = \overline{\omega_2^1}\,\overline{e_1} + \overline{\omega_2^3}\,\overline{e_3} \\ d\overline{e_3} = \overline{\omega_3^1}\,\overline{e_1} + \overline{\omega_3^2}\,\overline{e_2} \end{cases}$

由微分方程解的存在唯一性，只需五个 1-形式相对

$\therefore$ 只需证 $\boxed{\omega^1 = f^*\overline{\omega^1},\ \omega^2 = f^*\overline{\omega^2},\ \omega_1^3 = f^*\overline{\omega_1^3},\ \omega_2^3 = f^*\overline{\omega_2^3}}$ if $\det(f_*) > 0$

$\omega^1 = f^*\overline{\omega^1},\ \omega^2 = f^*\overline{\omega^2},\ \omega_1^3 = -f^*\overline{\omega_1^3},\ \omega_2^3 = -f^*\overline{\omega_2^3}$ if $\det(f_*) < 0$

$\det(f_*) > 0$：$f^*\overline{\omega^\alpha}(e_\beta) = \overline{\omega^\alpha}(f_* e_\beta) = \overline{\omega^\alpha}(\overline{e_\beta}) = \delta^\alpha_\beta = \omega^\alpha(e_\beta) \Rightarrow f^*\overline{\omega^1} = \omega^1,\ f^*\overline{\omega^2} = \omega^2 \Rightarrow f^*\overline{\omega_1^2} = \omega_1^2$

$\begin{pmatrix}\omega_1^3 \\ \omega_2^3\end{pmatrix} = \begin{pmatrix}h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\omega^1 \\ \omega^2\end{pmatrix}$，$\begin{pmatrix}\overline{\omega_1^3} \\ \overline{\omega_2^3}\end{pmatrix} = \begin{pmatrix}\overline{h_{11}} & \overline{h_{12}} \\ \overline{h_{21}} & \overline{h_{22}}\end{pmatrix}\begin{pmatrix}\overline{\omega^1} \\ \overline{\omega^2}\end{pmatrix}$ $\quad \begin{pmatrix}f^*\overline{\omega_1^3} \\ f^*\overline{\omega_2^3}\end{pmatrix} = \begin{pmatrix}\overline{h_{11}}\circ f & \overline{h_{12}}\circ f \\ \overline{h_{21}}\circ f & \overline{h_{22}}\circ f\end{pmatrix}\begin{pmatrix}\omega^1 \\ \omega^2\end{pmatrix}$

需证 (i) if $\det(f_*) > 0$，$\begin{pmatrix}h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix} = \begin{pmatrix}\overline{h_{11}}\circ f & \overline{h_{12}}\circ f \\ \overline{h_{21}}\circ f & \overline{h_{22}}\circ f\end{pmatrix}$

(ii) if $\det(f_*) < 0$，$\begin{pmatrix}h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix} = -\begin{pmatrix}\overline{h_{11}}\circ f & \overline{h_{12}}\circ f \\ \overline{h_{21}}\circ f & \overline{h_{22}}\circ f\end{pmatrix}$

引理：设 $\begin{pmatrix}\lambda & \mu\\ \bar\mu & \nu\end{pmatrix}$ 和 $\begin{pmatrix}\bar\lambda & \bar\mu\\ \bar\mu & \bar\nu\end{pmatrix}$ 为两个正定矩阵，满足 $\lambda\nu-\mu^2=\bar\lambda\bar\nu-\bar\mu^2$ $(*)$.

则有 $\det\begin{pmatrix}\bar\lambda-\lambda & \bar\mu-\mu\\ \bar\mu-\mu & \bar\nu-\nu\end{pmatrix}\le 0$，且取"$=$" $\Leftrightarrow \lambda=\bar\lambda,\ \mu=\bar\mu,\ \nu=\bar\nu$

证：$\exists A$ s.t. $A\begin{pmatrix}\bar\lambda-\lambda & \bar\mu-\mu\\ \bar\mu-\mu & \bar\nu-\nu\end{pmatrix}A^T$ 为对角阵，记 $A\begin{pmatrix}\lambda & \mu\\ \bar\mu & \nu\end{pmatrix}A^T=\begin{pmatrix}\lambda^* & \mu^*\\ \bar\mu^* & \nu^*\end{pmatrix}$，$A\begin{pmatrix}\bar\lambda & \bar\mu\\ \bar\mu & \bar\nu\end{pmatrix}A^T=\begin{pmatrix}\bar\lambda^* & \bar\mu^*\\ \bar\mu^* & \bar\nu^*\end{pmatrix}$，故 $\mu^*=\bar\mu^*$

$(*)\Rightarrow \lambda^*\nu^*-(\mu^*)^2=\bar\lambda^*\bar\nu^*-(\bar\mu^*)^2\Rightarrow \lambda^*\nu^*=\bar\lambda^*\bar\nu^*$ $(\star)$

由 $\det\left(A\begin{pmatrix}\bar\lambda-\lambda & \bar\mu-\mu\\ \bar\mu-\mu & \bar\nu-\nu\end{pmatrix}A^T\right)=(\det A)^2\det\begin{pmatrix}\bar\lambda-\lambda & \bar\mu-\mu\\ \bar\mu-\mu & \bar\nu-\nu\end{pmatrix}$

$=\det\begin{pmatrix}\bar\lambda^*-\lambda^* & \bar\mu^*-\mu^*\\ \bar\mu^*-\mu^* & \bar\nu^*-\nu^*\end{pmatrix}=(\bar\lambda^*-\lambda^*)(\bar\nu^*-\nu^*)$

$\overset{(\star)}{=}(\bar\lambda^*-\lambda^*)\left(\frac{\lambda^*\mu^*}{\bar\lambda^*}-\nu^*\right)=\frac{-\nu^*}{\bar\lambda^*}(\bar\lambda^*-\lambda^*)^2\le 0$

$\Rightarrow \det\begin{pmatrix}\bar\lambda-\lambda & \bar\mu-\mu\\ \bar\mu-\mu & \bar\nu-\nu\end{pmatrix}\le 0$

"$=$"成立 $\Leftrightarrow \bar\lambda^*=\lambda^*$ $(\Rightarrow \bar\nu^*=\nu^*)$ $\Leftrightarrow A\begin{pmatrix}\lambda & \mu\\ \bar\mu & \nu\end{pmatrix}A^T=A\begin{pmatrix}\bar\lambda & \bar\mu\\ \bar\mu & \bar\nu\end{pmatrix}A^T\Leftrightarrow\begin{pmatrix}\lambda & \mu\\ \bar\mu & \nu\end{pmatrix}=\begin{pmatrix}\bar\lambda & \bar\mu\\ \bar\mu & \bar\nu\end{pmatrix}$ □

微分1-形式 $(r,n,dn)\overset{局部}{=}(r,e_3,\omega_3^1e_1+\omega_3^2e_2)$

考虑 M 上曲面以上的微分1-形式 $\phi=(r,n,f^*\omega_3^1e_1+f^*\omega_3^2e_2)$

问题：① ϕ 是否整体定义？

② 若 $\int_M d\phi=0\Rightarrow$?

计算 $d\phi=(dr,n,f^*\omega_3^1e_1+f^*\omega_3^2e_2)+(r,dn,f^*\omega_3^1e_1+f^*\omega_3^2e_2)$

$+(r,n,d(f^*\omega_3^1e_1+f^*\omega_3^2e_2))=①+②+③$

设 M 上另有标架 $\{r;e_1',e_2',e_3'\}$，其中 $\begin{pmatrix}e_1'\\ e_2'\end{pmatrix}=\underbrace{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}}_{R_\theta}\begin{pmatrix}e_1\\ e_2\end{pmatrix}$

$\bar M$ 上相差 θ 处，有

$\begin{pmatrix}\bar e_1'\\ \bar e_2'\end{pmatrix}=\begin{pmatrix}f_*e_1'\\ f_*e_2'\end{pmatrix}=R_\theta\begin{pmatrix}f_*e_1\\ f_*e_2\end{pmatrix}=R_\theta\begin{pmatrix}\bar e_1\\ \bar e_2\end{pmatrix}$

$\begin{pmatrix}\bar\omega_3^{1'}\\ \bar\omega_3^{2'}\end{pmatrix}\overset{\bar e_3'=\bar e_3}{\underline{\qquad\qquad}}\begin{pmatrix}\langle d\bar e_3,\bar e_1'\rangle\\ \langle d\bar e_3,\bar e_2'\rangle\end{pmatrix}=R_\theta\begin{pmatrix}\langle d\bar e_3,\bar e_1\rangle\\ \langle d\bar e_3,\bar e_2\rangle\end{pmatrix}=R_\theta\begin{pmatrix}\bar\omega_3^1\\ \bar\omega_3^2\end{pmatrix}\Rightarrow\begin{pmatrix}f^*\bar\omega_3^{1'}\\ f^*\bar\omega_3^{2'}\end{pmatrix}=R_\theta\begin{pmatrix}f^*\bar\omega_3^1\\ f^*\bar\omega_3^2\end{pmatrix}$

$(f^*\bar\omega_3^{1'}\ \ f^*\bar\omega_3^{2'})\begin{pmatrix}e_1'\\ e_2'\end{pmatrix}=(f^*\omega_3^1\ \ f^*\omega_3^2)\underbrace{R_\theta^TR_\theta}_{I}\begin{pmatrix}e_1\\ e_2\end{pmatrix}$

∴ ϕ 不依赖于标架的选取，故是整体定义

$①=(\omega^1e_1+\omega^2e_2,n,f^*\bar\omega_3^1e_1+f^*\bar\omega_3^2e_2)$

$=\omega^1\wedge f^*\bar\omega_2^3+f^*\bar\omega_1^3\wedge\omega^2$

$=\omega^1\wedge(\bar h_{121}\circ f\,\omega^1+\bar h_{122}\circ f\cdot\omega^2)+(\bar h_{111}\circ f\omega^1+\bar h_{122}\circ f\omega^2)\wedge\omega^2$

$=(\bar h_{22}^2\circ f+\bar h_{11}^2\circ f)\,\omega^1\wedge\omega^2$

$=2H\circ f\,\omega^1\wedge\omega^2$

② $= (r,\ \omega_3^1 e_1 + \omega_3^2 e_2,\ f^*\overline{\omega_3^1} e_1 + f^*\overline{\omega_3^2} e_2)$

$= -\varphi(\omega_1^3 \wedge f^*\overline{\omega_2^3} + f^*\overline{\omega_1^3} \wedge \omega_2^3)$

$= -\varphi\left[\det\begin{pmatrix} h_{11} & h_{12} \\ \overline{h_{12}}\circ f & \overline{h_{22}}\circ f\end{pmatrix} + \det\begin{pmatrix} \overline{h_{11}}\circ f & \overline{h_{12}}\circ f \\ h_{21} & h_{22}\end{pmatrix}\right]\omega^1\wedge\omega^2$

$= -\varphi\left[2K - \det\begin{pmatrix} h_{11}-\overline{h_{11}}\circ f & h_{12}-\overline{h_{12}}\circ f \\ h_{21}-\overline{h_{21}}\circ f & h_{22}-\overline{h_{22}}\circ f\end{pmatrix}\right]\omega^1\wedge\omega^2$

$\det\begin{pmatrix} h_{11}-\overline{h_{11}}\circ f & h_{12}-\overline{h_{12}}\circ f \\ h_{21}-\overline{h_{21}}\circ f & h_{22}-\overline{h_{22}}\circ f\end{pmatrix} = \det\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22}\end{pmatrix} - \det\begin{pmatrix} h_{11} & h_{12} \\ \overline{h_{21}}\circ f & \overline{h_{22}}\circ f\end{pmatrix}$
$- \det\begin{pmatrix} \overline{h_{11}}\circ f & \overline{h_{12}}\circ f \\ h_{21} & h_{22}\end{pmatrix} + \underbrace{\det\begin{pmatrix} \overline{h_{11}}\circ f & \overline{h_{12}}\circ f \\ \overline{h_{21}}\circ f & \overline{h_{22}}\circ f\end{pmatrix}}_{K\circ f = K}$

③ $= (r,\ n,\ de_1\wedge(f^*\omega_3^1) + e_1 d(f^*\overline{\omega_3^1}) + de_2\wedge f^*\overline{\omega_3^2} + e_2 d(f^*\overline{\omega_3^2}))$

性质：M 上一个 1-形式 $\bar\omega$ 有 $\boxed{d(f^*\bar\omega) = f^*(d\bar\omega)}$ （练习）

$= (r,\ n,\ (\omega_1^2 e_2 + \omega_1^3 e_3)\wedge(f^*e_3^1) + e_1 f^*(d\overline{\omega_3^1}) + (\omega_2^1 e_1 + \omega_2^3 e_3)\wedge f^*\overline{\omega_3^2} + e_2 f^*(d\overline{\omega_3^2}))$

$f^*(\overline{\omega_3^2}\wedge\overline{\omega_2^1}) = f^*\overline{\omega_3^2}\wedge f^*\overline{\omega_2^1} = f^*\overline{\omega_3^2}\wedge\omega_2^1$ $\qquad f^*(\overline{\omega_3^1}\wedge\overline{\omega_1^2}) = f^*\overline{\omega_3^1}\wedge f^*\overline{\omega_1^2} = f^*\overline{\omega_3^1}\wedge\omega_1^2$

$= (r,\ n,\ *e_3)$

$= 0$

总结，$0 = \int_M d\delta = 2\int_M \overline{H}\circ f\, dV - 2\underbrace{\int_M K\varphi dV}_{\int_M H dV} + \int_M \varphi \det\begin{pmatrix} h_{11}-\overline{h_{11}}\circ f & h_{12}-\overline{h_{12}}\circ f \\ h_{21}-\overline{h_{21}}\circ f & h_{22}-\overline{h_{22}}\circ f\end{pmatrix} dV$

调整使 $\varphi > 0$，h，$\overline{h}$ 正定（$\overline{h}$ 正定蕴 $\det(f_*) > 0$）

$\Rightarrow 2\int_M (H - \overline{H}\circ f)\, dV = \int_M \varphi \det(\quad)\, dV \le 0$

同理，$2\int_M (\overline{H} - H\circ f^{-1})\, dV \le 0 \Rightarrow 2\int_M (\overline{H}\circ f - H)\circ f^{-1} dV = 2\int_M (\overline{H}\circ f - H)\, dV \le 0$

$\Rightarrow \int_M (H - \overline{H}\circ f) dV = 0$

$\Rightarrow \det(\quad) = 0$

$\Rightarrow h_{\alpha\beta} = \overline{h_{\alpha\beta}}\circ f,\ \forall \alpha, \beta$

$\det(f_*) < 0$. $\begin{pmatrix} -\overline{h_{11}}\circ f & -\overline{h_{12}}\circ f \\ -\overline{h_{21}}\circ f & -\overline{h_{22}}\circ f\end{pmatrix}$ 正定

考查 $\det\begin{pmatrix} h_{11}+\overline{h_{11}}\circ f & h_{12}+\overline{h_{12}}\circ f \\ h_{21}+\overline{h_{21}}\circ f & h_{22}+\overline{h_{22}}\circ f\end{pmatrix} = A + 2K$

$0 = 2\int_M \overline{H}\circ f\, dV + 2\underbrace{\int_M K\varphi dV}_{\int_M H dV} - \int_M \varphi \det(h_{\alpha\beta} + \overline{h_{\alpha\beta}}\circ f)\, dV$

$\Rightarrow 2\int_M (\overline{H}\circ f + H)\, dV = \int_M \underbrace{\tilde\varphi}_{\ge 0} \det(h_{\alpha\beta} + \overline{h_{\alpha\beta}}\circ f)\, dV \le 0$

对称地，$2\int_M (H\circ f^{-1} + \overline{H})\, d\bar V = \int_M \underbrace{\bar\varphi}_{\le 0} \det(h_{\alpha\beta}\circ f^{-1} + \overline{h_{\alpha\beta}})\, d\bar V \ge 0$ ...

$2\int_M (H + \overline{H}\circ f) dV$

□

$K\circ g^{-1}$：球面上的函数

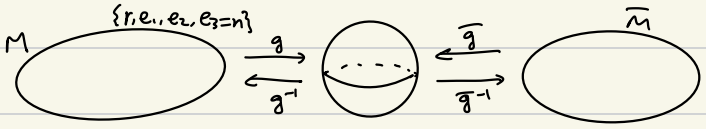
Minkowski 问题：给定 $k: S\to\mathbb{R}_{>0}$ 光滑，是否存在一个卵形面 s.t. $K\circ g^{-1}=k$？

存在性：Nirenberg 1953　　Pogorelov 1952

唯一性？

$\xrightarrow{K,g\ \ \bar K,\bar g}$ 定理 设 $M,\bar M$ 为 $\mathbb{E}^3$ 中两个卵形面，若存在函数 $k: S^2\to\mathbb{R}_{>0}$ s.t. $K\circ g^{-1}=k=\bar K\circ\bar g^{-1}$，

则 $M,\bar M$ 相差一个平移



记：　M.　$\omega^1,\omega^2,\omega_1^3,\omega_2^3$

S^2.　$\theta^1,\theta^2,\theta_1^3,\theta_2^3$

$\bar M$.　$\bar\omega^1,\bar\omega^2,\bar\omega_1^3,\bar\omega_2^3$

需证 $\begin{cases}\omega^1=(\bar g^{-1}\circ g)^*\bar\omega^1\\ \omega^2=(\bar g^{-1}\circ g)^*\bar\omega^2\end{cases}$，$\begin{cases}\omega_1^3=(\bar g^{-1}\circ g)^*\bar\omega_1^3\\ \omega_2^3=(\bar g^{-1}\circ g)^*\bar\omega_2^3\end{cases}$

注意到 $\begin{cases}(\bar g^{-1}\circ g)^*=g^*\circ(\bar g^{-1})^*\\ (g^*)^{-1}=(g^{-1})^*\end{cases}$，即证 (I) $\begin{cases}(g^{-1})^*\omega^1=(\bar g^{-1})^*\bar\omega^1\\ (g^{-1})^*\omega^2=(\bar g^{-1})^*\bar\omega^2\end{cases}$　(II) $\begin{cases}(g^{-1})^*\omega_1^3=(\bar g^{-1})^*\bar\omega_1^3\\ (g^{-1})^*\omega_2^3=(\bar g^{-1})^*\bar\omega_2^3\end{cases}$

(II)：M 上 $\frac{d}{dt}e_3(c(t))=\frac{d}{dt}e_3(g^{-1}(c(t)))$

$\Rightarrow\langle\frac{d}{dt}e_3(c(t)),e_\alpha(c(t))\rangle=\langle\frac{d}{dt}e_3(g^{-1}(c(t))),e_\alpha(g^{-1}(c(t)))\rangle$，$\alpha=1,2$　（$\frac{d}{dt}\big|_{t=0}g^{-1}(c(t))=v$）

$\Rightarrow\omega_3^\alpha((g^{-1})_*v)=\theta_3^\alpha(v)$，$\forall p\in S^2$，$\forall v\in T_pS^2\Rightarrow(g^{-1})^*\omega_3^\alpha=\theta_3^\alpha$，$\alpha=1,2$

(I)：M 上 $\begin{pmatrix}\omega_1^3\\ \omega_2^3\end{pmatrix}=\begin{pmatrix}h_{11}&h_{12}\\ h_{21}&h_{22}\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$　$K=\det\begin{pmatrix}h_{11}&h_{12}\\ h_{21}&h_{22}\end{pmatrix}>0$

逆矩阵 $\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}=\begin{pmatrix}\lambda_{11}&\lambda_{12}\\ \lambda_{21}&\lambda_{22}\end{pmatrix}\begin{pmatrix}\omega_1^3\\ \omega_2^3\end{pmatrix}$　总可假定 $(\lambda_{\alpha\beta})>0$

拉到 S^2 上有 $\begin{pmatrix}(g^{-1})^*\omega^1\\ (g^{-1})^*\omega^2\end{pmatrix}=\begin{pmatrix}\lambda_{11}\circ g^{-1}&\lambda_{12}\circ g^{-1}\\ \lambda_{21}\circ g^{-1}&\lambda_{22}\circ g^{-1}\end{pmatrix}\begin{pmatrix}(g^{-1})^*\omega_1^3\\ (g^{-1})^*\omega_2^3\end{pmatrix}$（$=\theta_1^3,\theta_2^3$）

类似地 $\begin{pmatrix}(\bar g^{-1})^*\bar\omega^1\\ (\bar g^{-1})^*\bar\omega^2\end{pmatrix}=\begin{pmatrix}\bar\lambda_{11}\circ\bar g^{-1}&\lambda_{12}\circ\bar g^{-1}\\ \lambda_{21}\circ\bar g^{-1}&\lambda_{22}\circ\bar g^{-1}\end{pmatrix}\begin{pmatrix}\theta_1^3\\ \theta_2^3\end{pmatrix}$

$\Rightarrow$ 只需证 $\lambda_{\alpha\beta}\circ g^{-1}=\bar\lambda_{\alpha\beta}\circ\bar g^{-1}$，$\alpha,\beta=1,2$

$\det(\lambda_{\alpha\beta}\circ g-\bar\lambda_{\alpha\beta}\circ g^{-1})\not\equiv 0$

积分公式：考虑 1-形式 $\psi=(r\circ g^{-1},\bar r\circ\bar g^{-1},d(\bar r\circ\bar g^{-1}))$，$S^2$ 上整体定义

计算 $d\psi=(d(r\circ g^{-1}),\bar r\circ\bar g^{-1},d(\bar r\circ\bar g^{-1}))+(r\circ g^{-1},d(\bar r\circ\bar g^{-1}),d(\bar r\circ\bar g^{-1}))=①+②$

$d(r\circ g^{-1})(v)=\frac{d}{dt}\big|_{t=0}(r\circ g^{-1})(c(t))=\frac{d}{dt}\big|_{t=0}r(g^{-1}(c(t)))=dr((g^{-1})_*(v))=(g^{-1})^*dr(v)$

$\Rightarrow d(r\circ g^{-1})=(g^{-1})^*dr=(g^{-1})^*(\omega^1e_1+\omega^2e_2)=(g^{-1})^*\omega^1e_1+(g^{-1})^*\omega^2e_2$（平移到 S^2 上 $e_i\circ g^{-1}$）

$①=((g^{-1})^*\omega^1e_1+(g^{-1})^*\omega^2e_2,\bar r\circ\bar g^{-1},(\bar g^{-1})^*\bar\omega^1e_1+(\bar g)^{-1}\bar\omega^2e_2)$

$$= (\bar{g}^{-1})^*\omega^1 \wedge (\bar{g})^{-1}\bar{\omega}^2(\underbrace{e_1, \bar{r}\circ\bar{g}^{-1}, e_2}_{-\langle \bar{r}\circ\bar{g}^{-1}, e_1, e_2\rangle}) + (\bar{g}^{-1})^*\omega^2 \wedge (\bar{g}^{-1})^*\bar{\omega}^1(\underbrace{e_2, \bar{r}\circ\bar{g}^{-1}, e_1}_{-\bar{\varphi}\circ\bar{g}^{-1}})$$

$$-\langle \bar{r}\circ\bar{g}^{-1}, e_1\wedge e_2\rangle = -\langle \bar{r}\circ\bar{g}^{-1}, \bar{n}\circ\bar{g}^{-1}\rangle = \bar{\varphi}\circ\bar{g}^{-1}$$

$$= \bar{\varphi}\circ\bar{g}^{-1}\left((\bar{g}^{-1})^*\omega^1 \wedge (\bar{g}^{-1})^*\omega^2 - (\bar{g}^{-1})^*\omega^2 \wedge (\bar{g}^{-1})^*\omega^1\right)$$

$$= \bar{\varphi}\circ\bar{g}^{-1}\left(\underbrace{\det\begin{pmatrix}\lambda_{11}\circ g^{-1} & \lambda_{12}\circ g^{-1}\\ \lambda_{21}\circ g^{-1} & \lambda_{22}\circ g^{-1}\end{pmatrix}}_{c} + \det\begin{pmatrix}\overline{\lambda_{11}\circ g^{-1}} & \overline{\lambda_{12}\circ g^{-1}}\\ \overline{\lambda_{21}\circ g^{-1}} & \overline{\lambda_{22}\circ g^{-1}}\end{pmatrix}\right)\theta_1^3\wedge\theta_2^3$$

$$\det(\lambda_{\alpha\beta}\circ g^{-1} - \overline{\lambda_{\alpha\beta}\circ\bar{g}^{-1}}) = -c + \det(\lambda_{\alpha\beta}\circ g^{-1}) + \det(\overline{\lambda_{\alpha\beta}\circ g^{-1}})$$

$$= -\varphi + \frac{1}{k\circ g^{-1}} + \frac{1}{\bar{k}\circ\bar{g}^{-1}} = -\varphi + \frac{2}{k}$$

$$② = (r\circ g^{-1}, (\bar{g}^{-1})^*\bar{\omega}^1 e_1 + (\bar{g})^{-1}\bar{\omega}^2 e_2, (\bar{g}^{-1})^*\bar{\omega}^1 e_1 + (\bar{g})^{-1}\bar{\omega}^2 e_2)$$

$$= (\bar{g}^{-1})^*\bar{\omega}^1 \wedge (\bar{g}^{-1})^*\bar{\omega}^2(\underbrace{r\circ g^{-1}, e_1, e_2}_{-\varphi\circ g^{-1}}) + (\bar{g}^{-1})^*\bar{\omega}^2 \wedge (\bar{g}^{-1})^*\bar{\omega}^1(\underbrace{r\circ g^{-1}, e_2, e_1}_{\varphi\circ g^{-1}})$$

$$= -2\varphi\circ g^{-1}\left((\bar{g}^{-1})^*\bar{\omega}^1 \wedge (\bar{g}^{-1})^*\bar{\omega}^2\right) - \varphi\circ g^{-1}$$

$$= -2\varphi\circ g^{-1}\frac{1}{\bar{k}\circ\bar{g}^{-1}} = -\varphi\circ g^{-1}\frac{2}{k}$$

$$d\mu = \left[\frac{2}{k}(\bar{\varphi}\circ\bar{g}^{-1} - \varphi\circ g^{-1}) - \bar{\varphi}\circ\bar{g}^{-1}\det(\lambda_{\alpha\beta}\circ g^{-1} - \overline{\lambda_{\alpha\beta}\circ\bar{g}^{-1}})\right]\theta_1^3\wedge\theta_2^3$$

积分，$\int_{S^2}\frac{2}{k}(\bar{\varphi}\circ\bar{g}^{-1} - \varphi\circ g^{-1})\theta_1^3\wedge\theta_2^3 = \int_{S^2}\underbrace{\bar{\varphi}\circ\bar{g}^{-1}}_{>0}\underbrace{\det(\lambda_{\alpha\beta}\circ g^{-1} - \overline{\lambda_{\alpha\beta}\circ\bar{g}^{-1}})}_{\leq 0}\theta_1^3\wedge\theta_2^3 \leq 0$$

对称地，$\int_{S^2}\frac{2}{k}(\varphi\circ g^{-1} - \bar{\varphi}\circ\bar{g}^{-1})\theta_1^3\wedge\theta_2^3 \leq 0 \Rightarrow = 0$ □

Liebmann定理 M $\mathbb{E}^3$中嵌入(完备)连通常高斯曲率曲面必为球面

完备曲面

设M为$\mathbb{E}^3$中C^∞光滑曲面，设P, Q为M上任两点，定义P, Q之间C^∞距离为$d(P,Q) = \inf_{\gamma\text{为连接}P,Q\text{的}C^\infty\text{分段光滑曲线}} l(\gamma)$

引理1：$\mathbb{E}^3$中光滑连通曲面M上任两点都可由曲面上C^∞分段光滑曲线相连接

证：给定P，考查$A_P := \{Q\in M \mid \text{在}M\text{上存在分段光滑曲线连接}P, Q\}$，只证$A_P$既开且闭，非空 □

引理2：d是度量

证：只需证当$P\neq Q$，则$d(P,Q)>0$

总是存在充分小的 ε s.t. 以 p 为心，ε 为半径的测地球 $B(0,\varepsilon)$ s.t. $Q\notin B(0,\varepsilon)$

$\forall$ 连接 P,Q 的分段光滑曲线 $\gamma:[0,l]\to M$，令 $t_0=\inf\{t\in[0,l]\mid\gamma(t)\notin B(0,\varepsilon)\}$

有 $l(\gamma)\geq l(\gamma|_{[0,t_0]})\geq\varepsilon>0$

$\Rightarrow d(P,Q)\geq\varepsilon>0$ □

定义. $\mathbb{E}^3$ 中曲面 M 称为完备的若 (M,d) 作为度量空间完备，即关于 d 的任何 Cauchy 列均收敛

性质：若 M 是 $\mathbb{E}^3$ 中完备曲面，则其上测地线可无限延伸，i.e. $\forall p\in M$，指数映射 $\exp_p:T_pM\to M$ 定义在整个切平面上 $\gamma_v:(-\infty,+\infty)\to M$ （→ 测地完备性）

证：M 上任何测地线可"无穷次延伸"若不能无限延伸，由完备性知有极限点 Q，…… □

事实上，测地完备性 $\Leftrightarrow\forall p\in M$，$\exp_p$ 定义在整个 T_pM 上 $\Leftrightarrow\exists p\in M$，$\exp_p$ 定义在整个 T_pM 上

⇕ Hopf-Rinow 定理

度量完备

定理 (Hilbert 1901) $\mathbb{E}^3$ 中不存在常负 Gauß 曲率的完备曲面

E. Holmgren 1902 不同的证明

注记：总可以假设 $K\equiv-1$ ($r\to cr$，$K=\frac{LN-M^2}{EF-G^2}=\frac{c^2\cdots}{c^4\cdots}=\frac{1}{c^2}\cdots$)

定义：$r:(a,b)\times(c,d)\to M$，$(u,v)\mapsto r(u,v)$ 的参数化称为一个 Chebyshev 网，如果所有参数曲线是弧长参数的 (i.e. $|r_u|=|r_v|=1$)，称 r 为一个渐近 Chebyshev 网，如果一个 Chebyshev 网的每条参数曲线均为渐近线

引理3：设 M 为 $\mathbb{E}^3$ 中 $K\equiv-1$ 的曲面，对 $\forall p\in M$，存在曲面片 $r:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\to M$，$r(0,0)=p$，其中 r 为渐近 Chebyshev 网

证：由 $K\equiv-1=k_1k_2$ ($k_1>0$, $k<0$) 知 M 无脐点，在 p 的邻域存在曲面片 $r:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\to M$，$r(0,0)=p$ 且 r_u, r_v 为主方向 (r 是曲率线网)，$\langle r_u,r_v\rangle\equiv0$

$e_1=\frac{r_u}{|r_u|}$，$e_2=\frac{r_v}{|r_v|}$，$e_3=e_1\wedge e_2$，一个渐近方向 $\cos\varphi e_1+\sin\varphi e_2$，$\varphi\in(0,\frac{\pi}{2})$

Euler: $k_1\cos^2\varphi+k_2\sin^2\varphi=0\Rightarrow k_1=\tan\varphi$

希望重新参数化 $\begin{cases}\xi=\xi(u,v)\\ \eta=\eta(u,v)\end{cases}$ 使得 $\begin{cases}r_\xi=\cos\varphi\frac{r_u}{|r_u|}=\frac{\cos\varphi}{\sqrt{E}}r_u\\ r_\eta=\sin\varphi\frac{r_v}{|r_v|}=\frac{\sin\varphi}{\sqrt{G}}r_v\end{cases}$

希望得出 $\begin{cases}d\xi=\frac{\sqrt{E}}{\cos\varphi}du\\ d\eta=\frac{\sqrt{G}}{\sin\varphi}dv\end{cases}$ 可积性条件：$d^2=0$ $\begin{cases}0=d^2\xi=\left(\frac{\sqrt{E}}{\cos\varphi}\right)_v dv\wedge du\\ 0=d^2\eta=\left(\frac{\sqrt{G}}{\sin\varphi}\right)_u du\wedge dv\end{cases}$

$\{e_1,e_2,e_3\}$ $r_u=\sqrt{E}e_1$, $r_v=\sqrt{G}e_2$

$\begin{cases}\omega^1=\sqrt{E}du\\ \omega^2=\sqrt{G}dv\end{cases}$ $\omega_1^2=-\frac{(\sqrt{E})_v}{\sqrt{EG}}\omega^1+\frac{(\sqrt{G})_u}{\sqrt{EG}}\omega^2=-\frac{(\sqrt{E})_v}{\sqrt{G}}du+\frac{(\sqrt{G})_u}{\sqrt{E}}dv$ $\begin{cases}\omega_1^3=k_1\omega^1\\ \omega_2^3=k_2\omega^2\end{cases}$

Codazzi 方程 $\begin{cases}d\omega_1^3=\omega_1^2\wedge\omega_2^3 & \langle1\rangle\\ d\omega_2^3=\omega_2^1\wedge\omega_1^3 & \langle2\rangle\end{cases}$

看方程$\langle1\rangle$，左端 $d\omega_1^3=d(k_1\omega^1)=d(k_1\sqrt{E}du)=(k_1\sqrt{E})_v dv\wedge du$

右端 $\omega_1^2\wedge\omega_2^3=k_2\omega_1^2\wedge\omega^2=k_2\left(-\frac{(\sqrt{E})_v}{\sqrt{G}}du+\frac{(\sqrt{G})_u}{\sqrt{E}}dv\right)\wedge(\sqrt{G}dv)=-k_2(\sqrt{E})_v du\wedge dv$

$\Rightarrow (k_1\sqrt{E})_v-k_2(\sqrt{E})_v=0$

$\Rightarrow (k_1)_v\sqrt{E}+(k_1-k_2)(\sqrt{E})_v=0$ $(*)$

同理 $(k_2)_u\sqrt{G}-(k_1-k_2)(\sqrt{G})_u=0$ $(**)$

代入 $k_1=\tan\varphi$, $k_2=-\cot\varphi$

$(*)\Rightarrow 0=\frac{\varphi_v}{\cos^2\varphi}\sqrt{E}+\frac{(\sqrt{E})_v}{\sin\varphi\cos\varphi}=\frac{1}{\sin\varphi}\left(\frac{\varphi_v\sin\varphi\sqrt{E}}{\cos^2\varphi}+\frac{(\sqrt{E})_v\cos\varphi}{\cos^2\varphi}\right)=\frac{1}{\sin\varphi}\left(\frac{\sqrt{E}}{\cos\varphi}\right)_v$

$(**)\Rightarrow 0=-\frac{1}{\cos\varphi}\left(\frac{\sqrt{E}}{\sin\varphi}\right)_u$

也即 Codazzi 方程 $\Rightarrow\left(\frac{\sqrt{E}}{\cos\varphi}\right)_v=0$, $\left(\frac{\sqrt{G}}{\sin\varphi}\right)_u=0$，因此$(*)$式可积性条件满足

再对(ξ,η)做参数变换 $\begin{cases}\xi=s+t\\ \eta=s-t\end{cases}$

$\begin{cases}\bar r_s=r_\xi\frac{d\xi}{ds}+r_\eta\frac{d\eta}{ds}=r_\xi+r_\eta\\ \bar r_t=r_\xi\frac{d\xi}{dt}+r_\eta\frac{d\eta}{dt}=r_\xi-r_\eta\end{cases}\Rightarrow \bar r(s,t)$ 是渐近 Chebyshev 网 □

$r=r(\xi,\eta)$ 正交活动标架 $\{e_1,e_2,e_3\}$ $\omega^1,\omega^2,\omega_1^2,\omega_1^3,\omega_2^3$ $\begin{cases}\omega^1=\cos\varphi d\xi\\ \omega^2=\sin\varphi d\eta\end{cases}$ $\begin{cases}\omega_1^3=k_1\omega^1=\tan\varphi\omega^1=\sin\varphi d\xi\\ \omega_2^3=k_2\omega^2=-\cot\varphi\omega^2=-\cos\varphi d\eta\end{cases}$

$\omega_1^2=-\frac{(\sqrt{E})_\eta}{\sqrt{EG}}\omega^1+\frac{(\sqrt{G})_\xi}{\sqrt{EG}}\omega^2=-\frac{(\cos\varphi)_\eta}{\cos\varphi\sin\varphi}\omega^1+\frac{(\sin\varphi)_\xi}{\cos\varphi\sin\varphi}\omega^2=\frac{\varphi_\eta}{\cos\varphi}\omega^1+\frac{\varphi_\xi}{\sin\varphi}\omega^2=\varphi_\eta d\xi+\varphi_\xi d\eta$

- Codazzi 方程 $\begin{cases}d\omega_1^3=\omega_1^2\wedge\omega_2^3\\ d\omega_2^3=\omega_2^1\wedge\omega_1^3\end{cases}\Leftrightarrow(\sin\varphi)_\eta d\eta\wedge d\xi=(\varphi_\eta d\xi+\varphi_\xi d\eta)\wedge(-\cos\varphi d\eta)=-\cos\varphi\,\varphi_\eta d\xi\wedge d\eta$ 自然成立

  还没有提供约束条件

· Gauss方程 $d\omega_1^2=\omega_1^3\wedge\omega_3^2$

$$右=\sin\varphi\, d\xi\wedge\cos\varphi\, d\eta=\sin\varphi\cos\varphi\, d\xi\wedge d\eta=\frac{1}{2}\sin(2\varphi)\, d\xi\wedge d\eta$$

$$左=d(\varphi_\eta d\xi+\varphi_\xi d\eta)=(\varphi_\eta)_\eta\, d\eta\wedge d\xi+(\varphi_\xi)_\xi d\xi\wedge d\eta=(\varphi_{\xi\xi}-\varphi_{\eta\eta})\, d\xi\wedge d\eta$$

Gauss方程 $\Leftrightarrow$ $\boxed{(2\varphi)_{\xi\xi}-(2\varphi)_{\eta\eta}=\sin 2\varphi}$ $\quad \alpha_{\xi\xi}-\alpha_{\eta\eta}=\sin\alpha \quad \alpha\in(0,\pi)$ sine-Gordon equation

($\varphi_{tt}-\varphi_{xx}+\varphi=0$ Klein-Gordon equation)

$$\alpha(\xi,\eta)=\alpha(\xi(s,t),\eta(s,t))=\bar{\alpha}(s,t)$$

$$\bar{\alpha}_{st}=(\bar{\alpha}_s)_t=\left(\alpha_\xi\frac{d\xi}{ds}+\alpha_\eta\frac{d\eta}{ds}\right)_t=(\alpha_\xi+\alpha_\eta)_t=\frac{\partial\alpha_\xi}{\partial\xi}-\frac{\partial\alpha_\xi}{\partial\eta}+\frac{\partial\alpha_\eta}{\partial\xi}-\frac{\partial\alpha_\eta}{\partial\eta}=\alpha_{\xi\xi}-\alpha_{\eta\eta}$$

∴ $\vec{r}(s,t)$ 满足in约束条件 (sine-Gordon方程) 为 $\alpha_{st}=\sin\alpha$, $\alpha=2\varphi$

定理in证明：先证 E^3 中 $K\equiv-1$ in完备曲面上存在渐近 Chebyshev 网，即

$$f:\mathbb{R}^2\to M \quad (s,t)\mapsto f(s,t)$$

s.t. $s\longmapsto f(s,t_0)$ ($s\in\mathbb{R}$) 是渐近线，$\forall t_0\in\mathbb{R}$

$t\longmapsto f(s_0,t)$ ($t\in\mathbb{R}$) 是渐近线，$\forall s_0\in\mathbb{R}$

构造：① $\forall p\in M$，$v\in T_pM$ 为一个单位渐近方向，完备性 ⟹ (仿照测地线片版本in证明，留作练习)

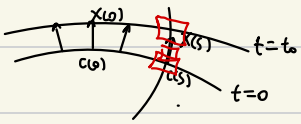
$\exists\, c:(-\infty,\infty)\to M$ s.t. $c(0)=p$, $c'(0)=v$ $\quad c$ 为弧长参数渐近线

② 在每一点 $c(s)\in M$ 处取与 $c(s)$ 线性无关in单位渐近方向 $X(s)$，

构造 $f:\mathbb{R}^2\to M$ $\quad f(0,0)=c(0)=p$

$f(s,0)=c(s)$, $\forall s\in\mathbb{R}$

$\forall s,t$, $f(s,t)=\gamma_s(t)$



问题：给定 t_0，$s\mapsto f(s,t_0)$ 是否为弧长参数渐近线？

只须证 $\forall (s_0,s_1)\in\mathbb{R}^2$，存在小区间 $(s_0-\varepsilon_0,s_0+\varepsilon_0)$ 使得 $s\mapsto f(s,t_0)$ $s\in(s_0-\varepsilon_0,s_0+\varepsilon_0)$ 为弧长参数渐近线。

紧合 $\{f(s_0,t)\in M\mid 0\le t\le t_0\}$ 是紧in，所以存在有限多in曲面片 $\underset{\varepsilon_0}{\square}$, …, $\underset{\varepsilon_n}{\square}$ 覆盖上述紧合，且 $\square_i\cap\square_{i+1}\neq\emptyset$

$\exists\,\varepsilon=\min\{\varepsilon_0,\cdots,\varepsilon_n\}>0$ s.t. $s\mapsto f(s,t_0)$, $s\in(s_0-\varepsilon,s_0+\varepsilon)$ 为弧长参数渐近线，得证

∴ $\exists$ 参数 $\alpha:\mathbb{R}^2\to(0,\pi)$ 满足 $\alpha_{st}=\sin\alpha$

引理：上述函数不存在

证：$\alpha\in(0,\pi)\Rightarrow\sin\alpha>0\Rightarrow t\mapsto\alpha_s(t)$ 是严格增函数，特别地 α_s 不恒为0，不妨 $\alpha_s(0,0)\neq0$，

由于 $\left.\frac{\partial^2}{\partial s\partial t}\alpha(s,t)\right|_{(0,0)}=\left.\frac{\partial^2}{\partial s\partial t}\alpha(-s,-t)\right|_{(0,0)}$，不妨 $\alpha_s(0,0)>0$

$\Rightarrow \exists s_3>0$ 使得 $\alpha_s(s,0)$ 在 $s\in[0,s_3]$ 上严格大于 0 $\Rightarrow \alpha_s$ 在 $[0,s_3]\times[0,+\infty)$ 上严格大于 0

存在 $0<s_1<s_2<s_3$ 使得 $s\in[s_1,s_2]$ 时 $\alpha(s,0)\in[\alpha(s_1,0),\alpha(s_2,0)]\subseteq(\delta,\pi-\delta)$

定义 $\delta=\min\{\alpha(s_1,0)-\alpha(0,0),\ \alpha(s_3,0)-\alpha(s_2,0)\}>0 \Rightarrow \sin\alpha(s,0)\geq\sin\delta>0$ for $s\in[s_1,s_2]$

对 $\forall t>0$ 有 $\alpha(s_1,t)-\alpha(0,t)=\int_0^{s_1}\alpha_s(s,t)\,ds \geq \int_0^{s_1}\alpha_s(s,0)\,ds=\alpha(s_1,0)-\alpha(0,0)$

$\alpha(s_3,t)-\alpha(s_2,t)\geq\alpha(s_3,0)-\alpha(s_2,0)$

因为 $\alpha(s,t)\in[\alpha(s_1,t),\alpha(s_2,t)]$, $\forall s\in[s_1,s_2]$, $\forall t>0$

$\Rightarrow \alpha(s,t)\in(\delta,\pi-\delta)$, $\forall s\in[s_1,s_2]$, $t>0$

积分 $\int_0^T\int_{s_1}^{s_2}\frac{\partial^2\alpha}{\partial s\partial t}\,ds\,dt=\int_0^T\int_{s_1}^{s_2}\sin\alpha(s,t)\,ds\,dt\geq\sin\delta\,(s_2-s_1)\cdot T$ 无界　　矛盾

$\int_0^T(\alpha_t(s_2,t)-\alpha_t(s_1,t))\,dt$

$\alpha(s_2,T)-\alpha(s_2,0)-\alpha(s_1,T)+\alpha(s_1,0)$ 有界

□

$\mathbb{E}^3$ 中是否存在高斯曲率 $K<0$ 的完备曲面？

单叶双曲面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

旋转面 $(f(u),g(u))=(\sqrt{u^2+1},u)$, $r(u,v)=(f(u)\cos v,f(u)\sin v,g(u))$

$$K(u,v)=\frac{(f'g''-f''g')g'}{((f')^2+(g')^2)^2 f}=-\frac{1}{(2u^2+1)^2}\longrightarrow 0 \text{ as } u\to\infty$$

$\mathbb{E}^3$ 中是否存在高斯曲率 $K\leq-\delta<0$ 的完备曲面？　(Cohn-Vossen 的猜想, 1936)

N.V. Efimov, Appearance of Singularities on Surfaces of Negative Curvature

Willmore 能量 1965

M 光滑、紧致无边、可定向曲面, $f:M\to\mathbb{E}^3$ 嵌入

Gauss-Bonnet: $\frac{1}{2\pi}\int_{f(M)}K\,dV=\chi(M)$

$\tau(f):=\frac{1}{2\pi}\int_{f(M)}H^2\,dV$

定义 $\tau(M)=\inf_{\substack{f:M\to\mathbb{E}^3\\ \text{光滑嵌入}}}\tau(f)$　　拓扑量

问 $\tau(M)$ 和 $\chi(M)$ 的关系？

定理(Willmore 1965) 设 M 是光滑浸没可定向，$g=0$ 的曲面，则有 $\tau(M)=\chi(M)$

若 $\tau(f)=\tau(M)=\chi(M)=2 \Leftrightarrow f(M)$ 是一个标准球面

证：$\tau(f)=\frac{1}{2\pi}\int_{f(M)}H^2dV=\frac{1}{2\pi}\int_{f(M)}KdV+\frac{1}{8\pi}\int_{f(M)}(k_1-k_2)^2dV \geq \chi(M)$ □

"$=$" $\Leftrightarrow k_1=k_2 \Leftrightarrow$ 全脐点曲面

$g\neq 1$，$\tau(M)=?$

环面 $f_{a,b}(T)=r(u,v)=((a+b\cos u)\cos v,(a+b\sin u)\sin v,\ b\sin u)\quad (0<b<a)$

练习. $H=\frac{a+2b\cos u}{2b(a+b\cos u)}$

$$\begin{aligned}\tau(f_{ab})&=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{a+2b\cos u}{2b(a+b\cos u)}\right)^2\sqrt{EG-F^2}\,du\,dv\\&=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{(a+2b\cos u)^2}{4b(a+b\cos u)}\,du\,dv \quad \text{令}\frac{b}{a}=c<1\\&=\frac{1}{8\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{(1+2c\cos u)^2}{c(c+\cos u)}\,du\,dv\\&=\frac{\pi}{2c\sqrt{1-c^2}}\end{aligned}$$

观察：$c\to 0$，$\tau(f_{a,b})\to\infty$

$c\to 1$，$\tau(f_{a,b})\to\infty$

当 $c=\frac{1}{\sqrt{2}}$ 时，$\tau(f_{a,b})$ 达最小值 π

$g=1$，$\tau(M)\geq\pi$ Willmore 猜想 Masques-Neves 2014

Mutz-Bensiman 1991 Observation of toroidal vesicles