PDE Summary

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March 7, 2023

Abstract

Semester: 2022 fall
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Reference: *Partial Differencial Equations* by Zhou Shulin

1 WAVE EQUATION

1.1 Method of Characteristics

Given a first order linear PDE

$$\begin{cases} \partial_t u + a(x.t)\partial_x u + b(x,t)u = f(x,t) \\ u(x,0) = \Phi(x) \end{cases} \tag{1}$$

Assume $x = x(t)$,while $u(x,t) = u(x(t),t) = U(t)$, and

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \partial_t u + \partial_x u \cdot x(t)$$

Let $x'(t) = a(x(t),t)$, consequently

$$\frac{\mathrm{d}U}{\mathrm{d}t} = b(x(t),t)U(t) = f(x(t),t)$$

Let $x(0) = c$, then

$$U(0) = u(x(0),0) = u(c,0)U(0) = \Phi(c)$$

Solve following first order linear ODEs:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a(x(t),t) \\ x(0) = c \end{cases} \tag{2}$$

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + b(x(t),t)U(t) = f(x(t),t) \\ U(0) = \Phi(c) \end{cases} \tag{3}$$

Eliminating constant c, we deduce the solution.

1.2 Cauchy Problem

1.2.1 Basic conceptions

Initial value problem (Cauchy problem) :

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t), x \in \Omega \subset R^n \\ u(x,0) = \phi(x),\ x \in \Omega \\ u_t(x,0) = \psi(x),\ x \in \Omega \end{cases} \tag{4}$$

With boundary values:

- Dirichlet:
$$u(x,t) = g(t),\ x \in \partial\Omega \tag{5}$$
- Neumann:
$$\frac{\partial}{\partial \boldsymbol{n}} u(x,t) = g(t),\ x \in \partial\Omega \tag{6}$$
- Robin:
$$u(x,t) + \alpha(x,t)\frac{\partial}{\partial \boldsymbol{n}} u(x,t) = g(t),\ x \in \partial\Omega,\ \alpha(x,t) > 0 \tag{7}$$

1.2.2 Decomposition of solution

Theorem 1 (Linear combination) *if u_1 satisfies*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = 0 \end{cases} \tag{8}$$

u_2 satisfies

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0) = 0 \\ u_t(x,0) = \psi(x) \end{cases} \tag{9}$$

u_3 satisfies

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \tag{10}$$

Then $u = u_1 + u_2 + u_3$ is a solution to Cauchy problem (7)

Theorem 2 *If $u_2 = M_\psi(x,t)$ is a solution to equation (9), we have*

$$u_1 = \partial_t M_\phi \tag{11}$$

$$u_3 = \int_0^t M_{f_\tau}(x, t-\tau)\ \mathrm{d}\tau \tag{12}$$

where $f_\tau = f(x,\tau)$.

Theorem 3 (Fourier transform) *If u satisfies equation*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases} \tag{13}$$

We take Fourier transform of x:

$$\begin{cases} \partial_t^2 \hat{u}(\xi,t) - 4\pi^2|\xi|^2\hat{u}(\xi,t) = 0 \\ \hat{u}(\xi,0) = \hat{\phi}(\xi) \\ \hat{u}_t(\xi,0) = \hat{\psi}(\xi) \end{cases} \tag{14}$$

Solving second order linear equation (14),we deduce the solution to equation (17) by Fourier inversion.

1.3 One Dimension Cauchy Problem

We have the form:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases} \tag{15}$$

As $(\partial_t^2 - \partial_x^2)u = (\partial_t + \partial_x)(\partial_t - \partial_x)u$, we can divide equqaion(18) into two first order linear PDEs. Applying **method of characteristcs** and **theorem 2** , we have:

Theorem 4 (D'Alembert's formula)

$$u(x,t) = \frac{1}{2}(\phi(x+t)+\phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t}\psi(y)\ \mathrm{d}y + \frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau} f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau \tag{16}$$

When $f(x,t) = 0$, let

$$F(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\int_0^x \psi(y)\ \mathrm{d}y \qquad G(x) = \frac{1}{2}\phi(x) - \frac{1}{2}\int_0^x \psi(y)\ \mathrm{d}y$$

We have

$$u(x,t) = F(x+t) + G(x-t) \tag{17}$$

where $F(x+t)$ is called **leftgoing wave** while $G(x-t)$ **rightgoing wave**.

Theorem 5 *If $\phi(x) \in C^2(R)$, $\psi(x) \in C^1(R)$, then **theorem 4** gives a $u(x,t) \in C^2(R \times R_+)$, and u is a solution to equation (15).*

Theorem 6 *When ϕ, ψ, f are all odd/even/periodic functions, u is also a(n) odd/even/periodic function.*

1.4 One Dimension Half-line Problem

Assume u satisfies the following equation (by default, $g(t) = 0$):

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t), x > 0, t > 0 \\ u(x,0) = \phi(x), x \geq 0 \\ u_t(x,0) = \psi(x), x \geq 0 \\ u(0,t) = g(t), t \geq 0 \end{cases} \tag{18}$$

Prolongate ϕ, ψ and f by odd reflection:

$$\bar{\phi}(x) = \begin{cases} \phi(x), & x > 0 \\ -\phi(-x), & x < 0 \end{cases}$$

$$\bar{\psi}(x) = \begin{cases} \psi(x), & x > 0 \\ -\psi(-x), & x < 0 \end{cases}$$

$$\bar{f}(x,t) = \begin{cases} f(x,t), & x > 0 \\ -f(-x,t), & x < 0 \end{cases}$$

Let $\bar{u}$ satisfies one dimension equation

$$\begin{cases} \partial_t^2 \bar{u} - \partial_x^2 \bar{u} = \bar{f}(x,t) \\ \bar{u}(x,0) = \bar{\phi}(x) \\ \bar{u}_t(x,0) = \bar{\psi}(x) \end{cases}$$

Applying **theorem 4**, we have

$$\bar{u} = \frac{1}{2}(\bar{\phi}(x+t) + \bar{\phi}(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \bar{\psi}(y)\ \mathrm{d}y + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} \bar{f}(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau$$

When $x \geq t$,

$$u(x,t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\ \mathrm{d}y + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau \tag{19}$$

When $0 < x < t$,

$$\begin{aligned} u(x,t) = &\frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{t-x}^{x+t} \psi(y)\ \mathrm{d}y \\ &+ \frac{1}{2}\int_{t-x}^{t} \int_{x-t+\tau}^{x+t-\tau} f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau + \frac{1}{2}\int_0^{t-x} \int_{t-\tau-x}^{x+t-\tau} f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau \end{aligned} \tag{20}$$

If u is a solution to equation 22, it satisfies compatibility condition at corner point.

Theorem 7 (compatibility condition)

$$\begin{cases} \phi(0) = \lim\limits_{x\to 0^+} u(x,0) = u(0,0) = \lim\limits_{t\to 0^+} u(0,t) = g(0) = 0 \\ \psi(0) = \lim\limits_{x\to 0^+} u_t(x,0) = u_t(0,0) = \lim\limits_{t\to 0^+} u_t(0,t) = g'(0) = 0 \\ \phi''(0) = \lim\limits_{x\to 0^+} (\partial_x^2 u(x,0) + f(x,0)) = \lim\limits_{x\to 0^+} u_{tt}(x,0) = u_{tt}(0,0) = \lim\limits_{t\to 0^+} u_{tt}(0,t) = g''(0) = 0 \end{cases} \tag{21}$$

Theorem 8 *If $\phi(x) \in C^2(\bar{R}_+)$, $\psi(x) \in C^1(\bar{R}_+)$, $f(x,t) \in C^1(\bar{R}_+ \times \bar{R}_+)$ satisfy compatibility condition,then formula (28) and (29) give a $u(x,t) \in C^2(\bar{R}_+ \times \bar{R}_+)$, and u is a solution to equation (18).*

If $g(t) \neq 0$, let $v(x,t) = u(x,t) - g(t)$, v is a solution to a one dimension half-line problem whose boundary value is 0.

1.5 Multidimensional Wave Equations

1.5.1 transform

Time transform:

$$u(x,t) \to u(x, t+t_0)$$

Space transform:

$$u(x,t) \to u(x+x_0, t)$$

Scaling transform:

$$u(x,t) \to u(\frac{x}{\lambda} + \frac{t}{\lambda}) = u^\lambda(x,t)$$

Lorentz transform:

$$u(x,t) \to u(x - x_0 + \frac{x_0 - vt}{\sqrt{1-|v|^2}}, \frac{t - v\cdot x}{\sqrt{1-|v|^2}})$$

1.5.2 three dimensions

Laplacian operater on a three dimensional sphere:

$$\Delta u = \partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u \tag{22}$$

As for Δ_{S^2}, we have

$$\int_{S^2} \Delta_{S^2} u \ \mathrm{d}\omega = \int_{S^2} \mathrm{div}\nabla u \ \mathrm{d}\omega = \int_{\partial S^2} \frac{\partial u}{\partial \boldsymbol{n}} \ \mathrm{d}S = 0$$

Theorem 9 (sphere average) *Let*

$$\begin{cases} \bar{u}(t,r) = \frac{1}{4\pi}\int_{S^2} u \ \mathrm{d}\omega \\ \bar{\phi}(t,r) = \frac{1}{4\pi}\int_{S^2} \phi \ \mathrm{d}\omega \\ \bar{\psi}(t,r) = \frac{1}{4\pi}\int_{S^2} \psi \ \mathrm{d}\omega \end{cases} \tag{23}$$

Integrate equation (22) on S^2, and plug Δu into three dimensional equation (f=0):

$$\partial_t^2 \bar{u} - (\partial_r^2 \bar{u} + \frac{2}{r}\partial_r \bar{u}) = 0$$

Let $\bar{v} = r\bar{u}$, then $\bar{v}$ satisfies

$$\begin{cases} \partial_t^2 \bar{u} - \partial_x^2 \bar{u} = 0 \\ \bar{v}(r,0) = r\bar{u}(r,0) = r\bar{\phi}(r) \\ \bar{v}_t(x,0) = r(\partial_t \bar{u})(r,0) = r\bar{\psi}(r) \end{cases} \tag{24}$$

Theorem 10 (Kirchhoff's formula) *Through the observation $u(0,t) = \bar{u}(0,t) = \partial_r(r\bar{u}(r,t))|_{r=0}]$, we deduce(let $y = x + t\omega$)*

$$\begin{aligned} u(x,t) &= \frac{\mathrm{d}}{\mathrm{d}t}(\frac{t}{4\pi}\int_{S^2}\phi(x+t\omega)\ \mathrm{d}\omega) + \frac{t}{4\pi}\int_{S^2}\psi(x+t\omega)\ \mathrm{d}\omega \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{4\pi t}\int_{|y-x|=t}\phi(y)\ \mathrm{d}S(y)) + \frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\ \mathrm{d}S(y) \end{aligned} \tag{25}$$

Theorem 11 *If $f \neq 0$, according to **theorem 2**, we have*

$$\begin{aligned} u(x,t) = &\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{4\pi t}\int_{|y-x|=t}\phi(y)\ \mathrm{d}S(y)) + \frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\ \mathrm{d}S(y) \\ &+ \int_0^t \frac{1}{4\pi(t-\tau)}\int_{|x-y|=t-\tau} f(y,\tau)\ \mathrm{d}y\ \mathrm{d}\tau \end{aligned} \tag{26}$$

1.5.3 two dimensions

Theorem 12 (Poisson's formula) *Regarding two dimensional wave equation as a particular three dimensional one as $f = 0$, we have*

$$u(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2\pi}\int_{|z-x|\leq t}\frac{\phi(z)}{\sqrt{t^2-|z-x|^2}}\ \mathrm{d}z) + \frac{1}{2\pi}\int_{|z-x|\leq t}\frac{\psi(z)}{\sqrt{t^2-|z-x|^2}}\ \mathrm{d}z \tag{27}$$

1.5.4 characteristic cone

Theorem 13 *The value of $u(x_0,t_0)$ is only concerned with values on $B(x_0,t_0)$. This region is called the **domain of dependence**.*

*As for $B(x_0,t_0)$, the cone composed by $u(x_0,t_0)$ and $B(x_0,t_0)$ is called the **domain of determinant**.*

*If the characteristic cone of a point (x_0,t_0) intersects $B(x_0,t_0)$, the set including all such points is called the **domain of influence***

Theorem 14 (support of a function)

$$supp\ f = \overline{\{x_0 : f(x_0) \neq 0\}}$$

Theorem 15 (limited velocity of propagation) *Abstract supp $u \subset D$, D is a region. Then supp u enlarges as t increases, while the velocity of enlarging is constant 1.*

1.6 Energy Estimation

1.6.1 energy conservation

Abstract a wave equation without f:

$$\partial_t^2 u - \Delta u = 0 \tag{28}$$

Multiple the equation by $\partial_t u$, we have

$$\partial_t u(\partial_t^2 u) = \frac{1}{2}\partial_t u(\partial_t u)^2$$

$$\begin{aligned}\partial_t u \Delta u &= \sum_{i=1}^{n} \partial_t u(\partial_{x_i}^2 u) = \sum_{i=1}^{n} (\partial_{x_i}(\partial_t u \partial_{x_i}) - \partial_t(\partial_{x_i} u)\partial_{x_i} u) \\ &= \sum_{i=1}^{n} (\partial_{x_i}(\partial_t u \partial_{x_i}) - \frac{1}{2}\partial_t u(\partial_{x_i} u)^2) \\ &= \mathrm{div}(\partial_t u \nabla u) - \frac{1}{2}\partial_t |\nabla u|^2\end{aligned}$$

Therefore, we deduce **energy conservation in differential form**:

$$0 = \frac{1}{2}\partial_t u(\partial_t u)^2 - (\mathrm{div}(u_t \nabla u) - \frac{1}{2}\partial_t |\nabla u|^2) = \partial_t(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}\partial_t |\nabla u|^2) - \mathrm{div}(u_t \nabla u) \tag{29}$$

Assuming u and its derivatives of various orders go to 0, when $|x|$ goes to $+\infty$, in another word:

$$\int_{R^n} \mathrm{div}(\partial_t u \nabla u) = \int_{\partial R^n} \partial_t u \frac{\partial u}{\partial \boldsymbol{n}} = 0$$

Intergate x in R^n, we deduce **energy conservation in integrated form**:

$$0 = \partial_t \int_{R^n} (\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2) \tag{30}$$

Let

$$E(t) = \int_{R^n} (\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2) \tag{31}$$

And $E(t)$ is a constant of variable t, called $E(t)$ the energy of equation (28). As for wave equations in $\Omega \subset R^n$, we can similarly define energy $E(t)$ as

$$E(t) = \int_{\Omega} (\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2) \tag{32}$$

1.6.2 uniqueness of solution

Assuming u_1 and u_2 are two different solutions to equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t), x \in \Omega, t > 0 \\ u(x,0) = \phi(x), x \in \Omega \\ u_t(x,0) = \psi(x), x \in \Omega \\ u(0,t)|_{\partial\Omega} = g(x,t), t \geq 0 \end{cases} \tag{33}$$

Then $u = u_1 - u_2$ satisfies

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, x \in \Omega, t > 0 \\ u(x,0) = 0, x \in \Omega \\ u_t(x,0) = 0, x \in \Omega \\ u(0,t)|_{\partial\Omega} = 0, t \geq 0 \end{cases} \tag{34}$$

Therefore, $E(t) = E(0) = 0$, $\partial_t u = |\nabla u| = 0, x \in \Omega$, so u is constant in Ω.

Let $u \to \partial\Omega$, with restriction of boundary value, $u = 0$.

1.6.3 stability of solution

L^2 norm:

$$||f(x)||_{L^2(\Omega)} = (\int_\Omega |f(x)|^2 \text{ d}x)^{\frac{1}{2}} \tag{35}$$

$$||f(x)||_{L^2(0,T,\Omega)} = (\int_0^T \int_\Omega |f(x)|^2 \text{ d}x \text{ d}t)^{\frac{1}{2}} \tag{36}$$

$\forall \epsilon > 0$, $\exists \eta > 0$, s.t. two wave equation satisfies

$$\begin{cases} ||\phi_1(x) - \phi_2(x)||_{L^2(\Omega)} \leq \eta \\ ||\nabla\phi_1(x) - \nabla\phi_2(x)||_{L^2(\Omega)} \leq \eta \\ ||\psi_1(x) - \psi_2(x)||_{L^2(\Omega)} \leq \eta \\ ||f_1(x) - f_2(x)||_{L^2(0,T,\Omega)} \leq \eta \end{cases} \tag{37}$$

Then

$$\begin{cases} ||u_1(x) - u_2(x)||_{L^2(\Omega)} \leq \epsilon \\ ||\nabla u_1(x) - \nabla u_2(x)||_{L^2(\Omega)} \leq \epsilon \\ ||\partial_t u_1(x) - \partial_t u_2(x)||_{L^2(\Omega)} \leq \epsilon \end{cases} \tag{38}$$

Factually, let $u = u_1 - u_2$, $\phi = \phi_1 - \phi_2$, $\psi = \psi_1 - \psi_2$, $f = f_1 - f_2$, apply energy estimation,

$$\partial_t \int_\Omega (\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2) \text{ d}x = \int_\Omega f \partial_t u \text{ d}x \tag{39}$$

Apply mean inequality to right hand side:

$$\int_\Omega f \partial_t u \text{ d}x \leq \frac{1}{2}\int_\Omega (|f|^2 + |\partial_t u|^2) \text{ d}x \leq \frac{1}{2}\int_\Omega |f|^2 \text{ d}x + E(t)$$

scilicet

$$\frac{\text{d}}{\text{d}t}E(t) \leq \frac{1}{2}\int_\Omega |f|^2 \text{ d}x + E(t) \tag{40}$$

Apply Gronwall's inequality(C_T is a function of just T):

$$
\begin{aligned}
E(t) &\le e^t(E(0) + \frac{1}{2}\int_0^t e^{-\tau}\int_\Omega |f|^2 \text{ d}x \text{ d}\tau) \\
&\le e^T(E(0) + \frac{1}{2}\int_0^T \int_\Omega |f|^2 \text{ d}x \text{ d}\tau) \\
&\le C_T(E(0) + \int_0^T \int_\Omega |f|^2 \text{ d}x \text{ d}\tau)
\end{aligned} \tag{41}
$$

Moreover, let $g(t) = \int_\Omega u^2\partial_t u \text{ d}x$,

$$
\begin{aligned}
g'(t) = &\int_\Omega u\partial_t u \text{ d}x \le \frac{1}{2}\int_\Omega \partial_t u^2 \text{ d}x + \frac{1}{2}\int_\Omega u^2 \text{ d}x \\
&\le g(t) + 2E(t) \\
&\le g(t) + 2C_T(E(0) + \int_0^T \int_\Omega |f|^2 \text{ d}x \text{ d}\tau) \\
&\forall\ 0 \le t \le T
\end{aligned} \tag{42}
$$

Apply Gronwall's inequality again:

$$
g(t) \le C_T(g(0) + E(0) + \int_0^T \int_\Omega |f|^2 \text{ d}x \text{ d}\tau) \tag{43}
$$

Equation (41) and (43) result in stability.

1.6.4 limited propagating velocity(proved by energy)

Intergrate equation (29) in circular truncated cone $|x - x_0| \le R - t$, we have ($e(t) = \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2$ is called **energy density**):

$$
\begin{aligned}
0 &= \int_V \partial_t e(u) - \text{div}(u_t\nabla u) \text{ d}x \text{ d}t \\
&= \int_V (\partial_t, \text{div})\cdot(e(u), -u_t\nabla u) \text{ d}x \text{ d}t \\
&= \int_{\partial V} (\partial_t, \text{div})\cdot \boldsymbol{n} \text{ d}x \text{ d}t \\
&= -\int_B e(u)(0) \text{ d}x + \int_T e(u)(t_0) \text{ d}x \\
&+ \int_K (e(u), -u_t\nabla u)\cdot \frac{1}{\sqrt{2}}(\frac{R-t}{|R-t|}, \frac{x-x_0}{|x-x_0|}) \text{ d}S
\end{aligned}
$$

After calculation,

$$
\int_B e(u)(0) \text{ d}x = \int_T e(u)(t_0) \text{ d}x + \text{Flux}[0, t_0] \tag{44}
$$

where

$$\text{Flux}[0,t_0] = \frac{1}{2\sqrt{2}}\int_K (\partial_t u - \frac{x-x_0}{|x-x_0|}\cdot\nabla u)^2 + (|\nabla u|^2 - |\frac{x-x_0}{|x-x_0|}\cdot\nabla u|^2)\ \mathrm{d}S \quad (45)$$

That is to say, bottom energy = top energy + laterally wasted energy. Because T, Flux ≥ 0, when $B = 0$, we have $T = 0$, Flux $= 0$.

1.7 Method of Variable Separation

1.7.1 Sturm-Liouville Boundary Value Problem

Abstract equation:

$$\begin{cases} X'' + \lambda X = 0 \\ -\alpha_1 X'(0) + \beta_1 X(0) = 0, \alpha_1 \geq 0, \beta_1 \geq 0, \alpha_1 + \beta_1 > 0 \\ \alpha_2 X'(l) + \beta_2 X(l) = 0, \alpha_2 \geq 0, \beta_2 \geq 0, \alpha_2 + \beta_2 > 0 \end{cases} \quad (46)$$

λ is an eigenvalue of operator $(-\frac{\mathrm{d}^2}{\mathrm{d}x^2})$.

Theorem 16 (Sturm-Liouville) *Sturm-Liouville boundary value problem has following properties:*

1. *All eigenvalues ≥ 0. Particularly, when $\beta_1 + \beta_2 > 0$, all eigenvalues > 0.*
2. *Eigenfunctions corresponding to different eigenvalues are orthogonal. i.e. $\int_0^l X_\lambda(x)X_\mu(x)\ \mathrm{d}x = 0, \forall \lambda \neq \mu$.*
3. *All eigenvalues make up a monotonic increasing series $\{\lambda_n\}, \lim_{n\to\infty} = +\infty$.*
4. *Every function can expand according to system of eigenfunctions. i.e. $f(x) = \sum_{n=1}^{\infty} C_n X_{\lambda_n}(x)$.*

1.7.2 Solution to Wave Equations in a Bounded Set

Abstract equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t), 0 \leq x \leq l, t > 0 \\ u(x,0) = \phi(x), x \in \Omega, u_t(x,0) = \psi(x), 0 \leq x \leq l \\ u(0,t) = g_1(t), u(l.t) = g_2(t), t \geq 0 \end{cases} \quad (47)$$

If $f = g_1 = g_2 = 0$, $f \neq 0$, we claim that

$$u(x,t) = T(t)X(x) \quad (48)$$

Plug into equation (47), we get $T''(t)X(x) - T(t)X''(x) = 0$, i.e. $\frac{T''(t)}{T(t)} = \frac{X''(t)}{X(t)} = -\lambda$, which leads to a Sturm-Liouville problem:

$$\begin{cases} X'' + \lambda X = 0 \\ T(t)X(0) = 0, T(t)X(l) = 0, t > 0 \end{cases} \quad (49)$$

Immediately, we notice $X(0) = X(l) = 0$. Through discussion, it is easy to confirm $\lambda > 0$.

Solve second order linear equation $X'' + \lambda X = 0$, and calculate 2 constants with boundary values of PDE (equivalent to primary values of ODE), we have:

$$\lambda_n = (\frac{n\pi}{l})^2,\ X_n(x) = \sin(\frac{n\pi}{l}x), n = 1, 2, \dots \tag{50}$$

Additionally solve the equation of $T(t)$:

$$T_n(t) = C_n \cos(\frac{n\pi}{l}t) + D_n \sin(\frac{n\pi}{l}t)$$

Therefore,

$$u_n(x,t) = (C_n \cos(\frac{n\pi}{l}t) + D_n \sin(\frac{n\pi}{l}t)) \sin(\frac{n\pi}{l}x) \tag{51}$$

As solutions' property of linear superposition,

$$u(x,t) = \sum_{n=1}^{\infty}(C_n \cos(\frac{n\pi}{l}t) + D_n \sin(\frac{n\pi}{l}t)) \sin(\frac{n\pi}{l}x) \tag{52}$$

Apply **theorem 16** to ϕ and ψ:

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l}x)$$
$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin(\frac{n\pi}{l}x)$$

where

$$\phi_n = \frac{\int_0^l \phi(x) sin(\frac{n\pi}{l}x)\ \mathrm{d}x}{\int_0^l sin^2(\frac{n\pi}{l}x)\ \mathrm{d}x} = \frac{2}{l}\int_0^l \phi(x) sin(\frac{n\pi}{l}x)\ \mathrm{d}x$$
$$\psi_n = \frac{\int_0^l \psi(x) sin(\frac{n\pi}{l}x)\ \mathrm{d}x}{\int_0^l sin^2(\frac{n\pi}{l}x)\ \mathrm{d}x} = \frac{2}{l}\int_0^l \psi(x) sin(\frac{n\pi}{l}x)\ \mathrm{d}x$$

Consider initial values of u, we deduce:

$$C_n = \frac{2}{l}\int_0^l \phi(x) sin(\frac{n\pi}{l}x)\ \mathrm{d}x,\ D_n = \frac{2}{n\pi}\int_0^l \psi(x) sin(\frac{n\pi}{l}x)\ \mathrm{d}x \tag{53}$$

Combining (52) and (53), equation (47) is solved.

If $f \neq 0$, we can similarly let

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(\frac{n\pi}{l}x) \tag{54}$$

Plug it into original equation:

$$\begin{aligned}\sum_{n=1}^{\infty} f_n(t)\sin(\frac{n\pi}{l}x) &= \partial_t^2 u - \partial_x^2 u \\ &= \sum_{n=1}^{\infty} T_n''(t)X_n(x) - \sum_{n=1}^{\infty} T_n(t)X_n''(x) \\ &= \sum_{n=1}^{\infty} (T_n''(t) + \lambda T_n(t))X_n(x)\end{aligned} \tag{55}$$

Compare coefficients,

$$T_n''(t) + \lambda T_n(t) = f_n(t) \tag{56}$$

Considering initial values,

$$T_n(0) = \phi_n,\ T_n'(0) = \psi_n \tag{57}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(\frac{n\pi}{l}x) \tag{58}$$

where

$$T_n(t) = \phi_n \sin(\frac{n\pi}{l}x) + \frac{l}{n\pi}\psi_n \sin(\frac{n\pi}{l}x) + \frac{l}{n\pi}\int_0^t f_n(\tau)\sin(\frac{n\pi}{l}(t-\tau))\ \mathrm{d}\tau \tag{59}$$

If $g_1,\ g_2 \neq 0$, let

$$v(x,t) = u(x,t) - \frac{(l-x)g_1(t) + xg_2(t)}{l} \tag{60}$$

Then boundary values of v comes to 0.

Theorem 17 (compatibility condition) *Under the condition $g_1 = g_2 = 0$ and $f = 0$, if ϕ, ψ in region $Q = (0,l) \times (0,+\infty)$ satisfy $\phi(x) \in C^3([0,l])$, $\psi(x) \in C^2([0,l])$, $\phi(0) = \phi(l) = \phi''(0) = \phi''(l) = \psi(0) = \psi(l) = 0$, then $u(x,t) \in C(\overline{Q})$ is a classical solution to equation (47).*

2 POTENTIAL EQUATION

2.1 Harmonic Function

We call

$$-\Delta u = f \tag{61}$$

a **potential equation**. When $f = 0$, it is a **harmonic equation** or **Laplacian equation**. Otherwise, we call it a **Poission's equation**.

2.1.1 mean-value property

If $u : \Omega \to R$ has continuous second partial derivatives, and $\Delta u = 0$, u is called a **harmonic function**.

Theorem 18 (integrate in polar coordinates) *Abstract $f : R^n \to R$,*

$$\int_{R^n} f(x)\ \mathrm{d}x = \int_0^\infty \int_{\partial B_r(x_0)} f(y)\ \mathrm{d}S(y)\ \mathrm{d}r = \int_0^\infty r^{n-1}\ \mathrm{d}r \int_{\omega=1} f(x_0+r\omega)\ \mathrm{d}S(\omega) \tag{62}$$

$$\frac{\mathrm{d}}{\mathrm{d}r} \int_{B_r(x_0)} f(y)\ \mathrm{d}y = \int_{\partial B_r(x_0)} f(y)\ \mathrm{d}S(y) \tag{63}$$

Theorem 19 (mean-value property) *Abstract $u \in C(\Omega), \Omega \subset R^3$,*

$$u(x) = \frac{1}{B_r(x)} \int_{B_r(x)} u(y)\ \mathrm{d}y = \frac{3}{4\pi r^3} B_r(x) \int_{B_r(x)} u(y) \tag{64}$$

$$u(x) = \frac{1}{\partial B_r(x)} \int_{\partial B_r(x)} u(y)\ \mathrm{d}y = \frac{1}{4\pi r^2} \partial B_r(x) \int_{\partial B_r(x)} u(y) \tag{65}$$

Theorem 20 *Two items of mean-value properties are equivalent.*

Theorem 21 *u is a harmonic function $\Leftrightarrow$ u has mean-value properties.*

2.1.2 value estimation

Theorem 22 (Harnack's inequality) *Abstract $\Delta u = 0, u(x) \geq 0, \forall x \in \Omega \subset R^n$. $\forall$ connected compact sets V, $\exists C = C(dist(V, \partial\Omega), n)$, s.t.*

$$\sup_V u \leq C \inf_V u \tag{66}$$

Theorem 23 (gradient estimation) *Abstract $u \in C(\overline{B}_R(x_0))$ is harmonic, then*

$$|(\nabla u)(x_0)| \leq \frac{n}{R} \max_{\overline{B}_R(x_0)} |u| \tag{67}$$

if we add a condition $u \geq 0$, there will be a stronger conclusion:

$$|(\nabla u)(x_0)| \leq \frac{n}{R} u(x_0) \tag{68}$$

Theorem 24 (Liouville) *A upper bounded or lower bounded harmonic function in R^n is constant.*

2.2 Fundamental Solution and Green's Function

2.2.1 Fundamental Solution

If $\Delta u = \delta$, u is called a fundamental solution. Particularly, if u is a fundamental solution, $\Delta u = 0, \forall x \neq 0$.

Theorem 25 (radial solution) *If $u = u(r)$, then*

$$\Delta u = \partial_r^2 u + \frac{n-1}{r}\partial_r u + \frac{\Delta_{S^{n-1}} u}{r^2} = \partial_r^2 u + \frac{n-1}{r}\partial_r u = 0$$

let $v(r) = \partial_r u$, we have

$$\partial_r v + \frac{n-1}{r} v = 0$$

Solve the equation, we deduce

$$u(r) = \begin{cases} C_1 r^{-(n-2)} + C^2, & n \geq 3 \\ C_1 \ln r + C_2, & n = 2 \end{cases} \tag{69}$$

Theorem 26 (fundamental solution of Laplacian equation)

$$\Gamma(x) = \begin{cases} \frac{1}{\omega_n |x|^{n-2}}, & n \geq 3 \\ -\frac{1}{2\pi} \ln |x|, & n = 2 \end{cases} \tag{70}$$

where ω_n presents the surface area of an n-dimensional unit sphere. It has a singularity $x = 0$.

Theorem 27 (Green's formula) *Abstract $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$*

$$\int_\Omega u \Delta v \ \mathrm{d}x = \int_{\partial\Omega} u \frac{\partial v}{\partial \boldsymbol{n}} \ \mathrm{d}S - \int_\Omega \nabla u \nabla v \ \mathrm{d}x \tag{71}$$

Swapping u, v in Green's formula and substract the other, we derive

Theorem 28 (Green's second formula)

$$\int_\Omega (u\Delta v - v\Delta u) \ \mathrm{d}x = \int_{\partial\Omega} (u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}) \ \mathrm{d}S \tag{72}$$

Theorem 29 *Abstract $n = 3$, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a harmonic function in region Ω, then $\forall x_0 \in \Omega$:*

$$u(x) = \frac{1}{4\pi} \int_{\partial\Omega} (-u\frac{\partial}{\partial \boldsymbol{n}}(\frac{1}{|y-x|}) + \frac{1}{|y-x|}\frac{\partial u}{\partial \boldsymbol{n}}) \ \mathrm{d}S(y) \tag{73}$$

Theorem 30 (Poisson's formula) *Abstract $\Delta u = \Delta g = 0$, and $g|_{\partial\Omega} = \frac{1}{4\pi|x-x_0|}$. Apply **Green's second formula** to u and g in Ω:*

$$\int_{\partial\Omega} (u\frac{\partial g}{\partial \boldsymbol{n}} - g\frac{\partial u}{\partial \boldsymbol{n}}) \ \mathrm{d}S = 0$$

*Subtract it by **theorem 29**:*

$$u(x_0) = \int_{\partial\Omega} (u\frac{\partial}{\partial \boldsymbol{n}}(g - \frac{1}{4\pi|x-x_0|}) + (g - \frac{1}{4\pi|x-x_0|})\frac{\partial u}{\partial \boldsymbol{n}}) \, \mathrm{d}S(x)$$
$$= \int_{\partial\Omega} (u\frac{\partial}{\partial \boldsymbol{n}}(g - \frac{1}{4\pi|x-x_0|}) \, \mathrm{d}S(x)$$

Let $G(x, x_0) = -g(x) + \frac{1}{4\pi|x-x_0|}$ and u satisfies boundary value $u|_{\partial\Omega} = \phi(x)$. Therefore, we derive a formula present u by only its Dirichlet's boundary value:

$$u(x_0) = -\int_{\partial\Omega} \phi(x)\frac{\partial G}{\partial \boldsymbol{n}}(x, x_0) \, \mathrm{d}S(x) \tag{74}$$

Theorem 31 (Green's function) *Green's function of operater $-\Delta$ in Ω satisfies*

1. *$G(x)$ is second order continuously differentiable and harmonic.*
2. *$G(x) = 0, \forall x \in \partial\Omega$.*
3. *$-G(x) + \frac{1}{4\pi|x-x_0|}$ is finite at x_0, second order continuously differentiable and harmonic everywhere.*

Theorem 32 *Green's function satisfies*

$$G(x, x_0) = G(x_0, x) \tag{75}$$

Theorem 33 *Green's function in half-space ($x_3 > 0$):*

$$G(x, x_0) = \frac{1}{4\pi|x-x_0|} - \frac{1}{4\pi|x-x_0^*|} \tag{76}$$

where x_0^* and x_0 are symmetric about plane $x_3 = 0$.

Theorem 34 *Green's function in $B_R(0)$:*

$$G(x, x_0) = \frac{1}{4\pi|x-x_0|} - \frac{R}{4\pi|x - \frac{R^2}{|x_0|}x_0||x_0|} \tag{77}$$

Theorem 35 (Poisson's formula) *Combine equation (74) and (77), we can solve Laplacian equation in $B_R(0)$:*

$$u(x) = \frac{R^2 - |x|^2}{4\pi R}\int_{|y|=R} \frac{\phi(y)}{|y-x|^3} \, \mathrm{d}S(y) \tag{78}$$

Theorem 36 (Harnack's inequality) *Abstract u is harmonic in $B_R(x_0)$ and $u \geq 0$, we have an inequality:*

$$\frac{R}{R+r}\frac{R-r}{R+r}u(x_0) \leq u(x) \leq \frac{R}{R-r}\frac{R+r}{R-r}u(x_0) \tag{79}$$

where $r = |x - x_0| < R$.

2.3 Extremum and Modulus Estimation

Abstract equation

$$\mathcal{L}u = -\Delta u + c(x)u = f(x), c(x) \geq 0, x \in \Omega \tag{80}$$

2.3.1 extremum principle

Theorem 37 (weak maximum principle) *If $c(x) \geq 0$, $f(x) < 0$, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies equation (80), then u(x) doesn't reach its non-negative maximum in Ω, i.e. $u(x)$ reaches its maximum at the boundary.*

Constructing auxiliary function $u_\epsilon(x) = u(x) + \epsilon(\text{diam}(\Omega)^2 - |x|^2)$, we can derive following conclusion from former theorem:

Theorem 38 *If $c(x) \geq 0$, $f(x) < 0$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ statisfies equation (80) and has positive maximum in $\overline{\Omega}$, then u(x) reach its non-negative maximum of $\overline{\Omega}$ on $\partial\Omega$, i.e.*

$$\max_{x \in \overline{\Omega}} u(x) \leq \max_{x \in \partial\Omega} u^+(x) \tag{81}$$

where $u^+(x) = \max\{u(x), 0\}$.

Constructing auxiliary function $v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}$, we can derive following conclusion from former theorem:

Theorem 39 (Hopf's lemma) *Abstract $B_R \subset R^n, n = 2, 3$. Bounded function $c(x) \geq 0$, in B_R. If $u \in C^2(B_R) \cap C(\overline{B_R})$ satisfies:*

1. *$\mathcal{L}u \leq 0$*
2. *$\exists x_0 \in \partial B_R$, s.t. $u(x)$ reaches its non-negative maximum in $\overline{B_R}$ at x_0 and $\forall x \neq x_0, u(x) < u(x_0)$*

we have:

$$\frac{\partial u}{\partial \boldsymbol{\nu}}|_{x=x_0} > 0 \tag{82}$$

as long as the angle between $\boldsymbol{\nu}$ and $\boldsymbol{n}$(the unit outer normal vector of ∂B_R at x_0) $< \frac{\pi}{2}$.

Theorem 40 (strong maximum principle) *Abstract $\Omega \subset R^n$ is a bounded region, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\mathcal{L}u \geq 0$, reaching its maximum of $\overline{\Omega}$ in Ω, then u is constant in $\overline{\Omega}$.*

Theorem 41 *Abstract $\Omega \subset R^n$ is a bounded region, and $u \in C(\overline{\Omega})$ satisfies mean-value property, then u reaches its maximum and minimum on $\partial\Omega$, unless u is constant.*

This theorem also implies if $-\Delta u \leq 0$ as $x \in \Omega$, u reaches its maximum on $\partial\Omega$.

2.3.2 maximum modulus estimation

Abstract potential equation

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases} \tag{83}$$

Constructing auxiliary function $w = u(x) - (G + \frac{F}{2n}(\mathrm{diam}(\Omega)^2 - |x|^2)$ and $\widetilde{w} = -u(x) - (G + \frac{F}{2n}(\mathrm{diam}(\Omega)^2 - |x|^2)$, we can derive a significant conclusion for elliptic equations:

Theorem 42 (maximum modulus estimation) *If u is a solution to equation (83), let $G = \max_{\partial\Omega} |g|$, $F = \max_{\partial\Omega} |f|$, we have:*

$$\max_{\overline{\Omega}} |u(x)| \leq G + CF \tag{84}$$

where C is a constant concerned with dimension and diameter of Ω.

Theorem 43 (stability of solution) *Abstract equations*

$$\begin{cases} -\Delta u_i = f_i \\ u_i|_{\partial\Omega} = g_i \end{cases} \tag{85}$$

where $i = 1, 2$.

*Let $u = u_1 - u_2$, $f = f_1 - f_2$, $g = g_1 - g_2$. Apply **theorem 42**, and we have:*

$$\max_{\overline{\Omega}} |u_1 - u_2| \leq \max_{\partial\Omega} |g_1 - g_2| + C \max_{\partial\Omega} |f_1 - f_2| \tag{86}$$

That is to say, when $g_1 - g_2$ and $f_1 - f_2$ are small enough, $u_1 - u_2$ is arbitrarily small.

*Furthermore, if $g_1 = g_2$, $f_1 = f_2$, there must be $u_1 = u_2$, i.e. **uniqueness of solution**.*

2.3.3 energy modulus estimation

Theorem 44 (Friedrichs's inequality) *Abstract $u \in C_0^1(\Omega)$, we have:*

$$\int_\Omega |u(x)|^2 \, \mathrm{d}x \leq 4d^2 \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x \tag{87}$$

where d is the diameter of Ω

Theorem 45 (energy modulus estimation) *Abstract equation (83) when $g = 0$, multiple the equation by u and integrate its both sides in Ω:*

$$\int_\Omega u(-\Delta u) \, \mathrm{d}x = \int_\Omega fu \, \mathrm{d}x$$

Integrate by parts, we have

$$-u\Delta u = -u(\nabla \cdot \nabla u) = -div(u\nabla u) + |\nabla u|^2$$

Therefore,

$$\int_\Omega u(-\Delta u)\ \mathrm{d}x = -\int_{\partial\Omega} u\frac{\partial u}{\partial \boldsymbol{n}}\ \mathrm{d}S + \int_{\partial\Omega} |\nabla u|^2\ \mathrm{d}x = \int_{\partial\Omega} |\nabla u|^2\ \mathrm{d}x$$

Meanwhile,

$$|fu| \le \epsilon|u|^2 + \frac{1}{4\epsilon}|f|^2$$

Therefore,

$$\int_\Omega |\nabla u|^2\ \mathrm{d}x = |\int_\Omega fu\ \mathrm{d}x| \le \int_\Omega |fu|\ \mathrm{d}x \le \epsilon\int_\Omega |u|^2\ \mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega |f|^2\ \mathrm{d}x$$

Apply Friedrish's inequality and make ϵ small enough:

$$\int_\Omega |\nabla u|^2\ \mathrm{d}x \le 4\epsilon d^2\int_\Omega |\nabla u|^2\ \mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega |f|^2\ \mathrm{d}x$$

We deduce:

$$\int_\Omega |\nabla u|^2\ \mathrm{d}x \le C\int_\Omega |f|^2\ \mathrm{d}x \tag{88}$$

Additionally, we can apply Friedrish's inequality again:

$$\int_\Omega |u|^2\ \mathrm{d}x \le 4d^2C\int_\Omega |f|^2\ \mathrm{d}x \tag{89}$$

That is to say, both $|u|$ and $|\nabla u|$ are controled by $|f|$.

2.3.4 variable separation

Abstract a Laplacian equation in a boundary region $\Omega = \{(x,y)|x^2+y^2=1\}$

$$\begin{cases} \Delta u = 0, (x,y)\in R^n \\ u = \phi(x), (x,y)\in R^n \end{cases} \tag{90}$$

Let $x = r\cos\theta$, $y = r\sin\theta$, we have:

$$\begin{cases} \Delta u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u \\ u|_{r=1} = \phi(\cos\theta,\sin\theta) = \widetilde{\phi}(\theta) \end{cases} \tag{91}$$

Plug $u(r,\theta) = R(r)\Theta(\theta)$ into equation (90) and separate variables,

$$-r^2\frac{R''(r)+\frac{1}{r}R(r)}{R(r)} = \frac{\Theta(\theta)}{\Theta(\theta)} = -\lambda$$

Therefore,

$$\Theta(\theta) = \begin{cases} C_1e^{-\sqrt{-\lambda}\theta} + C_2e^{\sqrt{-\lambda}\theta}, & \lambda > 0 \\ C_1\theta + C_2, & \lambda = 0 \\ C_1\cos(\sqrt{\lambda}\theta) + C_2\sin(\sqrt{\lambda}\theta), & \lambda < 0 \end{cases} \tag{92}$$

Noticing boundary value $\Theta(\theta + 2\pi) + \Theta(\theta)$, there must be $\lambda > 0$. Additionally, $\lambda = n^2$, where $n = 1, 2, \cdots$, i.e.

$$\Theta_n(\theta) = C_1 \cos(n\theta) + C_2 \sin(n\theta)$$

As for $R(r)$,

$$r^2 R''(r) + rR(r) - n^2 R(r) = 0 \tag{93}$$

Let $r = e^t$, then $\partial_t R = r\partial_r R$ and $\partial_t^2 R = r^2 \partial_r^2 + r\partial_r R$. Plug into equation (93), we deduce an ODE

$$\partial_t^2 R_n - n^2 R_n = 0$$

whose solution is

$$R_n(r) = \begin{cases} C_1 e^{nt} + C_2 e^{-nt} = C_1 r^n + C_2 r^{-n}, & n \geq 1 \\ C_1 t + C_2 = C \ln r + C_2, & n = 0 \end{cases} \tag{94}$$

To make R_n continuously derivative at origin, let

$$R_n(r) = \begin{cases} r^n, & n \geq 1 \\ 1, & n = 0 \end{cases} \tag{95}$$

Now u is presented as

$$u(r, \theta) = \sum_{n=1}^{\infty} r^n (C_n \cos(n\theta) + D_n \sin(n\theta)) + C_0 \tag{96}$$

Consider boundary value:

$$\widetilde{\phi}(\theta) = u(1, \theta) = \sum_{n=1}^{\infty} (C_n \cos(n\theta) + D_n \sin(n\theta)) + C_0 \tag{97}$$

Due to orthogonality of trigonometric series, it is easy to determine

$$C_n = \frac{1}{\pi} \int_0^{2\pi} \widetilde{\phi}(\theta) \cos(n\theta) \ \mathrm{d}\theta \tag{98}$$

$$D_n = \frac{1}{\pi} \int_0^{2\pi} \widetilde{\phi}(\theta) \sin(n\theta) \ \mathrm{d}\theta \tag{99}$$

$$C_0 = \int_0^{2\pi} \widetilde{\phi}(\theta) \ \mathrm{d}\theta \tag{100}$$

Pluging them into equation (96), we can present the solution.

3 HEAT EQUATION

3.1 Cauchy Problem of Free Heat Equation

Abstract equation

$$\begin{cases} \partial_t u - \Delta u = f(x,t), x \in R^n, t > 0 \\ u(x,0) = \phi(x) \end{cases} \tag{101}$$

3.1.1 Fourier transform

Theorem 46 (Fourier transform in R^n) *If $f \in L^1(R^n)$, i.e. $\int_{R^n} |f(x)| \, \mathrm{d}x < +\infty$, we define*

$$\hat{f}(\xi) = \int_{R^n} f(x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x \tag{102}$$

Theorem 47 (Schwartz space) *We use $\mathcal{S}(R^n)$ to represent Schwartz space:*

$$f(x) \in \mathcal{S}(R^n) \Leftrightarrow f \in C^\infty(R^n), \sup \langle x \rangle^\beta |\partial_x^\alpha f| < +\infty$$

Theorem 48 (properties of Fourier transform) *Fourier transform has following properties:*

1. *(translation) If $(\tau_{x_0} f)(x) = f(x - x_0)$:*

$$\widehat{(\tau_{x_0} f)}(\xi) = e^{-2\pi i x_0 \cdot \xi} \hat{f}(\xi)$$

2. *(scaling) If $(S_\lambda f)(x) = f(\lambda x)$:*

$$\widehat{(S_\lambda f)}(\xi) = \lambda^{-n} \hat{f}(\lambda^{-1} \xi)$$

3. *(derivation) If $\alpha = (\alpha_1, \cdots, \alpha_n)$, we define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, then:*

$$\widehat{(\partial^\alpha f)}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$$

$$(\widehat{(-2\pi i \xi)^\alpha f})(\xi) = \partial_\xi^\alpha \hat{f}(\xi)$$

4. *(convolution):*

$$(\widehat{f * g})(\xi) = \hat{f}(\xi) \hat{g}(\xi)$$

5. *(Laplacian):*

$$(\widehat{-\Delta f})(\xi) = 4\pi^2 |\xi|^2 \hat{f}(\xi)$$

Theorem 49 (Fourier inversion) *If $f \in \mathcal{S}(R^n)$, we define:*

$$\check{f}(x) = \int_{R^n} f(\xi) e^{2\pi i x \cdot \xi} \, \mathrm{d}\xi \tag{103}$$

Theorem 50 *If $f \in \mathcal{S}(R^n)$:*

$$\check{\hat{f}}(x) = f(x) \tag{104}$$

Here we calculate an significant Fourier transform in advance.

Abstract $f(x) = e^{-x^2}$, let

$$F(\xi) = \hat{f}(\xi) = \int_{R^n} e^{-x^2} e^{-2\pi i x \cdot \xi} \, \mathrm{d}x$$

Derivate F,

$$\begin{aligned} F'(\xi) &= \int_{R^n} e^{-x^2} (-2\pi i x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x \\ &= (\pi i) \int_{R^n} (e^{-x^2})' e^{-2\pi i x \cdot \xi} \, \mathrm{d}x \\ &= (\pi i) \hat{f}'(\xi) \\ &= (-w\pi^2 \xi) \hat{F}(\xi) \end{aligned}$$

Solve the ODE above, and notice $F(0) = \int_{R^n} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi}$, we deduce:

$$\hat{f}(\xi) = e^{-\pi^2 \xi^2} \sqrt{\pi}$$

Additionally, former process lead to a deduction:

When $n = 1$,

$$(e^{-4\check{\pi}^2 \xi^2 t})(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \tag{105}$$

When $n \geq 2$, consider each component,

$$(e^{-4\check{\pi}^2 |\xi|^2 t})(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \tag{106}$$

3.1.2 Solving free heat equation

Apply Fourier transform to equation (101) and let $f = 0$:

$$\begin{cases} \partial_t \hat{u}(\xi) + 4\pi^2 |\xi|^2 \hat{u}(\xi) = 0 \\ \hat{u}(\xi, 0) = \hat{\phi}(\xi) \end{cases} \tag{107}$$

A PDE is transformed into an ODE. Therefore,

$$\hat{u}(\xi, t) = e^{-4\pi |\xi|^2 t} \hat{\phi}(\xi) \tag{108}$$

Apply Fourier inversion:

$$\begin{aligned} u(x,t) &= \check{(}e^{-4\pi^2 |\xi|^2 t}) * \phi(x) \\ &= (\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}) * \phi(x) \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{R^n} e^{\frac{|x-y|^2}{4t}} \phi(y) \, \mathrm{d}y \end{aligned} \tag{109}$$

Theorem 51 (heat kernel) *Define heat kernel as*

$$K(x) = \frac{1}{4(\pi)^{\frac{n}{2}}} e^{\frac{|x|^2}{4}}, K_t(x) = t^{-\frac{n}{2}} K(t^{-\frac{1}{2}} x) \tag{110}$$

Then, the solution of a free heat equation can be presented as

$$u(x,t) = \int_{R^n} K_t(x-y)\phi(y) \text{ d}y = \int_{R^n} K_t(y)\phi(x-y) \text{ d}y \tag{111}$$

Theorem 52 (properties of heat kernel) *Abstract $K(t), t > 0$,*

1. $\int_{R^n} K_t(x) \text{ d}x = \int_{R^n} K(x) \text{ d}x = 1$
2. $\int_{R^n} |K_t(x)| \text{ d}x = 1$, *i.e.* $K_t(x) > 0$
3. $\forall \eta > 0, \int_{|x|>\eta} K_t(x) \text{ d}x \to 0, t \to 0^+$

Actually, $\{K_t(x)\}_{t>0}$ is a family of ***approximations of the identity****.*

Theorem 53 (smoothness of solution) *If bounded function $\phi \in C(R^n)$,*

$$u(x,t) = K_t(x) * \phi(x) \to \phi(x), t \to 0^+ \tag{112}$$

This conclusion also illustrates that $\phi(x)$ is definitely the initial value of $u(x,t)$.

Theorem 54 (properties of solution to free heat equation) *If u is a solution to free heat equation when $f = 0$, u has following properties:*

1. $u \in C^\infty(R^n), \forall t > 0$
2. $\sup_x |u(x,t)| \le \sup_x(\phi(x))$
3. *Heat equation doesn't satisfy time reversal.*
4. *Limitless propagating velocity.*

Item 4 implies that even if ϕ has an intense support, there's no longer $x_0 \in R^n$ s.t $u(x_0) = 0$ immediately after time starts to pass.

Theorem 55 (solution to nonhomogeneous equation) *Apply Fourier transform to equation (101):*

$$\begin{cases} \partial_t \hat{u}(\xi) + 4\pi^2|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi,t) \\ \hat{u}(\xi,0) = \hat{\phi}(\xi) \end{cases} \tag{113}$$

A PDE is transformed into an ODE once again. Therefore,

$$\hat{u}(\xi,t) = e^{-4\pi|\xi|^2 t}\hat{\phi}(\xi) + \int_0^t e^{-4\pi|\xi|^2(t-s)} \hat{f}(\xi,s) \text{ d}s \tag{114}$$

Apply Fourier inversion:

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{R^n} e^{\frac{|x-y|^2}{4t}} \phi(y) \text{ d}y + \int_0^t \frac{1}{4(\pi(t-s))^{\frac{n}{2}}} \int_{R^n} e^{\frac{|x-y|^2}{4(t-s)}} f(y,s) \text{ d}y \text{ d}s \tag{115}$$

3.2 Extremum and Modulus Estimation

3.2.1 Dirichlet boundary value problem

Abstract equation

$$\begin{cases} \partial_t u - \Delta u = f(x,t), x \in \Omega, t > 0 \\ u(x,0) = \phi(x), x \in \Omega \\ u(x,t) = h(x,t), x \in \partial\Omega, t \geq 0 \end{cases} \tag{116}$$

Let $Q_T = \Omega \times (0,T]$, we call $\Gamma_T = \bar{Q}_T \backslash Q_T$ a parabolic boundary.

Theorem 56 (extremum principle) *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ satisfies equation $\mathcal{L}u = u_t - u_{xx} = f \leq 0$, $u(x,t)$ reaches it maximum of $\bar{Q}_T$ on parabolic boundary, i.e.*

$$\max_{\bar{Q}_T} u(x,t) = \max_{\Gamma_T} u(x,t) \tag{117}$$

Naturally, we have a deduction:

Theorem 57 *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ satisfies equation $\mathcal{L}u = u_t - u_{xx} = f \geq 0$, $u(x,t)$ reaches it maximum of $\bar{Q}_T$ on parabolic boundary, i.e.*

$$\min_{\bar{Q}_T} u(x,t) = \min_{\Gamma_T} u(x,t) \tag{118}$$

Theorem 58 (camparison principle) *If u, $v \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ satisfies equation $\mathcal{L}u \leq \mathcal{L}v$ and $u|_{\Gamma_T} \leq v|_{\Gamma_T}$, we always have $u(x,t) \leq v(x,t)$ in $\bar{Q}_T$.*

Constructing auxiliary function $v = Ft + B - u$, we can derive maximum modulus estimation in comparison with that of elliptic equations:

Theorem 59 (maximum modulus estimation) *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a solution to equation:*

$$\begin{cases} \mathcal{L} = f, x \in (0,l) \times t > 0 \\ u|_{t=0} = \phi(x), x \in [0,l] \\ u|_{x=0} = g_1(t), u|_{x=l} = g_2(t) \end{cases} \tag{119}$$

let

$$F = \max_{\bar{Q}_T} |f|, \quad B = \max\{\max_{x\in[0,l]} |\phi|, \max_{0<t\leq T} |g_1(t)|, \max_{0<t\leq T} |g_2(t)|\}$$

and we have a conclusion:

$$\max_{\bar{Q}_T} |u(x,t)| \leq FT + B \tag{120}$$

3.2.2 Robin and Neumann boundary value problem

Theorem 60 (uniqueness of solution to Robin boundary value problem) *The solution to equation*

$$\begin{cases} \partial_t - \partial_x^2 = f(x,t), x \in (0,l) \times t > 0 \\ u(x,0) = \phi(x), x \in [0,l] \\ u(0,t) = g_1(x), (u_x + hu)(l,t) = g_2(x), h > 0 \end{cases} \tag{121}$$

is unique.

Constructing auxiliary function $\tilde{u} = (-x + l + 1)u$ and $v = e^{-\lambda t}\tilde{u}$, we can prove maximum principle of u. Imatating the proof of Robin boundary value problem, we deduce:

Theorem 61 (uniqueness of solution to Neumann boundary value problem) *The solution to equation*

$$\begin{cases} \partial_t - \partial_x^2 = f(x,t), x \in (0,l) \times t > 0 \\ u(x,0) = \phi(x), x \in [0,l] \\ u(0,t) = g_1(x), u_x(l,t) = g_2(x) \end{cases} \tag{122}$$

is unique.

3.2.3 modulus estimation

Considering $\omega(x,t) = Ft + \Phi + v_L(x,t) \pm u(x,t)$, where $v_L(x,t) = \frac{M}{L^2}(x^2 + 2t)$, we deduce a similar conclusion to **theorem 59**:

Theorem 62 (maximum modulus estimation in R) *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a bounded solution to equation:*

$$\begin{cases} \mathcal{L} = f, x \in R, 0 < t \leq T \\ u|_{t=0} = \phi(x), x \in R \end{cases} \tag{123}$$

let

$$F = \sup_{\bar{Q}_T} |f|, \Phi = \sup_{x \in R} |\phi|, M = \sup_{\bar{Q}_T} |u| \tag{124}$$

and we have a conclusion:

$$M \leq FT + \Phi \tag{125}$$

Theorem 63 (energy modulus estimation) *If $u \in C^{1,0}(\bar{Q}_T) \cap C^{2,1}(\bar{Q}_T)0$ is a solution to equation*

$$\begin{cases} \partial_t u - \partial_x^2 = f, x \in (0,l) \times (0,T] \\ u|_{t=0} = \phi(x), x \in [0,l] \\ u|_{x=0} = u|_{x=l} = 0 \end{cases} \tag{126}$$

multiple the equation by u, we have

$$\frac{1}{2}\frac{\partial}{\partial t}u^2 - \partial_x(uu_x) + u_x^2 = fu$$

Integrate two sides on $[0, l]$ *and apply mean value inequality:*

$$\frac{1}{2}\int_0^l \frac{\partial}{\partial t}u^2 \ \mathrm{d}x \leq \frac{1}{2}\int_0^l \frac{\partial}{\partial t}u^2 \ \mathrm{d}x + \int_0^l u_x^2 \ \mathrm{d}x \leq \frac{1}{2}\int_0^l f^2 \ \mathrm{d}x + \frac{1}{2}\int_0^l u^2 \ \mathrm{d}x$$

Apply Gronwall's inequality to $y(t) = \int_0^l u^2 \ \mathrm{d}x$,

$$y(t) \leq e^t y(0) + e^t \int_0^t \int_0^l f^2(x,s) \ \mathrm{d}x \ \mathrm{d}s$$

Additionally,

$$\int_0^l u(x,t)^2 \ \mathrm{d}x \leq e^T(\int_0^l \phi(x)^2 \ \mathrm{d}x + \int_0^T \int_0^l f^2(x,s) \ \mathrm{d}x \ \mathrm{d}s)$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\int_0^l u^2 \ \mathrm{d}x + \int_0^l u_x^2 \ \mathrm{d}x \leq \frac{1}{2}\int_0^l f^2 \ \mathrm{d}x + \frac{1}{2}\int_0^l u^2 \ \mathrm{d}x \\ &\leq \frac{1}{2}\int_0^l f^2 \ \mathrm{d}x + e^T(\int_0^l \phi(x)^2 \ \mathrm{d}x + \int_0^T \int_0^l f^2(x,s) \ \mathrm{d}x \ \mathrm{d}s)\end{aligned}$$

Integrate two sides on $[0, t]$,

$$\begin{aligned}&\frac{1}{2}\int_0^l u^2(x,t) \ \mathrm{d}x + \int_0^t \int_0^l u_x^2(x,s) \ \mathrm{d}x \ \mathrm{d}t \\ &\leq \int_0^l \phi(x)^2 \ \mathrm{d}x + \frac{1}{2}\int_0^t \int_0^l f^2(x,s) \ \mathrm{d}x \ \mathrm{d}s \\ &+ (e^t - 1)(\int_0^l \phi(x)^2 \ \mathrm{d}x + \int_0^T \int_0^l f^2(x,s) \ \mathrm{d}x \ \mathrm{d}s) \\ &\leq \frac{1}{2}e^T(\int_0^l \phi(x)^2 \ \mathrm{d}x + \int_0^T \int_0^l f^2(x,s) \ \mathrm{d}x \ \mathrm{d}s)\end{aligned}$$

Due to arbitrariness of t, *we have*

$$\sup_{0\leq t\leq T}\int_0^l u^2(x,t) \ \mathrm{d}x + 2\int_0^T \int_0^l u_x(x,t) \ \mathrm{d}x \ \mathrm{d}t \leq M(\int_0^l \phi^2 \ \mathrm{d}x + \int_0^T \int_0^l f^2(x,t) \ \mathrm{d}x \ \mathrm{d}t) \tag{127}$$

3.2.4 variable separation

Abstract equation:

$$\begin{cases} \partial_t u - \partial_x^2 = 0, x \in (0,l) \times (0,T] \\ u(x,0) = \phi(x), x \in [0,l] \\ u(0,t) = 0, u_x(l,t) + hu(l,t) = 0, h > 0 \end{cases} \tag{128}$$

Plug $u(x,t) = T(t)X(x)$ into the equation,

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{129}$$

New boundary values are

$$T(t)X(0) = 0, \quad T(t)X'(l) + hT(t)X(l) = 0$$

i.e.

$$X(0) = 0, \quad X'(l) + hX(l) = 0$$

Solving equation (129), we deduce

$$X(x) = \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, & \lambda < 0 \\ C_1 x + C_2, & \lambda = 0 \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \end{cases} \tag{130}$$

To satisfy boundary values, there must be $C_1 = 0$, $\lambda > 0$. Moreover,

$$\tan(\sqrt{\lambda}l) = -\frac{\sqrt{\lambda}}{n} \tag{131}$$

Analysing the graph, we figure out a increasing positive series of $\{\lambda_n\}$ satisfying equation (130). Let $X_n(x) = \sin(\sqrt{\lambda_n}x)$, then $T(t) = Ae^{-\lambda_n t}$ according to former equations, We have

$$u(x,t) = \sum_{n=1}^{\infty} Ae^{-\lambda_n t} \sin(\sqrt{\lambda_n}x) \tag{132}$$

Expand initial value $\phi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\sqrt{\lambda_n}x)$$

where

$$\phi_n = \frac{\int_0^l \phi(x) \sin(\sqrt{\lambda_n}x)\ \mathrm{d}x}{\int_0^l \sin^2(\sqrt{\lambda_n}x)\ \mathrm{d}x} \tag{133}$$

Compare parameters:

$$A_n = \phi_n e^{\lambda_n t}$$

Ultimately, we deduce

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n \sin(\sqrt{\lambda_n}x) \tag{134}$$