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第四章 波动方程

方程的来源：均匀细弦/薄膜/弹性体的自由/受迫振动

$u(x,t)$ 未知　　$\Delta u = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 u$.

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & x\in\Omega,\ t>0 \\ u(x,0) = f(x) & x\in\Omega \\ u_t(x,0) = g(x) & x\in\Omega \\ \text{边值} \end{cases}$$

初值，为关于 x 的函数

Rmk. 物理意义

$f(x,t)$ 表示单位质量所受外力

第一类边值（Dirichlet）　$u(x,t) = h(x,t)$　$\forall x\in\partial\Omega$

第二类边值（Newmann）　$\frac{\partial u}{\partial \vec{n}}(x,t) = h(x,t)$　$\forall x\in\partial\Omega$

第三类边值（Robin）　$\frac{\partial u}{\partial \vec{n}}(x,t) + \alpha(x,t)\,u(x,t) = h(x,t)$　$\forall x\in\partial\Omega$　$\alpha(x,t)>0$

Rmk. 边值意义

（Dirichlet）边界点位移变化

若 $h(x,t) \equiv h(x)$，则边界点固定

（Newmann）边界点受力情况

若 $h(x,t)\equiv 0$，则无外力通过 $\partial\Omega$ 对弹性体作用

（Robin）位移与受力线性组合

若 $h(x,t)\equiv 0$，则 $\partial\Omega$ 固定在支架

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§4.1 初值问题

§4.1.1 一阶偏微分方程的解

考虑:
$$\begin{cases} \frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} + b(x,t)u = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

$u(x,t)$ 为未知函数 $-\infty<x<+\infty,\ t>0$

若 $x=x(t)$, 令 $u(x(t),t) = U(t)$

$$\frac{dU}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} x'(t)$$

x(t) 称为特征线

若 $x'(t) = a(x(t),t)$, $\frac{dU}{dt} + b(x(t),t)U(t) = f(x(t),t)$

令 $x(0)=C$, 则 $U(0) = u(x(0),0) = \phi(C)$

$$\Rightarrow \begin{cases} \frac{dx}{dt} = a(x(t),t) \\ x(0)=C \end{cases} \qquad \begin{cases} \frac{dU}{dt} + b(x(t),t)U(t) = f(x(t),t) \\ U(0) = \phi(C) \end{cases}$$

转化为2个 ode

ex.
$$\begin{cases} \frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dx}{dt} = -a \\ x(0) = C \end{cases} \Rightarrow x(t) = -at + C$$

$$\begin{cases} \frac{dU}{dt} = 0 \\ U(0) = \phi(C) \end{cases} \Rightarrow U = \phi(C)$$

$U(t) = \phi(x(t)+at) = u(x(t),t)$.

取 $u(x,t) = \phi(x+at)$.

ex.
$$\begin{cases} \frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dx}{dt} = -a \\ x(0) = C \end{cases} \Rightarrow x(t) = -at+C$$

$$\begin{cases} \frac{dU}{dt} = f(x(t),t) \\ U(0) = \phi(C) \end{cases}$$

x(t) 带入, $\frac{dU}{dt} = f(-at+C,t)$

$$\Rightarrow U(t) = \phi(C) + \int_0^t f(-a\tau+C,\tau)d\tau$$

$$u(x,t) = \phi(x+at) + \int_0^t f(x+a(t-\tau),\tau)d\tau$$

ex. $\begin{cases} \frac{\partial u}{\partial t} + (x+t)\frac{\partial u}{\partial x} + u = x \\ u|_{t=0} = x \end{cases}$

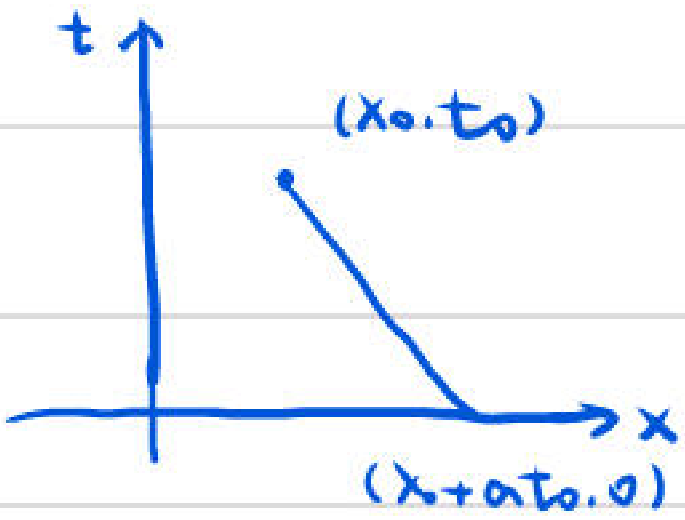
$\begin{cases} \frac{dx}{dt} = x+t \\ x(0) = C \end{cases}$ (一阶线性) $\Rightarrow x(t) = Ce^{t} - t - 1$

$\begin{cases} \frac{dU}{dt} + U(t) = Ce^{t} + e^{t} - t - 1 \\ U(0) = C \end{cases} \Rightarrow U(t) = -t + \frac{1}{2}(e^{t} - e^{-t}) + \frac{C}{2}(e^{t} - e^{-t})$

$\Rightarrow u(x,t) = \frac{1}{2}(x-t+1) - e^{-t} + \frac{1}{2}(x+t+1)e^{-2t}$

Rmk. 对第一个方程有如下较为几何的解释

在 $x(t) = -at + C$ 上，有 $\frac{dU}{dt} = 0$



依据初值 $u(x_0+at_0, 0) = \phi(x_0+at_0) = u(x_0, t_0) \quad \forall x_0, t_0$

$\Rightarrow u(x,t) = \phi(x+at)$

同样地，对第二个方程

$u(x_0+at_0, 0) = \phi(x_0+at_0)$

$u(x_0, t_0) = u(x_0+at_0, 0) + \int_0^{t_0} f(x(t), t)\,dt$

$= \phi(x_0+at_0) + \int_0^{t_0} f(x_0 + a(t_0 - t), t)\,dt \quad \forall x_0, t_0$

$\Rightarrow u(x,t) = \phi(x+at) + \int_0^{t} f(x + a(t-\tau), \tau)\,d\tau$

特征线法将PDE转化为ODE，最终结果即在特征线上对时间积分

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§4.1.2 问题的简化

对 R^n，仅需提出初值条件（无边界）

考察初值问题

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = \varphi(x) \qquad x\in R^n,\ t>0 \qquad (4.6) \\ u_t(x,0) = \psi(x) \end{cases}$$

用构造的办法给出初值问题的解，暂未说明唯一性

令 u_1 满足 $\begin{cases} \partial_t^2 u_1 - \Delta u_1 = 0 \\ u_1(x,0) = \varphi(x) \quad (4.7) \\ \partial_t u_1(x,0) = 0 \end{cases}$ "位移"　令 u_2 满足 $\begin{cases} \partial_t^2 u_2 - \Delta u_2 = 0 \\ u_2(x,0) = 0 \quad (4.8) \\ \partial_t u_2(x,0) = \Psi(x) \end{cases}$ "速度"

令 u_3 满足 $\begin{cases} \partial_t^2 u_3 - \Delta u_3 = f(x,t) \\ u_3(x,0) = 0 \quad (4.9) \\ \partial_t u_3(x,0) = 0 \end{cases}$ "外力"

则 $u_1+u_2+u_3$ 为 (4.6) 的解

thm 1. $u_2 = M_\psi(x,t)$ 为 (4.8) 解，则 (4.7)、(4.9) 的解为

$$u_1 = \frac{\partial}{\partial t} M_\varphi, \quad u_3 = \int_0^t M_{f_\tau}(x, t-\tau)\,d\tau$$

pr. 令 $\tilde{u}_1 = M_\varphi$，则 $\begin{cases} \partial_t^2 \tilde{u}_1 - \Delta \tilde{u}_1 = 0 \\ \tilde{u}_1(x,0) = 0 \\ \partial_t \tilde{u}_1(x,0) = \varphi(x) \end{cases}$

令 $v = \partial_t \tilde{u}_1$，则 $\begin{cases} \partial_t^2 v - \Delta v = 0 \\ v(x,0) = \varphi(x) \\ \partial_t v(x,0) = \partial_t^2 \tilde{u}_1(x,0) = \Delta \tilde{u}_1(x,0) = 0 \end{cases}$

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Rmk. 初值为先偏求导数，如 Δu、u_t、u_{tt}，再赋值　先赋值，再偏求导数

在上述过程中 $v(x,0)=\hat{u}_t(x,0)=\frac{\partial}{\partial t}\hat{u}(x,0)\times\frac{\partial}{\partial t}0=0$

对其余变量求偏导时，可交换偏求导数与赋值顺序

令 $\hat{u}_1(x,t)=M_{f_\tau}(x,t)$，则 $\begin{cases}\partial_t^2\hat{u}_1-\Delta\hat{u}_1=0\\ \hat{u}_1(x,0)=0\\ \partial_t\hat{u}_1(x,0)=f_\tau=f(x,\tau)\end{cases}$　τ 为参数

令 $v(x,t)=M_{f_\tau}(x,t-\tau)=\hat{u}_1(x,t-\tau)$，则

$$\begin{cases}\partial_t^2 v-\Delta v=(\partial_t^2\hat{u}_1-\Delta\hat{u}_1)(x,t-\tau)=0\\ v\big|_{t=\tau}=\hat{u}_1(x,0)=0\\ \partial_t v\big|_{t=\tau}=\partial_t\hat{u}_1(x,0)=f(x,\tau)\end{cases}$$

$$u_3=\int_0^t M_{f_\tau}(x,t-\tau)d\tau$$

$$\partial_t u_3=M_{f_t}(x,0)+\int_0^t\frac{\partial}{\partial t}M_{f_\tau}(x,t-\tau)d\tau$$
$$=\int_0^t\frac{\partial}{\partial t}M_{f_\tau}(x,t-\tau)d\tau$$

$$\partial_t^2 u_3=\frac{\partial}{\partial t}M_{f_t}(x,0)+\int_0^t\frac{\partial^2}{\partial t^2}M_{f_\tau}(x,t-\tau)d\tau$$
$$=f(x,t)+\int_0^t\Delta M_{f_\tau}(x,t-\tau)d\tau$$
$$=f(x,t)+\Delta\int_0^t M_{f_\tau}(x,t-\tau)d\tau$$
$$=f(x,t)+\Delta u_3$$

且 $u_3(x,0)=0$，$\partial_t u_3(x,0)=0$

Rmk. “Δ”可拿出来，由于为对 x 的导数，积分限不含 x

该过程被称作冲量原理（Duhamel）　非齐次方程 → 具有初速度齐次方程解的和

$$u_3(x,t)=\lim_{\|\Delta\|\to0}\sum_{i=0}^{n-1}M_{f_{t_i}}(x,t-t_i)\Delta t_i=\lim_{\|\Delta\|\to0}\sum_{i=0}^{n-1}M_{f_{t_i}\Delta t_i}(x,t-t_i)$$

故问题转化为如何考查方程 $\begin{cases} \partial_t^2 u-\Delta u=0 \\ u(x,0)=0,\ \partial_t u(x,0)=\psi(x) \end{cases}$

的解，thm 1. 指出了"速度"方程具有本质性.

Recall：傅立叶变换 $\hat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$　ξ 为参向量

$$\widehat{\Delta f}(\xi)=\int_{\mathbb{R}^n}\Delta f(x)e^{-2\pi i x\cdot\xi}dx=\sum_j\int_{\mathbb{R}^n} f_{x_jx_j}e^{-2\pi i x\cdot\xi}dx$$

$$=-\sum_j\int_{\mathbb{R}^n} f_{x_j}\cdot(-2\pi i\xi_j)e^{-2\pi i x\cdot\xi}dx$$

$f,\ f_{x_i}$ 在无穷远处 $\to 0$

$$=-\sum_j(-2\pi i\xi_j)\int_{\mathbb{R}^n} f_{x_j}e^{-2\pi i x\cdot\xi}dx$$

$$=\sum_j(-2\pi i\xi_j)^2\int_{\mathbb{R}^n} f e^{-2\pi i x\cdot\xi}dx$$

$$=-4\pi^2|\xi|^2\hat{f}(\xi)$$

将分析运算转化为代数运算

方程 $\begin{cases} \partial_t^2 u-\Delta u=0 \\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x) \end{cases}$

两边关于 x 的傅立叶变换，$\begin{cases} \partial_t^2\hat{u}(\xi,t)+4\pi^2|\xi|^2\hat{u}(\xi,t)=0 \\ \hat{u}(\xi,0)=\hat{\varphi}(\xi),\ \partial_t\hat{u}(\xi,0)=\hat{\psi}(\xi) \end{cases}$

ξ 视作参数，则为二阶常微分方程，$\lambda^2+4\pi^2|\xi|^2=0 \Rightarrow \lambda=\pm 2\pi|\xi| i$

$$\Rightarrow \hat{u}(\xi,t)=C_1\cos(2\pi t|\xi|)+C_2\sin(2\pi t|\xi|)$$

$$\hat{u}(\xi,0)=C_1=\hat{\varphi}(\xi)$$

$$\partial_t\hat{u}(\xi,0)=C_2\cdot 2\pi|\xi|=\hat{\psi}(\xi)$$

$$\Rightarrow \hat{u}(\xi,t)=\cos(2\pi t|\xi|)\hat{\varphi}(\xi)+\frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}\hat{\psi}(\xi)$$

Rmk. 也说明了 u_1,u_2 解的关系

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方程 $\begin{cases} \partial_t^2 u_3 - \Delta u_3 = f(x,t) \\ u_3(x,0)=0,\ \partial_t u_3(x,0)=0 \end{cases}$

关于 x 取傅立叶变换，$\begin{cases} \partial_t^2 \hat{u}_3 + 4\pi^2 |\xi|^2 \hat{u}_3 = \hat{f}(\xi,t) \\ \hat{u}_3(\xi,0)=0,\ \partial_t \hat{u}_3(\xi,0)=0 \end{cases}$

$\Rightarrow \hat{u}_3(\xi,t) = \int_0^t \frac{\sin(2\pi|\xi|(t-\tau))}{2\pi|\xi|} \hat{f}(\xi,\tau)\,d\tau$

Rmk. 也说明了 u_2, u_3 的关系.

若对 x, t 同时变换　$x\to\xi$, $t\to s$

$-4\pi^2 s^2 \hat{u} + 4\pi^2 |\eta|^2 \hat{u} = 0$

$\Rightarrow (|\eta|^2 - s^2)\,\hat{u}(s,\xi) = 0$

$\Rightarrow s^2 = |\eta|^2 \quad (s,\eta)\in \mathbb{R}^{1+n}$ 构成锥面

表明 $\hat{u}$ 仅在锥面上可不为 0，波方程对时空的 Fourier 变换支在锥面上.

§4.1.3 一维初值问题

考虑R上波动方程
$$\begin{cases} \partial_t^2 u-\partial_x^2 u=f(x,t) & x\in\mathbb{R},\ t>0 \\ u(x,0)=\varphi(x) & x\in\mathbb{R} \qquad (4.13) \\ \partial_t u(x,0)=\psi(x) & x\in\mathbb{R} \end{cases}$$

Rmk. x的范围会影响解方程的方法与结果

$$\begin{cases} \partial_t^2 u_2-\Delta u_2=0 \\ u_2(x,0)=0 \qquad (4.8) \\ \partial_t u_2(x,0)=\Psi(x) \end{cases}$$
(由上节thm知只需求解该方程即得一维波动方程的解)

即 $(\partial_t+\partial_x)(\partial_t-\partial_x)u=0$

令 $v(x,t)=(\partial_t-\partial_x)u$，则 $\begin{cases} \partial_t v+\partial_x v=0 \\ v(x,0)=(\partial_t-\partial_x)u(x,0)=\Psi(x) \end{cases}$

$\Rightarrow v(x,t)=\Psi(x-t)$

$$\begin{cases} \partial_t u-\partial_x u=\Psi(x-t) \\ u(x,0)=0 \end{cases}$$

$$\Rightarrow u(x,t)=\int_0^t \Psi(\underbrace{x+t-2\tau}_{=:y})\,d\tau$$
$$=\frac{1}{2}\int_{x-t}^{x+t}\Psi(y)\,dy$$

故 $u_2(x,t)=\frac{1}{2}\int_{x-t}^{x+t}\Psi(y)\,dy$

$u_1(x,t)=\frac{d}{dt}\left(\frac{1}{2}\int_{x-t}^{x+t}\varphi(y)\,dy\right)$
$=\frac{1}{2}\left(\varphi(x+t)+\varphi(x-t)\right)$

$u_3(x,t)=\int_0^t\frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)\,dy\,d\tau$
$=\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)\,dy\,d\tau$

故(4.13)解为

$u_1+u_2+u_3 \qquad (4.20)$

thm 2 (D' Alembert公式)

若 $f\equiv 0$, 令 $F(x)=\frac{1}{2}\varphi(x)+\frac{1}{2}\int_0^x\psi(y)\,dy$

$$G(x)=\frac{1}{2}\varphi(x)+\frac{1}{2}\int_x^0\psi(y)\,dy$$

$\Rightarrow u(x,t)=F(x+t)+G(x-t)$

左行波　右行波

thm 3　给出形式解存在条件 (形式解 → 古典解)

$\varphi\in C^2(R)$, $\psi\in C^1(R)$, $f\in C^1(R\times\overline{R^+})$

则 (4.20) 给出的函数 $u\in C^2(R\times\overline{R_+})$, 且为初值问题 (4.13) 解

pr. $u(x,t)=\frac{1}{2}(\varphi(x+at)+\varphi(x-at))+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\,dy+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(y,\tau)\,dy\,d\tau$

① $u(x,t)\in C(R\times\overline{R_+})$ 为连续函数复合/变限积分

② $u_t(x,t)=\frac{a}{2}\varphi'(x+at)-\frac{a}{2}\varphi'(x-at)+\frac{1}{2a}\left(a\psi(x+at)+a\psi(x-at)\right)$

$$+\frac{1}{2a}\int_0^t af(x+a(t-\tau),\tau)+af(x-a(t-\tau),\tau)\,d\tau$$

$u_x(x,t)=\frac{1}{2}\varphi'(x+at)+\frac{1}{2}\varphi'(x-at)+\frac{1}{2a}(\psi(x+at)-\psi(x-at))$

$$+\frac{1}{2a}\int_0^t f(x+a(t-\tau),\tau)-f(x-a(t-\tau),\tau)\,d\tau$$

$u(x,t)\in C^1(R\times\overline{R_+})$, 由于 $u_t, u_x\in C(R\times\overline{R_+})$

③ $u_{tt}(x,t)=\frac{a^2}{2}\varphi''(x+at)+\frac{a^2}{2}\varphi''(x-at)+\frac{1}{2a}(a^2\psi'(x+at)-a^2\psi'(x-at))$

$$+\frac{1}{2a}\left[2af(x,t)+\int_0^t a^2f_x(x+a(t-\tau),\tau)-a^2f_x(x-a(t-\tau),\tau)\,d\tau\right]$$

$u_{xx}(x,t)=\frac{1}{2}\varphi''(x+at)+\frac{1}{2}\varphi''(x-at)+\frac{1}{2a}(\psi'(x+at)-\psi'(x-at))$

$$+\frac{1}{2a}\int_0^t\left[f_x(x+a(t-\tau),\tau)-f_x(x-a(t-\tau),\tau)\right]d\tau$$

$u_{xt}(x,t)=\frac{a}{2}\varphi''(x+at)-\frac{a}{2}\varphi''(x-at)+\frac{1}{2}(\psi'(x+at)+\psi'(x-at))$

$$+\frac{1}{2a}\int_0^t af_x(x+a(t-\tau),\tau)+af_x(x-a(t-\tau),\tau)\,d\tau$$

$\in C(R\times\overline{R_+})=u_{tx}(x,t)$

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$u(x,t) \in C^2(\mathbb{R}\times\overline{\mathbb{R}_+})$

④ $u_{tt} - a^2 u_{xx} = f(x,t)$

$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x)$

则u为初值问题的解

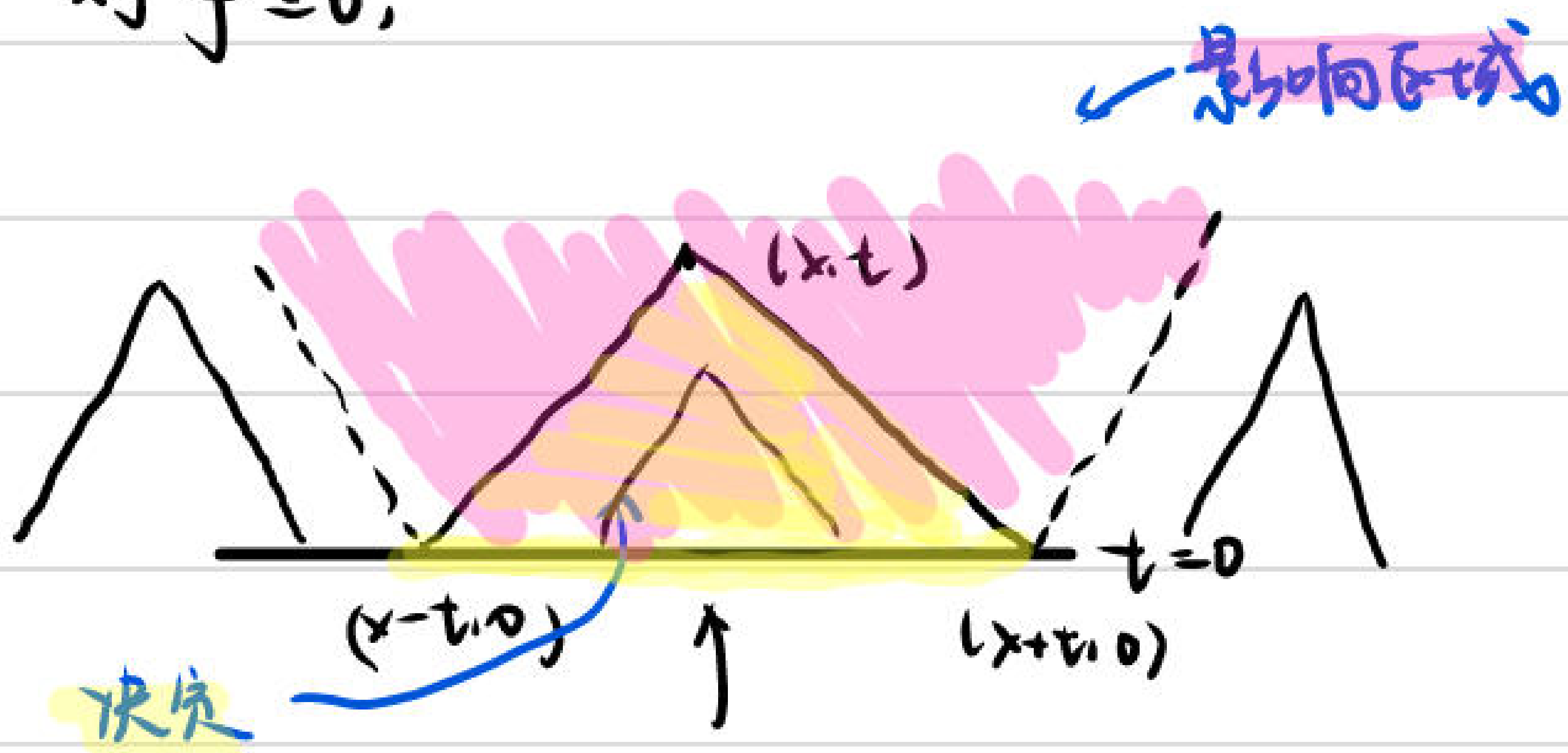
thm 4 利用形式解分析性质

若 φ、ψ、f 为x的偶/奇/周期为l函数

由(4.20)给出的解 u 为x的偶/奇/周期为l函数

Rmk. 几何性质

对 $f\equiv 0$,



$u(x,t)$ 只与初值在 $[x-t, x+t]$ 上取值有关

§4.1.4 一维半无界问题

考虑波动方程
$$\begin{cases} \partial_t^2 u-\partial_x^2 u=f(x,t) & x>0\ \ t>0 \\ u(x,0)=\varphi(x) & x\geq 0 \\ \partial_t u(x,0)=\psi(x) & x\geq 0 \\ u(0,t)=g(t) & t\geq 0\ \text{(边值)} \end{cases} \qquad (4.21)$$

1、若 $g(t)\equiv 0$，作奇延拓，

令 $\bar\varphi(x)=\begin{cases}\varphi(x) & x\geq 0\\ -\varphi(-x) & x<0\end{cases}$ $\quad \bar\psi(x)=\begin{cases}\psi(x) & x\geq 0\\ -\psi(-x) & x<0\end{cases}$ $\quad \bar f(x,t)=\begin{cases}f(x,t) & x\geq 0\\ -f(-x,t) & x<0\end{cases}$

Rmk. 奇延拓，则 $w(0)=0$；偶延拓，则 $w'(0)=0$

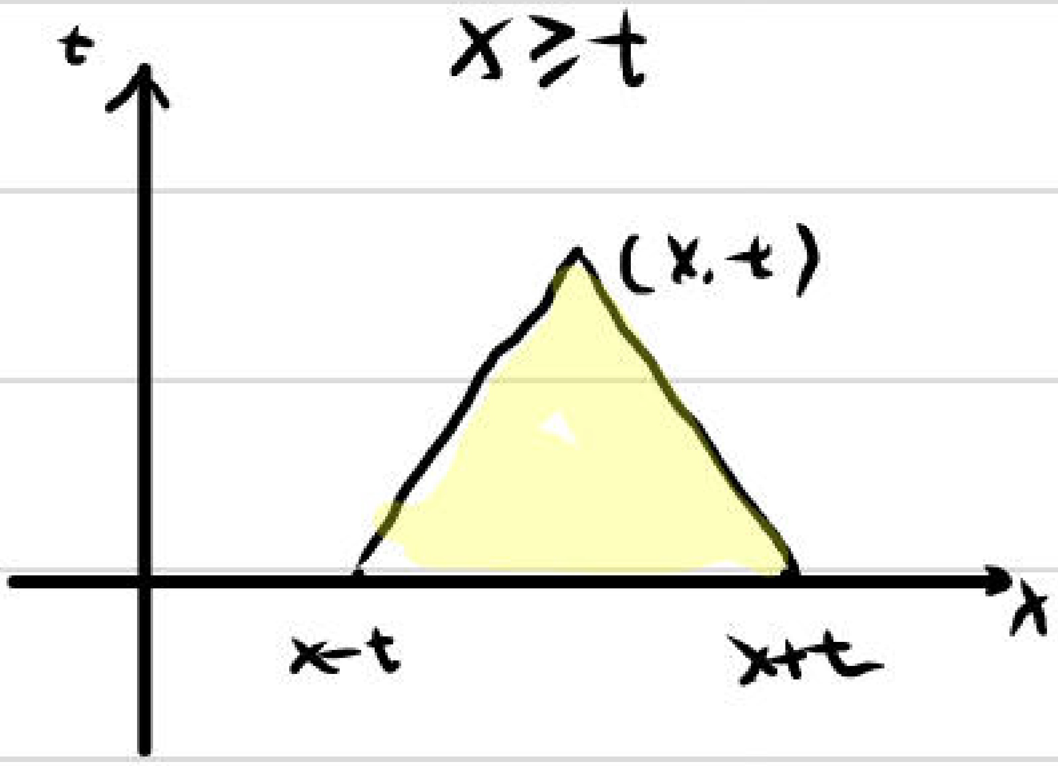
令 $\bar u(x,t)$ 为方程
$$\begin{cases} \partial_t^2 \bar u-\partial_x^2 \bar u=\bar f(x,t) & x\in\mathbb{R}\ \ t>0 \\ \bar u(x,0)=\bar\varphi(x) & x\in\mathbb{R} \\ \partial_t \bar u(x,0)=\bar\psi(x) & x\in\mathbb{R} \end{cases}$$

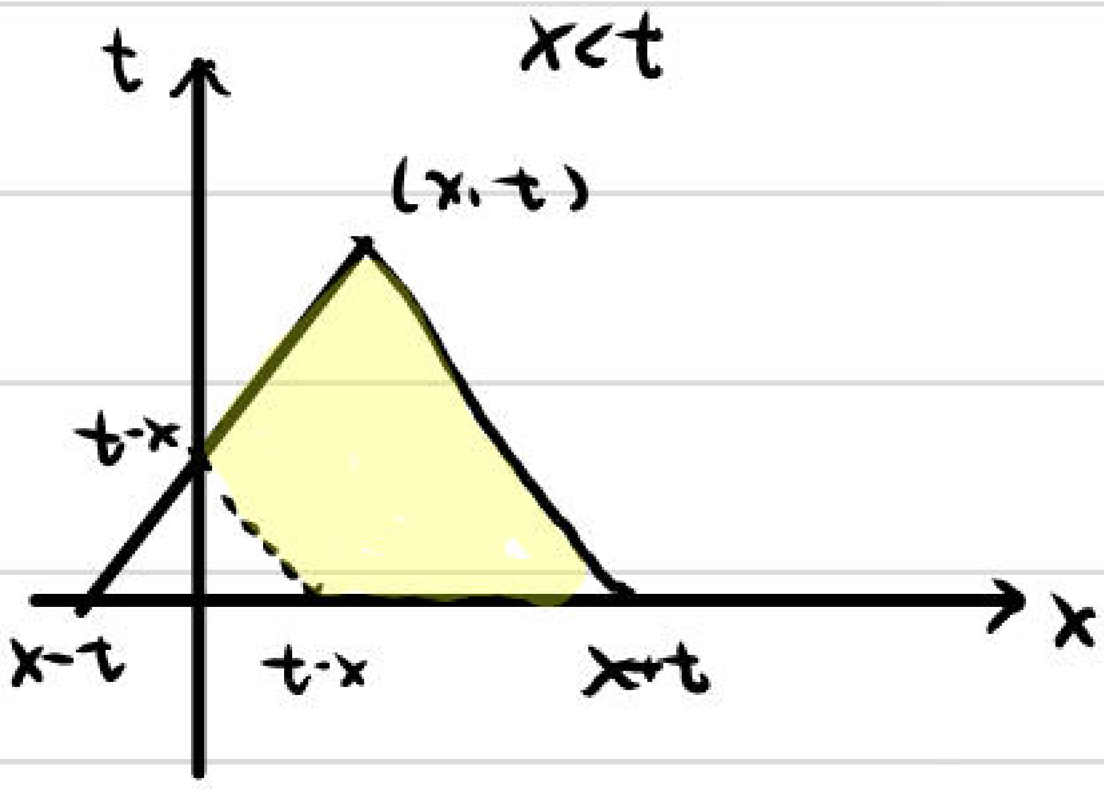
的解，由于对 x 的奇性，$\bar u(0,t)=0,\ t\geq 0$ 满足边值

由 (4.20)，$\bar u(x,t)=\frac12(\bar\varphi(x+t)+\bar\varphi(x-t))+\frac12\int_{x-t}^{x+t}\bar\psi(y)dy+\frac12\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}\bar f(y,\tau)\,dy\,d\tau$ $\quad x\in\mathbb{R},t>0$

$x>0$，$x\geq t$ 时，
$$u(x,t)=\frac12(\varphi(x+t)+\varphi(x-t))+\frac12\int_{x-t}^{x+t}\psi(y)dy+\frac12\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)\,dy\,d\tau \qquad (4.23)$$

$x>0$，$x<t$ 时，
$$u(x,t)=\frac12(\varphi(x+t)-\varphi(t-x))+\frac12\int_0^{x+t}\psi(y)dy+\frac12\int_{x-t}^0-\psi(-y)dy$$
$$+\frac12\int_{t-x}^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)dyd\tau+\frac12\int_0^{t-x}\int_0^{x+(t-\tau)}f(y,\tau)dyd\tau+\frac12\int_0^{t-x}\int_{x-(t-\tau)}^0-f(-y,\tau)dyd\tau$$
$$=\frac12(\varphi(x+t)-\varphi(t-x))+\frac12\int_{t-x}^{t+x}\psi(y)dy$$
$$+\frac12\int_{t-x}^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)\,dy\,d\tau+\frac12\int_0^{t-x}\int_{(t-\tau)-x}^{(t-\tau)+x}f(y,\tau)\,dy\,d\tau \qquad (4.24)$$





黄色部分所示为积分区域

如何完成从形式解到古典解过渡？ Rmk. 古典解：所需的各阶偏导数连续可微，且在边界处连续

相容性条件：

① $\lim\limits_{x\to0^+} u(x,0) = u(0,0) = \lim\limits_{t\to0^+} u(0,t)$

$\qquad \| \qquad\qquad\qquad\qquad \|$

$\quad \varphi(0) \qquad\qquad\qquad g(0)=0$ $\qquad \varphi(0)=0 \quad (4.25)$

② $\lim\limits_{x\to0^+} u_t(x,0) = u_t(0,0) = \lim\limits_{t\to0^+} u_t(0,t)$

$\qquad \| \qquad\qquad\qquad\qquad \|$

$\quad \psi(0) \qquad\qquad\qquad g'(0)=0$ $\qquad \psi(0)=0 \quad (4.26)$

③ $\lim\limits_{x\to0^+} u_{tt}(x,0) = u_{tt}(0,0) = \lim\limits_{t\to0^+} u_{tt}(0,t)$

$\qquad \| \qquad\qquad\qquad\qquad \|$

$\lim\limits_{x\to0^+}\left(\partial_x^2 u(x,0)+f(x,0)\right) \qquad g''(0)=0$ $\qquad \varphi''(0)+f(0,0)=0 \quad (4.27)$

$\qquad \|$

$\quad \varphi''(0)+f(0,0)$

thm 5

若 (4.21) 初值 $\varphi(x)\in C^2(R_+)$, $\psi(x)\in C^1(R_+)$, $f(x,t)\in C^1(\overline{R_+}\times\overline{R_+})$

满足相容性条件，且边值 $g(t)\equiv0$，(4.23), (4.24) 给出函数 $u\in C^2(\overline{R_+}\times\overline{R_+})$

且为 (4.21) 的解

2. 若 $g(t)\not\equiv0$，令 $v(x,t)=u(x,t)-g(t)$

则 $v(0,t)=u(0,t)-g(t)=0$

原方程转化为

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$$\begin{cases} \partial_t^2 v - \partial_x^2 v = f(x,t) - g''(t) \\ v(x,0) = u(x,0) - g(0) = \varphi(x) - g(0) \\ \partial_t v(x,0) = \partial_t u(x,0) - g'(0) = \psi(x) - g'(0) \end{cases}$$

即转化为情形1.

再给出更一般相容性条件

thm 6 相容性条件

$\varphi(0) = g(0)$, $\psi(0) = g'(0)$, $f(0,0) + \varphi''(0) = g''(0)$

Rmk、相容性条件保证了初值与边值在二阶可微时的自洽.

thm 7 第二类边值问题

给定 $u_x(0,t) = g(t)$. 则令 $u(x,t) = x g(t) + v(x,t)$

$v(x,t)$ 满足 $v_x(0,t) = 0$. 之后利用偶延拓得到 $v(x,t)$, 进而得到 $u(x,t)$

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§4.1.5 三维初值问题

$n=3$ $\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) \end{cases}$

在极坐标系下，$\Delta u = \partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u$ （Δ_{S^2} 表示 S^2 上 Laplace）

$\Delta \triangleq \mathrm{div}\,\nabla$

$$\partial_t^2 u - \left(\partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u\right) = f(x,t)$$

$$\int_{S^2} \Delta_{S^2} u\, d\omega = \int_{S^2} \mathrm{div}\,\nabla u\, d\omega = \int_{\partial S^2} \frac{\partial u}{\partial n}\, ds = 0$$

1、先考虑 $f(x,t)=0$ 情形

$$\partial_t^2 u - \left(\partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u\right) = 0$$

对方程两边在 S^2 上积分，S^2 为单位球面

$$\partial_t^2 \int_{S^2} u\,d\omega - \left(\partial_r^2 \int_{S^2} u\,d\omega + \frac{2}{r}\partial_r \int_{S^2} u\,d\omega\right) = 0$$

令 $\bar{u}(t,r) = \frac{1}{4\pi}\int_{S^2} u\,d\omega$

则 $\partial_t^2 \bar{u} - \partial_r^2 \bar{u} - \frac{2}{r}\partial_r \bar{u} = 0$

令 $\bar{u}(t,r) = r^{-k} v(t,r)$

$$\partial_r \bar{u} = -k r^{-k-1} v(t,r) + r^{-k}\partial_r v$$

$$\partial_r^2 \bar{u} = -k\left[(-k-1) r^{-k-2} v(t,r) + r^{-k-1}\partial_r v\right] + (-k) r^{-k-1}\partial_r v + r^{-k}\partial_r^2 v$$

$$= k(k+1) r^{-k-2} v - 2k r^{-k-1}\partial_r v + r^{-k}\partial_r^2 v$$

$$\partial_r^2 \bar{u} + \frac{2}{r}\partial_r \bar{u} = k(k+1) r^{-k-2} v - 2k r^{-k-1}\partial_r v + r^{-k}\partial_r^2 v - 2k r^{-k-2} v + 2 r^{-k-1}\partial_r v$$

取 $k=1$，$v(t,r) = r\bar{u}(t,r)$

有 $\begin{cases} \partial_t^2 v - \partial_r^2 v = 0 \\ v(r,0) = r\bar{u}(r,0) = r\bar{\varphi}(r) \\ \partial_t v(r,0) = r\partial_t \bar{u}(r,0) = r\bar{\psi}(r) \end{cases}$　　$r \geq 0$，"$-$"表示积分

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将 v 关于 r 作偶延拓，得到 $\bar{v}$

以下由"$\bar{\ }$"代表延拓后函数

$$\Rightarrow \bar{v}(r,t)=\frac{1}{2}\left((r+t)\bar{\varphi}(r+t)+(r-t)\bar{\varphi}(r-t)\right)+\frac{1}{2}\int_{r-t}^{r+t} y\,\bar{\Psi}(y)\,dy$$

Rmk,

$\partial_t^2 u-\Delta u=0$ 在以下变换中不变：

时间平移：$u(x,t)\mapsto u(x,t+t_0)$

空间平移：$u(x,t)\mapsto u(x+x_0,t)$

伸缩变换：$u(x,t)\mapsto u(\frac{x}{\lambda},\frac{t}{\lambda})\quad \lambda>0 \triangleq u^\lambda(x,t)$

$$\Big(\partial_t u^\lambda(x,t)=\partial_t\Big(u(\tfrac{x}{\lambda},\tfrac{t}{\lambda})\Big)=(\partial_t u)(\tfrac{x}{\lambda},\tfrac{t}{\lambda})\cdot\tfrac{1}{\lambda}$$

$$\partial_t^2 u^\lambda(x,t)=(\partial_t^2 u)(\tfrac{x}{\lambda},\tfrac{t}{\lambda})\cdot\tfrac{1}{\lambda^2}$$

$$\text{故 } \partial_t^2 u^\lambda-\Delta u^\lambda=(\partial_t^2 u)(\tfrac{x}{\lambda},\tfrac{t}{\lambda})\cdot\lambda^{-2}-\lambda^{-2}(\Delta u)(\tfrac{x}{\lambda},\tfrac{t}{\lambda})=0\Big)$$

Lorentz 变换：$u(x,t)\to u\left(x-x_v+\frac{x_v-vt}{\sqrt{1-|v|^2}},\ \frac{t-v\cdot x}{\sqrt{1-|v|^2}}\right)$

$$x_v\triangleq\left(x\cdot\frac{v}{|v|}\right)\frac{v}{|v|}$$

以上方程中 $\bar{v}\to v\to\bar{u}$，下考察 $\bar{u}$ 与 u 的关系

$$\bar{u}(r,t)=\frac{1}{4\pi}\int_{S^2} u\,d\omega=\frac{1}{4\pi}\int_{S^2} u(r,\omega)\,d\omega$$

$$\bar{u}(0,t)=u(0,t)=\partial_r\left(r\bar{u}(r,t)\right)\Big|_{r=0}$$

$$=\partial_r v\Big|_{r=0}$$

$$=\frac{1}{2}\left(\bar{\varphi}(t)+t\bar{\varphi}'(t)+\bar{\varphi}(-t)-t\bar{\varphi}'(-t)\right)$$

$$+\frac{1}{2}\left(t\bar{\Psi}(t)-(-t)\bar{\Psi}(-t)\right)$$

$$=\bar{\varphi}(t)+t\bar{\varphi}'(t)+t\bar{\Psi}(t)$$

$$=\frac{d}{dt}\left(t\bar{\varphi}(t)\right)+t\bar{\Psi}(t)$$

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$= \frac{d}{dt}(t\,\overline{\Phi}(t)) + t\,\Psi(t)$

$= \frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(t\omega)\,d\omega\right) + \frac{t}{4\pi}\int_{S^2}\psi(t\omega)\,d\omega$

对 $u(x+x_0, t)$ 应用于上一步，

其初值变为 $\varphi(x+x_0)$. $\psi(x+x_0)$

令 $x=0$,

$$u(x_0,t) = \frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(x_0+t\omega)\,d\omega\right) + \frac{t}{4\pi}\int_{S^2}\psi(x_0+t\omega)\,d\omega$$

$x_0 \to x$, $y = x + t\omega$

$$u(x,t) = \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)\,dS(y)\right) + \frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$

(Kirchhoff)

2、f 不恒为 0

$$u(x,t) = \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)\,dS(y)\right) + \frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$
$$+ \int_0^t \frac{1}{4\pi(t-\tau)}\int_{|x-y|=t-\tau} f(y,\tau)\,dy\,d\tau$$

Rmk 仅 n=3 时可良好转化为一维波方程问题

§4.1.6 二维初值问题

$n=2$，$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) \end{cases}$

1、$f(x,t)\equiv 0$

令 $\tilde{u}(\tilde{x},t) = u(\tilde{x},t)$ $\quad \tilde{x}=(x_1,x_2,x_3)$

$\tilde{\varphi}(\tilde{x}) = \varphi(x_1,x_2)$

$\tilde{\psi}(\tilde{x}) = \psi(x_1,x_2)$

则 $\tilde{u}(\tilde{x},t)$ 为三维波方程的解

即 $\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = 0 \\ \tilde{u}(\tilde{x},0) = \tilde{\varphi}(\tilde{x}),\ \partial_t \tilde{u}(\tilde{x},0) = \tilde{\psi}(\tilde{x}) \end{cases}$

由 Kirchhoff，

$$\underset{u(x,t)}{\overset{\parallel}{\tilde{u}(\tilde{x},t)}} = \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|\tilde{x}-\tilde{y}|=t} \tilde{\varphi}(\tilde{y})\,dS(\tilde{y})\right) + \frac{1}{4\pi t}\int_{|\tilde{x}-\tilde{y}|=t} \tilde{\psi}(\tilde{y})\,dS(\tilde{y})$$

$x_1, x_2, x_3 = 0$ 时，$\tilde{u}(0,t) = u(0,t) = \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y|=t}\varphi(y_1,y_2)\,dS(\tilde{y})\right)$

$$+\frac{1}{4\pi t}\int_{|y|=t}\psi(y_1,y_2)\,dS(\tilde{y})$$

注意到 $\int_{|y|=t}\psi(y_1,y_2)\,dS(\tilde{y})$

$$= 2\int_{y_3=\sqrt{t^2-y_1^2-y_2^2}} \psi(y_1,y_2)\,dS(\tilde{y})$$

$$\frac{\partial y_3}{y_1} = \frac{-y_1}{\sqrt{t^2-y_1^2-y_2^2}}$$

$$= 2\int_{y_1^2+y_2^2\le t^2} \psi(y_1,y_2)\sqrt{1+\left(\frac{\partial y_3}{y_1}\right)^2+\left(\frac{\partial y_3}{y_2}\right)^2}\,dy_1\,dy_2$$

$$= 2\int_{y_1^2+y_2^2\le t^2} \psi(y_1,y_2)\frac{t}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1\,dy_2$$

$$\Rightarrow u(0,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{B(0,t)}\frac{\varphi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2\right)$$
$$+\frac{1}{2\pi}\int_{B(0,t)}\frac{\psi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2$$

对 $u(x+x_0,t)$ 利用结论

$$u(x_0,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{|y|<t}\frac{\varphi(y+x_0)}{\sqrt{t^2-y^2}}\,dy\right)$$
$$+\frac{1}{2\pi}\int_{|y|<t}\frac{\psi(y+x_0)}{\sqrt{t^2-y^2}}\,dy$$

由 x_0 任意性 $u(x,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{|y-x|<t}\frac{\varphi(y)}{\sqrt{t^2-(y-x)^2}}\,dy\right)$
$$+\frac{1}{2\pi}\int_{|y-x|<t}\frac{\psi(y)}{\sqrt{t^2-(y-x)^2}}\,dy$$

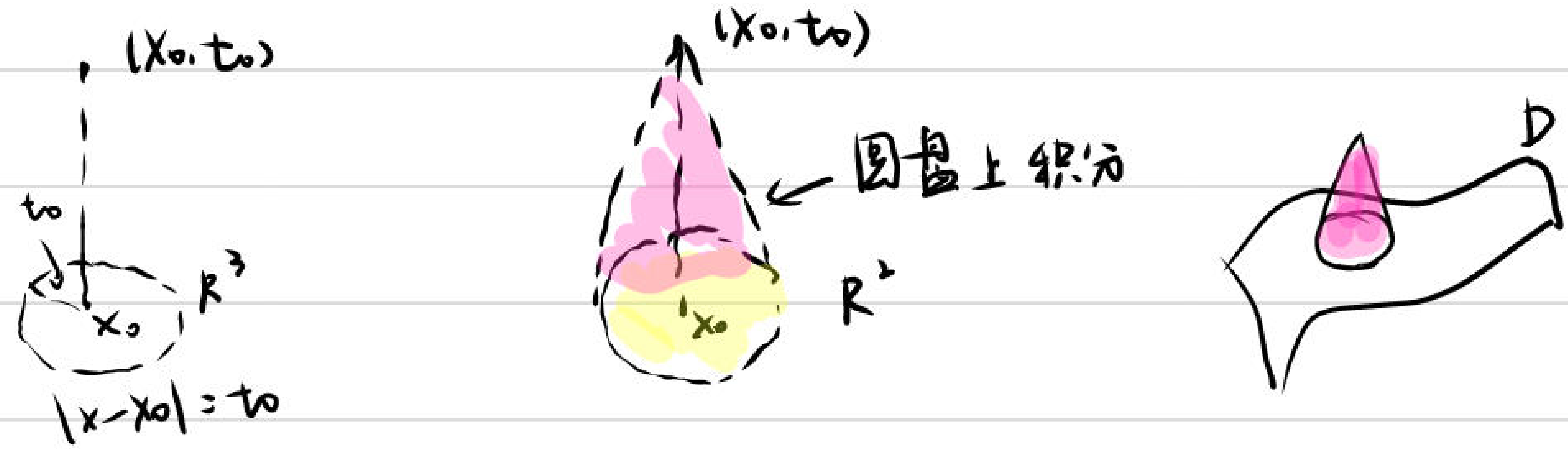
2、$f\neq 0$

3、性质

$$u(x,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$

(Kirchhoff)

(x_0,t_0) 仅与初值在球面上积分有关



依赖区域：(x_0,t_0) 依赖于 $\underbrace{\{x \mid |x-x_0|\le t_0\}}_{\triangleq D_{x_0,t_0}}$ 的值

决定区域：$\{(x,t) \mid D_{x,t}\subset D\}$ 为 D 的决定区域

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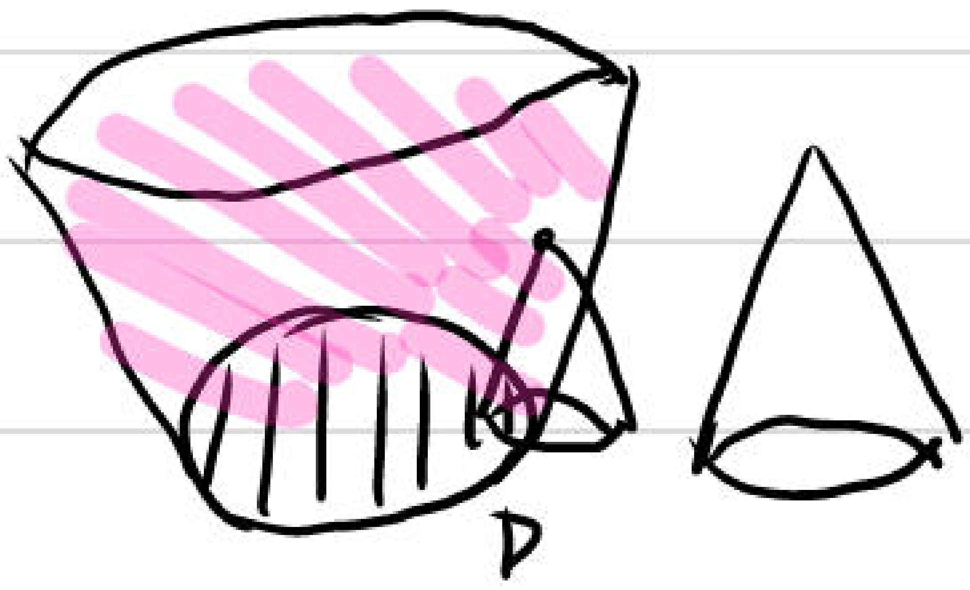


影响区域

Rmk. 几何性质

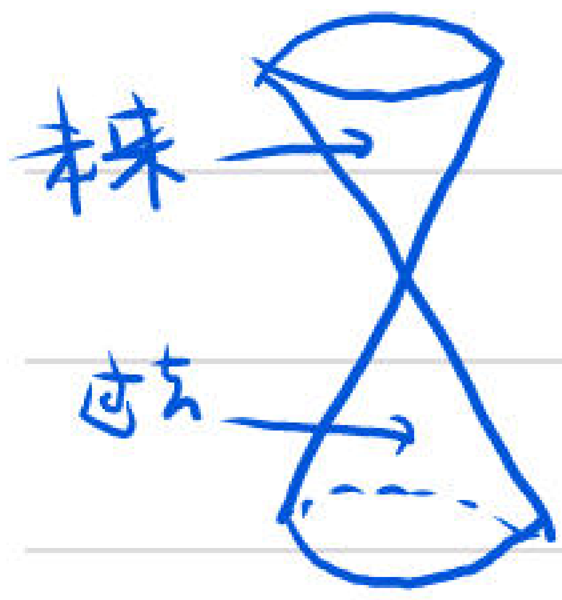
1、波方程具有有限传播速度

2、Huygens原理

$n=3$，依赖于球面

$n=2$，依赖于圆盘内

3、



光锥　因果律

§4.1.7 能量估计

$$u_{tt}-\Delta u=0 \Rightarrow u_t(u_{tt}-\Delta u)=0$$

$$u_t u_{tt}=\frac{1}{2}\partial_t (u_t)^2$$

$$u_t\Delta u=\sum_{i=1}^{n} u_t u_{x_i x_i}=\sum_{i=1}^{n}\left[\partial_{x_i}(u_t u_{x_i})-\partial_t\partial_{x_i}u\, u_{x_i}\right]$$

$$=\sum_{i=1}^{n}\left[\partial_{x_i}(u_t u_{x_i})-\frac{1}{2}\partial_t (u_{x_i})^2\right]$$

$$=\mathrm{div}(u_t\nabla u)-\frac{1}{2}\partial_t|\nabla u|^2$$

$$\Rightarrow \partial_t\left[\frac{1}{2}(u_t)^2+\frac{1}{2}|\nabla u|^2\right]-\mathrm{div}(u_t\nabla u)=0$$

称作能量守恒微分形式

u、低阶导数在空间无穷远 $\to 0$

则 $\partial_t\int_{R^n}\left(\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2\right)dx-\int_{R^n}\mathrm{div}(u_t\nabla u)=0$

散度 thm. $\int_\Omega \mathrm{div}\vec{F}\,dx=\int_{\partial\Omega}\vec{F}\cdot\vec{n}\,ds$

$$\Rightarrow \partial_t\int_{R^n}\left(\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2\right)dx=0$$

令 $E(t)=\int_{R^n}\underbrace{\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2}_{\text{能量密度 } e}dx$，则 $E(t)=E(0)$

称作能量守恒积分形式

考虑 $\begin{cases}\partial_t^2 u-\Delta u=0 & x\in\Omega\subset R^n,\ t>0\\ u|_{\partial\Omega}=0,\quad u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x)\end{cases}$

$$\partial_t\left[\frac{1}{2}(u_t)^2+\frac{1}{2}|\nabla u|^2\right]=\mathrm{div}(u_t\nabla u)$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(u_t)^2+\frac{1}{2}|\nabla u|^2\,dx=\int_\Omega \mathrm{div}(u_t\nabla u)\,dx$$

$$=\int_{\partial\Omega}u_t\nabla u\cdot\vec{n}\,ds$$

$$=\int_{\partial\Omega}u_t\frac{\partial u}{\partial n}\,ds$$

$u|_{\partial\Omega}=0\Rightarrow u_t|_{\partial\Omega}=0 \Rightarrow E(t)=\int_\Omega \frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2$，有 $E(t)=E(0)$

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考虑
$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & x\in\Omega,\ t>0 \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & x\in\Omega \quad \text{初值} \quad (※※) \\ u|_{\partial\Omega} = h(x,t) & t\geq 0 \quad \text{第一边值} \end{cases}$$

该方程最多有一个古典解.

pr: 设 (※※) 有两个解 u_1, u_2.

令 $u(x,t) = u_1(x,t) - u_2(x,t)$

则
$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0)=0,\ u_t(x,0)=0 \\ u|_{\partial\Omega}=0 \end{cases}$$

由能量估计, 令 $E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2\, dx$

则 $E(t) = E(0) = 0$

$\Rightarrow u_t = 0,\ \nabla u = \vec{0} \quad \forall x \text{ in } \Omega,\ t\geq 0$

$\Rightarrow u = \text{const} \quad \text{in } \Omega$

由于 u 边值为 0, 为使得 u 连续到边界, $u\equiv 0$

Rmk. 对第二类边值, 能量估计时

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2\, dx = \int_{\partial\Omega} u_t \frac{\partial u}{\partial n}\, ds$$

给出 $\frac{\partial u}{\partial n}=0$ 的条件可类比 u_t 处理

对第三类边值, $\frac{\partial u}{\partial n} + \alpha u = 0\ (\alpha>0),\ x\in\partial\Omega$

def. $E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} \alpha u^2 dx$

此时 $\frac{dE(t)}{dt} = 0$

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考虑 $\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & x\in\Omega,\ t>0 \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & x\in\Omega \quad \text{初值} \quad (※※) \\ u|_{\partial\Omega}=0 & t\geq 0 \quad \text{第一边值} \end{cases}$

(※※)的解在下述意义下关于初值和右端项稳定：

$\forall \varepsilon>0,\ \exists \eta=\eta(\varepsilon,T),\ \text{s.t.}$

若 $\|\varphi_1-\varphi_2\|_{L^2(\Omega)}\leq\eta,\ \|\nabla\varphi_1-\nabla\varphi_2\|_{L^2(\Omega)}\leq\eta,\ \|\psi_1-\psi_2\|_{L^2(\Omega)}\leq\eta$

$\|f_1-f_2\|_{L^2(0,T;\Omega)}\leq\eta$

则以 φ_1、ψ_1 为初值，f_1 为右端项的解 u_1

与以 φ_2、ψ_2 为初值，f_2 为右端项的解 u_2

其差在 $0\leq t\leq T$ 上满足 $\|u_1-u_2\|_{L^2(\Omega)}+\|\nabla u_1-\nabla u_2\|_{L^2(\Omega)}\leq\varepsilon$

$$\|\partial_t u_1-\partial_t u_2\|_{L^2(\Omega)}\leq\varepsilon$$

Rmk. $\|f\|_{L^2(\Omega)}=\left(\int_\Omega |f(x)|^2dx\right)^{\frac12}$

$\|f\|_{L^2(0,T;\Omega)}=\left(\int_0^T\int_\Omega |f(x,t)|^2dxdt\right)^{\frac12}$

pr: 令 $u(x,t)=u_1(x,t)-u_2(x,t)$

$f=f_1-f_2,\ \varphi=\varphi_1-\varphi_2,\ \psi=\psi_1-\psi_2$

则 $\begin{cases} \partial_t^2 u-\Delta u=f_1-f_2=f \\ u(x,0)=\varphi_1-\varphi_2=\varphi,\ u_t(x,0)=\psi_1-\psi_2=\psi \\ u|_{\partial\Omega}=0 \end{cases}$

$$u_t(\partial_t^2u-\Delta u)=\partial_t\left[\tfrac12(u_t)^2+\tfrac12|\nabla u|^2\right]-\operatorname{div}(u_t\nabla u)=u_t\cdot f$$

在 Ω 上积分，$\partial_t\int_\Omega \frac12(u_t)^2+\frac12|\nabla u|^2dx=\int_\Omega u_tf\,dx\leq\int_\Omega\frac12f^2+\frac12(u_t)^2dx$

$$\frac{d}{dt}E(t)\leq\frac12\int_\Omega f^2dx+\frac12\int_\Omega(u_t)^2dx\leq\int_\Omega\frac12f^2dx+E(t)$$

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$$\frac{d}{dt}(e^{-t}E(t)) \le \frac{1}{2}e^{-t}\int_\Omega f^2 dx$$

由Gronwall, $e^{-t}E(t) - E(0) \le \frac{1}{2}\int_0^t e^{-s}\int_\Omega f^2(x,s)dxds$

$$\Rightarrow E(t) \le e^t(E(0) + \frac{1}{2}\int_0^t e^{-s}\int_\Omega f^2(x,s)dxds)$$

$$\le e^t(E(0) + \frac{1}{2}\int_0^t\int_\Omega f^2(x,s)dxds)$$

$$\le e^T(E(0) + \frac{1}{2}\int_0^T\int_\Omega f^2(x,s)dxds)$$

$$\le C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds) \quad \forall\, 0\le t\le T \quad C_{1T} = const, 与T有关$$

$E(0)$ 表示 $\int_\Omega \frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2 dx$ 的初值

令 $y(t) = \int_\Omega |u|^2 dx$

$$y'(t) = 2\int_\Omega u u_t dx \le \int_\Omega u^2 + u_t^2\, dx \le y(t) + 2E(t)$$

与T有关常数

$$\le y(t) + 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$$

$$\frac{d}{dt}(e^{-t}y(t)) \le 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$$

由Gronwall, $e^{-t}y(t) - y(0) \le t\cdot 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$

$$\Rightarrow y(t) \le e^t[y(0) + t\cdot 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)]$$

$$\le e^t[y(0) + T\cdot 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)]$$

$$\le C_{2T}(y(0) + E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$$

故 $\|u(t)\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$

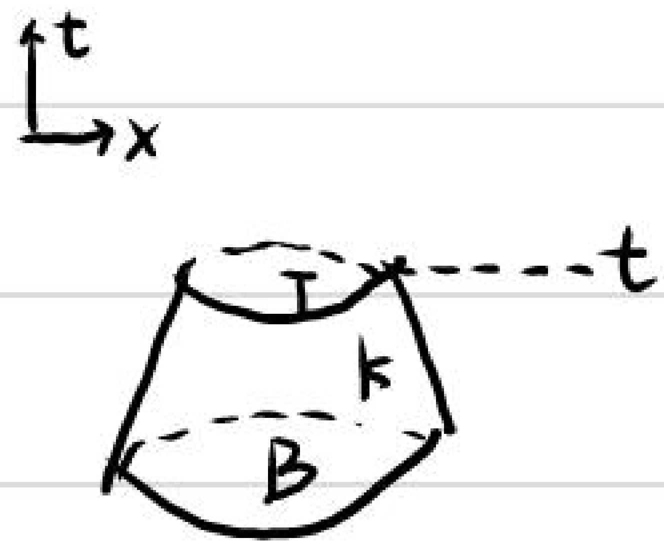
$$\le C_T(\|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)} + \|\nabla\varphi\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega f^2(x,s)dxds)$$

∃与T有关的常数

$$\le 4\eta^2 C_T \triangleq \frac{\varepsilon}{2} < \varepsilon$$

考虑 $\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$ $\quad x \in R^3$

两边同乘 u_t，有 $\partial_t\, e(u) - \mathrm{div}(u_t \nabla u) = 0$



在锥台 $|x-x_0| \le R-t$ 上积分

$$\iint_{\triangle} \partial_t\, e(u) - \mathrm{div}(u_t \nabla u)\, dx\, dt = 0$$

$K:\ |x-x_0|^2 = |R-t|^2$

$\varphi(x,t) = |x-x_0|^2 - |R-t|^2 = 0$

$\nabla\varphi = (2(R-t),\ 2(x-x_0))$

$n = \dfrac{\pm\nabla\varphi}{\|\nabla\varphi\|} = \pm\dfrac{1}{\sqrt{2}}\left(\dfrac{R-t}{|R-t|},\ \dfrac{x-x_0}{|x-x_0|}\right)$

$$\begin{aligned}
\text{左} &= \iint_{\triangle} (\partial_t, \nabla_x)\cdot(e(u),\ -u_t\nabla u)\, dx\, dt \\
&= \int_{\partial\triangle} (e(u),\ -u_t\nabla u)\cdot \vec{n}\, ds \\
&= -\int_B e(u)(0)\, dx + \int_T e(u)(t_0)\, dx
\end{aligned}$$

↑ 表示时间取值

$$+\int_K (e(u),\ -u_t\nabla u)\cdot\frac{1}{\sqrt{2}}\left(\frac{R-t}{|R-t|},\ \frac{x-x_0}{|x-x_0|}\right)$$

$$\begin{aligned}
\Rightarrow \int_B e(u)(0)\, dx &= \int_T e(u)(t_0)\, dx + \frac{1}{\sqrt{2}}\int_K \left(\frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2\right)\frac{R-t}{|R-t|} - u_t\nabla u\cdot\frac{x-x_0}{|x-x_0|}\, ds \\
&= \int_T e(u)(t_0)\, dx + \frac{1}{2\sqrt{2}}\int_K (u_t)^2 + |\nabla u|^2 - 2u_t\nabla u\cdot\frac{x-x_0}{|x-x_0|}\, ds \\
&= \int_T e(u)(t_0)\, dx + \underbrace{\frac{1}{2\sqrt{2}}\int_K \left|u_t - \frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2 + |\nabla u|^2 - \left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2 ds}_{\mathrm{Flux}\,[0,t]\ \ge 0}
\end{aligned}$$

即 $\displaystyle\int_B e(u)(0)\, dx = \int_T e(u)(t_0)\, dx + \mathrm{Flux}\,[0,t]$

(t=0处能量)　　(t=t₀处能量)　　(能量溢出)

若 $(u, u_t)\big|_{t=0} = 0$，在B上能量为0，则在 (u, u_t) 在T上恒为0

§4.2 混合问题

混合问题即初边值问题

$$\begin{cases} \partial_t^2 u-\Delta u=f(x,t) \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & x\in\Omega,\ t>0 \\ u\big|_{\partial\Omega}=g(x,t) \,/\, \frac{\partial u}{\partial n}\big|_{\partial\Omega}=g(x,t) \,/\, \frac{\partial u}{\partial n}+\alpha u\big|_{\partial\Omega}=g(x,t) \end{cases}$$

4.2.1 常微分方程齐次边值问题

考虑
$$\begin{cases} X''(x)+\lambda X(x)=0 & x\in(0,l) \\ -\alpha_1 X'(0)+\beta_1 X(0)=0 & (3.15) \\ \alpha_2 X'(l)+\beta_2 X(l)=0 \end{cases}$$

$\alpha_i\geq 0,\ \beta_i\geq 0,\ \alpha_i+\beta_i>0,\ i=1,2$

称为 Sturm-Liouville 特征值问题，λ称为特征值

$\lambda\in\mathbb{R}$ 称为特征值，相应于λ的非零解 $X(x)$ 称为对应于这个特征值的特征函数

thm (1) 所有特征值为非负实数

$\beta_1+\beta_2>0$ 时，所有特征值为正数

(2) 不同特征值对应特征函数正交.

即 $\int_0^l X_\lambda(x)X_\mu(x)\,dx=0$

(3) $\lambda_1,\dots,\lambda_n\dots$ 为特征值

$0\leq\lambda_1<\lambda_2<\dots<\lambda_n<\dots$ $\quad \lim\limits_{n\to\infty}\lambda_n=+\infty$

(4) $f(x)\in L^2(0,l)$ 可按特征函数系展开为

$$f(x)=\sum_{n=1}^{\infty}C_nX_n(x)$$

$$C_n=\frac{\int_0^l f(x)X_n(x)\,dx}{\int_0^l X_n^2(x)\,dx}$$

pr：(1) $X''+\lambda X=0 \Rightarrow XX''+\lambda X^2=0$

$$XX''+\lambda X^2=(XX')'-(X')^2+\lambda X^2=0$$

$$\Rightarrow \lambda\int_0^l X^2(x)dx=\int_0^l (X'(x))^2dx-\int_0^l (X(x)X'(x))'dx$$

$$=\int_0^l (X'(x))^2dx-X(x)X'(x)\Big|_{x=0}^{x=l}$$

$$=\int_0^l (X'(x))^2dx+X(0)X'(0)-X(l)X'(l)$$

$$-\alpha_1X'(0)+\beta_1X(0)=0 \Rightarrow -\alpha_1X'(0)^2+\beta_1X(0)X'(0)=0$$

$$-\alpha_1X'(0)X(0)+\beta_1X(0)^2=0$$

$$\Rightarrow X(0)X'(0)=\frac{\alpha_1}{\alpha_1+\beta_1}X'(0)^2+\frac{\beta_1}{\alpha_1+\beta_1}X(0)^2$$

类似地 $X(l)X'(l)=-\dfrac{\alpha_2X'(l)^2+\beta_2X(l)^2}{\alpha_2+\beta_2}$

代入表达式有

$$\lambda\int_0^l X^2dx=\int_0^l (X'(x))^2dx+\frac{\alpha_1}{\alpha_1+\beta_1}X'(0)^2+\frac{\beta_1}{\alpha_1+\beta_1}X(0)^2+\frac{\alpha_2}{\alpha_2+\beta_2}X'(l)^2+\frac{\beta_2}{\alpha_2+\beta_2}X(l)^2$$

$\geq 0 \Rightarrow \lambda\geq 0$

$\lambda=0 \Leftrightarrow X'=0$，且 $\dfrac{\beta_1}{\alpha_1+\beta_1}X(0)^2+\dfrac{\beta_2}{\alpha_2+\beta_2}X(l)^2=0$

故 $\beta_1=\beta_2=0$ 时，$X(x)=\text{Const}$

若 β_1、β_2 中存在不为0，$X(x)=\text{Const}=0$，0不为特征值

(2) 设 X_λ、X_μ 为不同特征值 λ、μ 的特征函数

则 $X_\lambda''+\lambda X_\lambda=0$

$X_\mu''+\mu X_\mu=0$

$$\lambda\int_0^l X_\lambda X_\mu dx=-\int_0^l X_\mu X_\lambda''dx=-\int_0^l X_\mu d(X_\lambda')=\int_0^l X_\lambda'X_\mu'dx-X_\mu X_\lambda'\Big|_0^l$$

$$=X_\mu(0)X_\lambda'(0)-X_\mu(l)X_\lambda'(l)+\int_0^l X_\lambda'X_\mu'dx$$

同理有结果，$\mu\displaystyle\int_0^l X_\lambda X_\mu dx=X_\lambda(0)X_\mu'(0)-X_\mu'(l)X_\lambda(l)+\int_0^l X_\lambda'X_\mu'dx$

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相减可得

$$(\lambda-\mu)\int_0^l X_\lambda X_\mu dx = (X_\lambda'(0)X_\mu(0) - X_\mu(0)X_\mu'(0)) - (X_\mu(l)X_\lambda'(l) - X_\mu'(l)X_\lambda(l))$$

由边条件 $\begin{cases} -\alpha_1 X_\lambda'(0)+\beta_1 X_\lambda(0)=0 & (1) \\ \alpha_2 X_\lambda'(l)+\beta_2 X_\lambda(l)=0 & (2) \end{cases}$

$\begin{cases} -\alpha_1 X_\mu'(0)+\beta_1 X_\mu(0)=0 & (3) \\ \alpha_2 X_\mu'(l)+\beta_2 X_\mu(l)=0 & (4) \end{cases}$

(1)(3) 构成关于 α_1、β_1 的线性方程，有非零解，则

$$\begin{vmatrix} X_\lambda'(0) & X_\lambda(0) \\ X_\mu'(0) & X_\mu(0) \end{vmatrix} = 0 \Rightarrow X_\lambda'(0)X_\mu(0) - X_\lambda(0)X_\mu'(0) = 0$$

同理 $X_\mu(l)X_\lambda'(l) - X_\mu'(l)X_\lambda(l) = 0$

$\lambda \neq \mu \Rightarrow \int_0^l X_\lambda X_\mu dx = 0$

Rmk. (3)(4) 的证明在泛函分析中学到

构成 L^2 一组基是由于算子 $\frac{d^2}{dt^2}$ 为对称紧算子

§4.2.2 分离变量法

考虑
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) & 0<x<l,\ t>0 \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & 0\le x\le l \\ u(0,t)=g_1(t),\ u(l,t)=g_2(t) & t\geq 0 \end{cases}$$

(一维定长弦的振动)

1、$f\equiv 0$，$g_1(t)$、$g_2(t)\equiv 0$

则
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & 0<x<l,\ t>0 \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & 0\le x\le l \\ u(0,t)=0,\ u(l,t)=0 & t\geq 0 \end{cases}$$

① u 仅与 t 有关，则 $\partial_t^2 u=0 \Rightarrow u=C_1 t+C_2$

$u(0,t)=C_1 t+C_2$ 不满足边值

② u 仅与 x 有关，则 $\partial_t^2 x=0 \Rightarrow u=C_1 x+C_2$

$u(x,0)=C_1x+C_2=\varphi(x)$ 不容易满足初值

③ 令 $u(x,t)=T(t)X(x)$

则 $T''(t)X(x)-T(t)X''(x)=0$

$T(t)\cdot X(x)\neq 0 \Rightarrow \dfrac{T''(t)}{T(t)}=\dfrac{X''(x)}{X(x)} \triangleq -\lambda$

$$\Rightarrow \begin{cases} X''(x)+\lambda X(x)=0 \\ T''(t)+\lambda T(t)=0 \end{cases}$$

$$\begin{cases} u(0,t)=0 \\ u(l,t)=0 \end{cases} \Rightarrow \begin{cases} T(t)X(0)=0 \\ T(t)X(l)=0 \end{cases} \quad \forall t\geq 0$$

$\Rightarrow X(0)=X(l)=0$

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考察对于X的 Sturm-Liouville 边值问题

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0 \\ X(l)=0 \end{cases}$$

$$0=\int_0^l X(x)\left(X''(x)+\lambda X(x)\right)dx = X(x)X'(x)\Big|_0^l - \int_0^l (X'(x))^2dx + \lambda\int_0^l (X(x))^2dx$$

$$\Rightarrow \lambda\int_0^l (X(x))^2dx = \int_0^l (X'(x))^2dx$$

$$\Rightarrow \lambda \geq 0$$

若 $\lambda=0$，则 $X''(x)=0 \Rightarrow X(x)=C_1x+C_2$

$X(0)=C_2=0$，$X(l)=C_1l=0 \Rightarrow C_1=0 \Rightarrow X(x)\equiv 0$

若 $\lambda>0$，$X''(x)+\lambda X(x)=0$

$\Rightarrow X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$

$X(0)=C_1=0$，$X(l)=C_2\sin\sqrt{\lambda}l=0$　$\sqrt{\lambda}l=n\pi$　$n\in\mathbb{Z}^+$

$\Rightarrow \lambda_n=\left(\frac{n\pi}{l}\right)^2$　$n=1,2\cdots$

与之对应特征函数 $X_n(x)=\sin(\frac{n\pi}{l}x)$ (事实上为 $C_2\sin(\frac{n\pi}{l}x)$，但系数可被"C_n""D_n"吸收)

由于 $T_n''(t)+\lambda_nT_n(t)=0$

$T_n(t)=C_n\cos(\frac{n\pi}{l}t)+D_n\sin(\frac{n\pi}{l}t)$

令 $u(x,t)=\sum\limits_{n=1}^{\infty}\left[C_n\cos(\frac{n\pi}{l}t)+D_n\sin(\frac{n\pi}{l}t)\right]\sin(\frac{n\pi}{l}x)$

再利用初值条件，

$u(x,0)=\sum\limits_{n=1}^{\infty}C_n\sin(\frac{n\pi}{l}x)=\varphi(x)$

$u_t(x,0)=\sum\limits_{n=1}^{\infty}\frac{n\pi}{l}D_n\sin(\frac{n\pi}{l}x)=\psi(x)$

由 Sturm-Liouville，$\left\{\sin(\frac{n\pi}{l}x)\right\}_{n=1}^{\infty}$ 为 L^2 中一组正交基

$$\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin\left(\frac{n\pi}{l}x\right)$$

$$\varphi_n=\frac{\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\,dx}{\int_0^l\sin^2\left(\frac{n\pi}{l}x\right)dx}=\frac{2}{l}\int_0^l\varphi(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$\psi(x)=\sum_{n=1}^{\infty}\psi_n\sin\left(\frac{n\pi}{l}x\right)$$

$$\psi_n=\frac{\int_0^l\psi(x)\sin\left(\frac{n\pi}{l}x\right)dx}{\int_0^l\sin^2\left(\frac{n\pi}{l}x\right)dx}=\frac{2}{l}\int_0^l\psi(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$\Rightarrow\left.\begin{aligned}C_n&=\varphi_n=\frac{2}{l}\int_0^l\varphi(x)\sin\left(\frac{n\pi}{l}x\right)dx\\ D_n&=\frac{l}{n\pi}\psi_n=\frac{2}{n\pi}\int_0^l\psi(x)\sin\left(\frac{n\pi}{l}x\right)dx\end{aligned}\right\}(※)$$

故原方程解为

$$u(x,t)=\sum_{n=1}^{\infty}\left[C_n\cos\left(\frac{n\pi}{l}t\right)+D_n\sin\left(\frac{n\pi}{l}t\right)\right]\sin\left(\frac{n\pi}{l}x\right)$$

其中 C_n、D_n 由(※)给出

thm. 相容性条件

若 $\varphi\in C^3([0,l])$，$\psi\in C^2([0,l])$，$\varphi(x)$、$\psi(x)$ 在 $(0,l)\times(0,+\infty)\triangleq Q$

两端满足相容性条件 $\varphi(0)=\varphi(l)=\varphi''(0)=\varphi''(l)=\psi(0)=\psi(l)=0$

则 $u(x,t)\in C^2(\bar{Q})$ 为古典解

Rmk. 若 $u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$

T_i、X_i 均足够好，使级数求导与求和可换序

则 $$\partial_t^2u-\partial_x^2u=\sum_{n=1}^{\infty}T_n''(t)X_n(x)-T_n(t)X_n''(x)$$
$$=\sum_{n=1}^{\infty}\left(T_n''(t)+\lambda_nT_n(t)\right)X_n(x)=0$$

2. $f\not\equiv0$，$g_1(t)$、$g_2(t)\equiv0$

令 $f(x,t)=\sum_{n=1}^{\infty}f_n(t)\sin\left(\frac{n\pi}{l}x\right)$

$$\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin\left(\frac{n\pi}{l}x\right),\quad\psi(x)=\sum_{n=1}^{\infty}\psi_n\sin\left(\frac{n\pi}{l}x\right)$$

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令 $u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin(\frac{n\pi}{l}x)$

代入 $\partial_t^2u-\partial_x^2u=f(x,t)$

有 $\sum_{n=1}^{\infty}(T_n''(t)+\lambda_nT_n(t))\sin(\frac{n\pi}{l}x)=\sum_{n=1}^{\infty}f_n(t)\sin(\frac{n\pi}{l}x)$

则 T_n 满足方程 $T_n''(t)+\lambda_nT_n(t)=f_n(t)$

$u(x,0)=\sum_{n=1}^{\infty}T_n(0)\sin(\frac{n\pi}{l}x)=\sum_{n=1}^{\infty}\varphi_n\sin(\frac{n\pi}{l}x)\Rightarrow T_n(0)=\varphi_n$

$u_t(x,0)=\sum_{n=1}^{\infty}T_n'(0)\sin(\frac{n\pi}{l}x)=\sum_{n=1}^{\infty}\psi_n\sin(\frac{n\pi}{l}x)\Rightarrow T_n'(0)=\psi_n$

$\Rightarrow T_n(t)=\varphi_n\cos(\frac{n\pi}{l}t)+\frac{l}{n\pi}\psi_n\sin(\frac{n\pi}{l}t)+\frac{l}{n\pi}\int_0^t f_n(\tau)\sin(\frac{n\pi}{l}(t-\tau))d\tau$ 　(※)

故原方程解为

$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin(\frac{n\pi}{l}x)$　T_n 由(※)给出

3. $f\neq0$，$g_1,g_2\neq0$

令 $v(x,t)=u(x,t)-\frac{(l-x)g_1(t)+xg_2(t)}{l}$

则 $\left.\begin{aligned}v(0,t)&=u(0,t)-g_1(t)=0\\ v(l,t)&=u(l,t)-g_2(t)=0\end{aligned}\right\}$ 边值化为0

$$\begin{cases}\partial_t^2v-\partial_x^2v=f(x,t)-\frac{(l-x)g_1''(t)+xg_2''(t)}{l}\\ v(x,0)=\varphi(x)-\frac{(l-x)g_1(0)+xg_2(0)}{l}\\ v_t(x,0)=\psi(x)-\frac{(l-x)g_1'(0)+xg_2'(0)}{l}\end{cases}$$

进而转化为零边值问题

Rmk. 本质为 S-L 边值问题的特征函数展开，与方程类型无关.

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ex.（热传导方程）

$$\begin{cases} u_t = u_{xx} & 0<x<l \\ u(x,0) = \varphi(x) \\ u(0,t)=0,\ u_x(l,t)+hu(l,t)=0 & h>0 \end{cases}$$

设 $u(x,t) = T(t)X(x)$

则 $T'(t)X(x) = T(t)X''(x)$

$\Rightarrow \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} \triangleq -\lambda$

$u(0,t)=0 \Rightarrow T(t)X(0)=0$

$u_x(l,t)+hu(l,t)=0 \Rightarrow T(t)X'(l)+hT(t)X(l)=0$

对 $\forall t$ 成立，只需 $X'(l)+hX(l)=0$

故 $X(x)$ 满足边值问题

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0 \\ X'(l)+hX(l)=0 \end{cases}$$

① $\lambda<0$ 时，$X(x) = C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x}$

$X(0)=0 \Rightarrow C_1+C_2=0$

$C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}l} - C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} + h(C_1e^{\sqrt{-\lambda}l}+C_2e^{-\sqrt{-\lambda}l})=0$

$C_1\left[(\sqrt{-\lambda}e^{\sqrt{-\lambda}l}+\sqrt{-\lambda}e^{-\sqrt{-\lambda}l})+h(e^{\sqrt{-\lambda}l}-e^{-\sqrt{-\lambda}l})\right]=0$

$\Leftrightarrow C_1=0$ 或 $\begin{cases}\sqrt{-\lambda}+h=0 \\ \sqrt{-\lambda}-h=0\end{cases} \Rightarrow h=0$

故 $C_1=0$，$C_2=0$

Rmk. 利用本题方法同样可说明 $\lambda\geq 0$，同乘 X 为 S-L 问题的手段

② $\lambda=0$ 时，$X(x)=C_1x+C_2$

$X(0)=C_2=0$

$X'(l)+hX(l)=C_1+hC_1l=0 \Rightarrow C_1=0 \Rightarrow X(x)\equiv 0$

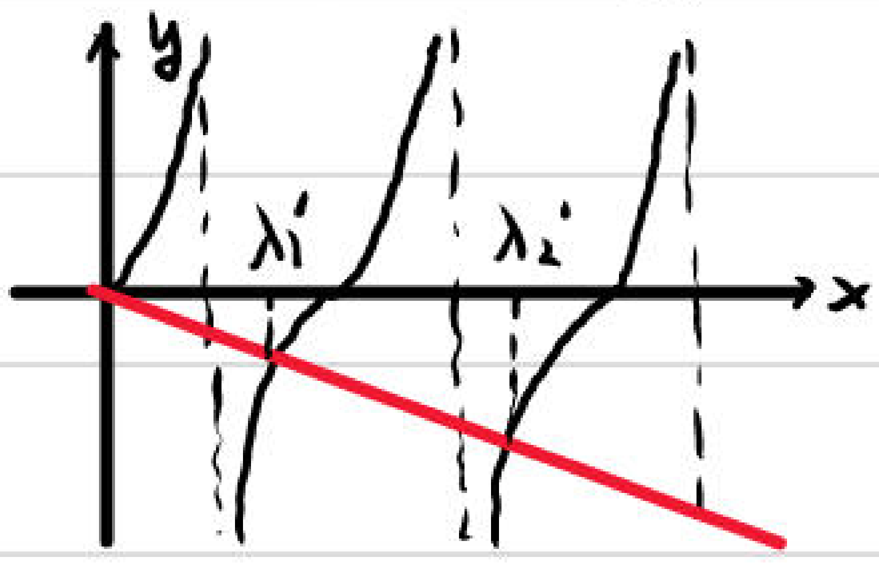
故 $\lambda>0$，则 $X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$

$X(0)=C_1=0$

$X'(l)+hX(l)=C_2\sqrt{\lambda}l\cos(\sqrt{\lambda}l)+hC_2\sin(\sqrt{\lambda}l)=0$

$C_2\neq 0 \Rightarrow \tan(\sqrt{\lambda}l)=\frac{-\sqrt{\lambda}}{h}$

考察方程 $\tan x=\frac{-x}{hl}$ 的解



$\exists\ 0<\lambda_1<\cdots<\lambda_n<\cdots$，满足 $\tan(\sqrt{\lambda}l)=\frac{-\sqrt{\lambda}}{h}$

令 $X_n(x)=\sin(\sqrt{\lambda_n}x)$ 为 λ_n 对应的特征函数

$T_n(t)$ 满足方程 $T_n'+\lambda_nT_n=0 \Rightarrow T_n(t)=A_ne^{-\lambda_nt}$

则 $u(x,t)=\sum_{n=1}^{\infty}A_ne^{-\lambda_nt}\sin(\sqrt{\lambda_n}x)$

而 $u(x,0)=\sum_{n=1}^{\infty}A_n\sin(\sqrt{\lambda_n}x)=\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin(\sqrt{\lambda_n}x)$

$$\Rightarrow A_n=\varphi_n=\frac{\int_0^l\varphi(x)\sin(\sqrt{\lambda_n}x)dx}{\int_0^l\sin^2(\sqrt{\lambda_n}x)dx}$$

ex.(位势方程)

令 $\Omega=\{(x,y)\mid x^2+y^2<1\}$

考虑 Ω 上位势方程:

$$\begin{cases}\Delta u=0 & \text{in } \Omega\\ u=\varphi & \text{in } \partial\Omega\end{cases}$$

令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$

$$\Rightarrow\begin{cases}\partial_r^2 u+\frac{1}{r}\partial_r u+\frac{1}{r^2}\partial_\theta^2 u=0 & \text{in } \Omega\\ u\big|_{r=1}=\varphi(\cos\theta,\sin\theta)\triangleq\tilde{\varphi}(\theta)\end{cases}$$

令 $u(r,\theta)=R(r)\Theta(\theta)$

则 $R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0$

$$\Rightarrow -r^2\frac{R''(r)+\frac{1}{r}R'(r)}{R(r)}=\frac{\Theta''(\theta)}{\Theta(\theta)}\triangleq-\lambda$$

考察 $\Theta(\theta)$ 满足的方程

$$\begin{cases}\Theta''(\theta)+\lambda\Theta(\theta)=0\\ \Theta(\theta)=\Theta(\theta+2\pi)\end{cases}$$

若 $\lambda<0$, $\Theta(\theta)=C_1e^{-\sqrt{\lambda}\theta}+C_2^{\sqrt{\lambda}\theta}$, 不以 2π 为周期

若 $\lambda=0$, $\Theta(\theta)=C_1\theta+C_2$, 不以 2π 为周期, 除非 $C_1=0$, 即 $\Theta(\theta)$ 为常数

若 $\lambda>0$, $\Theta(\theta)=C_1\cos(\sqrt{\lambda}\theta)+C_2\sin(\sqrt{\lambda}\theta)$

以 2π 为周期 $\Leftrightarrow \sqrt{\lambda}\in\mathbb{Z}^*$

则 $\Theta_n(\theta)=C_n\cos(n\theta)+D_n\sin(n\theta)$ 为 $\lambda_n=n^2$ 对应特征函数, $n=1,2\cdots$

可构成空间一组基 由于 $\frac{d^2}{dt^2}$ 为对称紧算子

(补充 $\Theta_0(\theta)=C_0$)

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考察 $R(r)$ 满足的方程

$$r^2 R_n''(r) + r R_n'(r) - n^2 R_n = 0$$

为 Euler 方程.

令 $r = e^t$, $\frac{dR_n}{dt} = R_n'(r) r$, $\frac{d^2R_n}{dt^2} = R_n''(r) r^2 + R_n'(r) r = R_n''(r) r^2 + \frac{dR_n}{dt}$

故 $\frac{d^2R_n}{dt^2} - n^2 R_n = 0$

$$\Rightarrow R_n(r) = \begin{cases} C_1 e^{nt} + C_2 e^{-nt} = C_1 r^n + C_2 r^{-n} & n \neq 0 \\ C_1 t + C_2 = C_1 \ln r + C_2 & n = 0 \end{cases}$$

为使 $r=0$ 处连续可微，应满足 $R_n(r) = \begin{cases} C_1 r^n & n \neq 0 \\ C_2 & n = 0 \end{cases}$

取 $C_1 = C_2 = 1$.

设 $u(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos(n\theta) + D_n \sin(n\theta))$

$u|_{r=1} = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\theta) + D_n \sin(n\theta) = \tilde{\varphi}(\theta)$

$C_n \int_0^{2\pi} \cos^2(n\theta)\, d\theta = \int_0^{2\pi} \tilde{\varphi}(\theta) \cos(n\theta)\, d\theta \Rightarrow C_n = \frac{1}{\pi} \int_0^{2\pi} \tilde{\varphi}(\theta) \cos(n\theta)\, d\theta$

$D_n \int_0^{2\pi} \sin^2(n\theta)\, d\theta = \int_0^{2\pi} \tilde{\varphi}(\theta) \sin(n\theta)\, d\theta \Rightarrow D_n = \frac{1}{\pi} \int_0^{2\pi} \tilde{\varphi}(\theta) \sin(n\theta)\, d\theta$

$C_0 \int_0^{2\pi} d\theta = \int_0^{2\pi} \tilde{\varphi}(\theta)\, d\theta \Rightarrow C_0 = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\varphi}(\theta)\, d\theta$

Rmk. 分离变量法对区域、算子较为敏感