Exercise 5 for 2022∼ 2023 USTC Course

'Introduction to Quantum Information'

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1. (1) You are given the stabilizer $S = \{XXIZ, YXIY, IZIX, ZZII, -YYIZ, XYIY, ZIIX, IIII\}$. Give a minimal list of elements of S which can be combined by multiplication to produce all the elements in S, i.e., the generators of S.

   (2) You are given that $\{XIX, ZIY\}$ generate S. Give a list of additional operators which are also in S.

   (3) If S acts on an n-qubit space, and the minimal generator set for S has d elements, then what is the dimension of the vector space which is stabilized by S?

   Answer:

   (1) $\{XXIZ, IZIX, ZIIX\}$.

   (2) $\{YIZ, III\}$.

   (3) 2^{n-d}.

2. (a) Give stabilizer generator sets for the following states.

(1) $(\|0\rangle + \mathrm{i}\,\|1\rangle)/\sqrt{2}$	(2) $\|1\rangle$
(3) $(\|00\rangle + \|11\rangle)/\sqrt{2}$	(4) $(\|00\rangle - \|11\rangle)/\sqrt{2}$
(5) $(\|01\rangle + \|10\rangle)/\sqrt{2}$	(6) $(\|01\rangle - \|10\rangle)/\sqrt{2}$
(7) $(\|000\rangle + \|111\rangle)/\sqrt{2}$	(8) $(\|{+0+}\rangle + \|{-1-}\rangle)/\sqrt{2}$,

   where $|\pm\rangle$ denote the eigenkets of Pauli X.

(b) Give stabilizer generator sets for the following vector spaces, specified by the basis sets given.

$$(1)\ \{|001\rangle, |110\rangle\}$$
$$(2)\ \left\{(|00\rangle + |11\rangle)/\sqrt{2}, (|01\rangle + |10\rangle)/\sqrt{2}\right\}.$$

Answer:

$$\begin{array}{lll} (a) & (1)\ \{Y\} & (2)\ \{-Z\} \\ & (3)\ \{XX, ZZ\} & (4)\ \{-XX, ZZ\} \\ & (5)\ \{XX, -ZZ\} & (6)\ \{-XX, -ZZ\} \\ & (7)\ \{XXX, ZZI, IZZ\} & (8)\ \{ZXZ, XZI, IZX\} \\ (b) & (1)\ \{ZZI, -IZZ\} & (2)\ \{XX\}. \end{array}$$

3. (1) For the 4-qubit state $|\psi\rangle = (|0011\rangle + |1100\rangle)/\sqrt{2}$, write down its stabilizer generator sets.

   (2) For 4-qubit cluster state $|\psi\rangle = (|+\rangle|0\rangle|+\rangle|0\rangle + |+\rangle|0\rangle|-\rangle|1\rangle + |-\rangle|1\rangle|-\rangle|0\rangle + |-\rangle|1\rangle|+\rangle|1\rangle)/2$, write down its stabilizer generator sets.

   Answer:

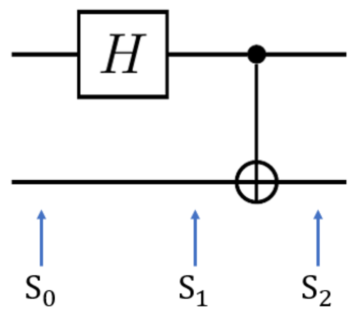
   (1) An example is as following: $g_1 = ZZII, g_2 = -IZZI, g_3 = IIZZ, g_4 = XXXX$.(Other answers are welcome.)

   (2) An example is as following: $g_1 = XZII, g_2 = ZXZI, g_3 = IZXZ, g_4 = IIZX$.

4. (a) Denote the controlled-NOT gate as U, calculate the following and express your results with only Pauli operators. The subscripts denote the labels of qubits.

$$(1)\ U(X_1I_2)U^\dagger \qquad (2)\ U(Z_1I_2)U^\dagger \qquad (3)\ U(I_1X_2)U^\dagger \qquad (4)\ U(I_1Z_2)U^\dagger.$$

   (b) Consider the following quantum circuit: The input state is $|00\rangle$, which is stabilized by $S_0 = \langle IZ, ZI\rangle$. Give the generators of the stabilizers describing the state after the Hadamard S_1 and after the controlled-NOT gate



S_2. Work this out by using the fact that U acting on a state stabilized by S produces a state stabilized by $USU^\dagger$.

Answer:

(a)

(1) X_1X_2 (2) Z_1I_2 (3) I_1X_2 (4) Z_1Z_2.

(b)

$$S_1 = \langle IZ, XI \rangle, \ S_2 = \langle ZZ, XX \rangle.$$

5. Please write down the difference between quantum error correction and classical error correction.

Answer:

Please read page 3-6 in the lecture "QIP2022chapt 5 Kai Chen.pdf" for reference.

6. Find a parity check matrix H for the $[6,2]$ repetition code defined by the generator matrix G. Then verify that $HG = 0$.

$$G = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Answer:

For example,

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(Other answers are welcome.)

7. Please give a parity check matrix H for the $[7, 4]$ Hamming code, and write down its distance.

Answer:

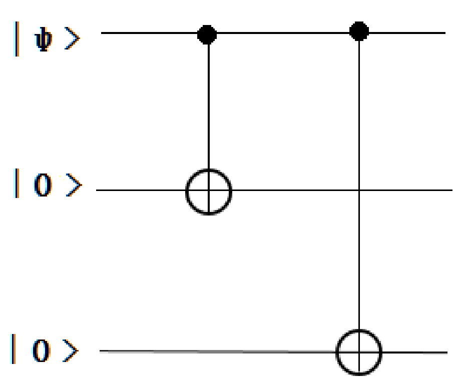
$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$d = 3.$

8. Please draw the quantum circuit of the 3-qubit bit flip code, and certify that it can encode the qubit $a\,|0\rangle + b\,|1\rangle$ to $a\,|000\rangle + b\,|111\rangle$.

Answer:

The quantum circuit is as following:



The encoding progress is as following:

$$
\begin{aligned}
|\psi\rangle|0\rangle|0\rangle &= (a|0\rangle + b|1\rangle)|0\rangle|0\rangle \\
&\xrightarrow{\text{C-NOT}} (a|00\rangle + b|11\rangle)|0\rangle \\
&\xrightarrow{\text{C-NOT}} (a|000\rangle + b|111\rangle)
\end{aligned}
$$

9. For 9-qubit Shor code, its logical bit code is

$$|0\rangle_L = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)/2\sqrt{2},$$

$$|1\rangle_L = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)/2\sqrt{2}.$$

(1) Please give all the generators of the stabilizers;

(2) Please draw the encoding quantum circuit;

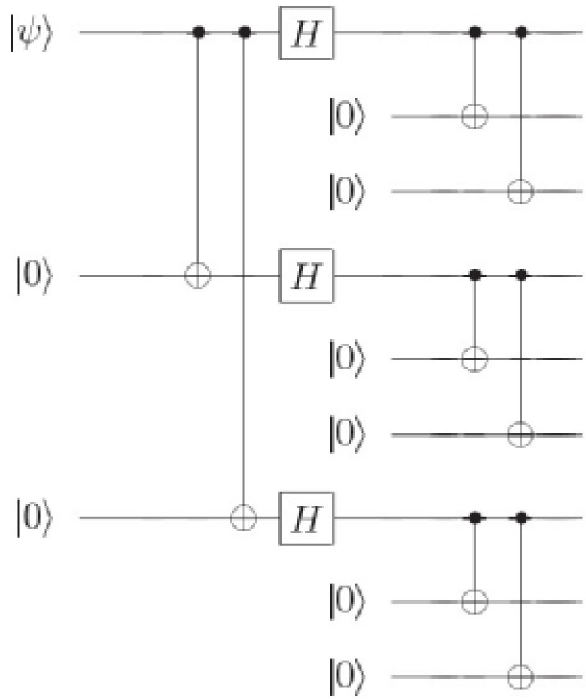
(3) For a bit/phase flip error of a certain bit, how to detect and correct it? Please take the bit flip error and phase flip error for example, write down the program of error detection and correction.

Answer:

(1) The generators of the stabilizers are as following:

Name	Operator
g_1	$Z\,Z\,I\,I\,I\,I\,I\,I\,I$
g_2	$I\,Z\,Z\,I\,I\,I\,I\,I\,I$
g_3	$I\,I\,I\,Z\,Z\,I\,I\,I\,I$
g_4	$I\,I\,I\,I\,Z\,Z\,I\,I\,I$
g_5	$I\,I\,I\,I\,I\,I\,Z\,Z\,I$
g_6	$I\,I\,I\,I\,I\,I\,I\,Z\,Z$
g_7	$X\,X\,X\,X\,X\,X\,I\,I\,I$
g_8	$I\,I\,I\,X\,X\,X\,X\,X\,X$
$\bar{Z}$	$X\,X\,X\,X\,X\,X\,X\,X\,X$
$\bar{X}$	$Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z$

(2) The encoding quantum circuit is as following:



(3) Please read page 433 of Nielsen's "Quantum Computation and Quantum Information" , or page 80 of the Chinese version translated by Qian-Chuan Zhao.

10. Single qubit quantum operations $\mathcal{E}(\rho)$ model quantum noise which is corrected by quantum error correction codes.

(1) Construct operation elements for $\mathcal{E}$ such that upon input of any state ρ replaces it with the completely randomized state $I/2$.

(2) The action of the bit flip channel can be described by the quantum operation $\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$. Show that this may be given an alternate operator-sum representation as $\mathcal{E}(\rho) = (1-2p)\rho + 2pP_+\rho P_+ + 2pP_-\rho P_-$, where P_+ and P_- are projectors onto the $+1$ and -1 eigenstates of X, $(|0\rangle + |1\rangle)/\sqrt{2}$ and $(|0\rangle - |1\rangle)/\sqrt{2}$, respectively.

Answer:

(1) We apply I, X, Y, Z gates with equal probability;

$$E_0 = \frac{I}{2}, E_1 = \frac{X}{2}, E_2 = \frac{Y}{2}, E_3 = \frac{Z}{2}.$$

We may describe the density matrix of one qubit as $\rho = I/2 + aX + bY + cZ$. Since I commutes with all operators, this component is left alone. Since X, Y, Z commute with half of the E_k and anticommute with half, these components will go to zero in the outputted density matrix, for example:

$$\mathcal{E}(X) = \sum_k E_k X E_k^\dagger = 0,$$

and likewise for Y, Z. Therefore, by linearity for all inputs $\rho : \mathcal{E}(\rho) = I/2$.

(2) We use the identities $X = P_+ - P_-, I = P_+ + P_-$. And then

$$\begin{aligned}\mathcal{E}(\rho) &= (1-p)\rho + pX\rho X \\ &= (1-2p)\rho + p(X\rho X + I\rho I) \\ &= (1-2p)\rho + 2pP_+\rho P_+ + 2pP_-\rho P_-,\end{aligned}$$

which is equivalent to measuring the state in the $|+\rangle, |-\rangle$ basis with probability $2p$.

11. Please see the next page about "Model single-photon imaging".

Answer: See the last two pages.

Model single-photon imaging

Figure 1 depicts the signal-acquisition model underlying a standard single-photon imaging setup. We index the scene pixels as (i, j), where $i, j = 1, 2, \ldots, n$. The reflectivity for pixel (i, j) is $\alpha_{i,j} \geq 0$. A periodically pulsed laser is used to illuminate the scene in a raster-scanned manner, and its repetition period is T_r and the waveform of a single pulse is denoted by $s(t)$. A single-photon avalanche diode (SPAD) detector provides time-resolved single-photon detections, called *clicks*. Its quantum efficiency η is the fraction of photons that are detected. Each pixel (i, j) is illuminated with N laser pulses. We record the total number of photon detections $k_{i,j}$. We also shine background light, with photon flux b_λ at the operating optical wavelength λ, onto the detector, and define the SPAD's dark noise rate as d. Note that the system is working at low-flux condition where the photon-flux per pixel per pulse-repetition period is much less than 1.

Question. Please derive the probability distribution of the numbers of detected photons, $K_{i,j}$, accross N illumination pulses, i.e., Pr$[K_{i,j} = k_{i,j};\, \alpha_{i,j}]$.

Answer. Illuminating pixel (i, j) with the pulse $s(t)$ results in backreflected light with photon flux $r_{i,j}(t) = \alpha_{i,j} s(t - 2z_{i,j}/c) + b_\lambda$ at the detector. The photon detections produced by the SPAD in response to the backreflected light from transmission of $s(t)$ constitute an inhomogeneous Poisson process with time-varying rate function $\eta r_{i,j}(t)$. To these photon detections we must add the detector dark counts, which come from an independent homogeneous Poisson process with rate d. Lumping the dark counts together with the background-generated counts yields the observation process at the SPAD's output, viz., as shown in Figure 1, an inhomogeneous Poisson process with rate function

$$
\begin{aligned}
\lambda_{i,j}(t) &= \eta r_{i,j}(t) + d \\
&= \eta \alpha_{i,j} s(t - 2z_{i,j}/c) + (\eta b_\lambda + d),
\end{aligned} \tag{1}
$$

when only a single pulse is transmitted. Figure 1 shows the rate function $\lambda_{i,j}(t)$ for the pulse-stream transmission.

Define $S = \int s(t)\, dt$ and $B = (\eta b_\lambda + d)T_r$ as the total signal and background count per pulse-repetition period, where we have used—and will use in all that follows—background counts to include dark counts as well as counts arising from ambient light. Using Poisson process properties, we have that the probability of the SPAD detector *not* recording a detection at pixel (i, j) from one illumination trial is

$$
P_0(\alpha_{i,j}) = \exp[-(\eta \alpha_{i,j} S + B)]. \tag{2}
$$

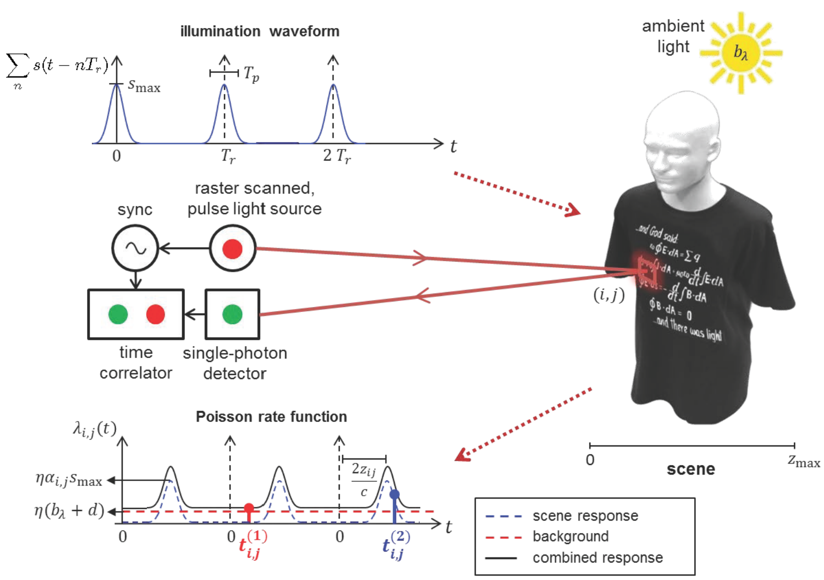


Fig. 1. The observation model of single-photon imaging. Here, $N = 2$ and $k_{i,j} = 2$. A background count (red) occurred after the second pulse was transmitted, and a signal count (blue) occurred after the third pulse was transmitted.

Because we illuminate with a total of N pulses, and the low-flux condition ensures that multiple detections per repetition interval can be neglected, the number of detected photons $K_{i,j}$ is *binomially distributed* with probability mass function

$$\Pr\left[K_{i,j} = k_{i,j}; \alpha_{i,j}\right] \tag{3}$$

$$= \binom{N}{k_{i,j}} P_0(\alpha_{i,j})^{N-k_{i,j}} \left[1 - P_0(\alpha_{i,j})\right]^{k_{i,j}}, \tag{4}$$

for $k_{i,j} = 0, 1, \ldots, N$.