磁流体力学中的解析方法

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绪论

1. 课程内容
   1. 静磁场
   2. 磁静平衡态及能量原理
   3. 磁流体力学中的相似解
2. 课程目的
   1. 解析分析和推导的方法
   2. 阅读文献的能力
3. 考核方式
   1. 作业 + 随堂练习 + 考试
   2. 开卷考试
4. 作业要求
   1. 必要的推导细节
   2. 独立完成

静磁场与无力场

- 磁场的结构及演化规律是空间物理研究的核心内容之一. 本章主要讨论无力场 (force-free field).
- 由磁流体力学的动量方程：

$$\rho\frac{d\boldsymbol{v}}{dt}=-\nabla p+\boldsymbol{j}\times\boldsymbol{B}+\boldsymbol{F}$$

如果电磁力占主导, 热压力和重力等其它作用力相比电磁力可以忽略；并且对于平衡系统 $d\boldsymbol{v}/dt=0$. 则可得此时磁场满足的条件：$\boldsymbol{j}\times\boldsymbol{B}=0$

- 这种情况被称为无力场 (force-free field), 本章主要讨论无力场.
- 对于日冕区域（特别是活动区上空）和行星际空间中的处于平衡系统中的等离子体, 等离子体 β（热压力和此压力之比）很小, 从而导致其他作用力相比电磁力可以忽略, 从而满足无力场条件.
- 电磁力也称为安培力, 其形式为 $f=\rho\boldsymbol{E}+\boldsymbol{j}\times\boldsymbol{B}$ (对于满足电中性的磁流体, $f=\boldsymbol{j}\times\boldsymbol{B}$), 不要与单电荷的洛伦兹力（$q\boldsymbol{E}+q\boldsymbol{v}\times\boldsymbol{B}$）混淆.
- 电磁力（安培力）做功的功率为 $P=\boldsymbol{j}\cdot\boldsymbol{E}$

无力场

1. 对于势场. $\boldsymbol{j}=0$, $\nabla\times\boldsymbol{B}=0$, 显然满足无力场条件, 本章不做详细讨论.
2. 带中性电流片的势场. 存在若干个孤立面, $\boldsymbol{B}=0$, 面电流 $\boldsymbol{i}$ 有限 ($\boldsymbol{j}\to\infty$). 其余部分为势场 ($\boldsymbol{j}=0$).
   - 磁场满足无散条件 $\nabla\cdot\boldsymbol{B}=0$, 因此在孤立面上磁场法向连续, 只能存在切向间断.
3. 无力场. $\boldsymbol{j}=\alpha\boldsymbol{B}$, α 称为无力因子 (force-free factor). $\alpha=\text{const.}$ 时为线性无力场. 一般地

$$\nabla^2\boldsymbol{B}+\alpha^2\boldsymbol{B}=0, \qquad (\mu=1),$$

$\alpha\neq\text{const.}$, $\boldsymbol{B}\cdot\nabla\alpha=0$.

4. 带奇异电流密度面的无力场. 无力场局限于有限空间, 在边界上 $\boldsymbol{j}\to\infty$, 但广义可积 ($\boldsymbol{i}=0$).
   - 即分界面上总电流密度为 0.

分区反转法 (适用于三维)

任给一个势场，找一个磁面，将其一侧 $\boldsymbol{B}$ 反向，或找两个磁面，将两磁面之间的 $\boldsymbol{B}$ 反向.

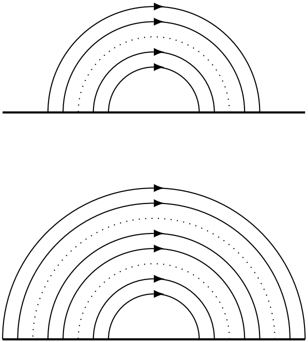


图 1.1: 分区反转磁场.

平面势场的复变函数表示 I

二维磁场

$$\boldsymbol{B} = (B_x, B_y), \tag{1.1}$$

满足 Cauchy-Riemann 条件

$$\begin{cases} \nabla \cdot \boldsymbol{B} = 0, & \dfrac{\partial B_x}{\partial x} = -\dfrac{\partial B_y}{\partial y}, \\ \nabla \times \boldsymbol{B} = 0, & \dfrac{\partial B_x}{\partial y} = \dfrac{\partial B_y}{\partial x}, \end{cases} \tag{1.2}$$

可以表示成

$$f(z) = B_x - iB_y, \qquad z = x + iy. \tag{1.3}$$

B_x 和 $-B_y$ 构成共轭调和函数. 有时需要给出磁势 ϕ 和磁通量函数 ψ ($\boldsymbol{A} = \psi \boldsymbol{e}_z$),

$$\begin{cases} \boldsymbol{B} = -\nabla\phi = \left(-\dfrac{\partial \phi}{\partial x}, -\dfrac{\partial \phi}{\partial y}\right), \\ \boldsymbol{B} = \nabla\psi \times \boldsymbol{e}_z = \left(\dfrac{\partial \psi}{\partial y}, -\dfrac{\partial \psi}{\partial x}\right). \end{cases} \tag{1.4}$$

平面势场的复变函数表示 II

可证

$$f(z)=\frac{dw}{dz},\quad \text{或}\quad w=\int f(z)dz,\qquad w(z)=-\phi+i\psi. \tag{1.5}$$

证明如下

$$\begin{aligned}
\int f(z)dz &= \int (B_x-iB_y)(dx+idy)=\int [(B_xdx+B_ydy)+i(B_xdy-B_ydx)] \\
&= \int \left[-\frac{\partial\varphi}{\partial x}dx-\frac{\partial\varphi}{\partial y}dy\right]+i\int\left[\frac{\partial\psi}{\partial y}dy+\frac{\partial\psi}{\partial x}dx\right] \\
&= -\int d\varphi+i\int d\psi \\
&= \int d(-\varphi+i\psi)
\end{aligned}$$

平面势场的复变函数表示 III

另有

$$\frac{\partial w}{\partial x} = f(z)$$

$$\frac{\partial w}{\partial y} = if(z)$$

静磁场的 Gauss 定理和环路定理的复数表示 I

环路积分

$$
\begin{aligned}
\oint_C f(z)dz &= \oint_C (B_x - iB_y)(dx + idy) \\
&= \oint_C (B_x dx + B_y dy) + i\oint_C (B_x dy - B_y dx) \\
&= \oint_C \boldsymbol{B}\cdot d\boldsymbol{l} + i\oint_C \boldsymbol{B}\cdot \boldsymbol{n} dl \\
&= \mu_0 I + i\lambda_m.
\end{aligned}
$$

式中 I 表示 (线) 电流密度, λ 表示 (线) 磁荷密度. 由复变函数的留数定理

$$
\oint_C f(z)dz = 2\pi i \sum_k \mathrm{Res}\left[f(z_k)\right]
$$

静磁场的 Gauss 定理和环路定理的复数表示 II

有

$$\mathrm{Im}\left\{\sum_k \mathrm{Res}\left[f(z_k)\right]\right\} = -\frac{\mu_0 I}{2\pi}, \tag{1.6}$$

$$\mathrm{Re}\left\{\sum_k \mathrm{Res}\left[f(z_k)\right]\right\} = \frac{\lambda_m}{2\pi}. \tag{1.7}$$

场源和极点 I

1. 一阶极点

$$f(z)=\frac{\lambda}{2\pi(z-z_0)},\qquad \mathrm{Res}f(z_0)=\frac{\lambda}{2\pi}.$$

   1. λ 是实数，由 (1.7), $\lambda_m=\lambda$ 表示线磁荷密度.
   2. λ 是纯虚数，由 (1.6), $I=i\lambda/\mu_0$ 表示线电流密度.
   3. λ 是复数，$\lambda=\lambda_1+i\lambda_2$, $\lambda_m=\lambda_1$ 为线磁荷密度，$I=-\lambda_2/\mu_0$ 为线电流密度.

场源和极点 II

2. 二阶极点和多阶极点. 考虑两线磁荷, 密度为 $\pm\lambda$, 分别位于 $z = z_0 \pm \delta_z/2$,

$$
\begin{aligned}
f(z) &= \frac{\lambda}{2\pi(z - z_0 - \delta_z/2)} - \frac{\lambda}{2\pi(z - z_0 + \delta_z/2)} \\
&= \frac{\lambda\delta_z}{2\pi\left[(z - z_0)^2 - \delta_z^2/4\right]}.
\end{aligned}
$$

令 $\lim\limits_{\substack{\delta_z\to 0\\ \lambda\to\infty}} \lambda\delta_z = D$, D 表示线磁偶极矩密度, 则有

$$
\lim_{\delta_z\to 0} f(z) = \frac{D}{2\pi(z - z_0)^2}. \tag{1.8}
$$

同理, 三阶极点表示线四极子, 四阶极点表示线八极子, 等等.

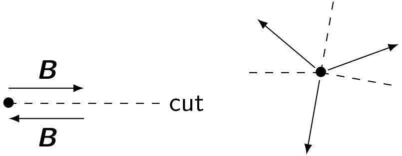
磁中性点与零点, 支点 I

磁场零点很重要，经常存在于磁场发生剧烈变化的区域.
在磁零点处,

$$\boldsymbol{B}\big|_{z=z_0} = 0, \qquad (f(z_0) = 0).$$

$$f(z) \propto (z - z_0)^{\frac{1}{m+1}}, \qquad (m \geq 1).$$

z_0 称为 $f(z)$ 的 m 阶支点, 绕支点旋转一周, $f(z)$ 的幅角改变 $2\pi/(m+1)$.[1]



(a) 一阶支点 (b) 二阶支点

图 1.2: 支点.

磁中性点与零点, 支点 II

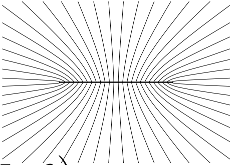
例

日珥与多阶支点间的切割 – 以二阶支点为例, 旋转一周幅角变化 $2\pi/3$.

图 1.3: 太阳日冕中的 cusp 结构.

[1] 中性电流片 – 作为一阶支点间与磁场平行的切割.

位于边界上的均匀磁荷片 I



在 $a \le x \le b$ 间均匀分布面磁荷

$$f(z) = \frac{\sigma_m}{2\pi} \ln\left(\frac{z-a}{z-b}\right),$$

$$\ln\left(\frac{z-a}{z-b}\right) = \ln\left|\frac{z-a}{z-b}\right| + i\arg\left(\frac{z-a}{z-b}\right), \quad \left(0 \le \arg\left(\frac{z-a}{z-b}\right) < 2\pi\right).$$

$$\arg(z-a) = \begin{cases} \frac{\pi}{2} - \tan^{-1}\left(\frac{x-a}{y}\right), & y \ge 0, \\ \frac{3\pi}{2} - \tan^{-1}\left(\frac{x-a}{y}\right), & y < 0, \end{cases}$$

$$-\pi/2 \le \tan^{-1} f \le \pi/2,$$

磁场

$$B_x = \frac{\sigma_m}{4\pi} \ln\left[\frac{(x-a)^2+y^2}{(x-b)^2+y^2}\right],$$

$$B_y = \frac{\sigma_m}{2\pi}\left[\tan^{-1}\left(\frac{x-a}{y}\right) - \tan^{-1}\left(\frac{x-b}{y}\right)\right],$$

位于边界上的均匀磁荷片 II

$y = 0$ 处,

$$B_y\big|_{y=0^+} = \begin{cases} 0, & x > b, \\ +\sigma_m/2, & a \le x \le b, \\ 0, & x < a, \end{cases}$$

$$B_y\big|_{y=0^-} = \begin{cases} 0, & x > b, \\ -\sigma_m/2, & a \le x \le b, \\ 0, & x < a, \end{cases}$$

$$B_y\big|_{y=0^+} - B_y\big|_{y=0^-} = \begin{cases} \sigma_m, & a \le x \le b, \\ 0, & x < a \text{ 或 } x > b. \end{cases}$$

其他量

$$w(z) = \int f(z)dz = \frac{\sigma_m}{2\pi}\left[(z-a)\ln(z-a) - (z-b)\ln(z-b)\right],$$

$$\psi = \mathrm{Im}(w)$$

位于边界上的均匀磁荷片 III

$$= \frac{\sigma_m y}{4\pi} \ln \left[\frac{(x-a)^2 + y^2}{(x-b)^2 + y^2} \right]$$
$$- \frac{\sigma_m (x-a)}{2\pi} \tan^{-1} \left(\frac{x-a}{y} \right) + \frac{\sigma_m (x-b)}{2\pi} \tan^{-1} \left(\frac{x-b}{y} \right) + \psi_0,$$
$$\psi_0 = \begin{cases} \frac{\sigma_m}{4}(b-a), & y > 0, \\ \frac{3\sigma_m}{4}(b-a), & y < 0, \end{cases}$$

如限于 $y > 0$, 可取 $\psi_0 = 0$. 若将 σ_m 换为面电流 i', 只需令 $\sigma_m = -i\mu_0 i'$.

中性电流片的计算 I

磁力线方程

$$\frac{dy}{dx} = \frac{B_y}{B_x}, \tag{1.9}$$

或

$$\begin{cases} \frac{dx}{ds} = \frac{B_x}{B}, \\ \frac{dy}{ds} = \frac{B_y}{B}. \end{cases} \tag{1.10}$$

找到支点处临界磁力线的斜率,

1. 写出 $B_x - iB_y = f(z)$ 在某一阶支点 $z = z_c$ 附近的 Taylor 展开式，令 $z = z_c + re^{i\theta}$, r 为小量. 当取 z 在临界磁力线上时, θ 即为临界磁力线的方向.

中性电流片的计算 II

2. 由

$$\begin{cases} \frac{B_y}{B_x} = -\tan\left[\arg f(z)\right], \\ \frac{B_y}{B_x} = \tan\theta \end{cases}$$

可推断

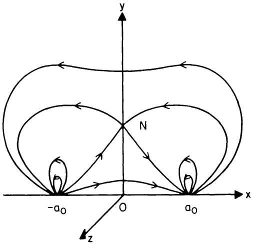
$$-\arg f(z) = \theta + n\pi. \tag{1.11}$$

式中 $\tan\theta$ 为临界磁力线的斜率, n 为整数.

3. 以 $\tan\theta$ 为斜率, 从该支点出发引磁力线, 使某端点脱离支点, 从而可解 (1.10), 继续进行常规积分. 更多相关内容见Hu et al. (1982); Low et al. (1983); 胡友秋 (1985, 1984); 胡友秋 等 (1986).

实例 I

1. 双线偶极子势场



$$
\begin{aligned}
B_{px} - iB_{py} &= \frac{D}{(z-a_0)^2} + \frac{D}{(z+a_0)^2} \\
&= \frac{2D(z^2+a_0^2)}{(z^2-a_0^2)^2}.
\end{aligned}
\tag{1.12}
$$

其线偶极矩密度为 $2\pi D/\mu$.

1. 磁中性点

$$
z_N = ia_0, \qquad y_N = a_0. \tag{1.13}
$$

实例 II

2. ON 之间的磁通

$$A_N = \int_0^{y_N} B_x|_{x=0}\, dy = \int_0^{y_N} \left.\frac{\partial \psi}{\partial y}\right|_{x=0} dy = \psi(0, y_N) - \psi(0,0),$$

而

$$w = \int f(z) dz = -\phi + i\psi = -\frac{D}{z-a_0} - \frac{D}{z+a_0},$$

$$\psi = \mathrm{Im} w = \frac{yD}{(x-a_0)^2+y^2} + \frac{yD}{(x+a_0)^2+y^2},$$

$$\psi(0,y) = \frac{2yD}{a_0^2+y^2}, \qquad \psi(0,0) = \psi(0,\infty) = 0.$$

因此

$$A_N = \psi(0, y_N) = \frac{D}{a_0}. \tag{1.14}$$

实例 III

2. 带中性电流片的势场

$$f(z) = B_x - iB_y = \frac{E(z^2 - z_1^2)^{1/2}(z^2 - z_2^2)^{1/2}}{(z^2 - a^2)^2}. \qquad (1.15)$$

3. 如何确定参数 E, z_1, z_2 和 a
    1. 在 $z = \pm a$ 处, $f(z) = \frac{D}{(z \mp a)^2}$, 由 (1.15),

$$f \to \frac{E(a^2 - z_1^2)^{1/2}(a^2 - z_2^2)^{1/2}}{4a^2(z \mp a)^2} \to \frac{D}{(z \mp a)^2},$$

那么

$$E = \frac{4a^2 D}{(a^2 - z_1^2)^{1/2}(a^2 - z_2^2)^{1/2}}. \qquad (1.16)$$

实例 IV

2. 取环路 $C - y$ 轴 $(-\infty < y < +\infty)$ 和右侧无穷大半圆周构成

$$\oint_C f(z)dz = 0, \qquad \sum \mathrm{res} f(z) = 0.$$

相关极点 $z = a$, 为二阶极点, 因此

$$\mathrm{res} f(a) = \frac{d}{dz}\left[f(z)(z-a)^2\right]_{z=a} = 0,$$

即

$$\frac{d}{dz}\left[\frac{(z^2 - z_1^2)^{1/2}(z^2 - z_2^2)^{1/2}}{(z+a)^2}\right]_{z=a} = 0,$$

而

$$z_1 z_2 = -a^2. \tag{1.17}$$

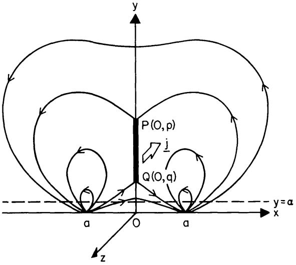
z_1 和 $-z_2$ 一定互为共轭复数, 或同为纯虚数.

实例 V

3. 在中性片下方的磁通为

$$A_N = D/a_0. \tag{1.18}$$

4. 垂直电流片 $(a < a_0)$



$$z_1 = ip, \quad z_2 = iq \quad (p > q), \tag{1.19}$$

$$pq = a^2. \tag{1.20}$$

由 (1.8),

$$E = \frac{4a^2 D}{(p^2 + a^2)^{1/2}(q^2 + a^2)^{1/2}}, \tag{1.21}$$

$$B_x - iB_y = \frac{4a^2 D(z^2 + p^2)^{1/2}(z^2 + q^2)^{1/2}}{(p^2 + a^2)^{1/2}(q^2 + a^2)^{1/2}(z^2 - a^2)^2}. \tag{1.22}$$

实例 VI

由 (1.18),

$$\frac{D}{a_0} = \frac{4a^2 D}{(p^2+a^2)^{1/2}(q^2+a^2)^{1/2}} \int_0^q (p^2-y^2)^{\frac{1}{2}}(q^2-y^2)^{\frac{1}{2}}(a^2+y^2)^{-2} dy. \tag{1.23}$$

计算 (迭代), 对给定 $a < a_0$, 预猜 q ($q < a_0$), 验证 (1.23) 是否满足, 调整 q, 直至 (1.23) 满足为止.

- 解析分析. 初始时 $a = a_0$, $q = p = a_0$, 尔后 $a \downarrow \Rightarrow q \downarrow$, 因而 $\frac{dE}{da} > 0$.

实例 VII

- 求支点处临界曲线的斜率. 令 $z = iq + re^{i\theta}$,

$$f(z) = B_x - iB_y \approx \frac{E\left[(iq + re^{i\theta})^2 + q^2\right]^{1/2}\left[p^2 - q^2\right]^{1/2}}{(-q^2 - a^2)^2}.$$

下一步求

$$\begin{aligned}
\arg f(z) &= \arg\left\{\left[(iq + re^{i\theta})^2 + q^2\right]^{1/2}\right\} \\
&= \arg\left\{\left[2iqre^{i\theta} + r^2 e^{i2\theta}\right]^{1/2}\right\} \\
&= \arg\left\{\left[2iqre^{i\theta}\right]^{1/2}\right\} \\
&\approx \arg\left\{e^{i(\theta + \frac{\pi}{2})/2}\right\} = \frac{\theta}{2} + \frac{\pi}{4}.
\end{aligned}$$

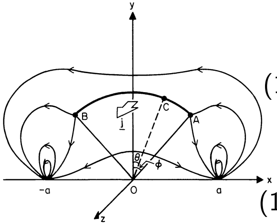
而 $-\arg f(z) = \theta + n\pi$,

$$\theta = -\frac{2}{3}\left(\frac{\pi}{4} + n\pi\right) = -\frac{\pi}{6}(4n + 1).$$

实例 VIII

$$\begin{cases} n=-1, & \theta=\pi/2, \\ n=-2, & \theta=\frac{7}{6}\pi, \\ n=-3, & \theta=\frac{11}{6}\pi. \end{cases}$$

5. 横向电流片 $(a > a_0)$



$$z_1 = ae^{i(\pi/2-\phi)}, \qquad z_2 = ae^{i(\pi/2+\phi)}, \tag{1.24}$$

满足 (1.17) 式, (1.15) 和 (1.16) 化为

$$E = 2D/\cos\phi, \tag{1.25}$$

$$f(z) = B_x - iB_y = \frac{2D}{\cos\phi}\frac{(z^2+a^2e^{2i\phi})^{1/2}(z^2+a^2e^{-2i\phi})^{1/2}}{(z^2-a^2)^2}. \tag{1.26}$$

定理

过支点 z_1 和 z_2 的圆周 (半径为 a) 是临界磁力线.

实例 IX

令 $z = z_c + re^{i\delta} = ae^{i\theta} + re^{i\delta}$, r 为小量，$\delta$ 为临界磁力线方向，则

$$\pi/2 - \phi < \theta < \pi/2 + \phi.$$

由此推出 $|\theta - \phi| < \pi/2$ 和 $\pi/2 < |\theta + \phi| < 3\pi/2$. 同时

$$z^2 = a^2 e^{2i\theta} + 2e^{i(\theta+\delta)} ar + r^2 e^{2i\delta},$$

可得

$$\arg f(z) \approx \arg \left[\frac{\left(e^{2i\theta} + e^{2i\phi}\right)^{1/2} \left(e^{2i\theta} + e^{-2i\phi}\right)^{1/2}}{\left(e^{2i\theta} - 1\right)^2} \right],$$

其中各因子

$$\arg\left[\left(e^{2i\theta} - 1\right)^2\right] = \arg\left\{\left[e^{i\theta}\left(e^{i\theta} - e^{-i\theta}\right)\right]^2\right\}$$
$$= \arg\left\{\left[e^{i(\theta+\pi/2)}\right]^2\right\} = 2\theta + \pi,$$

实例 X

$$\arg\left[\left(e^{2i\theta}+e^{2i\phi}\right)^{1/2}\right]=\arg\left\{e^{i(\theta+\phi)}\left[e^{i(\theta-\phi)}+e^{-i(\theta-\phi)}\right]\right\}^{1/2}$$
$$=\arg\left[2e^{i(\theta+\phi)}\cos(\theta-\phi)\right]^{1/2}=(\theta+\phi)/2,$$
$$\arg\left[\left(e^{2i\theta}+e^{-2i\phi}\right)^{1/2}\right]=\arg\left\{e^{i(\theta-\phi)}\left[e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}\right]\right\}^{1/2}$$
$$=\arg\left[2e^{i(\theta-\phi)}\cos(\theta+\phi)\right]^{1/2}=(\theta-\phi+\pi)/2.$$

那么

$$\arg f(z)=(\theta+\phi)/2+(\theta-\phi+\pi)/2-2\theta-\pi=-\theta-\pi/2,$$

由

$$-\arg f(z)=\delta+n\pi$$

实例 XI

可得

$$\delta = -\left(n - \frac{1}{2}\right)\pi + \theta$$

表明 z_c 是一阶极点，且临界磁力线方向与 OC 垂直，因此过 z_1 和 z_2 的圆周为临界磁力线.

由 (1.26) 求出 $B_x|_{x=0}$, 然后沿 y 轴积分 (从 0 到 a) 求磁通, 该磁通应等于 $A_N = D/a_0$.

$$1 = \frac{2a_0}{\cos\phi}\int_0^a \frac{(y^4 + a^4 - 2a^2y^2\cos 2\phi)^{1/2}}{(y^2+a^2)^2}dy \quad (a > a_0) \tag{1.27}$$

实例 XII

6. 带中性电流片势场的磁自由能
   1. 磁自由能 (Hu et al., 1982)

定义

系统磁能与对应势场磁能之差.

对应势场指的是在边界上 B_n 与考察磁场相同, 在给定 $B_n|_S$ 的前提下, 势场能量最低

$$W = \int_V \frac{B^2}{2\mu_0} dV,$$

$$\delta W = \delta \int_V \frac{B^2}{2\mu_0} dV = 0.$$

$\delta W = 0$ 是驻立值 (默认为极小值, 证明可由二阶变分给出). 边界约束为

$$B_n|_S \text{ 给定}, \ \delta\, B_n|_S = 0$$

实例 XIII

内约束为

$$\nabla \cdot \boldsymbol{B} = 0.$$

用 Lagrange 乘子法解约束

$$\delta \int \left(\frac{B^2}{2\mu_0} + \lambda \nabla \cdot \boldsymbol{B} \right) dV = 0,$$

λ 是位置的函数 $\lambda = \lambda(\boldsymbol{r})$，由

$$\delta \left(\frac{B^2}{2\mu_0} + \lambda \nabla \cdot \boldsymbol{B} \right) = \left(\frac{\boldsymbol{B}}{\mu_0} - \nabla \lambda \right) \cdot \delta \boldsymbol{B} + \nabla \cdot (\lambda \delta \boldsymbol{B}),$$

得

$$\int \left(\frac{\boldsymbol{B}}{\mu_0} - \nabla \lambda \right) \cdot \delta \boldsymbol{B} dV + \oint_S \lambda \delta B_n dS = 0.$$

实例 XIV

而

$$\oint_S \lambda\delta B_n dS = 0,$$

由 $\delta \boldsymbol{B}$ 的任意性可知，$\frac{\boldsymbol{B}}{\mu_0} - \nabla\lambda = 0$. 即 $\boldsymbol{B} = \mu_0\nabla\lambda$，磁场为势场. □

磁自由能

$$F = \frac{1}{2\mu_0}\int_V (B^2 - B_p^2) dV, \qquad (\,B_n|_S = B_{pn}|_S). \tag{1.28}$$

$\boldsymbol{B}_p$ 为对应的势场. 对二维磁场，一般定义

$$W_s = \frac{dF}{dz}. \tag{1.29}$$

$$W_s = \frac{1}{2\mu_0}\int_S (B^2 - B_p^2) dx\, dy. \tag{1.30}$$

为了将 (1.30) 化简，先做一个预推导

$$\frac{1}{2\mu_0}\int_V B^2 dV = \frac{1}{2\mu_0}\int_V \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) dV$$

实例 XV

$$
\begin{aligned}
&= \frac{1}{2\mu_0}\int_V \left[\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) + (\nabla\times\boldsymbol{B})\cdot\boldsymbol{A}\right] dV \\
&= \frac{1}{2\mu_0}\oint_S (\boldsymbol{A}\times\boldsymbol{B})\cdot d\boldsymbol{S} + \frac{1}{2}\int_V \boldsymbol{j}\cdot\boldsymbol{A}dV
\end{aligned}
$$

二维情况下，上半空间平面

$$
\frac{1}{2\mu_0}\int B^2\,dx\,dy = -\frac{1}{2\mu_0}\int_{-\infty}^{\infty}(\psi B_x)_{y=0}\,dx + \frac{1}{2}\int_{-\infty}^{\infty}\left(\int_0^{\infty} j\psi\,dy\right)dx,
$$

因此

$$
W_s = \frac{1}{2}\int_{-\infty}^{\infty}\left(\int_0^{\infty} j\psi\,dy\right)dx - \frac{1}{2\mu_0}\int_{-\infty}^{\infty}\left[\psi(B_x - B_{px})\right]_{y=0}dx. \tag{1.31}
$$

实例 XVI

2. 中性电流片势场的磁能. 电流仅限于电流片中, 沿电流片 $\psi = \psi_c = \text{const.}$.

$$W_s = \sum \frac{1}{2} I_c \psi_c - \frac{1}{2\mu_0} \int_{-\infty}^{\infty} \left[\psi(B_x - B_{px})\right]_{y=0} dx, \tag{1.32}$$

设法省去计算 $\boldsymbol{B}_p$ 之烦. 将 $\boldsymbol{B}$ 分解为两部分 $\boldsymbol{B} = \boldsymbol{B}_c + \boldsymbol{B}_0$, $\boldsymbol{B}_c$ 为电流片产生的场 (Biot-Savart 定律), $\boldsymbol{B}_0$ 为余下部分, 在上半平面为势场.

$$\boldsymbol{B} = \boldsymbol{B}_c + \boldsymbol{B}_{c'} + \boldsymbol{B}_0 - \boldsymbol{B}_{c'},$$

$\boldsymbol{B}_{c'}$ 为像电流的场

$$\boldsymbol{B}_{c'x}|_{y=0} = \boldsymbol{B}_{cx}|_{y=0},$$
$$(\boldsymbol{B}_{c'y} + \boldsymbol{B}_{cy})_{y=0} = 0.$$

令 $\boldsymbol{B}_p = \boldsymbol{B}_0 - \boldsymbol{B}_{c'}$, 那么 $\boldsymbol{B} = \boldsymbol{B}_c + \boldsymbol{B}_{c'} + \boldsymbol{B}_p$.

- $\boldsymbol{B}_p$ 为势场 $(y > 0)$.

实例 XVII

- $B_{pn}|_{y=0} = B_n|_{y=0}$. $\boldsymbol{B}_p$ 即为所要对应的势场.

$$
\begin{aligned}
B_x|_{y=0} &= 2B_{cx}|_{y=0} + B_{px}|_{y=0}\,, \\
(B_x - B_{px})_{y=0} &= 2B_{cx}|_{y=0}\,.
\end{aligned}
$$

于是 (1.32) 化为

$$
W_s = \sum \frac{1}{2} I_c \psi_c - \frac{1}{\mu_0} \int_{-\infty}^{\infty} (\psi B_{cx})_{y=0}\, dx. \tag{1.33}
$$

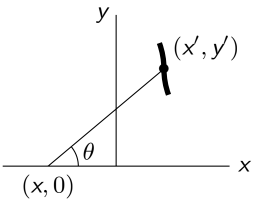
实例 XVIII

3. 磁自由能的计算. 由 (1.33) 出发, 已知 $\boldsymbol{B}$, 计算 J_c (I_c 和 $B_{cx}|_{y=0}$), 及 ψ_c 和 $\psi|_{y=0}$, 最后计算 W_s. 在中性电流片上, 由

$$\boldsymbol{j} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B}. \tag{1.34}$$

可计算得到体电流密度 J_c.
再将 J_c 沿电流片积分求得 I_c,

$$I_c = \int J_c dl'. \tag{1.35}$$



由 Biot-Savart 定律可求得 x 轴上的 B_{cx},

$$B_{cx}(x,0) = \frac{\mu_0}{2\pi} \int \frac{J_c y' dl'}{(x'-x)^2 + y'^2}, \tag{1.36}$$

对于 $\psi|_{y=0}$, 可通过 $\psi = \mathrm{Im}\, w(z)$ 来获得.

实例 XIX

7. 垂直电流片势场的磁自由能

$$J(y) = -\frac{8Da^2(p^2 - y^2)^{1/2}(y^2 - q^2)^{1/2}}{\mu_0(p^2 + a^2)^{1/2}(q^2 + a^2)^{1/2}(y^2 + a^2)} \quad (q \le y \le p)$$

$$I_c = -\frac{8D}{\mu_0 a(P^2 + Q^2 + 2)^{1/2}} \int_Q^P \frac{(P^2 - Y^2)^{1/2}(Y^2 - Q^2)^{1/2}}{(1 + Y^2)^2} dY.$$

这里 $P = p/a$, $Q = q/a$, $Y = y/a$, $X = x/a$. 而

$$\begin{aligned} B_{cx}(a, 0) &= -\frac{\frac{PQ}{4D}}{\pi a^2(P^2 + Q^2 + 2)^{1/2}} \\ &\quad \times \int_Q^P \frac{(P^2 - Y^2)^{1/2}(Y^2 - Q^2)^{1/2}}{(1 + Y^2)^3} dY, \\ &= -\frac{D}{4a^2}\left[\frac{a^2 - q^2}{a^2 + q^2}\right]^2. \end{aligned}$$

实例 XX

$PQ=1$. 上述积分用到分部积分及如下不定积分公式

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx+2c}{x\sqrt{b^2-4ac}}\right), \quad (c<0, b^2-4ac>0),$$

$$\int \frac{dx}{x^2\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{1}{2c}\int \frac{dx}{x\sqrt{ax^2+bx+c}}.$$

可得磁自由能

$$W_s = \frac{D}{2a_0}I_c + \frac{\pi D^2}{2a^2\mu_0}\left[\frac{a^2-q^2}{a^2+q^2}\right]^2.$$

实例 XXI

8. 横向电流片势场的磁自由能

$$J(\theta) = \frac{2D}{\mu_0 a^2 \cos\phi} \frac{(\sin^2\phi - \sin^2\theta)^{1/2}}{\cos^2\theta}.$$

θ 为考察点与 y 轴的夹角，$|\theta| \le \phi$.

$$I_c = \frac{4D}{\mu_0 a \cos\phi} \int_0^{\phi} \frac{(\sin^2\phi - \sin^2\theta)^{1/2}}{\cos^2\theta} d\theta,$$

$$B_{cx}(a,0) = -\frac{D}{4a^2} \tan^2\phi,$$

$$W_s = \frac{D}{2a_0} I_c - \frac{\pi D^2}{\mu_0 a^2} \tan^2\phi.$$

三维空间中的保角变换 I

保角变换又称为逆矢径变换 (梁昆淼, 1978). 应用见Low (1986).

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=0. \tag{1.37}$$

若 $\Phi(r,\theta,\varphi)$ 为解, 则取某个半径为 R 的球, 作逆矢径变换 (Kelvin 变换)

$$r_1=R^2/r. \tag{1.38}$$

可证

$$\Phi_1(r_1,\theta,\varphi)=r\Phi(r,\theta,\varphi)=\frac{R^2}{r_1}\Phi\left(\frac{R^2}{r_1},\theta,\varphi\right) \tag{1.39}$$

为 (r_1,θ,φ) 空间的调和函数. 即 Φ_1 满足

$$\frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left[r_1^2\frac{\partial \Phi_1}{\partial r_1}\right]+\frac{1}{r_1^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi_1}{\partial\theta}\right)+\frac{1}{r_1^2\sin^2\theta}\frac{\partial^2\Phi_1}{\partial\varphi^2}=0. \quad \square$$

三维空间中的保角变换 II

对轴对称问题，引入磁通量函数 A

$$\boldsymbol{B} = \nabla \times \left(\frac{A}{r\sin\theta} \boldsymbol{e}_\varphi \right) = \frac{1}{r\sin\theta} \nabla A \times \boldsymbol{e}_\varphi = \frac{1}{r\sin\theta} \left(\frac{1}{r} \frac{\partial A}{\partial \theta}, -\frac{\partial A}{\partial r}, 0 \right).$$

若 $\boldsymbol{B}$ 为势场，则 $\nabla \times \boldsymbol{B} = 0$.

$$\frac{\partial^2 A}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial A}{\partial \theta} \right) = 0. \tag{1.40}$$

同样可运用逆矢径变换

$$A_1(r_1, \theta) = \frac{1}{r} A(r, \theta) = \frac{r_1}{R^2} A\left(\frac{R^2}{r_1}, \theta \right), \tag{1.41}$$

可证 A_1 满足方程

$$\frac{\partial^2 A_1}{\partial r_1^2} + \frac{\sin\theta}{r_1^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial A_1}{\partial \theta} \right) = 0.$$

即 A_1 为逆矢径变换后的新坐标系下的磁通函数.

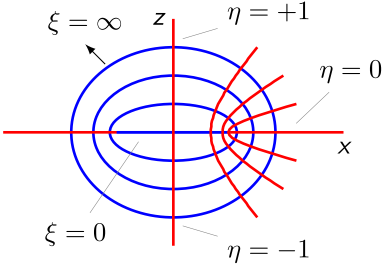
旋转椭球坐标系和势场解 I

按下式定义旋转椭球坐标系 (η, ξ, φ),

$$\begin{cases} x = a(1+\xi^2)^{1/2}(1-\eta^2)^{1/2}\cos\varphi, \\ y = a(1+\xi^2)^{1/2}(1-\eta^2)^{1/2}\sin\varphi, \\ z = a\xi\eta. \end{cases} \tag{1.42}$$

对应的三个坐标曲面方程为

$$\begin{cases} \frac{x^2+y^2}{a^2(1+\xi^2)} + \frac{z^2}{a^2\xi^2} = 1, \\ \quad \text{旋转椭球}, \\ \frac{x^2+y^2}{a^2(1-\eta^2)} - \frac{z^2}{a^2\eta^2} = 1, \\ \quad \text{旋转单叶双曲面}, \\ \frac{y}{x} = \tan\varphi. \end{cases} \tag{1.43}$$



$(\xi, \eta) = (0, 0)$ 为焦点, $0 \le \xi < \infty$, $-1 \le \eta \le 1$, $0 \le \varphi < 2\pi$. 由 (1.43) 可解出

$$\xi^2 = \frac{1}{2}\left[\left(1-\frac{r^2}{a^2}\right)^2 + \frac{4r^2}{a^2}\cos^2\theta\right]^{1/2} - \frac{1}{2}\left(1-\frac{r^2}{a^2}\right) \tag{1.44}$$

旋转椭球坐标系和势场解 II

$$\eta^2 = \frac{1}{2}\left[\left(1-\frac{r^2}{a^2}\right)^2 + \frac{4r^2}{a^2}\cos^2\theta\right]^{1/2} + \frac{1}{2}\left(1-\frac{r^2}{a^2}\right) \tag{1.45}$$

曲线坐标系 (u_1, u_2, u_3) 中的 Lame 系数为

$$h_i = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}$$

由 (1.42) 求得

$$\begin{cases} h_\xi = \frac{a(\xi^2+\eta^2)^{1/2}}{(1+\xi^2)^{1/2}}, \\ h_\eta = \frac{a(\xi^2+\eta^2)^{1/2}}{(1-\eta^2)^{1/2}}, \\ h_\varphi = a(1+\xi^2)^{1/2}(1-\eta^2)^{1/2}. \end{cases} \tag{1.46}$$

旋转椭球坐标系和势场解 III

Lapace 方程则为

$$\frac{\partial}{\partial \xi}\left[(1+\xi^2)\frac{\partial \Phi}{\partial \xi}\right]+\frac{\partial}{\partial \eta}\left[(1-\eta^2)\frac{\partial \Phi}{\partial \eta}\right]+\frac{\xi^2+\eta^2}{(1+\xi^2)(1-\eta^2)}\frac{\partial^2 \Phi}{\partial \varphi^2}=0.$$

其中 $\frac{\xi^2+\eta^2}{(1+\xi^2)(1-\eta^2)}=\frac{1}{1-\eta^2}-\frac{1}{1+\xi^2}$. 轴对称的情况下，$\frac{\partial}{\partial \varphi}=0$,

$$\frac{\partial}{\partial \xi}\left[(1+\xi^2)\frac{\partial \Phi}{\partial \xi}\right]+\frac{\partial}{\partial \eta}\left[(1-\eta^2)\frac{\partial \Phi}{\partial \eta}\right]=0. \tag{1.47}$$

磁通函数 A 有

$$\boldsymbol{B}=\nabla\times\left(\frac{A}{h_\varphi}\boldsymbol{e}_\varphi\right)=\nabla A\times\frac{1}{h_\varphi}\boldsymbol{e}_\varphi,$$

由 $\nabla\times\boldsymbol{B}=0$,

$$\frac{\partial}{\partial \xi}\left(\frac{h_\eta}{h_\xi h_\varphi}\frac{\partial A}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\frac{h_\xi}{h_\eta h_\varphi}\frac{\partial A}{\partial \eta}\right)=0,$$

旋转椭球坐标系和势场解 IV

得

$$(1+\xi^2)\frac{\partial^2 A}{\partial \xi^2}+(1-\eta^2)\frac{\partial^2 A}{\partial \eta^2}=0. \tag{1.48}$$

式 (1.47) 和 (1.48) 的分离变量解为

$$\Phi_n = P_n(\eta)Q_n(i\xi), \tag{1.49}$$

$$A_n = (1+\xi^2)^{1/2}(1-\eta^2)^{1/2}P_n'(\eta)Q_n'(i\xi). \tag{1.50}$$

其中 P_n 和 Q_n 分别为第 1 和第 2 类 n 阶 Legendre 多项式；$P_n'(\eta)$ 和 $Q_n'(i\xi)$ 定义为，

$$\begin{cases} P_n'(\eta) = (1-\eta^2)^{1/2}\frac{d}{d\eta}P_n(\eta), \\ Q_n'(i\xi) = (1+\xi^2)^{1/2}\frac{d}{d\xi}Q_n(i\xi), \end{cases} \tag{1.51}$$

为 n 阶 1 次伴随 Legendre 多项式. $Q_n(\eta)$ 在 $\eta=\pm 1$(极轴) 处奇异, 故舍去. 而 $P_n(i\xi)$ 在 $\xi=\pm\infty$ 处奇异, 也舍去. 同理舍去 $Q_n'(\eta)$ 和 $P_n'(i\xi)$.

旋转椭球坐标系和势场解 V

Legendre 多项式的一些性质如下

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dP_n(\eta)}{d\eta}\right]+n(n+1)P_n(\eta)=0,$$
$$\frac{d}{d\xi}\left[(1+\xi^2)\frac{dQ_n(i\xi)}{d\xi}\right]-n(n+1)Q_n(i\xi)=0,$$
$$(1-\eta^2)\frac{d^2}{d\eta^2}\left[(1-\eta^2)^{1/2}P_n'(\eta)\right]+n(n+1)(1-\eta^2)^{1/2}P_n'(\eta)=0,$$
$$(1+\xi^2)\frac{d^2}{d\xi^2}\left[(1+\xi^2)^{1/2}Q_n'(i\xi)\right]-n(n+1)(1+\xi^2)^{1/2}Q_n'(i\xi)=0,$$

而

$$P_0(\eta)=1,\quad P_1(\eta)=\eta,\quad P_2(\eta)=\frac{1}{2}(3\eta^2-1),\quad \cdots,$$
$$P_1'(\eta)=(1-\eta^2)^{1/2},\quad P_2'(\eta)=3\eta(1-\eta^2)^{1/2},$$

旋转椭球坐标系和势场解 VI

$$P_3'(\eta) = \frac{3}{2}(1-\eta^2)^{1/2}(5\eta^2-1), \quad \cdots .$$

和

$$Q_n(i\xi) = \frac{i^{n-1}}{2^{n-1}n!}\frac{d^n}{d\xi^n}\left[(1+\xi^2)^n \tan^{-1}\frac{1}{\xi}\right] + iP_n(i\xi)\tan^{-1}\frac{1}{\xi},$$

$$Q_1(i\xi) = \xi\tan^{-1}\frac{1}{\xi} - 1,$$

$$Q_2(i\xi) = \frac{i}{2}\left[(1+3\xi^2)\tan^{-1}\frac{1}{\xi} - 3\xi\right],$$

$$Q_3(i\xi) = -\frac{1}{2}(5\xi^2+3)\xi\tan^{-1}\frac{1}{\xi} - \frac{1}{6}(15\xi^2+4),$$

$$\cdots .$$

及

$$(1+\xi^2)^{1/2}Q_1'(i\xi) = (1+\xi^2)\tan^{-1}\frac{1}{\xi} - \xi,$$

旋转椭球坐标系和势场解 VII

$$(1+\xi^2)^{1/2}Q_2'(i\xi) = i\left[3\xi(1+\xi^2)\tan^{-1} - 2 - 3\xi^2\right],$$

$$(1+\xi^2)^{1/2}Q_3'(i\xi) = \frac{1}{2}\left[-3(1+5\xi^2)(1+\xi^2)\tan^{-1}\frac{1}{\xi} + \frac{15}{2}\xi^3 + \frac{13}{2}\xi\right],$$

$$\cdots$$

$A = \xi, \eta$ 也是势场解，未计入 (1.50) 中，则有

$$A = \xi, \quad B_\xi = 0, \quad B_\eta = -\frac{1}{a^2(\xi^2+\eta^2)^{1/2}(1-\eta^2)^{1/2}},$$

$$A = \eta, \quad B_\eta = 0, \quad B_\xi = \frac{1}{a^2(\xi^2+\eta^2)^{1/2}(1+\xi^2)^{1/2}}.$$

旋转椭球坐标系和势场解的奇异性 I

1. 坐标系本身的奇异性
   1. 圆盘内部, $x^2 + y^2 < a^2$

$$\eta_1 = -\eta_2, \quad \xi_1 = \xi_2,$$
$$\boldsymbol{e}_{\xi_1} = -\boldsymbol{e}_{\xi_2}, \quad \boldsymbol{e}_{\eta_1} = -\boldsymbol{e}_{\eta_2}$$

$$\begin{array}{ccc} & \uparrow \boldsymbol{e}_{\xi_1} & \\ \boldsymbol{e}_{\eta_1} & & \boldsymbol{e}_{\eta_1} \\ \rightarrow & & \leftarrow \\ \rightarrow & & \leftarrow \\ \boldsymbol{e}_{\eta_2} & & \boldsymbol{e}_{\eta_2} \\ & \downarrow \boldsymbol{e}_{\xi_2} & \end{array}$$

   2. 圆盘边缘

$$z = 0, \quad x^2 + y^2 = a^2, \quad \xi = \eta = 0,$$
$$h_\eta, h_\xi \propto (\xi^2 + \eta^2)^{1/2} \to 0, \quad h_\varphi \to a,$$

旋转椭球坐标系和势场解的奇异性 II

2. 势场解的奇异性. 磁场的形式推导可得:

$$\begin{cases} B_\eta = -\frac{P_n'(\eta)}{a^2(\xi^2+\eta^2)^{1/2}}\frac{d}{d\xi}\left[(1+\xi^2)^{1/2}Q_n'(i\xi)\right], \\ B_\xi = \frac{Q_n'(i\xi)}{a^2(\xi^2+\eta^2)^{1/2}}\frac{d}{d\eta}\left[(1-\eta^2)^{1/2}P_n'(\eta)\right]. \end{cases} \tag{1.52}$$

对于圆盘边缘处, 即 $\xi,\eta \to 0$ 时, 结合 Legendre 多项式的性质, 可知 B_η 和 B_ξ 的分子项不为 0, 因此在圆盘边缘附近有 $B \to \infty$.
对于圆盘面处, 圆盘上下两侧 η 反号, 由 (1.52), 可得

- n 为奇数时, $P_n'(\eta)$ 连续, B_η 连续, B_ξ 跳变. $B_\eta \mathbf{e}_\eta$ 跳变, $B_\xi \mathbf{e}_\xi$ 连续, 对应于电流片奇异性.
- n 为偶数时, $P_n'(\eta)$ 跳变, B_η 跳变, B_ξ 连续. $B_\eta \mathbf{e}_\eta$ 连续, $B_\xi \mathbf{e}_\xi$ 跳变, 对应于磁荷片奇异性.

带赤道中性电流片的势场解 I

1. 球坐标系下的势场解 (磁通函数). 满足 (1.40) 在无穷远正则的解

$$W_k = \frac{\sin\theta}{kr^k} P_k'(\cos\theta) = \frac{\sin^2\theta}{kr^k}\left[\frac{dP_k(\mu)}{d\mu}\right]_{\mu=\cos\theta}. \tag{1.53}$$

2. 椭球坐标系下带电流片奇异性的势场解. 由 (1.50) 出发, 取 n 为奇数的 A_n.
3. 再将 A_n 进行反演变换. $r \to r_1 = a^2/r$

$$\begin{cases} \xi^2 \to u^2 = \frac{1}{2}\left[\left(1-\frac{a^2}{r_1^2}\right)^2 + \frac{4a^2}{r_1^2}\cos^2\theta\right]^{1/2} - \frac{1}{2}\left(1-\frac{a^2}{r_1^2}\right), \\ \eta^2 \to v^2 = \frac{1}{2}\left[\left(1-\frac{a^2}{r_1^2}\right)^2 + \frac{4a^2}{r_1^2}\cos^2\theta\right]^{1/2} + \frac{1}{2}\left(1-\frac{a^2}{r_1^2}\right). \end{cases} \tag{1.54}$$

新坐标系中的磁通函数 S_n 的形式为

$$S_n = r_1(1+u^2)^{1/2}(1-v^2)^{1/2} Q_n'(iu) P_n'(v). \tag{1.55}$$

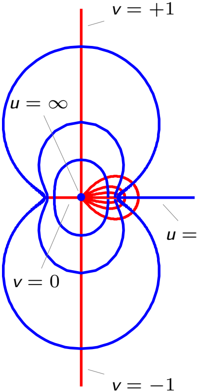
带赤道中性电流片的势场解 II

坐标的奇异性位于

$$
\begin{aligned}
z_1 &= 0, \quad x_1^2 + y_1^2 \geq a^2, \\
u|_1 &= u|_2, \quad v|_1 = -v|_2,
\end{aligned}
$$

坐标变换

$$
\begin{cases}
x_1 = \dfrac{a(1+u^2)^{1/2}(1-v^2)^{1/2}}{u^2-v^2+1}\cos\varphi, \\
y_1 = \dfrac{a(1+u^2)^{1/2}(1-v^2)^{1/2}}{u^2-v^2+1}\sin\varphi, \\
z_1 = \dfrac{auv}{u^2-v^2+1}, \\
r_1 = (x_1^2+y_1^2+z_1^2)^{1/2} = \dfrac{a}{\sqrt{u^2-v^2+1}}.
\end{cases}
\tag{1.56}
$$



Lame 系数

$$
\begin{cases}
h_u = \dfrac{a(u^2+v^2)^{1/2}}{(u^2-v^2+1)(1+u^2)^{1/2}}, \\
h_v = \dfrac{a(u^2+v^2)^{1/2}}{(u^2-v^2+1)(1-v^2)^{1/2}}, \\
h_\varphi = \dfrac{a(1+u^2)^{1/2}(1-v^2)^{1/2}}{(u^2-v^2+1)} = r_1\sin\theta.
\end{cases}
\tag{1.57}
$$

带赤道中性电流片的势场解 III

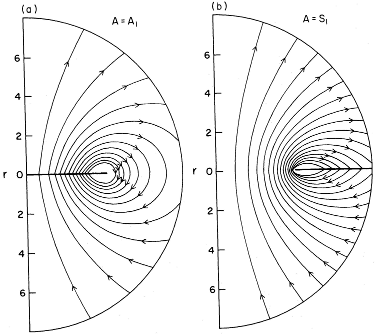


图 1.4: (1.55) 式的球反演.

带赤道中性电流片的势场解 IV

4. 对 S_n 进行修正. 简化起见, 以下推导中, 新坐标系中的 r_1 仍写为 r.
   1. 在赤道电流片上 ($\theta = \pi/2$, $r > a$ ($u = 0$, $v^2 = 1 - a^2/r^2$), $B_\theta|_{u=0} \neq 0$, 出现法向分量, 须设法消除.

$$B_u\big|_{\substack{u=0,\\ v=\pm(1-a^2/r^2)^{1/2}}} = \left[\frac{1}{h_v h_\varphi}\frac{\partial S_n}{\partial v}\right]_{\substack{u=0,\\ v=\pm(1-a^2/r^2)^{1/2}}}.$$

而

$$\frac{1}{h_v h_\varphi} = \frac{a^2}{r^4|v|},$$
$$\frac{\partial r}{\partial v} = \frac{r^3 v}{a^2},$$
$$\frac{d}{dv}\left[(1-v^2)^{1/2}P_n'(v)\right] = -n(n+1)P_n(v),$$

故

$$B_u\big|_{\substack{u=0,\\ v=\pm(1-a^2/r^2)^{1/2}}} = \frac{v}{|v|}\frac{a^2}{r^3}\left(\frac{dQ_n(iu)}{du}\right)_{u=0}$$

带赤道中性电流片的势场解 V

$$\left[\frac{dP_n(v)}{dv} - \frac{n(n+1)}{v} P_n(v)\right]_{v=\pm(1-a^2/r^2)^{1/2}}. \tag{1.58}$$

中括弧的项为 v^2 的 $\frac{n-1}{2}$ 次多项式，即 $\frac{1}{r^2}$ 的 $\frac{n-1}{2}$ 次多项式. 考察球坐标下的势场解 (1.53)，由它求得

$$B_{\theta k} = -\frac{1}{r\sin\theta}\frac{\partial W_k}{\partial r} = \frac{(1-u^2)^{1/2}}{r^{k+2}}\frac{dP_k(\mu)}{d\mu},$$

$$B_{rk} = \frac{1}{r^2\sin\theta}\frac{\partial W_k}{\partial \theta} = \frac{k+1}{r^{k+2}} P_k(\mu).$$

$\mu = \cos\theta$，k 为奇数，在赤道上，P_k 为零，即 $B_r = 0$，而

$$B_{\theta k} = \frac{1}{r^{k+2}}\left[\frac{dP_k(\mu)}{d\mu}\right]_{\mu=0} = \frac{1}{r^3}\left[\frac{dP_k(\mu)}{d\mu}\right]_{\mu=0}\left(\frac{1}{r^2}\right)^{\frac{k-1}{2}}.$$

$k = 1, 3, 5, \cdots, n$，其系数正好与 (1.58) 相应 $\frac{1}{r^2}$ 的幂次之系数对应，取

$$T_n = \sum_{k=1,k\text{ 为奇数}}^{n} \alpha_k^n W_k, \tag{1.59}$$

带赤道中性电流片的势场解 VI

在电流片上，需要 T_n 的法向分量正好与 S_n 的法向场大小相等，符号相反，则

$$S_n + T_n \quad n \text{ 为奇数} \tag{1.60}$$

在电流片上法向场为零，系数 α_k^n 由下式决定

$$\begin{aligned}\sum_{k=1}^{n} \alpha_k^n \left[\frac{dP_k(\mu)}{d\mu}\right]_{\mu=0} \left(\frac{1}{r^2}\right)^{\frac{k-1}{2}} \\ = a^2 \left[\frac{dQ_n(iu)}{du}\right]_{u=0} \left[\frac{dP_n(v)}{dv} - \frac{n(n+1)}{v} P_n(v)\right].\end{aligned} \tag{1.61}$$

例 $n = 1$,

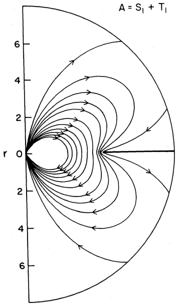
$$\left[\frac{dP_1(\mu)}{d\mu}\right]_{\mu=0} = 1,$$

$$\left[\frac{dQ_1(iu)}{du}\right]_{u=0} = \left[\tan^{-1}\frac{1}{u} - \frac{u}{1+u^2}\right]_{u=0} = \pi/2,$$

$$\left[\frac{dP_1(v)}{dv} - \frac{2P_1(v)}{v}\right]_{v=\pm\sqrt{1-a^2/r^2}} = -1,$$

带赤道中性电流片的势场解 VII

$$
\begin{aligned}
\alpha_1^1 &= -\frac{\pi a^2}{2}, \\
T_1 &= -\frac{\pi a^2(1-\mu^2)}{2r} = -\frac{\pi a^2 \sin^2\theta}{2r}, \\
S_1 &= r(1+u^2)^{1/2}(1-v^2)^{1/2} P_1'(v) Q_1'(iu) \\
&= r(1-v^2)\left[(1+u^2)\tan^{-1}\frac{1}{u} - u\right],
\end{aligned}
$$



最后

$$
\begin{aligned}
S_1 + T_1 = & r(1-v^2)\left[(1+u^2)\tan^{-1}\frac{1}{u} - u\right] \\
& -\frac{\pi a^2 \sin^2\theta}{2r}.
\end{aligned} \tag{1.62}
$$

带赤道中性电流片的势场解 VIII

2. 电流片内沿奇异性.

$$B_v = -\frac{1}{h_u h_\varphi}\frac{\partial S_n}{\partial u},$$

$$\frac{\partial S_n}{\partial u} = (1-v^2)\frac{dP_n(v)}{dv}\left\{\frac{\partial r}{\partial u}(1+u^2)\frac{dQ_n(iu)}{du} + r\frac{d}{du}\left[(1+u^2)\frac{dQ_n(iu)}{u}\right]\right\},$$

$$\frac{\partial r}{\partial u} = -\frac{au}{(u^2-v^2+1)^{3/2}},$$

$$\frac{d}{du}\left[(1+u^2)\frac{dQ_n(iu)}{u}\right] = n(n+1)Q_n(iu),$$

$$\left.\frac{1}{h_u h_\varphi}\right|_{u=0} = \frac{a}{r^3|v|} \quad (v = \pm\sqrt{1-a^2/r^2}),$$

因此

$$B_v|_{u=0,v=\pm\sqrt{1-a^2/r^2}} = -\frac{a}{r^4|v|}n(n+1)\frac{dP_n(v)}{dv}\left[Q_n(iu)\right]_{u=0}.$$

带赤道中性电流片的势场解 IX

可见，在赤道电流片内沿处，$B_v \propto \left.\frac{1}{v}\right|_{v=0} \to \infty$，存在奇异性．($B_u$ 不存在奇异性) B_v 跨过电流片连续，但 $\boldsymbol{e}_v$ 反向，

$$\left.B_r\right|_{u=0} = \mp \frac{a}{r^4|v|} n(n+1) \frac{dP_n(v)}{dv} \left. Q_n(iu)\right|_{u=0}. \tag{1.63}$$

$\mp$ 表示 $\theta = \pi/2 \mp 0$．为消除上述奇异性，考虑如下势场

$$A_p = \eta,$$

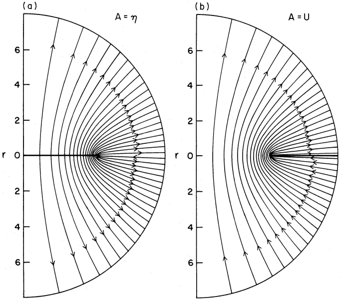
改成

$$U = |\eta| = \begin{cases} \eta, & \theta < \pi/2 - 0, \\ -\eta, & \theta > \pi/2 + 0. \end{cases}$$

与 U 对应的场在 $\eta = 0$ 处的奇异性

$$B_\xi = -\frac{1}{h_\eta h_\varphi} \frac{\partial U}{\partial \eta} = \mp \frac{1}{h_\eta h_\varphi}$$

$$= \frac{1}{a(\xi^2+\eta^2)^{1/2}(1+\xi^2)^{1/2}}$$



带赤道中性电流片的势场解 X

$$\frac{1}{\xi} = \frac{a}{\sqrt{r^2 - a^2}},$$
$$\frac{1}{|v|} = \frac{r}{\sqrt{r^2 - a^2}}.$$

那么

$$B_\xi = B_r = \mp - \frac{1}{a^2 \xi} \quad (\theta = \pi/2 \mp 0).$$

作 $\beta_n U$, 使

$$\beta_n / a^2 = - \left[\frac{a}{r^4}\right]_{r=a} n(n+1) \left[\frac{dP_n(v)}{dv}\right]_{v=0} Q_n(iu)|_{u=0}$$

或

$$\beta_n = \frac{n(n+1)}{a} \left[\frac{dP_n(v)}{dv}\right]_{v=0} Q_n(iu)|_{u=0}\,, \tag{1.64}$$

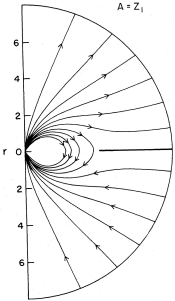
带赤道中性电流片的势场解 XI

则

$$Z_n = S_n + T_n + \beta_n U. \tag{1.65}$$

在 $r = a$, $\theta = \pi/2$ 处不存在奇异性，且该点处 $\boldsymbol{B} = 0$. 对 $n = 1$, $n(n+1) = 2$, $\beta_1 = 2a$,

$$\begin{aligned} Z_1 =& r(1 - v^2)\left[(1 + u^2)\tan^{-1}\frac{1}{a} - u\right] \\ & - \frac{\pi a^2 \sin^2\theta}{2r} + 2aU. \end{aligned} \tag{1.66}$$



对 $n = 3$, 可导出

$$\begin{aligned} Z_3 =& \frac{3}{4} r(1 - v^2)(5v^2 - 1) \\ & \left[-3(1 + u^2)(5u^2 + 1)\tan^{-1}\frac{1}{u} \right. \\ & \left. + 15u^3 + 13u \right] \end{aligned}$$

带赤道中性电流片的势场解 XII

$$+\frac{45\pi a^4\sin^2\theta(5\cos^3\theta-1)}{8r^3}+\frac{9\pi a^2\sin^2\theta}{2r}+12aU. \tag{1.67}$$

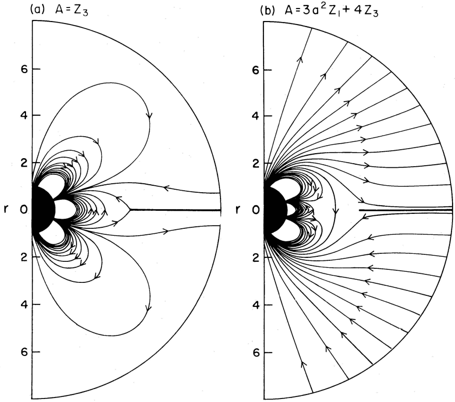


图 1.5: Z_3 和 $3a^2Z_1+4Z_3$ $(a=4.0)$.

一维非线性无力场 I

1. 可忽略坐标 (ignorable coordinate) – 全部 Lame 系数与之无关的坐标. 一维问题限于至少具有两个可忽略坐标的坐标系. 直角坐标 (3 个), 圆柱坐标 (2 个).
2. 直角坐标系下的一维无力场. 设与 x 有关

$$\nabla \cdot \boldsymbol{B} = 0, \quad B_x = \text{const.}.$$

非平凡解 $B_x = 0$ ($\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \alpha B_x = 0$, 而 $\alpha \neq 0$).

$$\frac{dB_y}{dx} = \alpha B_z,$$
$$\frac{dB_z}{dx} = -\alpha B_y,$$

或

$$B_y \frac{dB_y}{dx} + B_z \frac{dB_z}{dx} = 0.$$

一维非线性无力场 II

得

$$B^2 = B_0^2 = \text{const.}. \tag{1.68}$$

考虑到

$$\sin^2\Phi(x) + \cos^2\Phi(x) = 1,$$
$$\cosh^2\Phi(x) - \sinh^2\Phi(x) = 1,$$

有

$$\boldsymbol{B} = \boldsymbol{B}_0(t)\left(0, \sin\Phi(x), \cos\Phi(x)\right), \tag{1.69}$$

$$\alpha = \frac{d\Phi}{dx}, \tag{1.70}$$

或者

$$\boldsymbol{B} = \boldsymbol{B}_0(t)\left(0, \tanh\Psi(x), 1/\cosh\Psi(x)\right), \tag{1.71}$$

一维非线性无力场 III

$$\alpha = \frac{1}{\cosh\Psi}\frac{d\Psi}{dx}. \tag{1.72}$$

这里 $\tan\Phi = \sinh\Psi$.

3. 圆柱坐标系 (r, θ, z) ($h_r = 1,\ h_\theta = r,\ h_z = 1$) 下的一维无力场 (Lüst et al., 1954).

$$\begin{aligned}
&\boldsymbol{B} = \boldsymbol{B}(r), \\
&\nabla\cdot\boldsymbol{B} = 0, \\
&rB_r = \text{const.}, \\
&\frac{1}{r}\left(\frac{\partial B_z}{\partial\theta} - r\frac{\partial B_\theta}{\partial z}\right) = \alpha B_r = 0.
\end{aligned}$$

非平凡解 $B_r = 0$,

$$\frac{d}{dr}\left(\frac{B_\theta^2 + B_z^2}{2}\right) + \frac{B_\theta^2}{r} = 0.$$

一维非线性无力场 IV

令 $F(r) = (B_\theta^2 + B_z^2)/2$, 则

$$B_\theta^2 = -r\frac{dF}{dr}, \tag{1.73}$$

$$B_z^2 = 2F + r\frac{dF}{dr}. \tag{1.74}$$

$$\alpha = \frac{1}{rB_z}\frac{d(rB_\theta)}{dr} = -\frac{3\frac{dF}{dr} + r\frac{d^2F}{dr^2}}{\left[-4r\frac{dF}{dr}\left(2F + r\frac{dF}{dr}\right)\right]^{1/2}}. \tag{1.75}$$

对 F 的要求

$$\begin{cases} \text{总压力为正}, & F(r) > 0, \\ B_\theta^2 > 0, & \frac{dF}{dr} < 0, \\ B_z^2 > 0, & \frac{d(r^2F)}{dr} > 0. \end{cases} \tag{1.76}$$

一维非线性无力场 V

例

1. $F(r) = \frac{a^2+R^2}{a^2+r^2}$,

$$\begin{cases} B_\theta = \frac{\sqrt{2(a^2+R^2)}}{a^2+r^2} r, \\ B_z = \frac{\sqrt{2(a^2+R^2)}}{a^2+r^2} a, \\ \alpha = \frac{2a}{a^2+r^2}. \end{cases}$$

2. $F(r) = \frac{1}{2}a^2 + \frac{1}{4}e^{-2r^2}$ $(a^2 > \frac{1}{2e^2})$.
3. $F(r) = \frac{1}{2(2\nu-1)(1+r^2)^{2\nu-1}}$ $(\frac{1}{2} \le \nu \le 1)$.
4. $F(r) = \frac{1}{2} + \sigma e^{-r}(1+r)$.
5. $F(r) = \frac{1}{2} + re^{-r}$.
6. $F(r) = r_0^2 - \frac{1}{2}r^2$ (Low, 1993a).
7. $F(r) = \frac{1}{2}[J_0^2(\alpha r) + J_1^2(\alpha r)]$ (线性无力场) (Hood et al., 1980; Cargill et al., 1986).

二维非线性无力场 I

限于至少具有一个可忽略坐标的坐标系 (x_1, x_2, x_3)，设 x_3 为可忽略坐标，$\boldsymbol{B} = \boldsymbol{B}(x_1, x_2)$，令

$$\boldsymbol{B} = \nabla A \times \frac{1}{h_3}\boldsymbol{e}_3 + G\frac{1}{h_3}\boldsymbol{e}_3, \tag{1.77}$$

那么

$$\nabla \times \boldsymbol{B} = -h_3^2 \nabla \cdot \left(\frac{1}{h_3^2}\nabla A\right)\frac{1}{h_3}\boldsymbol{e}_3 + \nabla G \times \frac{1}{h_3}\boldsymbol{e}_3.$$

定义

$$\begin{aligned}\mathcal{L} &= h_3^2 \nabla \cdot \left(\frac{1}{h_3^2}\nabla\right) \\ &= \frac{h_3}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1 h_3}\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial x_2}\right)\right],\end{aligned} \tag{1.78}$$

二维非线性无力场 II

则

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\nabla A \nabla \cdot \left(\frac{1}{h_3^2} \nabla A \right) - \frac{1}{h_3^2} G \nabla G + \frac{1}{h_3^2} \nabla G \times \nabla A = 0.$$

用 $\boldsymbol{e}_3$ 点乘上式，有 $\nabla A \times \nabla G = 0$，即 $\frac{\partial(A,G)}{\partial(x_1,x_2)} = 0$，或

$$G = G(A) \tag{1.79}$$

代回原式，

$$\mathcal{L}A + G\frac{dG}{dA} = 0, \tag{1.80}$$

即 Grad–Shafranov 方程.[2]

- 直角坐标 – $\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
- 圆柱坐标 – $\mathcal{L} = \frac{\partial^2}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)$ 或 $\mathcal{L} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.
- 球坐标 – $\mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$.

[2] 例 $G = c_1 + c_2 A$, 方程简化为 Helmholtz 方程; 或 $G = \lambda\sqrt{A}$.

三维无力场 I

引入正交曲线坐标 (x_1, x_2, x_3), 其中 x_1 为磁面坐标

$$\alpha = \alpha(x_1), \tag{1.81}$$
$$\boldsymbol{B} = (0, B_2, B_3). \tag{1.82}$$

设 Lame 系数为 (h_1, h_2, h_3),

$$\nabla \cdot \boldsymbol{B} = 0,$$
$$\frac{\partial}{\partial x_2}(h_3 h_1 B_2) + \frac{\partial}{\partial x_3}(h_1 h_2 B_3) = 0. \tag{1.83}$$
$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B},$$
$$\frac{\partial}{\partial x_2}(h_3 B_3) - \frac{\partial}{\partial x_3}(h_2 B_2) = 0, \tag{1.84}$$
$$-\frac{1}{h_1 h_3}\frac{\partial}{\partial x_1}(h_3 B_3) = \alpha B_2, \tag{1.85}$$
$$\frac{1}{h_1 h_2}\frac{\partial}{\partial x_1}(h_2 B_2) = \alpha B_3. \tag{1.86}$$

三维无力场 II

式 (1.83) 和 (1.84) 是关于 x_2 和 x_3 的偏微分方程, (1.85) 和 (1.86) 是关于 x_1 的偏微分方程. 为方便求解, 作如下代换

$$H_2 = \frac{h_1 h_3}{h_2}, \quad H_3 = \frac{h_1 h_2}{h_3}, \tag{1.87}$$

$$\beta_2 = h_2 B_2, \quad \beta_3 = h_3 B_3, \tag{1.88}$$

方程化为

$$\frac{\partial}{\partial x_2}(H_2\beta_2) + \frac{\partial}{\partial x_3}(H_3\beta_3) = 0, \tag{1.89}$$

$$\frac{\partial \beta_3}{\partial x_2} - \frac{\partial \beta_2}{\partial x_3} = 0, \tag{1.90}$$

$$-\frac{1}{H_2\alpha}\frac{\partial \beta_3}{\partial x_1} = \beta_2, \tag{1.91}$$

$$\frac{1}{H_3\alpha}\frac{\partial \beta_2}{\partial x_1} = \beta_3. \tag{1.92}$$

三维无力场 III

讨论特例 (Ko et al., 1996)

$$\begin{cases} h_1 = h_1(x_1), \\ \frac{\partial}{\partial x_1}\left(\frac{h_2}{h_3}\right) = 0. \end{cases} \tag{1.93}$$

满足 (1.93) 的坐标系只可能是直角坐标系和球坐标系，方程 (1.89)–(1.92) 化为

$$\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_2}\beta_2\right) + \frac{\partial}{\partial x_3}\left(\frac{h_2}{h_3}\beta_3\right) = 0, \tag{1.94}$$

$$\frac{\partial \beta_3}{\partial x_2} - \frac{\partial \beta_2}{\partial x_3} = 0, \tag{1.95}$$

$$-\frac{1}{h_1\alpha}\frac{\partial \beta_3}{\partial x_1} = \frac{h_3\beta_2}{h_2}, \tag{1.96}$$

$$\frac{1}{h_1\alpha}\frac{\partial \beta_2}{\partial x_1} = \frac{h_2\beta_3}{h_3}. \tag{1.97}$$

三维无力场 IV

令

$$h_1\alpha = \frac{dA}{dx_1} \qquad (A = A(x_1)), \tag{1.98}$$

将 (1.96) 和 (1.97) 求解

$$\beta_2 = b_1(x_2, x_3)\cos A + b_2(x_2, x_3)\sin A, \tag{1.99}$$

$$\beta_3 = \frac{h_3}{h_2}\left[b_2(x_2, x_3)\cos A - b_1(x_2, x_3)\sin A\right]. \tag{1.100}$$

将其代入 (1.94) 和 (1.95),

$$\cos A\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_2}b_1\right) + \sin A\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_2}b_2\right) + \cos A\frac{\partial b_2}{\partial x_3} - \sin A\frac{\partial b_1}{\partial x_3} = 0,$$

$$\cos A\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_2}b_2\right) - \sin A\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_2}b_1\right) - \cos A\frac{\partial b_1}{\partial x_3} - \sin A\frac{\partial b_2}{\partial x_3} = 0,$$

三维无力场 V

即

$$\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_2}b_1\right)+\frac{\partial b_2}{\partial x_3}=0, \tag{1.101}$$

$$\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_2}b_2\right)-\frac{\partial b_1}{\partial x_3}=0. \tag{1.102}$$

引入函数 $F(x_1,x_2)$,

$$b_2=\frac{h_2}{h_3}\frac{\partial F}{\partial x_3}, \quad b_1=\frac{\partial F}{\partial x_2}, \tag{1.103}$$

则

$$\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_2}\frac{\partial F}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_3}\frac{\partial F}{\partial x_3}\right)=0. \tag{1.104}$$

三维无力场 VI

1. 球坐标

$$
\begin{aligned}
x_1 &= r, \quad x_2 = \theta, \quad x_3 = \varphi, \\
h_1 &= 1, \quad h_2 = r, \quad h_3 = r\sin\theta, \\
h_2/h_3 &= 1/\sin\theta.
\end{aligned}
$$

(1.104) 式化为

$$
\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{\partial^2 F}{\partial\varphi^2}=0. \tag{1.105}
$$

令

$$
\eta = -\ln\left(\tan\frac{\theta}{2}\right), \tag{1.106}
$$

三维无力场 VII

那么 $\sin\theta\frac{\partial}{\partial\theta}=-\frac{\partial}{\partial\eta}$, (1.105) 化为

$$\frac{\partial^2 F}{\partial\eta^2}+\frac{\partial^2 F}{\partial\varphi^2}=0. \tag{1.107}$$

解

$$\begin{aligned}F&=-a_0\eta+\sum_{n=1}^{\infty}\left(a_{1n}\cos n\varphi+a_{2n}\sin n\varphi\right)\left(a_{3n}\cot^n\frac{\theta}{2}+a_{4n}\tan^n\frac{\theta}{2}\right)\\&=-a_0\eta+\sum_{n=1}^{\infty}\left[a_{1n}\cos n\varphi+a_{2n}\sin n\varphi\right]\left(a_{3n}e^{n\eta}+a_{4n}e^{-n\eta}\right),\end{aligned}$$

和

$$b_1=\frac{\partial F}{\partial\theta}=\frac{1}{\sin\theta}$$

三维无力场 VIII

$$\left[a_0+\sum_{n=1}^{\infty} n\left(a_{1n}\cos n\varphi+a_{2n}\sin n\varphi\right)\left(-a_{3n}\cot^n\frac{\theta}{2}+a_{4n}\tan^n\frac{\theta}{2}\right)\right],$$

$$b_2=\frac{1}{\sin\theta}\frac{\partial F}{\partial\varphi}=\frac{1}{\sin\theta}$$

$$\sum_{n=1}^{\infty} n\left(-a_{1n}\sin n\varphi+a_{2n}\cos n\varphi\right)\left(a_{3n}\cot^n\frac{\theta}{2}+a_{4n}\tan^n\frac{\theta}{2}\right),$$

而

$$B_\theta=\frac{1}{r\sin\theta}\left\{\left[a_0+\sum_{n=1}^{\infty} n(a_{1n}\cos n\varphi+a_{2n}\sin n\varphi)\left(-a_{3n}\cot^n\frac{\theta}{2}+a_{4n}\tan^n\frac{\theta}{2}\right)\right]\right.$$
$$\cos A+\left[\sum_{n=1}^{\infty} n(-a_{1n}\sin n\varphi+a_{2n}\cos n\varphi)\left(a_{3n}\cot^n\frac{\theta}{2}+a_{4n}\tan^n\frac{\theta}{2}\right)\right]$$
$$\left.\sin A\right\}, \tag{1.108}$$

三维无力场 IX

$$B_\varphi = \frac{1}{r\sin\theta}\left\{\left[\sum_{n=1}^{\infty} n(-a_{1n}\sin n\varphi + a_{2n}\cos n\varphi)\left(a_{3n}\cot^n\frac{\theta}{2} + a_{4n}\tan^n\frac{\theta}{2}\right)\right]\right.$$
$$\cos A - \left[a_0 + \sum_{n=1}^{\infty} n(a_{1n}\cos n\varphi + a_{2n}\sin n\varphi)\left(-a_{3n}\cot^n\frac{\theta}{2} + a_{4n}\tan^n\frac{\theta}{2}\right)\right]$$
$$\left.\sin A\right\}. \tag{1.109}$$

2. 直角坐标

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad h_1 = h_2 = h_3 = 1.$$

那么

$$\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0. \tag{1.110}$$

$$B_y = \frac{\partial F}{\partial y}\cos A(x) + \frac{\partial F}{\partial z}\sin A(x), \tag{1.111}$$

三维无力场 X

$$B_z = \frac{\partial F}{\partial z} \cos A(x) - \frac{\partial F}{\partial y} \sin A(x), \tag{1.112}$$

$$\alpha = \frac{dA}{dx}. \tag{1.113}$$

F 可由复变数或 B_y, B_z 复变数法直接决定 (Low, 1988).

第一次作业 I

作业要求：下次上课 (11 月 22 日) 前，将作业电子版发送到助教邮箱 (wangyubao@mail.ustc.edu.cn).

1. 求位于底部 ($y=0$) $x=-b,\ -a,\ a,\ b$ ($a<b$) 处，线磁荷密度为 $-2\pi\lambda$, $2\pi\lambda$, $-2\pi\lambda$, $2\pi\lambda$ 的四根无限长平行线磁荷的磁场表达式 $B_x - iB_y = f(z)$, 以及相应的磁势函数和磁通量函数.
2. 对磁场

$$B_x - iB_y = \frac{2\lambda(b-a)(z-z_1)^{1/2}(z-z_1^*)^{1/2}(z-z_2)^{1/2}(z-z_2^*)^{1/2}}{\left[1-4ab\sin^2\varphi/(a+b)^2\right]^{1/2}(z^2-a^2)(z^2-b^2)}$$

式中

$$z_1 = y_N e^{i\theta_1}, \qquad z_2 = y_N e^{i\theta_2},$$
$$\theta_1 = \pi/2-\varphi, \qquad \theta_2 = \pi/2+\varphi, \qquad y_N^2 = ab$$

求磁中性点 z_1 处所有临界磁力线的斜率.

第一次作业 II

3. 取 $F(r) = re^{-r^2}$, 求圆柱坐标系 (r, θ, z) 中一维无力场的磁场和无力因子表达式.
4. 在圆柱坐标系 (r, θ, z) 下, 满足 $\partial/\partial z = 0$. 设如下局地无力场解

$$
\begin{aligned}
A =& \frac{\lambda^2 r_0^2}{8}\left(1 - \frac{r^2}{r_0^2}\right), \quad B_z = \lambda A^{1/2}, \quad (r \leq r_0) \\
A =& A_p(r, \theta), \quad B_z = 0, \quad (r > r_0),
\end{aligned}
$$

其中在 $(r \leq r_0)$ 的区域为无力场，在 $(r > r_0)$ 的区域为势场，在边界 $r = r_0$ 上满足 A 和 $\boldsymbol{B}$ 连续的条件. 试给出势场磁通量函数 A_p 满足的方程和边界条件, 并给出 A_p 的表达式.

基本方程

静平衡态的磁流体力学方程组为

$$-\nabla p+\frac{1}{\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}-\rho\nabla\psi=0, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{B}=0, \tag{2.2}$$

$$p=\rho RT, \tag{2.3}$$

$$\nabla\cdot(\mathbb{K}\cdot\nabla T)=-L+R. \tag{2.4}$$

$\mathbb{K}$ 为热传导张量, L 为加热率, R 为辐射损失.

1. 电阻问题

$$\frac{\partial\boldsymbol{B}}{\partial t}=\eta\nabla^2\boldsymbol{B}+\nabla\times(\boldsymbol{v}\times\boldsymbol{B}).$$

   $\eta\neq 0$ 时, 磁静平衡态很难达到. 当 η 很小时, 若满足 $\left|\frac{\partial\boldsymbol{B}}{\partial t}\right|\ll 1$, 准静态演化.

2. $\nabla\cdot\boldsymbol{B}$ 体现了对 $\boldsymbol{B}$ 的约束.

简化 I

1. 力平衡方程之简化[3]
   1. 略去重力, $\nabla\psi = 0$ (实验室等离子体).
   2. 冷等离子体近似, $\nabla p = 0$ (日珥平衡/重力和磁力平衡).
   3. 强磁场 (太阳活动区), $\beta = 2\mu p/B^2 \ll 1$, $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = 0$.
2. 能量方程之简化
   1. 过程方程近似

$$p = p(\rho).$$

   - 给定温度

$$p = RT\rho. \tag{2.5}$$

   特例为 $T = \text{const}$. (等温磁静平衡态).
   - 多方过程

$$p = c\rho^{\gamma}. \tag{2.6}$$

   $1 < \gamma \le 5/3$.
   2. 给定 $L(\rho, T)$ 和 $R(\rho, T)$ (经验关系).
   3. 从对 $\boldsymbol{B}$ 的附加约束或部分给定, 回避能量方程, 仅对其进行 “事后分析”.

[3]本章不作此简化.

对磁场的要求 I

引入 Euler 位 u 和 v,

$$\boldsymbol{B} = \nabla u \times \nabla v, \tag{2.7}$$

u 和 v 沿磁力线为常数. 取 u, v 和 ψ 为坐标,

$$\nabla p = \frac{\partial p}{\partial u}\nabla u + \frac{\partial p}{\partial v}\nabla v + \frac{\partial p}{\partial \psi}\nabla \psi,$$

代入 (2.1), 并利用

$$\begin{aligned}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} &= [\nabla \times (\nabla u \times \nabla v)] \times (\nabla u \times \nabla v) \\ &= \{[\nabla \times (\nabla u \times \nabla v)] \cdot \nabla v\} \nabla u - \{[\nabla \times (\nabla u \times \nabla v)] \cdot \nabla u\} \nabla v\end{aligned}$$

有

$$-\frac{\partial p}{\partial u}\nabla u - \frac{\partial p}{\partial v}\nabla v - \frac{\partial p}{\partial \psi}\nabla \psi$$

对磁场的要求 II

$$
\begin{aligned}
&+\frac{1}{\mu}\left\{[\nabla\times(\nabla u\times\nabla v)]\cdot\nabla v\right\}\nabla u-\frac{1}{\mu}\left\{[\nabla\times(\nabla u\times\nabla v)]\cdot\nabla u\right\}\nabla v\\
&-\rho\nabla\psi=0.
\end{aligned}
$$

由 u, v 和 ψ 彼此独立, 得

$$
\mu\frac{\partial p}{\partial u}=\nabla v\cdot\left[\nabla\times(\nabla u\times\nabla v)\right], \tag{2.8}
$$

$$
\mu\frac{\partial p}{\partial v}=-\nabla u\cdot\left[\nabla\times(\nabla u\times\nabla v)\right], \tag{2.9}
$$

$$
\frac{\partial p}{\partial\psi}=-\rho, \tag{2.10}
$$

而 $\frac{\partial^2 p}{\partial u\partial v}=\frac{\partial^2 p}{\partial v\partial u}$, 即

$$
\frac{\partial}{\partial u}\left\{\nabla u\cdot\left[\nabla\times(\nabla u\times\nabla v)\right]\right\}+\frac{\partial}{\partial v}\left\{\nabla v\cdot\left[\nabla\times(\nabla u\times\nabla v)\right]\right\}=0. \tag{2.11}
$$

对磁场的要求 III

对无力场

$$\nabla u\cdot[\nabla\times(\nabla u\times\nabla v)]=\nabla v\cdot[\nabla\times(\nabla u\times\nabla v)]=0,$$

条件 (2.11) 自动满足.

一维磁静平衡 I

直角坐标, 圆柱坐标.

1. 直角坐标 (z)

$$\frac{\partial B_z}{\partial z} = 0, \qquad B_z = \text{const.}.$$

均匀重力.

1. 重力沿 z 方向, $\boldsymbol{g} = g_0 \boldsymbol{e}_z$.

$$B_z \frac{\partial B_y}{\partial z} = 0,$$
$$B_z \frac{\partial B_x}{\partial z} = 0,$$

- $B_z \neq 0$, $B_x, B_y = \text{const.}$. 均匀场 + 静压力平衡.
- $B_z = 0$, $B_x = B_x(z)$, $B_y = B_y(z)$.

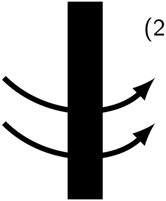
$$\frac{d}{dz}\left(p + \frac{B^2}{2\mu}\right) = -\rho g_0.$$

一维磁静平衡 II

2. 重力沿 y 方向，$g = g_0 \boldsymbol{e}_y$. 此时 $B_z = \text{const.}$ ($B_z \neq 0$). 由 $B_z \frac{\partial B_x}{\partial z} = 0$, $B_x = \text{const.}$,

$$B_z \frac{dB_y}{dz} = \mu \rho g_0, \tag{2.12}$$

$$\frac{d}{dz}\left(p + \frac{B^2}{2\mu}\right) = 0. \tag{2.13}$$



等温情况 ($T = T_0$) 下 (Kippenhahn et al., 1957),

$$\begin{aligned}
B_z &= B_0, \\
B_y &= 2\alpha H B_0 \tanh(\alpha z), \\
p &= \frac{2\alpha^2 H^2}{\mu} B_0^2 \mathrm{sech}^2(\alpha z), \\
\rho &= \frac{p}{RT_0} = \frac{2\alpha^2 H^2}{\mu R T_0} B_0^2 \mathrm{sech}^2(\alpha z), \\
H &= RT_0/g_0.
\end{aligned}$$

一维磁静平衡 III

2. 圆柱坐标 (r)

$$\frac{d}{dr}(rB_r) = 0, \qquad rB_r = \text{const.}. \tag{2.14}$$

重力沿 r 方向，$\boldsymbol{g} = g(r)\boldsymbol{e}_r$.

$$B_r \frac{dB_z}{dr} = 0,$$
$$B_r \frac{d}{dr}(rB_\varphi) = 0.$$

1. $B_r \neq 0$, $B_z = \text{const.}$, $rB_\varphi = \text{const.}$, 势场解.
2. $B_r = 0$,

$$B_z = B_z(r), \quad B_\varphi = B_\varphi(r), \tag{2.15}$$

$$\frac{d}{dr}\left(p + \frac{B_z^2 + B_\varphi^2}{2\mu}\right) + \frac{B_\varphi^2}{\mu r} = -\rho g. \tag{2.16}$$

二维磁静平衡 I

1. 二维磁静平衡方程. 引入正交曲线坐标系, 单位基矢量为 $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ 和 $\boldsymbol{e}_3$. x^3 为可忽略坐标, 以致有

$$\nabla \cdot (h_3 \boldsymbol{e}_3) = \nabla \times (\boldsymbol{e}_3 / h_3) = 0.$$

引入 F 和 G

$$F = F(x^1, x^2), \qquad G = G(x^1, x^2), \tag{2.17}$$

使

$$\boldsymbol{B} = \nabla F \times (\boldsymbol{e}_3 / h_3) + G(\boldsymbol{e}_3 / h_3).$$

从而

$$\nabla \times \boldsymbol{B} = \nabla G \times (\boldsymbol{e}_3 / h_3) - \nabla \cdot (\nabla F / h_3^2)(h_3 \boldsymbol{e}_3).$$

二维磁静平衡 II

代入 (2.1),

$$-\mu\nabla p - \nabla F\nabla\cdot\left(\frac{1}{h_3^2}\nabla F\right) + \frac{1}{h_3^2}\nabla G\times\nabla F - \frac{G\nabla G}{h_3^2} - \mu\rho\nabla\psi = 0. \tag{2.18}$$

x^3 方向的方程则为

$$\nabla G\times\nabla F = 0, \quad \text{即} \quad G = G(F), \tag{2.19}$$

一般 $\frac{\partial(F,\psi)}{\partial(x^1,x^2)} \neq 0$, 以致可用 (F,ψ) 取代 (x^1,x^2) 作为自变量, 那么

$$\nabla p = \left(\frac{\partial p}{\partial F}\right)_\psi \nabla F + \left(\frac{\partial p}{\partial\psi}\right)_F \nabla\psi,$$

二维磁静平衡 III

代入 (2.18), 并利用 $-\frac{G}{h_3^2}\left(\frac{\partial G}{\partial\psi}\right)_F\nabla\psi=0$,

$$
\begin{aligned}
&-\left[\mu\left(\frac{\partial p}{\partial F}\right)_\psi+\nabla\cdot\left(\frac{1}{h_3^2}\nabla F\right)+\frac{G}{h_3^2}\frac{dG}{dF}\right]\nabla F\\
&\quad-\mu\left[\left(\frac{\partial p}{\partial\psi}\right)_F+\rho\right]\nabla\psi+\frac{1}{h_3^2}\nabla G\times\nabla F=0.
\end{aligned}
$$

可得

$$h_3^2\nabla\cdot\left(\frac{1}{h_3^2}\nabla F\right)+G\frac{dG}{dF}+\mu h_3^2\left(\frac{\partial p}{\partial F}\right)_\psi=0. \tag{2.20}$$

$$\left(\frac{\partial p}{\partial\psi}\right)_F+\rho=0. \tag{2.21}$$

取 Euler 位

$$u=F(x^1,x^2),\qquad v=x^3+w(x^1,x^2), \tag{2.22}$$

二维磁静平衡 IV

那么 $\boldsymbol{B}=\nabla F\times \boldsymbol{e}_3/h_3+\nabla F\times\nabla w$. 与 (2.17) 比较,

$$G=\frac{h_3}{h_1h_2}\frac{\partial(F,w)}{\partial(x^1,x^2)}. \tag{2.23}$$

按 (2.22), ∇u 和 ∇v 与 x^3 无关. 可证 u 和 v 满足 (2.11).
(2.20) 通常称为 Grad–Shafranov 方程, 简称 G–S 方程. 涉及到两个待定函数 $G=G(F)$ (可任意给定) 和 $p=p(\psi,F)$(满足 (2.21)).

2. G–S 方程定解问题的适定性 (存在/唯一/稳定). 方程 (2.20) 可写成如下形式

$$\nabla^2F+S(x^1,x^2,F,\nabla F)=0, \tag{2.24}$$

式中

$$S=-\frac{2}{h_3}\nabla h_3\cdot\nabla F+G\frac{dG}{dF}+\mu h_3^2\left(\frac{\partial p}{\partial F}\right)_\psi. \tag{2.25}$$

二维磁静平衡 V

在边界上给定第一类边值问题 (Dirichlet 问题), 其解存在的充分条件是 (Courant et al., 1962, Vol. II, Chapter 4)

$$\frac{\partial S}{\partial F} \leq 0. \tag{2.26}$$

如果 $\frac{\partial S}{\partial F} > 0$, 有四种可能 (Birn et al., 1978; Low, 1982a)

- 无解
- 有唯一解
- 有两个解
- 有无穷多解

压强项的处理 I

考虑无剪切情况，$G = 0$，

$$h_3^2 \nabla \cdot \left(\frac{1}{h_3^2} \nabla F \right) + \mu h_3^2 \left(\frac{\partial p}{\partial F} \right)_\psi = 0. \tag{2.27}$$

1. 多方过程近似

$$p = \left(\frac{\gamma - 1}{\gamma} \right)^\gamma [p_1(F)]^{1-\gamma} \rho^\gamma, \tag{2.28}$$

γ 是多方指数，将 (2.28) 代入 (2.21)，积分得

$$p = p_1(F) \left[C(F) - \psi \right]^{\gamma/(\gamma-1)}, \tag{2.29}$$

$C(F)$ 是“积分常数”.

2. 给定温度分布，$T = T(F, \psi)$. 再由状态方程得到 ρ，代入 (2.21)，积分可得

$$p = p_0(F) \exp\left[-\int_{\psi_0}^{\psi} \frac{d\psi}{RT(F, \psi)} \right]. \tag{2.30}$$

直角坐标 I

均匀重力情况, 设平衡态与 z 无关, y 轴与重力方向相反

$$\psi = gy, \tag{2.31}$$

对等温大气, 取 $\psi_0 = 0$, 由 (2.30),

$$p = p_0(F)e^{-y/H}, \tag{2.32}$$
$$H = RT/g. \tag{2.33}$$

常数 $R = 1.653 \times 10^4 \mathrm{J/kg \cdot K}$, $g = 271 \mathrm{m/s^2}$. (2.27) 化为

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + f(F)e^{-y/H} = 0, \tag{2.34}$$
$$f(F) = \mu \frac{dp_0}{dF}. \tag{2.35}$$

直角坐标 II

1. 情况 I (Zweibel et al., 1982).

$$f(F) = \alpha^2 F, \tag{2.36}$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \alpha^2 F e^{-y/H} = 0. \tag{2.37}$$

用分离变量法求解

$$F = X(x)Y(y), \tag{2.38}$$

$$\frac{d^2 X}{dx^2} + k^2 X = 0, \tag{2.39}$$

$$\frac{d^2 Y}{dy^2} + (\alpha^2 e^{-y/H} - k^2)Y = 0. \tag{2.40}$$

对 y 的方程作自变量替换

$$\eta = e^{-\frac{y}{2H}}, \tag{2.41}$$

直角坐标 III

$$\eta^2 \frac{d^2 Y}{d\eta^2} + \eta \frac{dY}{d\eta} + (4\alpha^2 H^2 \eta^2 - 4k^2 H^2) Y = 0. \tag{2.42}$$

其解为

$$Y(\eta) = aJ_{2kH}(2\alpha H\eta) + bN_{2kH}(2\alpha H\eta), \qquad (\alpha^2 > 0), \tag{2.43}$$

$$Y(\eta) = aI_{2kH}(2|\alpha|H\eta) + bK_{2kH}(2|\alpha|H\eta), \qquad (\alpha^2 < 0). \tag{2.44}$$

对半无限域 $(0 \le y < \infty)$, 只需保留到 (2.43) 和 (2.44) 右边的第一项, 且系数 a 由边值 ($y = 0$ 处) 决定. 对有限域 $(0 \le y \le y_m)$, 两项均需保留, 系数 a 和 b 由 $y = 0$, y_m 处的边界条件决定. (2.39) 为周期解 (舍去 $k^2 < 0$ 的解)

$$X(x) = c\cos(kx) + d\sin(kx), \tag{2.45}$$

取半无限域 $(0 \le y < \infty)$, 并在 $y = 0$ 处提如下边界条件

$$B_y(x, 0) = B_0 \sin(\pi x/L), \quad k = \pi/L, \tag{2.46}$$

直角坐标 IV

得

$$F(x,y)=\frac{LB_0\cos\left(\frac{\pi x}{L}\right)Z_{2\pi H/L}(2|\alpha|He^{-y/2H})}{\pi Z_{2\pi H/L}(2|\alpha|H)}. \tag{2.47}$$

Z 表示 J 或 I. 由 (2.35) 和 (2.36),

$$p_0(F)=\frac{1}{\mu}\int f(F)dF=\frac{\alpha^2F^2}{2\mu}+C,$$

那么

$$p(x,y)=e^{y/H}p_0(F)=e^{-y/H}\left\{p(\pm L/2,0)\right.$$
$$\left.+\frac{\alpha^2B_0^2L^2\cos^2\left(\frac{\pi x}{L}\right)Z^2_{2\pi H/L}(2|\alpha|He^{-y/H})}{2\mu\pi^2Z^2_{2\pi H/L}(2|\alpha|H)}\right\}. \tag{2.48}$$

直角坐标 V

上述情况可以推广至 $G \neq 0$ 的情况，只需取 $B_z = \beta F$，求得如下方程

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + (\beta^2 + \alpha^2 e^{-y/H})F = 0. \quad (2.49)$$

所得的解与前面类似，只是 Bessel 函数的阶数由 $2kH$ 改为 $2\sqrt{k^2 - \beta^2}H$ (苏庆瑞, 1984).

2. 情况 II.

$$f(F) = 1, \quad (2.50)$$

(2.34) 化为

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + e^{-y/H} = 0. \quad (2.51)$$

上述方程的一个特解为 $F_1 = -H^2 e^{-y/H}$,

$$F(x, y) = \varphi(x, y) - H^2 e^{-y/H}. \quad (2.52)$$

直角坐标 VI

3. 情况 III (Webb, 1986).

$$f(F) = -\lambda F^n, \tag{2.53}$$

(2.34) 化为

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - \lambda F^n e^{-y/H} = 0. \tag{2.54}$$

4. 情况 IV.

$$f(F) = -F_0 \alpha^2 \exp\left(-\frac{2F}{F_0}\right), \tag{2.55}$$

(2.34) 化为

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - \alpha^2 F_0 \exp\left(\frac{-2F}{F_0} - \frac{y}{H}\right) = 0. \tag{2.56}$$

直角坐标 VII

作变换

$$F(x,y)=F_0\left[V(x,y)-\frac{y}{2H}\right], \tag{2.57}$$

那么

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}-\alpha^2\exp(-2V)=0. \tag{2.58}$$

其相似解 V 为

$$\ln\left[\frac{\alpha(1+\phi^2+\psi^2)}{2|\nabla\phi|}\right], \qquad (\alpha^2>0), \tag{2.59}$$

$$\ln\left[\frac{|\alpha||\phi^2+\psi^2-1|}{2|\nabla\phi|}\right], \qquad (\alpha^2<0), \tag{2.60}$$

直角坐标 VIII

或

$$\ln\left|\frac{\alpha\phi}{\nabla\phi}\right|, \qquad (\alpha^2<0). \tag{2.61}$$

ϕ 和 ψ 互为共轭调和函数, $f(z)=\phi+i\psi$.

例 1

$$f(z)=e^{\alpha z}, \tag{2.62}$$

那么

$$\phi=e^{\alpha x}\cos(\alpha y), \qquad \psi=e^{\alpha x}\sin(\alpha y).$$

代入 (2.59),

$$V=\ln\left[\cosh(\alpha x)\right],$$

直角坐标 IX

从而

$$F = F_0 \ln\left[\cosh(\alpha x)\right] - \frac{F_0 y}{2H}. \tag{2.63}$$

进一步由 (2.17), (2.32) 和 (2.35) 求得相应的 $\boldsymbol{B}$ 和 p (Kippenhahn et al., 1957),

$$B_x = -\frac{F_0}{2H}, \quad B_y = -\alpha F_0 \tanh(\alpha x), \quad p = \frac{\alpha^2 F_0^2}{2\mu}\mathrm{sech}^2(\alpha x).$$

直角坐标 X

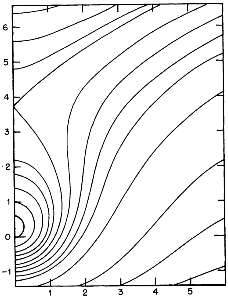
例 2 (Lerche et al., 1980)

$$f(z) = \frac{\alpha}{2} z,$$

那么

$$\phi = \frac{1}{2}\alpha x, \qquad \psi = \frac{1}{2}\alpha y$$

$$V = \ln\left(1 + \frac{\alpha^2 r^2}{4}\right), \tag{2.64}$$



从而

$$F = F_0 \ln\left(1 + \frac{\alpha^2 r^2}{4}\right) - \frac{F_0 y}{2H}. \tag{2.65}$$

最后有，

$$B_x = \frac{2F_0\alpha^2 y}{4 + \alpha^2 r^2} - \frac{F_0}{2H}, \quad B_y = -\frac{2F_0\alpha^2 x}{4 + \alpha^2 r^2}, \quad p = \frac{8\alpha^2 F_0^2}{\mu(4 + \alpha^2 r^2)^2}.$$

球坐标 I

反平方重力情况. 设平衡态与 φ 无关, 引力势为

$$\psi = -\frac{GM}{r}. \tag{2.66}$$

代入 (2.30) 得

$$p = p_0(F)\exp\left[-\int_{r_0}^{r}\frac{g}{C^2}dr\right], \tag{2.67}$$

式中

$$g = \left|\frac{d\psi}{dr}\right| = \frac{GM}{r^2}, \quad C^2 = RT. \tag{2.68}$$

进一步假定

$$T = T(r), \quad p_0(F) = W_0 + W_1 F, \tag{2.69}$$

球坐标 II

再将 (2.67) 代入 (2.27) 得

$$\frac{\partial^2 F}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial F}{\partial\theta}\right) + \mu\frac{dp_0}{dF}r^2\sin^2\theta\exp\left[-\int_{r_0}^{r}\frac{g}{C^2}dr\right] = 0. \tag{2.70}$$

则 (2.70) 可化为

$$\frac{\partial^2 F}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial F}{\partial\theta}\right) + r^2\sin^2\theta G(r) = 0, \tag{2.71}$$

$$G(r) = \mu W_1\exp\left[-\int_{r_0}^{r}\frac{g}{C^2}dr\right]. \tag{2.72}$$

球坐标 III

1. 特解 I

$$F = V(r)\sin^2\theta. \tag{2.73}$$

代入 (2.71),

$$\frac{d^2V}{dr^2} - \frac{2}{r^2}V + r^2G(r) = 0, \tag{2.74}$$

其通解为

$$V = V_1 + V_2, \tag{2.75}$$

其中 V_1 满足 (对应 (2.74) 的齐次方程)

$$\frac{d^2V_1}{dr^2} - \frac{2}{r^2}V_1 = 0,$$

球坐标 IV

可得

$$V_1 = \frac{c_1}{r} + c_2 r^2, \tag{2.76}$$

V_2 为 (2.74) 的特解，可用常数变易法求得. Wronskian 行列式

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

$y_1 = 1/r$ 和 $y_2 = r^2$ 是 (2.74) 对应的齐次方程的解.

$$y = \int \frac{\xi^2 G(\xi)\left[y_1(r)y_2(\xi) - y_1(\xi)y_2(r)\right]}{W(\xi)} d\xi$$

$W = 3$,

$$V_2 = \int \frac{\xi^2 G(\xi)\left[\frac{1}{\xi} r^2 - \xi^2 \frac{1}{r}\right]}{3} d\xi$$

球坐标 V

$$= -\frac{r^2}{3}\int^r \xi G(\xi)d\xi + \frac{1}{3r}\int^r \xi^4 G(\xi)d\xi \tag{2.77}$$

1. 反比于径向距离的温度分布

$$T = T_0 r_0/r. \tag{2.78}$$

代入 (2.72),

$$G(r) = \mu W_1 \left(\frac{r_0}{r}\right)^{r_0/H}, \tag{2.79}$$

$$H = \frac{RT_0 r_0^2}{GM}. \tag{2.80}$$

将 (2.79) 代入 (2.77),

$$V_2 = -\mu W_1 r_0^{r_0/H}\left[\left(5-\frac{r_0}{H}\right)\left(2-\frac{r_0}{H}\right)\right]^{-1} r^{4-r_0/H}. \tag{2.81}$$

球坐标 VI

2. 等温情况 (Hundhausen et al., 1981), $T = T_0$. 由 (2.72),

$$G(r) = \mu W_1 e^{-r_0/H} \exp\left(\frac{r_0^2}{Hr}\right), \tag{2.82}$$

代入 (2.77),

$$\begin{aligned} V_2 = \mu W_1 e^{-r_0/H} \Bigg[& -\frac{r^4}{10} e^{\frac{r_0^2}{Hr}} - \frac{3r^3 r_0^2}{20H} e^{\frac{r_0^2}{Hr}} + \frac{r^2 r_0^4}{6H^2} E_i\left(\frac{r_0^2}{Hr}\right) \\ & + \frac{r^2 r_0^4}{180H^2} e^{\frac{r_0^2}{Hr}} + \frac{r r_0^6}{360H^3} e^{\frac{r_0^2}{Hr}} + \frac{r_0^8}{360H^4} e^{\frac{r_0^2}{Hr}} \\ & - \frac{r_0^{10}}{360rH^5} E_i\left(\frac{r_0^2}{Hr}\right) \Bigg], \end{aligned} \tag{2.83}$$

$$E_i(x) = \int_{\infty}^{x} \frac{e^t}{t} dt. \tag{2.84}$$

球坐标 VII

3. 多方大气 (Low, 1984). 在 (2.29) 中取 $C(F)=0$, $p_0(F)=p_1(F)\left(\frac{GM}{r_0}\right)^{\frac{\gamma}{\gamma-1}}$, 则

$$\begin{cases} p = p_0(F)\left(\frac{r_0}{r}\right)^{\frac{\gamma}{\gamma-1}}, \\ \rho = \rho_0(F)\left(\frac{r_0}{r}\right)^{\frac{1}{\gamma-1}}, \\ T = T_0(F)\frac{r_0}{r}, \\ p_0(F) = R\rho_0(F)T_0(F). \end{cases} \tag{2.85}$$

与 (2.81) 类似, 本情况的解为

$$F = V(r)\sin^2\theta, \quad V(r) = \frac{c_1}{r} + c_2 r^2 + V_2,$$
$$V_2 = -\frac{\mu W_1(\gamma-1)^2}{(4\gamma-5)(\gamma-2)} r_0^{\frac{\gamma}{\gamma-1}} r^{\frac{3\gamma-4}{\gamma-1}}. \tag{2.86}$$

球坐标 VIII

2. 特解 II. 在特解 (2.73) 的基础上, 加上齐次方程的某个通解,

$$F = V(r)\sin^2\theta + F_p \tag{2.87}$$

$$\frac{\partial^2 F_p}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial F_p}{\partial\theta}\right) = 0. \tag{2.88}$$

$V(r)$ 可取 (2.86) 中的 V_2, 令 $\gamma = \frac{m+1}{m}$ (m 是整数), 则

$$V(r) = -\frac{\mu W_1}{(m-4)(m-1)} r_0^{m+1} r^{3-m}. \tag{2.89}$$

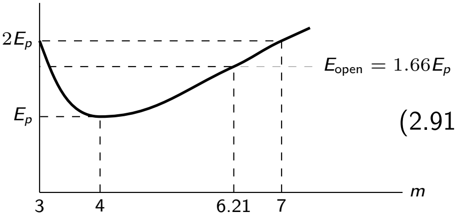
形如 (2.87) 的解是非常灵活的 (Low et al., 1993). 从 (2.89) 出发, 将其改写成

$$V(r) = \frac{1}{2}B_0 r_0^2\left(\frac{r_0}{r}\right)^{m-3}, \tag{2.90}$$

球坐标 IX

则

$$F = \frac{1}{2} B_0 r_0^2 \left(\frac{r_0}{r}\right)^{m-3} \sin^2\theta. \tag{2.91}$$



利用 (2.17), 可求得相应磁场

$$B_r = B_0 \left(\frac{r_0}{r}\right)^{m-1} \cos\theta, \tag{2.92}$$

$$B_\theta = B_0 \left(\frac{r_0}{r}\right)^{m-1} \left(\frac{m-3}{2}\right) \sin\theta. \tag{2.93}$$

以上解的特点是

1. 在 $r = r_0$ 处, $B_r = B_0 \cos\theta$, 与 m 无关, 对应偶极场的 B_r 分布.
2. $m = 4$, 正好是偶极场.
   $m < 4$, 磁场随距离衰减比偶极场慢.
   $m > 4$, 磁场随距离衰减比偶极场快.

二维问题 I

- 过去几节都是给定 $p(F,\psi)$, $G(F) \Rightarrow F$ (G–S 方程). 本节则事先给定 F 和 $G(F)$, 由 (2.20) 计算 p, 由 (2.21) 计算 ρ, 由 (2.3) 计算 T, 再代入 (2.4) 求 $L+R$.
- 任意二维磁场结构下的磁静平衡态 (Low, 1980, 1981). 由 (2.20),

$$p = p_0(\psi) - \int^F \frac{1}{\mu h_3^2}\left(\mathcal{L}F + G\frac{dG}{dF}\right) dF, \tag{2.94}$$

$$\rho = -\frac{dp_0}{d\psi} + \frac{1}{\mu}\int^F \frac{\partial}{\partial\psi}\left[\frac{1}{h_3^2}\left(\mathcal{L}F + G\frac{dG}{dF}\right)\right] dF. \tag{2.95}$$

$$T = p/\rho R. \tag{2.96}$$

式中

$$\mathcal{L}F \equiv h_3^2 \nabla\cdot\left(\frac{1}{h_3^2}\nabla F\right)$$

二维势场 I

二维势场几何下的磁静平衡 (Hu, 1988)

1. 二维势场

$$\boldsymbol{B}_p = \frac{1}{h_3}\nabla x^1 \times \boldsymbol{e}_3 = -\nabla x^2. \tag{2.97}$$

(x^1, x^2, x^3) 为正交曲线坐标系, x^3 为可忽略坐标.

$$\boldsymbol{B}_p = -\frac{1}{h_1 h_3}\boldsymbol{e}_2 = -\frac{1}{h_2}\boldsymbol{e}_2 = B_p \boldsymbol{e}_2,$$

那么

$$h_2 = -\frac{1}{B_p}, \quad h_1 = -\frac{1}{h_3 B_p}. \tag{2.98}$$

二维势场 II

2. 磁静平衡态. 给定

$$F = F(x^1), \quad G = G(x^1), \tag{2.99}$$

那么

$$\boldsymbol{B} = \nabla F \times \frac{1}{h_3}\boldsymbol{e}_3 + \frac{G}{h_3}\boldsymbol{e}_3 = \frac{dF}{dx^1}\boldsymbol{B}_p + \frac{1}{h_3}G\boldsymbol{e}_3. \tag{2.100}$$

而

$$\mathcal{L}F = h_3^2 B_p^2 F''.$$

可令 $\psi = \psi(x^2)$, 以致 (2.94) 和 (2.95) 化为

$$p = p_0(\psi) - \frac{1}{\mu}\int_0^{x^1}\left[B_p^2 F' F'' + \frac{1}{h_3^2} G G'\right] dx^1, \tag{2.101}$$

$$\rho = -\frac{dp_0}{d\psi} + \frac{1}{\mu}\int_0^{x^1}\left[F' F'' \frac{\partial B_p^2}{\partial \psi} + G G' \left(\frac{\partial h_3^{-2}}{\partial \psi}\right)_{x^1}\right] dx^1. \tag{2.102}$$

二维势场 III

3. 说明
   - 易于构成.
   - 稳定性易于证明

     K–S 日珥模型 (Kippenhahn et al., 1957) 的稳定性证明 (Zweibel, 1982).

三维二分量问题 I

仍设 x^3 为可忽略坐标，且假定 $B_3 = 0$，

$$\boldsymbol{B} = \nabla F \times \frac{1}{h_3}\boldsymbol{e}_3 = \frac{1}{h_2 h_3}\frac{\partial F}{\partial x^2}\boldsymbol{e}_1 - \frac{1}{h_1 h_3}\frac{\partial F}{\partial x^1}\boldsymbol{e}_2 = B_1\boldsymbol{e}_1 + B_2\boldsymbol{e}_2, \tag{2.103}$$

$F = F(x^1, x^2, x^3)$,

$$\nabla \times \boldsymbol{B} = -h_3\nabla_p \cdot \left[\frac{1}{h_3^2}\nabla_p F\right]\boldsymbol{e}_3 - \frac{1}{h_3}\frac{\partial B_2}{\partial x^3}\boldsymbol{e}_1 + \frac{1}{h_3}\frac{\partial B_1}{\partial x^3}\boldsymbol{e}_2,$$

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\nabla_p \cdot \left[\frac{1}{h_3^2}\nabla_p F\right]\nabla_p F - \frac{1}{2h_3}\frac{\partial B^2}{\partial x^3}\boldsymbol{e}_3.$$

其中 $B^2 = B_1^2 + B_2^2$, $\nabla_p = \frac{\boldsymbol{e}_1}{h_1}\frac{\partial}{\partial x^1} + \frac{\boldsymbol{e}_2}{h_2}\frac{\partial}{\partial x^2}$.

$$-\nabla p = -\left(\frac{\partial p}{\partial \psi}\right)_{F,x^3}\nabla\psi - \left(\frac{\partial p}{\partial F}\right)_{\psi,x^3}\nabla F - \left(\frac{\partial p}{\partial x^3}\right)_{F,\psi}\frac{1}{h_3}\boldsymbol{e}_3,$$

$$\nabla F = \nabla_p F + \frac{\partial F}{\partial x^3}\frac{1}{h_3}\boldsymbol{e}_3,$$

三维二分量问题 II

$$\left(\frac{\partial p}{\partial x^3}\right)_{x^1,x^2} = \left(\frac{\partial p}{\partial x^3}\right)_{F,\psi} + \left(\frac{\partial p}{\partial F}\right)_{\psi,x^3} \frac{\partial F}{\partial x^3} + \left(\frac{\partial p}{\partial \psi}\right)_{F,x^3} \frac{\partial \psi}{\partial x^3},$$

$$\left(\frac{\partial p}{\partial x^3}\right)_{F,\psi} = \left(\frac{\partial p}{\partial x^3}\right)_{x^1,x^2} - \left(\frac{\partial p}{\partial F}\right)_{\psi,x^3} \frac{\partial F}{\partial x^3} - \left(\frac{\partial p}{\partial \psi}\right)_{F,x^3} \frac{\partial \psi}{\partial x^3}.$$

以上诸式代入 (2.1) 得

$$\begin{aligned}
-\left(\frac{\partial p}{\partial \psi}\right)_{F,x^3} \nabla\psi - \left(\frac{\partial p}{\partial F}\right)_{\psi,x^3} \nabla_p F - \left(\frac{\partial p}{\partial x^3}\right)_{x^1,x^2} \frac{\boldsymbol{e}_3}{h_3} \\
+ \left(\frac{\partial p}{\partial \psi}\right)_{F,x^3} \frac{\partial \psi}{\partial x^3} \frac{\boldsymbol{e}_3}{h_3} - \frac{1}{\mu}\nabla_p \cdot \left[\frac{1}{h_3^2}\nabla_p F\right] \nabla_p F \\
- \frac{1}{2\mu h_3} \frac{\partial B^2}{\partial x^3} \boldsymbol{e}_3 - \rho\nabla\psi = 0.
\end{aligned}$$

三维二分量问题 III

求得三个分方程 $(\nabla_p F, \nabla\psi, \boldsymbol{e}_3)$,

$$\nabla_p \cdot \left[\frac{1}{h_3^2}\nabla_p F\right] + \mu\left(\frac{\partial p}{\partial F}\right)_{\psi,x^3} = 0. \tag{2.104}$$

$$\left(\frac{\partial p}{\partial \psi}\right)_{F,x^3} + \rho = 0. \tag{2.105}$$

$$\left[\frac{\partial\left(p + \frac{B^2}{2\mu}\right)}{\partial x^3}\right]_{x^1,x^2} + \rho\left(\frac{\partial \psi}{\partial x^3}\right)_{x^1,x^2} = 0. \tag{2.106}$$

假定

$$\left(\frac{\partial \psi}{\partial x^3}\right)_{x^1,x^2} = 0. \tag{2.107}$$

三维二分量问题 IV

(2.106) 可对 x^3 积分

$$p+\frac{B^2}{2\mu}=W(x^1,x^2). \tag{2.108}$$

可证明磁场的 Euler 位 $u=F(x^1,x^2,x^3)$, $v=x^3$ 满足磁静平衡需要的条件 (2.11)，证明如下。(2.11) 式化为

$$\frac{\partial}{\partial F}[\nabla F\cdot(\nabla\times\boldsymbol{B})]+\frac{\partial}{\partial x^3}\left[\frac{1}{h_3}\boldsymbol{e}_3\cdot(\nabla\times\boldsymbol{B})\right]=0.$$

而 $\nabla F=-h_3B_2\boldsymbol{e}_1+h_3B_1\boldsymbol{e}_2)+\frac{1}{h_3}\frac{\partial F}{\partial x^3}\boldsymbol{e}_3$, 代回上式,

$$\left\{\frac{\partial}{\partial F}\left[\frac{1}{2}\left(\frac{\partial B^2}{\partial x^3}\right)_{x^1,x^2}-\left(\frac{\partial F}{\partial x^3}\right)_{x^1,x^2}\nabla_p\cdot\left(\frac{1}{h_3^2}\nabla F\right)\right]\right\}_{\psi,x^3}$$

$$=\left\{\frac{\partial}{\partial x^3}\left[\nabla_p\cdot\left(\frac{1}{h_3^2}\nabla_pF\right)\right]\right\}_{F,\psi}.$$

三维二分量问题 V

利用 (2.104), 上式化为

$$\left\{\frac{\partial}{\partial F}\left[\frac{1}{2}\left(\frac{\partial B^2}{\partial x^3}\right)_{x^1,x^2}+\frac{1}{\mu}\left(\frac{\partial p}{\partial F}\right)_{\psi,x^3}\left(\frac{\partial F}{\partial x^3}\right)_{x^1,x^2}\right]\right\}_{\psi,x^3}$$

$$=\mu\left(\frac{\partial^2 p}{\partial x^3\partial F}\right)_{\psi}.$$

由 (2.105) 和 (2.106) 及如下恒等式

$$\left(\frac{\partial p}{\partial x^3}\right)_{x^1,x^2}=\left(\frac{\partial p}{\partial x^3}\right)_{F,\psi}+\left(\frac{\partial p}{\partial F}\right)_{\psi,x^3}\frac{\partial F}{\partial x^3}+\left(\frac{\partial p}{\partial \psi}\right)_{F,x^3}\frac{\partial \psi}{\partial x^3},$$

则左边可化为 $\mu\left(\frac{\partial^2 p}{\partial F\partial x^3}\right)_{\psi}$, 它等于右边 $\mu\left(\frac{\partial^2 p}{\partial x^3\partial F}\right)_{\psi}$. 证毕. □

三维二分量问题 VI

特别的，考虑 x^1 坐标面与 ψ 的等值面重合的情况

$$\psi = \psi(x^1). \tag{2.109}$$

于是由 (2.108) 求得

$$\begin{aligned}\mu\left(\frac{\partial p}{\partial F}\right)_{\psi,x^3} =&\mu\left(\frac{\partial p}{\partial F}\right)_{x^1,x^3} = -\frac{1}{2}\left(\frac{\partial B^2}{\partial F}\right)_{x^1,x^3} + \mu\left(\frac{\partial W}{\partial F}\right)_{x^1}\\ =&\frac{\partial x^2}{\partial F}\left(-\frac{1}{2}\frac{\partial B^2}{\partial x^2} + \mu\frac{\partial W}{\partial x^2}\right)\end{aligned}$$

即

$$\mu\frac{\partial F}{\partial x^2}\left(\frac{\partial p}{\partial F}\right)_{\psi,x^3} = \mu\left(\frac{\partial p}{\partial F}\right)_{x^1,x^3} = -\frac{1}{2}\frac{\partial B^2}{\partial x^2} + \mu\frac{\partial W}{\partial x^2}.$$

三维二分量问题 VII

代入 (2.104), 经化简得

$$\frac{\partial\left(\frac{h_2}{h_1 h_3}\frac{\partial F}{\partial x^1}, F\right)}{\partial(x^1, x^2)} + \frac{1}{h_2 h_3}\left(\frac{\partial F}{\partial x^2}\right)^2 \frac{\partial h_1}{\partial x^2} + \frac{1}{h_1 h_3}\left(\frac{\partial F}{\partial x^1}\right)^2 \frac{\partial h_2}{\partial x^2} + \mu h_1 h_2 h_3 \frac{\partial W}{\partial x^2} = 0. \tag{2.110}$$

当取

$$W = W_1(\psi) = W(x^1) \tag{2.111}$$

时, 即假定总压按引力等位面分层均匀, 则 (2.110) 化为

$$\frac{\partial\left(\frac{h_2}{h_1 h_3}\frac{\partial F}{\partial x^1}, F\right)}{\partial(x^1, x^2)} + \frac{1}{h_2 h_3}\left(\frac{\partial F}{\partial x^2}\right)^2 \frac{\partial h_1}{\partial x^2} + \frac{1}{h_1 h_3}\left(\frac{\partial F}{\partial x^1}\right)^2 \frac{\partial h_2}{\partial x^2} = 0. \tag{2.112}$$

三维二分量问题 VIII

对直角坐标 (z, x, y), 圆柱坐标 (z, r, φ), 引力沿负 z 方向. 对球坐标 (r, θ, φ), 引力沿负 r 方向, 相应 Lame 系数为

$$(h_1, h_2, h_3) = (1, 1, 1), \quad (1, 1, r), \quad (1, r, r\sin\theta).$$

凑巧能使 (2.112) 是左边两项为零, 以致

$$\frac{\partial\left(\frac{h_2}{h_1 h_3}\frac{\partial F}{\partial x^1}, F\right)}{\partial(x^1, x^2)} = 0. \tag{2.113}$$

从而

$$\frac{h_2}{h_1 h_3}\frac{\partial F}{\partial x^1} = f(F, x^3),$$

即

$$\int \frac{dF}{f(F, x^3)} = \int \frac{h_1 h_3}{h_2} dx^1 + G(x^2, x^3).$$

三维二分量问题 IX

令

$$\zeta(x^1, x^2, x^3) = \int \frac{h_1 h_3}{h_2} dx^1 + G(x^2, x^3), \tag{2.114}$$

则

$$F(x^1, x^2, x^3) = F(\zeta, x^3). \tag{2.115}$$

只要选定 $F(\zeta, x^3)$ 和 $G(x^2, x^3)$ 的函数形式, 就可由 (2.114) 和 (2.115) 确定三维二分量问题的解. 对上述三种坐标, 有如下结果 (Low, 1982b, 直角坐标, 暗条/日珥) (Hu, 1983, 大尺度日冕结构, 球坐标) (胡友秋 等, 1985, 圆柱坐标, 黑子)

$$F(z, x, y) = F(\zeta, y), \quad \zeta = z + G(x, y), \tag{2.116}$$
$$F(z, r, \varphi) = F(\zeta, \varphi), \quad \zeta = rz + G(r, \varphi), \tag{2.117}$$
$$F(r, \theta, \varphi) = F(\zeta, \varphi), \quad \zeta = r\sin\theta + G(\theta, \varphi). \tag{2.118}$$

三维二分量问题 X

关键的假定是

$$\psi = \psi(x^1),$$
$$\frac{\partial h_1}{\partial x^3} = \frac{\partial h_2}{\partial x^3} = 0.$$

三维三分量问题

1. 等总压面与引力垂直 (Low, 1984).
2. 电流与引力垂直 (Low, 1985; Bogdan et al., 1986).
3. 电流与引力垂直情况的推广 (Low, 1991, 1992).
4. 互相耦合的电流系统 (Low, 1993b,c).

第二次作业 I

作业要求：下次上课 (12 月 13 日) 前，将作业电子版发送到助教邮箱 (wangyubao@mail.ustc.edu.cn).

1. 在柱坐标 (r, θ, z) 下，考虑独立于 z 的二维二分量磁静平衡，
   1. 从推广的 G-S 方程出发，证明上述问题的平衡方程为

      $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}+\mu\left(\frac{\partial p}{\partial F}\right)_{\Psi}=0,$$

      式中 F 为磁通量函数，μ 为磁导率，Ψ 为引力势.
   2. 设大气等温，温度为 T，引力势 $\Psi=\Psi_0\ln(r/r_0)$，求压强 p 的表达式.
   3. 取 $\mu\left(\frac{\partial p}{\partial F}\right)_{\Psi}=\alpha^2\left(\frac{r_0}{r}\right)^{\frac{\Psi_0}{RT}}F$，分别讨论 $\alpha^2>0$, $\alpha^2<0$ 以及 $k>0$, $k<0$ 和 $k=0$ 情况下 F 的通解，该解满足无穷远 $(r\to\infty)$ 处的正则条件 $(k\equiv\frac{\Psi_0}{2RT}-1)$.

第二次作业 II

2. 已知 Liouville 方程

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=\lambda e^F \qquad (\lambda>0)$$

的相似解具有如下形式

$$F=\ln\left[\frac{8(\nabla\varphi)^2}{\lambda(\varphi^2+\psi^2-1)^2}\right],$$

式中 φ 和 ψ 为共轭调和函数. 从如下解析函数

$$f(z)=\varphi(x,y)+i\psi(x,y)=\frac{\sqrt{\lambda/8}(1-iz)}{(\sqrt{1+\lambda/8}-1)(1+iz)}, \qquad z=x+iy$$

出发导出 F 的表达式.

第二次作业 III

3. 对偶极子势场和引力场

$$\boldsymbol{B}_p = \frac{1}{r\sin\theta}\nabla x^1 \times \boldsymbol{e}_3 = -\nabla x^2, \qquad \psi = -\frac{GM_s}{r},$$

$\boldsymbol{e}_3$ 是 x^3 方向的单位向量. 引入如下正交曲线坐标

$$x^1 = \frac{\sin^2\theta}{r}, \quad x^2 = \frac{\cos\theta}{r^2}, \quad x^3 = \varphi$$

求

1. 该坐标系的 Lame 系数 h_1, h_2 和 h_3.
2. 当

$$\boldsymbol{B} = \frac{dF}{dx^1}\boldsymbol{B}_p, \qquad F = F_0 e^{Rx^1}$$

(F_0, R 为常数) 时的 p 和 ρ.

稳定性 I

1. 定义. 一个处于平衡或动态平衡的系统, 在加上任意扰动后, 扰动幅度保持有限或随时间衰减, 称该系统是稳定的. 反之, 若扰动随时间不断增长, 使系统偏离原来的平衡态越来越远, 则称系统是不稳定的.
2. 一个简单力学系统的稳定性 – 重力作用下的小球.

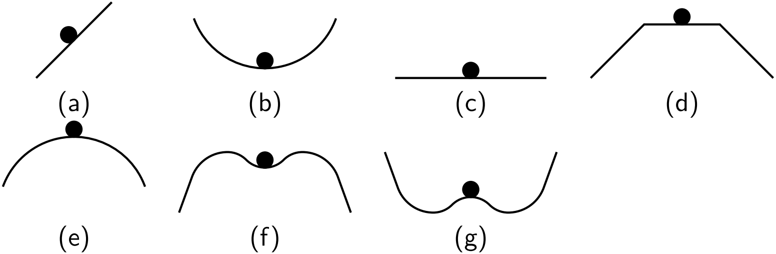


图 3.1: (a) 不平衡, 不存在稳定性问题; (b) 平衡, 稳定; (c) (随遇) 平衡, 临界稳定 (边际稳定) / marginal stable; (d) 平衡, 亚稳定; (e) 平衡, 不稳定; (f) 平衡, 线性稳定, 非线性不稳定; (g) 平衡, 线性不稳定, 非线性稳定 (非线性饱和).

等离子体稳定性的分类 I

1. 线性稳定和非线性稳定
   线性性 – 平衡态对小扰动的响应
   非线性性 – 平衡态对有限幅度扰动的响应
2. 宏观不稳定性和微观不稳定性
   宏观 – 宏观量在位置空间分布的不均匀 – MHD 不稳定性
   微观 – 粒子速度分布函数偏离平衡分布 – 等离子体不稳定性
3. 理想 MHD 和耗散 MHD 不稳定性
   理想 MHD – 以 Alfven 时间尺度发展 (快, 释能, 形成电流片)
   耗散 MHD – 以耗散时间尺度发展
4. 其他分类标准
   1. 按形成的物理机制
      压力, 电流, 漂移, 双流, 损失锥
   2. 按不稳定性的形态特征
      腊肠/sausage, 扭折/kink, 交换/interchange, 撕裂/tearing.
   3. 按发现者命名

等离子体稳定性的分类 II

Kelvin–Helmhotz – 速度剪切
Rayleigh–Taylor – 重力
Parker – 浮力 (重力)/buoyancy

研究稳定性问题的意义

1. 可控热核聚变
2. 空间等离子体现象的机制分析
    1. 各种稳定平衡结构
    2. 爆发现象机制
    3. 辐射机制
    4. 异常输运现象 (波-粒相互作用 $\cdots$)

理想 MHD 稳定性分析的基本方程 I

1. 理想 MHD 方程

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=0, \tag{3.1}$$

$$\rho\frac{\partial\boldsymbol{v}}{\partial t}+\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}=-\nabla p+\frac{1}{\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}-\rho\nabla\psi, \tag{3.2}$$

$$\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{v}\times\boldsymbol{B}), \tag{3.3}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right)+\boldsymbol{v}\cdot\nabla\left(\frac{p}{\rho^\gamma}\right)=0, \tag{3.4}$$

或

$$\frac{\partial p}{\partial t}+\boldsymbol{v}\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{v}=0. \tag{3.5}$$

理想 MHD 稳定性分析的基本方程 II

2. 磁静平衡方程

$$-\nabla p + \frac{1}{\mu}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \rho \nabla \psi = 0, \tag{3.6}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3.7}$$

$$p = \rho R T. \tag{3.8}$$

3. 小扰动方程 (线性化)

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v}_1) = 0, \tag{3.9}$$

$$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{\mu}(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_1 + \frac{1}{\mu}(\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 - \rho_1 \nabla \psi, \tag{3.10}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0), \tag{3.11}$$

$$\frac{\partial p_1}{\partial t} + \boldsymbol{v}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{v}_1 = 0. \tag{3.12}$$

理想 MHD 稳定性分析的基本方程 III

4. 小扰动方程的变形
引入等离子体位移 $\boldsymbol{\xi}(\boldsymbol{r}_0, t)$, $\boldsymbol{r}_0$ 为平衡位置, $\boldsymbol{\xi}$ 则为流体元偏离平衡位置的位移，故有 $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{\xi}$, 于是

$$\boldsymbol{v}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t}, \tag{3.13}$$

代入 (3.9),

$$\frac{\partial}{\partial t}\left[\rho_1 + \nabla\cdot(\rho_0\boldsymbol{\xi})\right] = 0,$$

$$\rho_1 + \nabla\cdot(\rho_0\boldsymbol{\xi}) = \left[\rho_1 + \nabla\cdot(\rho_0\boldsymbol{\xi})\right]_{t=0} = 0.$$

即

$$\rho_1 = -\nabla\cdot(\rho_0\boldsymbol{\xi}) \tag{3.14}$$

理想 MHD 稳定性分析的基本方程 IV

同理

$$\boldsymbol{B}_1 = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0), \tag{3.15}$$

$$p_1 = -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi}. \tag{3.16}$$

以上积分用到初条件

$$(\rho_1, p_1, \boldsymbol{B}_1)_{\boldsymbol{\xi}=0} = 0. \tag{3.17}$$

将 (3.13)–(3.16) 代入 (3.10), 得

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \boldsymbol{F}(\boldsymbol{\xi}), \tag{3.18}$$

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{\xi}) &= -\nabla p_1 + \frac{1}{\mu}(\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 + \frac{1}{\mu}(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_1 - \rho_1 \nabla \psi \\ &= \nabla(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}) + \frac{1}{\mu}(\nabla \times \boldsymbol{B}_0) \times [\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)] \\ &\quad + \frac{1}{\mu}\{\nabla \times [\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)]\} \times \boldsymbol{B}_0 + \nabla \psi \nabla \cdot (\rho_0 \boldsymbol{\xi}).\end{aligned} \tag{3.19}$$

理想 MHD 稳定性分析的基本方程 V

5. 方程的应用

   (3.6)–(3.8) – 找平衡态.

   (3.1)–(3.4) – 以平衡态为初始条件, 进行时变模拟, 研究非线性不稳定性.

   (3.9)–(3.12) – 由已知的平衡态, 可通过解析方法 (简正模方法) 或数值方法做线性稳定性分析.

   (3.18) – 由已知的平衡态, 转换为能量原理, 做 (线性) 稳定性分析.

能量原理的表述 (Bernstein et al., 1958)

能量

$$W = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi})dV. \tag{3.20}$$

对任意满足

$$\xi_n|_\sigma = 0, \quad B_n|_\sigma = 0 \tag{3.21}$$

的位移 ξ, 由 (3.20) 表示的能量积分恒正, 是系统稳定的充分必要条件. 对空间等离子体, (3.21) 式应代之以

$$\boldsymbol{\xi}|_\sigma = 0. \tag{3.22}$$

能量原理应用的一些相关文献

1. 一维磁静平衡态的稳定性 Newcomb (1960)
2. 二维磁静平衡态的能量原理 胡友秋 (1986). 给出了具有一个可忽略坐标的任意正交曲线坐标系下能量积分的普遍形式: 针对二维二分量磁静平衡态, 给出若干稳定性判据和判稳方法.
3. Zweibel 判据 Zweibel (1981, 1982). 直角坐标系下二维二分量磁静平衡态及均匀重力情况, 稳定的充分条件为

$$p+\frac{B^2}{2\mu}=\text{const.}.$$

   K–S 日珥模型 Kippenhahn et al. (1957) 正好满足, 故是稳定的.
4. 无力场的稳定性
   1. 磁面为球面和平面的无力场是稳定的 胡友秋 (1991)
   2. 圆柱坐标系下一维无力场的稳定性. 螺距及其径向分布作为稳定性判据 (定性) Hu (2001); Hu et al. (2002)

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