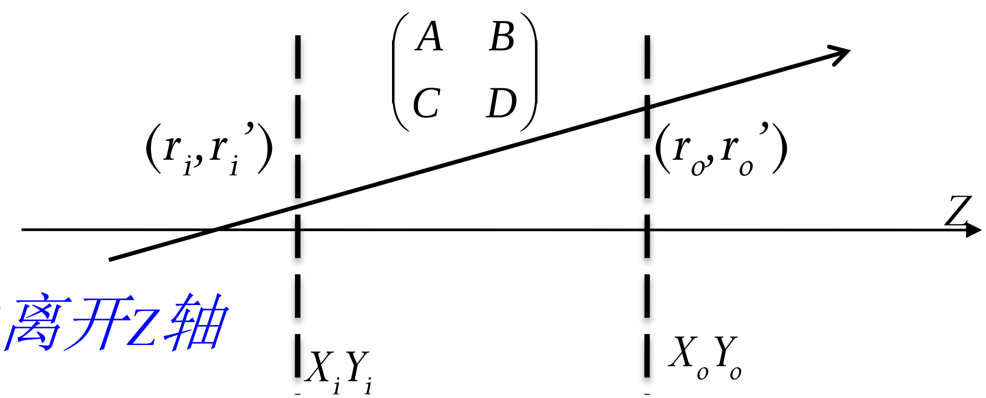
光束在均匀介质和类透镜介质中的传播 ----Gaussian beam

1.光线矩阵(Ray Matrix)

◆光线：在几何光学近似成立的条件下,光能量可以看作沿一定的曲线传播,该曲线被称为“光线”。

◆**傍轴近似(Paraxial-ray approximation)**下，光线在光学系统中传播、透射(或反射)的行为可以用一个2×2的矩阵来描述，该矩阵被称为“光线矩阵”

$$\begin{pmatrix} r_o \\ r_o' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_i \\ r_i' \end{pmatrix}$$



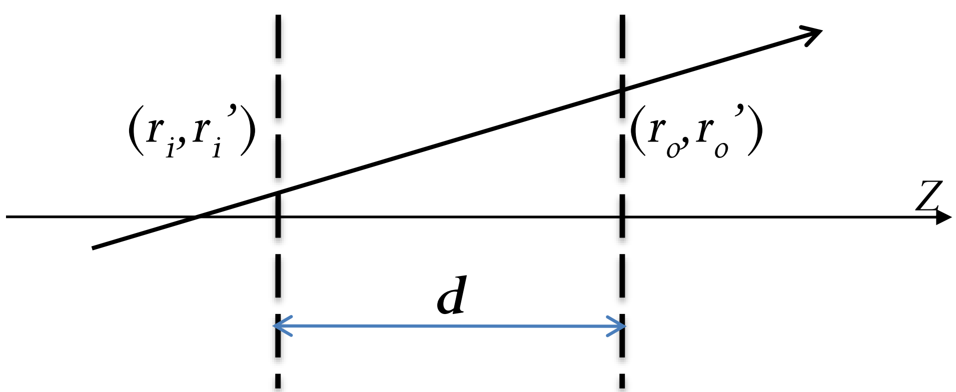
***r**, 光线在XY平面内的位置(离开Z轴的距离)；*

***r’ =dr/dz** 光线在在该位置的斜率*

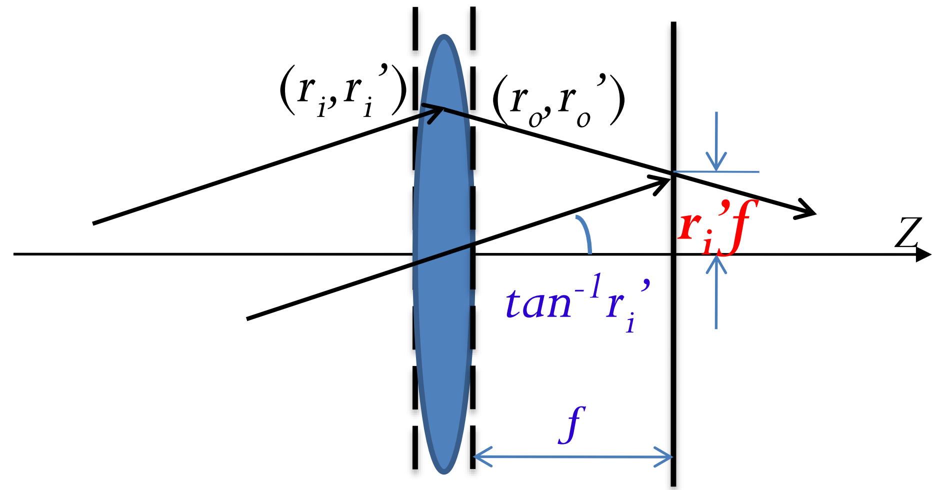
自由空间传播

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} r_o = r_i + d \cdot r_i' \\ r_o' = r_i' \end{cases}$$



薄透镜



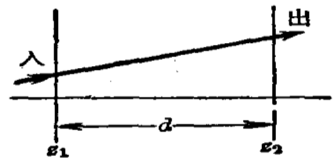
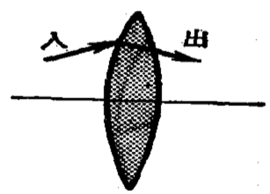
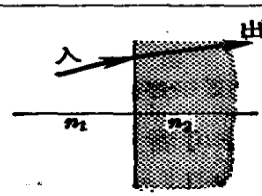
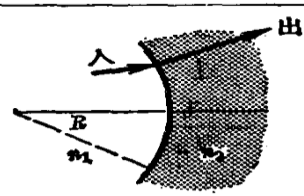
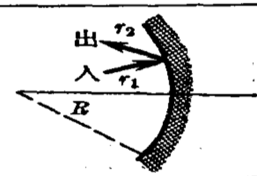
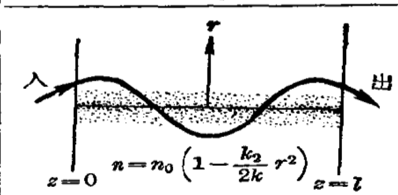
薄透镜: $r_o = r_i$

$$-r_o' = \frac{r_o - r_i' f}{f} = \frac{r_o}{f} - r_i'$$

$$r_o' = -\frac{r_o}{f} + r_i' = -\frac{r_i}{f} + r_i'$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

典型的几个光线矩阵

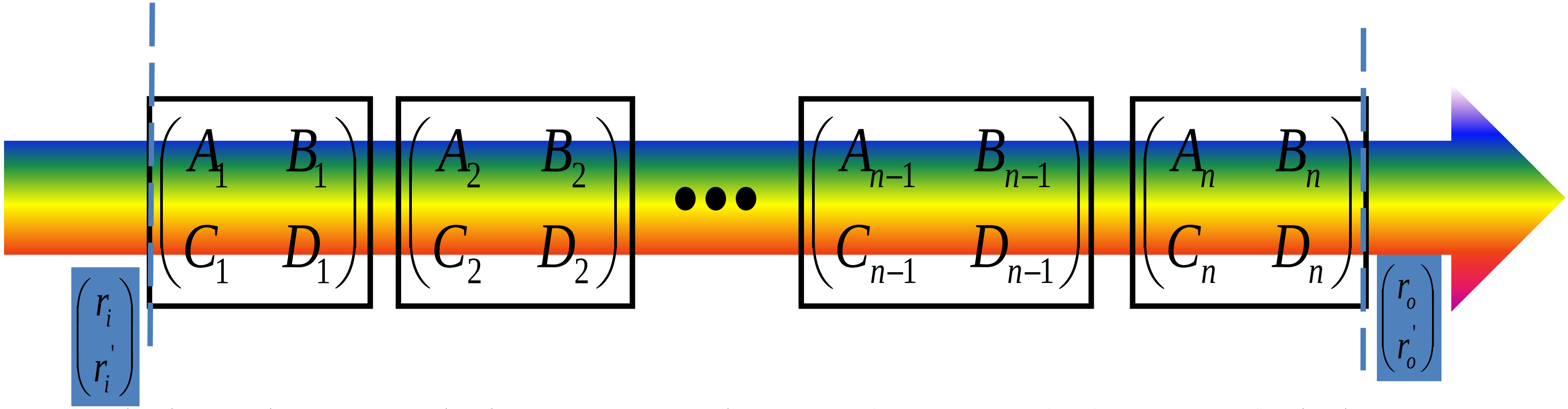
(1) 长度为 d 的直线段		$\begin{bmatrix}1 & d\\ 0 & 1\end{bmatrix}$
(2) 薄透镜：焦距 f（$f>0$，会聚；$f<0$，发散）		$\begin{bmatrix}1 & 0\\ \frac{-1}{f} & 1\end{bmatrix}$
(3) 电介质界面：折射率 n_1、n_2		$\begin{bmatrix}1 & 0\\ 0 & \frac{n_1}{n_2}\end{bmatrix}$
(4) 球面电介质界面：半径 R		$\begin{bmatrix}1 & 0\\ \frac{n_2-n_1}{n_2}\frac{1}{R} & \frac{n_1}{n_2}\end{bmatrix}$
(5) 球面反射镜：曲率半径 R		$\begin{bmatrix}1 & 0\\ -\frac{2}{R} & 1\end{bmatrix}$
(6) 有二次型折射率变化曲线的介质	 $n=n_0\left(1-\frac{k_2}{2k}r^2\right)$	$\begin{bmatrix}\cos\left(\sqrt{\frac{k_2}{k}}\,l\right) & \sqrt{\frac{k}{k_2}}\sin\left(\sqrt{\frac{k_2}{k}}\,l\right)\\ -\sqrt{\frac{k_2}{k}}\sin\left(\sqrt{\frac{k_2}{k}}\,l\right) & \cos\left(\sqrt{\frac{k_2}{k}}\,l\right)\end{bmatrix}$

$$\det\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\frac{n_1}{n_2}$$

两者等价

注意反射界面的符号规则！

复杂光学系统(多个元件)的光线矩阵

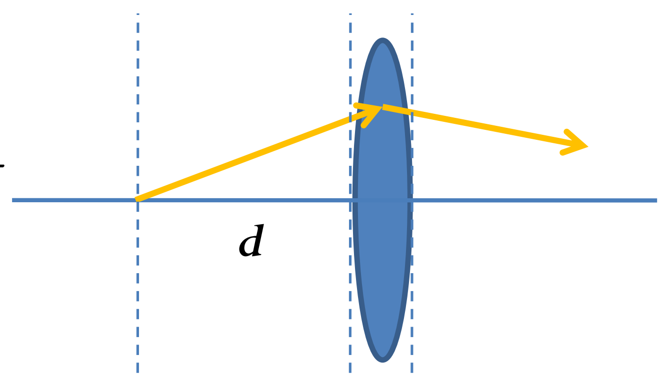


$$\begin{pmatrix} r_o \\ r_o' \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \begin{pmatrix} A_{n-1} & B_{n-1} \\ C_{n-1} & D_{n-1} \end{pmatrix} \cdots \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} r_i \\ r_i' \end{pmatrix}$$

！!注意矩阵相乘的次序：从出射面开始到入射面结束

简单举例:

光线在均匀介质空间传播距离d后通过焦距为f的透镜



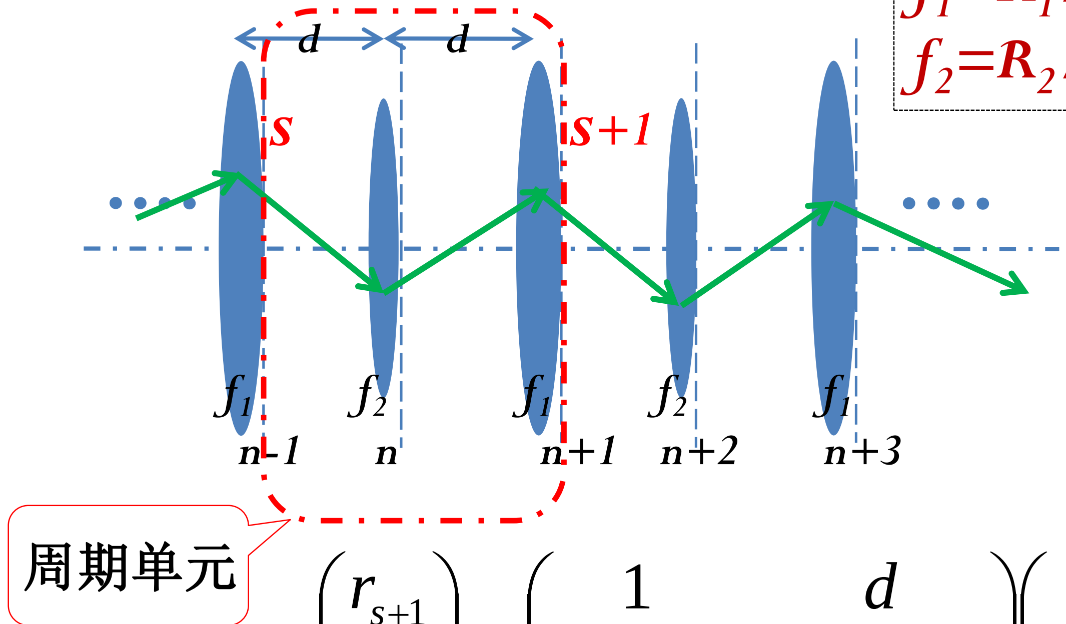
$$\begin{pmatrix} r_o \\ r_o' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_i \\ r_i' \end{pmatrix}$$

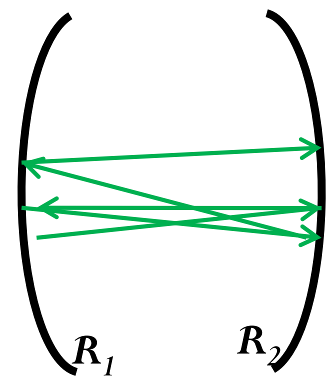
$$= \begin{pmatrix} 1 & d \\ -1/f & 1-d/f \end{pmatrix}$$

2.透镜波导阵列

二元透镜波导阵列 ⟷ 一对反射镜腔

$f_1 = R_1/2$
$f_2 = R_2/2$





$$\begin{pmatrix} r_{s+1} \\ r'_{s+1} \end{pmatrix} = \begin{pmatrix} 1 & d \\ -1/f_1 & 1-d/f_1 \end{pmatrix} \begin{pmatrix} 1 & d \\ -1/f_2 & 1-d/f_2 \end{pmatrix} \begin{pmatrix} r_s \\ r'_s \end{pmatrix}$$

$$\begin{pmatrix} r_{s+1} \\ r'_{s+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_s \\ r'_s \end{pmatrix}$$

$$\begin{pmatrix} r_{s+1} \\ r'_{s+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_s \\ r'_s \end{pmatrix}$$

$$A = 1 - \frac{d}{f_2}$$

$$B = d(2 - \frac{d}{f_2})$$

$$C = -\frac{1}{f_1} - \frac{1}{f_2}(1 - \frac{d}{f_1})$$

$$D = -\frac{d}{f_1} + (1 - \frac{d}{f_1})(1 - \frac{d}{f_2})$$

$$AD - BC = 1$$

$$\begin{cases} r_{s+1} = Ar_s + Br'_s \\ r'_{s+1} = Cr_s + Dr'_s \end{cases}$$

$$r'_s = \frac{1}{B}(r_{s+1} - Ar_s)$$

$$r'_{s+1} = \frac{1}{B}(r_{s+2} - Ar_{s+1})$$

$$r_{s+2} - (A+D)r_{s+1} + (AD-BC)r_s = 0$$

令$b = (A+D)/2$ $= 1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2f_1f_2}$

$$r_{s+2} - 2br_{s+1} + r_s = 0$$

差分方程

$\Leftrightarrow$ $r'' + Ar = 0$

微分方程

方程的解

$$r(z) = r(0)e^{\pm i\sqrt{A}z}$$

$$r_s = r_0 e^{i\theta s}$$

$$e^{i2\theta} - 2be^{i\theta} + 1 = 0$$

$$e^{\pm i\theta} = b \pm i\sqrt{1-b^2}$$

$$\cos\theta = b = (A+D)/2$$

实数解：$r_s = r_m \sin(\theta s + \delta)$

透镜波导阵列中的光线轨迹函数!

透镜波导中的光线稳定条件

光线稳定(光线被约束)条件：θ为实数，即 $|b|\le 1$

$$0 \le \left(1-\frac{d}{2f_1}\right)\left(1-\frac{d}{2f_2}\right) \le 1$$

如果|b|>1， θ为纯虚数！ $e^{\alpha\pm} = b \pm i\sqrt{1-b^2}$

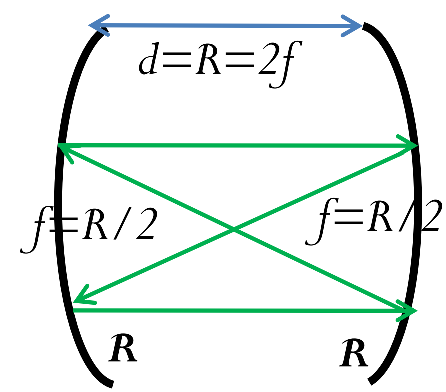
$$r_s = Ae^{(\alpha+)s} + Be^{(\alpha-)s}$$

r_s随s增加而发散

相同透镜构成的波导：$f_1=f_2=f$

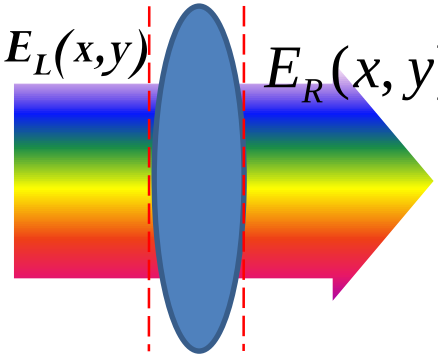
光线稳定条件简化为 $0 \le d \le 4f$

第n个透镜处的光线半径：$r_n = r_m \sin(\theta n + \delta)$



3.类透镜介质中光线的传播

透镜的物理性质：*对透过的波面产生二次位相弯曲* **(聚焦或发散作用)**



$$E_R(x,y)=E_L(x,y)e^{ik\frac{x^2+y^2}{2f}}$$

$$n(x,y)=n_0[1-\frac{k_2}{2k}(x^2+y^2)]$$

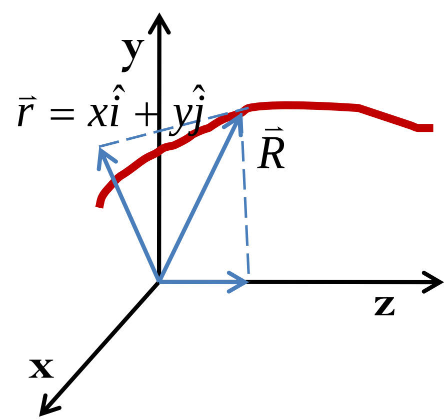
$k_2>0$ 会聚

$k_2<0$ 发散

类透镜介质： 通过二次型折射率分布实现类似透镜的功能

非均匀介质中的光线方程(见《光学原理》)

$$\frac{d}{ds}(n\frac{d\vec{R}}{ds})=\nabla n$$



代入上面类透镜介质的折射率公式，并且只考虑傍轴光线的情况下$(ds=dz)$，

$$\frac{d^2r}{dz^2}+(\frac{k_2}{k})r=0 \quad 这里, r=|\vec{r}|=|x\hat{i}+y\hat{j}|$$

求解方程 $\frac{d^2r}{dz^2}+(\frac{k_2}{k})r=0$　　假设 $k_2>0$

$$r(z)=c_1\cos\sqrt{\frac{k_2}{k}}z+c_2\sin\sqrt{\frac{k_2}{k}}z$$

c_1，c_2由$z=0$处的初始光线参数(r_0,r'_0)决定

$$r_0=r(0)=c_1 \qquad c_2=\sqrt{\frac{k}{k_2}}r_0'$$

$$r(z)=r_0\cos\sqrt{\frac{k_2}{k}}z+r_0'\sqrt{\frac{k}{k_2}}\sin\sqrt{\frac{k_2}{k}}z$$

$$r'(z)=-r_0\sqrt{\frac{k_2}{k}}\sin\sqrt{\frac{k_2}{k}}z+r_0'\cos\sqrt{\frac{k_2}{k}}z$$

光线在其中的行进的轨迹作正弦形振荡，振荡的周期l_p为：

$$l_p=2\pi\sqrt{\frac{k}{k_2}}$$

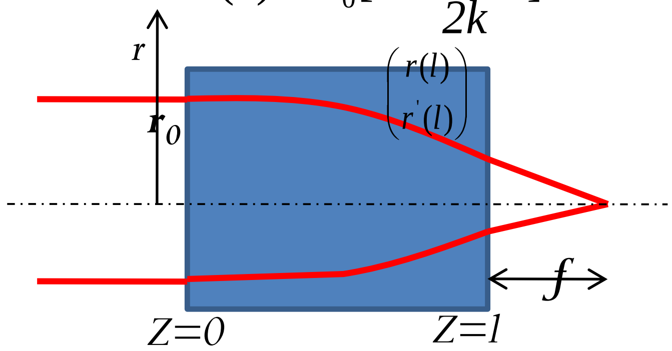
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} \cos\sqrt{\frac{k_2}{k}}z & \sqrt{\frac{k}{k_2}}\sin\sqrt{\frac{k_2}{k}}z \\ -\sqrt{\frac{k_2}{k}}\sin\sqrt{\frac{k_2}{k}}z & \cos\sqrt{\frac{k_2}{k}}z \end{pmatrix}$$

满足，$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}=1$

自聚焦棒，梯度折射率光纤

平行光垂直入射到长度为l的自聚焦棒的端面，则z=0处的光线参数为 $\begin{pmatrix} r_0 \\ 0 \end{pmatrix}$

$$n(r) = n_0[1 - \frac{k_2}{2k} r^2]$$



$$\begin{pmatrix} r(l) \\ r'(l) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_0 \\ 0 \end{pmatrix} = \begin{pmatrix} Ar_0 \\ Cr_0 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{\frac{k_2}{k}} l \\ -\sqrt{\frac{k_2}{k}} \sin\sqrt{\frac{k_2}{k}} l \end{pmatrix} r_0$$

如果焦点在介质内，则焦距为

$$f = \left|\frac{r}{r'}\right| = \sqrt{\frac{k}{k_2 k}} ctg\sqrt{\frac{k_2}{k}} l$$

如果焦点空气中，$r'_{(l)外} = n_0 r'_{(l)内}$

$$h = \left|\frac{r}{r'_{外}}\right| = \frac{1}{n_0} \sqrt{\frac{k}{k_2 k}} ctg\sqrt{\frac{k_2}{k}} l$$

二次型折射率分布的物理原因：

- 热效应
- 离子交换掺杂：梯度折射率光纤、波导
- 光Kerr效应

求解方程 $\frac{d^2r}{dz^2}+(\frac{k_2}{k})r=0$ 假设 $k_2<0$ $\frac{d^2r}{dz^2}-(\frac{|k_2|}{k})r=0$

$$r(z)=c_1e^{\sqrt{\frac{|k_2|}{k}}z}+c_2e^{-\sqrt{\frac{|k_2|}{k}}z}$$

c_1，c_2由$z=0$处的初始光线参数(r_0,r'_0)决定

$$c_1=\frac{r_0}{2}+\frac{1}{2}\sqrt{\frac{k}{k_2}}r_0' \qquad c_2=\frac{r_0}{2}-\frac{1}{2}\sqrt{\frac{k}{k_2}}r_0'$$

$$r(z)=r_0\cosh\sqrt{\frac{k_2}{k}}z+r_0'\sqrt{\frac{k}{k_2}}\sinh\sqrt{\frac{k_2}{k}}z$$

$$r'(z)=r_0\sqrt{\frac{k_2}{k}}\sinh\sqrt{\frac{k_2}{k}}z+r_0'\cosh\sqrt{\frac{k_2}{k}}z$$

假设平行光垂直入射$(r_0，0)$，

$$r(z)=r_0e^{\sqrt{\frac{|k_2|}{k}}z}$$

光线发散。该介质类负透镜

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}\cosh\sqrt{\frac{k_2}{k}}z & \sqrt{\frac{k}{k_2}}\sinh\sqrt{\frac{k_2}{k}}z\\ \sqrt{\frac{k_2}{k}}\sinh\sqrt{\frac{k_2}{k}}z & \cosh\sqrt{\frac{k_2}{k}}z\end{pmatrix}$$

4 .(二次型)折射率变化介质中的波动方程

Maxwell方程、波动方程与Helmhotz方程

Maxwell方程:

$$\begin{cases} \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{H} = \dfrac{\partial\vec{D}}{\partial t} + \vec{j} \\ \nabla\cdot\vec{D} = 0 \\ \nabla\cdot\vec{B} = 0 \end{cases}$$

物质方程:

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0(1+\chi)\vec{E}$$

$$= \varepsilon_0\varepsilon_r\vec{E} = \varepsilon\vec{E}$$

$$\vec{B} = \mu_0\vec{H}$$ ，*非磁材料*

$$\vec{j} = \sigma\vec{E}$$

$$\nabla\times\nabla\times\vec{E} = -\mu_0\frac{\partial}{\partial t}(\nabla\times\vec{H})$$

$$= -\mu_0\frac{\partial}{\partial t}\vec{j} - \mu_0\frac{\partial^2}{\partial t^2}\vec{D}$$

$$= -\mu_0\sigma\frac{\partial}{\partial t}\vec{E} - \mu_0\varepsilon\frac{\partial^2}{\partial t^2}\vec{E}$$

$$\nabla\times\nabla\times\vec{E} = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$$

$$= -\nabla^2\vec{E}$$

均匀介质 $\varepsilon = const.$

$$\nabla\cdot\vec{E} = 0$$

非均匀介质，即 $\varepsilon = \varepsilon(r)$

$$\nabla\cdot\vec{D} = \nabla\cdot(\varepsilon(r)\vec{E})$$

$$= \varepsilon(r)\nabla\cdot\vec{E} + \vec{E}\nabla\cdot\varepsilon(r)$$

$$\Downarrow$$

$$\nabla\cdot\vec{E} = -\frac{1}{\varepsilon(r)}\vec{E}\nabla\cdot\varepsilon(r)$$

慢变近似 $\nabla\cdot\varepsilon(r) \approx 0$

$$\nabla^2\vec{E} = \mu_0\sigma\frac{\partial}{\partial t}\vec{E} + \mu_0\varepsilon\frac{\partial^2}{\partial t^2}\vec{E}$$

$$\nabla^2 \vec{E} = \mu_0 \sigma \frac{\partial}{\partial t} \vec{E} + \mu_0 \varepsilon \frac{\partial^2}{\partial t^2} \vec{E}$$

假定单色谐波： $\vec{E}(x,y,z,t) = \mathrm{Re}[\vec{E}(x,y,z)e^{i\omega t}]$

$$\nabla^2 \vec{E}(x,y,z) - i\mu_0 \sigma \omega \vec{E}(x,y,z) + \mu_0 \varepsilon \omega^2 \vec{E}(x,y,z) = 0$$

记， $k^2 = \mu_0 \varepsilon \omega^2 (1 - i\frac{\sigma}{\varepsilon\omega}) = \mu_0 \omega^2 \tilde{\varepsilon}$

$\nabla^2 \vec{E}(x,y,z) + k^2 \vec{E}(x,y,z) = 0$ ，*Helmhotz方程*

非均匀介质，在慢变近似情况下，Helmhotz方程的形式保持不变，

$$\nabla^2 \vec{E}(x,y,z) + k(r)^2 \vec{E}(x,y,z) = 0$$

但$k(r)$是空间坐标的函数： $k(r)^2 = \mu_0 \varepsilon(r) \omega^2 (1 - i\frac{\sigma(r)}{\varepsilon\omega})$

σ>0，吸收介质； σ<0，增益介质

二次型折射率变化介质：$n(r)=n_0(1-\frac{k_2}{2k}r^2)$ 具有轴对称性！！

$$k^2(r)=[\frac{2\pi}{\lambda}n(r)]^2=(\frac{2\pi}{\lambda}n_0)^2[1-\frac{k_2}{k}r^2+(\frac{k_2}{2k})^2r^4]$$

$$\approx(\frac{2\pi}{\lambda}n_0)^2[1-\frac{k_2}{k}r^2]=k^2-kk_2r^2$$

考虑吸收，则 $k^2=k(0)^2=\mu_0\varepsilon(0)\omega^2(1-i\frac{\sigma(0)}{\varepsilon(0)\omega})$

求解Helmhotz方程(对二次型折射率变化介质或轴对称空间)

在轴对称性假设下，我们只考虑沿z方向传播的细光束！！

$$E(x,y,z)=\psi(x,y,z)e^{-ikz}=\psi(r,z)e^{-ikz}$$

代入Helmhotz方程：$\nabla^2\vec{E}(x,y,z)+k(r)^2\vec{E}(x,y,z)=0$

$$\nabla^2=\nabla_\perp^2+\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

$$\nabla_\perp^2\psi(r,z)e^{-ikz}+\frac{\partial^2}{\partial z^2}[\psi(r,z)e^{-ikz}]+k(r)^2\psi(r,z)e^{-ikz}=0$$

$$\frac{\partial^2}{\partial z^2}[\psi(r,z)e^{-ikz}] = \frac{\partial^2\psi}{\partial z^2}e^{-ikz} - 2ik\frac{\partial\psi}{\partial z}e^{-ikz} - k^2\psi e^{-ikz}$$

慢变近似： $\frac{\partial^2\psi}{\partial z^2} << ik\frac{\partial\psi}{\partial z}$

Helmhotz方程简化为：

$$\nabla_\perp^2\psi(r,z) - 2ik\frac{\partial\psi(r,z)}{\partial z} - k^2\psi(r,z) + k(r)^2\psi(r,z) = 0$$

对二次型折射率变化介质 $k^2(r) = k^2 - kk_2r^2$

$$\nabla_\perp^2\psi(r,z) - 2ik\frac{\partial\psi(r,z)}{\partial z} - kk_2r^2\psi(r,z) = 0$$

为求解上述方程，假定形式解： $\psi(r,z) = e^{-iP(z)}e^{-i\frac{1}{2}Q(z)r^2}$

$$\nabla_\perp^2\psi(r,z) = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})\psi(r,z) = e^{-iP(z)}e^{-i\frac{Q(z)}{2}r^2}[-2iQ(z) - Q^2(z)r^2]$$

$$\frac{\partial\psi(r,z)}{\partial z} = e^{-iP(z)}e^{-i\frac{Q(z)}{2}r^2}[-i\frac{\partial P(z)}{\partial z} - i\frac{r^2}{2}\frac{\partial Q(z)}{\partial z}]$$

$$r^2[-Q^2(z) - k\frac{\partial Q(z)}{\partial z} - kk_2] - [2k\frac{\partial P(z)}{\partial z} + 2iQ(z)] = 0$$

for all “r”

So, [. . .]=0 [. . .]=0

$$\begin{cases} Q^2(z) + k\dfrac{\partial Q(z)}{\partial z} + kk_2 = 0 \\ k\dfrac{\partial P(z)}{\partial z} + iQ(z) = 0 \end{cases}$$

考虑均匀介质中的情形（高斯光束）：$k_2=0$

$$Q^2(z) + k\frac{\partial Q(z)}{\partial z} = 0$$

$$\frac{\partial Q(z)}{\partial z} = -\frac{1}{k}Q^2(z)$$

$$\frac{1}{Q(z)} - \frac{1}{Q(0)} = \frac{z}{k}$$

$$\frac{k}{Q(z)} = \frac{k}{Q(0)} + z$$

令，$q(z) = \dfrac{k}{Q(z)}$

则，$q(z) = q_0 + z$

$$k\frac{\partial P(z)}{\partial z} + iQ(z) = 0$$

$$\frac{\partial P(z)}{\partial z} = -i\frac{1}{q(z)} = -i\frac{1}{q_0 + z}$$

$$P(z) = -i\int_0^z \frac{1}{q_0 + z} + C = -i\ln(q_0 + z)\Big|_0^z = -i\ln(1 + \frac{z}{q_0})$$

通过平移时间原点可以忽略！

$$\psi(r,z) = e^{-iP(z)} e^{-i\frac{1}{2}Q(z)r^2} = e^{-\ln(1+\frac{z}{q_0})} e^{-i\frac{1}{2}\frac{kr^2}{q(z)}}$$

$$= \frac{q_0}{z + q_0} e^{-i\frac{1}{2}\frac{kr^2}{q(z)}}$$

q_0参数待定

q_0为纯虚数

考察 z=0 的位置： $\psi(r,0)=e^{-i\frac{1}{2}\frac{kr^2}{q_0}}$

如果q_0=*Real*，

球面波位相，常数振幅。违背能量守恒！

因为，$q(z)=q_0+z$ 所以适当选择z轴的坐标原点，可以使q_0为纯虚数

记，$q_0=iz_0$

z=0平面： $\psi(r,0)=e^{-\frac{kr^2}{2z_0}}\equiv e^{-\frac{r^2}{w_0^2}}$

高斯型振幅分布！其中w_0是光斑半径

$$\frac{k}{2z_0}=\frac{1}{w_0^2}\Rightarrow z_0=\frac{1}{2}kw_0^2=\frac{n\pi w_0^2}{\lambda}$$

一般地：

$$\psi(r,z)=\frac{q_0}{z+q_0}e^{-i\frac{1}{2}\frac{kr^2}{q(z)}}=\frac{iz_0}{z+iz_0}e^{-i\frac{1}{2}\frac{kr^2}{z+iz_0}}=\frac{1}{\sqrt{1+(\frac{z}{z_0})^2}}e^{i\tan^{-1}(\frac{z}{z_0})}e^{\frac{-kr^2}{2z_0[(\frac{z}{z_0})^2+1]}}e^{\frac{-ikr^2}{2z[1+(\frac{z_0}{z})^2]}}$$

$$E(r,z)=\psi(r,z)e^{-ikz}=\frac{1}{\sqrt{1+(\frac{z}{z_0})^2}}e^{i\tan^{-1}(\frac{z}{z_0})}e^{-ikz}e^{\frac{-kr^2}{2z_0[(\frac{z}{z_0})^2+1]}}e^{\frac{-ikr^2}{2z[1+(\frac{z_0}{z})^2]}}$$

波前上振幅作高斯分布

$$\equiv E_0\frac{w_0}{w(z)}e^{-i[kz-\phi(z)]}e^{\frac{-r^2}{w^2(z)}}e^{\frac{-ikr^2}{2R(z)}}\equiv E_0\frac{w_0}{w(z)}e^{-i[kz-\phi(z)]}e^{\frac{-ikr^2}{2q(z)}}$$

传播相移

光束中心振幅

波前上相位类球面波分布

定义几个参数

$$w_{(z)}^2=w_0^2[1+(\frac{z}{z_0})^2]=w_0^2[1+(\frac{\lambda z}{n\pi w_0^2})^2]$$

$$R(z)=z[1+(\frac{z_0}{z})^2]=z[1+(\frac{n\pi w_0^2}{\lambda z})^2]$$

$$\phi(z)=\tan^{-1}(\frac{z}{z_0})$$

q参数

$$\frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{2}{kw_{(z)}^2}$$

$$=\frac{1}{R(z)}-i\frac{\lambda}{n\pi w_{(z)}^2}$$

高斯光波和平面波、球面波一样都是Maxwell方程的本征解

基模高斯光束的特性：

（1）波前相位特征（与球面波比较）：

沿z轴传播的傍轴球面波

$$\frac{A}{R}e^{-ikR}=\frac{A}{R}e^{-ik\sqrt{x^2+y^2+z^2}}\approx\frac{A}{R}e^{-ikz(1+\frac{x^2+y^2}{2z^2})}=\frac{A}{R}e^{-ikz}e^{-i\frac{kr^2}{2z}}$$

$$R=\sqrt{x^2+y^2+z^2};\quad r^2=x^2+y^2$$

高斯光束

$$Exp\{-ikz-i\frac{kr^2}{2R(z)}+i\phi(z)\}$$

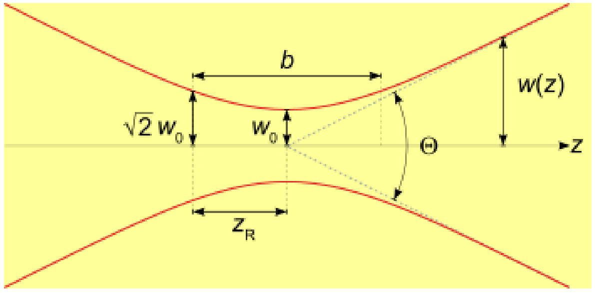
①高斯光束有附加相移$\phi(z)$

②除此之外，波面与球面波的情况类似，但球面波的波面曲率半径为z，高斯波的曲率半径为

$$R(z)=z[1+(\frac{z_0}{z})^2]=z[1+(\frac{n\pi w_0^2}{\lambda z})^2]$$

波面特征:

波面位置	Z=0	Z=Z_0	Z =∞
曲率半径	R=∞	R=2Z_0	R=∞
曲率中心	-∞	-Z_0	0



（2）振幅特征： $E_0\dfrac{w_0}{w(z)}e^{\frac{-r^2}{w^2(z)}}$

① 横截面内的振幅作高斯分布，中心振幅是 $E_0\dfrac{w_0}{w(z)}$

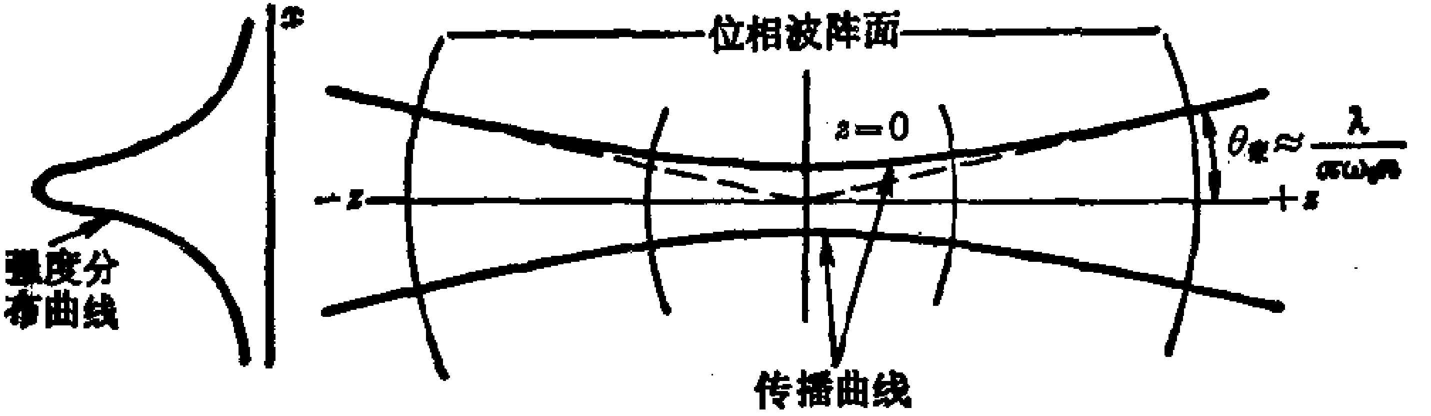
② 不同横截面的中心振幅随光斑半径增大而减小

③ 横截面内光斑半径为 $w(z)=w_0[1+(\dfrac{z}{z_0})^2]^{\frac{1}{2}}$

④ $z=0$处的光斑半径最小为w_0，被称为束腰半径

⑤ $w(z_0)=\sqrt{2}w_0$ ，z_0被称为瑞利距离

⑥ 高斯光束的半径从$z=0$向$\pm\infty$发散

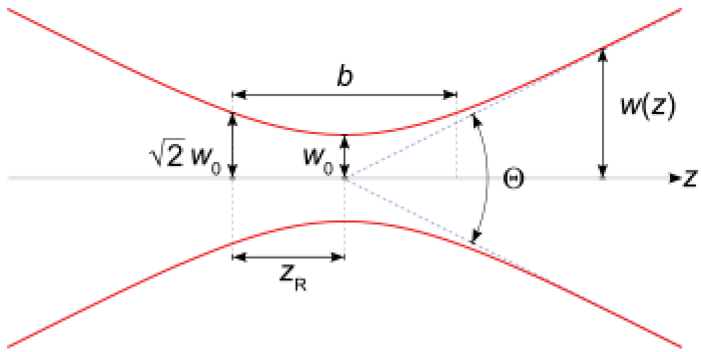


（3）高斯光束的发散特性：

定义高斯光束的发散角：

$$2\theta \equiv 2\frac{dw(z)}{dz} = 2w_0 \frac{z}{z_0^2} \Big/ [1+(\frac{z}{z_0})^2]^{\frac{1}{2}}$$

$$= 2\frac{w_0}{z_0} \Big/ [1+(\frac{z_0}{z})^2]^{\frac{1}{2}}$$



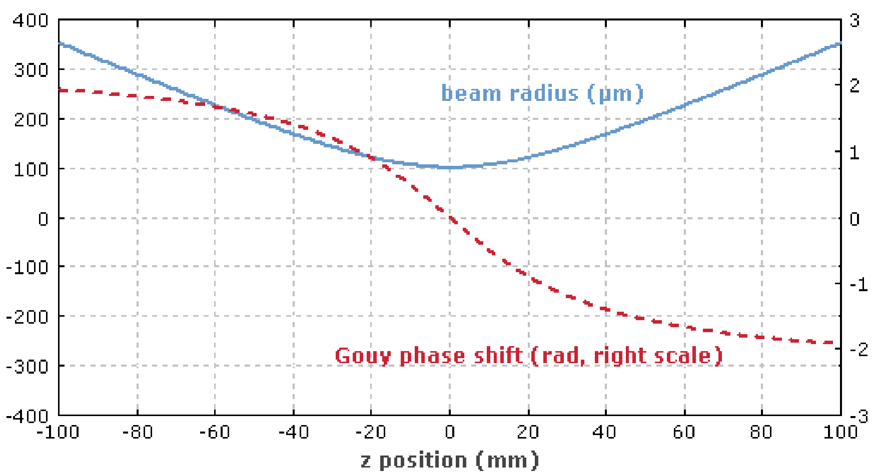
远场发散角：

$$2\theta \equiv \lim_{z\to\infty} it\, 2\frac{dw(z)}{dz} = 2\frac{w_0}{z_0} = 2\frac{\lambda}{n\pi w_0}$$

（4）附加相移——Gouy Phase Shift

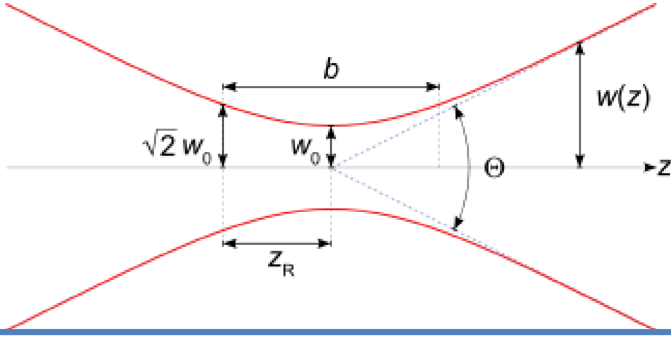
$$\phi(z) = \tan^{-1}(\frac{z}{z_0})$$

$$\phi(\infty) - \phi(-\infty) = \pi$$



高斯光束的特征参数—可确定高斯光束特性的基本参数

（1）束腰w_0或共焦参数z_0



$$w_{(z)}^2 = w_0^2[1+(\frac{z}{z_0})^2] = w_0^2[1+(\frac{\lambda z}{n\pi w_0^2})^2]$$

$$R(z) = z[1+(\frac{z_0}{z})^2] = z[1+(\frac{n\pi w_0^2}{\lambda z})^2]$$

$$\phi(z) = \tan^{-1}(\frac{z}{z_0})$$

束腰**半径**w_0和束腰的**位置**(z=0的确定)确定后，高斯光束就完全确定下来了

（2）任意位置的光斑半径$W(z)$和波面曲率半径$R(z)$

问题：从下面方程组中求解Z和W_0　　其中，$z_0 = \frac{1}{2}kw_0^2 = \frac{n\pi w_0^2}{\lambda}$

$$\begin{cases} w_{(z)}^2 = w_0^2[1+(\frac{z}{z_0})^2] \\ R(z) = z[1+(\frac{z_0}{z})^2] \end{cases}$$

$$\Rightarrow w_0^2 = w_{(z)}^2[1+(z/z_0)^2]^{-1} = w_{(z)}^2[1+(n\pi w_{(z)}^2/\lambda R(z))^2]^{-1}$$

$$\Rightarrow z = R(z)[1+(z_0/z)^2]^{-1} = R(z)[1+(\lambda R(z)/n\pi w_{(z)}^2)^2]^{-1}$$

$$\frac{w_{(z)}^2}{R(z)} = \frac{w_0^2}{z}\frac{[1+(z/z_0)^2]}{[1+(z_0/z)^2]} = \frac{2}{k}\frac{z}{z_0} \Rightarrow \frac{z}{z_0} = \frac{k}{2}\frac{w_{(z)}^2}{R(z)} = \frac{n\pi w_{(z)}^2}{\lambda R(z)}$$

（3）高斯光束的q参数----复参数

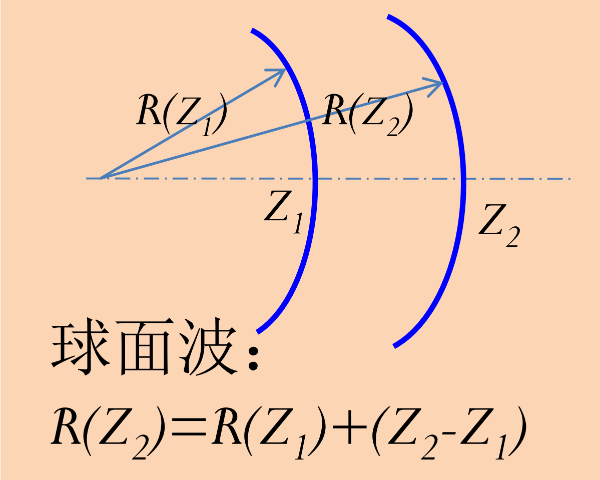
利用q参数，高斯光束遵从与球面波相同的传播规律

$$Define:\ \frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{2}{kw_{(z)}^2}=\frac{1}{R(z)}-i\frac{\lambda}{n\pi w_{(z)}^2}$$

①高斯光束在自由空间传播：

$$q(z)=q_0+z$$

$$z_1\rightarrow z_2:q(z_2)=q(z_1)+z_2-z_1$$



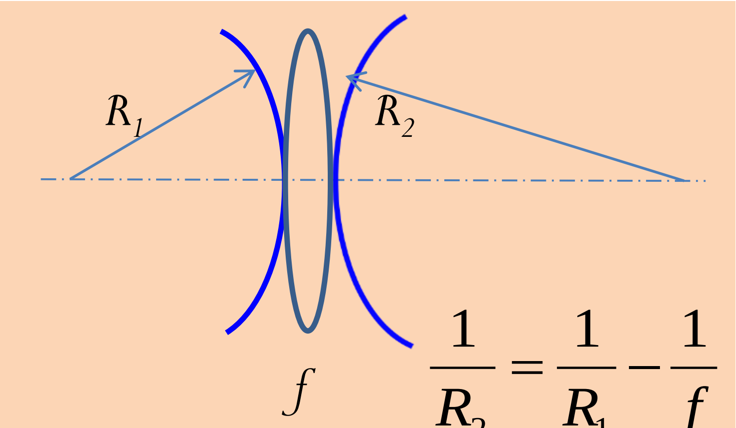
② 高斯光束经过透镜变换：

薄透镜的作用： $\exp[i\frac{kr^2}{2f}]$

对高斯光束：

$$e^{-i\frac{kr^2}{2R_1}}\rightarrow e^{-i\frac{kr^2}{2R_2}}=e^{-i\frac{kr^2}{2R_1}}\cdot e^{i\frac{kr^2}{2f}}\Rightarrow\frac{1}{R_2}=\frac{1}{R_1}-\frac{1}{f}$$

又， $w_2=w_1$ $\therefore\ \frac{1}{q_2}=\frac{1}{q_1}-\frac{1}{f}$

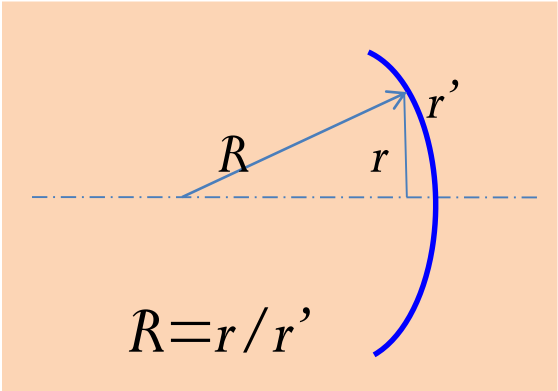


$$\frac{1}{R_2}=\frac{1}{R_1}-\frac{1}{f}$$

高斯光束传播的ABCD定律

用光线矩阵描述球面波传播问题:

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix} \Leftrightarrow \begin{cases} r_2 = Ar_1 + Br_1' \\ r_2' = Cr_1 + Dr_1' \end{cases}$$



$$R_2 = \frac{r_2}{r_2'} = \frac{Ar_1 + Br_1'}{Cr_1 + Dr_1'} = \frac{A\, r_1/r_1' + B}{C\, r_1/r_1' + D} = \frac{AR_1 + B}{CR_1 + D}$$

类比知高斯光束传播的ABCD规律: $q_2 = \dfrac{Aq_1 + B}{Cq_1 + D}$

自由空间传播, $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ $\quad q_2 = \dfrac{1\cdot q_1 + z}{0\cdot q_1 + 1} = q_1 + z$

透镜变换, $\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$ $\quad q_2 = \dfrac{1\cdot q_1 + 0}{-1/f\cdot q_1 + 1} \Rightarrow \dfrac{1}{q_2} = \dfrac{1}{q_1} - \dfrac{1}{f}$

N各光学元件, $\begin{pmatrix} A_T & B_T \\ C_T & D_T \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \cdots \begin{pmatrix} A_N & B_N \\ C_N & D_N \end{pmatrix}$ $\quad q_{N+1} = \dfrac{A_T q_1 + B_T}{C_T q_1 + D_T}$

类透镜介质中的高斯光束

类透镜介质，$k^2(r)=k^2-kk_2r^2$

$$\begin{cases} Q^2(z)+k\dfrac{\partial Q(z)}{\partial z}+kk_2=0 \qquad (*) \\ k\dfrac{\partial P(z)}{\partial z}+iQ(z)=0 \end{cases}$$

令，$Q(z)=k\dfrac{s'}{s}$

$$(k\frac{s'}{s})^2+k\frac{ks''}{s}-k\frac{ks's'}{s^2}+kk_2=0$$

$$\Rightarrow s''+\frac{k_2}{k}s=0$$

$$s=a\sin\sqrt{\frac{k_2}{k}}z+b\cos\sqrt{\frac{k_2}{k}}z$$

$$s'=a\sqrt{\frac{k_2}{k}}\cos\sqrt{\frac{k_2}{k}}z-b\sqrt{\frac{k_2}{k}}\sin\sqrt{\frac{k_2}{k}}z$$

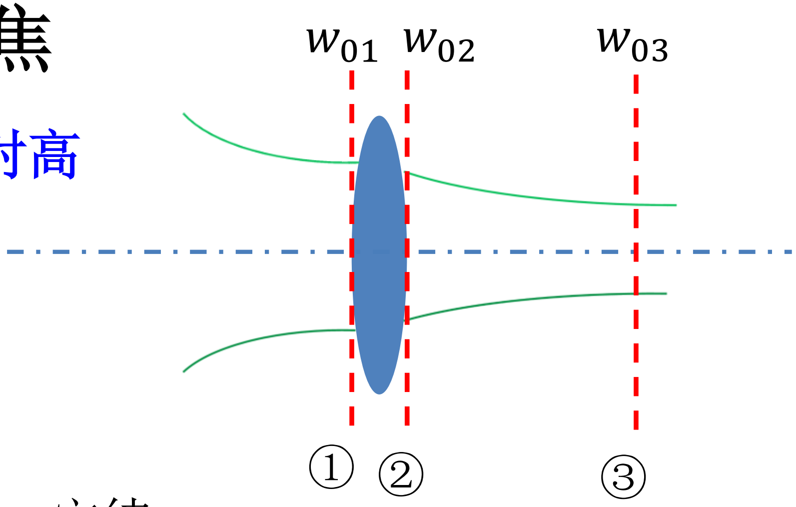
$$q(z)=\frac{k}{Q(z)}=\frac{s}{s'}=\frac{a\sin\sqrt{\frac{k_2}{k}}z+b\cos\sqrt{\frac{k_2}{k}}z}{a\sqrt{\frac{k_2}{k}}\cos\sqrt{\frac{k_2}{k}}z-b\sqrt{\frac{k_2}{k}}\sin\sqrt{\frac{k_2}{k}}z}$$

记，$q_0=q(0)=\dfrac{b}{a}\sqrt{\dfrac{k}{k_2}}$

$$q(z)=\frac{q_0\cos\sqrt{\frac{k_2}{k}}z+\sqrt{\frac{k}{k_2}}\sin\sqrt{\frac{k_2}{k}}z}{-q_0\sqrt{\frac{k_2}{k}}\sin\sqrt{\frac{k_2}{k}}z+\cos\sqrt{\frac{k_2}{k}}z}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} \cos\sqrt{\frac{k_2}{k}}z & \sqrt{\frac{k}{k_2}}\sin\sqrt{\frac{k_2}{k}}z \\ -\sqrt{\frac{k_2}{k}}\sin\sqrt{\frac{k_2}{k}}z & \cos\sqrt{\frac{k_2}{k}}z \end{pmatrix}$$

举例：高斯光束经过透镜聚焦



问题：高斯光的束腰入射到透镜上，求出射高斯光束的束腰位置和半径？

①位置是入射高斯光束的束腰位置，$R_1 = \infty$

所以，$$q_1 = iz_{01} = i\frac{n\pi w_{01}^2}{\lambda}$$

① → ② → ③的光线矩阵为：

$$\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1-l/f & l \\ -1/f & 1 \end{pmatrix}$$

由ABCD定律：

$$q_3 = \frac{(1-l/f)\cdot q_1 + l}{-1/f\cdot q_1 + 1} = \frac{(1-l/f)\cdot iz_{01} + l}{-1/f\cdot iz_{01} + 1}$$

③是出射高斯光束的束腰位置，所以$R_3 = \infty$

$$0 = \frac{1}{R_3} = \mathrm{Re}[\frac{1}{q_3}] = \frac{l - 1/f\cdot(1-l/f)z_{01}^2}{[(1-l/f)\cdot z_{01}]^2 + l^2} \Rightarrow l - 1/f\cdot(1-l/f)z_{01}^2 = 0$$

$$\Rightarrow l = \frac{z_{01}^2/f}{1 + z_{01}^2/f^2} = \frac{f}{1 + f^2/z_{01}^2}$$

由$\mathrm{Im}[1/q_3]$可以计算出束腰半径：

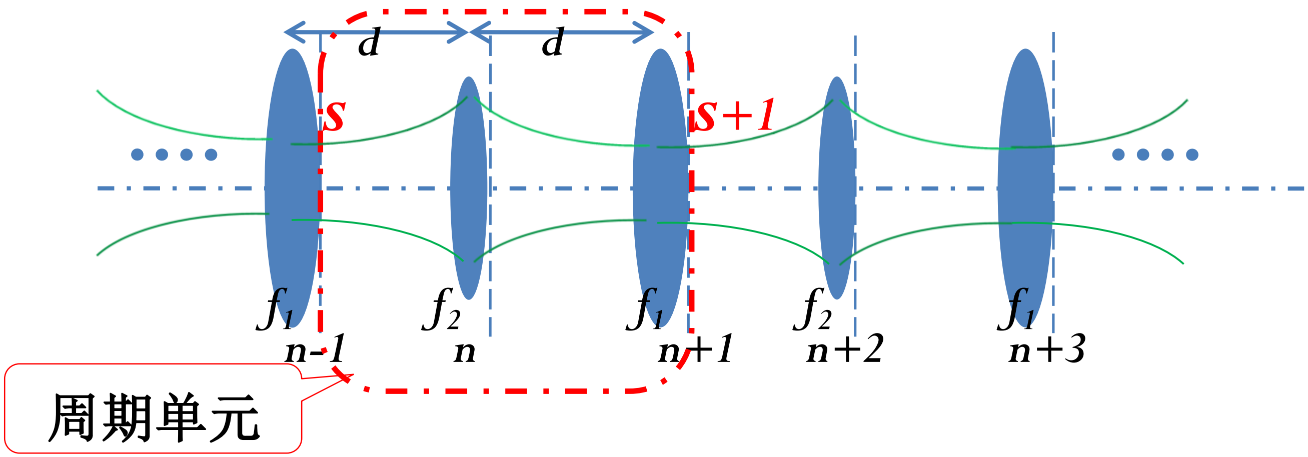
$$\mathrm{Im}[\frac{1}{q_3}]=\frac{-z_{01}}{[(1-l/f)\cdot z_{01}]^2+l^2}=-\frac{\lambda}{n\pi w_{03}^2}\Rightarrow\frac{n\pi w_{03}^2}{\lambda}=\frac{f^2/z_{01}}{1+f^2/z_{01}^2}\Rightarrow w_{03}^2=\frac{\lambda}{n\pi}\frac{f^2/z_{01}}{1+f^2/z_{01}^2}$$

$$\Rightarrow\frac{w_{03}^2}{w_{01}^2}=\frac{f^2/z_{01}^2}{1+f^2/z_{01}^2}\Rightarrow\frac{w_{03}}{w_{01}}=\frac{f/z_{01}}{\sqrt{1+f^2/z_{01}^2}}<f/z_{01}=\frac{\lambda f}{\pi w_{01}^2}$$

与平面波聚焦作比较：

	平面波	高斯光束
焦点位置	f	$l=\frac{f}{1+f^2/z_{01}^2}<f$
焦斑大小	$1.22\frac{\lambda f}{D}$，$Airy$斑	$w_{03}=\frac{\lambda f/\pi w_{01}}{\sqrt{1+f^2/z_{01}^2}}\Big\vert_{z_{01}>>f}\approx\frac{\lambda f}{\pi w_{01}}$ $=\frac{2}{\pi}\frac{\lambda f}{2w_{01}}<Airy$斑

透镜波导中的高斯光束



光场经过一个周期单元后可以再现，所以：

$$q_{s+1} = \frac{Aq_s + B}{Cq_s + D} = q_s \Rightarrow Cq_s^2 + Dq_s - Aq_s - B = 0$$

$$\Rightarrow Cq_s^2 + (D-A)q_s - B = 0$$

光场在透镜波导中受约束，则光束半径必须为有限值，即q_s的虚部不为0

所以
$$\left.\begin{array}{r}(D-A)^2 + 4BC < 0 \\ \text{又，} AD - BC = 1\end{array}\right\} \Rightarrow (D+A)^2 - 4 < 0 \Rightarrow \left|\frac{D+A}{2}\right| < 1$$

光束稳定条件！

高斯光束的高阶模

基模高斯光束：

$$E(r,z)=E_0\frac{w_0}{w(z)}e^{-i[kz-\phi(z)]}e^{\frac{-r^2}{w^2(z)}}e^{\frac{-ikr^2}{2R(z)}}$$

高阶模高斯光束也是Helmhotz方程的解

$$\nabla^2\vec{E}(x,y,z)+k^2\vec{E}(x,y,z)=0$$

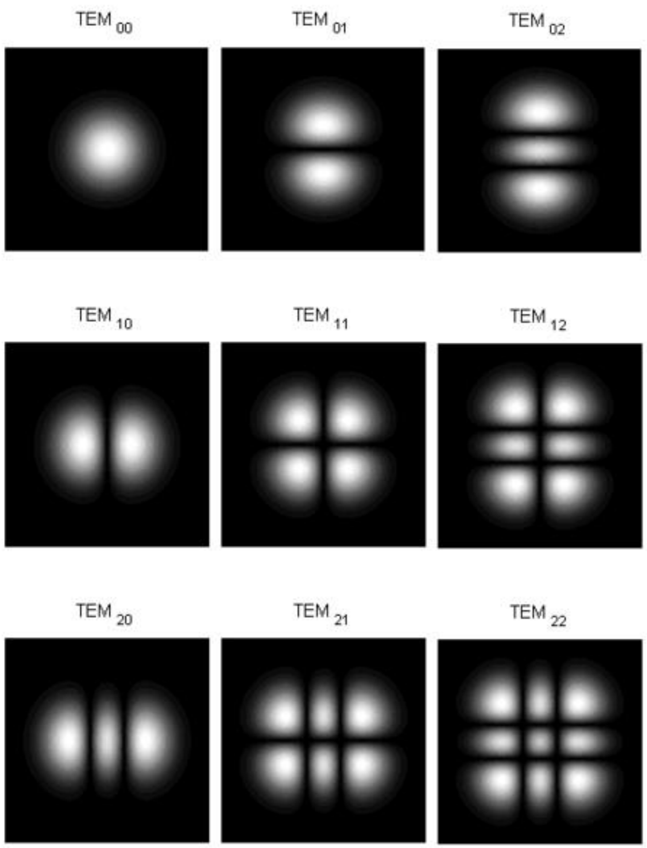
慢变振幅近似

$$E(x,y,z)=\psi(x,y,z)e^{-ikz}$$

$$\nabla_\perp^2\psi(x,y,z)-2ik\frac{\partial\psi(x,y,z)}{\partial z}=0$$

$$\psi(x,y,z)=\psi(r,z)=e^{-iP(z)}e^{-i\frac{1}{2}Q(z)r^2}$$

$$\psi(x,y,z)=f(x)g(y)e^{-iP(z)}e^{-i\frac{1}{2}Q(z)r^2}$$



$$E_{lm}(r,z)=E_0\frac{w_0}{w(z)}H_l(\sqrt{2}\frac{x}{w(z)})H_m(\sqrt{2}\frac{y}{w(z)})e^{-i[kz-(l+m+1)\varphi(z)]}e^{\frac{-ik(x^2+y^2)}{2q(z)}}$$

$$=E_0\frac{w_0}{w(z)}H_l(\sqrt{2}\frac{x}{w(z)})H_m(\sqrt{2}\frac{y}{w(z)})e^{\frac{-r^2}{w^2(z)}}e^{-ik[z+\frac{x^2+y^2}{2R(z)}]+i(l+m+1)\varphi(z)}$$

$$w_{(z)}^2 = w_0^2[1+(\frac{z}{z_0})^2] \qquad R(z) = z[1+(\frac{z_0}{z})^2] \qquad \phi(z) = \tan^{-1}(\frac{z}{z_0})$$

$$z_0 = \frac{n\pi w_0^2}{\lambda} \qquad \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi w_{(z)}^2}$$

$H_l(\sqrt{2}\frac{x}{w(z)})$、$H_m(\sqrt{2}\frac{y}{w(z)})$，厄密多项式；$l$、$m$是横模序数

$$H_0(x)=1 \quad H_1(x)=2x \quad H_2(x)=4x^2-2 \quad H_3(x)=8x^3-12x \quad \cdots$$

高阶模的特征

① 光斑半径$W(z)$以及波面的曲率半径$R(z)$与基模高斯光束相同

② 与基模相比较，振幅和附加相移不同

③ 附加相移与模序数有关：$\phi(l,m,z)=(l+m+1)\tan^{-1}(\frac{z}{z_0})$

④ 横截面光斑上有暗线：厄密多项式调制的高斯分布

$$E_0\frac{w_0}{w(z)}H_l(\sqrt{2}\frac{x}{w(z)})H_m(\sqrt{2}\frac{y}{w(z)})e^{\frac{-r^2}{w^2(z)}}$$

⑤ 高阶模的能量更分散，发散角更大

定义模斑半径（按光斑能量分布的加权平均）：

$$w^2 = 4\int_{-\infty}^{\infty} x^2 E^2(x)dx \Big/ \int_{-\infty}^{\infty} E^2(x)dx \quad \text{其中，} E(x) = H_m(\sqrt{2}\frac{x}{w_0})e^{-\frac{x^2}{w_0^2}}$$

归约化常数

照此定义：

基模：$w^2 = w_0^2$

1阶模：$w_{01}^2 = 3w_0^2 \Rightarrow w_{01} = \sqrt{3}w_0$

2阶模：$w_{02}^2 = 5w_0^2 \Rightarrow w_{01} = \sqrt{5}w_0$

m阶模：$w_{0m}^2 = (2m+1)w_0^2 \Rightarrow w_{01} = \sqrt{2m+1}w_0$

每个小模斑的平均半径：$w'_{01} = \dfrac{\sqrt{2m+1}}{m+1}w_0$

类透镜介质中的高斯光束高阶模

同样，解Helmhotz方程 $\nabla^2\vec{E}(x,y,z)+k_{(r)}^2\vec{E}(x,y,z)=0$

二次型折射率变化介质：$n(r)=n_0(1-\frac{k_2}{2k}r^2)$

$$n^2(r)=n_0^2(1-\frac{k_2}{2k}r^2)^2=n_0^2[1-\frac{k_2}{k}r^2+(\frac{k_2}{2k}r^2)^2]\approx n_0^2[1-\frac{k_2}{k}r^2]\equiv n_0^2(1-\frac{n_2}{n}r^2)$$

$$\nabla^2\vec{E}(x,y,z)+k^2(1-\frac{n_2}{n}r^2)\vec{E}(x,y,z)=0$$

由于高阶模的能量分散，各模斑沿Z方向的传播常数不同，故令

$$E(x,y,z)=\psi(x,y)e^{-i\beta z}$$

β对不同阶次的模斑取值不同！

分离变量 $\psi(x,y)=f(x)g(y)$ 可解得，

$$E(x,y,z)=E_0H_l(\sqrt{2}\frac{x}{w})H_m(\sqrt{2}\frac{y}{w})e^{\frac{-(x^2+y^2)}{w^2}}e^{-i\beta_{l,m}z}$$

其中，$\beta_{l,m}=k[1-\frac{2}{k}\sqrt{\frac{n_2}{n}}(l+m+1)]^{\frac{1}{2}}$

模式色散！！

模式色散所引起的群速度色散

通常，n_2很小，$\frac{2}{k}\sqrt{\frac{n_2}{n}}(l+m+1) << 1$

$$\beta_{l,m} = k[1-\frac{2}{k}\sqrt{\frac{n_2}{n}}(l+m+1)]^{\frac{1}{2}} = k[1-\frac{1}{k}\sqrt{\frac{n_2}{n}}(l+m+1)-\frac{1}{2}(\frac{1}{k}\sqrt{\frac{n_2}{n}}(l+m+1))^2]$$

$$= \frac{n\omega}{c} - \sqrt{\frac{n_2}{n}}(l+m+1) - \frac{n_2 c}{2\omega n^2}(l+m+1)^2$$

群速度： $(v_g)_{l,m} = \frac{d\omega}{d\beta_{l,m}} = \frac{1}{d\beta_{l,m}/d\omega} = \frac{1}{\frac{n}{c}+\frac{n_2 c}{2\omega^2 n^2}(l+m+1)^2}$

$$= \frac{c/n}{1+\frac{n_2 c^2}{2\omega^2 n^3}(l+m+1)^2} \approx \frac{c}{n}[1-\frac{n_2 c^2}{2\omega^2 n^3}(l+m+1)^2]$$

$$k = \frac{\omega n}{c}$$

群速度色散： $\frac{dv_g}{d\omega} = \frac{n_2 c^3}{\omega^3 n^4}(l+m+1)^2 = \frac{n_2}{k^3 n}(l+m+1)^2$

模式色散所对应的脉冲展宽

脉宽为τ的光脉冲在类透镜介质中传播L的距离，光场的模式为(l,m)

宽度为τ的脉冲对应的光谱宽度为 $\Delta\omega = \dfrac{1}{\pi\tau}$

单频光传播L距离所用的时间为 $t = \dfrac{L}{v_g}$

脉冲展宽： $\Delta t = \dfrac{dt}{d\omega}\Delta\omega = \left|\dfrac{d(L/v_g)}{d\omega}\right|\Delta\omega = \dfrac{L}{v_g^2}\dfrac{dv_g}{d\omega}\Delta\omega$

$$= \frac{L}{\left[\dfrac{c/n}{1+\dfrac{n_2 c^2}{2\omega^2 n^3}(l+m+1)^2}\right]^2}\frac{n_2}{k^3 n}(l+m+1)^2\frac{1}{\pi\tau}$$

$$= \frac{Lnn_2}{c^2 k^3 \pi\tau}\left[1+\frac{n_2/n}{2k^2}(l+m+1)^2\right]^2 \underline{(l+m+1)^2}$$

二次型增益变化介质中的高斯光束传播

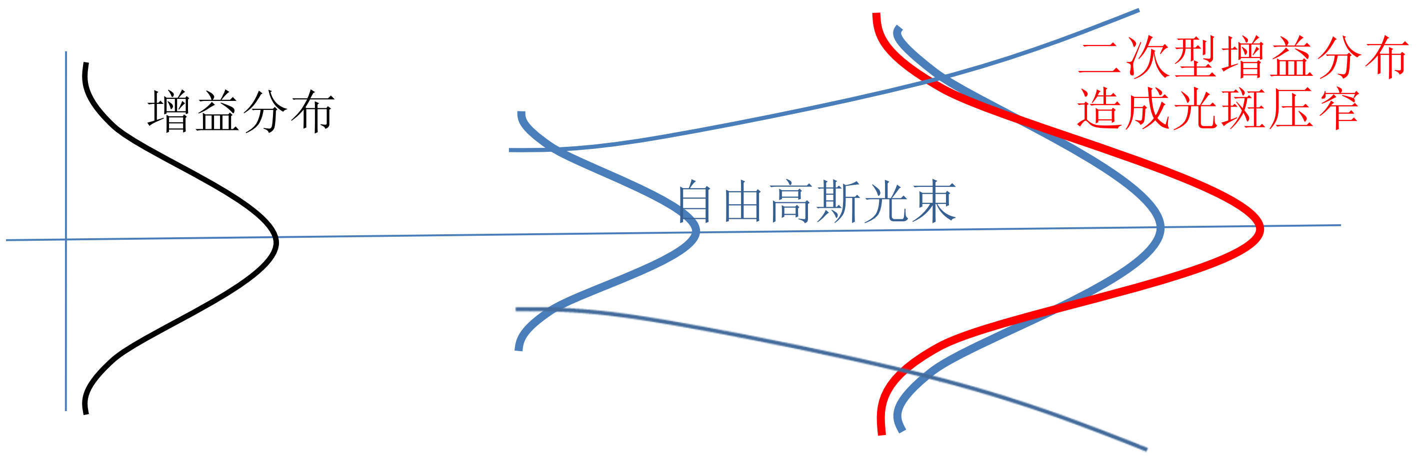
二次型增益分布的物理原因：

① 气体激光器中，高能等离子区电子的径向分布引起粒子数反转的径向分布

② 固体激光器中，泵浦光束光强的径向分布

③ 增益饱和引起的反转粒子数的径向分布

二次型增益变化介质中的高斯光束传播的**简单图像**



稳定光斑解

稳定光斑条件：$\dfrac{dQ(z)}{dz}=0$

$$\therefore\ Q^2(z)+k\frac{\partial Q(z)}{\partial z}+kk_2=0 \Rightarrow Q^2(z)+kk_2=0 \Rightarrow Q(z)=\sqrt{-kk_2}$$

二次型折射率变化介质：$k^2(r)=k^2-kk_2r^2 \Rightarrow k(r)=\sqrt{k^2-kk_2r^2}\approx k-\dfrac{k_2}{2}r^2$

对二次型吸收变化介质***k***为复数 $k(r)=k+i(\alpha_0-\dfrac{\alpha_2}{2}r^2)=k+i\alpha_0-i\dfrac{\alpha_2}{2}r^2$

类比可知，若 $k_2\rightarrow i\alpha_2$

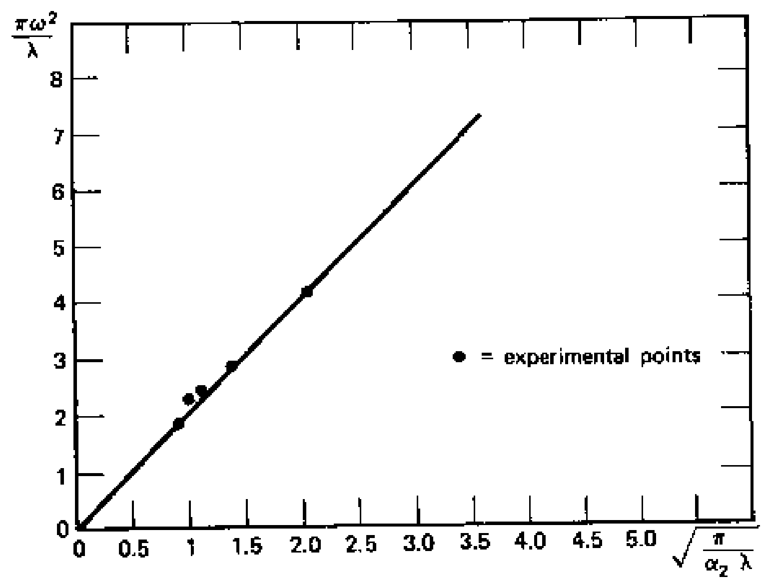
对二次型折射率变化介质的讨论可以完全移植到二次型吸收介质上。

所以，$\dfrac{1}{q(z)}=\dfrac{Q(z)}{k}=\dfrac{\sqrt{-kk_2}}{k}=\sqrt{-\dfrac{k_2}{k}}$

$$\rightarrow\sqrt{-\frac{i\alpha_2}{k}}=\frac{\sqrt{2}}{2}\sqrt{\frac{\alpha_2}{k}}(1-i)$$

$$\frac{1}{R}=\frac{\sqrt{2}}{2}\sqrt{\frac{\alpha_2}{k}}\Rightarrow R=\sqrt{\frac{2k}{\alpha_2}}=2\sqrt{\frac{\pi n}{\alpha_2\lambda}}$$

$$\frac{\lambda}{\pi n w^2}=\frac{\sqrt{2}}{2}\sqrt{\frac{\alpha_2}{k}}\Rightarrow w^2=\frac{\lambda}{\pi n}\sqrt{\frac{2k}{\alpha_2}}=2\sqrt{\frac{\lambda}{\pi n\alpha_2}}$$



高斯光束的高阶模、椭圆高斯光束

$$e^{\frac{-ikr^2}{2q(z)}} = e^{\frac{-ik(x^2+y^2)}{2q(z)}} \to e^{-ik[\frac{x^2}{2q_x(z)}+\frac{y^2}{2q_y(z)}]}$$

$$E(x,y,z) = E_0 \frac{\sqrt{w_{0x}w_{0y}}}{\sqrt{w_x(z)w_y(z)}} e^{-i[kz-\phi(z)]} e^{-ik[\frac{x^2}{2q_x(z)}+\frac{y^2}{2q_y(z)}]}$$

$$= E_0 \frac{\sqrt{w_{0x}w_{0y}}}{\sqrt{w_x(z)w_y(z)}} e^{-i[kz-\phi(z)]} e^{-x^2[\frac{1}{w_x^2(z)}-\frac{ik}{2R_x(z)}]} e^{-y^2[\frac{1}{w_y^2(z)}-\frac{ik}{2R_y(z)}]}$$

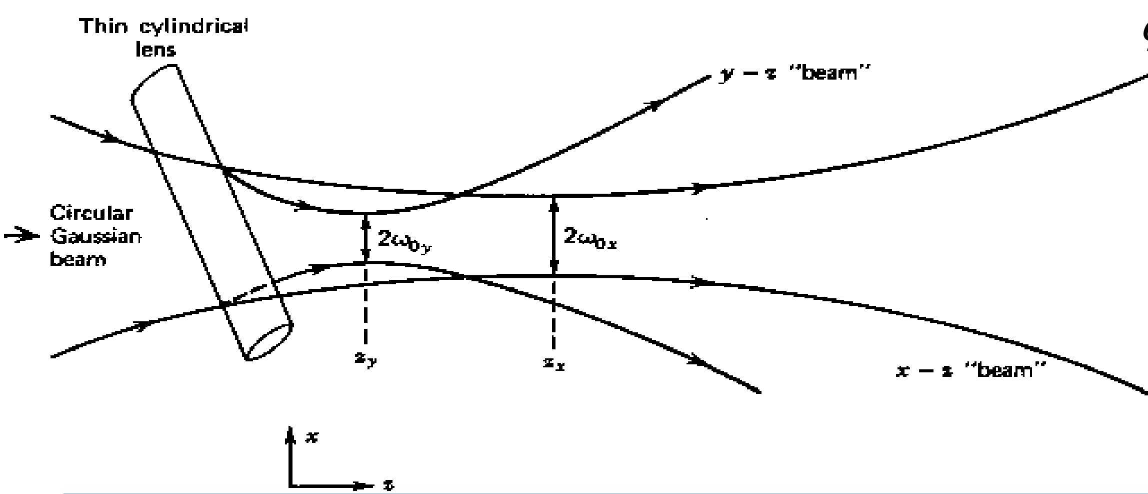
$$w_{(z)}^2 = w_{0x}^2[1+(\frac{\lambda(z-z_{0x})}{n\pi w_{0x}^2})^2]$$

$$R_x(z) = z[1+(\frac{n\pi w_{0x}^2}{\lambda(z-z_{0x})})^2]$$

$$\frac{1}{q_x(z)} = \frac{1}{R_x(z)} - i\frac{\lambda}{n\pi w_{x(z)}^2}$$

$$q_{0x} = i\frac{n\pi w_{0x}^2}{\lambda}$$

$$\phi(z) = \frac{1}{2}\tan^{-1}(\frac{z-z_{0x}}{z_{0x}}) + \frac{1}{2}\tan^{-1}(\frac{z-z_{0y}}{z_{0y}})$$



椭圆高斯光的成因：

① 圆对称高斯光经过柱透镜聚焦

② 像散腔(腔镜的曲率半径在XZ面和YZ面内不同)

③ 波导的n_x和n_y不同

椭圆高斯光束可看作是在XZ面和YZ面内两个独立的高斯"光束"，其光束的复参数q_x和q_y各自按照ABCD规律传播演化

Laguerre-Gausian mode

$$\nabla^2 \vec{E}(x,y,z) + k^2 \vec{E}(x,y,z) = 0$$

慢变振幅近似 $E(x,y,z) = \psi(x,y,z)e^{-ikz}$

$$\nabla_\perp^2 \psi(x,y,z) - 2ik\frac{\partial \psi(x,y,z)}{\partial z} = 0$$

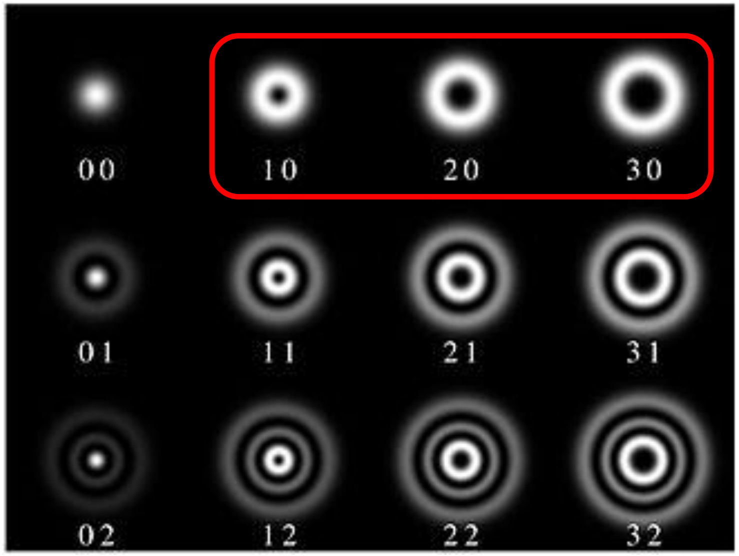
柱坐标下, $(x,y,z) \to (\rho,\phi,z)$

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} - 2ik\frac{\partial\psi}{\partial z} = 0$$

$$\psi(\rho,\phi,z) = F(\rho,z)\exp[-\mathrm{i}(\mathrm{p}+\frac{k}{2q}\rho^2)]e^{il\phi}$$

$$\psi(\rho,\phi,z) = \frac{C_{pl}}{w(\mathrm{z})}[\frac{\sqrt{2}\rho}{w(\mathrm{z})}]^l \exp[-\frac{\rho^2}{w^2(\mathrm{z})}]\mathrm{L}_p^l[\frac{2\rho^2}{w^2(\mathrm{z})}]e^{il\phi} \times \exp\{-\mathrm{i}[\mathrm{kz}+\frac{k\rho^2}{2R(\mathrm{z})} - (2\mathrm{p}+l+1)\arctan(\frac{\mathrm{z}}{\mathrm{z}_0})]\}$$

$$\rho_{\max} = \sqrt{\frac{l}{2}}w(\mathrm{z}) \quad \text{for} \quad \mathrm{p}=0$$



The intensity profiles of Laguerre-Gaussian modes (*l*, *p*)

伴随拉盖尔函数:

$$\mathrm{L}_p^l[\mathrm{x}] = \sum_{k=0}^{p}\frac{(\mathrm{p}+\mathrm{l})!(-\mathrm{x})^k}{(\mathrm{l}+\mathrm{k})!\mathrm{k}!(\mathrm{p}-k)!}$$

Open Question:

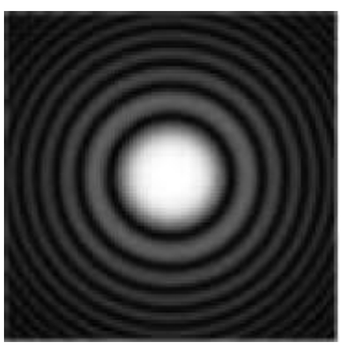
Any other beams can also be solutions for Helmhotz Equation?

What properties do they have?

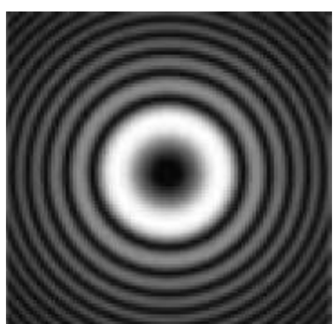
Bessel beam:

Ref, Contemporary Physics, Vol. 46, No. 1, January–February 2005, 15 – 28

$$E(r,\phi,z) = A_0 \exp(ik_z z) J_n(k_r r) \exp(in\phi)$$



J_0



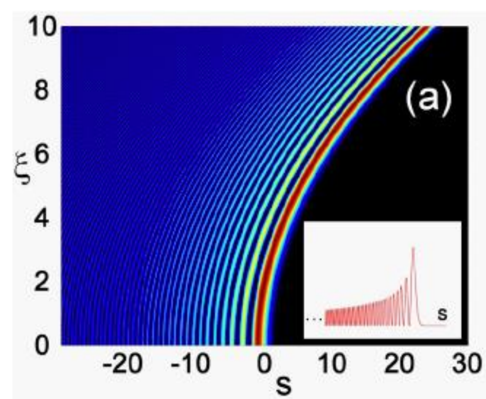
J_1

Airy beam:

*ref "PRL **99,** 213901 (2007)"*

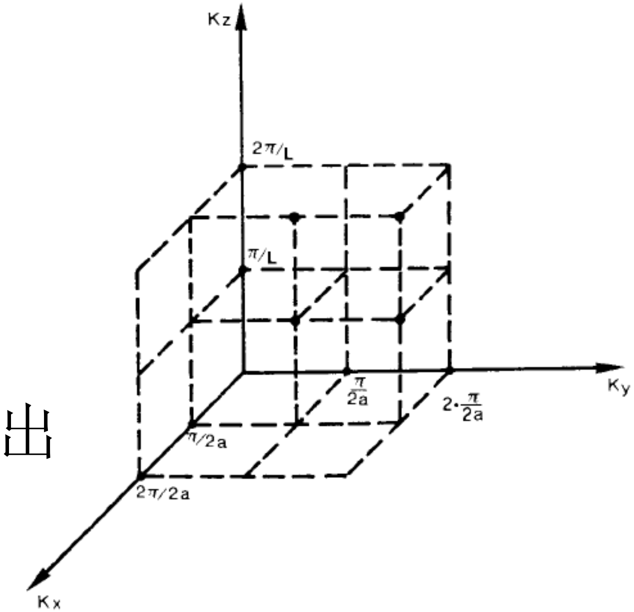
$$i\frac{\partial \phi}{\partial \xi} + \frac{1}{2}\frac{\partial^2 \phi}{\partial s^2} = 0$$

$$\phi(\xi, s) = Ai(s - (\xi/2)^2)\exp(i(s\xi/2) - i(\xi^3/12))$$



Ch7 光学谐振腔

一、谐振腔简介



激光器中光学谐振腔的作用：

①空间和频率滤波

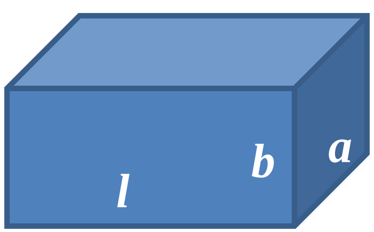
②对光能量提供正反馈以获得强的功率输出

封闭腔 ⇒ 开放腔

谐振腔的模式： 特定的电场分布（横模）；确定的频率和损耗（纵模）。

如图的封闭腔的稳定场分布要求在相对腔壁之间形成驻波，即：

$$k_x \cdot 2a = m \cdot 2\pi \Rightarrow k_x = m \cdot \frac{\pi}{a}, k_y = n \cdot \frac{\pi}{b}, k_z = q \cdot \frac{\pi}{l}$$



在K空间，模体元体积：

$$V_{体元} = \frac{\pi}{a} \cdot \frac{\pi}{b} \cdot \frac{\pi}{l} = \frac{\pi^3}{V}$$

考虑到偏振，单一频率ν对应的模式数为：

$$N_\nu = 2 \times \frac{1}{8} \times \frac{\frac{4}{3}\pi(\frac{2\pi\nu}{c})^3}{\pi^3 / V} = \frac{8}{3}\frac{\pi\nu^3}{c^3}V$$

单位腔体积（V=1），一定频率间隔Δν的封闭腔的模式数为：

$$\Delta N = \frac{8\pi\nu^2}{c^3}\Delta\nu$$

举例： Maser，封闭腔

$\nu = 3\times10^9 Hz \qquad (\lambda = 10cm)$

腔体积 $V = 100cm^3$

工作介质谱宽, $\Delta\nu = 10^9 Hz$

腔内模式数

$$\Delta N = V\frac{8\pi\nu^2}{c^3}\Delta\nu$$

$$= 100cm^3 \times \frac{8\pi(3\times10^9)^2}{(3\times10^8)^3}\times10^9 \approx 1$$

Example 5.1. *Number of modes in closed and open resonators.* Consider a He-Ne laser oscillating at the wavelength of $\lambda = 633$ nm, with a Doppler-broadened gain linewidth of $\Delta\nu_0^* = 1.7\times10^9$ Hz. Assume a resonator length $L = 50$ cm and consider first an open resonator. According to Eq. (5.1.3) the number of longitudinal modes which fall within the laser linewidth is $N_{open} = 2L\Delta\nu_0^*/c \cong 6$. Assume now that the resonator is closed by a cylindrical lateral surface with a cylinder diameter of $2a = 3$ mm. According to Eq. (2.2.16) the number of modes of this closed resonator which fall within the laser linewidth $\Delta\nu_0^*$ is $N_{closed} = 8\pi\nu^2 V\Delta\nu_0^*/c^3$, where $\nu = c/\lambda$ is the laser frequency and $V = \pi a^2 L$ is the resonator volume. From the previous expressions and data we readily obtain $N_{closed} = (2\pi a/\lambda)^2 N_{open} \cong 1.2\times 10^9$ modes.

举例： **Laser，开放腔**

PHYSICAL REVIEW VOLUME 112, NUMBER 6 DECEMBER 15, 1958

First laser paper!!

Infrared and Optical Masers

封闭腔 → 开放腔： 泄模

A. L. SCHAWLOW AND C. H. TOWNES*
Bell Telephone Laboratories, Murray Hill, New Jersey

(Received August 26, 1958)

The extension of maser techniques to the infrared and optical region is considered. It is shown that by using a resonant cavity of centimeter dimensions, having many resonant modes, maser oscillation at these wavelengths can be achieved by pumping with reasonable amounts of incoherent light. For wavelengths much shorter than those of the ultraviolet region, maser-type amplification appears to be quite impractical. Although use of a multimode cavity is suggested, a single mode may be selected by making only the end walls highly reflecting, and defining a suitably small angular aperture. Then extremely monochromatic and coherent light is produced. The design principles are illustrated by reference to a system using potassium vapor.

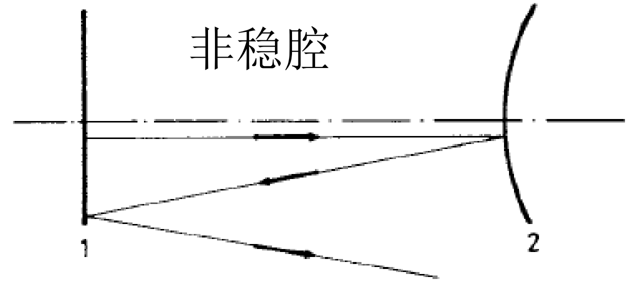
二、球面谐振腔

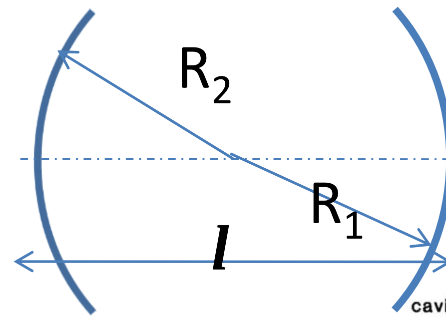
(1)谐振腔类型（按结构）

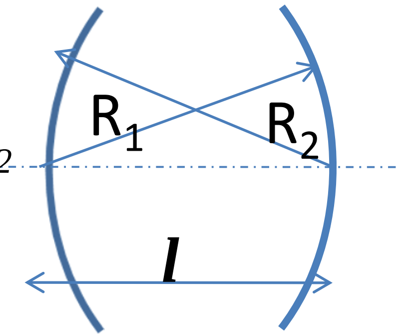
- 平面腔
- 球面腔
  - 稳定腔
  - 非稳定腔

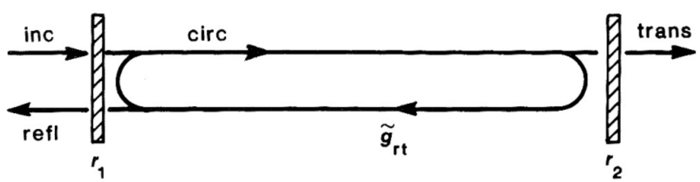
共心腔 $l=2R$ $R_1=R_2=R$

共焦腔 $l=R_1=R_2$

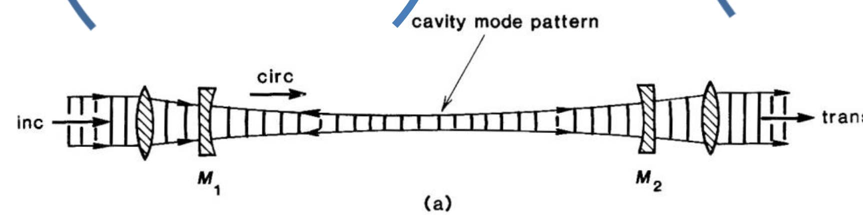




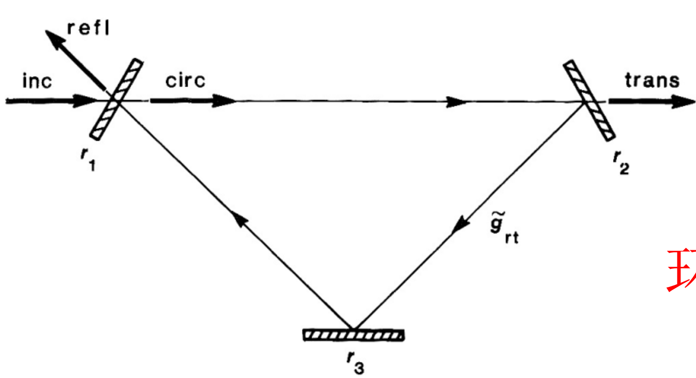




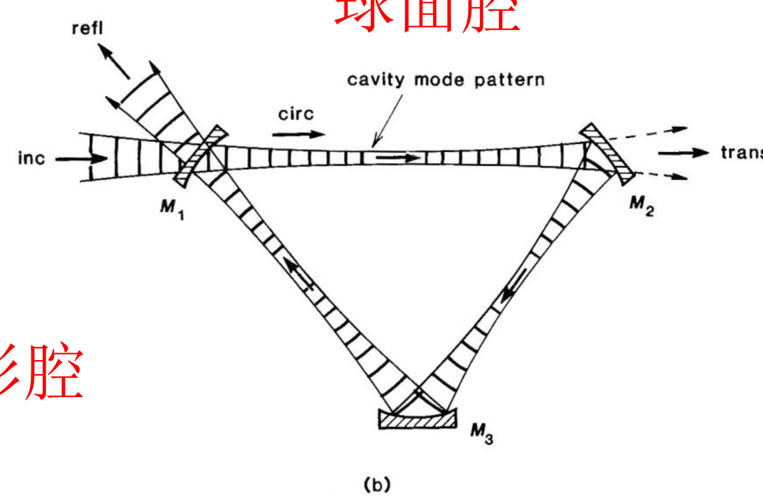
平平腔， $R_1=R_2=\infty$



球面腔



环形腔



球面谐振腔维持激光低损耗振荡应该满足下面两个条件：

①几何光学要求：近轴光线多次往返不溢出反射镜

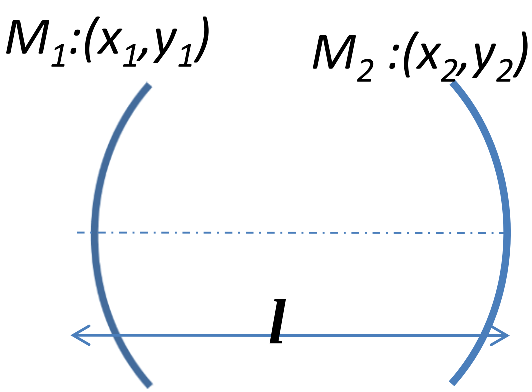
②物理光学要求：反射镜尺寸应使衍射损耗足够小，即 $\frac{a_1 a_2}{\lambda l} \geq 1$

*a1 , a2*分别是M_1, M_2反射镜的半径, *l*是M_1, M_2间的距离。M_1在M_2反射镜位置处形成的衍射斑大小为$\lambda l/a_1$ ，M_2的直径应该大于该衍射斑，故 $a_2 > \lambda l/a_1$

(2)球面腔横模的求解方法

① 标量衍射法

利用基尔霍夫衍射积分计算M_1反射镜到M_2反射镜的衍射场，此衍射场再被M_1反射镜衍射回M_2反射镜，……，如此反复，直到衍射场与初始场之间只相差一个复常数。



对M_1=M_2的情况下，自洽场方程可写为：

$$\tilde{\alpha} \cdot E(x_2, y_2) = \iint_{M_1} E(x_1, y_1) e^{-ik\rho(x_1, y_1; x_2, y_2)} dx_1 dy_1$$

解此积分方程的本征解就是该球面所支持的模式！

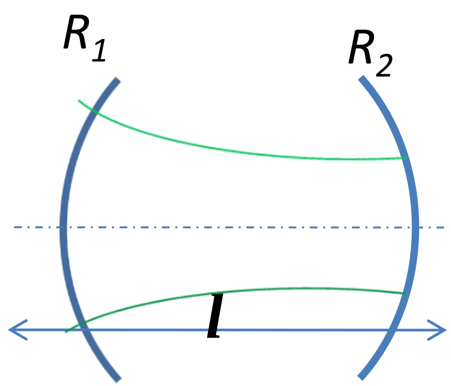
② 高斯光束解-----让腔镜的反射面与高斯光束的波阵面完全重合

$$E_{lm}(r,z)=E_0\frac{w_0}{w(z)}H_l(\sqrt{2}\frac{x}{w(z)})H_m(\sqrt{2}\frac{y}{w(z)})e^{\frac{-r^2}{w^2(z)}}e^{-ik[z+\frac{x^2+y^2}{2R(z)}]+i(l+m+1)\varphi(z)}$$

其中，$w_{(z)}^2=w_0^2[1+(\frac{z}{z_0})^2]$ $R(z)=z[1+(\frac{z_0}{z})^2]$ $\phi(z)=\tan^{-1}(\frac{z}{z_0})$ $z_0=\frac{n\pi w_0^2}{\lambda}$

基于此，球面腔设计的基本问题包括

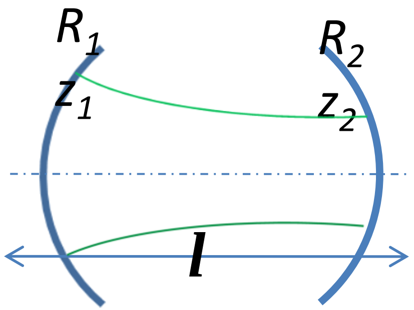
a) 已知光束的基本特性(比如w_0)，设计腔特性参数R_1、R_2、l



b) 已知腔特性参数R_1、R_2、l，求光束特性

(3)光学谐振腔的代数运算

① 问题：已知w_0，求位置z_1、z_2处腔镜的曲率半径R_1、R_2



$$R_1 = z_1[1+(\frac{z_0}{z_1})^2] = z_1 + \frac{z_0^2}{z_1}$$

$$R_2 = z_2 + \frac{z_0^2}{z_2} \qquad z_0 = \frac{n\pi w_0^2}{\lambda}$$

② 问题：R_1、R_2和w_0已知，求腔镜位置z_1、z_2

$$R_1 = z_1 + \frac{z_0^2}{z_1} \Rightarrow z_1 = \frac{1}{2}[R_1 \pm \sqrt{R_1^2 - 4z_0^2}]$$

$$R_2 = z_2 + \frac{z_0^2}{z_2} \Rightarrow z_2 = \frac{1}{2}[R_2 \pm \sqrt{R_2^2 - 4z_0^2}]$$

③ 问题：R_1、R_2和l已知，求w_0及腔镜上的光斑$w_1=w(M_1)$ $w_2=(M_2)$

$$\left.\begin{aligned} 2z_1 &= R_1 + \sqrt{R_1^2 - 4z_0^2} \\ 2z_2 &= R_2 + \sqrt{R_2^2 - 4z_0^2} \\ z_2 - z_1 &= l \end{aligned}\right\}$$

$$\Rightarrow 2l = 2(z_2 - z_1) = (R_2 - R_1) + \sqrt{R_2^2 - 4z_0^2} - \sqrt{R_1^2 - 4z_0^2}$$

$$\Rightarrow (2l - R_2 + R_1)^2 = [\sqrt{R_2^2 - 4z_0^2} - \sqrt{R_1^2 - 4z_0^2}]^2$$

$$\Rightarrow 4l^2 + 4l(R_1 - R_2) - 2R_1R_2 + 8z_0^2 = -2\sqrt{(R_2^2 - 4z_0^2)(R_1^2 - 4z_0^2)}$$

$$\Rightarrow 4l^2+4l(R_1-R_2)-2R_1R_2+8z_0{}^2=-2\sqrt{(R_2{}^2-4z_0{}^2)(R_1{}^2-4z_0{}^2)}$$

$$\Rightarrow [4l(l+R_1-R_2)-2R_1R_2+8z_0{}^2]^2=4(R_2{}^2-4z_0{}^2)(R_1{}^2-4z_0{}^2)$$

$$\Rightarrow 16l^2(l+R_1-R_2)^2-16R_1R_2l(l+R_1-R_2)+16z_0{}^2[4l(l+R_1-R_2)-2R_1R_2+R_2{}^2+R_1{}^2]=0$$

$$\Rightarrow l(l+R_1-R_2)[l(l+R_1-R_2)-R_1R_2]+z_0{}^2(2l+R_1-R_2)^2=0$$

$$\Rightarrow \boxed{z_0{}^2=\frac{l(R_2-R_1-l)(l+R_1)(l-R_2)}{(2l+R_1-R_2)^2}}$$

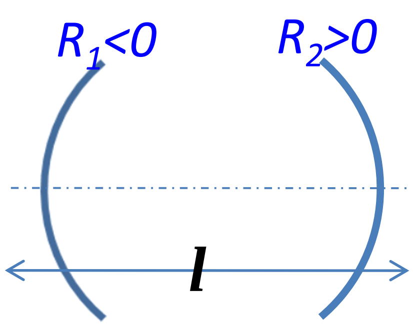
光斑半径：$w_0=\sqrt{\dfrac{\lambda z_0}{n\pi}}$

腔镜M_1位置的光斑：$w_1^2=w_0^2[1+(\dfrac{z_1}{z_0})^2]$　其中，$z_1=\dfrac{1}{2}[R_1\pm\sqrt{R_1{}^2-4z_0{}^2}]$

*** 对称腔：** $z_2=-z_1=l/2$，$R_2=-R_1=R$

$$z_0{}^2=\frac{l(2R-l)}{4}=\frac{l}{2}(R-\frac{l}{2})$$

束腰半径：$w_0=(\dfrac{\lambda z_0}{n\pi})^{\frac{1}{2}}=(\dfrac{\lambda}{n\pi})^{\frac{1}{2}}(\dfrac{l}{2})^{\frac{1}{4}}(R-\dfrac{l}{2})^{\frac{1}{4}}$



镜面光斑：

$$w_{1,2}=w_0[1+(\frac{z_{1,2}}{z_0})^2]^{1/2}=w_0[\frac{R}{R-l/2}]^{1/2}$$

$$=(\frac{\lambda l}{2\pi n})^{1/2}[\frac{2R^2}{l(R-l/2)}]^{1/4}$$

If R>>l(类平平腔)：

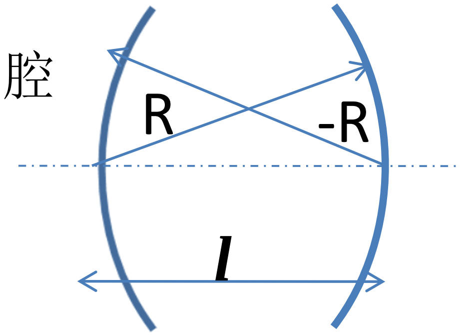
束腰半径：$w_0 = (\frac{\lambda z_0}{n\pi})^{\frac{1}{2}} = (\frac{\lambda}{n\pi})^{\frac{1}{2}}(\frac{l}{2})^{\frac{1}{4}}(R-\frac{l}{2})^{\frac{1}{4}} \approx (\frac{\lambda}{n\pi})^{\frac{1}{2}}(\frac{l}{2})^{\frac{1}{4}}R^{\frac{1}{4}}$

$$w_{1,2} = (\frac{\lambda l}{2\pi n})^{1/2}[\frac{2R^2}{l(R-l/2)}]^{1/4} \approx (\frac{\lambda l}{2\pi n})^{1/2}[\frac{2R^2}{lR}]^{1/4}$$

$$= (\frac{\lambda}{\pi n})^{1/2}(\frac{l}{2})^{1/4}R^{1/4} = w_0$$

光斑在整个腔内几乎不扩展！！

*** 对称共焦腔：** $R_2 = -R_1 = R = l$



对称共焦腔

$l=-R_1=R_2$

$$z^2_{0,confocal} = \frac{l(2R-l)}{4} = \frac{l^2}{4} = \frac{R^2}{4}$$

束腰半径：$w_{0,confocal} = (\frac{\lambda z_0}{n\pi})^{\frac{1}{2}} = (\frac{\lambda l}{2\pi n})^{\frac{1}{2}}$

镜面光斑：$w_{1,2,confocal} = (\frac{\lambda l}{2\pi n})^{1/2}[\frac{2R^2}{l(R-l/2)}]^{1/4} = (\frac{\lambda l}{\pi n})^{1/2} = \sqrt{2}w_{0,confocal}$

三、模式稳定性判据和谐振腔的自洽解

谐振腔稳定性的判断：

① 模斑大小判断：腔镜上的光斑越大损耗越大；

② g因子判断：⇔ 透镜波导的稳定条件

③ 自洽场判断：光场在腔内走一个来回后能够保持自洽

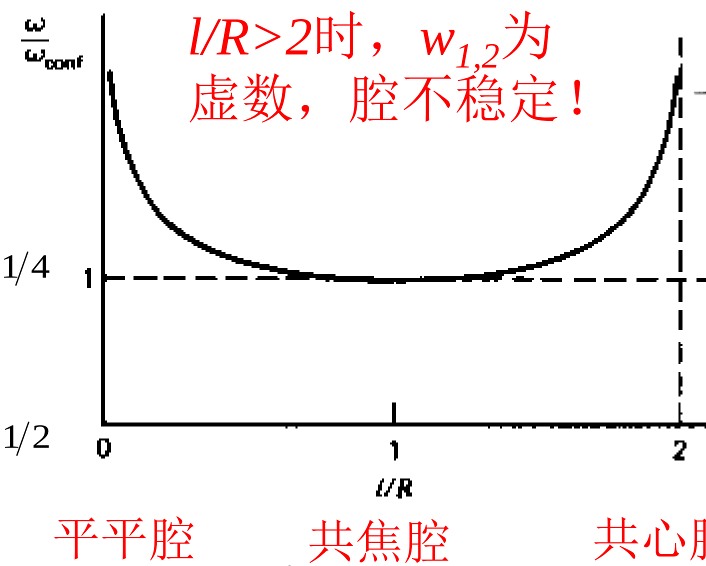
① 模斑大小判断

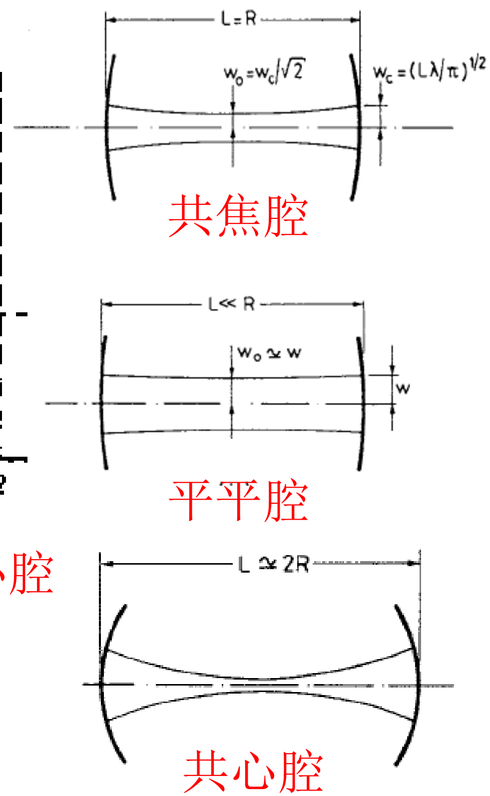
以对称腔为例

$$w_{1,2} = \left(\frac{\lambda l}{2\pi n}\right)^{1/2}\left[\frac{2R^2}{l(R-l/2)}\right]^{1/4}$$

共焦腔：$w_{1,2,confocal} = \left(\frac{\lambda l}{\pi n}\right)^{1/2}$

$$\frac{w_{1,2}}{w_{1,2,confocal}} = \frac{1}{\sqrt{2}}\left[\frac{2R^2}{l(R-l/2)}\right]^{1/4} = \left[\frac{1}{\frac{l}{R}(2-\frac{l}{R})}\right]^{1/4}$$





② g因子判断

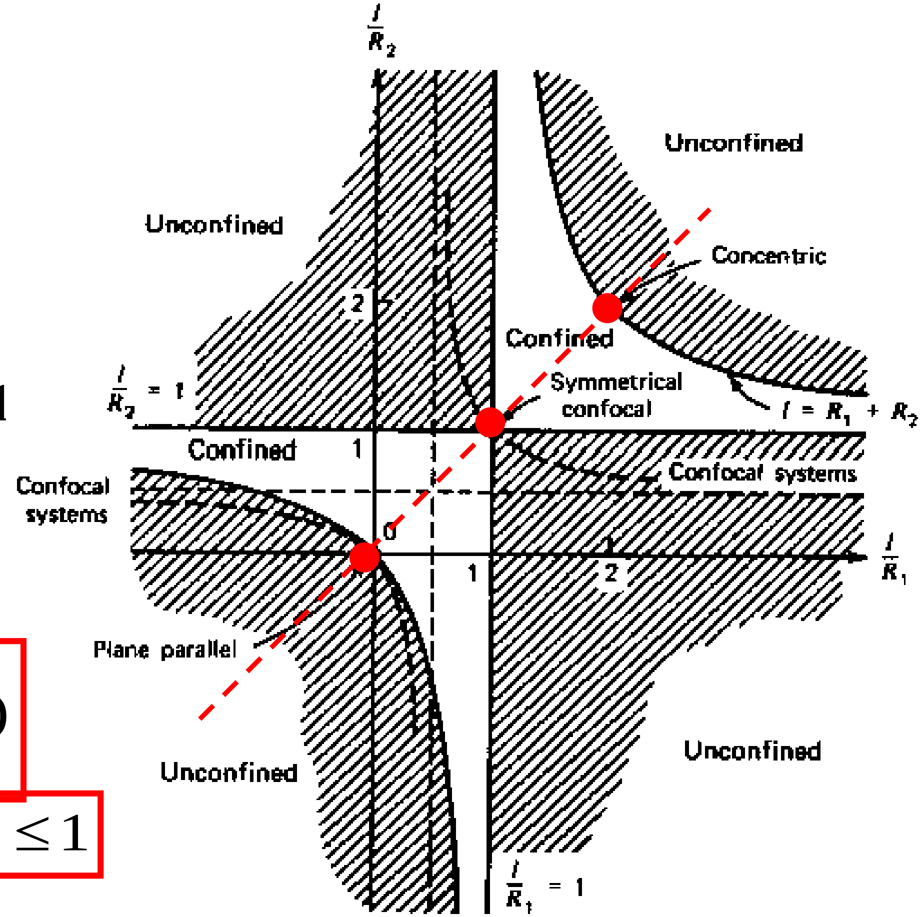
$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} : f \to \frac{R}{2}$$

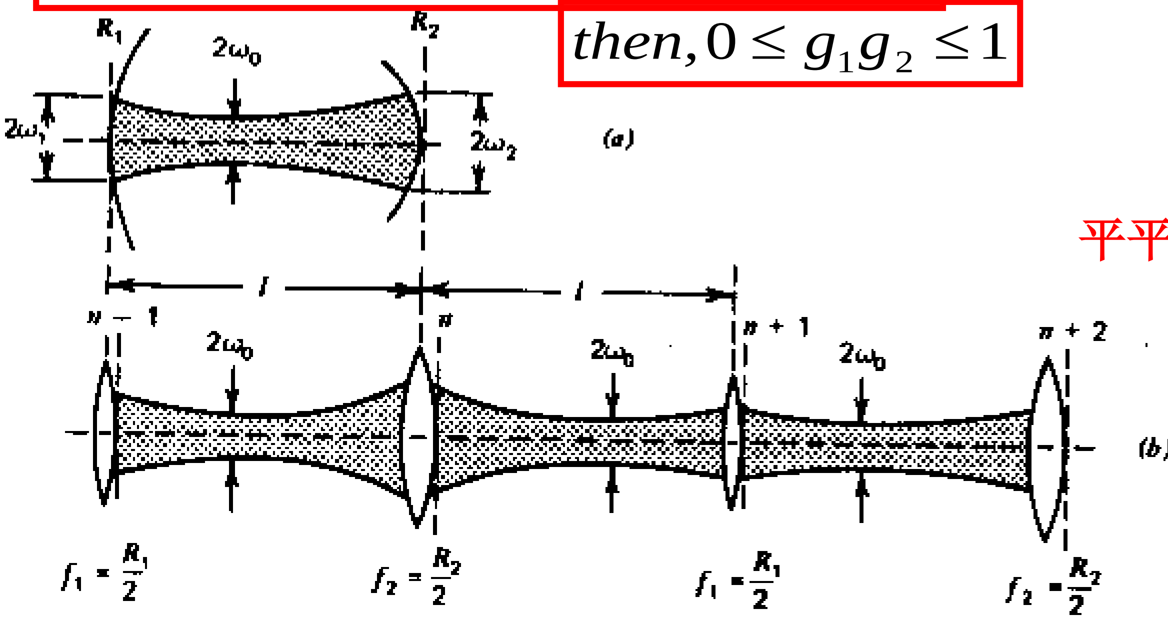
稳定条件 $: 0 \le (1-\frac{l}{2f_1})(1-\frac{l}{2f_2}) \le 1$

$$\Rightarrow 0 \le (1-\frac{l}{R_1})(1-\frac{l}{R_2}) \le 1$$

$$Define: g_1 \equiv (1-\frac{l}{R_1});\ g_2 \equiv (1-\frac{l}{R_2})$$

$$then, 0 \le g_1 g_2 \le 1$$





平平腔 $R = \infty, g \equiv (1-\frac{l}{R}) = 1,$

$g_1 g_2 = 1$

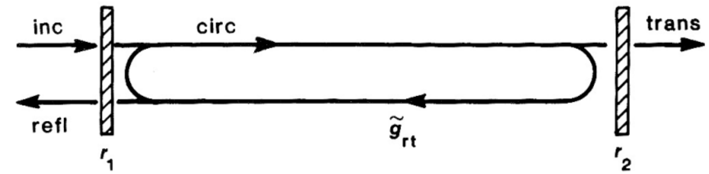
共心腔 $l = R_1 + R_2, g_1 g_2 = 1,$

$g = -1$ 对称共心腔

共焦腔 $l = R_1/2 + R_2/2$

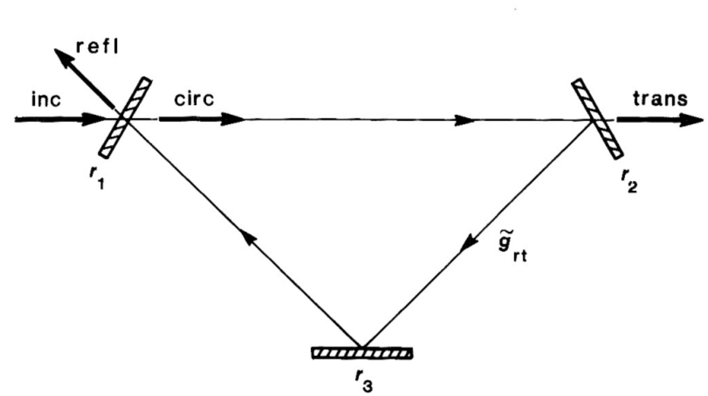
$R_1 = R_2 = l, g = 0$ 对称共焦

③ 广义谐振腔——自洽场



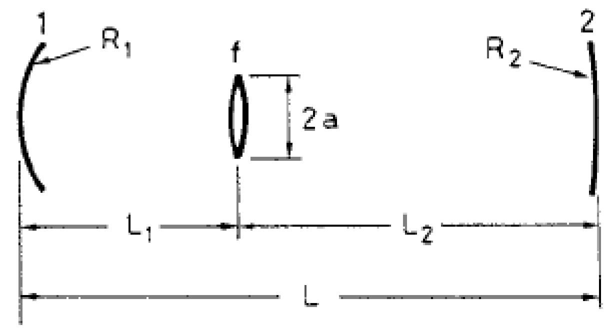
广义谐振腔（多元谐振腔）：

对于如右图所示的复杂结构的谐振腔，其模式稳定条件可以通过自洽场方法获得。



用 q 参数描述腔内的高斯光束，假定光束在腔内传播一个闭合路径的光线矩阵为：

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$



光场自洽条件要求：

$$\frac{Aq+B}{Cq+D}=q \Rightarrow C+(D-A)\frac{1}{q}-B(\frac{1}{q})^2=0$$

$$\Rightarrow \frac{1}{q}=\frac{1}{2B}[(D-A)\pm\sqrt{(D-A)^2+4BC}]$$

$$\Rightarrow R=\frac{2B}{D-A};\ w=(\frac{\lambda}{n\pi})^{1/2}\frac{|B|^{1/2}}{[1-(\frac{D+A}{2})^2]^{1/4}}$$

稳定条件： $\left|\frac{D+A}{2}\right|<1$

四、谐振腔共振频率

光场在谐振腔中经过一个“闭合路径”后“自再现”的条件包括：（a）光斑半径 w 和波面曲率半径 R“再现”；（b）相位“再现”（相位延迟为 2π整数倍）

① 纵模共振频率

高斯光束的传播相位延迟因子：$\theta_{m,n}(z)=kz-(m+n+1)\tan^{-1}\frac{z}{z_0}$

考虑相位“再现”，对球面谐振腔的单程相移应该是的π整数倍，即

$$\theta_{m,n}(z_2)-\theta_{m,n}(z_1)=k_q(z_2-z_1)-(m+n+1)(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})=q\pi,\ q=0,1,2,\cdots$$

先不考虑横模的影响（m， n 保持不变），纵模频率间隔为，

$$(k_{q+1}-k_q)(z_2-z_1)=(k_{q+1}-k_q)l=\pi$$

$$(\nu_{q+1}-\nu_q)\frac{2\pi n_0 l}{c}=\pi\Rightarrow\Delta\nu_L=\frac{c}{2n_0 l}$$

与 F P 腔的情况相同！

② **横模共振频率** $k_q l-(m+n+1)(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})=q\pi,\ q=0,1,2,\cdots$

m, n变化但(m+n)保持不变的模式，具有相同的共振频率(简并)

(m+n)变化但q保持不变，则

$$\begin{cases} k_1 l-(m+n+1)_1(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})=q\pi \\ k_2 l-(m+n+1)_2(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})=q\pi \end{cases}$$

$$\Rightarrow \frac{2\pi n_0}{c}\Delta\nu l=\Delta k l=\Delta(m+n)(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})$$

$$\boxed{\Delta\nu=\frac{c}{2\pi n_0 l}\Delta(m+n)(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})}$$

横模频率间隔！

举例1：对称共焦腔　$z_2=z_0,\ z_1=-z_0$

$$\Delta\nu_{conf}=\frac{c}{2\pi n_0 l}\Delta(m+n)\times\frac{\pi}{2}$$

$$=\frac{c}{4n_0 l}\Delta(m+n)$$

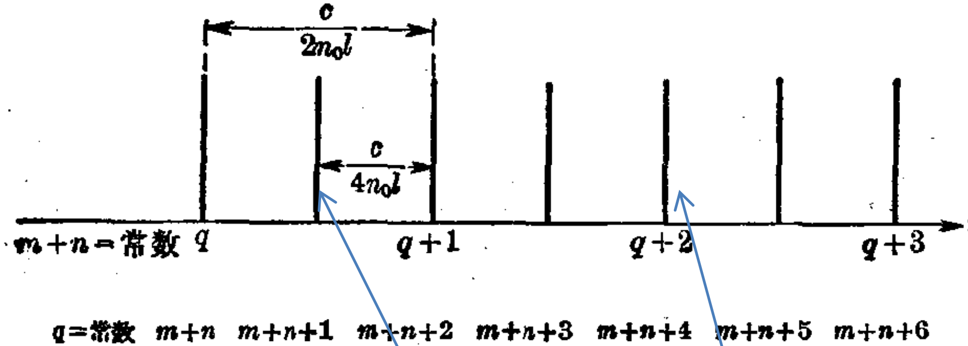


图 7.5　共焦($R=l$)光学谐振腔共振频率的位置随模式的阶次 m, n 和 q 的变化

横模频率

纵模频率

举例2：类平平腔　$R>>l$, 从而$z_0>>z_{2,1}, l$

$$\tan^{-1}\frac{z_2}{z_0}\approx\frac{z_2}{z_0}$$

$$\Delta\nu_{PP}=\frac{c}{2\pi n_0 l}\Delta(m+n)\left(\frac{z_2}{z_0}-\frac{z_1}{z_0}\right)$$

$$=\frac{c}{2\pi n_0 z_0}\Delta(m+n)<<\frac{c}{2\pi n_0 l}$$

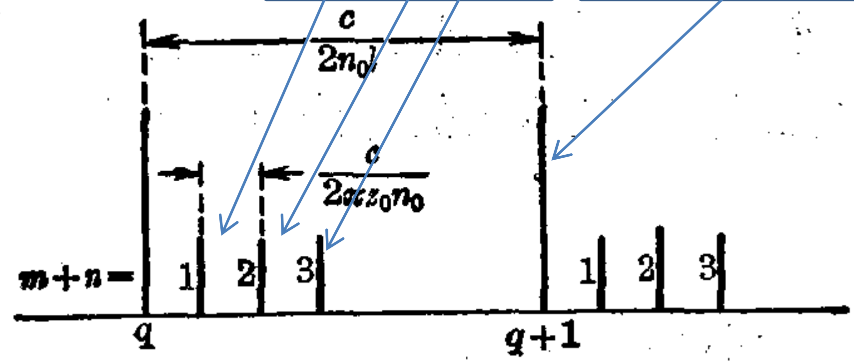


图 7.6　近似平面的($R\gg l$)光学谐振腔的共振频率随模式的阶次 m, n 和 q 的变化

五、光学谐振腔的损耗

"损耗"的作用：

① 决定激光振荡的阈值

② 由于增益饱和效应，谐振腔的损耗决定了稳定振荡时激光的输出强度

"损耗"的种类：

① 反射镜的透射及腔镜材料的吸收、散射

② 激光增益介质的吸收、散射

③ 衍射损耗

* 基模高斯光的单程衍射损耗

基模高斯光强分布：$I(\rho) = I_0 e^{-2\rho^2/w^2}$

$$\delta_D = \frac{P'}{P} = \frac{\int_a^\infty I(\rho)\cdot\pi 2\rho d\rho}{\int_0^\infty I(\rho)\cdot\pi 2\rho d\rho} = \frac{\frac{\pi}{2}w^2 I_0 e^{-2a^2/w^2}}{\frac{\pi}{2}w^2 I_0} = e^{-2a^2/w^2}$$

对对称共焦腔：

$$w = \sqrt{2}w_0 = \left(\frac{\lambda l}{n\pi}\right)^{\frac{1}{2}}$$

$$\delta_D = e^{-2a^2/w^2} = e^{-2\pi n\frac{a^2}{\lambda l}} = e^{-2\pi nN}$$

N是菲涅尔数！

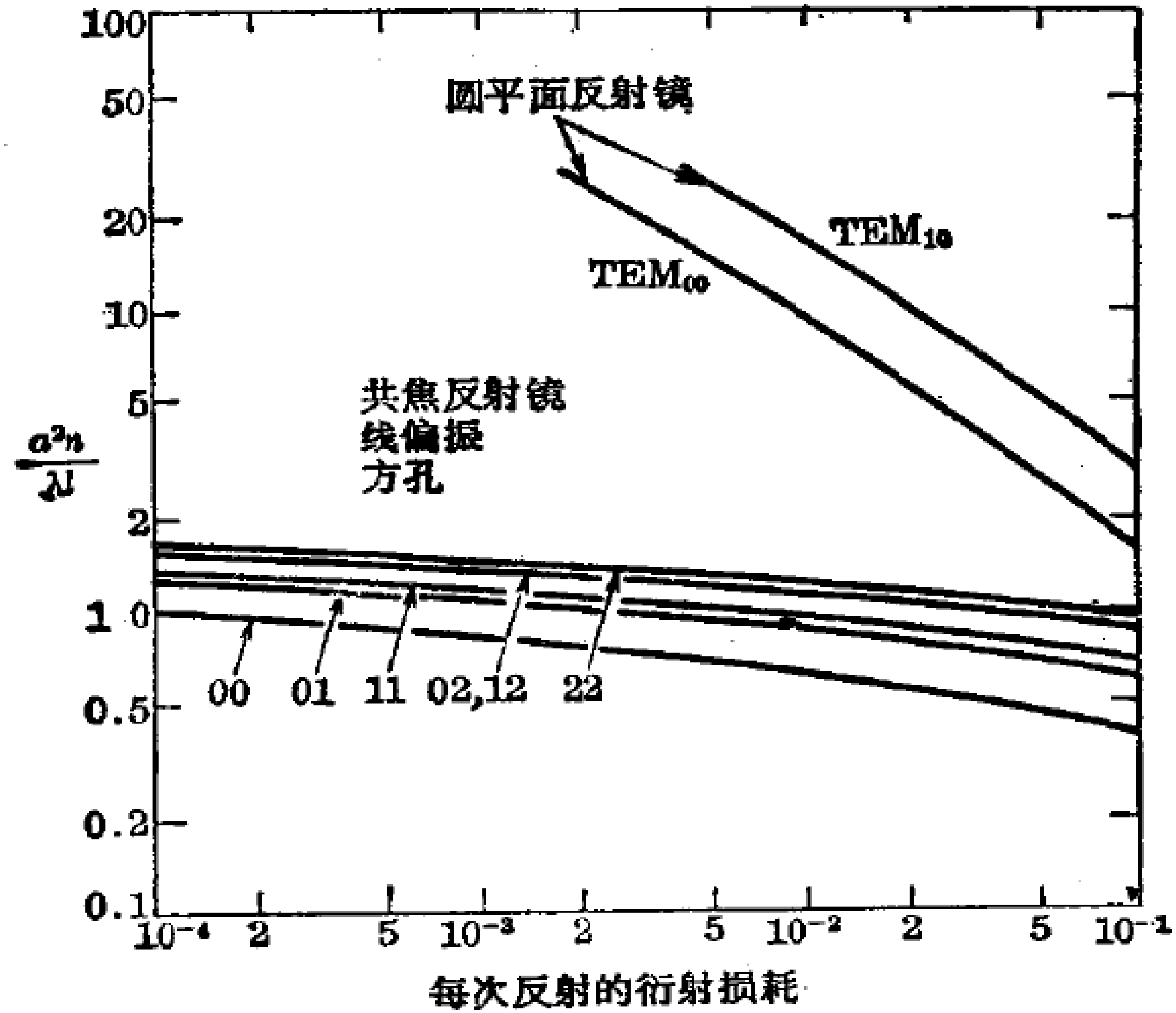


图 7.7 平面平行腔和共焦腔几个低阶模式的衍射损耗；a 是反射镜半径，l 是反射镜间隔。箭头所指的一对数字是横模的阶次 m, n[3]

谐振腔“损耗”的参数表征：

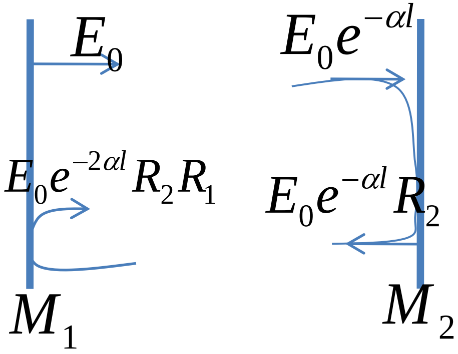
① 单程损耗因子(loss per pass)L：

光在谐振腔内经过一个单程的能量损耗比例

$$\left(\frac{1-L}{1}\right)^2=\frac{E_0e^{-2\alpha l}R_2R_1}{E_0}=R_1R_2e^{-2\alpha l}$$

$$\Rightarrow L=1-\sqrt{R_1R_2}e^{-\alpha l}=1-e^{-\alpha l+\ln\sqrt{R_1R_2}}$$

$$\approx 1-(1-\alpha l+\ln\sqrt{R_1R_2})=\alpha l-\ln\sqrt{R_1R_2}$$



② 光子寿命(photon lifetime)t_c一光子在腔内滞留的平均时间

$$n(t)=n_0e^{-t/t_c}$$

$$\mathrm{E}(t)=n_0hve^{-t/t_c}$$

$$\frac{d\mathrm{E}(t)}{dt}=-\frac{E(t)}{t_c}$$

$$L=\frac{n_0hv-n_0hve^{-t/t_c}}{n_0hv}=1-e^{-t/t_c}=1-e^{-\frac{nl}{ct_c}}$$

$$\approx 1-\left(1-\frac{nl}{ct_c}\right)=\frac{nl}{ct_c}$$

$$\Rightarrow t_c=\frac{nl}{cL}=\frac{nl}{c(\alpha l-\ln\sqrt{R_1R_2})}\approx\frac{nl}{c[\alpha l+(1-\sqrt{R_1R_2})]}$$

$\ln x\approx x-1$

在x=1附近一阶展开

$$t_c = \frac{nl}{cL} = \frac{t_T}{L},$$

其中,

$$t_T = \frac{nl}{c}$$

渡越时间(transit time)

Example 5.2. *Calculation of the cavity photon lifetime.* We will assume $R_1 = R_2 = R = 0.98$ and $T_i \cong 0$. From Eq. (5.3.7) we obtain $\tau_c = \tau_T/[-\ln R] = 49.5\,\tau_T$, where τ_T is the transit time of the photons for a single-pass in the cavity. From this example we note that the photon lifetime is much longer than the transit time, a result which is typical of low loss cavities. If we now assume $L = 90$ cm, we get $\tau_T = 3$ ns and $\tau_c \cong 150$ ns.

Electric field inside the cavity:

$$\mathrm{E}(t) = E_0 e^{-t/2t_c} e^{i\omega t}$$

take the Fourier transform of this field, we find that the power spectrum of the emitted light has a Lorentzian line shape with linewidth (FWHM) given by

$$\Delta\nu_c = \frac{1}{2\pi t_c} = \frac{c(\alpha - \frac{1}{l}\ln\sqrt{R_1 R_2})}{2\pi n}$$

Example 5.3. *Linewidth of a cavity resonance.* If we take again $R_1 = R_2 = 0.98$ and $T_i = 0$, from Eqs. (5.3.10) and (5.3.7) we get $\Delta\nu_c \cong 6.4307 \times 10^{-3} \times (c/2L)$, while from Eq. (4.5.12) we get $\Delta\nu_c \cong 6.4308 \times 10^{-3} \times (c/2L)$. For the particular case $L = 90$ cm, we then obtain $\Delta\nu_c \cong 1.1$ MHz. Even at the relatively low reflectivity values of $R_1 = R_2 = 0.5$, the discrepancy is not large. In fact from Eqs. (5.3.10) and (5.3.7) we get $\Delta\nu_c \cong 0.221 \times (c/2L)$, while from Eq. (4.5.12) $\Delta\nu_c \cong 0.225 \times (c/2L)$. Again for $L = 90$ cm we then obtain $\Delta\nu_c \cong 37.5$ MHz. Thus, in typical cases, $\Delta\nu_c$ may range from a few to a few tens of MHz.

③ 品质因子Q

$$Def: Q = \omega \frac{E}{-dE/dt} = 2\pi\nu \frac{\text{腔内存储能量}}{\text{单位时间消耗能量}}$$

$$= 2\pi \frac{\text{total energy stored}}{\text{energy dispersed/cycle}}$$

$$Q = \omega \frac{E}{-dE/dt} = \omega \frac{E}{E/t_c} = \omega t_c = \omega \frac{nl}{cL}$$

$$Q = \frac{\nu}{\Delta\nu_c}$$

Example 5.4. *Q-factor of a laser cavity* According to example 5.2 we will again take $t_c \cong 150$ ns and assume $\nu \cong 5 \times 10^{14}$ Hz (i.e. $\lambda \cong 630$ nm). From $Q = 2\pi\nu t_c$ we obtain $Q = 4.7 \times 10^8$. Thus, very high Q-values can be achieved in a laser cavity and this means that a very small fraction of the energy is lost during one oscillation cycle

物理意义：

$$\frac{Q}{2\pi} = \frac{t_c}{T}$$ 表示腔内存储的能量所能维持震荡的周期数

六、“非稳”光学谐振腔

(1) 稳定腔的“弱点”

稳定腔衍射损耗小，一般适用于低增益、小功率激光器。其“弱点”主要有：模体积小、功率小、光束质量差(衍射损耗小，选模质量差)。

举例：共焦腔基模的模体积计算

设，$R = l = 2m,\ \lambda = 1\mu m,$

束腰 $w_0 = (\frac{\lambda l}{2\pi})^{1/2} = 0.56mm,$ 镜面光斑 $w_{1,2} = \sqrt{2} w_0 = 0.8mm$

模体积：$V_{00} = \frac{1}{2} l\pi (\frac{w_1 + w_2}{2})^2 \approx 2cm^3$

假设实际气体激光腔的腔镜直径为 $D = 10mm,$ 则腔的几何体积为

$$V_{cavity} = l\pi(\frac{D}{2})^2 = 2m \times \pi \times (\frac{10mm}{2})^2 \approx 157cm^3$$

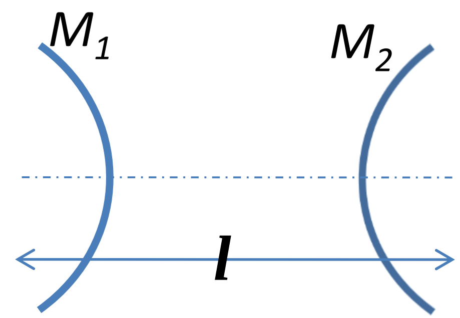
腔内工作介质的利用率：$\frac{V_{00}}{V_{cavity}} = \frac{2}{157} \approx 1.3\%$

(2) 非稳定腔的构成

$$g_1 g_2 = (1-\frac{l}{R_1})(1-\frac{l}{R_2}) < 0 \quad or \quad g_1 g_2 = (1-\frac{l}{R_1})(1-\frac{l}{R_2}) > 1$$

常见非稳定腔：

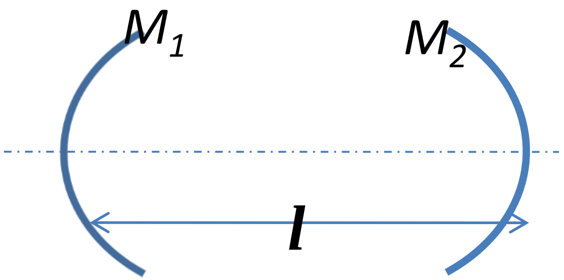
① 双凸腔(包括平凸腔)



$$R_1 < 0,\ R_2 < 0 \Rightarrow g_1 > 1,\ g_2 > 1$$

$$\Rightarrow g_1 g_2 > 1$$

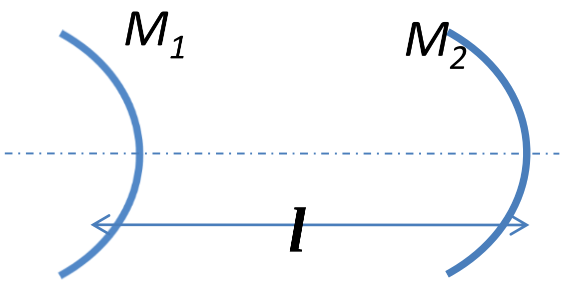
② 双凹腔



$R_1 > 0,\ R_2 > 0$,要求

$$g_1 g_2 = (1-\frac{l}{R_1})(1-\frac{l}{R_2})\begin{cases} < 0 \Rightarrow R_1 > l, R_2 < l \\ > 1 \Rightarrow R_1 << l, R_2 << l \end{cases}$$

③ 凹凸腔



$R_1 < 0,\ R_2 > 0$, 要求

$$g_1 g_2 = (1-\frac{l}{R_1})(1-\frac{l}{R_2}) < 0 \ \Rightarrow R_2 << l$$

(3) 非稳定腔特性—以双凸腔为例

按光束自再现条件的要求：

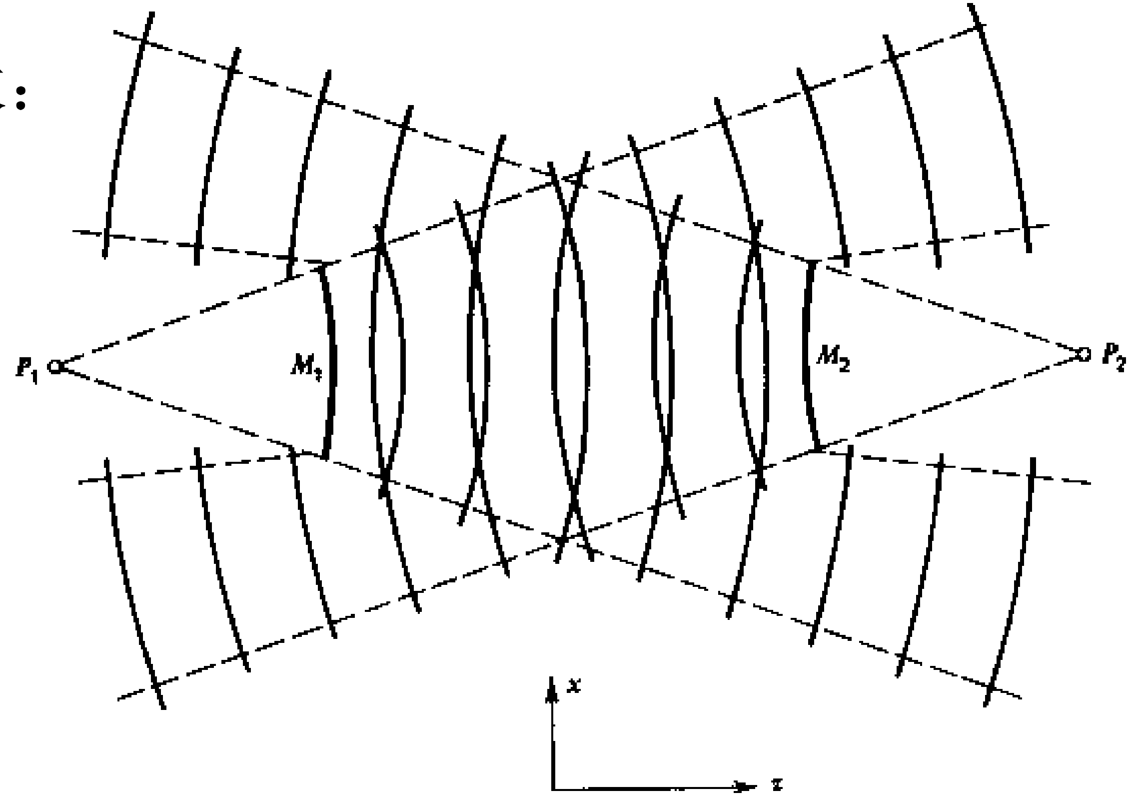
对 M_1, P_2是物；P_1是像

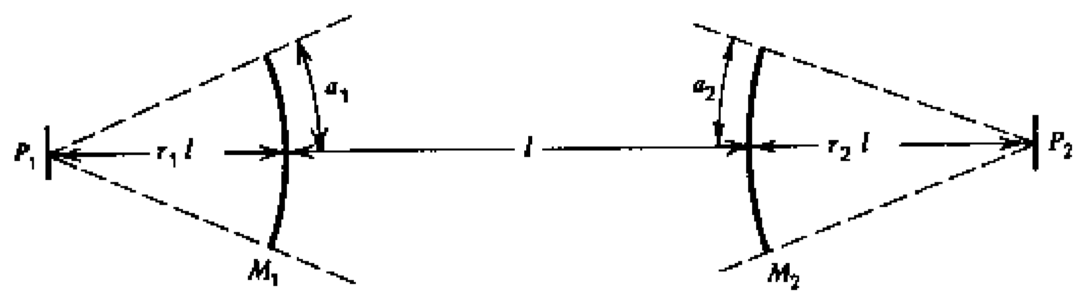
对 M_2, P_1是物；P_2是像

$$\begin{cases} \dfrac{1}{r_2 l + l} - \dfrac{1}{r_1 l} = \dfrac{2}{R_1} \\ \dfrac{1}{r_1 l + l} - \dfrac{1}{r_2 l} = \dfrac{2}{R_2} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{1}{r_1} - \dfrac{1}{r_2 + 1} = -\dfrac{2l}{R_1} = 2(g_1 - 1) \\ \dfrac{1}{r_2} - \dfrac{1}{r_1 + 1} = -\dfrac{2l}{R_2} = 2(g_2 - 1) \end{cases}$$

$$\Rightarrow \begin{cases} r_1 = \dfrac{\pm\sqrt{g_1 g_2 (g_1 g_2 - 1)} - g_1 g_2 + g_2}{2 g_1 g_2 - g_1 - g_2} \\ r_2 = \dfrac{\pm\sqrt{g_1 g_2 (g_1 g_2 - 1)} - g_1 g_2 + g_1}{2 g_1 g_2 - g_1 - g_2} \end{cases}$$





双凸腔的几何损耗：

$$(\Gamma_1)_x = \frac{a_2}{\frac{r_1 l + l}{r_1 l} a_1} = \frac{r_1 a_2}{(r_1+1)a_1}; \quad (\Gamma_2)_x = \frac{r_2 a_1}{(r_2+1)a_2}$$

$$\Rightarrow (\Gamma_{12})_x = \frac{r_1 r_2}{(r_1+1)(r_2+1)}$$

$$2D: \quad (\Gamma_{12})_{2D} = (\Gamma_{12})_x^2 = \frac{(r_1 r_2)^2}{(r_1+1)^2 (r_2+1)^2}$$

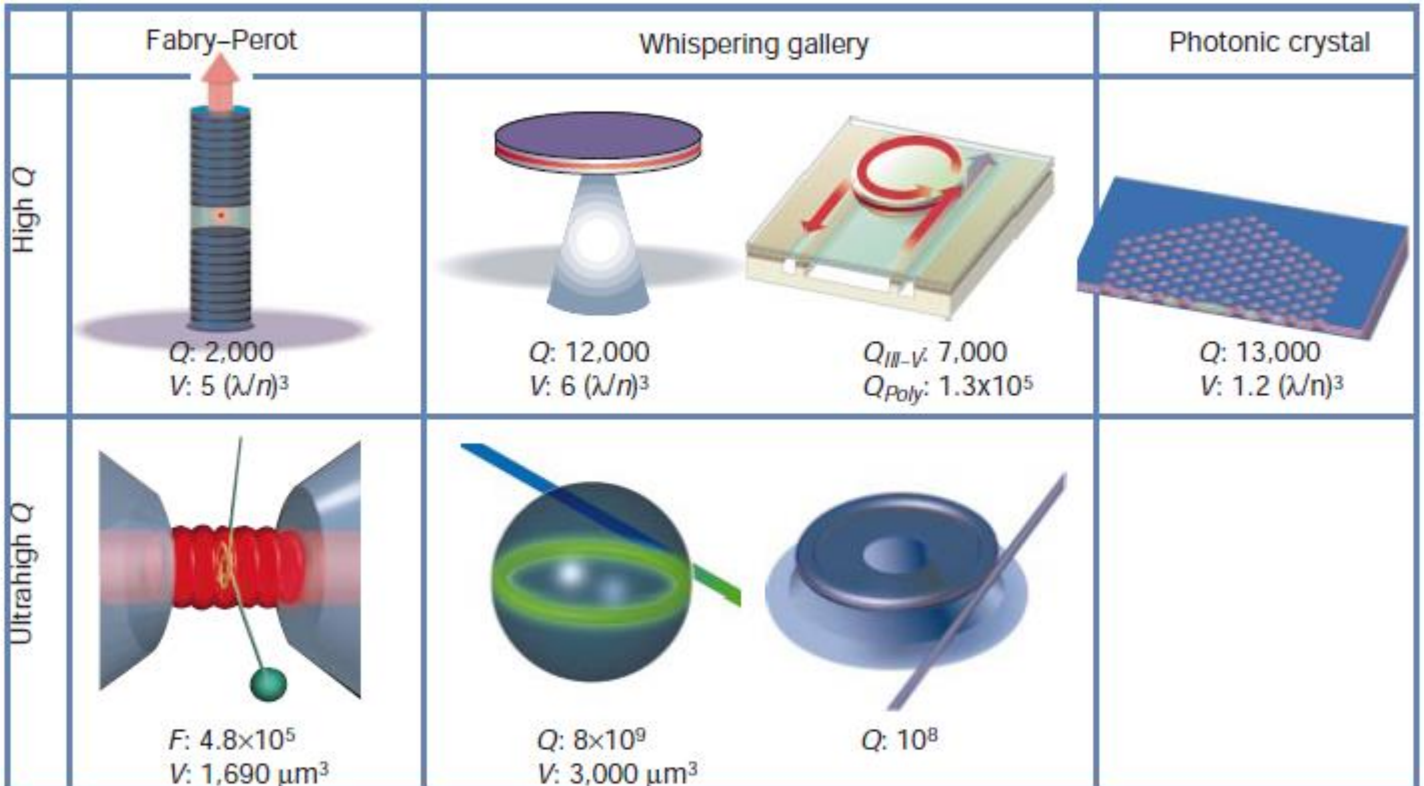
往返损耗: $\delta_{往返} = 1 - (\Gamma_{12})_{2D} = 1 - \frac{(r_1 r_2)^2}{(r_1+1)^2 (r_2+1)^2}$，不显含$R_1, R_2, l$!!

非稳腔的特点：

模体积大、易实现单模、光束分散角小、腔内光束均匀不易破坏工作介质

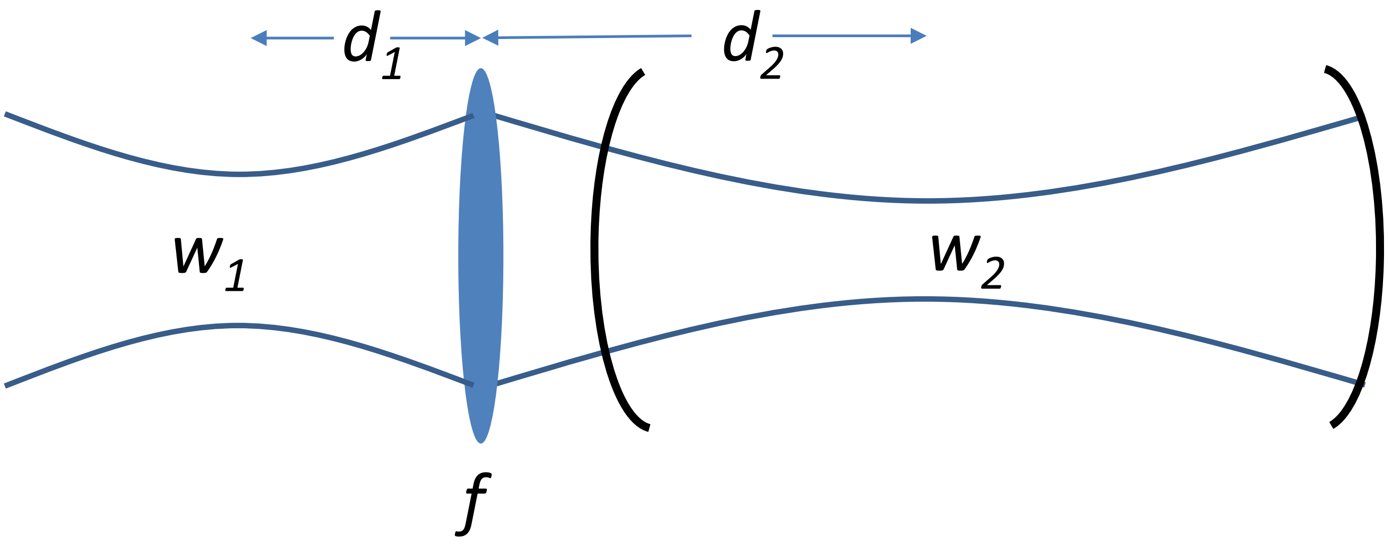
Open question: Why Microcavities?

Ref:"optical microcavities" Kerry J Vahala, Nature 424, 839-846 (2003)



Upper row: micropost48, microdisk52, semiconductor103, polymer104 add/drop filter, photonic crystal cavity62. Lower row: Fabry-Perot bulk optical cavity21,31, microsphere29, microtoroid6. *n is the material refractive* index, and, *V, if not indicated, was not available.* Microsphere volume *V was inferred using the diameter* noted in the cited reference and finesse (*F) is given for* the ultrahigh-*Q Fabry–Perot as opposed to Q. Two Q* values are cited for the add/drop filter: one for a polymer design, *QPoly, and the second for a III–V semiconductor* design, *QIII–V.*

模式耦合问题



Ch8 辐射场与原子系统的相互作用

一、密度矩阵

密度矩阵方法是在系统的精确波函数不知道的情况下计算算符平均值的一种方法。

假设一个由N个粒子组成的系综，粒子的l个能量本征波函数$\{u_1(\vec{r}), u_2(\vec{r}), \cdots u_l(\vec{r})\}$构成正交完备集。粒子的某个量子态在$\{u_n(\vec{r})\}$上展开：$\psi(\vec{r},t) = \sum_n C_n(t) u_n(\vec{r})$，其中$C_n(t) = (u_n(\vec{r}), \psi(\vec{r},t))$

一般情况下$\psi(\vec{r},t)$的精确状态未知，即展开系数$C_n(t)$测不准，但可以通过系综平均求得。

力学量A的量子力学平均：

其中, $A_{mn} = < u_m | \mathrm{A} | \mathrm{u}_n >$

$$< A > = < \psi | \mathrm{A} | \psi > = \sum_{m,n} c_m^* < u_m | \mathrm{A} | \mathrm{u}_n > c_n = \sum_{m,n} c_m^* c_n A_{mn}$$

力学量A的系综平均： $< \overline{A} > = \sum_{m,n} \overline{c_m^* c_n} A_{mn}$

Def 密度矩阵：$\rho_{nm} = \overline{c_m^* c_n}$ $\quad < \overline{A} > = \sum_n \rho_{nm} A_{mn} = Tr(\rho \mathrm{A})$

ρ是厄米矩阵：$\rho_{nm} = \rho_{mn}^*$

$\rho_{nn} = \overline{c_n^* c_n} = |c_n|^2$ 系统处于$\{u_n(r)\}$态的几率

ρ_{nm}与系统的辐射偶极矩有关

密度矩阵的时间演化

粒子波函数: $\psi^k(t,r) = \sum_l c_l^k(t) u_l(r)$

$$\rho_{nm} = \overline{c_m^* c_n} = \frac{1}{N}\sum_k c_m^{k*} c_n^k$$

代入薛定谔方程: $i\hbar \frac{\partial \psi}{\partial t} = H\psi$,

并两边乘以$u_m^*(r)$并积分得

$$i\hbar \frac{\partial c_m^k}{\partial t} = \sum_l H_{ml} c_l^k$$

其中, $H_{ml} = \int u_m^* H u_l d\vec{r}$

$$\frac{\partial \rho_{nm}}{\partial t} = \frac{1}{N}\sum_k \left(\frac{\partial c_m^{k*}}{\partial t} c_n^k + \frac{\partial c_n^k}{\partial t} c_m^{k*}\right)$$

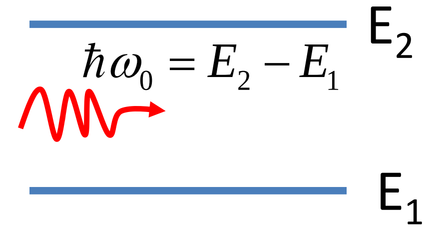
$$\frac{\partial \rho_{nm}}{\partial t} = \frac{1}{N}\sum_k \sum_l \{-\frac{1}{i\hbar} \mathrm{H}_{ml}^* c_l^{k*} c_n^k + \frac{1}{i\hbar} \mathrm{H}_{nl} c_l^k c_m^{k*}\}$$

$$= \frac{1}{i\hbar}\sum_l (\mathrm{H}_{nl}\,\rho_{lm} - \rho_{nl}\,\mathrm{H}_{lm})$$

$$i\hbar \frac{\partial \rho}{\partial t} = \mathrm{H}\rho - \rho\mathrm{H} = [\mathrm{H}, \rho]$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[\mathrm{H}, \rho] = \frac{i}{\hbar}[\rho, H]}$$

二、原子极化率



考虑二能级原子系统与光场的相互作用

在能量本征态表象中 $\mathrm{H}_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$

偶极相互作用哈密顿量 $\mathrm{H}' = -\mu E(\mathrm{t}) = -\begin{pmatrix} \mu_{11}E & \mu_{12}E \\ \mu_{21}E & \mu_{22}E \end{pmatrix} = -\begin{pmatrix} 0 & \mu E(\mathrm{t}) \\ \mu E(\mathrm{t}) & 0 \end{pmatrix}$

其中μ是偶极作用算符，考虑宇称守恒有 $\mu_{11}=\mu_{22}=0$

一般地，选择合适的相位可以使$\mu_{12}=\mu_{21}=\mu$

根据密度矩阵方法，$<\overline{\mu}>= Tr(\rho\mu) = (\rho_{12}+\rho_{21})\mu$

密度矩阵的时间演化：$\dfrac{d\rho_{21}}{dt} = -\dfrac{i}{\hbar}[\mathrm{H}_0+\mathrm{H}',\rho]_{21} = -\dfrac{i}{\hbar}\{[\mathrm{H}_0,\rho]_{21}+[\mathrm{H}',\rho]_{21}\}$

$$= -\frac{i}{\hbar}\{(\mathrm{H}_0\,\rho)_{21} - (\rho\,\mathrm{H}_0)_{21} + (\mathrm{H}'\,\rho)_{21} - (\rho\,\mathrm{H}')_{21}\}$$

$$= -\frac{i}{\hbar}\{E_2\rho_{21} - \rho_{21}E_1 + (-\mu E(\mathrm{t})\rho_{11} + \mu E(\mathrm{t})\rho_{22})\}$$

$$\frac{d\rho_{21}}{dt} = -i\omega_0\rho_{21} + i\frac{\mu E(\mathrm{t})}{\hbar}(\rho_{11}-\rho_{22})$$

类似的， $$\frac{\partial\rho_{22}}{\partial t}=-\frac{i}{\hbar}[\mathrm{H}_0+\mathrm{H}^{'},\rho]_{22}=-\frac{i}{\hbar}\{[\mathrm{H}_0,\rho]_{22}+[\mathrm{H}^{'},\rho]_{22}\}$$

$$=-\frac{i}{\hbar}\{E_2\rho_{22}-\rho_{22}E_2+(-\mu E(\mathrm{t})\rho_{12}+\mu E(\mathrm{t})\rho_{21})\}$$

$$=-\frac{i}{\hbar}\mu E(\mathrm{t})(\rho_{21}-\rho_{12})=-\frac{i}{\hbar}\mu E(\mathrm{t})(\rho_{21}-\rho_{21}^{*})$$

又, $\rho_{11}+\rho_{22}=1$ 所以

$$\frac{d}{dt}(\rho_{11}-\rho_{22})=2i\frac{\mu}{\hbar}E(\mathrm{t})(\rho_{21}-\rho_{21}^{*})$$

引入“碰撞项”，

$$\begin{cases}\dfrac{d\rho_{21}}{dt}=-i\omega_0\rho_{21}+i\dfrac{\mu E(\mathrm{t})}{\hbar}(\rho_{11}-\rho_{22})-\dfrac{\rho_{21}}{T_2}\\[2ex]\dfrac{d}{dt}(\rho_{11}-\rho_{22})=2i\dfrac{\mu}{\hbar}E(\mathrm{t})(\rho_{21}-\rho_{21}^{*})-\dfrac{(\rho_{11}-\rho_{22})-(\rho_{11}-\rho_{22})_0}{\tau}\end{cases}$$

T_2: “碰撞”引起的位相退相干驰豫

τ: 上下能级粒子数差$N(\rho_{11}-\rho_{22})$驰豫到平衡值所用的时间

假设外场E(t)=0, $\frac{d\rho_{21}}{dt} = -i\omega_0\rho_{21} - \frac{\rho_{21}}{T_2} \Rightarrow \rho_{21}(\mathrm{t}) = \rho_{21}(0)e^{-i\omega_0 t}e^{-t/T_2}$

振荡衰减！

假设外场为时谐场，$E(\mathrm{t}) = E_0\cos(\omega t) = \frac{1}{2}E_0(e^{-i\omega t} + e^{i\omega t})$

且$\omega \approx \omega_0$时，可定义 $\rho_{21} = \sigma_{21}e^{-i\omega t}$ 则，$\rho_{12} = \rho_{21}^* = \sigma_{12}e^{i\omega t}$

$\sigma_{12}(\mathrm{t}) = \sigma_{21}^*(\mathrm{t})$是t的慢变函数

$$\frac{d\rho_{21}}{dt} = \frac{d}{dt}(\sigma_{21}e^{-i\omega t}) = \frac{d\sigma_{21}}{dt}e^{-i\omega t} - i\omega\sigma_{21}e^{-i\omega t}$$

$$= -i\omega_0\sigma_{21}e^{-i\omega t} + i\frac{\mu E_0}{2\hbar}(e^{-i\omega t} + e^{i\omega t})(\rho_{11} - \rho_{22}) - \frac{\sigma_{21}e^{-i\omega t}}{T_2}$$

$$\Rightarrow \frac{d\sigma_{21}}{dt} = i(\omega - \omega_0)\sigma_{21} + i\frac{\mu E_0}{2\hbar}(1 + e^{i2\omega t})(\rho_{11} - \rho_{22}) - \frac{\sigma_{21}}{T_2}$$

：旋转波近似RWA

$$2i\frac{\mu}{\hbar}E(\mathrm{t})(\rho_{21} - \rho_{21}^*) = 2i\frac{\mu}{\hbar}\cdot\frac{1}{2}E_0(e^{-i\omega t} + e^{i\omega t})(\sigma_{21}e^{-i\omega t} - \sigma_{21}^*e^{i\omega t})$$

$$= i\frac{\mu}{\hbar}E_0(\sigma_{21}e^{-i2\omega t} + \sigma_{21} - \sigma_{21}^* - \sigma_{21}^*e^{i2\omega t}) = i\frac{\mu}{\hbar}E_0(\sigma_{21} - \sigma_{21}^*)$$

$$\begin{cases} \dfrac{d\sigma_{21}}{dt} = i(\omega-\omega_0)\sigma_{21} + i\dfrac{\mu E_0}{2\hbar}(\rho_{11}-\rho_{22}) - \dfrac{\sigma_{21}}{T_2} \\ \dfrac{d}{dt}(\rho_{11}-\rho_{22}) = i\dfrac{\mu}{\hbar}E_0(\sigma_{21}-\sigma_{21}^*) - \dfrac{(\rho_{11}-\rho_{22})-(\rho_{11}-\rho_{22})_0}{\tau} \end{cases}$$

求上面方程的稳态解,

$$\begin{cases} 0 = i(\omega-\omega_0)\sigma_{21} + i\dfrac{\mu E_0}{2\hbar}(\rho_{11}-\rho_{22}) - \dfrac{\sigma_{21}}{T_2} & (1) \\ 0 = i\dfrac{\mu}{\hbar}E_0(\sigma_{21}-\sigma_{21}^*) - \dfrac{(\rho_{11}-\rho_{22})-(\rho_{11}-\rho_{22})_0}{\tau} & (2) \end{cases}$$

$$Def: \quad \Omega \equiv \frac{\mu E_0}{2\hbar}$$

$$(1)+(1)^*: \; 0 = i(\omega-\omega_0)(\sigma_{21}-\sigma_{21}^*) - \frac{(\sigma_{21}+\sigma_{21}^*)}{T_2} = -(\omega-\omega_0)\cdot 2\,\mathrm{Im}\,\sigma_{21} - \frac{1}{T_2}\cdot 2\,\mathrm{Re}\,\sigma_{21}$$

$$\Rightarrow \; \mathrm{Re}\,\sigma_{21} = -(\omega-\omega_0)T_2\cdot \mathrm{Im}\,\sigma_{21} \qquad \text{.......} (3)$$

$$(1)-(1)^*: \; 0 = i(\omega-\omega_0)(\sigma_{21}+\sigma_{21}^*) + 2i\Omega(\rho_{11}-\rho_{22}) - \frac{1}{T_2}(\sigma_{21}-\sigma_{21}^*)$$

$$= i(\omega-\omega_0)2\,\mathrm{Re}\,\sigma_{21} + 2i\Omega(\rho_{11}-\rho_{22}) - \frac{1}{T_2}i2\,\mathrm{Im}\,\sigma_{21} \quad \text{.....}(4)$$

$$(2) \;\Rightarrow\; (\rho_{11}-\rho_{22}) = -i4\Omega\tau\,\mathrm{Im}\,\sigma_{21} + (\rho_{11}-\rho_{22})_0 \quad \text{......}(5)$$

将(3)(5)代入(4)得，
$$\operatorname{Im}\sigma_{21}=\frac{\Omega T_2(\rho_{11}-\rho_{22})_0}{(\omega-\omega_0)^2T_2^2+4\Omega^2T_2\tau+1}$$

$$\operatorname{Re}\sigma_{21}=-(\omega-\omega_0)\operatorname{Im}\sigma_{21}T_2=\frac{(\omega_0-\omega)\Omega T_2^2(\rho_{11}-\rho_{22})_0}{(\omega-\omega_0)^2T_2^2+4\Omega^2T_2\tau+1}$$

$$\rho_{11}-\rho_{22}=-4\Omega\operatorname{Im}\sigma_{21}\tau+(\rho_{11}-\rho_{22})_0=(\rho_{11}-\rho_{22})_0\frac{1+(\omega-\omega_0)^2T_2^2}{(\omega-\omega_0)^2T_2^2+4\Omega^2T_2\tau+1}$$

$$\Delta N=N(\rho_{11}-\rho_{22})=N(\rho_{11}-\rho_{22})_0\frac{1+(\omega-\omega_0)^2T_2^2}{(\omega-\omega_0)^2T_2^2+4\Omega^2T_2\tau+1}$$

$$=\Delta N_0\frac{1+(\omega-\omega_0)^2T_2^2}{(\omega-\omega_0)^2T_2^2+4\Omega^2T_2\tau+1}$$

记 $\Delta N_0=N(\rho_{11}-\rho_{22})_0$

宏观激化矢量，

$$P=N<\bar{\mu}>=N\mu(\rho_{21}+\rho_{12})=2N\mu\operatorname{Re}\rho_{21}=2N\mu(\operatorname{Re}\sigma_{21}\cos\omega t+\operatorname{Im}\sigma_{21}\sin\omega t)$$

$$=2N\mu\Omega T_2(\rho_{11}-\rho_{22})_0\frac{(\omega_0-\omega)T_2\cos\omega t+\sin\omega t}{(\omega-\omega_0)^2T_2^2+4\Omega^2T_2\tau+1}$$

对比电动力学中P的定义，

$P=\operatorname{Re}[\varepsilon_0\chi E_0e^{i\omega t}]=E_0(\varepsilon_0\chi'\cos\omega t+\varepsilon_0\chi''\sin\omega t)$，其中$\chi=\chi'-i\chi''$

$$\chi''(\omega) = \frac{\mu^2 T_2 \Delta N_0}{\varepsilon_0 \hbar} \frac{1}{(\omega - \omega_0)^2 T_2^2 + 4\Omega^2 T_2 \tau + 1} \quad \propto \ \Delta N \cdot g(\nu)$$

$$\chi'(\omega) = \frac{\mu^2 T_2 \Delta N_0}{\varepsilon_0 \hbar} \frac{(\omega_0 - \omega) T_2}{(\omega - \omega_0)^2 T_2^2 + 4\Omega^2 T_2 \tau + 1} \quad \propto \ \Delta N \cdot (\nu_0 - \nu) g(\nu)$$

$$\Delta N = \Delta N_0 \frac{1 + (\omega - \omega_0)^2 T_2^2}{(\omega - \omega_0)^2 T_2^2 + 4\Omega^2 T_2 \tau + 1}$$

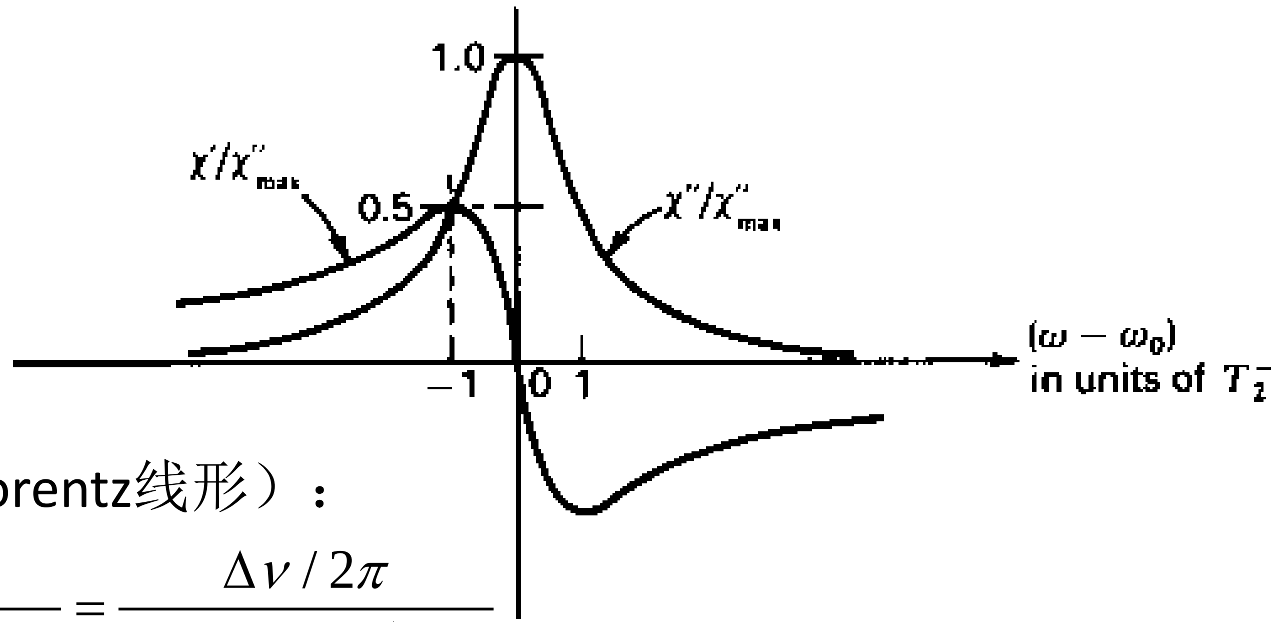
饱和效应： $E_0 \uparrow \ \Rightarrow \ \Delta N, \ \chi, \ \chi'' \downarrow$

饱和效应显著的判据：

$$4\Omega^2 T_2 \tau > (\omega - \omega_0)^2 T_2^2 + 1, \text{ 即 } \frac{\mu^2 E_0^2 T_2 \tau}{\hbar^2} > (\omega - \omega_0)^2 T_2^2 + 1$$

$$\chi''(\omega) = \frac{\mu^2 T_2 \Delta N_0}{\varepsilon_0 \hbar} \frac{1}{(\omega-\omega_0)^2 T_2^2 + 4\Omega^2 T_2 \tau + 1} \quad \propto \ \Delta N \cdot g(\nu)$$

$$\chi'(\omega) = \frac{\mu^2 T_2 \Delta N_0}{\varepsilon_0 \hbar} \frac{(\omega_0-\omega) T_2}{(\omega-\omega_0)^2 T_2^2 + 4\Omega^2 T_2 \tau + 1} \quad \propto \ \Delta N \cdot (\nu_0 - \nu) g(\nu)$$



定义线形函数（Lorentz线形）：

$$g(\nu) = \frac{2T_2}{4\pi^2(\nu-\nu_0)^2 T_2^2 + 1} = \frac{\Delta\nu / 2\pi}{(\nu-\nu_0)^2 + (\frac{\Delta\nu}{2})^2}$$

其中, $\Delta\nu \equiv \frac{1}{\pi T_2}$ 是线形函数的宽度

$$\int_{-\infty}^{\infty} g(\nu) d\nu = 1$$ 线形函数归一化

$\chi(\nu)$的物理意义

$$\vec{D}=\varepsilon_0\vec{E}+\vec{P}+\vec{P}_{跃迁}=\varepsilon\vec{E}+\varepsilon_0\chi(\nu)\vec{E}=\varepsilon[1+\frac{\varepsilon_0\chi(\nu)}{\varepsilon}]\vec{E}=\varepsilon'(\nu)\vec{E}$$

$$\varepsilon'(\nu)\equiv\varepsilon[1+\frac{\varepsilon_0\chi(\nu)}{\varepsilon}]$$

$$k'=\omega\sqrt{\mu\varepsilon'}=\omega\sqrt{\mu\varepsilon[1+\frac{\varepsilon_0\chi(\nu)}{\varepsilon}]}=\omega\sqrt{\mu\varepsilon}(1+\frac{\varepsilon_0\chi(\nu)}{\varepsilon})^{\frac{1}{2}}$$

$$\approx k[1+\frac{\varepsilon_0\chi(\nu)}{2\varepsilon}]=k[1+\frac{\chi'(\nu)}{2n^2}]-\mathrm{i}\frac{k\chi''(\nu)}{2n^2}$$

非共振吸收介质的传播常数：

$$k=\omega\sqrt{\mu\varepsilon}$$

$$\varepsilon=n^2$$

$$\Delta k=k\frac{\chi'(\nu)}{2n^2}$$

$$\boxed{\gamma(\nu)=-k\frac{\chi''(\nu)}{n^2}}$$

平面波经过该原子系统传播：

$$E(\mathrm{z},\mathrm{t})=\mathrm{Re}\{\mathrm{E}\,e^{i(\omega\mathrm{t}-k'\mathrm{z})}\}=\mathrm{Re}\{\mathrm{E}\,e^{i[\omega\mathrm{t}-(k+\Delta k)\mathrm{z}]+\frac{\gamma}{2}\mathrm{z}}\}$$

Δk 传播常数的改变量； γ 吸收或增益系数

Kramers-Kronig relations (K-K关系)

$$\int_{c'} \frac{\chi(\omega')}{\omega'-\omega}\,d\omega' + \int_{-R}^{\omega-\varepsilon} \frac{\chi(\omega')}{\omega'-\omega}\,d\omega' + \int_{\omega+\varepsilon}^{R} \frac{\chi(\omega')}{\omega'-\omega}\,d\omega' + \int_{c} \frac{\chi(\omega')}{\omega'-\omega}\,d\omega' = 0$$

留数定理

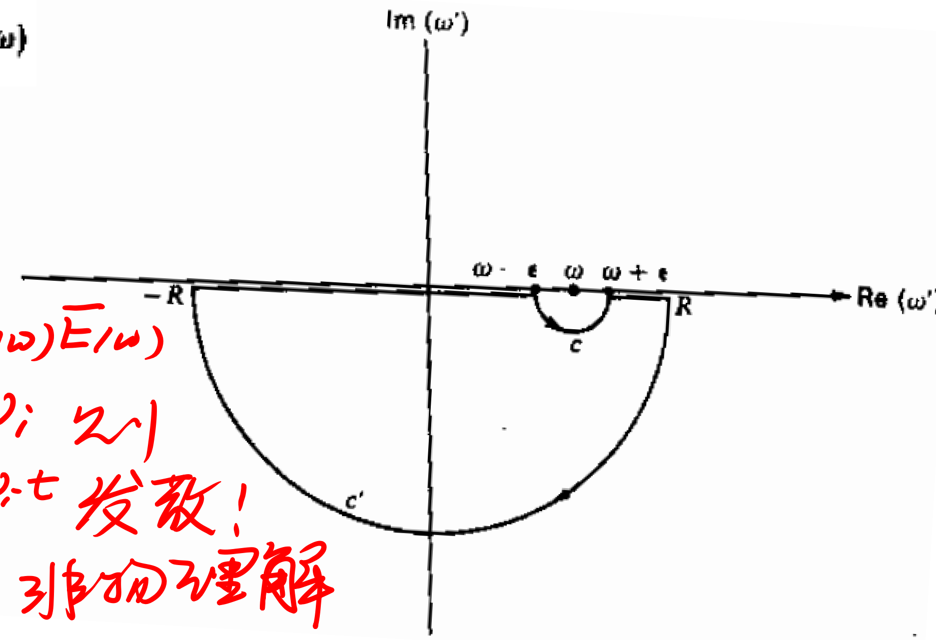
$R\to\infty$时 $\chi(\omega)=0$: $\int_{C'} \frac{\chi(\omega')}{\omega'-\omega}d\omega' = 0$

→ $R\to\infty$, $\varepsilon\to 0$时 为主值积分: $\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega'-\omega}d\omega'$

$$\omega' = \omega + \varepsilon e^{i\phi}$$

$$\lim_{\varepsilon\to 0} \int_{c} \frac{\chi(\omega')}{\omega'-\omega}\,d\omega' = \lim_{\varepsilon\to 0} \int_{\pi}^{2\pi} \frac{\chi(\omega+\varepsilon e^{i\phi})\, i\varepsilon e^{i\phi}}{\varepsilon e^{i\phi}}\,d\phi = \pi i \chi(\omega)$$

$$\chi(\omega) = \frac{i}{\pi}\,\mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{\chi(\omega')}{\omega'-\omega}\,d\omega'$$



$$\chi'(\omega) = \frac{1}{\pi}\,\mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{\chi''(\omega')}{\omega'-\omega}\,d\omega'$$

$$\chi''(\omega) = -\frac{1}{\pi}\,\mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{\chi'(\omega')}{\omega'-\omega}\,d\omega'$$

$P(\omega)=\chi(\omega)E(\omega)$

若 $\omega_0 - i\omega_i$ 则 $e^{i\omega_0 t}e^{\omega_i t}$ 发散!

非物理解

$$\chi(\omega) = \chi'(\omega) - i\chi''(\omega)$$

$$= -\frac{\mu^2 \Delta N_0}{\varepsilon_0 \hbar}\, \frac{\omega - [\omega_0 - (i/T_2)]}{\{\omega - [\omega_0 - (i/T_2)(1+s^2)^{1/2}]\}\{\omega - [\omega_0 + (i/T_2)(1+s^2)^{1/2}]\}}$$

: $s=0$时满足 K-K关系.

三、自发与感应跃迁、增益系数

模型： $|2,\ n_l> \ \rightarrow\ |1,\ n_l+1>$

偶极相互作用哈密顿量：$H' = -e\vec{E}_l(z,t)\cdot\vec{r} = -eE_l(z,t)y$ 类平面波，$\vec{E}_l // \hat{y}$

电磁场二次量子化： $H' = -iey\sqrt{\frac{\hbar\omega_l}{V\varepsilon}}(a_l^+ - a_l)\sin k_l z$ V是腔体积

产生、湮灭算符： $a^+|n> = \sqrt{n+1}|n+1>$ $[a,a^+]=1$

$a|n> = \sqrt{n}|n-1>$

|m> → |k> 的跃迁速率：

$$W' = \frac{2\pi}{\hbar}|H'_{km}|^2\ \delta(E_m - E_k)$$

$$= \frac{2\pi e^2\omega_l}{V\varepsilon}\left|<1,n_l+1|y(a_l^+ - a_l)|2,n_l>\right|^2 \sin^2 k_l z\cdot\delta(E_2 - E_1 - \hbar\omega_l)$$

$$= \frac{2\pi e^2\omega_l}{V\varepsilon}\left|<1,n_l+1|y\,a_l^+|2,n_l>\right|^2 \sin^2 k_l z\cdot\delta(E_2 - E_1 - \hbar\omega_l)$$

$$= \frac{2\pi e^2\omega_l y_{12}^2}{V\varepsilon}(n_l+1)\sin^2 k_l z\cdot\delta(E_2 - E_1 - \hbar\omega_l)$$

$$W_i' = \frac{2\pi e^2 \omega_l y_{12}^2}{V\varepsilon} n_l \sin^2 \mathrm{k}_l\, z \cdot \delta(\mathrm{E}_2 - \mathrm{E}_1 - \hbar\omega_l)$$

$$W'_{自发} = \frac{2\pi e^2 \omega_l y_{12}^2}{V\varepsilon} \sin^2 \mathrm{k}_l\, z \cdot \delta(\mathrm{E}_2 - \mathrm{E}_1 - \hbar\omega_l)$$

吸收：|1> → |2>跃迁 $\left| < 2, n_l - 1 \,|\, y(\mathrm{a}_l^+ - \mathrm{a}_l) \,|\, 1, n_l > \right|^2 = n_l y_{12}^2$

$$W'_{1\to 2} = W_i'_{2\to 1} = \frac{2\pi e^2 \omega_l y_{12}^2}{V\varepsilon} n_l \sin^2 \mathrm{k}_l\, z \cdot \delta(\mathrm{E}_2 - \mathrm{E}_1 - \hbar\omega_l)$$

① 受激发射和吸收的跃迁速率表达式相同，都 $\propto n_l$

② 受激发射的光子和激发光子属于相同的模式

$$\left| < 1, n_l + 1 \,|\, y\, \mathrm{a}_l^+ \,|\, 2, n_l > \right|^2 = (n_l + 1) y_{12}^2$$

③ 自发辐射与腔内*l*模的光子数无关，另外， $\frac{W_i'}{W'_{自发}} = n_l$

自发辐射寿命：|2>到连续模的寿命被称为|2>能级的自发辐射寿命

模式能量密度(单位能量间隔的模式数): $P(\mathrm{E}=\mathrm{h}\nu_l)=\dfrac{8\pi\nu_l^2 Vn^3}{hc^3}$

$$W_{自发}=\frac{1}{t_{自发}}=\int W'_{自发}P\cdot d\,\mathrm{E}=\frac{2n^3e^2y_{12}^2\omega^3g_1}{\varepsilon hc^3}=\frac{2n^3\mu^2\omega^3g_1}{\varepsilon hc^3}$$

$\mu\equiv ey_{12}$

$\hbar\omega\equiv \mathrm{E}_2-\mathrm{E}_1$

单色场的感应跃迁

体系的能级准连续分布,辐射场具有一定的线形*g(ν)*

E_2-E_1的间隔为$E\rightarrow E+dE$

光强: $I_\nu=\dfrac{n_lh\nu_l}{V}\dfrac{c}{n}=\dfrac{cn_lh\nu_l}{nV}$

$$(W_{21})_i=\int_E W_i'\frac{1}{h}g\left(\frac{E_2-E_1}{h}\right)\mathrm{dE}=\frac{\pi e^2\omega_l y_{12}^2 n_l g_1}{hV\varepsilon}g(\nu_l)$$

$$(W_{21})_i=\frac{\lambda^2 I_\nu}{8\pi h\nu n^2 t_{自发}}g(\nu)$$

$$(W_{12})_i=(W_{21})_i\frac{g_2}{g_1}=\frac{g_2}{g_1}\frac{\lambda^2 I_\nu}{8\pi h\nu n^2 t_{自发}}g(\nu)$$

增益系数

问题： 频率为ν的单色光波通过二能级原子体系。|2>能级的原子密度为N_2，|1>能级的原子密度为N_1，单位时间、单位面积内|2> → |1>的感应跃迁与|1> → |2>的跃迁之差对应于感应辐射

$$\frac{\text{功率}}{\text{体积}}=\frac{P}{V}=[\mathrm{N}_2(W_{21})_i-\mathrm{N}_1(W_{12})_i]\cdot \mathrm{h}\nu=[\mathrm{N}_2-\mathrm{N}_1\frac{g_2}{g_1}]\frac{\lambda^2 I_\nu}{8\pi n^2 t_{\text{自发}}}g(\nu)$$

$$\frac{\mathrm{dI}_\nu(\mathrm{z})}{\mathrm{dz}}=\gamma(\nu)\mathrm{I}_\nu(\mathrm{z})=\frac{dW}{dz\cdot ds\cdot dt}=\frac{dW/dt}{dV}=\frac{dP}{dV}$$

$$Def:\ \Delta N=[\mathrm{N}_2-\mathrm{N}_1\frac{g_2}{g_1}]$$

$$\gamma(\nu)=[\mathrm{N}_2-\mathrm{N}_1\frac{g_2}{g_1}]\frac{\lambda^2}{8\pi n^2 t_{\text{自发}}}g(\nu)=\frac{\lambda^2}{8\pi n^2 t_{\text{自发}}}\cdot\Delta\mathrm{N}\cdot g(\nu)$$

对比半经典理论结果：

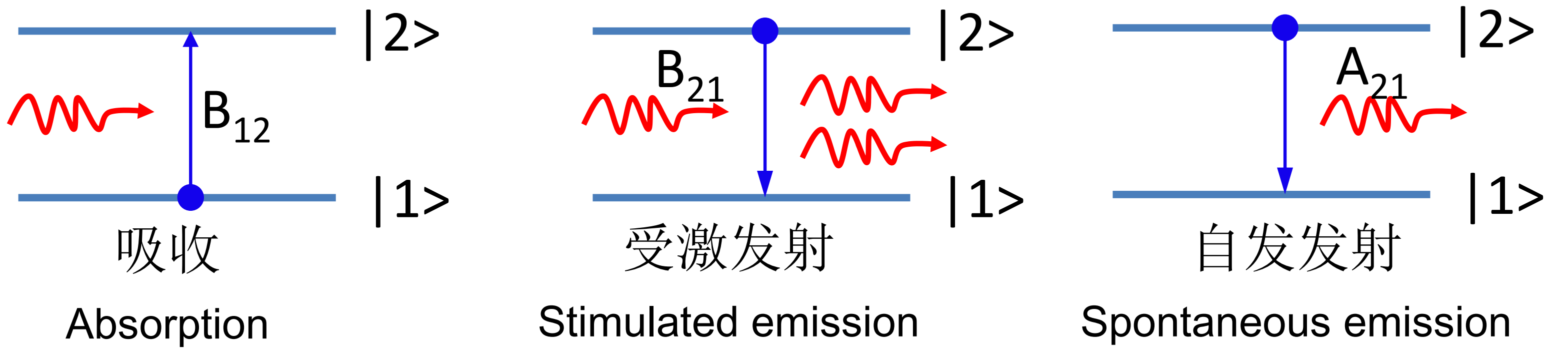
$$\gamma(\nu)=-k\frac{\varepsilon_0\chi''(\nu)}{n^2}$$

$$g(\nu)=\frac{\Delta\nu/2\pi}{(\nu-\nu_0)^2+(\frac{\Delta\nu}{2})^2}\quad \text{其中}, \Delta\nu\equiv\frac{1}{\pi T_2}$$

$$\chi''(\omega)=\frac{\mu^2 T_2\Delta N_0}{\varepsilon_0\hbar}\frac{1}{(\omega-\omega_0)^2T_2^2+4\Omega^2T_2\tau+1}\propto\Delta N\cdot g(\nu)$$

四、Einstein系数

用经典理论对自发跃迁、感应跃迁的唯象描述



|1>、|2>能级的粒子数变化

$$\begin{cases} \dfrac{dN_2}{dt} = N_1\rho(\nu)B_{12} - N_2\rho(\nu)B_{21} - N_2A \\ \dfrac{dN_1}{dt} = -N_1\rho(\nu)B_{12} + N_2\rho(\nu)B_{21} + N_2A \end{cases}$$

感应跃迁速率：

$$(W_{21})_i = \rho(\nu)B_{21}$$

$$(W_{12})_i = \rho(\nu)B_{12}$$

显然，$\dfrac{d(N_1+N_2)}{dt} = 0$，粒子数守恒！

稳态时，$\frac{dN_1}{dt}=\frac{dN_2}{dt}=0 \Rightarrow \frac{N_2}{N_1}=\frac{\rho(\nu)B_{12}}{\rho(\nu)B_{21}+A_{21}}$

假设入射场的能量密度为热平衡时的黑体辐射能量密度

$$\rho(\nu)=\frac{8\pi n^3 h\nu^3}{c^3}\frac{1}{e^{h\nu/kt}-1}$$

$T\rightarrow\infty$时，$\rho(\nu)\rightarrow\infty$；$\frac{N_2/g_2}{N_1/g_1}\rightarrow 1$

$$\therefore \quad \frac{B_{12}}{B_{21}}=\frac{g_2}{g_1} \Rightarrow B_{12}=\frac{g_2}{g_1}B_{21}$$

若不考虑简并，则$B_{12}=B_{21}$

一般温度下，粒子数分布满足Boltzman分布

$$\frac{N_2}{N_1}=\frac{g_2}{g_1}e^{-h\nu/kt}=\frac{\rho(\nu)B_{12}}{\rho(\nu)B_{21}+A} \Rightarrow \frac{A}{B_{21}}=\frac{8\pi n^3 h\nu^3}{c^3} \propto \nu^3$$

$$(W_{21})_i=\rho(\nu)B_{21}=\frac{c^3 A}{8\pi n^3 h\nu^3}\rho(\nu)=\frac{c^3}{8\pi n^3 h\nu^3 t_{自发}}\rho(\nu)$$

与量子理论的比较：

单色场作用下的感应跃迁：

$$(W_{21})_i = \frac{\lambda^2 I_\nu}{8\pi h\nu n^2 t_{自发}} g(\nu) = \frac{\lambda^2 \frac{c}{n}\rho_\nu}{8\pi h\nu n^2 t_{自发}} g(\nu) = \frac{c^3\rho_\nu}{8\pi h\nu^3 n^3 t_{自发}} g(\nu)$$

光强与辐射能量密度之间的关系： $I_\nu = \frac{c}{n}\rho_\nu$

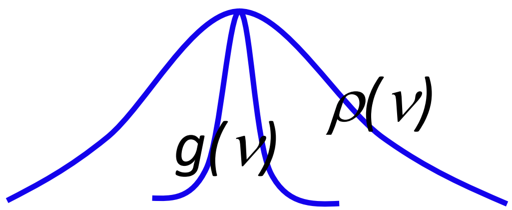
单色场 $\Rightarrow$ 非单色场： $\rho_\nu \rightarrow \rho(\nu)d\nu$

$$(W'_{21})_i = \int_\nu (W_{21})_i = \int_\nu \frac{c^3\rho(\nu)}{8\pi h\nu^3 n^3 t_{自发}} g(\nu)\,\mathrm{d}\nu = \frac{c^3}{8\pi h n^3 t_{自发}} \int_\nu \frac{\rho(\nu)}{\nu^3} g(\nu)\,\mathrm{d}\nu$$

$$= \frac{c^3}{8\pi h n^3 t_{自发}} \frac{\rho(\nu)}{\nu^3} \int_\nu g(\nu)\,\mathrm{d}\nu = \frac{c^3}{8\pi h\nu^3 n^3 t_{自发}} \rho(\nu)$$

$$\int_\nu g(\nu)\,\mathrm{d}\nu = 1$$

与Einstein处理的结果相同！



一般地，$g(\nu)$的谱宽<< $\rho(\nu)$的谱宽

五、均匀加宽和非均匀加宽

谱线的线型因子g(ν, ν$_0$):

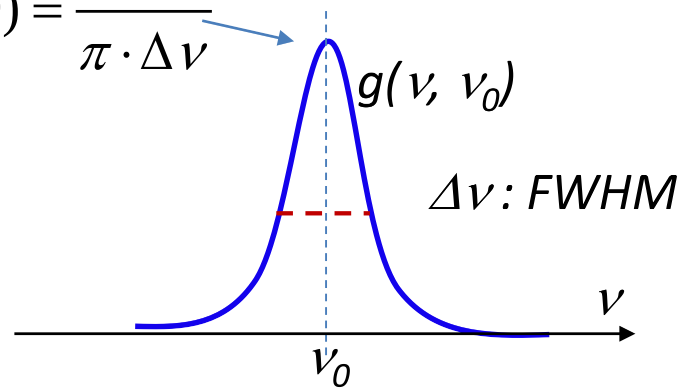
Def: 粒子系统（原子、分子或离子）自发辐射或对外界入射光场的响应（吸收或增益）都呈现频率的一定统计分布，称为谱线线型。一般地，我们引入一个归一化函数——线型因子g(ν, ν$_0$)来描述

$$\int_0^{\infty} g(\nu,\nu_0)\,\mathrm{d}\nu=1,$$ 其中ν_0是谱线的中心频率.

以Lorentz线型为例，

$$g(\nu-\nu_0)=\frac{\Delta\nu/2\pi}{(\nu-\nu_0)^2+(\frac{\Delta\nu}{2})^2}$$

$$g(0)=\frac{2}{\pi\cdot\Delta\nu}$$



谱线的自然宽度: 是自发辐射跃迁的结果，自然宽度是谱线变窄的极限

t=0时刻|2>能级的粒子数为N_{20}，t>0开始$N_2(t)$由于自发辐射而衰减，

$$\frac{dN_2(t)}{dt} = -AN_2(t) \Rightarrow N_2(t) = N_{20}e^{-At}$$

自发辐射产生的光场能量，

$$P(t) = N_2(t)A \cdot h\nu = N_{20}A \cdot h\nu e^{-At} = P_0 e^{-At}$$

经典谐振子: $\ddot{x} - i\frac{\gamma}{2}\dot{x} + \omega_0^2 x = 0$

$x = x_0 e^{-i\omega_0 t} e^{-\frac{\gamma}{2}t}$　振子强度: $|x|^2 = x_0^2 e^{-\gamma t}$

振子强度谱:

$$I(\omega) = |\mathrm{FT}\{x\}|^2 \propto \frac{1}{(\omega-\omega_0)^2 + \frac{\gamma^2}{4}}$$

对比经典谐振子可知:

光场的衰减系数 **γ=A**

光场的强度谱

$$I(\nu) = \frac{KI_0}{(\nu-\nu_0)^2 + (\frac{A}{4\pi})^2} = I_0 \cdot g(\nu,\nu_0)$$

$$g(\nu,\nu_0) = \frac{K}{(\nu-\nu_0)^2 + (\frac{A}{4\pi})^2}$$

$$= \frac{A/4\pi^2}{(\nu-\nu_0)^2 + (\frac{A}{4\pi})^2}$$

$$= \frac{\Delta\nu_N/2\pi}{(\nu-\nu_0)^2 + (\frac{\Delta\nu_N}{2})^2}$$

由归一化条件确定

$$K = \frac{A}{4\pi^2}$$

$$\Delta\nu_N = \frac{A}{2\pi}$$

$$\Delta\nu_N = \frac{A}{2\pi}$$

自然线宽意味着原子能级的不确定性，即能级具有一定的宽度。

若能级|1>是基态(寿命无限长)，则$\Delta\nu_N$表示能级|2>的宽度。

$$\Delta E_2 = h\Delta\nu_N = h\frac{A}{2\pi} = \hbar\frac{1}{\tau_{21}} \Rightarrow \Delta E_2 \cdot \tau_{21} = \hbar$$

Heisenberg不确定关系。

若|1>、 |2>都是非基态能级，则|2> →|1>跃迁的谱线宽度：

$$\Delta\nu = \frac{1}{\tau_1} + \frac{1}{\tau_2}$$

谱线的均匀加宽:

原子是不可分的，谱线的宽度源自所有原子的共同作用

均匀加宽的类型：

① 原子与声子或其他原子之间的非弹性碰撞（碰撞加宽）

② 自发辐射或无辐射跃迁（寿命加宽）

③ 破坏相位的弹性碰撞（驰豫加宽）

④ 与电磁场相互作用的加宽 （功率加宽）

$$\chi''(\omega)=\frac{\mu^2T_2\Delta N_0}{\varepsilon_0\hbar}\frac{1}{(\omega-\omega_0)^2T_2^2+\frac{\mu^2E_0^2}{\hbar^2}T_2\tau+1}$$

无外场E_0=0时：

$$\chi''(\omega)=\frac{\mu^2T_2\Delta N_0}{\varepsilon_0\hbar}\frac{1}{(\omega-\omega_0)^2T_2^2+1}=\frac{\mu^2\Delta N_0}{\varepsilon_0\hbar}\frac{1/T_2}{(\omega-\omega_0)^2+1/T_2^2}\propto\frac{\Delta\omega}{(\omega-\omega_0)^2+(\frac{\Delta\omega}{2})^2}$$

其中，$\Delta\omega=\frac{2}{T_2}$ 或 $\Delta\nu=\frac{1}{\pi T_2}$ 驰豫加宽

外场不为0时：

$$\Delta\nu_{饱和}=\Delta\nu\sqrt{\frac{\mu^2E_0^2}{\hbar^2}T_2\tau+1}$$ 功率加宽或饱和加宽

谱线的非均匀加宽（多普勒加宽）：

光谱线型中不同的频率对应于不同运动速度的粒子

由于气体分子的运动产生多普勒效应： $\nu=\nu_0+\frac{\mathrm{v}_x}{c}\nu_0$

气体分子运动在平衡温度T的分布满足Maxwell分布函数：

$$f(\mathrm{v}_x,\mathrm{v}_y,\mathrm{v}_z)=(\frac{M}{2\pi kT})^{3/2}\exp\{-\frac{M}{2kT}(\mathrm{v}_x^2+\mathrm{v}_y^2+\mathrm{v}_z^2)\}$$

跃迁频率在ν→ν+dν之间的几率g(ν)dν等价于

速度分量在 $\mathrm{v}_x=\frac{c}{\nu_0}(\nu-\nu_0)\ \rightarrow\ \frac{c}{\nu_0}(\nu+d\nu-\nu_0)$之间的几率

$$g(\nu)\,\mathrm{d}\nu=(\frac{M}{2\pi kT})^{3/2}\iint e^{-\frac{M}{2kT}(\mathrm{v}_y^2+\mathrm{v}_z^2)}\,\mathrm{dv}_y\mathrm{dv}_z\cdot e^{-\frac{M}{2kT}\frac{c^2}{\nu_0^2}(\nu-\nu_0)^2}\,\frac{c}{\nu_0}d\nu$$

$$=(\frac{M}{2\pi kT})^{\frac{1}{2}}e^{-\frac{M}{2kT}\frac{c^2}{\nu_0^2}(\nu-\nu_0)^2}\,\frac{c}{\nu_0}d\nu\quad\Rightarrow g(\nu)=\frac{c}{\nu_0}(\frac{M}{2\pi kT})^{\frac{1}{2}}e^{-\frac{M}{2kT}\frac{c^2}{\nu_0^2}(\nu-\nu_0)^2}$$

半高谱宽 $\Delta\nu_D=\nu_0\sqrt{\frac{2kT}{c^2M}\ln 2}$

$$g(\nu)=\frac{2(\ln 2)^{1/2}}{\pi^{1/2}\Delta\nu_D}e^{-[4\ln 2(\nu-\nu_0)^2/\Delta\nu_D]}$$

高斯线型

六、增益饱和效应

概念： 一定泵浦强度下，反转粒子数（或增益系数）随光强的增加而减小的现象

增益系数： $\gamma(\nu)=[\mathrm{N}_2-\mathrm{N}_1\frac{g_2}{g_1}]\frac{\lambda^2}{8\pi n^2 t_{\text{自发}}}g(\nu)=\frac{\lambda^2}{8\pi n^2 t_{\text{自发}}}\cdot\Delta\mathrm{N}\cdot g(\nu)$

反转粒子数： $\Delta N=\Delta N_0\frac{1+(\omega-\omega_0)^2T_2^2}{(\omega-\omega_0)^2T_2^2+4\Omega^2T_2\tau+1}$

$$g(\nu)=\frac{2T_2}{4\pi^2(\nu-\nu_0)^2T_2^2+1}=\frac{2T_2}{(\omega-\omega_0)^2T_2^2+1}$$

（1）均匀加宽的增益饱和

$$\Delta N=\Delta N_0\frac{1+(\omega-\omega_0)^2T_2^2}{(\omega-\omega_0)^2T_2^2+1+4\Omega^2T_2\tau}$$

$$=\Delta N_0\frac{1}{1+\frac{\mu^2E_0^2}{\hbar^2}T_2\tau\Big/[1+(\omega-\omega_0)^2T_2^2]}$$

$$=\Delta N_0\frac{1}{1+\frac{\mu^2E_0^2\tau}{2\hbar^2}g(\nu)}=\Delta N_0\frac{1}{1+I_\nu/I_s(\nu)}$$

光强： $I_\nu=\frac{cn\varepsilon_0E_0^2}{2}$

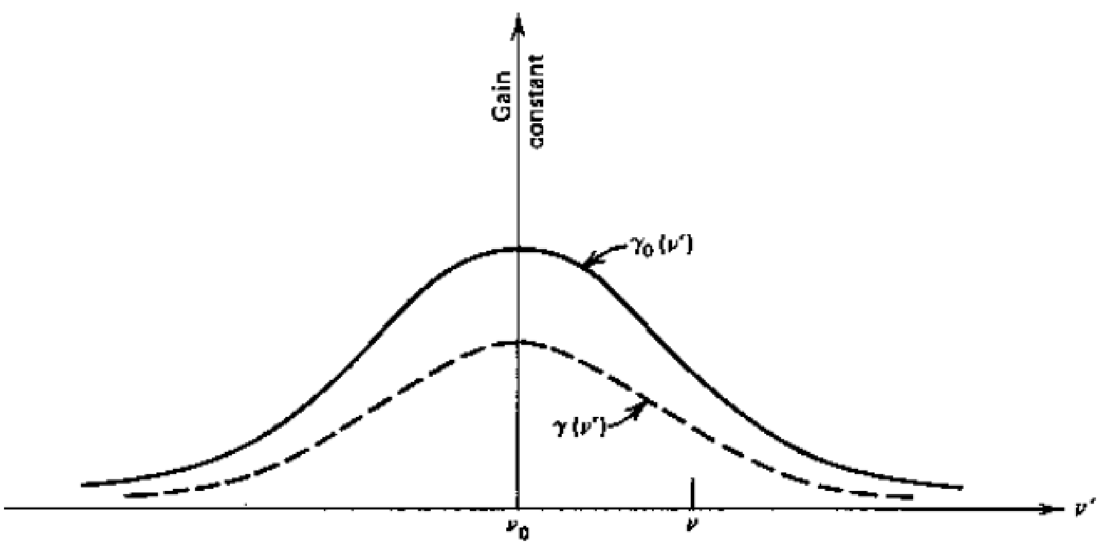
饱和光强：

$$I_s(\nu)=\frac{4\pi n^2h\nu}{(\tau/\mathrm{t}_{\text{自发}})\lambda^2g(\nu)}=I_{so}/g(\nu)$$

$$\gamma(\nu)=\frac{\lambda^2}{8\pi n^2 t_{自发}}\cdot \Delta \mathrm{N}\cdot g(\nu)=\frac{\lambda^2}{8\pi n^2 t_{自发}}\cdot \Delta N_0\cdot g(\nu)\frac{1}{1+I_\nu/I_s(\nu)}=\frac{\gamma_0}{1+I_\nu/I_s(\nu)}$$

小信号增益**:** $\gamma_0=\Delta N_0\dfrac{\lambda^2 g(\nu)}{8\pi n^2 t_{自发}}$

当 $I_\nu=I_s(\nu)$时**,** $\gamma(\nu)=\gamma_0/2$; $\Delta N=\Delta N_0$ **/2,** 称为"饱和"**!**



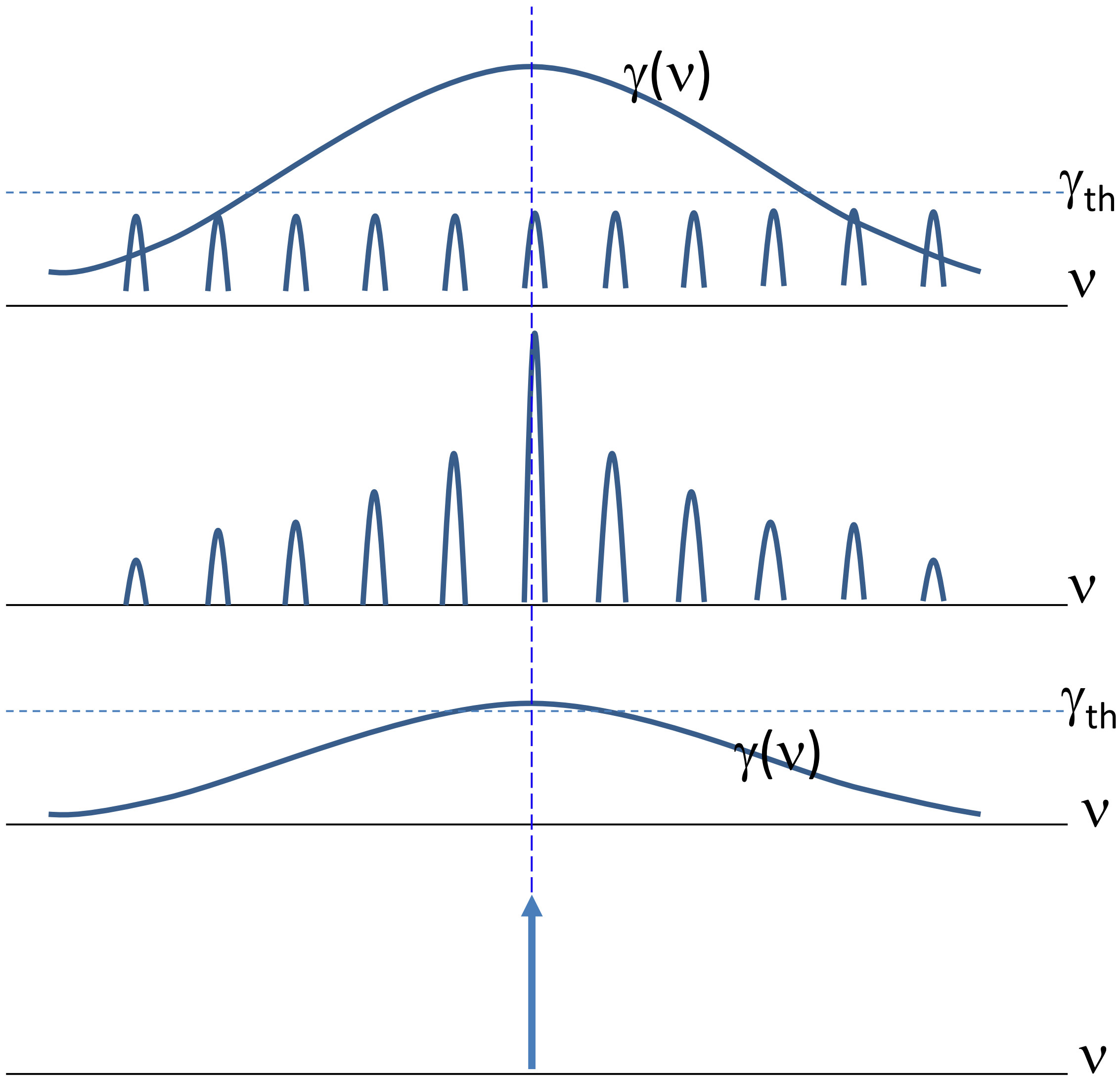
① 饱和光强**:** $I_s(\nu)=I_{so}/g(\nu)$ 增益曲线的中心下降快,两边下降慢

② 模式竞争和单模输出

自发辐射形
成多模振荡

模式竞争
增益饱和

单模运转



（2）非均匀加宽的增益饱和

分析思路:

$$\int P(\nu_\xi)\,\mathrm{d}\nu_\xi = 1$$

① 将原子按辐射中心频率ν_ξ分类;

② 定义几率函数P(ν_ξ)表示中心频率为ν_ξ的原子所占的比例,
那么P(ν_ξ) d ν_ξ表示中心频率在ν_ξ ~ ν_ξ +dν_ξ之间的原子的比例;

③ 具有相同中心频率ν_ξ的原子按均匀加宽情形处理, 其线型函数为$g^\xi(\nu)$

$$\int g^\xi(\nu)\,\mathrm{d}\nu = 1$$

总的跃迁线型函数为: $g(\nu) = \int P(\nu_\xi) g^\xi(\nu)\,\mathrm{d}\nu_\xi$

对中心频率为ν_ξ的一类原子, 其原子数为ΔN_0P(ν_ξ) d ν_ξ

$$\gamma_\xi(\nu) = \frac{\Delta N_0 \lambda^2}{8\pi n^2 t_{自发}}\left[\frac{P(\nu_\xi)\,\mathrm{d}\nu_\xi}{\frac{1}{g^\xi(\nu)} + \frac{I_\nu \phi \lambda^2}{4\pi n^2 h\nu}}\right], \quad 其中,\ \phi = \frac{\tau}{t_{自发}}$$

总增益: $$\gamma(\nu) = \frac{\Delta N_0 \lambda^2}{8\pi n^2 t_{自发}} \int_0^\infty \frac{P(\nu_\xi)\,\mathrm{d}\nu_\xi}{\frac{1}{g^\xi(\nu)} + \frac{I_\nu \phi \lambda^2}{4\pi n^2 h\nu}}$$

讨论：

① 小信号近似下：$I_\nu \ll \dfrac{4\pi n^2 h\nu}{\phi\lambda^2 g^\xi(\nu)} = I^\xi(\nu)$ 饱和光强

$$\gamma(\nu) = \frac{\Delta N_0 \lambda^2}{8\pi n^2 t_{自发}} \int_0^\infty \frac{P(\nu_\xi)\mathrm{d}\nu_\xi}{\dfrac{1}{g^\xi(\nu)}} = \frac{\Delta N_0 \lambda^2}{8\pi n^2 t_{自发}} \int_0^\infty g^\xi(\nu) P(\nu_\xi)\mathrm{d}\nu_\xi = \frac{\Delta N_0 \lambda^2}{8\pi n^2 t_{自发}} g(\nu)$$

和均匀加宽的情形相同

② 强非均匀加宽情况下: $P(\nu_\xi)$是常数$P(\nu)$

假设"ξ类"中的原子是无差别的均匀加宽情形，即

$$g^\xi(\nu) = \frac{\Delta\nu / 2\pi}{(\nu - \nu_\xi)^2 + (\frac{\Delta\nu}{2})^2}$$

$\Delta\nu$是非均匀谱线中的均匀线宽

$$\gamma(\nu) = \frac{\Delta N_0 \lambda^2 \Delta\nu}{16\pi^2 n^2 t_{自发}} P(\nu) \int_0^\infty \frac{\mathrm{d}\nu_\xi}{(\nu - \nu_\xi)^2 + (\frac{\Delta\nu}{2})^2 + \dfrac{I_\nu \phi\lambda^2 \Delta\nu}{8\pi^2 n^2 h\nu}}$$

$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}$

$$= \frac{\Delta N_0 \lambda^2 P(\nu)}{8\pi n^2 t_{自发}} \frac{1}{\sqrt{1 + I_\nu \phi\lambda^2 / 2\pi^2 n^2 h\nu\Delta\nu}} = \gamma_0(\nu) \frac{1}{\sqrt{1 + I_\nu / I_s}}$$

饱和光强:

$$I_s = \frac{2\pi^2 n^2 h\nu\Delta\nu}{\phi\lambda^2}$$

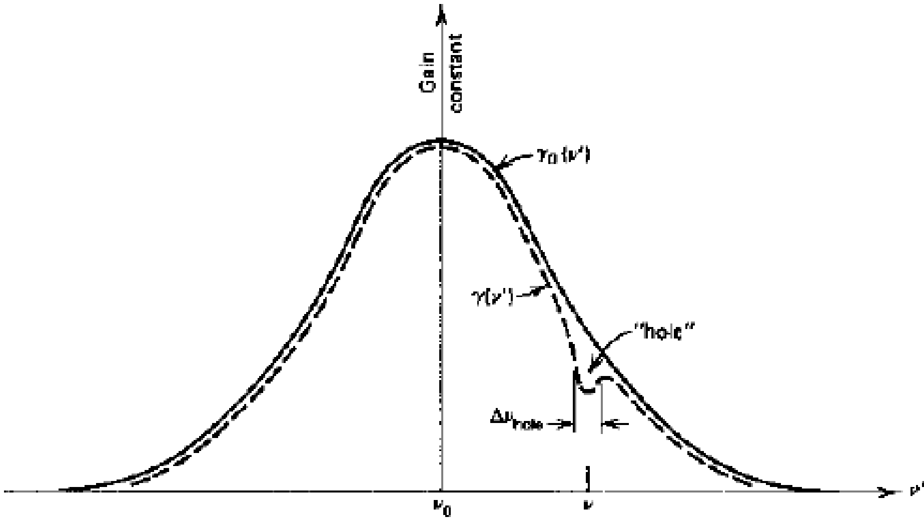
与均匀加宽相比:

① I_s与线型g无关

② "饱和"速度较慢

③ "烧孔"效应

频率为 ν的强光泵浦; 频率为ν′的弱光探测γ(ν′)



$$\gamma(\nu')=\frac{\Delta N_0\lambda^2}{8\pi n^2 t_{自发}}\int_0^{\infty}P(\nu_\xi)g^{\xi}(\nu)\,d\nu_\xi\cdot\left[\frac{(\nu'-\nu_\xi)^2+(\frac{\Delta\nu}{2})^2}{(\nu'-\nu_\xi)^2+(\frac{\Delta\nu}{2})^2+\frac{I_\nu\phi\lambda^2\Delta\nu}{8\pi^2 n^2 h\nu}}\right]$$

$\gamma(\nu)\propto\Delta N g(\nu)$　消去 ΔN

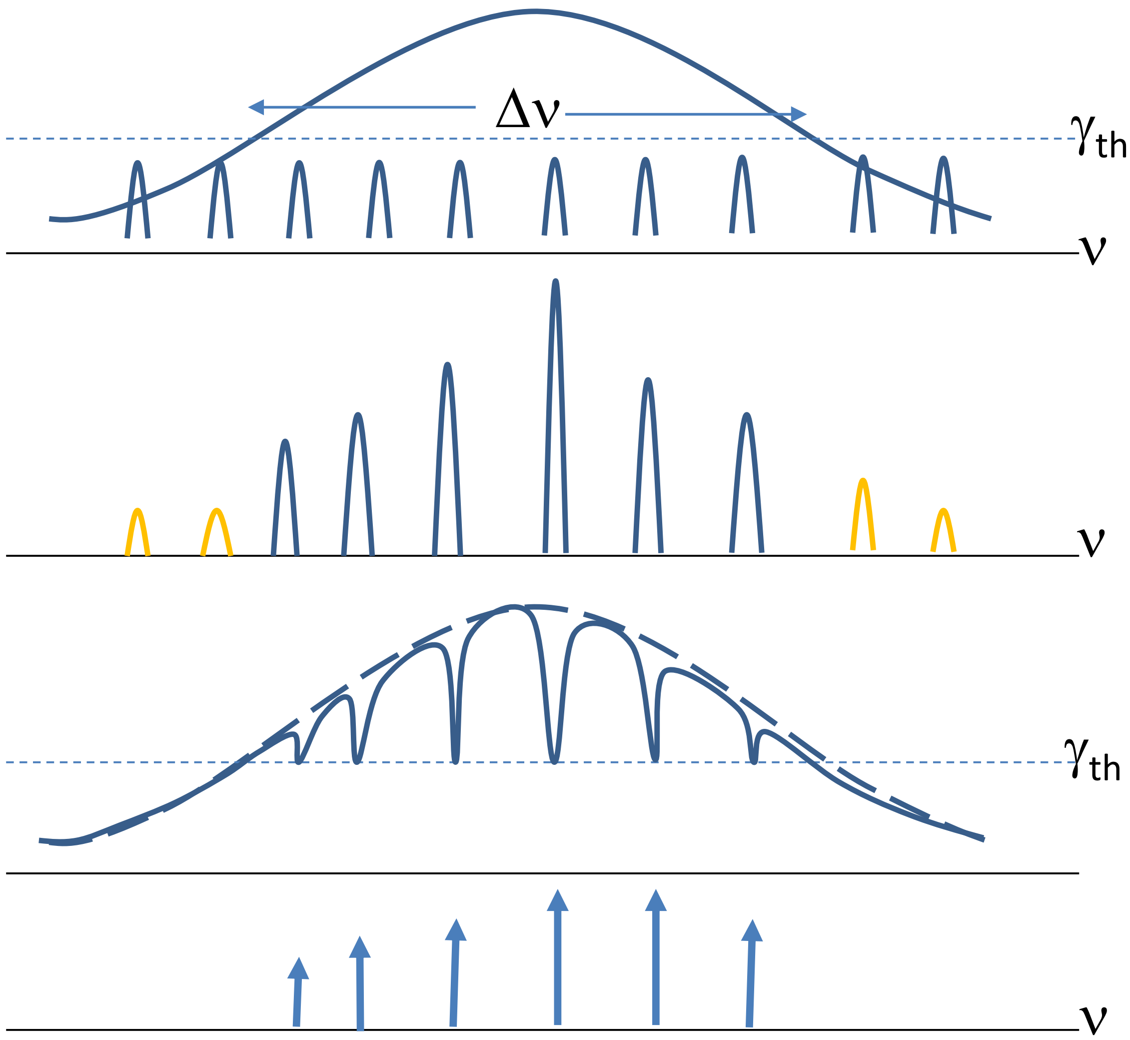
由于泵浦光的谱线宽很小，故上式中[…]里面的 ν_ξ 可用 ν 代替

$$=\frac{\Delta N_0\lambda^2}{8\pi n^2 t_{自发}}\left[\int_0^{\infty}P(\nu_\xi)g^{\xi}(\nu)\,d\nu_\xi\right]\cdot\left[\frac{(\nu'-\nu)^2+(\frac{\Delta\nu}{2})^2}{(\nu'-\nu)^2+(\frac{\Delta\nu}{2})^2+\frac{I_\nu\phi\lambda^2\Delta\nu}{8\pi^2 n^2 h\nu}}\right]$$

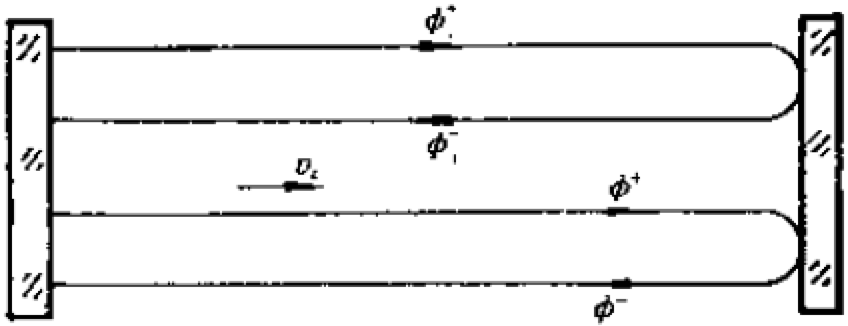
$$=\gamma_0(\nu')\left[\frac{(\nu'-\nu)^2+(\frac{\Delta\nu}{2})^2}{(\nu'-\nu)^2+(\frac{\Delta\nu}{2})^2+\frac{I_\nu\phi\lambda^2\Delta\nu}{8\pi^2 n^2 h\nu}}\right]$$

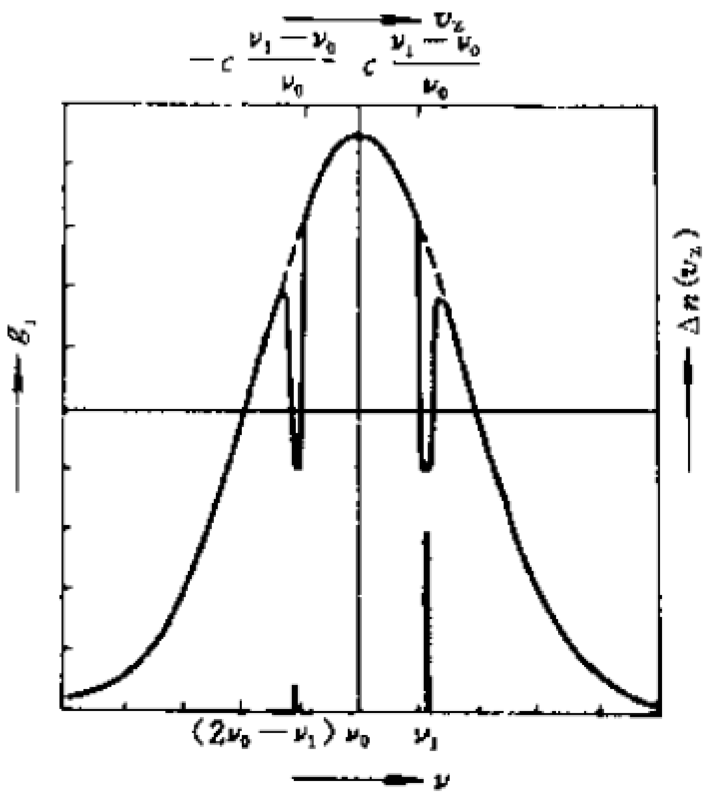
烧孔宽度: $\Delta\nu_{孔}=\Delta\nu\sqrt{1+\frac{I_\nu}{I_s}}$

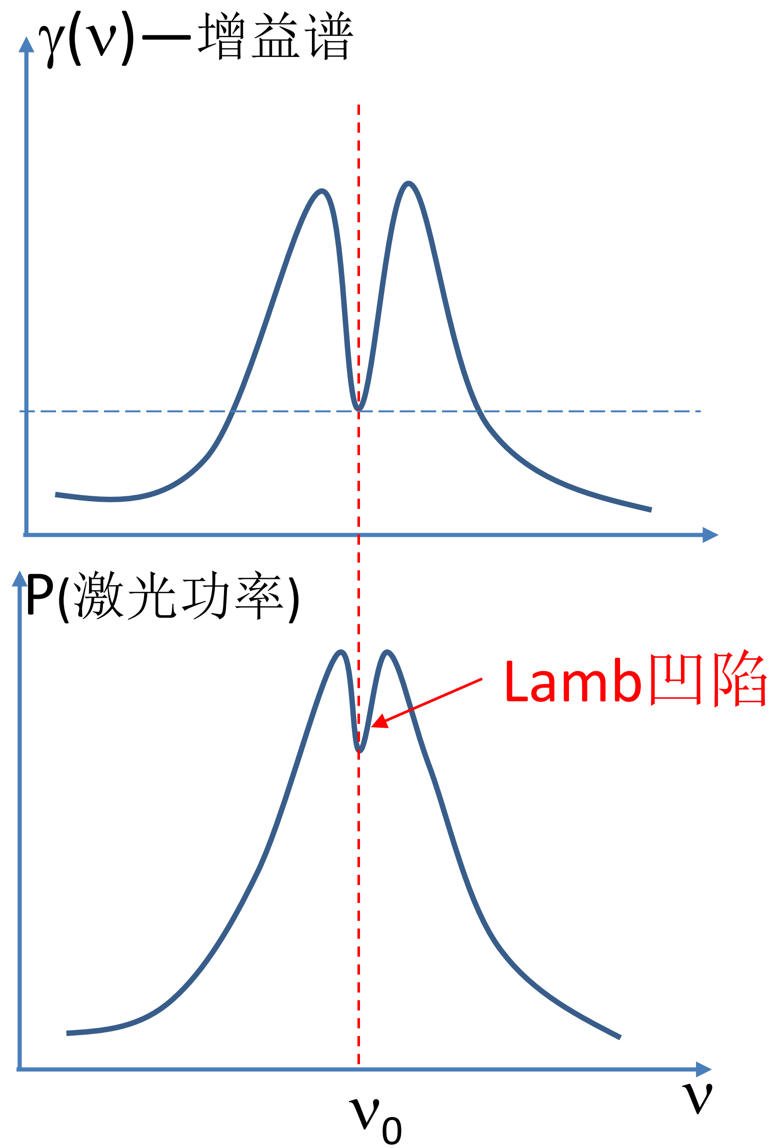
④ 多模输出



⑤ 兰姆(Lamb)凹陷: 多普勒效益的"双烧孔"







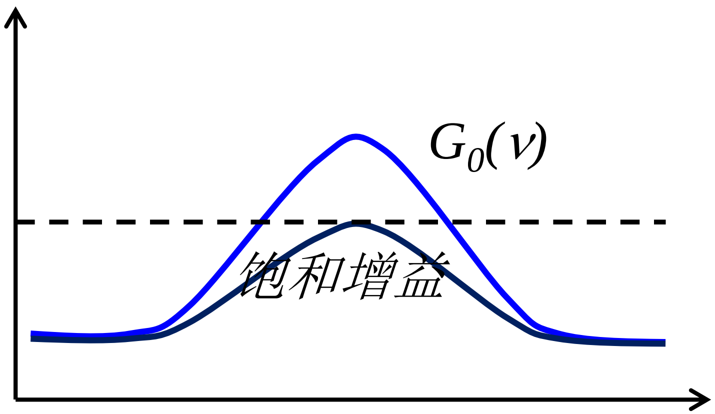
小结

- 概念：自发辐射、受激辐射、吸收
- 增益系数、洛仑兹线型
- 谱线展宽机制：均匀展宽、非均匀展宽
- 增益饱和现象及其在均匀、非均匀展宽下的不用物理表现

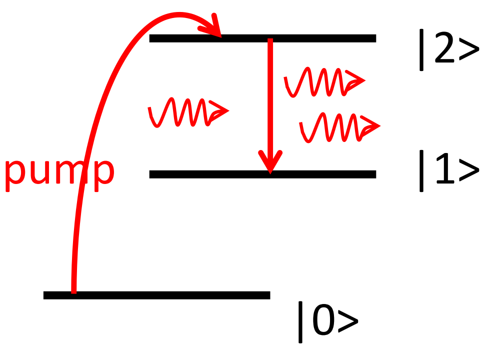
Ch9 激光振荡

激光形成示意图

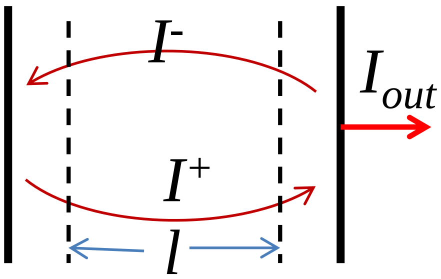
激光器结构示意图



增益曲线变化图



泵浦示意图

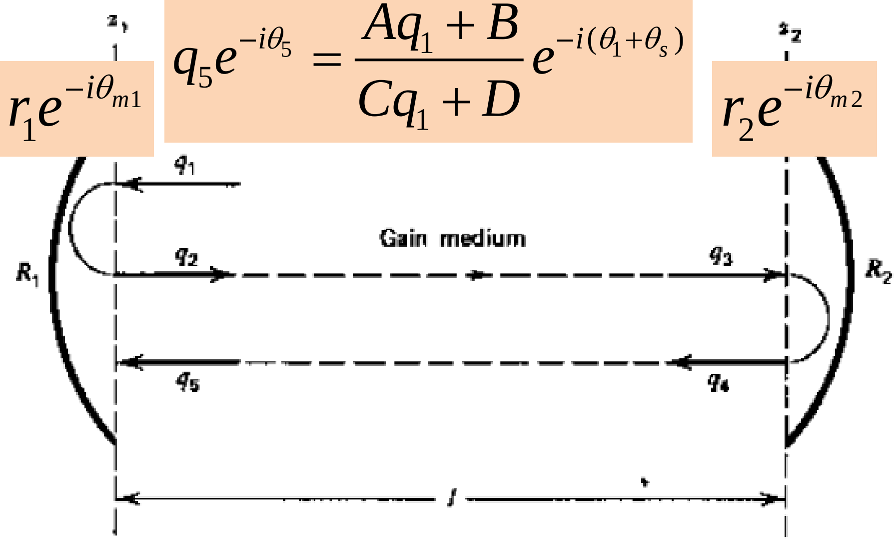


腔内光强传播意图

9.1 激光振荡条件

腔内光束的完整描述：$q(z)e^{-i\theta(z)}$

$r_1e^{-i\theta_{m1}}$ $\quad q_5e^{-i\theta_5}=\dfrac{Aq_1+B}{Cq_1+D}e^{-i(\theta_1+\theta_s)}$ $\quad r_2e^{-i\theta_{m2}}$



① q参数描述光束的复半径

$$\frac{1}{q(z)}=\frac{1}{R(z)}-i\frac{\lambda}{n\pi w_{(z)}^2}$$

*q(z)*满足ABCD规律：$q_2=\dfrac{Aq_1+B}{Cq_1+D}$

② 光束的传播位相及增益或吸收

$$e^{-i\theta_s}=e^{-i[2k'l-(n+m+1)(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})]}r_1r_2e^{-i(\theta_{m1}+\theta_{m2})}$$

$e^{-i\theta(z)}$，其中$\theta(z)=\theta_R+\theta_I$

光束自再现条件要求：$q_5e^{-i\theta_5}=\dfrac{Aq_1+B}{Cq_1+D}e^{-i(\theta_1+\theta_s)}=q_1e^{-i(\theta_1+2m\pi)}$

$$\Rightarrow\begin{cases} q_5=\dfrac{Aq_1+B}{Cq_1+D}=q_1 & \text{，腔稳定条件} \\ e^{-i\theta_s}=e^{-i2m\pi} & \text{，激光振荡条件}\end{cases}$$

对基模光束，m=n=0：

$$e^{-i\theta_s}=e^{-i[2k'l-(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})]}r_1r_2e^{-i(\theta_{m1}+\theta_{m2})}$$

$$=e^{-i2m\pi}$$

对基模光束, $m=n=0$: $e^{-i\theta_s}=e^{-i[2k'l-(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})]}r_1r_2e^{-i(\theta_{m1}+\theta_{m2})}=e^{-i2m\pi}$

阈值反转

$$\left|e^{-i\theta_s}\right|=1$$

$$k'=k[1+\frac{\chi'(\nu)}{2n^2}]-\mathrm{i}\frac{k\chi''(\nu)}{2n^2}-i\frac{\alpha}{2}$$

增益系数: $\gamma(\nu)=-k\dfrac{\chi''(\nu)}{n^2}$

$$e^{(\gamma_t-\alpha)l}r_1r_2=1\ \Rightarrow\ \boxed{\gamma_t=\alpha-\frac{1}{l}\ln(r_1r_2)}$$

$$\Delta\mathrm{N}_t=(\mathrm{N}_2-\mathrm{N}_1\frac{g_2}{g_1})=\frac{8\pi n^2t_{自发}}{g(\nu_0)\lambda^2}\gamma_t$$

$$=\frac{8\pi n^2t_{自发}}{g(\nu_0)\lambda^2}[\alpha-\frac{1}{l}\ln(\mathrm{r}_1\mathrm{r}_2)]$$

$$=\frac{8\pi n^3\nu^2t_{自发}\Delta\nu}{c^3t_c}$$

$\Delta\nu\equiv\dfrac{1}{g(\nu_0)}$ ：增益谱宽

$t_c=\dfrac{n}{c}\Big/[\alpha-\dfrac{1}{l}\ln(\mathrm{r}_1\mathrm{r}_2)]$ ：腔寿命

振荡频率

$$k[1+\frac{\chi'(\nu)}{2n^2}]l-(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0})+\frac{\theta_{m1}+\theta_{m2}}{2}=m\pi$$

空腔($\chi'(\nu)=0$)的振荡频率:

$$\nu_m=m\frac{c}{2nl}+\frac{c}{2\pi nl}(\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}-\frac{\theta_{m1}+\theta_{m2}}{2})$$

$$\nu[1+\frac{\chi'(\nu)}{2n^2}]=\nu_m$$

利用$\chi'(\nu)$和$\chi''(\nu)$之间的KK关系:

$$\chi'(\nu)=\frac{2(\nu_0-\nu)}{\Delta\nu}\chi''(\nu)\qquad\gamma(\nu)=-k\frac{\chi''(\nu)}{n^2}$$

$$\nu[1-\frac{(\nu_0-\nu)}{\Delta\nu}\cdot\frac{\gamma_t(\nu)}{k}]=\nu_m$$

腔模线宽

$$\frac{\nu-\nu_m}{\nu_0-\nu}=\frac{\gamma_t(\nu)\cdot\nu}{\Delta\nu\cdot k}=\frac{1}{\Delta\nu}\frac{\gamma_t(\nu)\cdot c}{2\pi n}=\frac{1/2\pi t_c}{\Delta\nu}=\frac{\Delta\nu_c}{\Delta\nu}\ll 1 \text{ 通常}$$

$$\gamma_t=\alpha-\frac{1}{l}\ln(r_1r_2)$$
$$=\frac{n}{c}\cdot\frac{1}{t_c}$$

腔内光子寿命

ν更接近于ν_m！

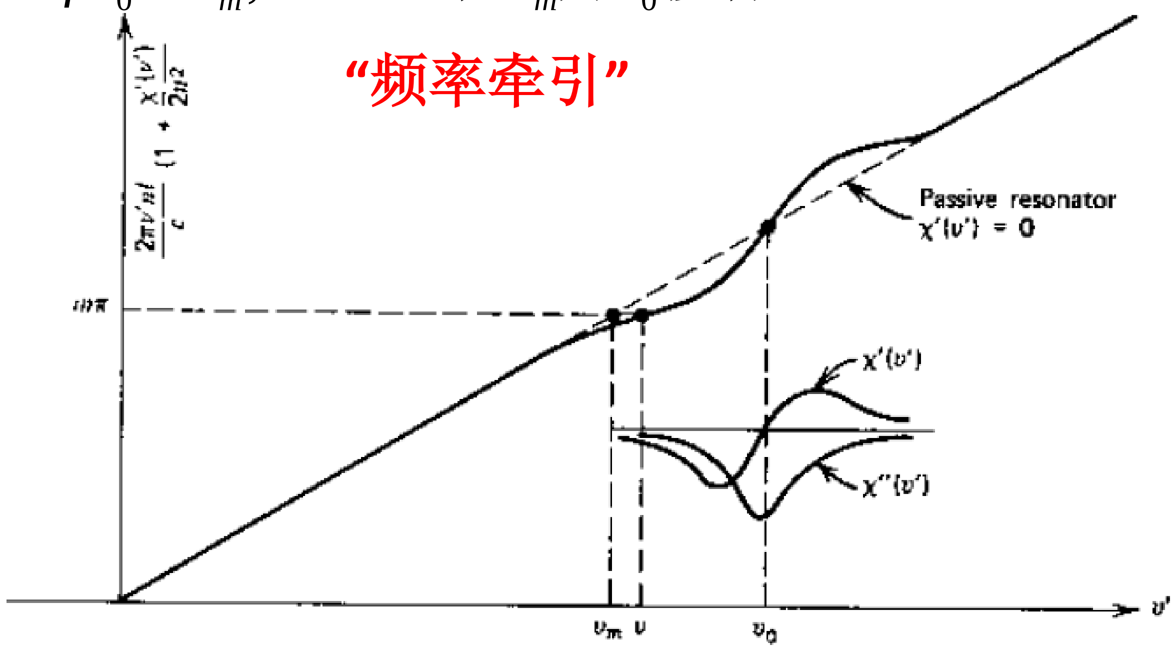
因此, 可以利用$\gamma_t(\nu_m)$代替$\gamma_t(\nu)$得

$$\nu\approx\nu_m-(\nu-\nu_0)\frac{c\gamma_t(\nu_m)}{2\pi n\Delta\nu}=\nu_m-(\nu-\nu_0)\frac{\Delta\nu_c}{\Delta\nu}$$

$if\ \nu_0=\nu_m,\quad then\ \ \nu=\nu_m=\nu_0$

$if\ \nu_0\neq\nu_m,\quad then\ \ \nu$从$\nu_m$向$\nu_0$移动！

“频率牵引”



9.2 激光振荡的一般形式

广义谐振腔内含有粒子数反转介质，
介质的作用可以描述为： $P_l(\mathrm{r},\mathrm{t}) = \varepsilon_0 \chi E_l(\mathrm{r},\mathrm{t})$

$$\mathbf{E}(\mathbf{r},\ t) = \sum_a \frac{1}{\sqrt{\varepsilon}} p_a(t)\mathbf{E}_a(\mathbf{r})$$

$$\nabla \times \mathbf{H}_a = k_a \mathbf{E}_a$$

$$\nabla \times \mathbf{E}_a = k_a \mathbf{H}_a$$

$$\mathbf{H}(\mathbf{r},\ t) = \sum_a \frac{1}{\sqrt{\mu}} \omega_a q_a(t)\mathbf{H}_a(\mathbf{r})$$

$$\nabla \times \mathbf{H} = \mathbf{i} + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P}_{非共振} + \mathbf{P}_{激光})$$

$$p_a = \dot{q}_a$$

$$= \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial t}\mathbf{P}_{激光}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\sum_a \frac{1}{\sqrt{\mu}} \omega_a q_a k_a \mathbf{E}_a$$

$$= -\sigma \sum_a \frac{1}{\sqrt{\varepsilon}} p_a \mathbf{E}_a - \sum_a \sqrt{\varepsilon}\, \dot{p}_a \mathbf{E}_a + \frac{\partial}{\partial t}\mathbf{P}_{激光}(\mathbf{r},\ t)$$

$$\omega_l^2 q_l + \frac{\sigma}{\varepsilon} p_l + \dot{p}_l - \frac{1}{\sqrt{\varepsilon}} \frac{\partial}{\partial t} \int_V \mathbf{P}_{激光} \cdot \mathbf{E}_l dv = 0$$

$$\omega_l^2 \dot{q}_l + \frac{\sigma}{\varepsilon} \dot{p}_l + \ddot{p}_l - \frac{1}{\sqrt{\varepsilon}} \frac{\partial^2}{\partial t^2} \int_V \mathbf{P}_{激光} \cdot \mathbf{E}_l dv = 0$$

$$\boxed{\omega_l^2 p_l + \ddot{p}_l + \frac{\sigma}{\varepsilon} \dot{p}_l = \frac{1}{\sqrt{\varepsilon}} \frac{\partial^2}{\partial t^2} \int_V \mathbf{P}_{激光}(\mathbf{r},\ t) \cdot \mathbf{E}_l(\mathbf{r}) dv}$$

$$p_l(t) = p_l(0) e^{-i\omega_l \left[1 - \frac{1}{8}(\sigma^2/\omega_l^2 \varepsilon^2)\right] t} e^{-(\sigma, 2\varepsilon)t}$$

$$t_c = \frac{\varepsilon}{\sigma} = \frac{Q}{\omega_l}$$

形式解：$p_l(t) = p_{l0}(t) e^{i\omega t}$

$$\left\{\left[(\omega_l^2 - \omega^2) + i\frac{\sigma\omega}{\varepsilon}\right] p_{l0}(t) + \left(2i\omega + \frac{\sigma}{\varepsilon}\right) \dot{p}_{l0}\right\} e^{i\omega t}$$

$$= \frac{1}{\sqrt{\varepsilon}} \frac{\partial^2}{\partial t^2} \int_V (\mathbf{P}_{激光} \cdot \mathbf{E}_l) dv$$

$$\left\{\left[(\omega_l^2-\omega^2)+i\frac{\sigma\omega}{\varepsilon}\right]p_{l0}(t)+\left(2i\omega+\frac{\sigma}{\varepsilon}\right)\dot{p}_{l0}\right\}e^{i\omega t}$$

$$=\frac{1}{\sqrt{\varepsilon}}\frac{\partial^2}{\partial t^2}\int_V(\mathbf{P}_{激光}\cdot\mathbf{E}_l)dv$$

假设单模起振: $\mathbf{E}(\mathbf{r},t)=-\frac{1}{\sqrt{\varepsilon}}p_l(t)\mathbf{E}_l(\mathbf{r})$

$$\mathbf{P}_{激光}(\mathbf{r},t)=-\frac{\varepsilon_0}{\sqrt{\varepsilon}}\chi(\omega)p_{l0}e^{i\omega t}\mathbf{E}_l(\mathbf{r})$$

对于稳态情况, $\dot{p}_{l0}=0$

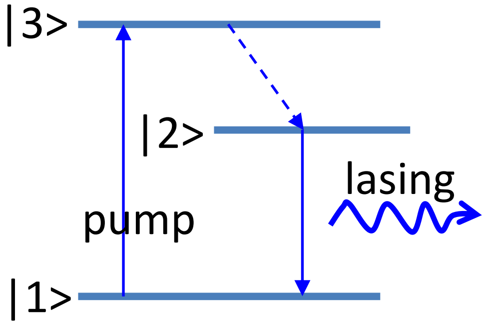
$$(\omega_l^2-\omega^2)+i\frac{\sigma\omega}{\varepsilon}=\frac{\omega^2\varepsilon_0 f}{\varepsilon}(\chi'-i\chi'')$$

$$f=\int_{V_{激光介质}}\mathbf{E}_l\cdot\mathbf{E}_l dv$$

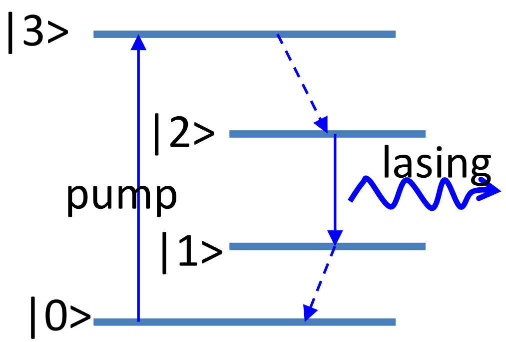
9.3 速率方程与激光输出功率

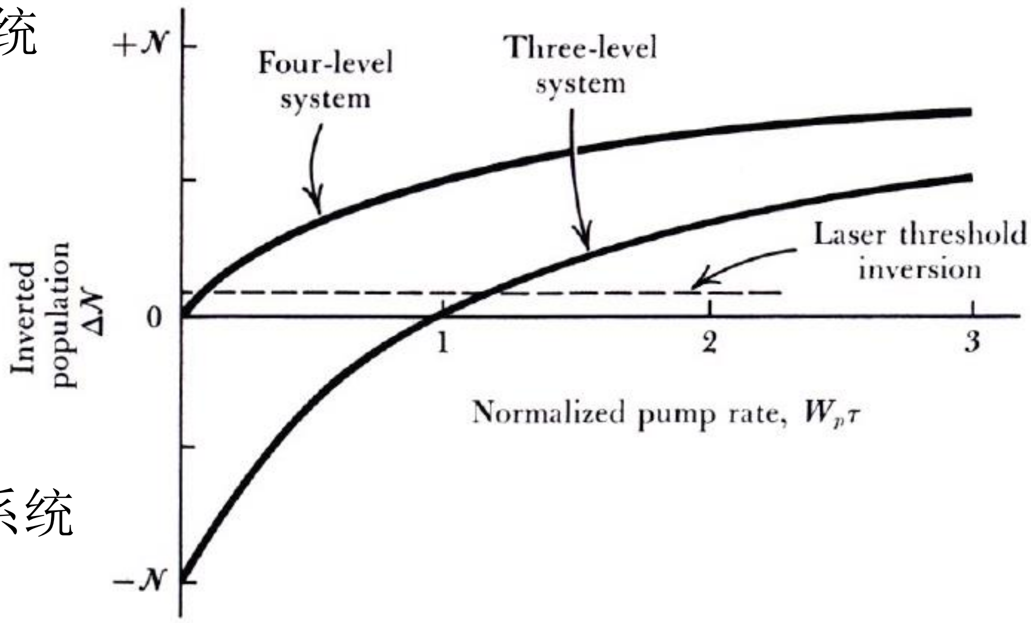
激光的能级系统

以基态为激光下能级的三能级系统



以激发态为激光下能级的四能级系统



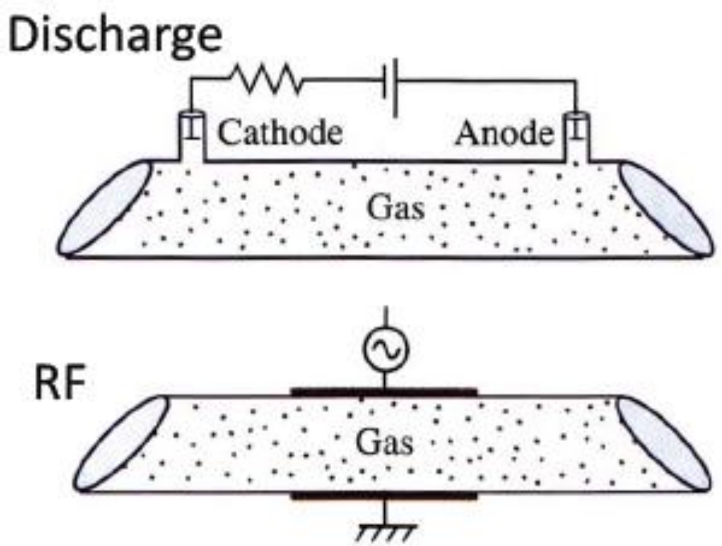


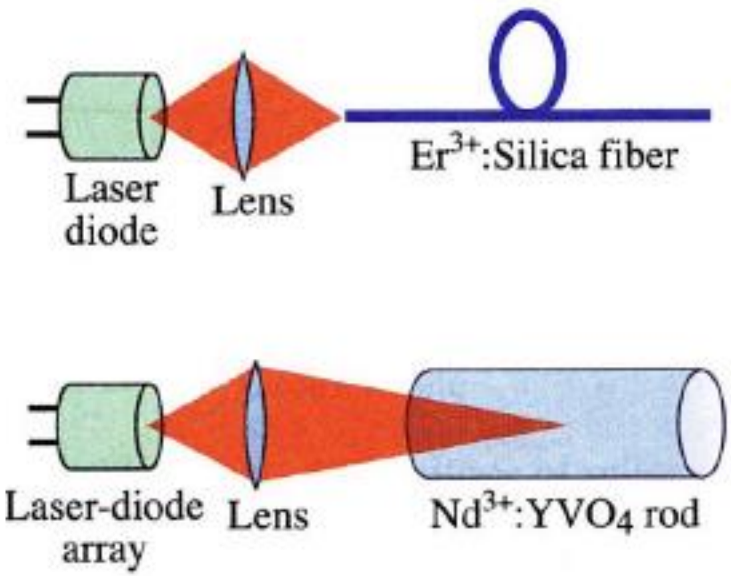
对三能级系统，由于基态的寿命很长，为获得粒子数反转至少需要将总粒子数的一半$N_0/2$泵浦到上能级才行，所以一般采用脉冲泵浦方式。

四能级系统中，激光下能级|1>的寿命很短，一般可认为$N_1=0$，泵浦效率较高

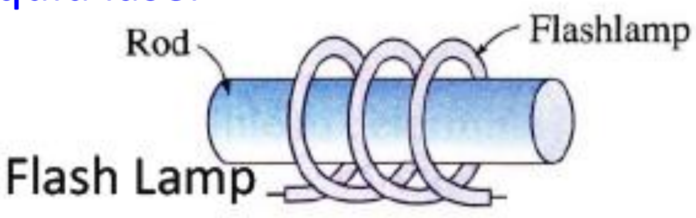
Pumping Mechanisms

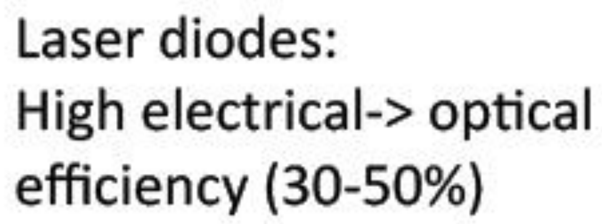
Electrical Pumping: suited for gas and semiconductor laser





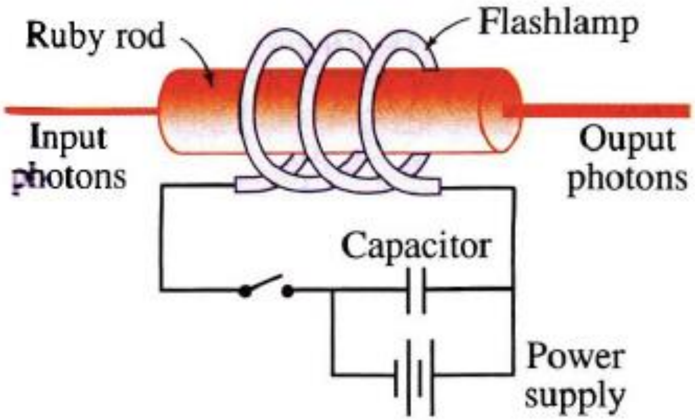
Optical Pumping : suited for solid-state and liquid laser

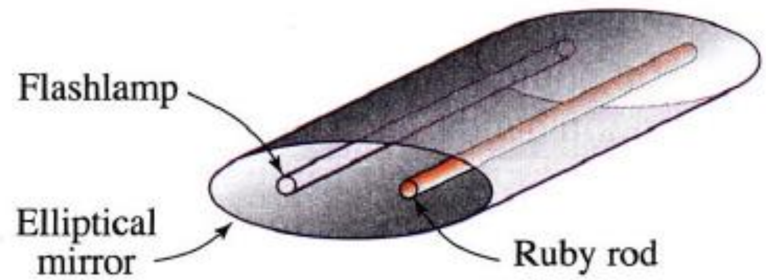




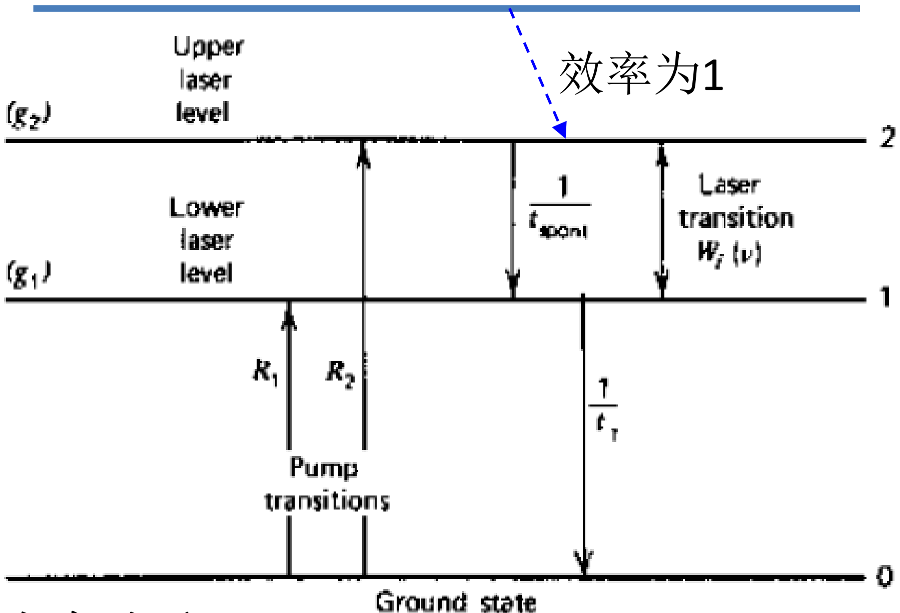
Pumping geometries

Flashlamps





以四能级系统为例



Def：
激光上能级寿命为t_2，下能级寿命为t_1，
上能级→下能级的寿命为t_{21}

$$\frac{1}{t_2} = \frac{1}{t_{21}} + (\text{到其他能级的跃迁速率})$$

$$\frac{1}{t_{21}} = \frac{1}{t_{\text{自发}}} + \frac{1}{(t_{21})_{\text{无辐射}}}$$

$$W_{2\to1} = W_i = \frac{\lambda^2 g(\nu)}{8\pi h\nu n^2 t_{\text{自发}}} I_\nu$$

$$W_{1\to2} = \frac{g_2}{g_1} W_i$$

速率方程：

$$\begin{cases} \dfrac{dN_2}{dt} = R_2 - \dfrac{N_2}{t_2} - (N_2 - \dfrac{g_2}{g_1} N_1) W_i \\ \dfrac{dN_1}{dt} = R_1 - \dfrac{N_1}{t_1} + \dfrac{N_2}{t_{21}} + (N_2 - \dfrac{g_2}{g_1} N_1) W_i \end{cases}$$

稳态：$$\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0 \Rightarrow \Delta N \equiv N_2 - \frac{g_2}{g_1} N_1 = \frac{R_2 t_2 - (R_1 + R_2\delta) t_1 \dfrac{g_2}{g_1}}{1 + [t_2 + (1-\delta) t_1 \dfrac{g_2}{g_1}] W_i}$$

$$\delta = \frac{1/t_{21}}{1/t_2} = \frac{t_2}{t_{21}}$$

令$W_i=0$得初始反转粒子数：$\Delta N_0 \equiv (N_2-\frac{g_2}{g_1}N_1)_0 = R_2 t_2-(R_1+R_2\delta)t_1\frac{g_2}{g_1}$

稳态反转粒子数可表示为： $\Delta N=\frac{\Delta N_0}{1+\phi t_{21}W_i}$

其中，$\phi=\delta[1+(1-\delta)\frac{t_1 g_2}{t_2 g_1}]$

讨论：

对确定的原子系统φ是常数。理想情况下，$t_2=t_{21}$，即$\delta=1$，并假设$R_1=0$，得 $\Delta N_0=R_2(t_2-t_1\frac{g_2}{g_1})$

① 产生粒子数反转（$\Delta N_0>0$）的条件：$t_2>t_1\frac{g_2}{g_1}$

if $g_2=g_1$则$t_2>t_1$，上能级寿命长，下能级寿命短！

if $g_2<g_1$，即使$t_2<t_1$(即$N_2<N_1$)时也可能产生增益($\Delta N_0>0$)

② 实际激光系统中，通常$\frac{t_1 g_2}{t_2 g_1}\ll 1$，故 $\phi\approx\delta=\frac{t_2}{t_{21}}$

所以稳态反转粒子数通常表示为

$$\Delta N=\frac{\Delta N_0}{1+t_2 W_i}$$

输出激光功率和最佳耦合

稳态反转粒子数：$\Delta N = \dfrac{\Delta N_0}{1+t_2 W_i}$

增益系数：$\gamma(\nu) = \Delta N \dfrac{\lambda^2}{8\pi n^2 t_{自发}} g(\nu) = \dfrac{\gamma_0}{1+t_2 W_i}$ $\gamma_0 = \Delta N_0 \dfrac{\lambda^2}{8\pi n^2 t_{自发}} g(\nu)$

稳态时，

增益取阈值增益，$\gamma_t = \alpha - \dfrac{1}{l}\ln(r_1 r_2)$ $\Delta N_t = \dfrac{8\pi n^2 t_{自发}}{g(\nu_0)\lambda^2}[\alpha - \dfrac{1}{l}\ln(r_1 r_2)]$

$$\frac{\gamma_0}{1+t_2 W_i} = \alpha - \frac{1}{l}\ln(r_1 r_2) \Rightarrow W_i = \frac{1}{t_2}\left[\frac{\gamma_0 l}{\alpha l - \ln(r_1 r_2)_i} - 1\right]$$ 稳态受激辐射速率

稳态激光辐射功率：

低增益对称腔 $R_1=R_2=R\approx 1$

$$-\ln r_1 r_2 = -\ln\sqrt{R_1 R_2} = -\ln R \approx 1 - R = T$$

$$P_e = \Delta N V_m W_i h\nu = \Delta N_t V_m W_i h\nu$$

$$= \frac{8\pi n^2 hc(V_m / l)}{g(\nu_0)\lambda^3 (t_2 / t_{自发})}[\alpha l - \ln(r_1 r_2)]\left[\frac{\gamma_0 l}{\alpha l - \ln(r_1 r_2)_i} - 1\right]$$

$$= \frac{8\pi n^2 hc(V_m / l)}{g(\nu_0)\lambda^3 (t_2 / t_{自发})}[g_0 - (L_i + T)]$$

$\alpha l \to L_i$ 腔内单程损耗

$-\ln r_1 r_2 \to T$ 耦合输出

$\gamma_0 l \to g_0$ 未饱和(小信号)增益

输出激光功率：

$$P_o=\frac{T}{L_i+T}P_e=\frac{8\pi n^2hc(V_m/l)}{g(\nu_0)\lambda^3(t_2/t_{自发})}[g_0-(L_i+T)]\frac{T}{L_i+T}$$

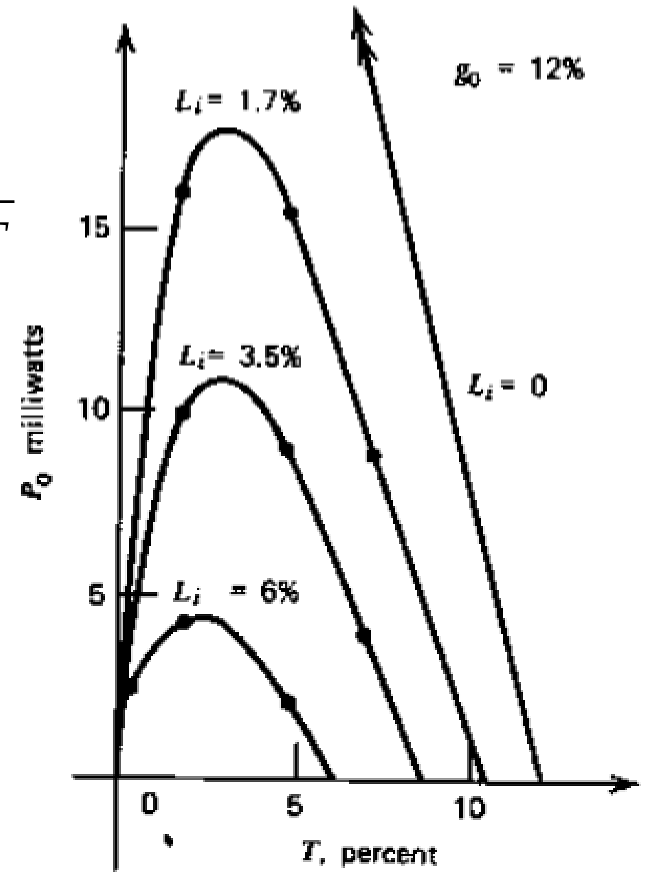
$$=\frac{8\pi n^2hcA}{g(\nu_0)\lambda^3(t_2/t_{自发})}[\frac{g_0}{(L_i+T)}-1]T$$

$$=2I_sA[\frac{g_0}{(L_i+T)}-1]T$$　　I_s：饱和光强

$A=V_m/l$，激光束的横截面面积

求P_o(T)的极值$\frac{\partial P_o(T)}{\partial T}=0$，得最佳耦合输出条件：

$$T_{opt}=-L_i+\sqrt{g_0L_i}$$



He-Ne laser

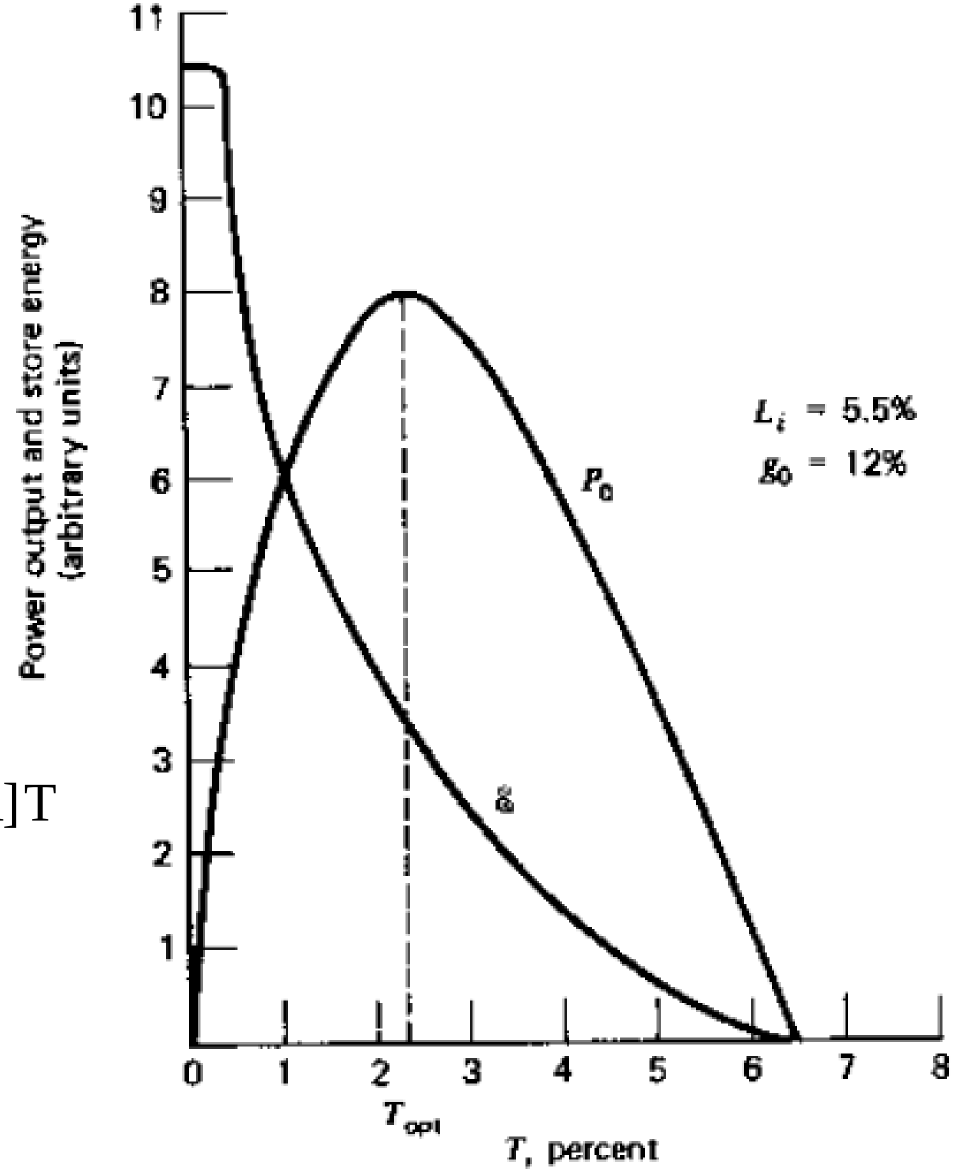
最佳输出功率： $$(P_o)_{opt}=\frac{8\pi n^2hcA}{g(\nu_0)\lambda^3(t_2/t_{自发})}[\sqrt{g_0}-\sqrt{L_i}]^2$$

$$=2I_sA[\sqrt{g_0}-\sqrt{L_i}]^2$$

腔内光场能量：

$$\xi(\mathrm{T}) = P_0 / T = 2 I_s A\left(\frac{g_0}{L_i + T} - 1\right)$$

$$P_o = \frac{T}{\mathrm{L}_i + T} P_e = 2 \mathrm{I}_s A\left[\frac{\mathrm{g}_0}{(\mathrm{L}_i + \mathrm{T})} - 1\right] \mathrm{T}$$



自发辐射的影响

对四能级系统，一般地，$t_2 >> t_1$, $\Delta N \approx N_2$

自发辐射功率：$P_{自发} = \kappa N_2 h\nu / t_{自发} \equiv K\Delta N$

受激+自发辐射功率：

$$P_e = \Delta N h\nu V_m W_i + K\Delta N = \Delta N(h\nu V_m W_i + K) = \frac{\Delta N_0}{1+W_i t_2}(h\nu V_m W_i + K)$$

$$\frac{\text{受激辐射速率/模式}}{\text{自发辐射速率/模式}} = \frac{h\nu V_m W_i}{K} = n_m$$

$$W_i = \frac{\lambda^2 g(\nu)}{8\pi h\nu n^2 t_{自发}} I_\nu$$

$$\Rightarrow K = \frac{h\nu V_m W_i}{n_m} = \frac{h\nu V_m}{n_m} \frac{\lambda^2 g(\nu_0)}{8\pi h\nu n^2 t_{自发}} \frac{n_m h\nu}{V_m} \cdot \frac{c}{n}$$

$$I_\nu = \frac{n_m h\nu}{V_m} \cdot \frac{c}{n}$$

$$= \frac{h\nu\lambda^2 c}{8\pi n^3 t_{自发}} g(\nu_0) = \frac{h\nu c^3}{8\pi n^3 \nu^2 \Delta\nu t_{自发}}$$

$$\Delta\nu \equiv \frac{1}{g(\nu_0)}$$

① 低于阈值 $\gamma_t = \alpha - \frac{1}{l}\ln(r_1 r_2)$ 时 $W_i = 0$

$$\Delta N_t = \frac{8\pi n^3 \nu^2 t_{自发} \Delta\nu}{c^3 t_c}$$

$$P_{e(\Delta N_0 < \Delta N_t)} = \Delta N_0 K = \Delta N_0 \frac{h\nu c^3}{8\pi n^3 \nu^2 \Delta\nu t_{自发}} = \frac{\Delta N_0}{\Delta N_t}\frac{h\nu}{t_c} = \frac{g_0}{L_i + T}\frac{h\nu}{t_c}$$

$\Delta N_0 = \Delta N_t$ 时，

$$P_e = \frac{h\nu}{t_c}$$

$$= \frac{\Delta N_0 V_m h\nu}{t_{自发}} \Big/ p$$

自发辐射光功率对模式数p的平均！

腔内储存一个光子！

$$p = \frac{8\pi n^3 \nu^2}{c^3} \cdot \Delta\nu \cdot V_m \quad 模式数$$

② 高于阈值 $h\nu V_m W_i \gg K$ 时， 此时，$\Delta N = \Delta N_t$

$$P_{e(\Delta N_0 > \Delta N_t)} = \Delta N_t h\nu V_m W_i + \frac{h\nu}{t_c}$$

ΔN=ΔNt 时的自发辐射

$$= \Delta N_t h\nu V_m (\frac{\Delta N_0}{\Delta N_t} - 1)\frac{1}{t_2} + \frac{h\nu}{t_c}$$

$$= \frac{\Delta N_t h\nu V_m}{t_2}[\frac{g_0}{L_i + T} - 1] + \frac{h\nu}{t_c}$$

$$\Delta N_t = \frac{\Delta N_0}{1 + t_2 W_i}$$

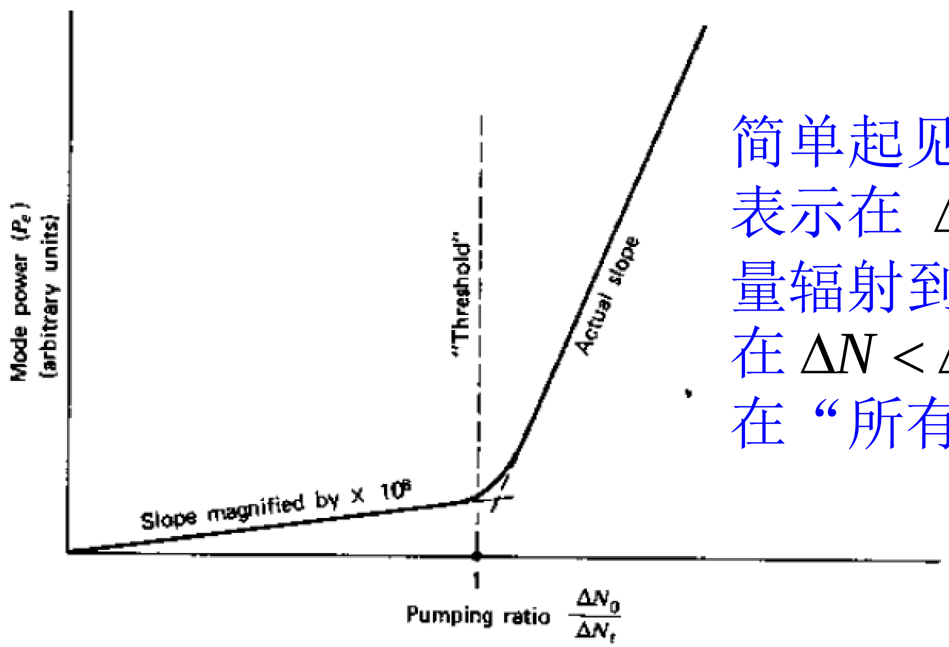
$$\Rightarrow W_i = (\frac{\Delta N_0}{\Delta N_t} - 1)\frac{1}{t_2}$$

斜率比：

$$\frac{[dP_e/d(\Delta N_0)]_{\Delta N_0>\Delta N_t}}{[dP_e/d(\Delta N_0)]_{\Delta N_0<\Delta N_t}}=\frac{h\nu V_m/t_2}{h\nu/(\Delta N_t t_c)}$$

$$=\frac{V_m t_c}{t_2}\Delta N_t=\frac{V_m t_c}{t_2}\frac{8\pi n^3\nu^2 t_{自发}\Delta\nu}{c^3 t_c}$$

$$=\frac{8\pi n^3\nu^2\Delta\nu V_m}{c^3}\Bigg/\frac{t_2}{t_{自发}}=\frac{p}{t_2/t_{自发}}$$



简单起见，令$t_2=t_{自发}$，该式表示在 $\Delta N>\Delta N_t$ 时，激光能量辐射到“一个模式”中；在 $\Delta N<\Delta N_t$ 时能量均匀分布在“所有模式(p个)”中！

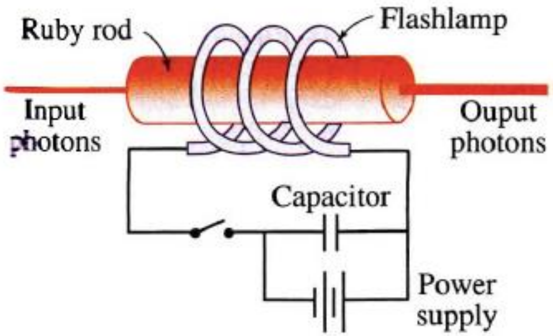
Common laser transitions

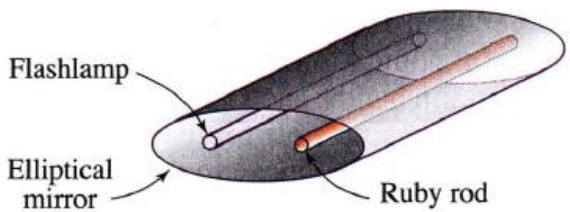
Table 14.3-1 Characteristics of common laser transitions.

Laser Medium	Transition Wavelength[a] λ_o (nm)	Transition Cross Section σ_0 (cm^2)	Spontaneous Lifetime t_{sp}	Transition Linewidth[b] $\Delta\nu$		Refractive Index n
C^{5+}	18.2	5×10^{-16}	12 ps	1 THz	I	≈ 1
ArF Excimer	193	3×10^{-16}	10 ns	10 THz	I	≈ 1
Ar^+	515	3×10^{-12}	10 ns	3.5 GHz	I	≈ 1
Rhodamine-6G dye	560–640	2×10^{-16}	5 ns	40 THz	H/I	1.40
He–Ne	633	3×10^{-13}	150 ns	1.5 GHz	I	≈ 1
$Cr^{3+}:Al_2O_3$	694	2×10^{-20}	3 ms	330 GHz	H	1.76
$Cr^{3+}:BeAl_2O_4$	700–820	1×10^{-20}	260 μs	25 THz	H	1.74
$Ti^{3+}:Al_2O_3$	700–1050	3×10^{-19}	3.9 μs	100 THz	H	1.76
Yb^{3+}:YAG	1030	2×10^{-20}	1 ms	1 THz	H	1.82
Nd^{3+}:Glass (phosphate)	1053	4×10^{-20}	370 μs	7 THz	I	1.50
Nd^{3+}:YAG	1064	3×10^{-19}	230 μs	150 GHz	H	1.82
$Nd^{3+}:YVO_4$	1064	8×10^{-19}	100 μs	210 GHz	H	2.0
InGaAsP[c]	1300–1600	2×10^{-16}	2.5 ns	10 THz	H	3.54
Er^{3+}:Silica fiber	1550	6×10^{-21}	10 ms	5 THz	H/I	1.46
CO_2	10 600	3×10^{-18}	3 s	60 MHz	I	≈ 1

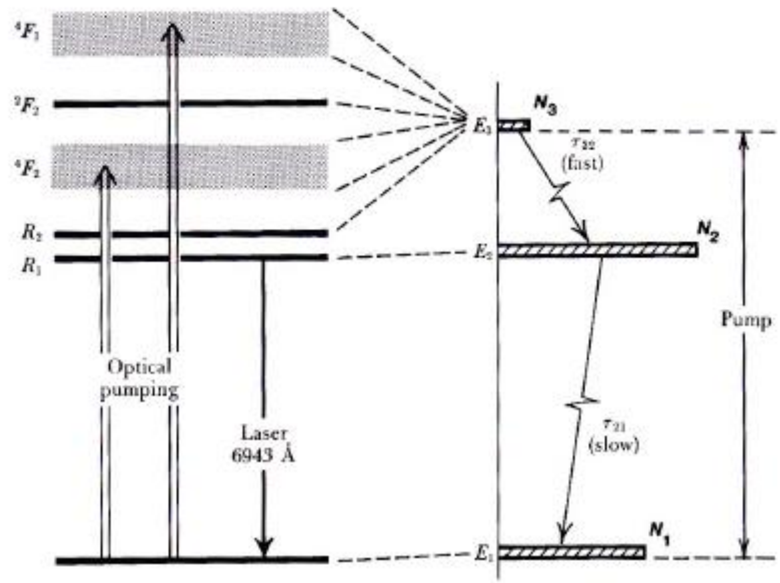
Ruby energy levels: 3 levels

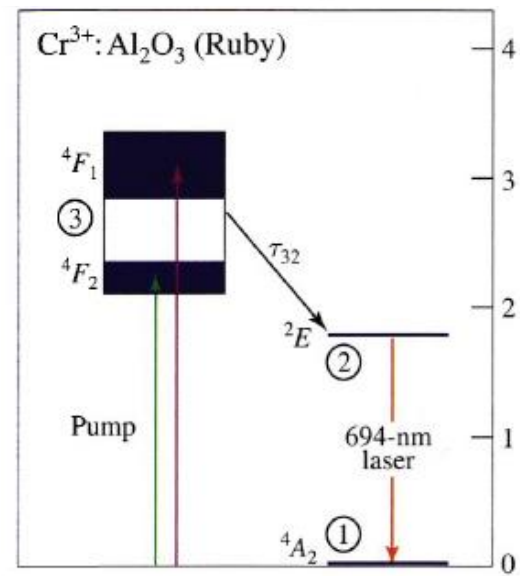
Flashlamps





Property	Values and units
Cr_2O_3 doping	0.05 wt. %
Cr^{3+} concentration	1.58×10^{19} ions/cm^3
Output wavelengths	694.3 nm (R_1 line)
	692.9 nm (R_2 line)
Upper laser level lifetime	3 ms
Linewidth of R_1 laser transition	11 cm^{-1}
Stim. emission cross-section σ_e	2.5×10^{-20} cm^2
Absorption cross section σ_a	1.22×10^{-20} cm^2
Refractive index ($\lambda = 694.3$ nm)	$n = 1.763$ ($E \perp c$)
	$n = 1.755$ ($E \| c$)





Nd-Yag system: 4 levels

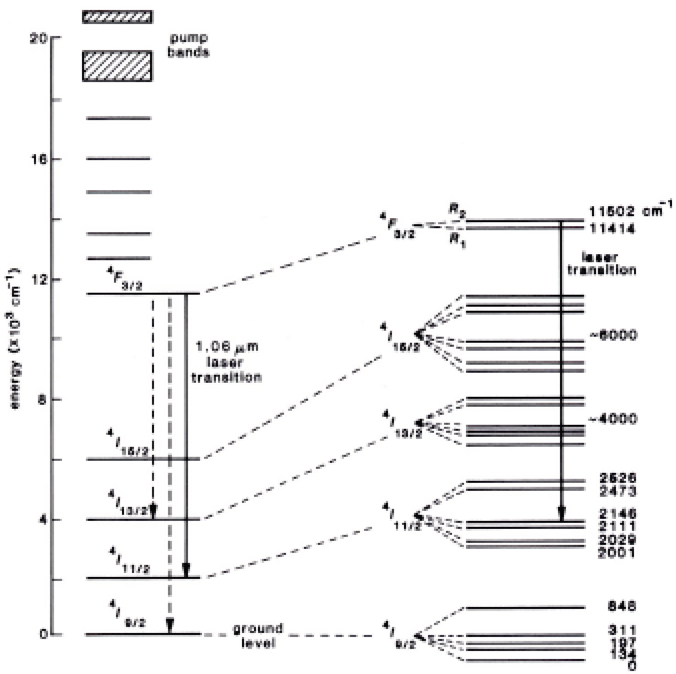
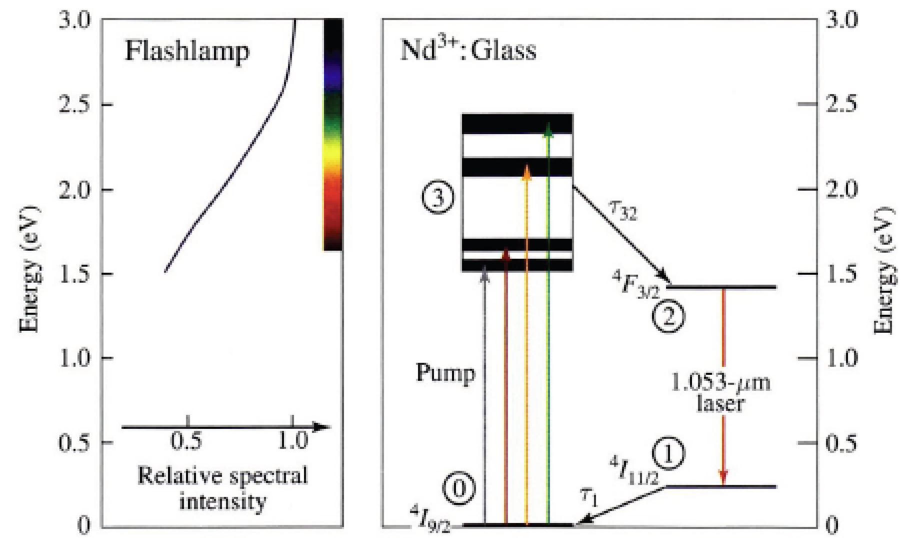


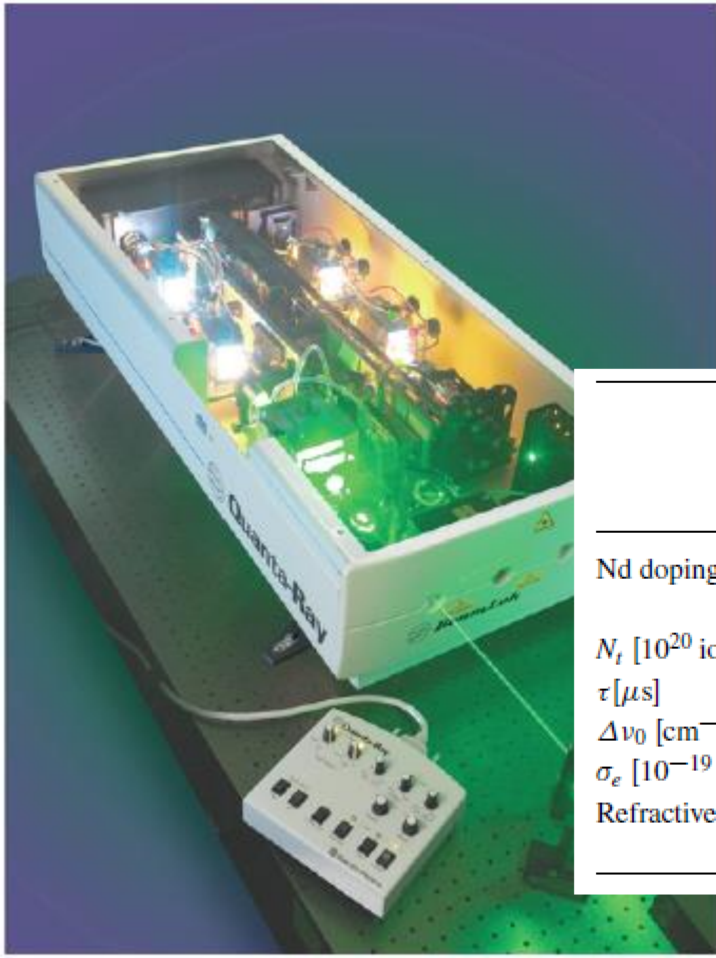
FIGURE 3.3
Quantum-mechanical energy levels of the Nd^{3+} ion in a Nd:YAG laser crystal.



Nd: YAG

Nd: Glass

Nd:Yag

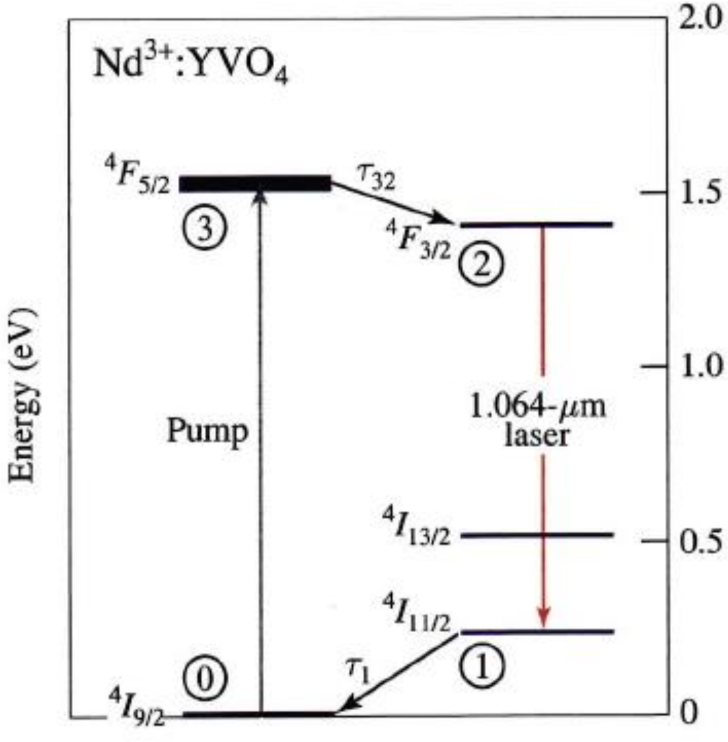


- Repetition rates from 1 to 100 Hz and energies in excess of 2500 mJ/pulse
- Gold-coated pump chamber for component longevity and lamp efficiency

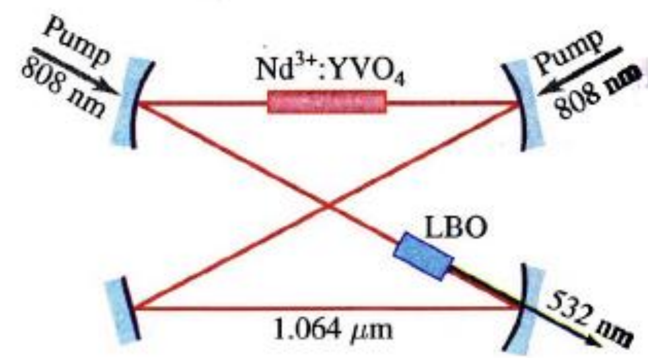
	Nd:YAG $\lambda = 1.064\,\mu$m	Nd : YVO_4 $\lambda = 1.064\,\mu$m	Nd:YLF $\lambda = 1.053\,\mu$m	Nd:glass $\lambda = 1.054\,\mu$m (Phosphate)
Nd doping [at. %]	1 at. %	1 at. %	1 at. %	3.8% Nd_2O_3 by weight
N_t [10^{20} ions/cm^3]	1.38	1.5	1.3	3.2
$\tau[\mu s]$	230	98	450	300
$\Delta\nu_0$ [cm^{-1}]	4.5	11.3	13	180
σ_e [10^{-19} cm^2]	2.8	7.6	1.9	0.4
Refractive index	$n = 1.82$	$n_o = 1.958$ $n_e = 2.168$	$n_0 = 1.4481$ $n_e = 1.4704$	$n = 1.54$

Quanta Ray, Newport

Vanadate laser



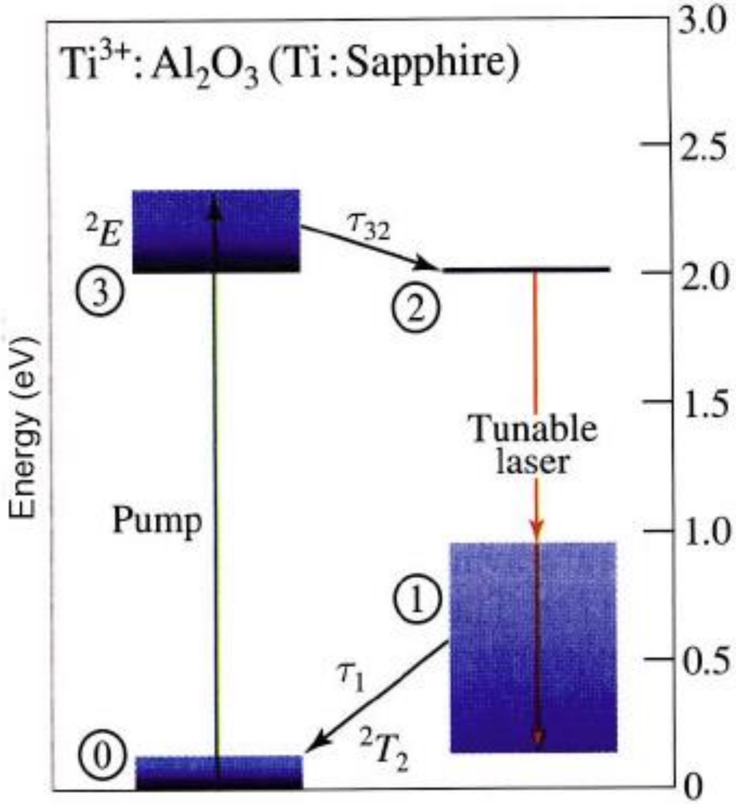
(a)



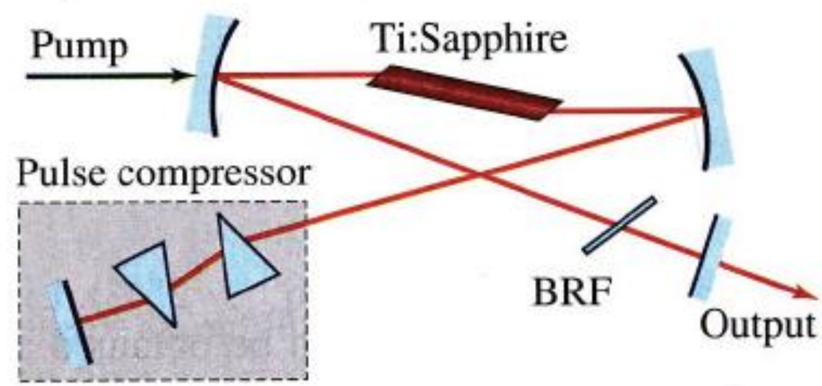
(b)

Ti-Sapphire laser

	Ti:sapphire
doping [at. %]	0.1 at. %
N_t [10^{19} ions/cm^3]	3.3
Peak wavelength [nm]	790
Tuning range [nm]	660–1,180
σ_e [10^{-20} cm^2]	28
τ[μs]	3.2
$\Delta\nu_0$ [THz]	100
Refractive indices	$n_o = 1.763$
	$n_e = 1.755$



Pump: laser 515/532nm, 5-10W



Fiber lasers

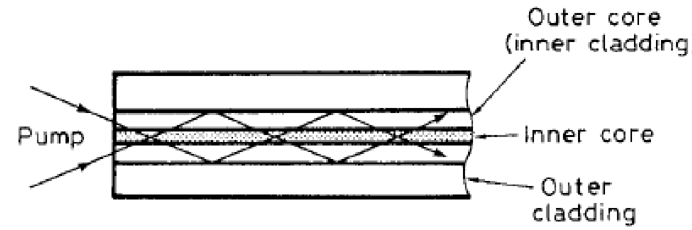
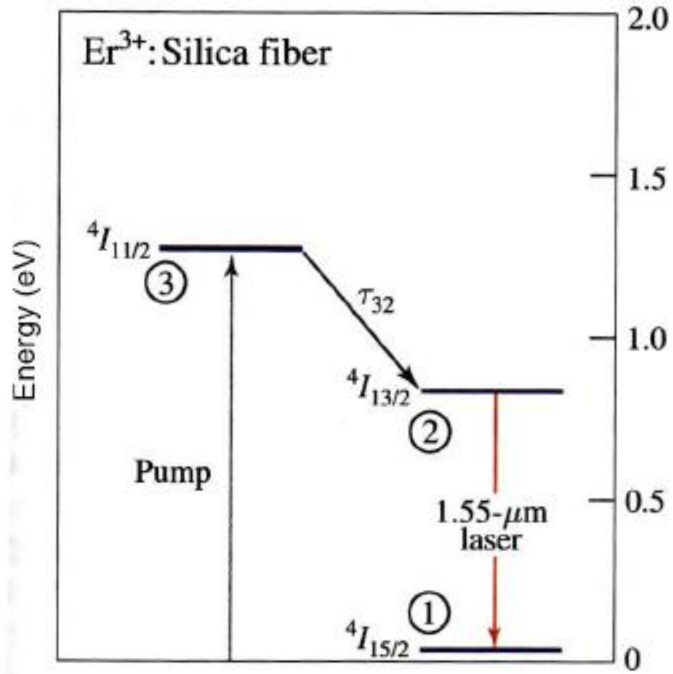
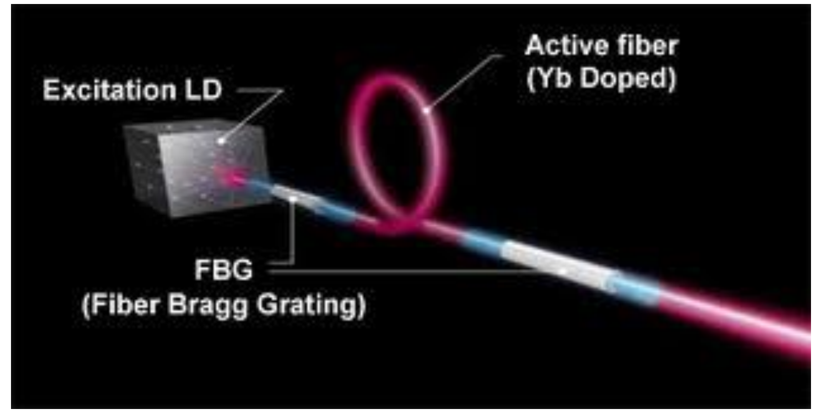
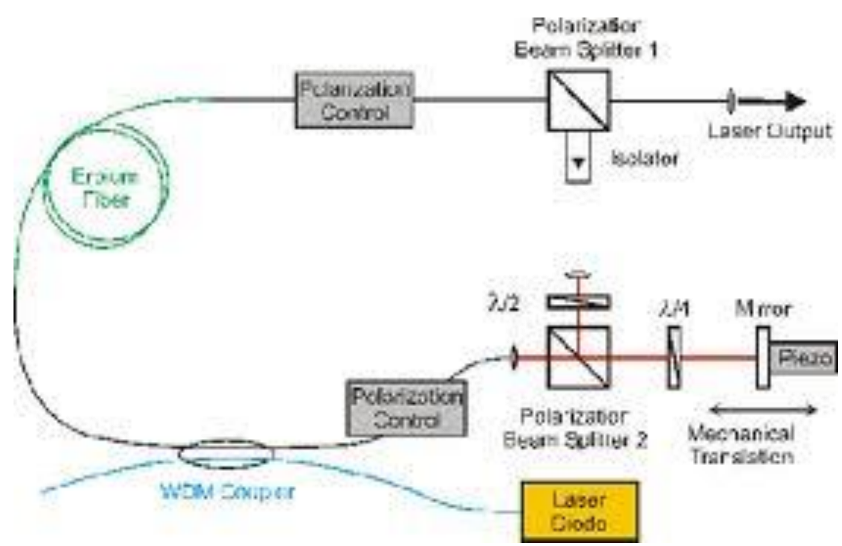


FIG. 9.6. Scheme of cladding pumping.





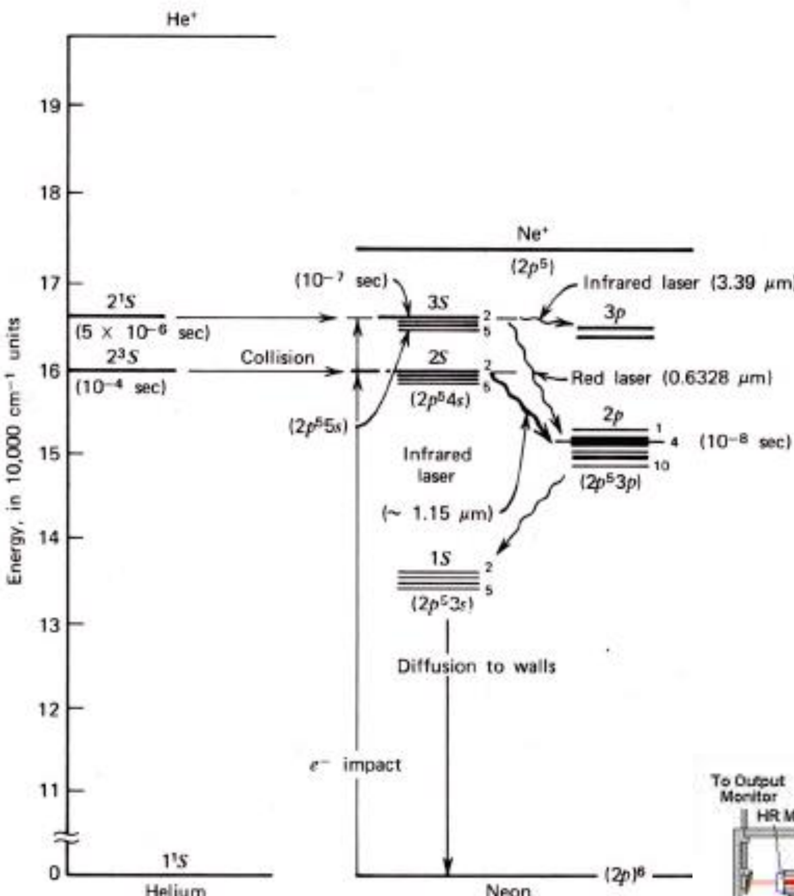


Gas Lasers

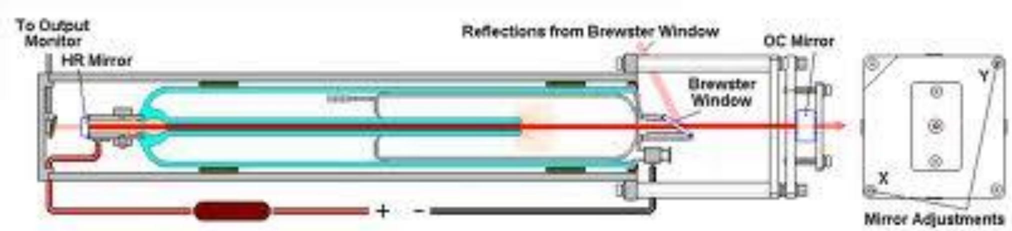
- Energy levels
  - electronic levels of atoms/ions Ne/Ar+
  - Vibrational levels (CO_2) at longer wavelength
- Pumping
  - interatomic collisions (He -> Ne)
  - collision with walls (des-excitation of the lower level of Ne)

HeNe

Laser type	He-Ne
Laser wavelength [nm]	633
Cross-section [10^{-14} cm^2]	30
Upper-state lifetime [ns]	150
Lower-state lifetime [ns]	10
Transition Linewidth [GHz]	1.5
Partial pressures of gas mixture [torr]	4 (He)
	0.8 (Ne)

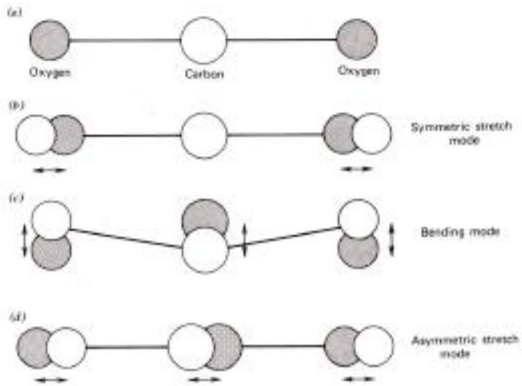


First c.w. laser

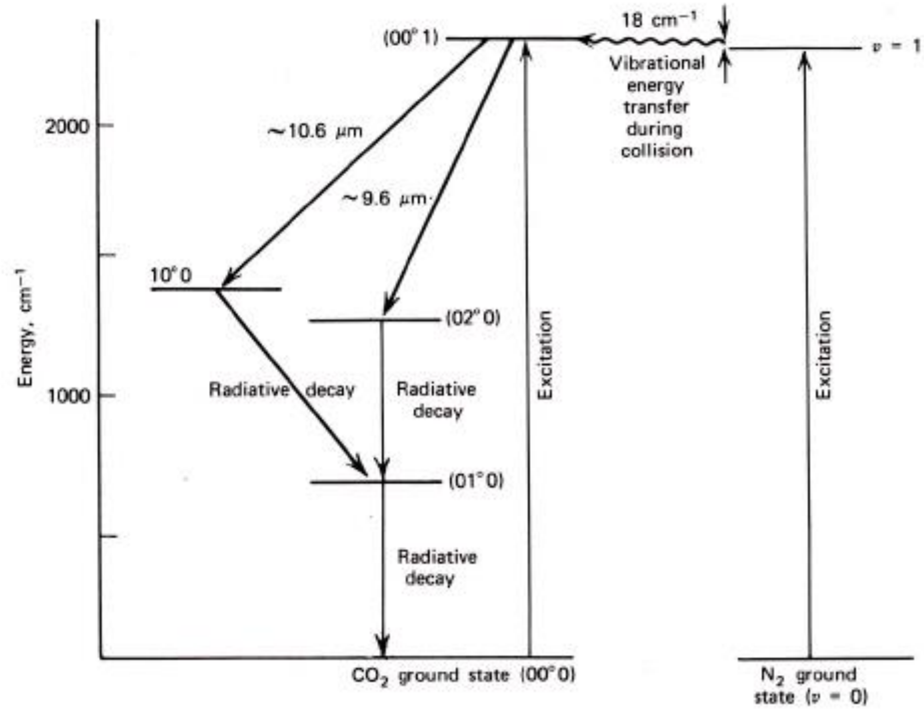


HeNe Laser Tube with Internal HR and Brewster Window with External OC

CO_2 laser



(a)
Oxygen
Carbon
Oxygen
(b)
Symmetric stretch mode
(c)
Bending mode
(d)
Asymmetric stretch mode



18 cm⁻¹
(00°1)
v = 1
Vibrational energy transfer during collision
2000
~10.6 μm
~9.6 μm
10°0
(02°0)
Energy, cm⁻¹
1000
Radiative decay
Radiative decay
Excitation
Excitation
(01°0)
Radiative decay
CO₂ ground state (00°0)
N₂ ground state (v = 0)

CO2 laser

400W
TEM_{00},98% Purity M^2<1.2±0.1
<1.2
<150μsec
4.5mm
4.0mR
10.2-10.7μm
±7%
±5%
Random
Water
8000W

Synrad

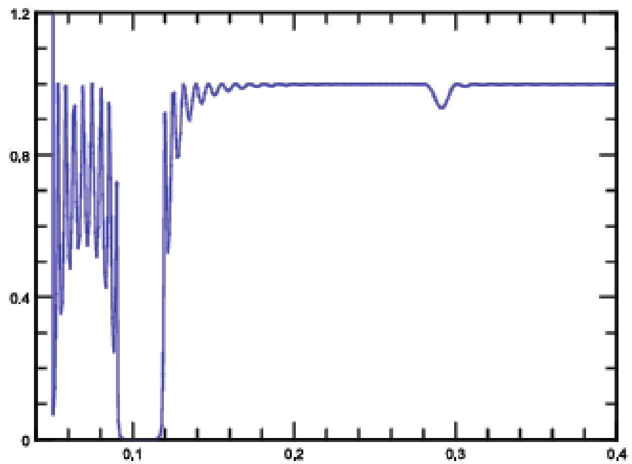
Plasmalab

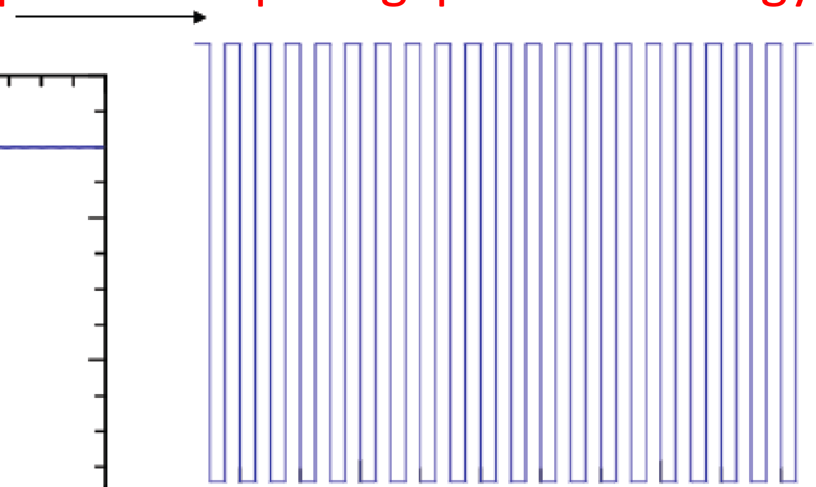
http://www.youtube.com/watch?v=2EFAGhIn8OQ&feature=related

半导体激光

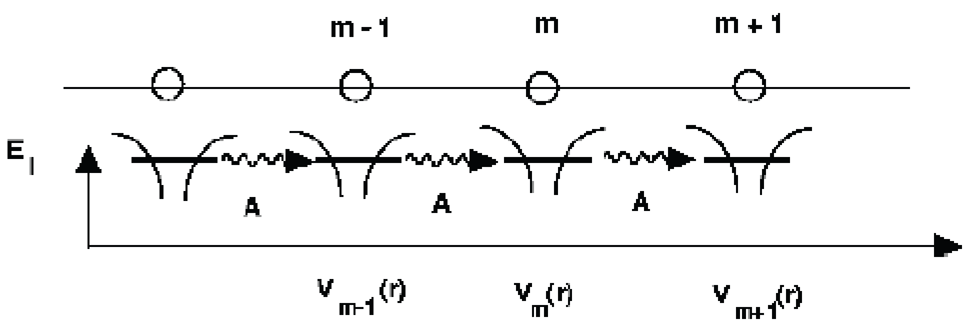
半导体的基本物理特性：能带与带隙

A periodic modulation of the potential opens gaps in the energy spectrum

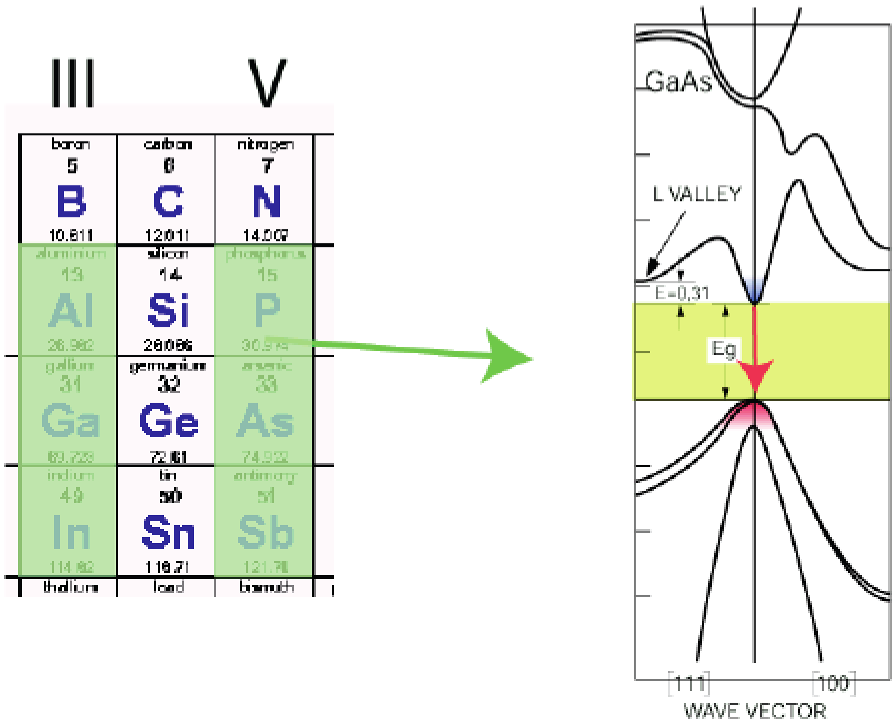




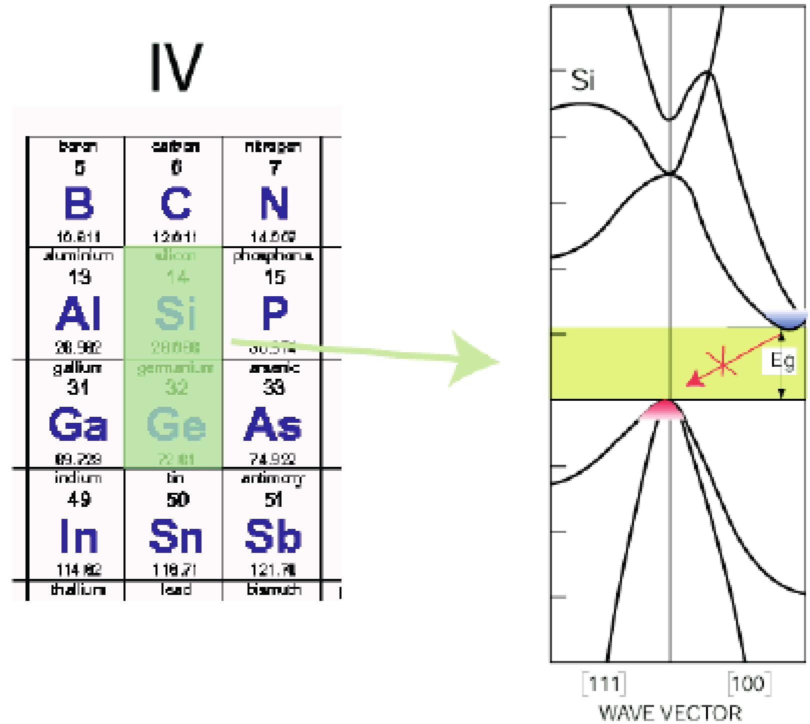
A periodic array of coupled isolated states forms bands



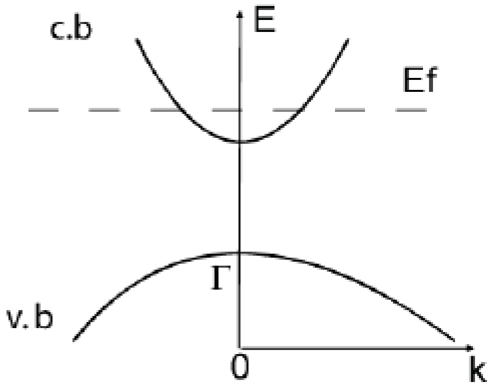
直接带隙半导体

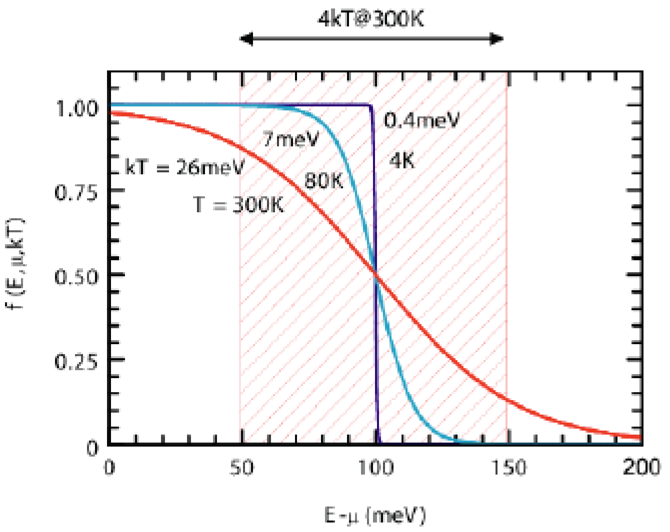


间接带隙半导体



Fermi-Dirac distribution

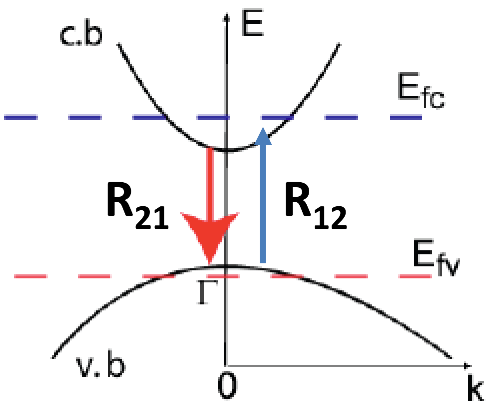




Electrons are distributed according to the Fermi-Dirac distribution function

$$f(E,T) = \frac{1}{\exp(\frac{E-E_f}{kT})+1}$$

Quasi-Fermi levels



Quasi-Fermi levels:

$$f_c(E,T) = \frac{1}{\exp(\frac{E-E_{fc}}{kT})+1}$$

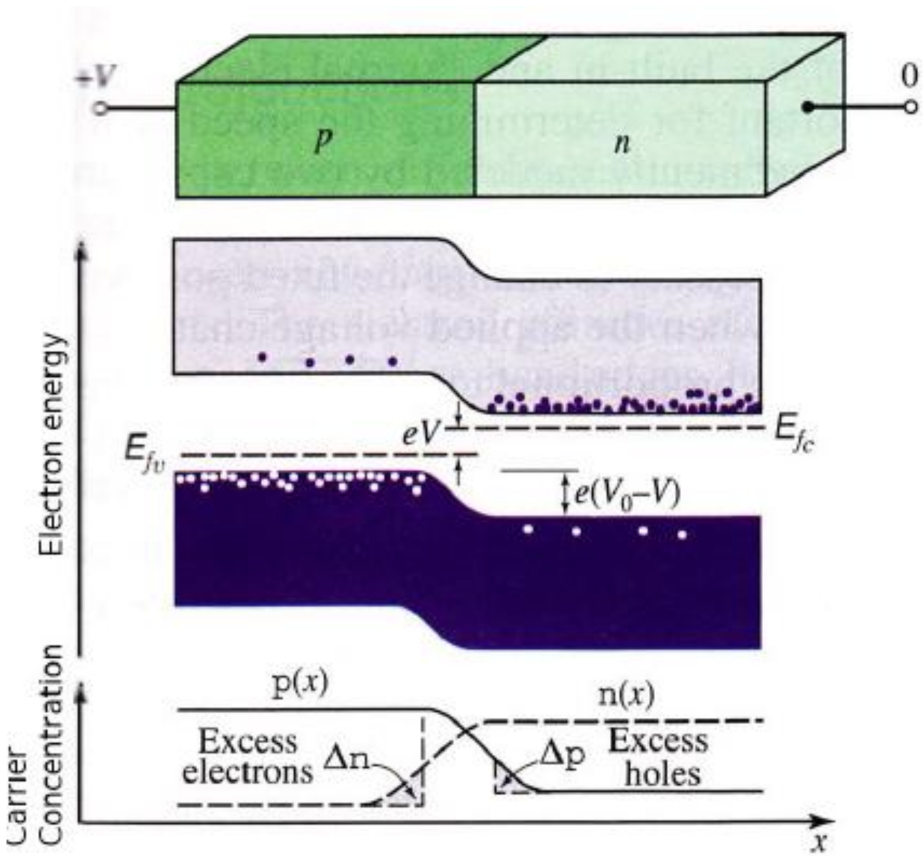
$$f_v(E,T) = \frac{1}{\exp(\frac{E-E_{fv}}{kT})+1}$$

Bernard-Durrafourg inversion condition

$$\frac{\mathcal{R}_{12}}{\mathcal{R}_{21}} = \frac{f_v(E_1,T)\,[1-f_c(E_2,T)]}{f_c(E_2,T)\,[1-f_v(E_1,T)]}$$
$$= \exp\left[\frac{\hbar\omega-(E_{fc}-E_{fv})}{k_B T}\right]$$

Gain condition: $E_{fc} - E_{fv} > h\nu$

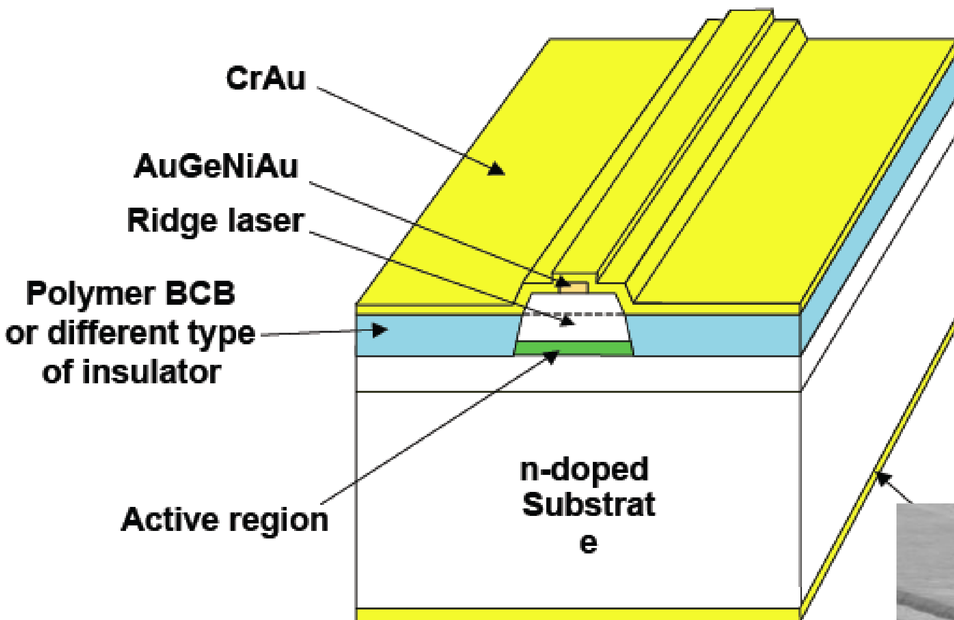
Pn结：少数载流子注入

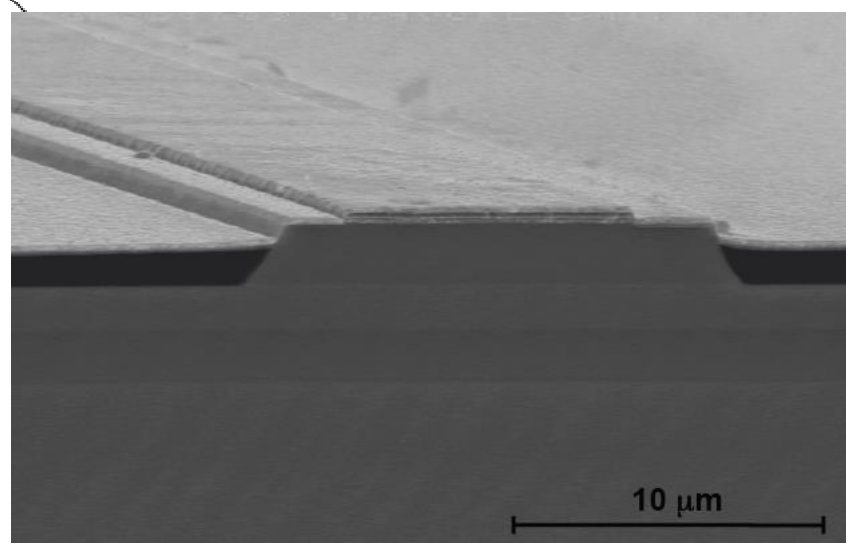


Key number:

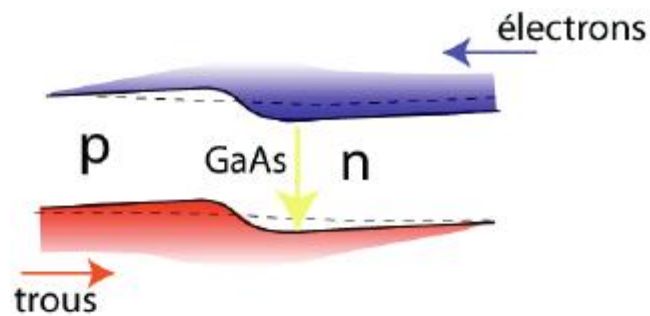
Diffusion length of electron and holes

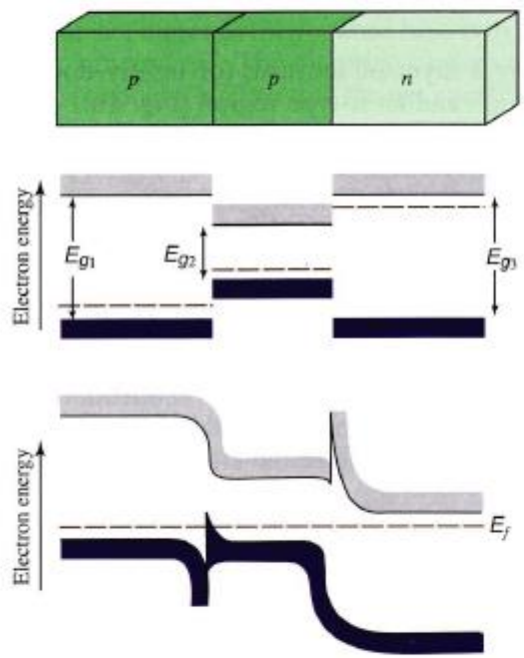
pn junction laser

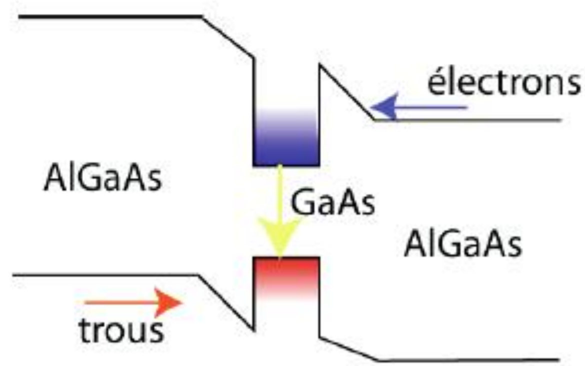


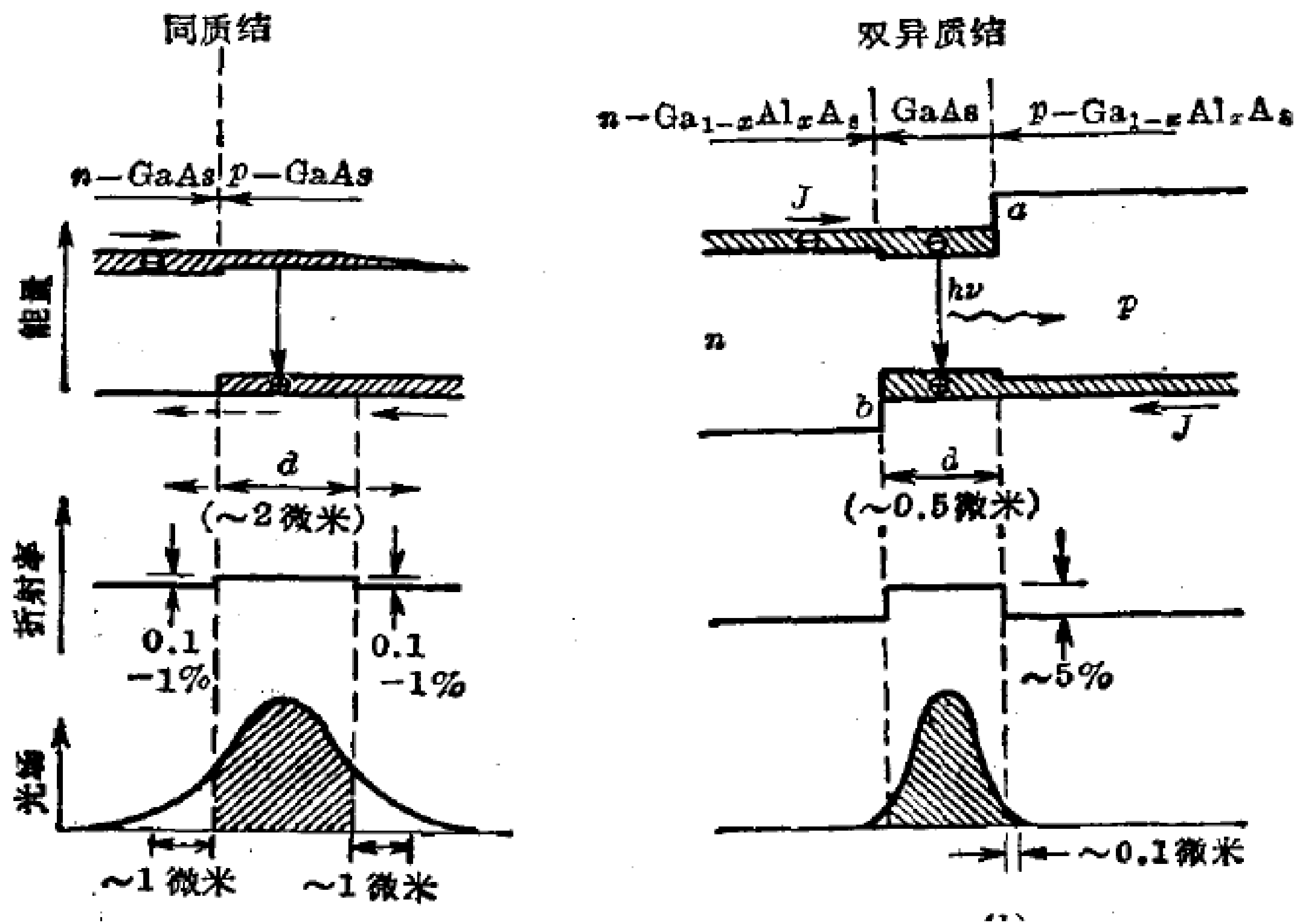


异质结(heterojunction)









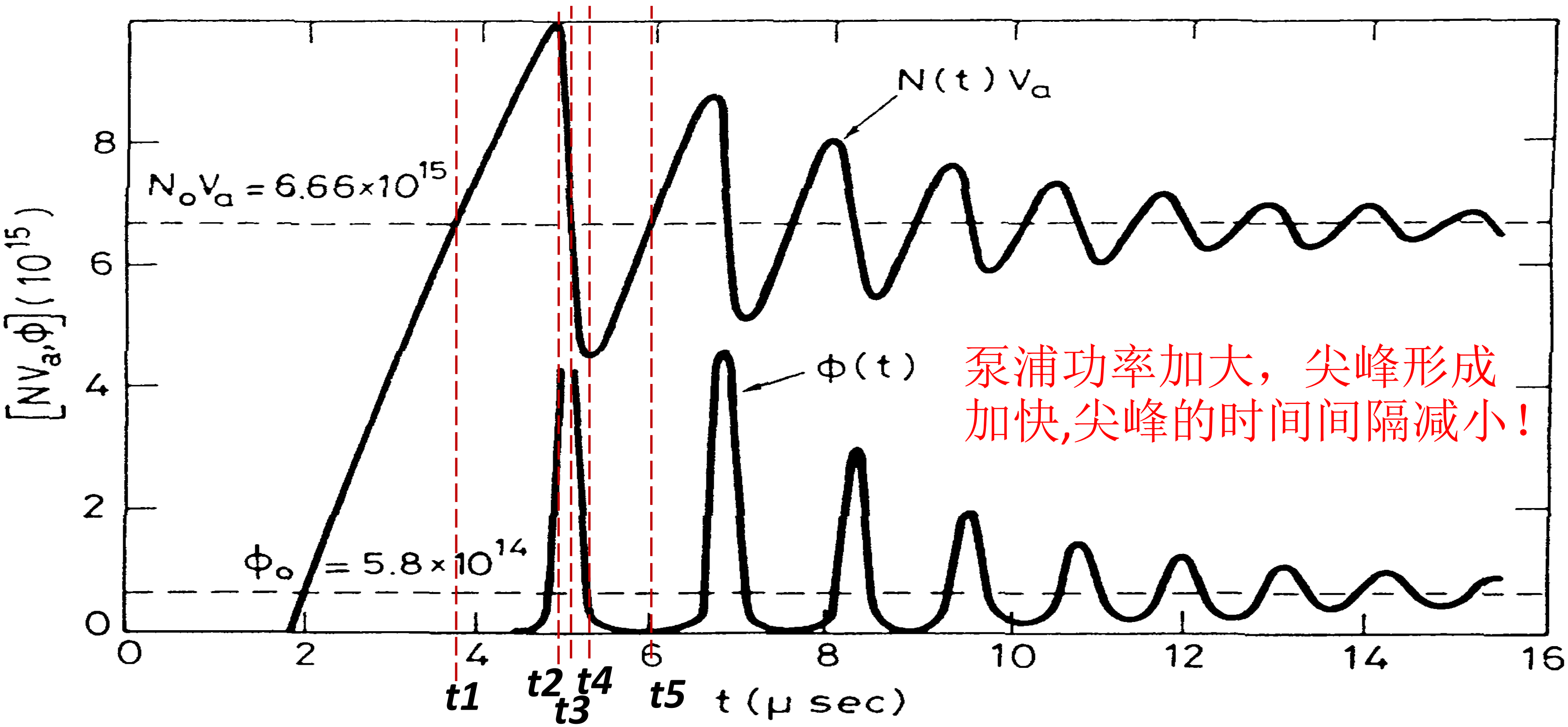
异质结：更高的注入效率、更低的阈值；

更好的波导约束

脉冲激光

-------激光器的调Q与锁模

11.1 激光器的驰豫振荡



t_1~t_2: t_1时$\Delta N=\Delta N_t$, $\phi=0$, 随后由于泵浦作用ΔN增加,光子数φ增长;

t_2~t_3: t_2时光子数增加造成的ΔN下降超过泵浦造成的ΔN 增加, ΔN达到峰值后下降,但仍然大于ΔN_t, 故φ仍上升;

t_3~t_4: $\Delta N<\Delta N_t$, φ急剧下降;

t_4~t_5: 由于光子数的减少及泵浦作用, ΔN增加直至 ΔN_t, 下一个尖峰开始形成

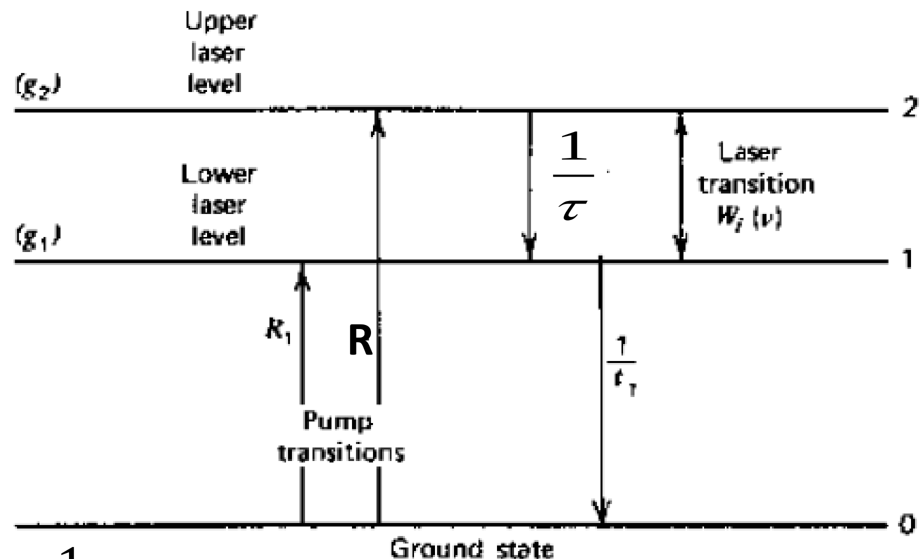
驰豫振荡的物理基础是腔内振荡光场与反转粒子数的相互作用
仍然采用理想的四能级模型，并假设

$$\frac{1}{t_1} \gg \frac{1}{t_2}、\frac{1}{W_i}，即\ N_1 \approx 0,\ 则 \Delta N \equiv N = N_2$$

腔内光子数密度记为ϕ， $W_i \propto \phi$，记 $W_i = B\phi$

速率方程：

$$\frac{dN}{dt} = R - W_i N - \frac{N}{\tau} = R - B\phi N - \frac{N}{\tau}$$



腔内光子数的速率方程：

$$\frac{d\phi}{dt} = B\phi N - \frac{\phi}{t_c},$$ t_c是腔内光子寿命

平衡态时，$\frac{d\phi}{dt} = B\phi N - \frac{\phi}{t_c} = 0 \Rightarrow N_0 = \frac{1}{Bt_c}$

$$\phi_0 = 0 \Rightarrow R_t = \frac{1}{Bt_c\tau}$$

记过阈值泵浦系数 $r = R/R_t$，

$$\frac{dN}{dt} = 0 \Rightarrow \phi_0 = \frac{RBt_c - \frac{1}{\tau}}{B}$$

则，$\phi_0 = \frac{r-1}{B\tau}$

非平衡小扰动情况下，

$$N(t)=N_0+N_1(t);\ \phi(t)=\phi_0+\phi_1(t),\ N_1 \ll N_0;\quad \phi_1 \ll \phi_0$$

$$\begin{aligned}\frac{dN}{dt}&=R-B\phi N-\frac{N}{\tau}\\ \frac{d\phi}{dt}&=B\phi N-\frac{\phi}{t_c}\end{aligned} \Rightarrow \begin{cases}\dfrac{dN_1}{dt}+\dfrac{dN_0}{dt}=R-B(\phi_0+\phi_1)(N_0+N_1)-\dfrac{N_0+N_1}{\tau}\\ \dfrac{d\phi_1}{dt}+\dfrac{d\phi_0}{dt}=B(\phi_0+\phi_1)(N_0+N_1)-\dfrac{\phi_0+\phi_1}{t_c}\end{cases}$$

$$\Rightarrow\begin{cases}\dfrac{dN_1}{dt}=-RBt_cN_1-\dfrac{\phi_1}{t_c}\\ \dfrac{d\phi_1}{dt}=(RBt_c-\dfrac{1}{\tau})N_1\end{cases} \Rightarrow \frac{dN_1}{dt}=\frac{1}{(RBt_c-\frac{1}{\tau})}\frac{d^2\phi_1}{dt^2}=-RBt_cN_1-\frac{\phi_1}{t_c}$$

$$N_1=\frac{1}{(RBt_c-\frac{1}{\tau})}\frac{d\phi_1}{dt}$$

$$=\frac{-RBt_c}{(RBt_c-\frac{1}{\tau})}\frac{d\phi_1}{dt}-\frac{\phi_1}{t_c}$$

$$\Rightarrow \frac{d^2\phi_1}{dt^2}+RBt_c\frac{d\phi_1}{dt}+(RBt_c-\frac{1}{\tau})\frac{\phi_1}{t_c}=0 \Rightarrow \frac{d^2\phi_1}{dt^2}+\frac{r}{\tau}\frac{d\phi_1}{dt}+\frac{1}{\tau t_c}(r-1)\phi_1=0$$

$$\frac{d^2\phi_1}{dt^2}+\frac{r}{\tau}\frac{d\phi_1}{dt}+\frac{1}{\tau t_c}(r-1)\phi_1=0$$

解: $\phi_1(\mathrm{t})=e^{pt}$, 则特征方程为 $p^2+\frac{r}{\tau}p+\frac{1}{\tau t_c}(r-1)=0 \Rightarrow p=-\alpha\pm i\omega_m$

其中, $\alpha=\frac{r}{2\tau}$, $\omega_m=\sqrt{\frac{1}{\tau t_c}(r-1)-(\frac{r}{2\tau})^2}\approx\sqrt{\frac{1}{\tau t_c}(r-1)}$ $\quad if\ \frac{1}{\tau t_c}(r-1)\gg(\frac{r}{2\tau})^2$

腔内光子密度的波动(即光强波动)：$\phi_1(\mathrm{t})\propto e^{-\alpha t}\cos(\omega_m t)$

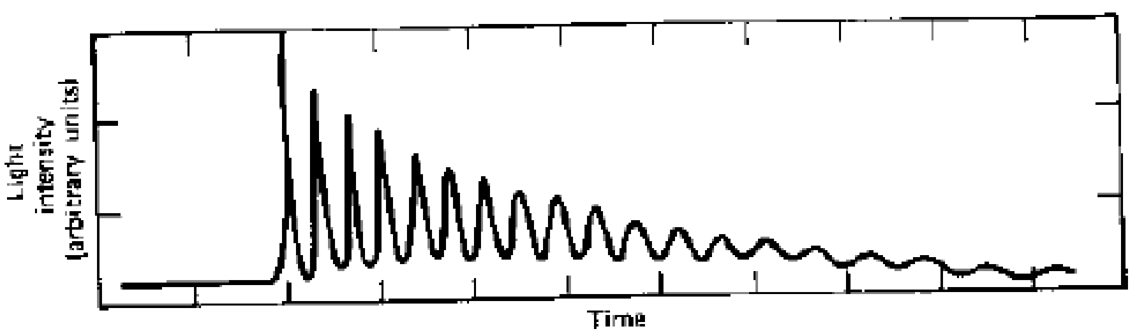


FIGURE 20.17 Intensity relaxation oscillation in a $CaWO_4:Nd^{3+}$ laser at 1.06 μm. Horizontal scale = 20 μsec/div. *Source:* Reference 20.

考虑泵浦速率R随时间变化： $R(\mathrm{t}) = R_0 + R_1(\mathrm{t})$

$$\begin{cases} \dfrac{dN_1}{dt} = R_1 - RBt_cN_1 - \dfrac{\phi_1}{t_c} \\ \dfrac{d\phi_1}{dt} = (\mathbf{R}_0 Bt_c - \dfrac{1}{\tau})N_1 \end{cases} \Rightarrow \frac{d^2\phi_1}{dt^2} + \frac{r}{\tau}\frac{d\phi_1}{dt} + \frac{1}{\tau t_c}(r-1)\phi_1 = \frac{1}{\tau}(\mathrm{r}-1)\mathrm{R}_1 \qquad \textbf{(*)}$$

对上面的(*)作傅立叶变换, 并记 $FT\{\phi_1(\mathrm{t})\} = \phi(\omega)$, $FT\{R_1(\mathrm{t})\} = R(\omega)$

$$\phi(\omega) = \frac{-\dfrac{1}{\tau}(r-1)\mathrm{R}(\omega)}{\omega^2 - i\dfrac{\omega r}{\tau} - \dfrac{1}{\tau t_c}(r-1)} = \frac{-\dfrac{1}{\tau}(r-1)\mathrm{R}(\omega)}{(\omega - \omega_m - \mathrm{i}\alpha)(\omega + \omega_m - \mathrm{i}\alpha)}$$

其中, $\alpha = \dfrac{r}{2\tau}$, $\omega_m = \sqrt{\dfrac{1}{\tau t_c}(r-1) - (\dfrac{r}{2\tau})^2} \approx \sqrt{\dfrac{1}{\tau t_c}(r-1)}$

Typical time behavior of early cw-pumped solid-state lasers. Time scale is 50 μs/div.

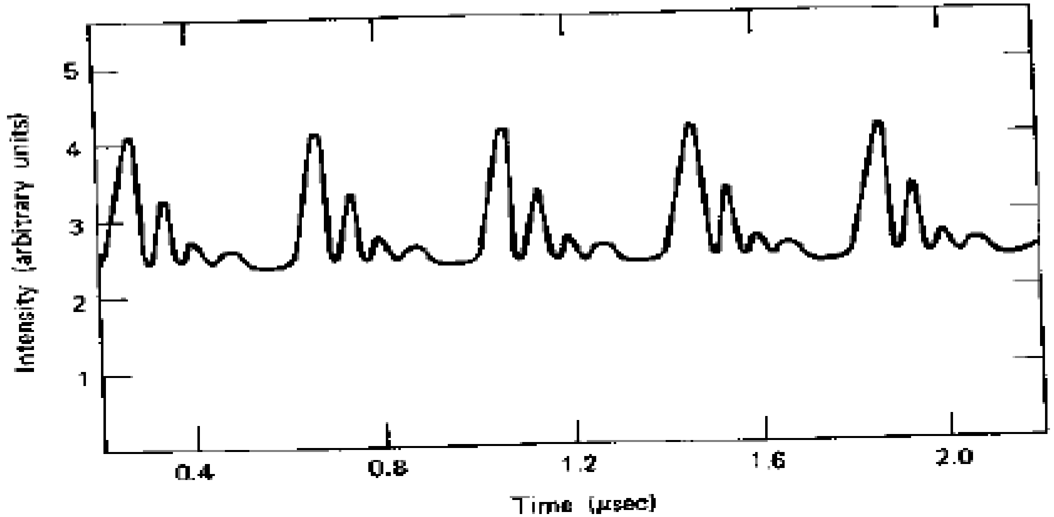


FIGURE 20.18 Intensity relaxation oscillation in a xenon 3.51-μm laser. *Source:* Reference 21.

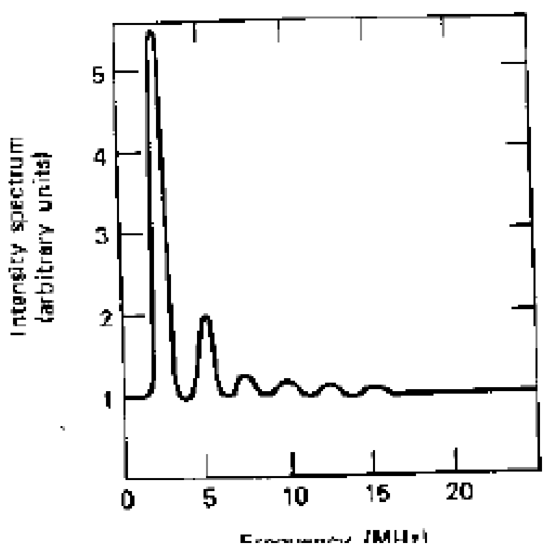


FIGURE 20.19 The intensity fluctuation spectrum of the laser output shown in Figure 20.18. *Source:* Reference 21.

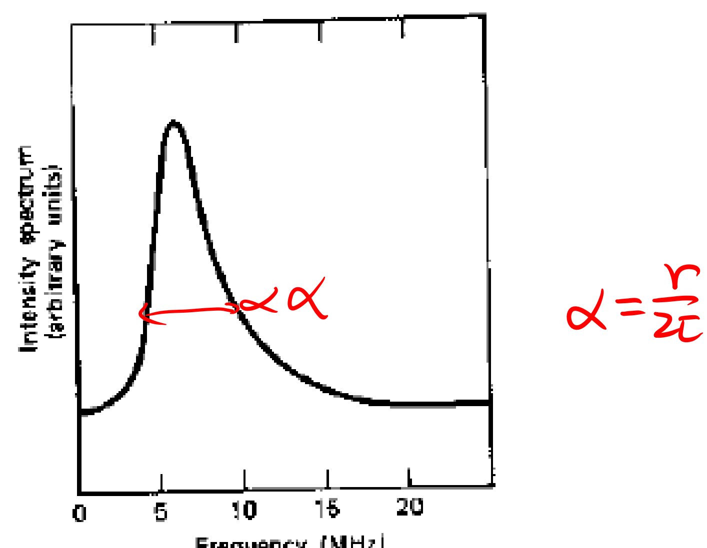


FIGURE 20.20 Same as Figure 20.19 except at increased pumping. *Source:* Reference 21.

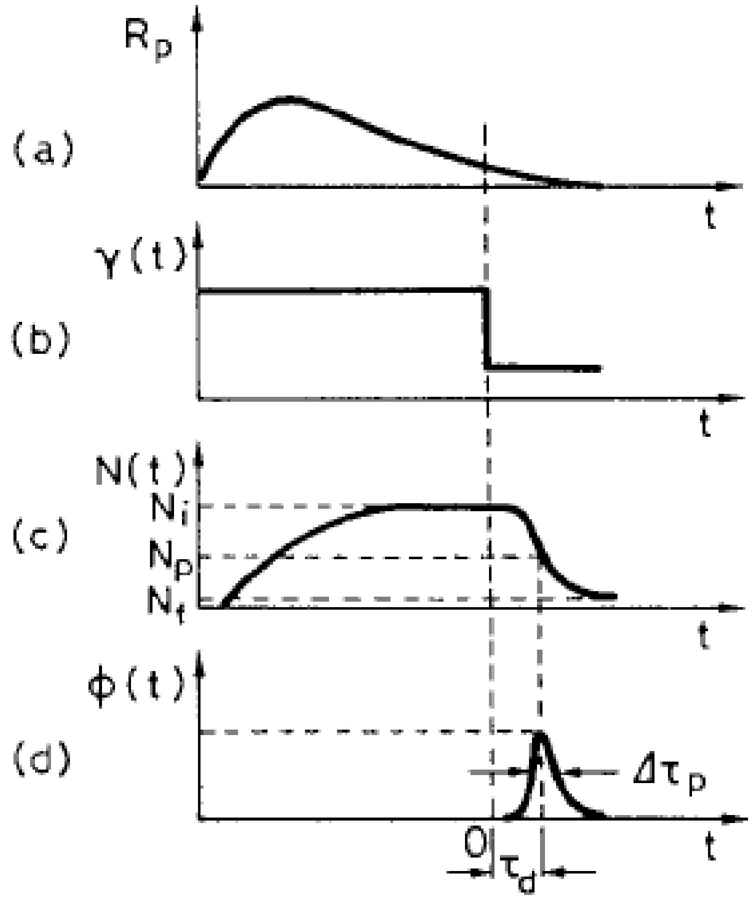
11.2 Q开关

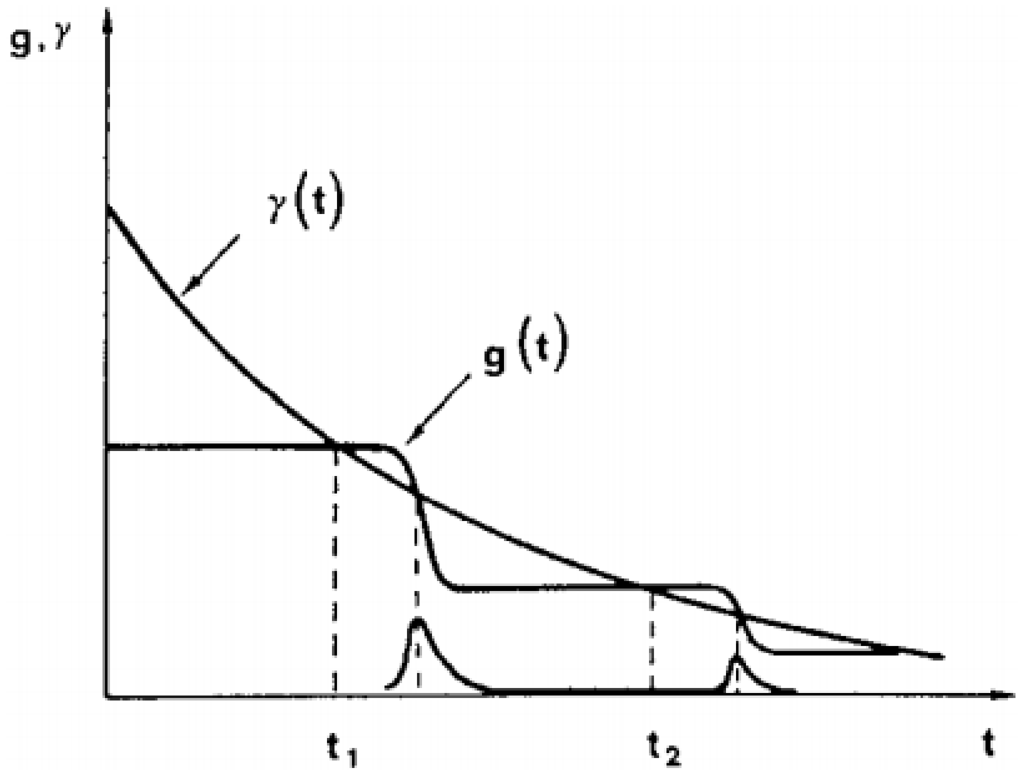
调Q原理

初始时刻低Q（高损耗），泵浦抽运使增益增大，反转粒子数达到峰值后，Q值迅速升高（低损耗），受激辐射造成反转粒子数耗尽产生“巨脉冲”

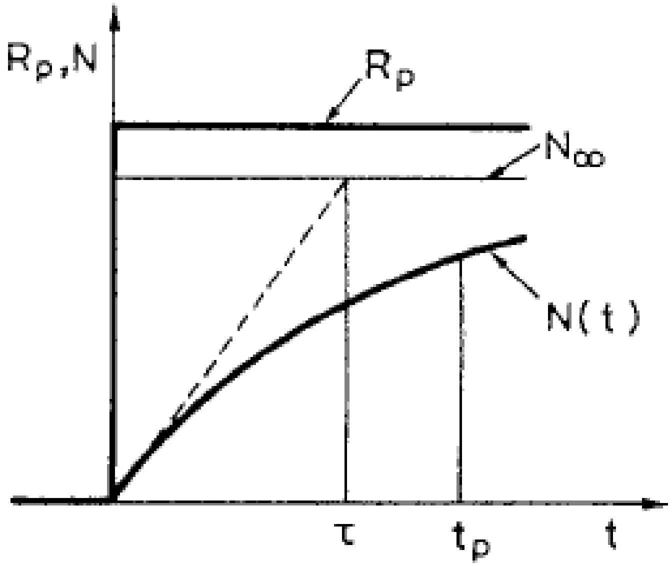
腔内光子寿命（输出脉冲的衰减时间）

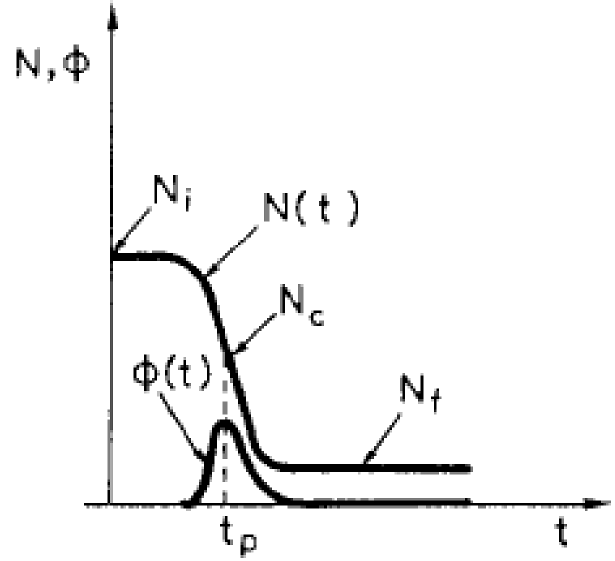
$$t_c = \frac{n_0 l}{c(\alpha l - \ln\sqrt{R_1 R_2})} \approx 10^{-8} s$$





慢速开关将导致多脉冲产生





初始时刻低Q（高损耗），泵浦抽运使增益增大

$$\frac{dN}{dt}=R_p-\frac{N}{\tau} \Rightarrow N(\mathrm{t})=N_\infty(1-e^{-t/\tau}) \text{，其中} N_\infty=R_p\tau$$

① 假设矩形泵浦脉冲，当$t>>\tau$时，反转粒子数不会无限增长而是趋近于一个常数值N_∞。

② 为获得大的N_∞一般要求τ比较大（比如Nd, Yb, Er, Ho 掺杂在不同基质中，其上能级寿命通常在ms量级）。

假设脉冲期间忽略粒子数驰豫和抽运，并且Q开关无限快

增益系数γ的定义: $\frac{dI}{dz}=\gamma I \quad \Rightarrow \frac{dI}{dt}=\frac{dI}{dz}\frac{dz}{dt}=\gamma I\cdot\frac{c}{n_0}=\frac{\gamma c}{n_0}I$

激光腔长l，工作介质长度L，所以填充因子为L/l

腔内光子数变化的速率方程: $\frac{d\phi}{dt}=(\gamma\frac{cL}{n_0 l}-\frac{1}{t_c})\phi$

用t_c对时间t做归约化: $\tau=t/t_c$，则 $\gamma_t=n_0 l/cLt_c$

$$\frac{d\phi}{d\tau}=t_c(\gamma\frac{cL}{n_0 l}-\frac{1}{t_c})\phi=(\frac{\gamma}{n_0 l/cLt_c}-1)\phi=(\frac{\gamma}{\gamma_t}-1)\phi$$

又 $\gamma\propto n$，故

$$\frac{d\phi}{d\tau}=(\frac{n}{n_t}-1)\phi \qquad (1)$$

$$\frac{dn}{d\tau}=-2\frac{n}{n_t}\phi \qquad (2)$$

反转粒子数减少的速率是腔内光子数产生速率的2倍！

$$\frac{(1)}{(2)}\Rightarrow\frac{d\phi}{dn}=-\frac{1}{2}+\frac{n_t}{2n}$$

$$\Rightarrow\phi-\phi_i=\frac{1}{2}[n_t\ln\frac{n}{n_i}-(\mathrm{n}-n_i)]$$

$$\phi - \phi_i = \frac{1}{2}[n_t \ln \frac{n}{n_i} - (\mathrm{n} - n_i)]$$

通常ϕ_i=0 (假设初始无光子)

所以, $\phi = \frac{1}{2}[n_t \ln \frac{n}{n_i} - (\mathrm{n} - n_i)]$

$\tau \to \infty$即$t \gg t_c$时, $\phi = 0$

即, $0 = \frac{1}{2}[n_t \ln \frac{n_f}{n_i} - (\mathrm{n}_f - n_i)]$

$$\Rightarrow \frac{n_f}{n_i} = \exp(\frac{\mathrm{n}_f - n_i}{n_t})$$

when, $\frac{n_i}{n_t} \uparrow$

能量利用率$\eta = \frac{\mathrm{n}_i - n_f}{n_i} \to 1$

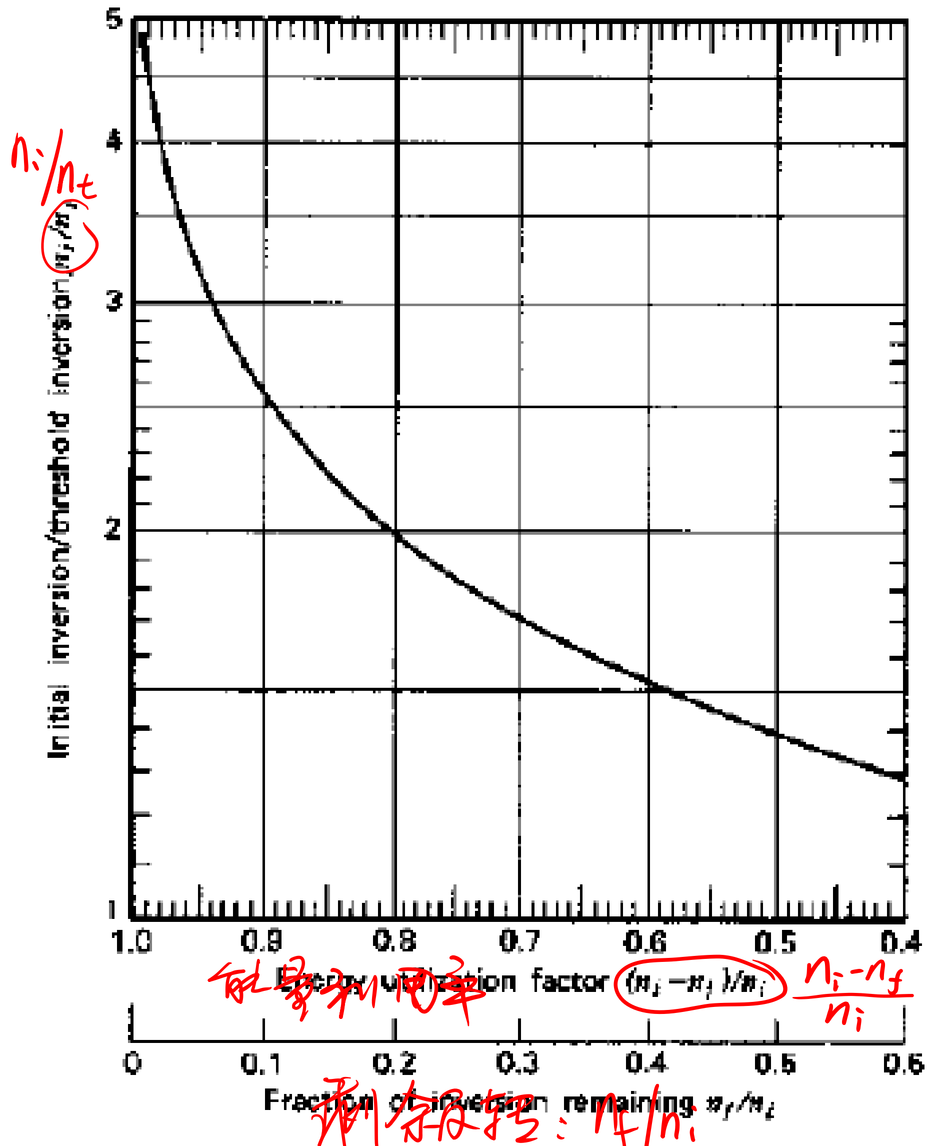


FIGURE 20.1 Energy utilization factor $(n_i - n_f)/n_i$ and inversion remaining after the giant pulse. *Source:* Reference 4.

(瞬时)激光输出功率:

$$P=\frac{\phi h\nu}{t_c}=\frac{h\nu}{2t_c}\left[n_t\ln\frac{n}{n_i}-(\mathrm{n}-n_i)\right]$$

$$峰值:\frac{\partial P}{\partial n}=0\ \Rightarrow n_t\frac{n_i}{n}\frac{1}{n_i}-1=0$$

$$\Rightarrow\quad n=n_t\ \ 时\ P=P_{\max}$$

$$P_{\max}=\frac{h\nu}{2t_c}\left[n_t\ln\frac{n_t}{n_i}-(\mathrm{n}_t-n_i)\right]$$

$$if\ \ n_i\gg\ \mathrm{n}_t:P_{\max}=\frac{n_i h\nu}{2t_c}$$

$$\frac{h\nu}{2t_c}n_i\left[\frac{n_t}{n_i}\ln\frac{n_t}{n_i}-\frac{n_t}{n_i}+1\right]$$

$$\lim_{x\to 0}(x\ln x-x)=\lim_{x\to 0}\frac{\ln x-1}{1/x}$$

$$=\lim_{x\to 0}\frac{1/x}{-1/x^2}=\lim_{x\to 0}(-x)=0$$

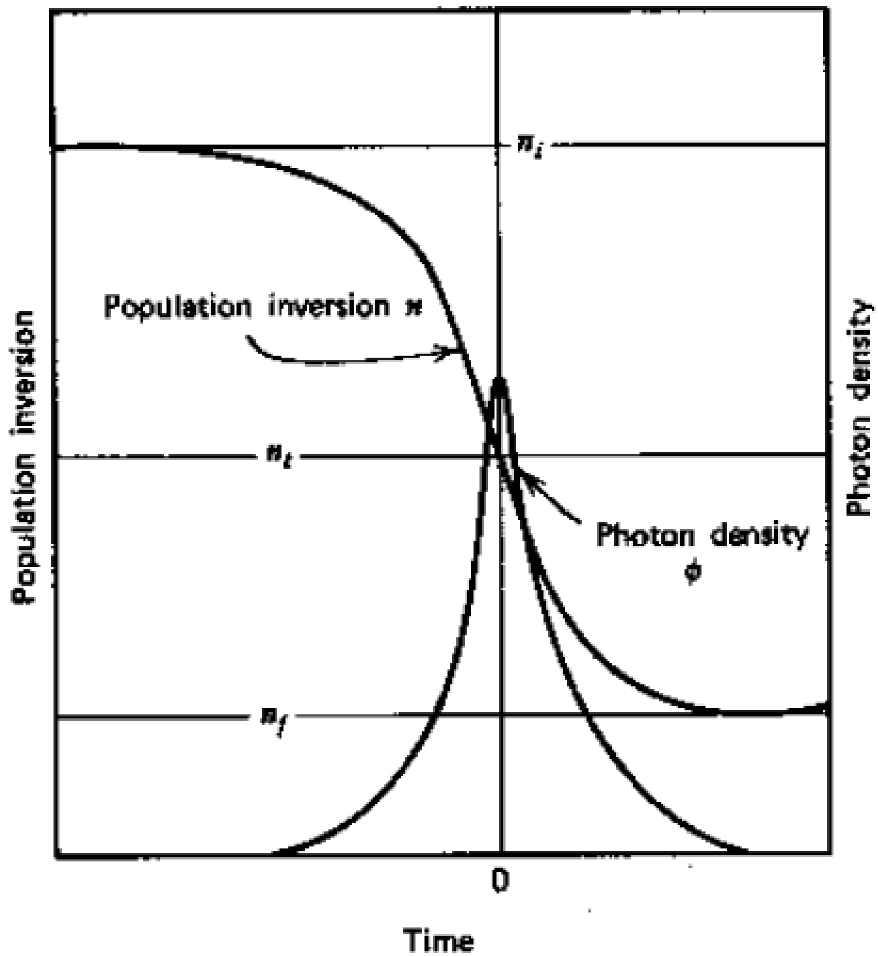


FIGURE 20.2 Inversion and photon density during a giant pulse. *Source:* Reference 4.

调Q脉冲的时间特性：

$$\begin{cases} \dfrac{d\phi}{d\tau} = (\dfrac{n}{n_t} - 1)\phi \\ \dfrac{dn}{d\tau} = -2\dfrac{n}{n_t}\phi \end{cases} \Rightarrow$$

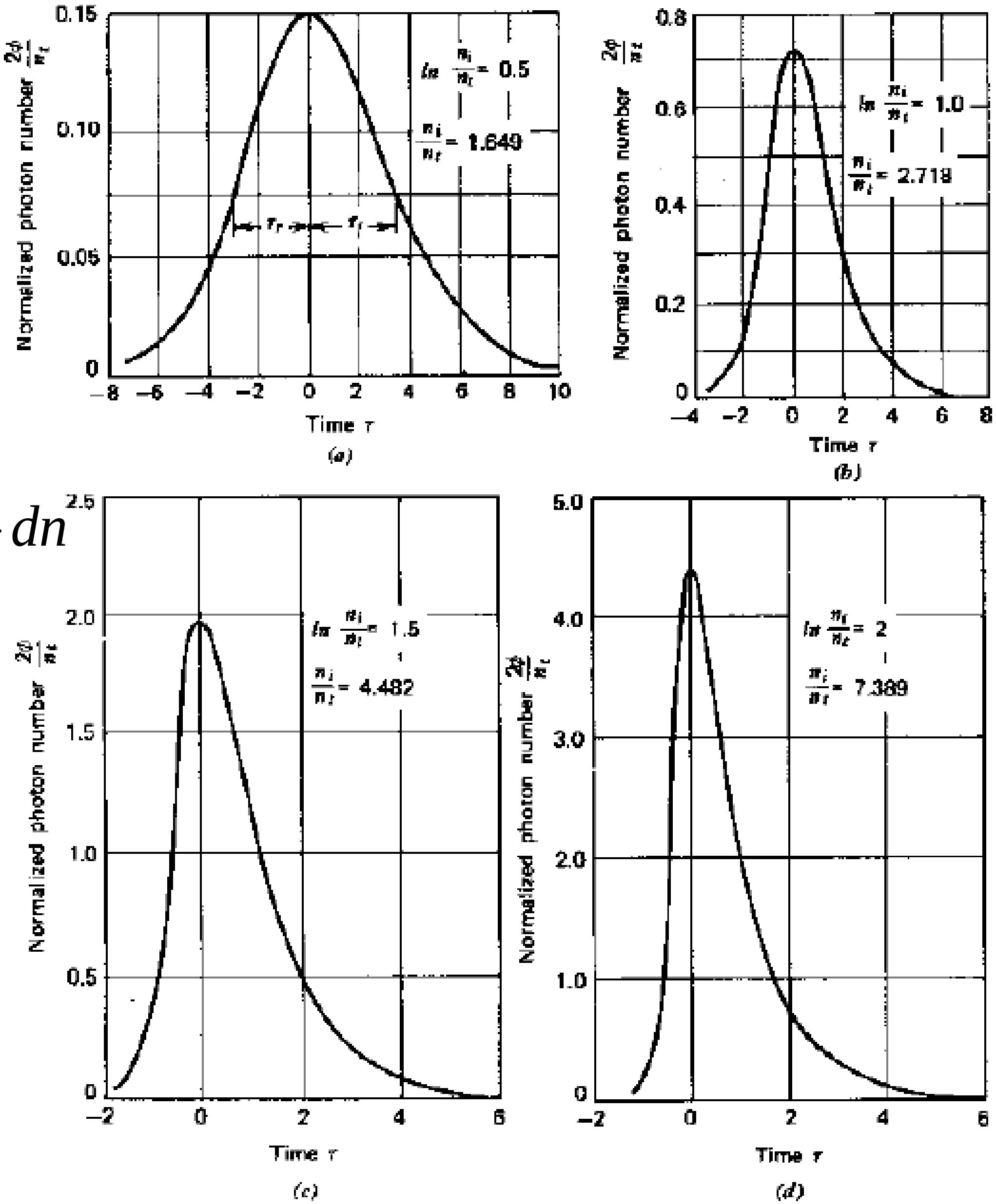
$$\phi = \frac{1}{2}[n_t \ln \frac{n}{n_i} - (\mathrm{n} - n_i)]$$

$$d\tau = -\frac{n_t}{2n\phi} dn = -\frac{n_t}{n[n_t \ln \frac{n}{n_i} - (\mathrm{n} - n_i)]} dn$$

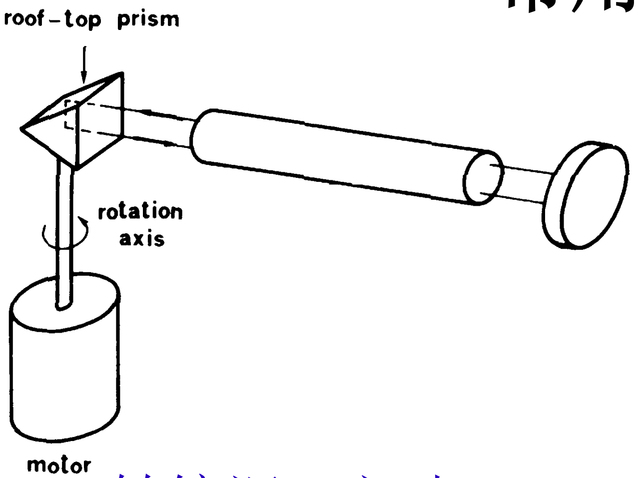
$$\tau_1 = -\int_{n_i}^{n_t} \frac{n_t}{n[n_t \ln \frac{n}{n_i} - (\mathrm{n} - n_i)]} dn$$

$$\tau_2 = -\int_{n_t}^{n_f} \frac{n_t}{n[n_t \ln \frac{n}{n_i} - (\mathrm{n} - n_i)]} dn$$

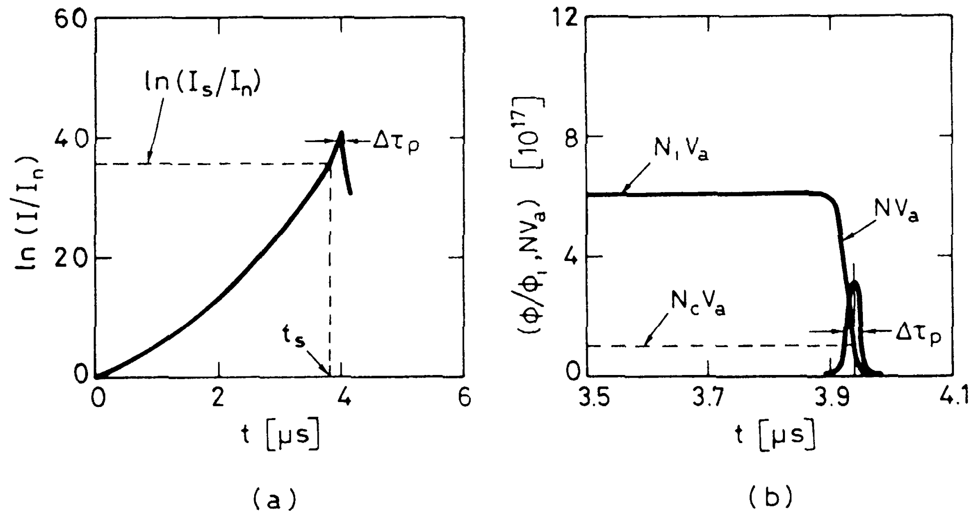
$$\tau = \tau_1 + \tau_2$$



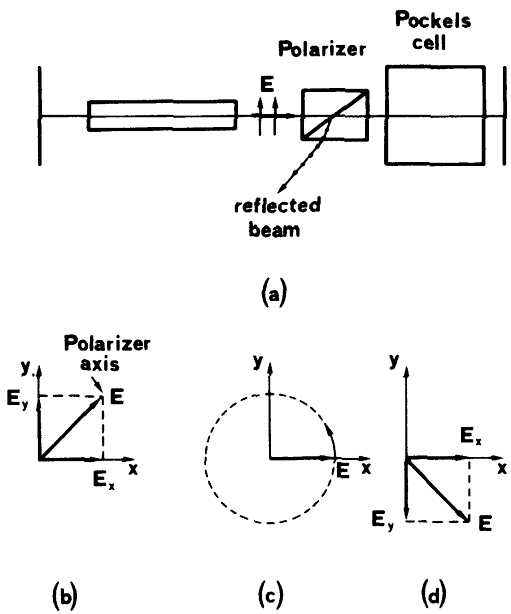
常用调Q方法

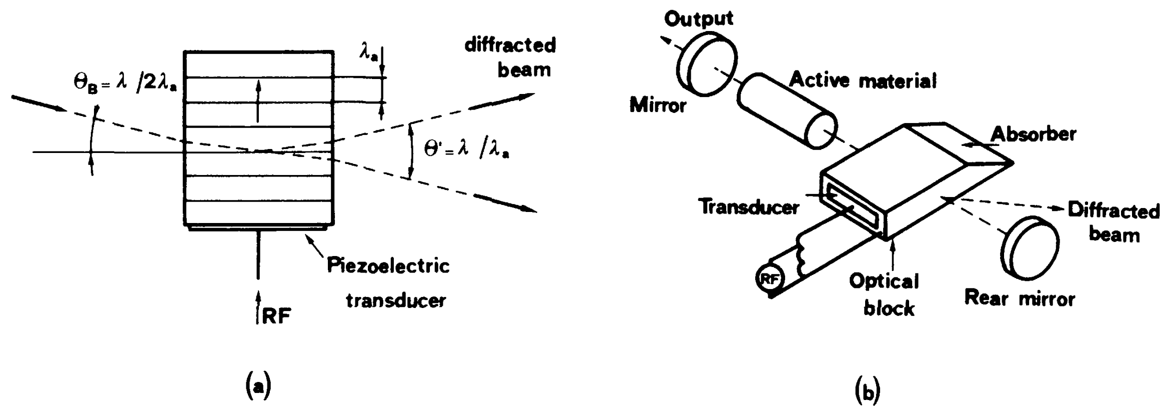


转镜调Q(主动)



可饱和吸收调Q(被动)



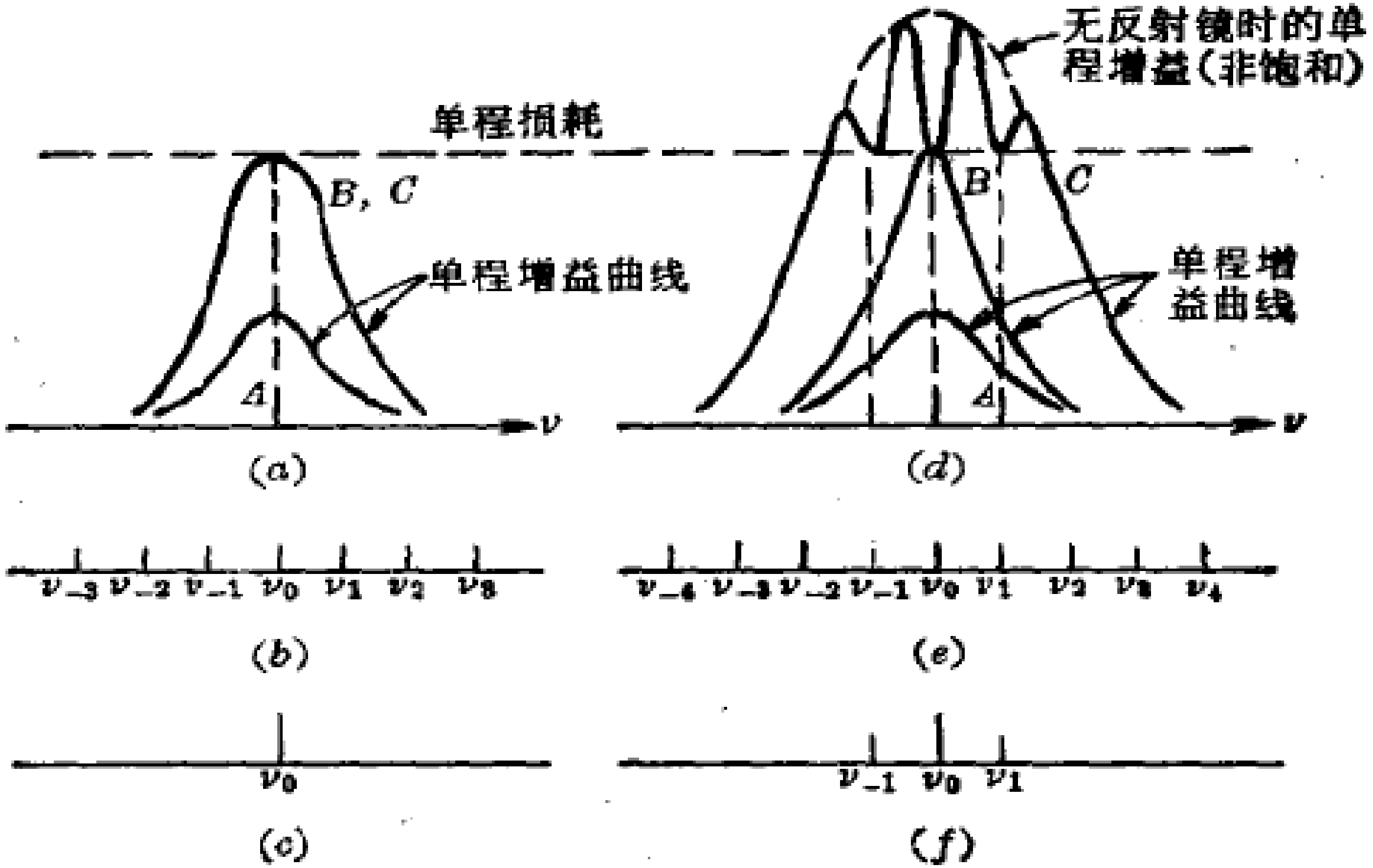


电光调Q(主动)

声光调Q(主动)

11.3 激光锁模

非均匀加宽介质中的多模激光运转



多模自由运转激光特性：

假设激光腔长L，纵模间隔 $\Delta\nu_q = \dfrac{c}{2nL}$ 各纵模间非相干，具有独立、随机的初位相

光强：

$$E(t)=\sum_{q=1}^{N} E_q \cos(\omega_q t+\varphi_q)\;;\quad \text{随机相位: } \varphi_q-\varphi_k=random$$

$$I(t)=E^2(t)=\sum_{q=1}^{N} E_q^2\cos^2(\omega_q t+\varphi_q)+\sum_{k=1}^{N}\sum_{q=1}^{N} E_qE_k\cos(\omega_q t+\varphi_q)\cos(\omega_k t+\varphi_k)$$

平均光强:

$$<I(t)>=<E^2(t)>=\frac{1}{T}\int_0^T E^2(t)dt$$

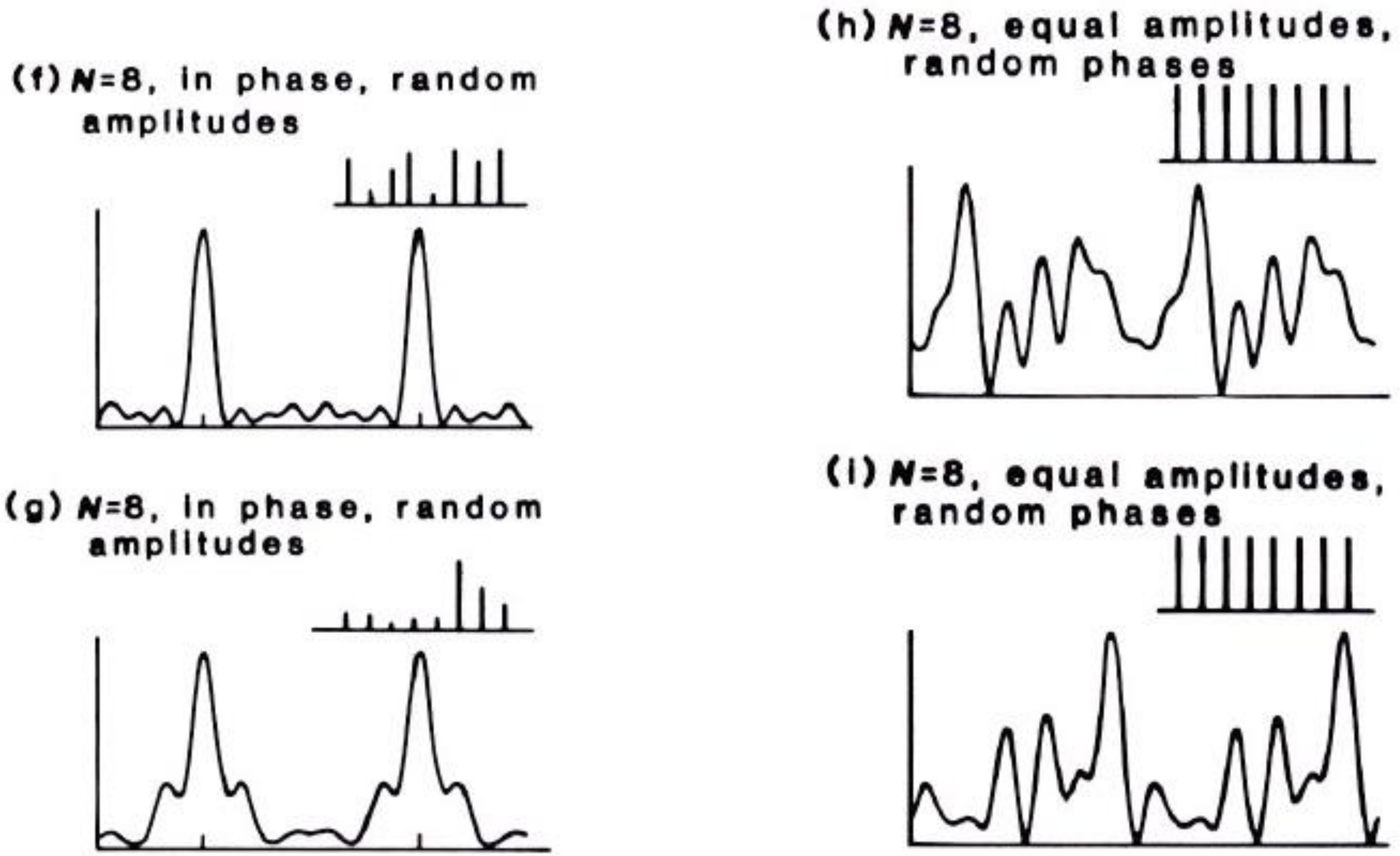
$$=\frac{1}{T}\sum_{q=1}^{N}\int_0^T E_q^2\cos^2(\omega_q t+\varphi_q)dt+\frac{1}{T}\sum_{k=1}^{N}\sum_{q=1}^{N}\int_0^T E_qE_k\cos(\omega_q t+\varphi_q)\cos(\omega_k t+\varphi_k)dt$$

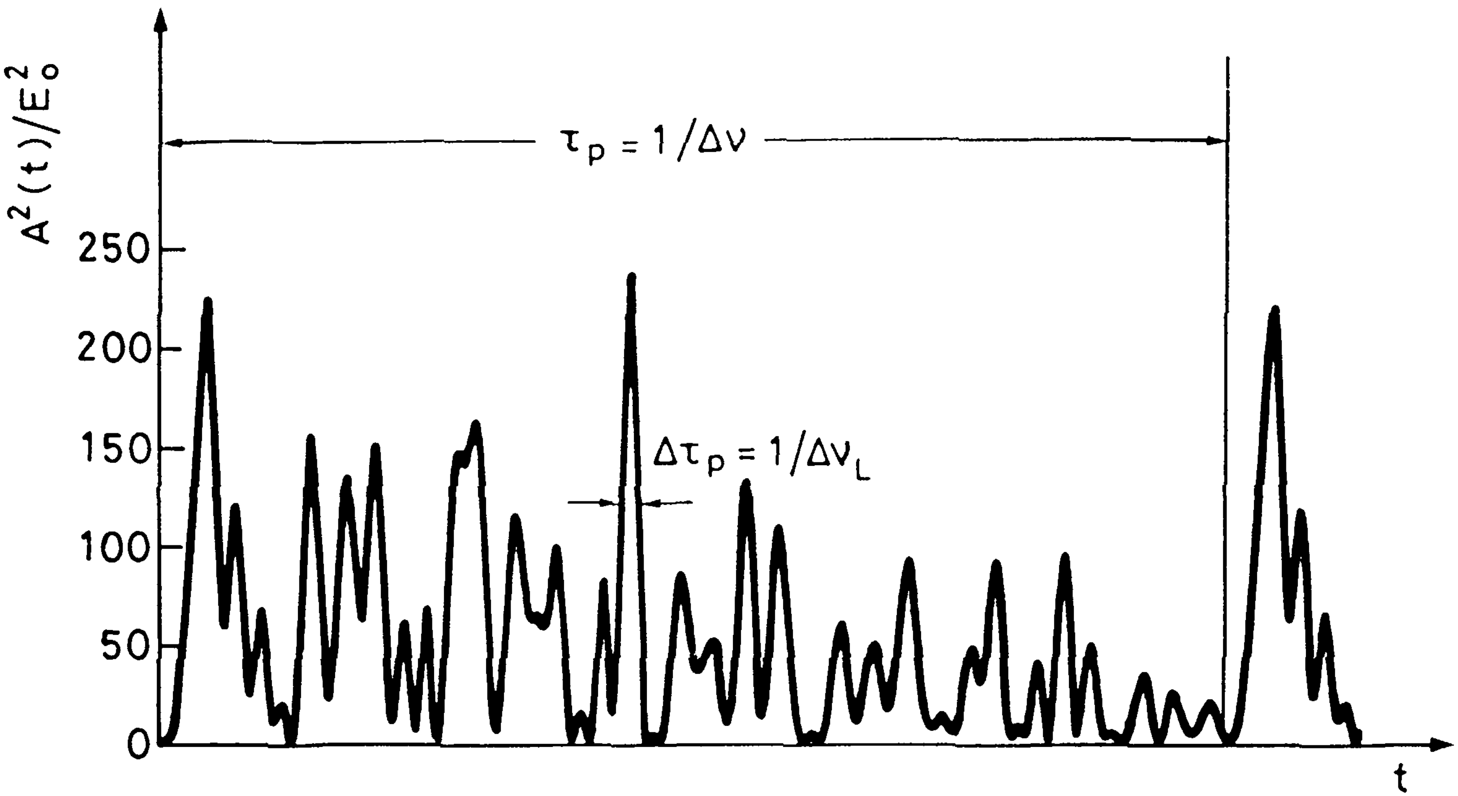
$=0$

$$=\frac{1}{2}\sum_{q=1}^{N}E_q^2=\sum_{q=1}^{N}I_q$$

各纵模光强之和！

随机相位的作用


(f) N=8, in phase, random amplitudes
(g) N=8, in phase, random amplitudes
(h) N=8, equal amplitudes, random phases
(i) N=8, equal amplitudes, random phases



振幅为E_0, 均匀频率间隔为$\Delta\nu$的N=51个纵模频率(随机初位相)构成的时间波前。其中 $\Delta\nu_L = N \cdot \Delta\nu$

锁模的基本原理：

多模激光器中，各振荡模具有相同的振幅E_0，共2N+1个模式，中心频率为ω_0，纵模间隔为 $\Delta\nu_q = \frac{c}{2nL}$，相邻模之间的相位差恒定为$\alpha$，即

$$E_q(t) = E_0 e^{i[(\omega_0 + q\Delta\omega)t + q\alpha]}$$

多模相干叠加:

$$E(t) = \sum_{q=-N}^{N} E_q(t) = E_0 e^{i\omega_0 t} \sum_{q=-N}^{N} e^{i(q\Delta\omega t + q\alpha)} = E_0 e^{i\omega_0 t} \frac{1 - e^{i(2N+1)(\Delta\omega t + \alpha)}}{1 - e^{i(\Delta\omega t + \alpha)}} e^{-iN(\Delta\omega t + \alpha)}$$

$$= E_0 e^{i\omega_0 t} \frac{\sin[\frac{1}{2}(2N+1)(\Delta\omega t + \alpha)]}{\sin[\frac{1}{2}(\Delta\omega t + \alpha)]}$$

特例, $\alpha = 0$:

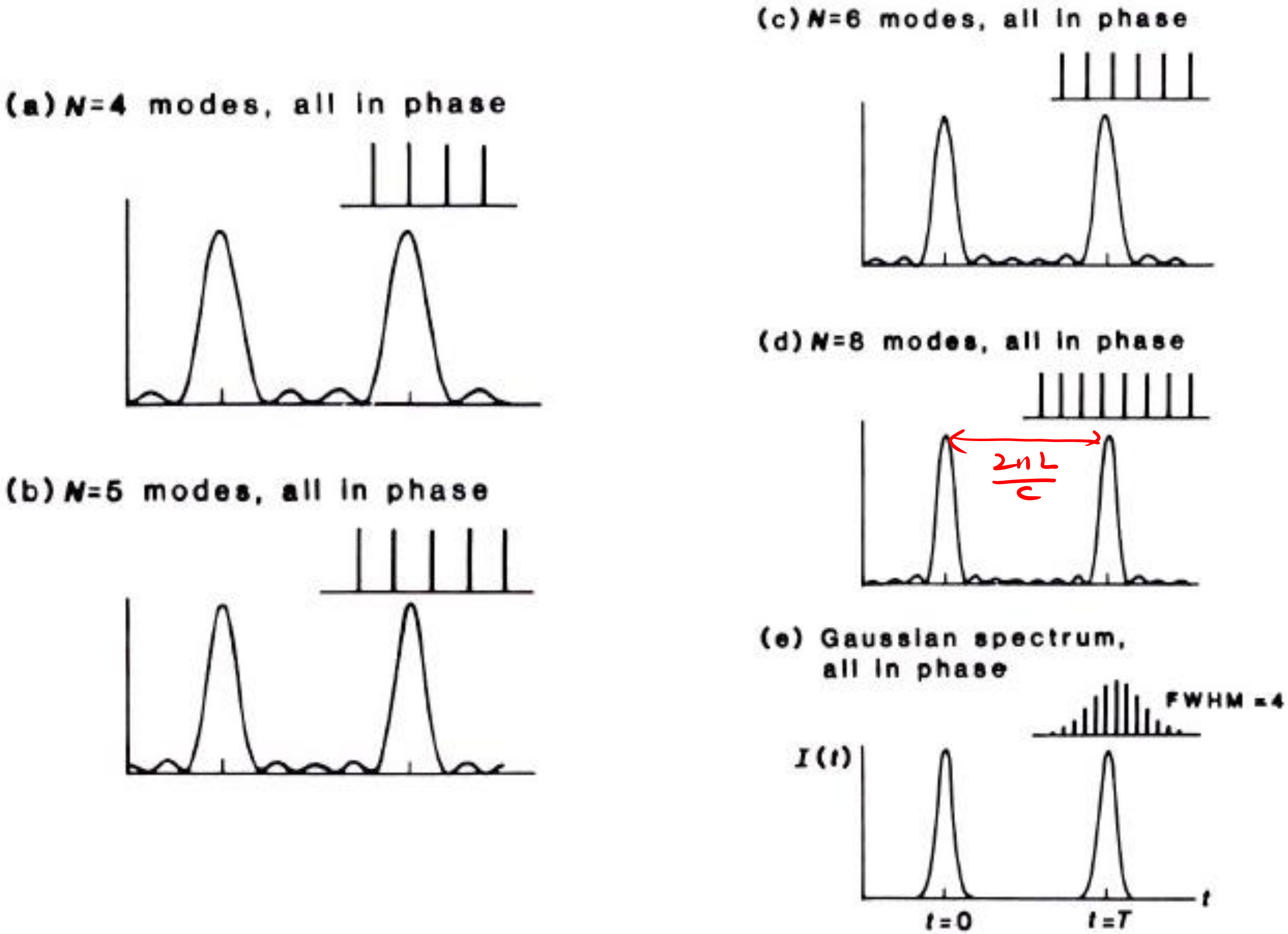
$$E(t) = E_0 e^{i\omega_0 t} \frac{\sin[\frac{1}{2}(2N+1)\Delta\omega t]}{\sin\frac{1}{2}\Delta\omega t}$$

光强: $I(t) = |E(t)|^2 = E_0^2 \{\frac{\sin[\frac{1}{2}(2N+1)\Delta\omega t]}{\sin\frac{1}{2}\Delta\omega t}\}^2$

极值条件: $\frac{1}{2}\Delta\omega t = n\pi$

相邻脉冲间隔: $\Delta T = \frac{2\pi}{\Delta\omega} = \frac{2nL}{c}$

N模”锁定”的结果


(a) N=4 modes, all in phase
(b) N=5 modes, all in phase
(c) N=6 modes, all in phase
(d) N=8 modes, all in phase
(e) Gaussian spectrum, all in phase
FWHM =4
I(t)
t
t=0
t=T

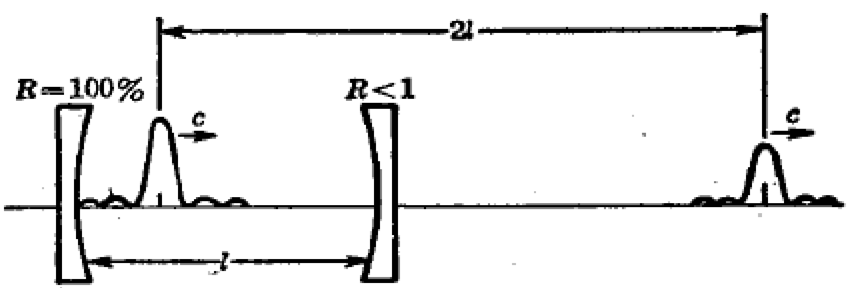
锁模脉冲的特点：

① 激光脉冲是周期为T的序列 $T=\dfrac{2\pi}{\Delta\omega}=\dfrac{2nL}{c}$

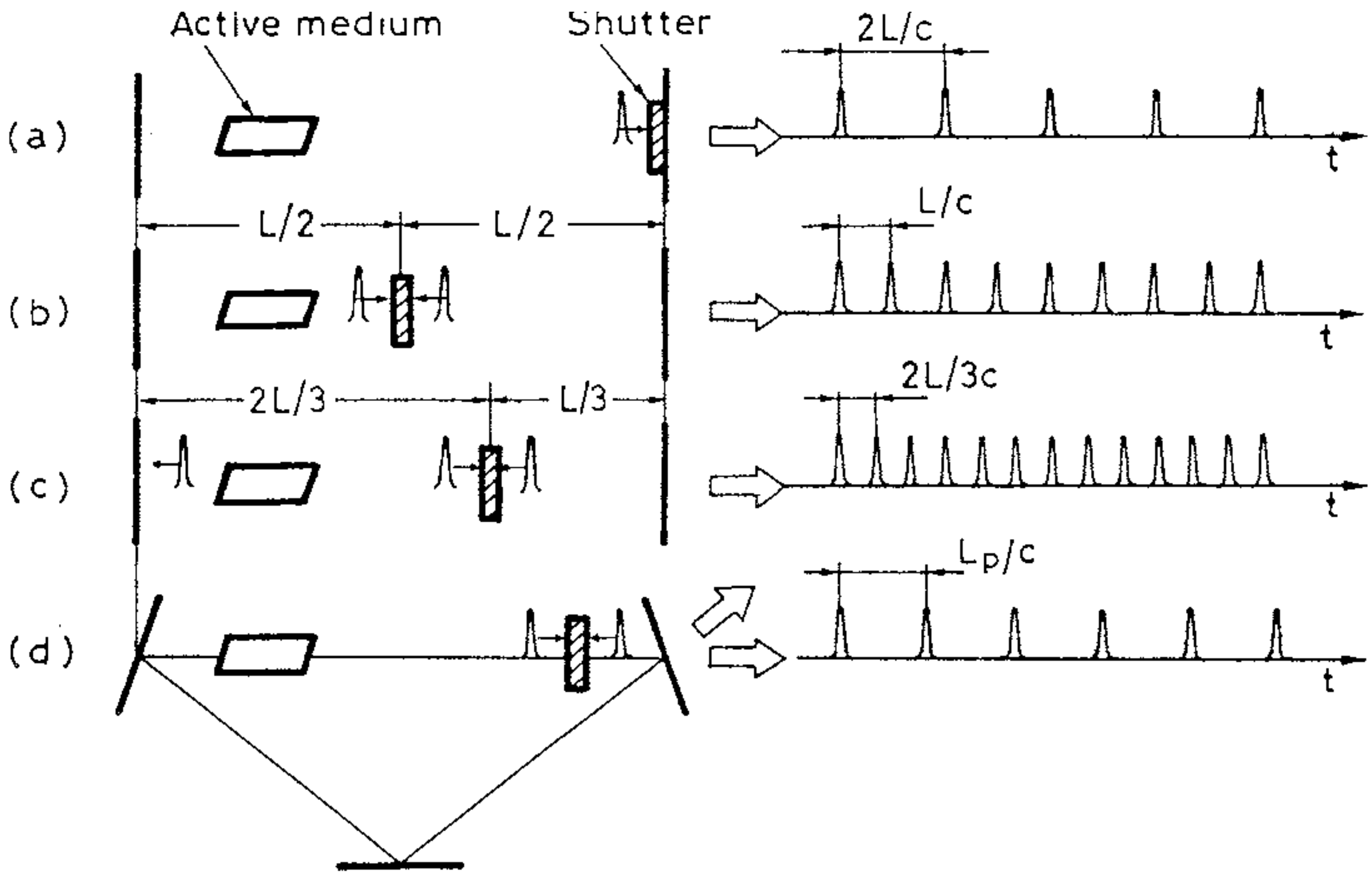
② 峰值功率 $P=(2N+1)^2\,\mathrm{E}_0^2$ 非相干叠加时 $P=(2N+1)\,\mathrm{E}_0^2$

③ 脉冲宽度 $\Delta\tau=\dfrac{1}{2N+1}\dfrac{1}{\Delta\nu_q}=\dfrac{1}{\Delta\nu_g}$

④ 多纵模激光器锁模后，各振动模式之间发生功率耦合，不再独立。



锁模脉冲的时域图像



Fundamental mode-locking

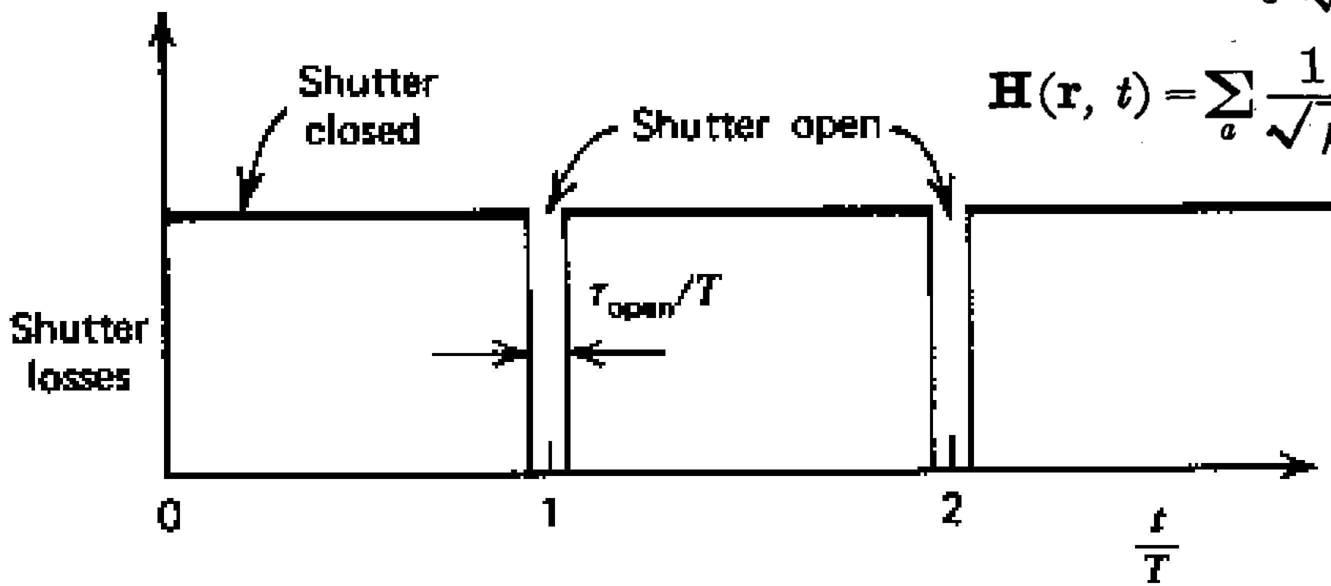
Harmonic mode-locking

锁模方法---内损耗调制

损耗调制的振荡谐振腔Maxwell方程求解

$$\mathbf{E}(\mathbf{r},t) = -\sum_a \frac{1}{\sqrt{\varepsilon}} p_a(t)\mathbf{E}_a(\mathbf{r})$$

$$\mathbf{H}(\mathbf{r},t) = \sum_a \frac{1}{\sqrt{\mu}} \omega_a q_a(t)\mathbf{H}_a(\mathbf{r})$$



令谐振腔介质的有效电导率σ在空间和时间上变化就可引导出损耗调制，因此麦克斯韦方程可写为

$$\nabla\times\mathbf{H} = \sigma(\mathbf{r},t)\mathbf{E} + \varepsilon\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \qquad (11.2\text{-}10)$$

按式(5.5-11)的展开式来代替 $\mathbf{H}$ 和 $\mathbf{E}$，并利用式(5.5-3, 4)，则式(11.2-10)的第一个方程为

$$\sum_a \frac{1}{\sqrt{\mu}} \omega_a q_a k_a \mathbf{E}_a = -\frac{\sigma(\mathbf{r}, t)}{\sqrt{\varepsilon}} \sum_a p_a \mathbf{E}_a - \sqrt{\varepsilon} \sum_a \dot{p}_a \mathbf{E}_a \tag{11.2-11}$$

而第二个方程则为

$$\dot{q}_b = p_b \tag{11.2-12}$$

用 $\mathbf{E}_b$ 点乘式(11.2-11)，并在腔体积范围内积分，可得

$$\omega_b^2 q_b = -\sum_a S_{b,a}(t) p_a - \dot{p}_b \tag{11.2-13}$$

式中

$$S_{b,a}(t) = \frac{1}{\varepsilon} \int_{腔} \sigma(\mathbf{r}, t) \mathbf{E}_a \cdot \mathbf{E}_b dv \tag{11.2-14}$$

方程式(11.2-12)和(11.2-13)是 p_b 和 q_b 的运动方程。在该点上，很方便就可以引进简正模振幅

$$c_a(t) = (2\omega_a)^{-1/2}[\omega_a q_a(t) + i p_a(t)] \tag{11.2-15}$$

利用式(11.2-15)及其在式(11.2-12, 13)的复共轭

$$\frac{dc_a^*}{dt} = i\omega_a c_a^* - \sum_b \varkappa_{b,a}(t)(c_b^* - c_b)$$

$$\frac{dc_a}{dt} = -i\omega_a c_a + \sum_b \varkappa_{b,a}(t)(c_b^* - c_b) \tag{11.2-16}$$

式中 $$\varkappa_{b,a}(t) = \frac{1}{2} S_{b,a}(t) \sqrt{\frac{\omega_b}{\omega_a}}$$

若电导率取为一平均项和一谐波微扰项的总和

$$\sigma(\mathbf{r},\ t)=\sigma_0+\sigma_1(\mathbf{r})\cos(\omega_m t+\phi)$$

利用式(11.2-14)，$\varkappa_{b,a}(t)$ 的表示式变为

$$\varkappa_{b,a}(t)=\frac{\sigma_0}{2\varepsilon}\delta_{a,b}+\frac{\varkappa_{b,a}}{2}\left[e^{i(\omega_m t+\phi)}+e^{-i(\omega_m t+\phi)}\right] \quad (11.2\text{-}17)$$

还有

$$\varkappa_{b,a}=\frac{1}{2\varepsilon}\sqrt{\frac{\omega_b}{\omega_a}}\int_{腔}\sigma_1(\mathbf{r})\mathbf{E}_b\cdot\mathbf{E}_a\,dv \quad (11.2\text{-}18)$$

将式(11.2-17)代入运动方程(11.2-16)，可得

$$\frac{dc_a^*}{dt}=i\omega_a c_a^*-\frac{\sigma_0}{2\varepsilon}(c_a^*-c_a)$$

$$-\sum_b\frac{\varkappa_{b,a}}{2}\left[e^{i(\omega_m t+\phi)}+e^{-i(\omega_m t+\phi)}\right](c_b^*-c_b) \quad (11.2\text{-}19)$$

这是 dc_a/dt 的复数共轭。这就是主要的通用方程。

今定义“失调参数” $\Delta\omega_m$ 为

$$\omega_{a+1}-\omega_a=\pi c/l=\omega_m-\Delta\omega_m \quad (11.2\text{-}20)$$

因此 $\Delta\omega$ 为调制频率对模之间的间距的偏离值。定义绝热变量 $D_a^*(t)$ 为

$$c_a^*=D_a^*(t)e^{i[(\omega_a+a\Delta\omega_m)t+a\phi+a\pi/2]}e^{-(\sigma_0/2\varepsilon)t}$$

代入式(11.2-19)并利用式(11.2-20)，可得

$$\frac{dD_a^*}{dt}+ia\Delta\omega D_a^*=-i\frac{\varkappa}{2}D_{a+1}^*+i\frac{\varkappa}{2}D_{a+1}^* \quad (11.2\text{-}21)$$

式中 $\varkappa \equiv \varkappa_{a,a+1} \simeq \varkappa_{a,a-1}$。稳态解 $(dD_a^*/dt=0)$ 为

$$D_a^* = I_a\left(\frac{\varkappa}{\Delta\omega}\right) \tag{11.2-22}$$

式中 I_a 为 a 阶双曲线贝塞尔(Bessel)函数。$c_a^*(t)$ 则由下式表示

$$c_a^*(t) = I_a\left(\frac{\varkappa}{\Delta\omega}\right) e^{i[(\omega_a + a\Delta\omega)t + a\phi + a\pi/2]} e^{-\sigma_0 t/2\varepsilon} \tag{11.2-23}$$

式中，按式 (11.2-20)，并取 $\omega_a = \omega_0 + (a\pi c/l)$，则 $\omega_a + a\Delta\omega = \omega_0 + a\omega$。对于 $\varkappa/\Delta\omega \gg 1$ 的情况，用 $[2\pi(\varkappa/\Delta\omega)]^{-1/2}$ 来代替 $I_a(\varkappa/\Delta\omega)$，于是

$$c_a^*(t) = \left(2\pi \frac{\varkappa}{\Delta\omega}\right)^{-1/2} e^{i[(\omega_0 + a\omega)t + a\phi + a\pi/2]} \tag{11.2-24}$$

式中衰减项 $\exp(-\sigma_0 t/2\varepsilon)$ 已消去，因为激光介质的增益会引起稳态模的振荡。

均匀展宽介质的锁模

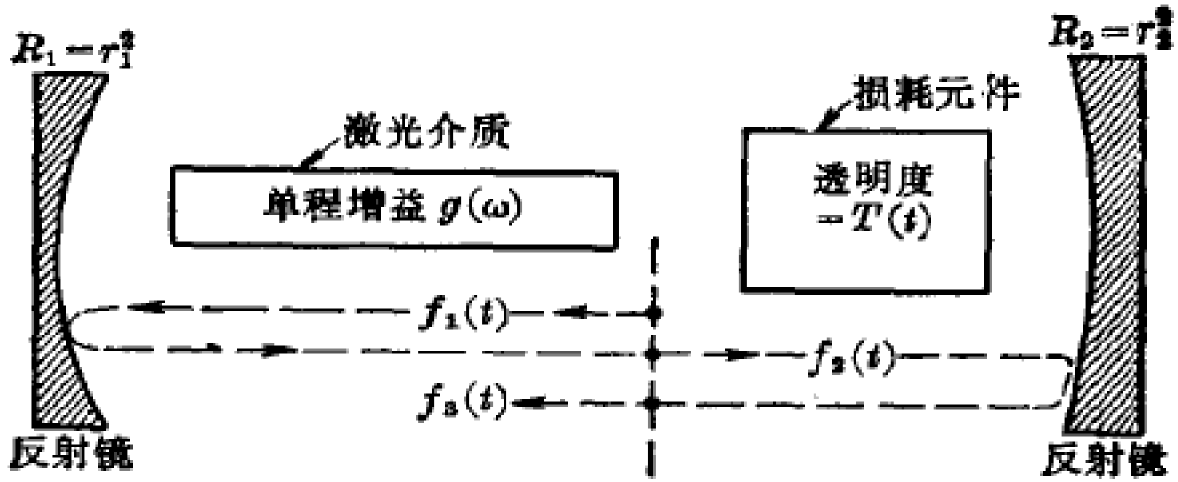


图 11.13　均匀加宽激光器锁模的理论分析所采用的实验装置

增益介质的传递函数

$$E_{出}(\omega)=E_{入}(\omega)g(\omega)$$

利用式(8.2-4)和式(8.1-19)，可得

$$g(\omega)=\exp\left\{-ikl\left[1+\frac{1}{2n^2}(\chi'-i\chi'')\right]\right\}$$

$$=\exp\left\{-ikl-\frac{kl\mu^2T_2\Delta N_0}{2n^2\varepsilon_0\hbar}\left[\frac{1}{1+i(\omega-\omega_0)T'_2}\right]\right\}$$

$$\simeq\exp\left\{-ikl+\frac{\gamma_{极大}l}{2}[1-i(\omega-\omega_0)T_2-(\omega-\omega_0)^2T_2^2]\right\}$$

对于 $(\omega-\omega_0)T_2\ll1$ 的情况能得到很好的近似，同时要记起，对于

增益有 $\Delta N_0<0$。由于脉冲是两次通过损耗元件的，于是有

$$\frac{E_{出}(\omega)}{E_{入}(\omega)}=[g(\omega)]^2$$

$$=\exp\{-i2kl+\gamma_{极大}l[1-i(\omega-\omega_0)T_2-(\omega-\omega_0)^2T_2^2]\}$$

指数中的虚数项对应于时间延迟(由于脉冲的有限群速度引起的)

$$\tau_d=\frac{2l}{c}+l\gamma_{极大}T_2$$

这里只需考虑脉冲形状的影响，因此可略去虚数项[8]，于是

$$[g(\omega)]^2=\exp\{\gamma_{极大}l[1-(\omega-\omega_0)^2T_2^2]\}\qquad(11.3\text{-}1a)$$

损耗元件的传递函数

假设损耗元件的单程振幅透过系数 $T(t)$ 由下式表示

$$E_{出}(t)=E_{入}(t)T(t)=E_{入}(t)\exp[-2\delta_1^2\sin^2(\pi\Delta\nu_{轴}t)] \tag{11.3-2}$$

式中 $\Delta\nu_{轴}$ 为纵模间距，可表示为

$$\Delta\nu_{轴}=\frac{c}{2l_0}$$

$$T(t)\simeq\exp\left[-\frac{1}{4}(\Gamma_m^2\omega_m^2t^2)\right]=\exp[-2\delta_1^2(\pi\Delta\nu_{轴}t)^2]$$

现在，回过头来讨论主要分析。图 11.13 中的起始脉冲可取为

$$f_1(t)=Ae^{-\alpha_1 t^2}e^{i(\omega_0 t+\beta_1 t^2)} \tag{11.3-4}$$

这相当于一"啁啾"频率

$$\omega(t)=\omega_0+2\beta_1 t \tag{11.3-5}$$

它的傅里叶变换为

$$\begin{aligned}F_1(\omega)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}f_1(t)e^{-i\omega t}dt\\&=\frac{A}{2}\sqrt{\frac{1}{\pi(\alpha_1-i\beta_1)}}\exp[-(\omega-\omega_0)^2/4(\alpha_1-i\beta_1)]\end{aligned} \tag{11.3-6}$$

以传递系数 $[g(\omega)]^2r_1$ 乘 $F_1(\omega)$ 就可计算出两次通过放大器并经一个反射镜反射(r_1)后的傅里叶变换为

$$\begin{aligned}F_2(\omega)=F_1(\omega)[g(\omega)]^2r_1&=\frac{r_1A}{2}e^{g_0}\sqrt{\frac{1}{\pi(\alpha_1-i\beta_1)}}\\&\times\exp\left\{[-(\omega-\omega_0)^2]\left[\frac{1}{4(\alpha_1-i\beta_1)}+g_0T_2^2\right]\right\}\end{aligned} \tag{11.3-7}$$

式中 $g_0=\gamma_{最大}l$，而 $[g(\omega)]^2$ 可由式 (11.3-1a) 来表示。将其改回时域

$$f_2(t)=\int_{-\infty}^{\infty}F_2(\omega)e^{i\omega t}d\omega$$

$$=\frac{r_1Ae^{g_0}}{2\pi}\sqrt{\frac{\pi}{\alpha_1-i\beta_1}}\,e^{-\omega_0^2Q}\sqrt{\frac{\pi}{Q}}\exp[-(2i\omega_0Q-t)^2/4Q] \tag{11.3-8}$$

式中

$$Q=\frac{1}{4(\alpha_1-i\beta_1)}+g_0T_2^2 \tag{11.3-9}$$

从反射镜 2 反射回来并通过损耗元件，根据式 (11.3-3) 可得

$$f_3(t)=r_2f_2(t)\exp[-2\delta_l^2\pi^2(\Delta\nu_{轴})^2t^2]$$

$$=\frac{r_1r_2Ae^{g_0}}{2}\sqrt{\frac{1}{(\alpha_1-i\beta_1)Q}}\,e^{i\omega_0t}$$

$$\times\exp\{-[2\delta_l^2(\pi\Delta\nu_{轴})^2+(1/4Q)]t^2\} \tag{11.3-10}$$

由于自洽，要求 $f_3(t)$ 与 $f_1(t)$ 一模一样。于是令式 (11.3-10) 与式 (11.3-4) 的指数相等，可得

$$\alpha_1=2\delta_l^2(\pi\Delta\nu_{轴})^2+\mathrm{Re}\left(\frac{1}{4Q}\right)$$

$$\beta_1=-\mathrm{Im}\left(\frac{1}{4Q}\right) \tag{11.3-11}$$

利用式(11.3-9)和(11.3-11)中的第二个式子，求得

$$\beta_1=\frac{\beta_1}{(1+4g_0T_2^2\alpha_1)^2+(4g_0T_2^2\beta_1)^2}$$

因此自治解要求

$$\beta_1=0$$

这就意味着不存在啁啾。当 $\beta_1=0$ 时式(11.3-11)的第一个式子可写成

$$2\delta_i^2(\pi\Delta\nu_{轴})^2+\frac{\alpha_1}{(1+4g_0T_2^2\alpha_1)}=\alpha_1 \qquad (11.3\text{-}12)$$

假设

$$4g_0T_2^2\alpha_1\ll 1 \qquad (11.3\text{-}13)$$

结果可得

$$\alpha_1=\left(\frac{\delta_1^2}{2g_0}\right)^{1/2}\frac{\pi\Delta\nu_{轴}}{T_2}$$

由式(11.3-4)可得半强度点上的脉冲宽度为

$$\tau_p=(2\ln 2)^{1/2}\alpha_1^{-1/2}$$

因此自洽脉冲的宽度为

$$\tau_p=\frac{(2\ln 2)^{1/2}}{\pi}\left(\frac{2g_0}{\delta_1^2}\right)^{1/4}\left(\frac{1}{\Delta\nu_{轴}\Delta\nu}\right)^{1/2} \qquad (11.3\text{-}14)$$

式中 $\Delta\nu=(\pi T_2)^{-1}$。条件(11.3-13)式现在可解释为要求

$$\tau_p\gg 2\sqrt{g_0}T_2$$

在大多数情况下这个要求是正确的。

锁模技术分类

① 主动锁模（振幅调制/损耗调制、位相调制）
② 被动锁模（可饱和吸收）
③ 自锁模（利用工作物质本身的非线性）
④ 同步泵浦锁模

主动锁模

在腔内插入调制器， 且调制频率等于纵模间隔2nL/c

① 振幅调制

调制信号: $a(\mathrm{t}) = A_m \sin(\frac{1}{2}\omega_m \mathrm{t})$

损耗变化: $\alpha(\mathrm{t}) = \alpha_0 - \Delta\alpha_0 \cos(\omega_m \mathrm{t})$

调制器透过率: $T(\mathrm{t}) = \mathrm{T}_0 + \Delta T \cos(\omega_m \mathrm{t})$

调制前光场: $E(\mathrm{t}) = E_c \sin(\omega_c \mathrm{t} + \varphi_c)$

调制后光场:

$$E(\mathrm{t}) = E_c T(\mathrm{t}) \sin(\omega_c \mathrm{t} + \varphi_c) = E_c \{\mathrm{T}_0 + \Delta T \cos(\omega_m \mathrm{t})\} \sin(\omega_c \mathrm{t} + \varphi_c)$$

$$= A_c \{1 + m \cos(\omega_m \mathrm{t})\} \sin(\omega_c \mathrm{t} + \varphi_c)$$

$$= A_c \sin(\omega_c \mathrm{t} + \varphi_c) + \frac{1}{2} m A_c \sin[(\omega_c - \omega_m)\mathrm{t} + \varphi_c] + \frac{1}{2} m A_c \sin[(\omega_c + \omega_m)\mathrm{t} + \varphi_c]$$

振幅调制产生上下两个“边频”$\omega_c \pm \omega_m$，且它们的位相与ω_c光场的相同！

② 位相调制

调制前光场: $E(\mathrm{t}) = E_c \cos(\omega_c \mathrm{t} + \varphi_c)$

位相调制:

$$E(\mathrm{t}) = E_c \cos[\omega_c \mathrm{t} + \mathrm{m}_\phi \cos(\omega_m \mathrm{t}) + \varphi_c]$$

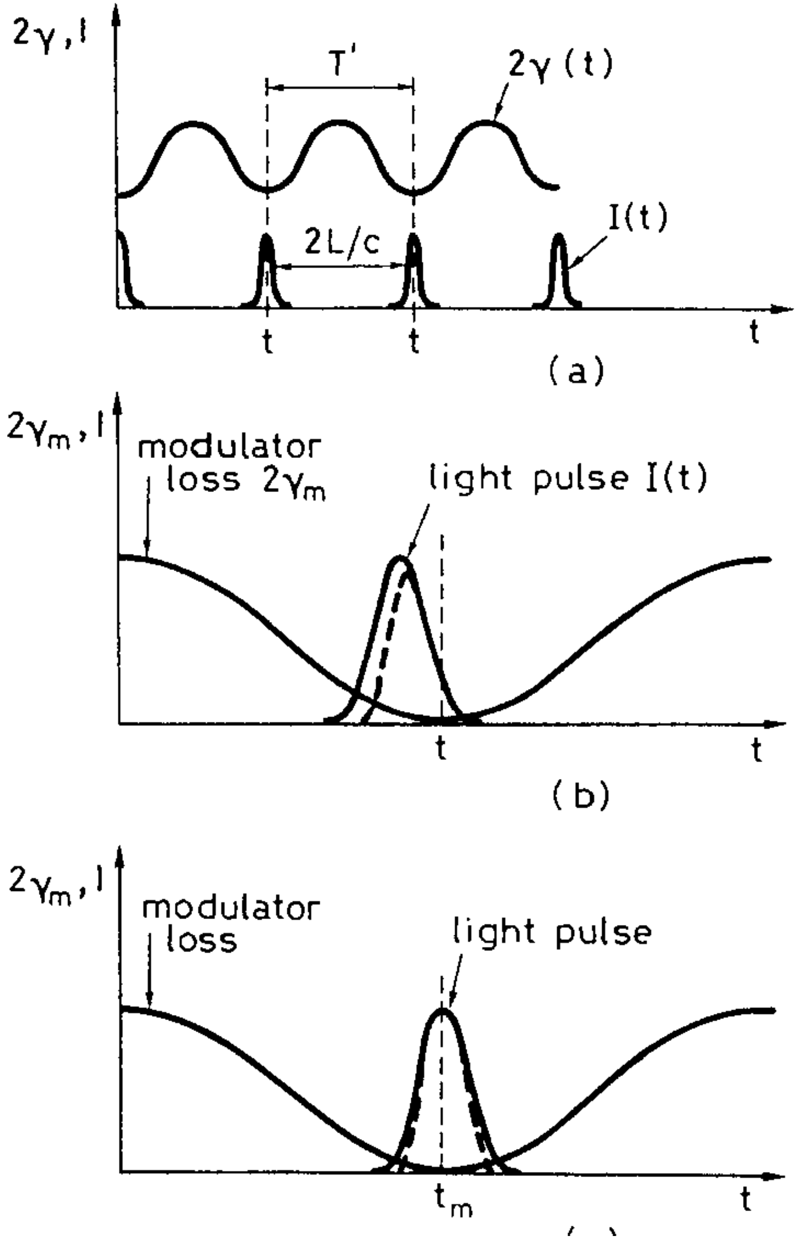
$$= E_c \cos(\omega_c \mathrm{t} + \varphi_c) + \frac{E_c}{2} \mathrm{m}_\phi \cos[(\omega_c + \omega_m)\mathrm{t} + \varphi_c] - \frac{E_c}{2} \mathrm{m}_\phi \cos[(\omega_c - \omega_m)\mathrm{t} + \varphi_c], \quad \text{when} \quad \mathrm{m}_\phi \ll 1$$

$$= E_c \{\mathrm{J}_0(\mathrm{m}_\phi) \cos(\omega_c \mathrm{t} + \varphi_c) + \mathrm{J}_1(\mathrm{m}_\phi)[\cos((\omega_c + \omega_m)\mathrm{t} + \varphi_c) - \cos((\omega_c - \omega_m)\mathrm{t} + \varphi_c)]$$

$$+ \mathrm{J}_2(\mathrm{m}_\phi)[\cos((\omega_c + 2\omega_m)\mathrm{t} + \varphi_c) - \cos((\omega_c - 2\omega_m)\mathrm{t} + \varphi_c)]$$

$+......\}$, 调制深度m_ϕ比较大时.

振幅调制(AM)的时域脉冲演化：



调制器设计要点：

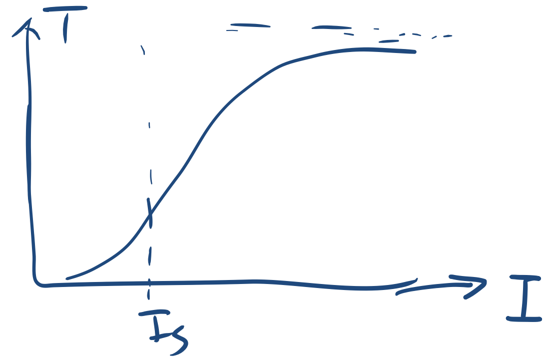
① 器件设计更严格，端面反射率控制到最小，否则会减少纵模数量

② 调制器应放置在腔内尽量靠近反射镜的位置

③ 锁模调制频率要严格调谐到 $f_m = \Delta\nu_q = \dfrac{c}{2L}$ (位相调制);

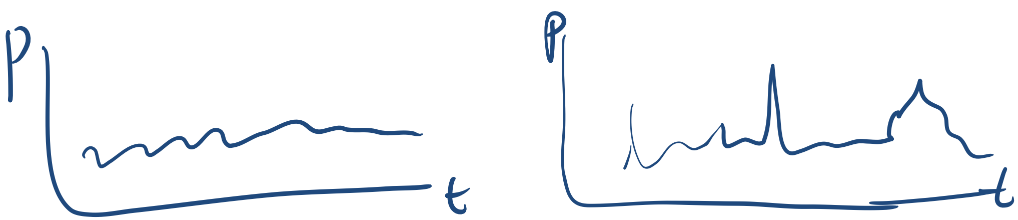
$$f_m = \frac{\Delta\nu_q}{2} = \frac{c}{4L}$$ (振幅调制)

饱和吸收体

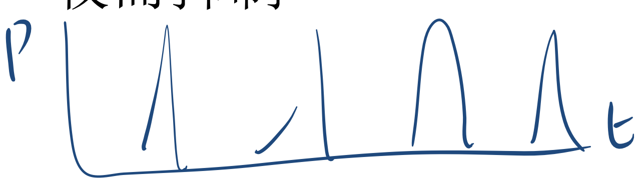


被动锁模过程

① 线性放大：抑制弱脉冲，放大强脉冲，对自发辐射荧光选模



② 非线性吸收：强脉冲使吸收介质“漂白”而增强，弱脉冲被吸收而抑制



③ 非线性放大：脉冲中心放大的多，前后沿放大少，脉冲宽度进一步压窄

常见锁模激光

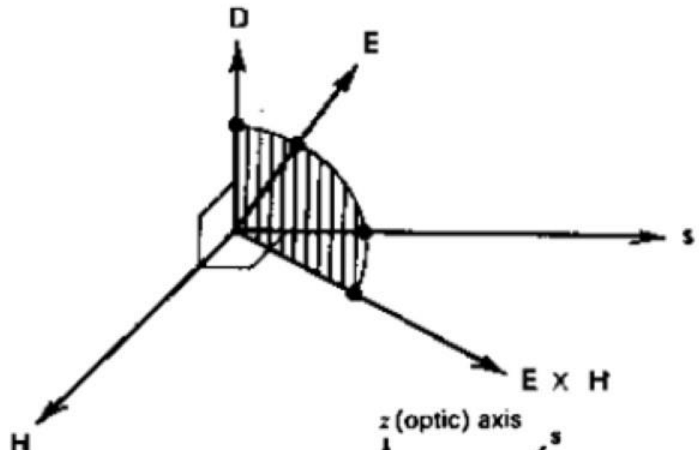
Laser Medium		Transition Linewidth[a] $\Delta\nu$	Calculated Pulse Duration $\tau_{pulse} = 1/\Delta\nu$	Observed Pulse Duratio
$Ti^{3+}:Al_2O_3$	H	100 THz	10 fs	10 fs
Rhodamine-6G dye	H/I	40 THz	25 fs	27 fs
Nd^{3+}:Glass (phosphate)	I	7 THz	140 fs	150 fs
Er^{3+}:Silica fiber	H/I	5 THz	200 fs	200 fs
Nd^{3+}:YAG	H	150 GHz	7 ps	7 ps
Ar^+	I	3.5 GHz	286 ps	150 ps
He–Ne	I	1.5 GHz	667 ps	600 ps
CO_2	I	60 MHz	16 ns	20 ns

Ch14 光辐射的调制

电光效应

---晶体在外加*电场*作用下*折射率发生变化*的效应

各项异性介质与折射率椭球



$\vec{D}=\vec{\varepsilon}\cdot\vec{E}$

一般地, $\vec{\varepsilon}$是张量, $\vec{D}$不平行于 $\vec{E}$

$$\frac{D_x^2}{\varepsilon_x'}+\frac{D_y^2}{\varepsilon_y'}+\frac{D_z^2}{\varepsilon_z'}=2w_e\varepsilon_0$$

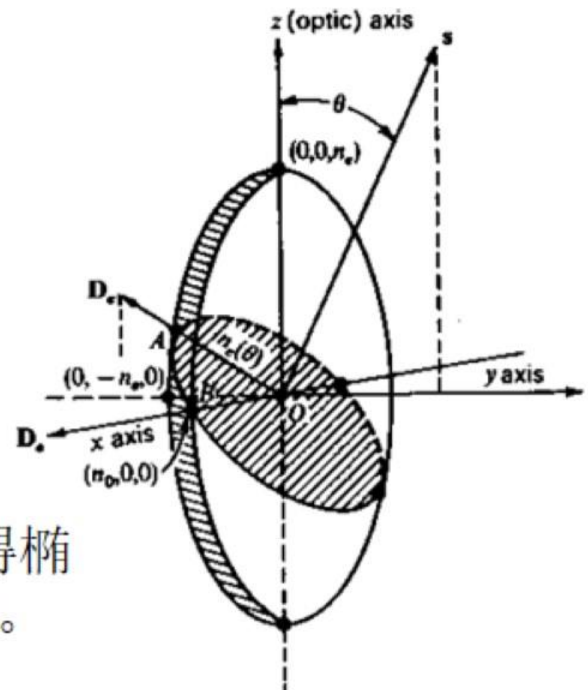
令 $\vec{r}=\vec{D}/2w_e\varepsilon_0$ 得

$$\frac{x^2}{n_x^2}+\frac{y^2}{n_y^2}+\frac{z^2}{n_z^2}=1$$

$n_x \neq n_y \neq n_z$ 双轴晶体

$n_x = n_y \neq n_z$ 单轴晶体

n_x, n_y, n_z是主轴折射率



晶体中沿k方向传播电磁波折射率的确定方法:
过原点作垂直于k方向的平面与折射率椭球相交得椭圆，其长短轴分别对应于两个本征振动的折射率。

不同晶体结构的光学性质

- 各向同性：立方晶系
  NaCl、GaAs等

- 单轴晶体： 三角、四方、六角晶系
  冰、石英、方解石、KDP等

- 双轴晶体：单斜、三斜、正交晶系
  石膏、云母等

线性电光效应（Pockels effect）只存在于非中心反演对称的晶体中。电光晶体在外加电场E的作用下“变形”为：

$$\left(\frac{1}{n^2}\right)_1 x^2+\left(\frac{1}{n^2}\right)_2 y^2+\left(\frac{1}{n^2}\right)_3 z^2+2\left(\frac{1}{n^2}\right)_4 yz+2\left(\frac{1}{n^2}\right)_5 zx+2\left(\frac{1}{n^2}\right)_6 xy=1$$

系数$\left(\frac{1}{n^2}\right)_i$，$i=1,2,3,\ldots,6$线性依赖于外加电场：

$$\Delta\left(\frac{1}{n^2}\right)_i=\sum_{j=1}^{3}\gamma_{ij}E_j$$

γ_{ij}：电光系数张量

$$\begin{vmatrix} \left(\frac{1}{n^2}\right)_1 & \left(\frac{1}{n^2}\right)_6 & \left(\frac{1}{n^2}\right)_5 \\ \left(\frac{1}{n^2}\right)_6 & \left(\frac{1}{n^2}\right)_2 & \left(\frac{1}{n^2}\right)_4 \\ \left(\frac{1}{n^2}\right)_5 & \left(\frac{1}{n^2}\right)_4 & \left(\frac{1}{n^2}\right)_3 \end{vmatrix}$$

$$\begin{vmatrix} \Delta\left(\frac{1}{n^2}\right)_1 \\ \Delta\left(\frac{1}{n^2}\right)_2 \\ \Delta\left(\frac{1}{n^2}\right)_3 \\ \Delta\left(\frac{1}{n^2}\right)_4 \\ \Delta\left(\frac{1}{n^2}\right)_5 \\ \Delta\left(\frac{1}{n^2}\right)_6 \end{vmatrix} = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \\ r_{41} & r_{42} & r_{43} \\ r_{51} & r_{52} & r_{53} \\ r_{61} & r_{62} & r_{63} \end{vmatrix} \begin{vmatrix} E_1 \\ E_2 \\ E_3 \end{vmatrix}$$

一般地，折射率椭球的主轴方向与坐标轴（x,y,z）不重合。为获得椭球的主轴方向和主轴折射率可以通过二次型主轴变换求上面矩阵的本征值获得。

以KDP(KH_2PO_4)晶体举例: 四方晶体$\bar{4}2m$点群,

负单轴晶, $n_x = n_y = n_o$, $n_z = n_e$, $n_o > n_e$

$$\frac{x^2+y^2}{n_o^2}+\frac{z^2}{n_e^2}=1$$

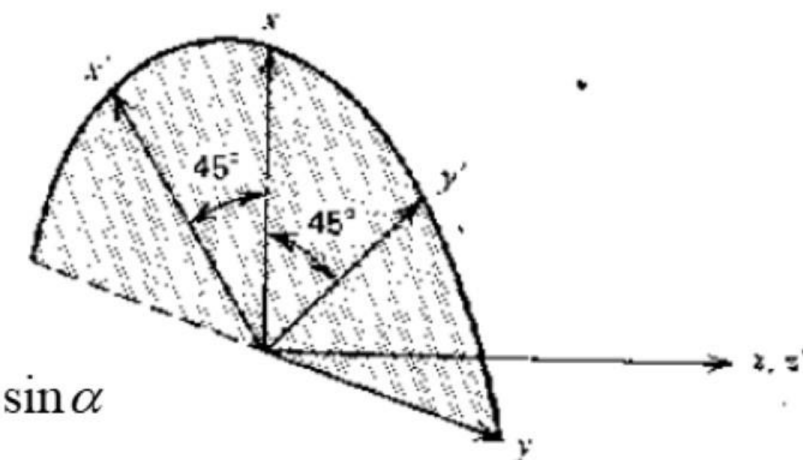
电光张量: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_{41} & 0 & 0 \\ 0 & \gamma_{41} & 0 \\ 0 & 0 & \gamma_{63} \end{pmatrix}$

外加电场$\vec{E}$=(E_x, E_y, E_z)时的折射率椭球:

$$\frac{x^2+y^2}{n_o^2}+\frac{z^2}{n_e^2}+2\gamma_{41}E_x yz+2\gamma_{41}E_y zx+2\gamma_{63}E_z xy=1$$

假定外加电场沿z轴方向, $\vec{E}=E_z\,\hat{z}$:

$$\frac{x^2+y^2}{n_o^2}+\frac{z^2}{n_e^2}+2\gamma_{63}E_z xy=1$$



新主轴坐标系(x', y', z')

$$\frac{x'^2}{n_{x'}^2}+\frac{y'^2}{n_{y'}^2}+\frac{z'^2}{n_{z'}^2}=1$$

作如下坐标变换

$$\begin{cases} x = x'\cos\alpha + y'\sin\alpha \\ y = x'\sin\alpha - y'\cos\alpha \end{cases}$$

x, y坐标轴绕z轴旋转α角

$$(\frac{1}{n_o^2}+\gamma_{63}E_z\sin 2\alpha)x'^2+(\frac{1}{n_o^2}-\gamma_{63}E_z\sin 2\alpha)y'^2+\frac{z'^2}{n_e^2}+2\gamma_{63}E_z\cos 2\alpha x'y'=1$$

(x', y', z')是主轴坐标系, 即上式的交叉项为0: $\cos 2\alpha=0\Rightarrow\ \alpha=\pi/4$

$$(\frac{1}{n_o^2}+\gamma_{63}E_z)x'^2+(\frac{1}{n_o^2}-\gamma_{63}E_z)y'^2+\frac{z'^2}{n_e^2}=1$$

$$\begin{cases}\dfrac{1}{n_{x'}^2}=\dfrac{1}{n_o^2}+\gamma_{63}E_z\\ \dfrac{1}{n_{y'}^2}=\dfrac{1}{n_o^2}-\gamma_{63}E_z\\ \dfrac{1}{n_{z'}^2}=\dfrac{1}{n_e^2}\end{cases}\xrightarrow{\text{一般地},\gamma_{63}E_z\ll\frac{1}{n_o^2}}\begin{cases}\Delta n_x\equiv n_{x'}-n_o\approx-\dfrac{1}{2}n_o^3\gamma_{63}E_z\\ \Delta n_y\equiv n_{y'}-n_o\approx\dfrac{1}{2}n_o^3\gamma_{63}E_z\\ \Delta n_z=0\end{cases}$$

对KDP晶体，外加电场的作用

① 折射率椭球从单轴变为双轴

② 折射率椭球的主轴绕z轴旋转π/4角度 （电场加在z轴方向）

③ 折射率的改变$\propto E_z$

符号

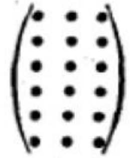
• 零矩阵元　　　等非零矩阵元

● 非零矩阵元　　　等非零矩阵元，但符号相反

每一张最左上角的符号是一般的对称群的名称

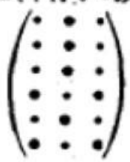
中心对称——所有矩阵元为零

三斜晶系

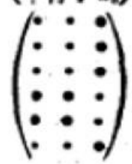


单斜晶系

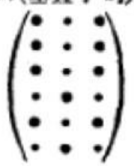
2（平行于 x_2）



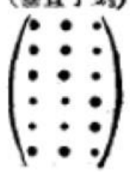
（平行于 x_3）



m（垂直于 x_2）

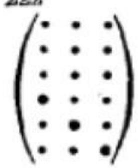


（垂直于 x_3）

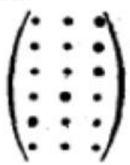


正交斜方

222

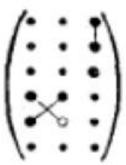


$mm2$



正方晶系

4



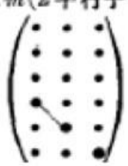
$\bar{4}$

422

$4mm$

例：($BaTiO_3$)

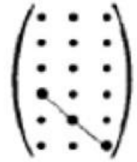
$\bar{4}2m$（2 平行于 x_1）



例：KH_2PO_4（KDP）

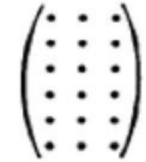
立方晶系

$\bar{4}3m$, 23



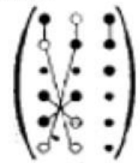
例：（闪锌矿类晶体：GaAs, InAs, CdTe）

432

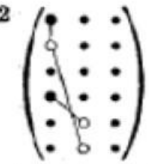


三角晶系

3



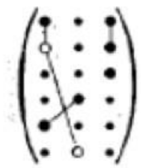
32



例：（Te，石英）

$3m$（垂直于 x_1）标准取向

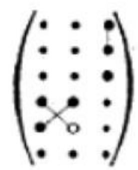
$3m$（垂直于 x_2）



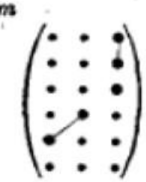
例：($LiNbO_3$, $LiTaO_3$)

六角晶系

6



$6mm$

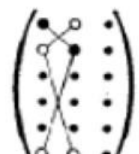


例：(CdS)

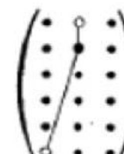
（与 $4mm$ 相同）

622

$\bar{6}$



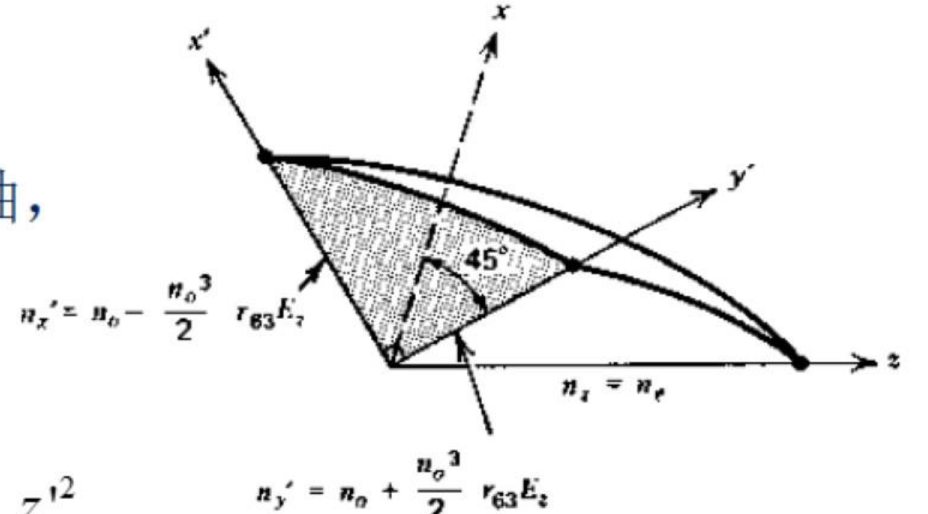
$\bar{6}m2$（m 垂直于 x_1 标准取向）



（垂直于 x_2）

电光位相延迟

对KDP晶体，外加电场E平行于z轴，光场沿z轴传播，偏振方向沿x轴



折射率椭球: $\dfrac{x^2+y^2}{n_o^2}+\dfrac{z^2}{n_e^2}+2\gamma_{63}E_z xy=1$

主轴化: $(\dfrac{1}{n_o^2}+\gamma_{63}E_z)x'^2+(\dfrac{1}{n_o^2}-\gamma_{63}E_z)y'^2+\dfrac{z'^2}{n_e^2}=1$

$$n_{x'}=n_o-\frac{1}{2}n_o^3\gamma_{63}E_z$$

$$n_{y'}=n_o+\frac{1}{2}n_o^3\gamma_{63}E_z$$

入射面(z = 0平面)

$$\begin{cases} E_{x'}=Ae^{i(\omega t-\frac{\omega}{c}n_{x'}z)}=Ae^{i\omega t} \\ E_{y'}=Ae^{i(\omega t-\frac{\omega}{c}n_{y'}z)}=Ae^{i\omega t} \end{cases}$$

出射面($z=l$平面)

$$\begin{cases} E_{x'}=Ae^{i(\omega t-\frac{\omega}{c}n_{x'}l)} \\ E_{y'}=Ae^{i(\omega t-\frac{\omega}{c}n_{y'}l)} \end{cases}$$

x',y'偏振光的位相差:

$$\Gamma=\phi_{x'}-\phi_{y'}=-\frac{\omega}{c}n_{x'}l-(-\frac{\omega}{c}n_{y'}l)=\frac{\omega n_0^3\gamma_{63}}{c}E_z l=\frac{\omega n_0^3\gamma_{63}}{c}V=\pi\frac{V}{V_\pi}$$

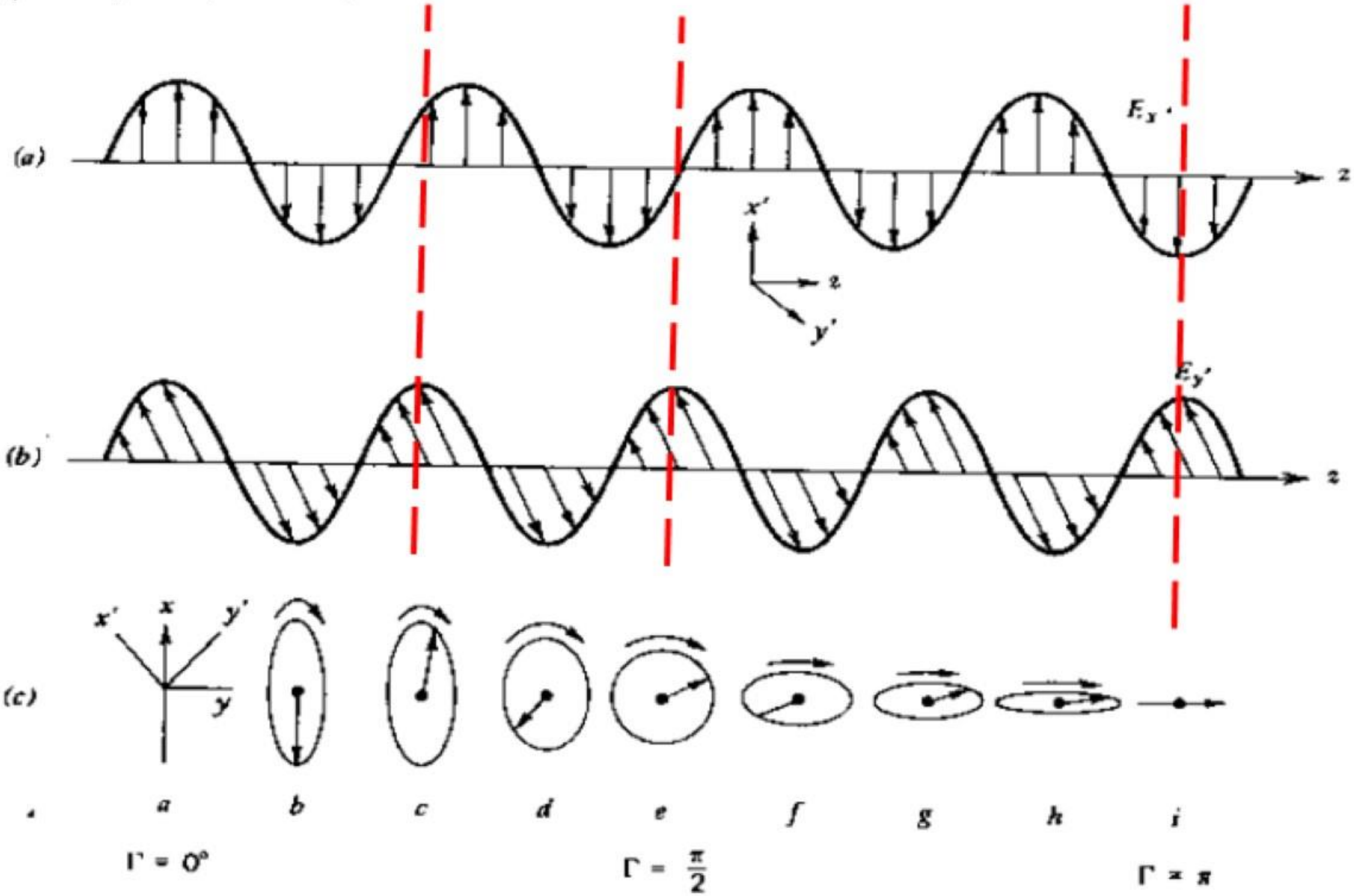
半波电压V_π: $\dfrac{\omega n_0^3\gamma_{63}}{c}V_\pi=\pi \Rightarrow V_\pi=\dfrac{\lambda}{2n_0^3\gamma_{63}}$

椭圆轨迹方程: $\frac{E_{x'}^2}{A_1^2}+\frac{E_{y'}^2}{A_2^2}-2\frac{E_{x'}E_{y'}}{A_1A_2}\cos(\Delta\varphi)=\sin^2(\Delta\varphi)$

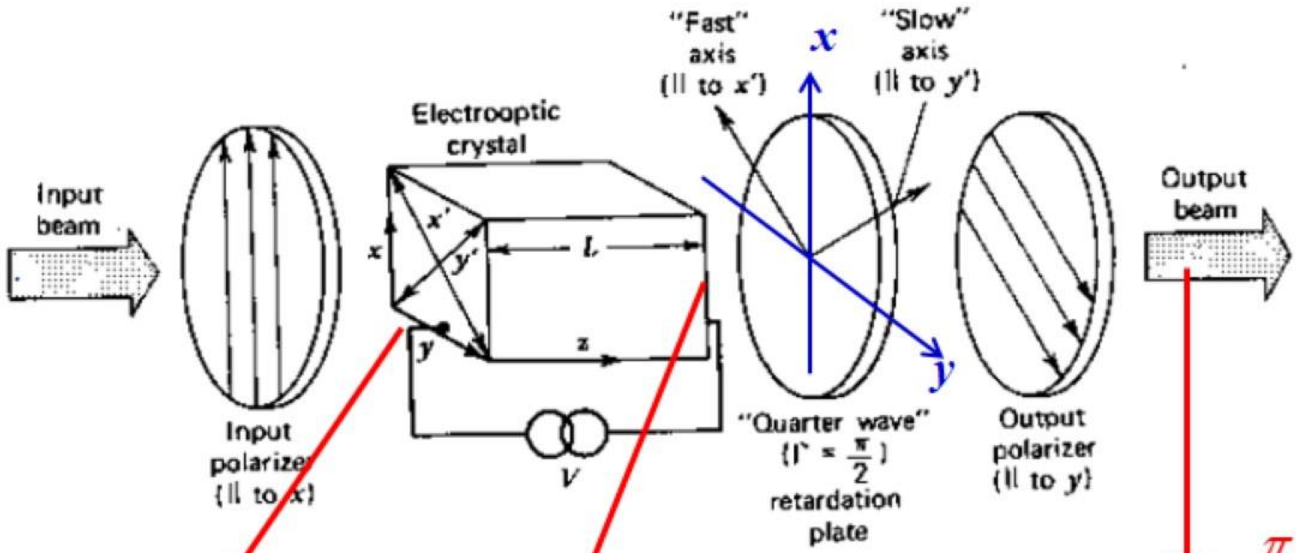
① $\Delta\varphi=n\cdot 2\pi$, 一三象限线偏振光

② $V_{\lambda/2}$, $\Delta\varphi=(2\mathrm{n}+1)\cdot\pi$, 二四象限线偏振光

③ $V_{\lambda/4}$, $\Delta\varphi=(\mathrm{n}+1/2)\cdot\pi$, 椭偏或圆偏振光



电光调幅



恒定相移到线性区！

$$\Gamma = \frac{\pi}{2} + \pi\frac{V}{V_\pi}$$

入射面(z = 0平面)

$$\begin{cases} E_{x'} = Ae^{i\omega t} \\ E_{y'} = Ae^{i\omega t} \end{cases}$$

$$I_i = |E_{x'}|^2 + |E_{y'}|^2 = 2A^2$$

晶体出射面(z = l平面)

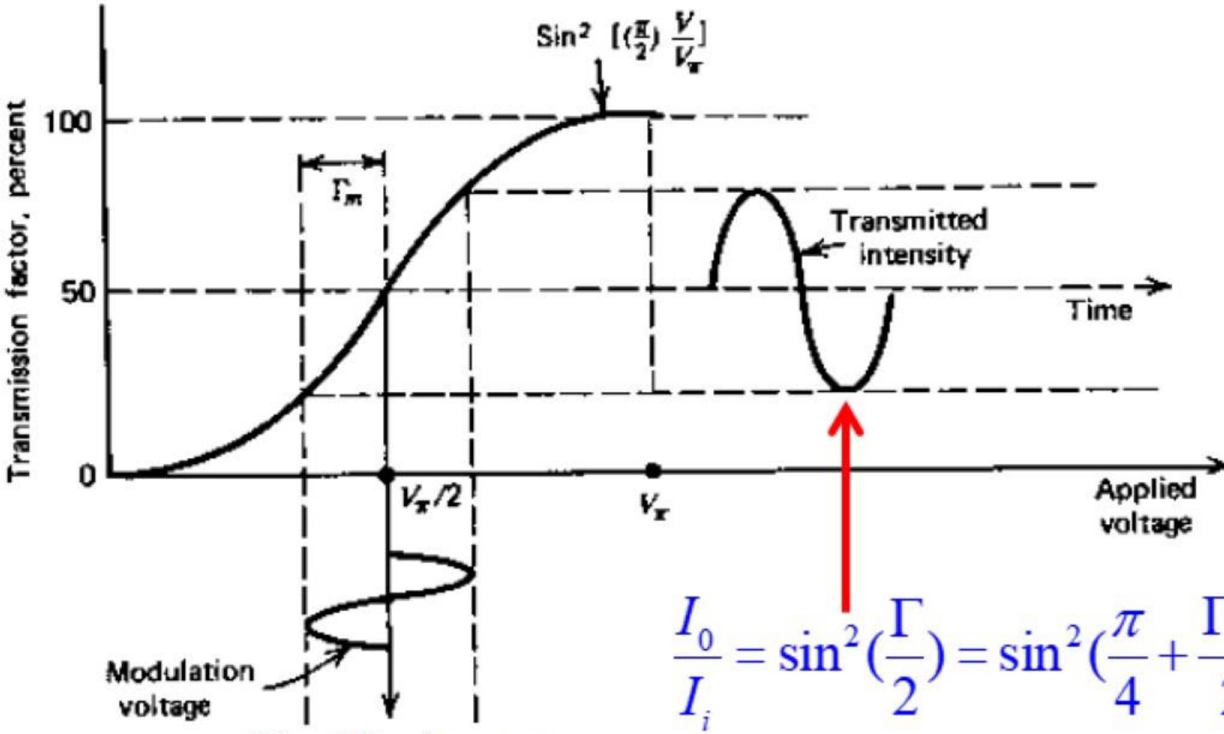
$$\begin{cases} E_{x'} = Ae^{i(\omega t - \frac{\omega}{c} n_{x'} l)} \\ E_{y'} = Ae^{i(\omega t - \frac{\omega}{c} n_{y'} l)} \end{cases}$$

出射光(检偏器后)

$$E_y = \frac{A}{\sqrt{2}}(e^{-i\Gamma} - 1)$$

$$I_o = |E_y|^2 = 2A^2 \sin^2\frac{\Gamma}{2}$$

$$I_o = |E_y|^2 = 2A^2 \sin^2\frac{\Gamma}{2} = 2A^2 \sin^2(\frac{\pi}{4} + \frac{\pi}{2}\frac{V}{V_\pi})$$



$$V = V_m \sin \omega_m t$$

$$\Gamma_m = \pi \frac{V_m}{V_\pi}$$

$$\frac{I_0}{I_i} = \sin^2(\frac{\Gamma}{2}) = \sin^2(\frac{\pi}{4} + \frac{\Gamma_m}{2}\sin \omega_m t)$$

$$= \frac{1}{2}[1 - \cos(\frac{\pi}{2} + \Gamma_m \sin \omega_m t)]$$

$$= \frac{1}{2}[1 + \sin(\Gamma_m \sin \omega_m t)]$$

$$\approx \frac{1}{2}[1 + \Gamma_m \sin \omega_m t] \quad For \quad \Gamma_m << 1$$

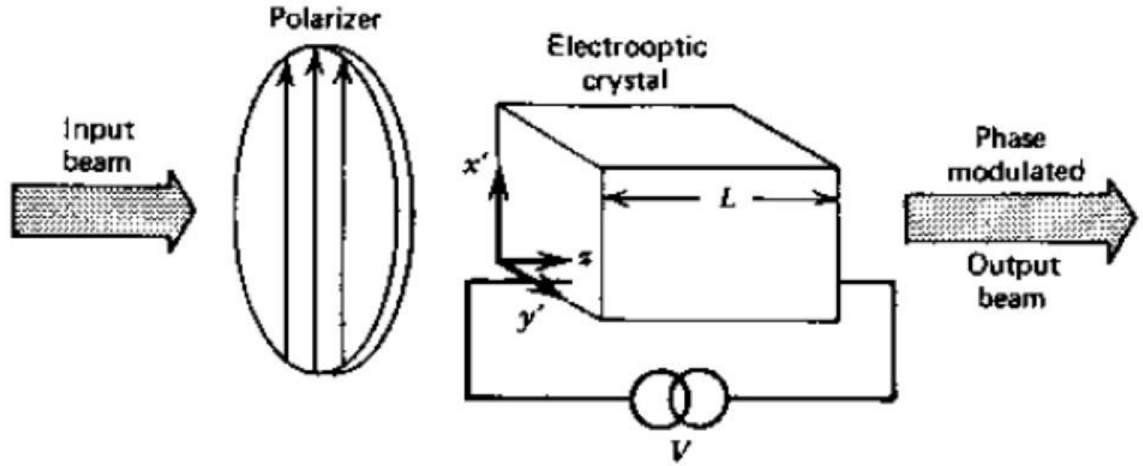
考虑四分之一玻片的作用，$\Gamma = \frac{\pi}{2} + \pi \frac{V}{V_\pi}$

对深调制，$\Gamma_m << 1$不再满足，

$$\sin(\Gamma_m \sin \omega_m t) = \sum_{n=0}^{\infty} 2J_{2n+1}(\Gamma_m) \sin[(2n+1)\omega_m t]$$

出现奇数阶高次谐波！

位相调制



$$\phi_{x'} = -\frac{\omega}{c} l \Delta n_{x'}$$
$$= \frac{\omega n_0^3 r_{63}}{2c} E_z l$$

Carrier wave

Phase–modulated wave

外加电场

$$E_z = E_m \sin\omega_m t$$

晶体前端入射场：$E_{in} = A\cos\omega t$

出射光场：

$$E_{出} = A\cos\left[\omega t - \frac{\omega}{c}\left(n_o - \frac{n_o^3}{2} r_{63} E_m \sin(\omega_m t)\right) l\right]$$
$$= A\cos\left[\omega t - \frac{\omega}{c} n_o l + \underbrace{\frac{\omega}{c}\frac{n_o^3 r_{63}}{2} V_m}_{\delta} \sin\omega_m t\right]$$

略去恒定位相因子

$$E_{出} = A\cos[\omega t + \delta\sin\omega_m t)$$
$$= A[\cos\omega t\cos(\delta\sin\omega_m t) - \sin\omega t\sin(\delta\sin\omega_m t)]$$

$$\cos(\delta\sin\omega_m t) = J_0(\delta) + 2J_2(\delta)\cos 2\omega_m t + \cdots$$
$$\sin(\delta\sin\omega_m t) = 2J_1(\delta)\sin\omega_m t + 2J_3(\delta)\sin 3\omega_m t + \cdots$$

$$E_{出} = AJ_0(\delta)\cos\omega t + A\sum_{n=1}^{\infty} J_n(\delta)[\cos(\omega + n\omega_m)t + (-1)^n\cos(\omega - n\omega_m)t]$$

调相波出现边带！

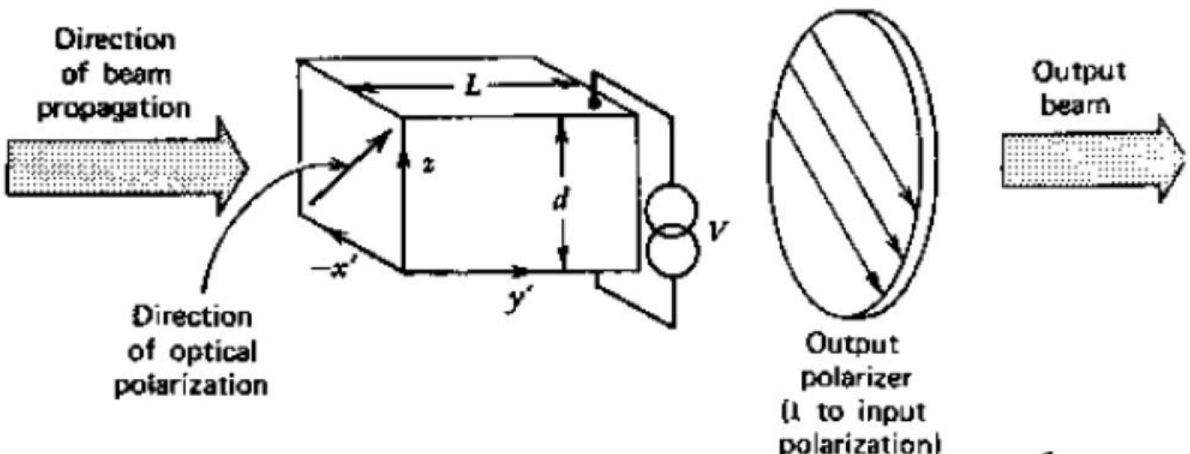
横向调制

纵向电光调制的问题:

加电场方向与光传播方向相同；电光位相延迟与所加电压V成正比，与晶体长度无关；电极面与通光面重叠，损耗大。

横向调制:

调制电场方向垂直于光传播方向；电光位相延迟与晶体长度成正比。



$$n_{x'} = n_o - \frac{1}{2} n_o^3 \gamma_{63} E_z$$

$$n_{y'} = n_o + \frac{1}{2} n_o^3 \gamma_{63} E_z$$

$$n_{z'} = n_e$$

$$\Gamma = \phi_{z'} - \phi_{x'} = -\frac{\omega}{c} n_{z'} l - (-\frac{\omega}{c} n_{x'} l) = [(n_o - n_e) - \frac{1}{2} n_0^3 \gamma_{63} E_z] \frac{\omega l}{c}$$

$$= (n_o - n_e)\frac{\omega l}{c} - \frac{1}{2} n_0^3 \gamma_{63} \frac{\omega}{c} (\frac{l}{d}) V$$

晶体的纵横比

自然双折射，可能随环境温度发生漂移

自然双折射的不利影响：

① 温度漂移；

② 对光束发散角的限制

KDP晶体: $\dfrac{n_e - n_o}{\Delta T} \approx 1.1\times 10^{-5}$ /度

假设晶体长度 $L=30mm$, 波长 $\lambda=632.8nm$, $\Delta T=1$度，

附加相移: $\Delta\varphi = \dfrac{2\pi}{\lambda}\Delta n \cdot L = 1.1\pi\ rad$

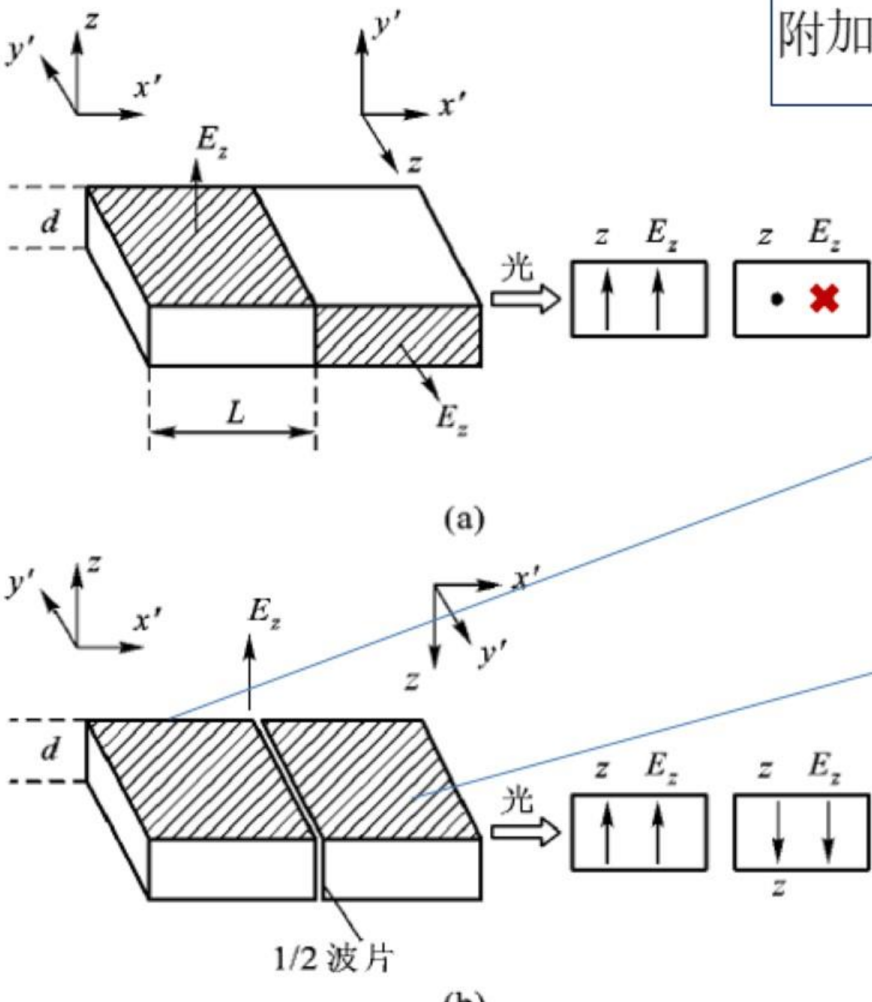


图 5.12　γ_{63}横向电光效应的两种补偿方法

"组合调制器"补偿温漂

$$\Gamma_1 = \phi_z - \phi_{y'} = -\frac{\omega}{c} n_z l - (-\frac{\omega}{c} n_{y'} l)$$

$$= [(n_o - n_e) + \frac{1}{2} n_0^3 \gamma_{63} E_z]\frac{\omega l}{c}$$

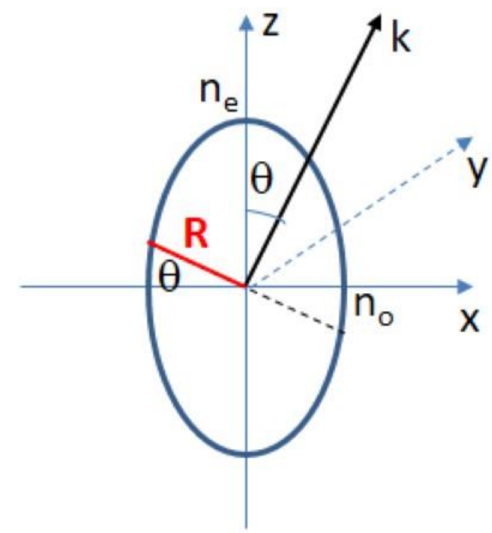
$$\Gamma_2 = \phi_{y'} - \phi_z = [(n_e - n_o) + \frac{1}{2} n_0^3 \gamma_{63} E_z]\frac{\omega l}{c}$$

$$\Gamma = \Gamma_1 + \Gamma_2$$

$$= n_0^3 \gamma_{63} E_z \frac{\omega l}{c} = \frac{2\pi}{\lambda} n_0^3 \gamma_{63} (\frac{l}{d}) \mathrm{V}$$

半波电压: $(V_{\lambda/2})_{横} = (\dfrac{d}{l})(V_{\lambda/2})_{纵}$

自然双折射对光束发散角的限制



光线与z轴成θ角传播,

折射率椭球: $$\frac{x^2}{n_o^2}+\frac{z^2}{n_e^2}=\frac{R^2\cos^2\theta}{n_o^2}+\frac{R^2\sin^2\theta}{n_e^2}=1$$

$$\Rightarrow R=\frac{n_o n_e}{\sqrt{n_e^2+(n_o^2-n_e^2)\sin^2\theta}}=\frac{n_o}{\sqrt{1+(\frac{n_o^2}{n_e^2}-1)\sin^2\theta}}$$

$$=\frac{n_o}{\sqrt{1+(\frac{n_o^2}{n_e^2}-1)\theta^2}}\approx n_o[1-(\frac{n_o^2}{n_e^2}-1)\frac{\theta^2}{2}]=n_e(\theta)$$

$$\therefore\ n_o-n_e(\theta)=\frac{n_o\theta^2}{2}(\frac{n_o^2}{n_e^2}-1)$$

> 举例: KDP晶体,
> n_o=1.51, n_e=1.47, $\lambda=1\mu m$, $l=1cm$
> 则, θ<$1.71\times10^{-2}\,rad\sim1^\circ$

自然双折射相位差对入射角的依赖关系:

$$\Delta\varphi=\frac{\omega l}{c}[n_o-n_e(\theta)]=\frac{\omega l n_o}{2c}(\frac{n_o^2}{n_e^2}-1)\theta^2$$

假设$\Delta\varphi$的可以接受范围$\Delta\varphi<\pi/4$,

则, $$\theta<[\frac{\lambda}{4n_o l(\frac{n_o^2}{n_e^2}-1)}]^{\frac{1}{2}}$$

立方$\bar{4}3m$晶体的电光效应（如GaAs，CuCl等）

其非零电光张量元为$r_{41}=r_{52}=r_{63}$，无电场时为各向同性晶体。

① 外电场沿z轴方向：$E=E_z$，$E_x=E_y=0$

$$\frac{x^2+y^2+z^2}{n_o^2}+2r_{41}E_z xy=1$$

同KDP晶体的情况。

主轴化变换：z轴不变，x,y转45°。

$$\begin{cases} x=\frac{1}{\sqrt{2}}(x'-y') \\ y=\frac{1}{\sqrt{2}}(x'+y') \end{cases}$$

主轴坐标系中：
(x',y',z')

$$x'^2\left(\frac{1}{n_o^2}+r_{41}E\right)+y'^2\left(\frac{1}{n_o^2}-r_{41}E\right)+\frac{z'^2}{n_o^2}=1$$

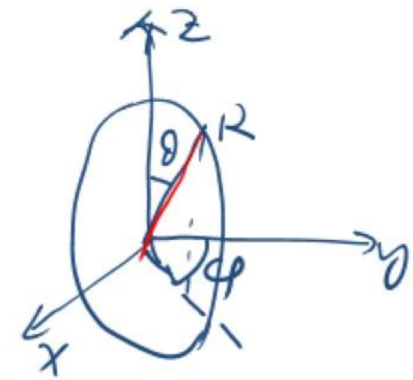
$$\begin{cases} n_{x'}=n_o-\frac{1}{2}n_o^3 r_{41}E \\ n_{y'}=n_o+\frac{1}{2}n_o^3 r_{41}E \\ n_{z'}=n_o \end{cases}$$

② $E_x = E_y = E/\sqrt{2}$, $E_z = 0$

$$\frac{x^2+y^2+z^2}{n_o^2} + \sqrt{2}\, r_{41} E (yz + zx) = 1$$

⇓

$$R^2\left[\frac{1}{n_o^2} + \sqrt{2} r_{41} E(\sin\theta\cos\theta\cos\varphi + \sin\theta\cos\theta\sin\varphi)\right] = 1$$



$$\begin{cases} x = R\sin\theta\sin\varphi \\ y = R\sin\theta\cos\varphi \\ z = R\cos\theta \end{cases}$$

$$\Rightarrow R^2 = 1/\left\{\frac{1}{n_o^2} + \sqrt{2} r_{41} E \sin\theta\cos\theta(\cos\varphi + \sin\varphi)\right\}$$

$$\frac{\partial(R^2)}{\partial\varphi} = -\frac{\sqrt{2} r_{41} E \sin\theta\cos\theta(\cos\varphi - \sin\varphi)}{\{\cdots\}^2} = 0$$

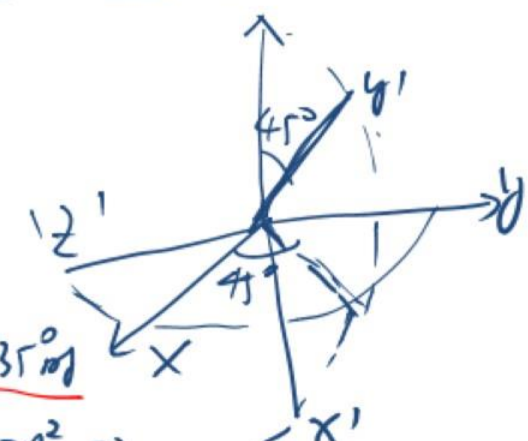
$$\Rightarrow \cos\varphi = \sin\varphi \Rightarrow \varphi = 45° \ (\text{or } 225°)$$

令 $\varphi = 45°$, $\frac{\partial(R^2)}{\partial\theta} = -\frac{\sqrt{2} r_{41} E \cdot 2\cos 2\theta}{\{\cdots\}} = 0$

$$\Rightarrow \cos 2\theta = 0 \Rightarrow \theta = 45° \text{ or } 135°.$$

$\varphi = 45°$, $\theta = 45°$ 时 $R^2 = 1/\{\frac{1}{n_o^2} + r_{41}E\} = n_{y'}^2$

y'轴 $\therefore n_{y'} = n_o - \frac{1}{2} n_o^3 r_{41} E$



x'轴 $\varphi = 45°, \theta = 135°$ 时

$R^2 = \frac{1}{\frac{1}{n_o^2} - r_{41}E} = n_{x'}^2 \Rightarrow n_{x'} = n_o + \frac{1}{2} n_o^3 r_{41} E$

z'轴. $\theta = 90°, \varphi = -45°$.

$\Rightarrow R^2 = n_o^2 \Rightarrow n_z = n_o$

③ $E_x = E_y = E_z = E/\sqrt{3}$

$$\frac{x^2+y^2+z^2}{n_0^2} + \frac{2}{\sqrt{3}} r_{41} E (yz + xz + xy) = 1$$

$$R^2 \left\{ \frac{1}{n_0^2} + \frac{2}{\sqrt{3}} r_{41} E \left(\sin\theta\cos\varphi\cos\theta + \sin\theta\sin\varphi\cos\theta + \sin^2\theta\sin\varphi\cos\varphi \right) \right\} = 1$$

$\sin\theta\cos\theta(\cos\varphi + \sin\varphi) + \sin^2\theta\sin\varphi\cos\varphi$

$$\Rightarrow R^2 = 1/\{\cdots\}$$

$$\frac{\partial(R^2)}{\partial\varphi} = \frac{1}{\{\cdots\}} \cdot \frac{2}{\sqrt{3}} r_{41} E \left(\sin\theta\cos\theta(-\sin\varphi + \cos\varphi) + \sin^2\theta(\cos^2\varphi - \sin^2\varphi) \right) = 0$$

$$\cos\varphi = \sin\varphi \Rightarrow \varphi = 45^\circ \text{ or } 135^\circ$$

$$\varphi = 45^\circ:\ R^2 = 1 / \left\{ \frac{1}{n_0^2} + \frac{2}{\sqrt{3}} r_{41} E \left(\sqrt{2}\sin\theta\cos\theta + \frac{1}{2}\sin^2\theta \right) \right.$$

$$\frac{\partial(R^2)}{\partial\theta} = \frac{1}{\{\cdots\}} \cdot \frac{2}{\sqrt{3}} r_{41} E \left[\sqrt{2}(\cos^2\theta - \sin^2\theta) + \frac{1}{2} \cdot 2\sin\theta\cos\theta \right] = 0$$

$$\Rightarrow \sqrt{2}\cos 2\theta + \frac{1}{2}\sin 2\theta = 0 \Rightarrow \tan 2\theta = -2\sqrt{2}$$

$$\tan 2\theta = -2\sqrt{2} \Rightarrow \frac{\sin^2 2\theta}{\cos^2 2\theta} = 8$$

$$\frac{1-\cos^2 2\theta}{\cos^2 2\theta} = 8 \Rightarrow \cos^2 2\theta = \frac{1}{9}$$

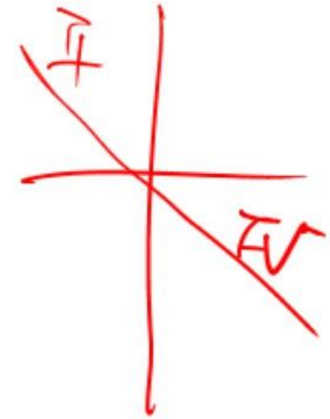
$$\Rightarrow \cos 2\theta = \frac{1}{3} = 2\cos^2\theta - 1$$

$$\Rightarrow \cos^2\theta = \frac{1}{3} \Rightarrow \cos\theta = 1/\sqrt{3}$$

$$2/3 \qquad \cos\theta = \sqrt{\frac{2}{3}}$$

主轴① $\varphi = 45^\circ$, $\cos\theta = \frac{1}{\sqrt{3}}$.

主轴② $\varphi = 45^\circ$ $\cos\theta = \sqrt{\frac{2}{3}}$



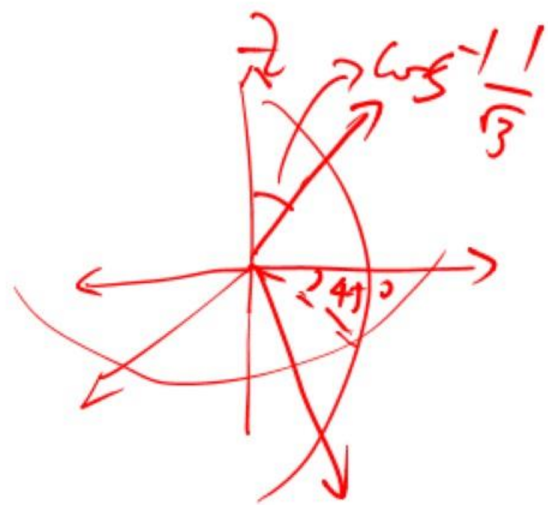
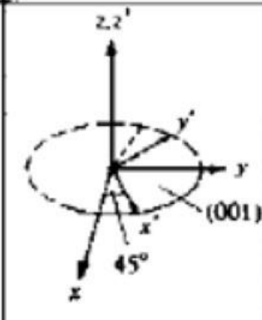
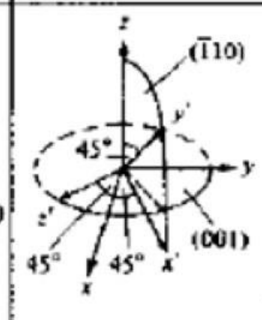
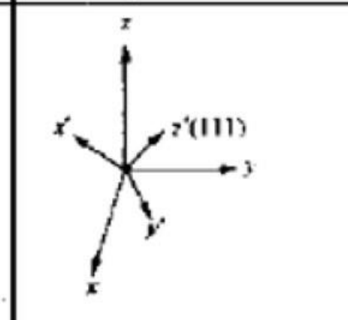
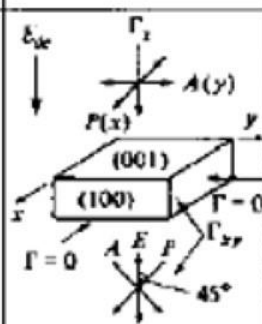
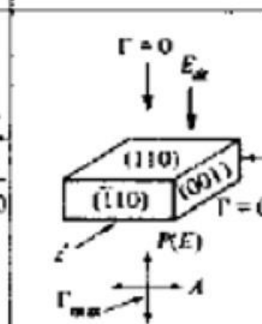
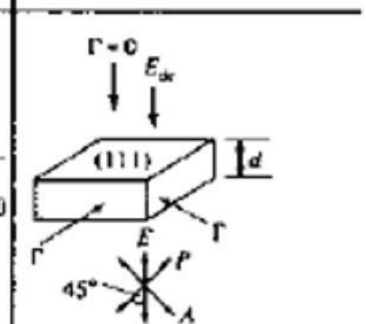


TABLE 14.3. Electrooptical Properties and Retardation in $\bar{4}3m$ (Zinc Blende Structure) Crystals for Three Directions of Applied Field.

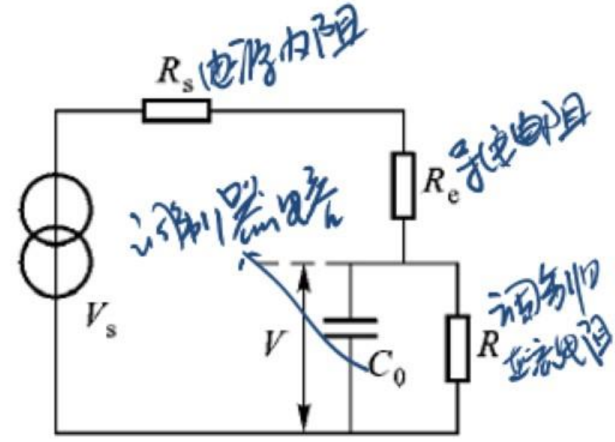
	$E \perp (001)$ plane $E_x = E_y = 0,\ E_z = E$	$E \perp (\bar{1}10)$ plane $E_x = E_y = \frac{E}{\sqrt{2}},\ E_z = 0$	$E \perp (111)$ plane $E_x = E_y = E_z = \frac{E}{\sqrt{3}}$
Index ellipsoid	$\frac{x^2+y^2+z^2}{n_o^2} + 2 r_{41} E xy = 1$	$\frac{x^2+y^2+z^2}{n_o^2} + \sqrt{2}\, r_{41} E (yz + zx) = 1$	$\frac{x^2+y^2+z^2}{n_o^2} + \frac{2}{\sqrt{3}} r_{41} E (yz + zx + xy) = 1$
n_x'	$n_o + \frac{1}{2} n_o^3 r_{41} E$	$n_o + \frac{1}{2} n_o^3 r_{41} E$	$n_o + \frac{1}{2\sqrt{3}} n_o^3 r_{41} E$
n_y'	$n_o - \frac{1}{2} n_o^3 r_{41} E$	$n_o - \frac{1}{2} n_o^3 r_{41} E$	$n_o + \frac{1}{2\sqrt{3}} n_o^3 r_{41} E$
n_z'	n_o	n_o	$n_o - \frac{1}{\sqrt{3}} n_o^3 r_{41} E$
$x'y'z'$ coordinates			
Directions of optical path and axes of crossed polarizer			
Retardation phase difference $\Gamma(V = Ed)$	$\Gamma_z = \frac{2\pi}{\lambda} n_o^3 r_{41} V$ $\Gamma_{xy} = \frac{\pi}{\lambda} \frac{l}{d} n_o^3 r_{41} V$	$\Gamma_{max} = \frac{2\pi}{\lambda} \frac{l}{d} n_o^3 r_{41} V$	$\Gamma = \frac{\sqrt{3}\pi}{\lambda} \frac{l}{d} n_o^3 r_{41} V$

高频调制

$$V=\frac{V_s\left[\frac{1}{(1/R)+i\omega C_0}\right]}{R_s+R_e+\frac{1}{(1/R)+i\omega C_0}}=\frac{V_sR}{R_s+R_e+R+i\omega C_0(R_sR+R_eR)}$$

低频时, $R\gg R_s+R_e$, $i\omega c$小, $\therefore V\approx V_s$

高频时, $R\approx 0$, $R_s>1/\omega c$, 电压主要降在R_s上.



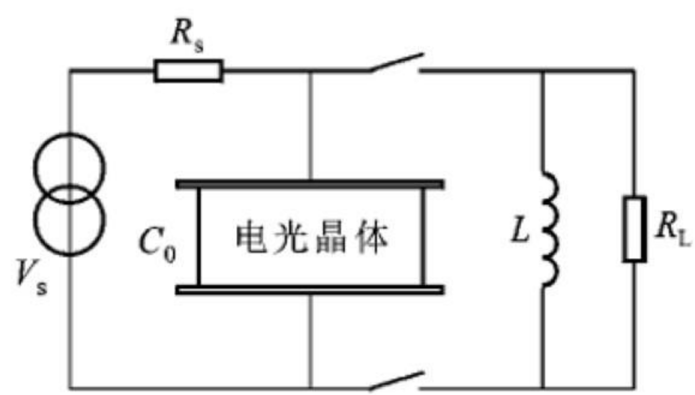
调制器等效电路

*RLC*谐振电路的阻抗,

$$\frac{1}{Z}=\frac{1}{R_L}+\frac{1}{i\omega L}+i\omega c=\frac{1}{R_L}+i(\omega c-\frac{1}{\omega L})$$

$$\Rightarrow Z=\frac{1}{1/R_L+i(\omega c-1/\omega L)}$$

$$=-i\frac{\omega/c}{(\omega^2-1/Lc)-\mathrm{i}\,\omega/R_Lc}$$



谐振电路

谐振频率 : $\omega_0=\frac{1}{\sqrt{LC}}$, 带宽 $\Delta\omega=\frac{1}{R_Lc}$, 谐振阻抗 $Z=R_L$

带宽有限

调制带宽决定调制驱动功率

峰值相移(对KDP晶体):

$$\Gamma_m = \frac{\omega n_0^3 \gamma_{63}}{c} V_m \quad \Rightarrow \quad V_m = \frac{\Gamma_m c}{\omega n_0^3 \gamma_{63}}$$

带宽 $\Delta\omega = \dfrac{1}{R_L c}$

$$\Rightarrow R_L = \frac{1}{\Delta\omega c},$$

$$c = \varepsilon \frac{A}{l}$$

驱动功率:

$$P = V_m^2 \Big/ 2R_L = \frac{\Gamma_m^2 c^2}{(\omega n_0^3 \gamma_{63})^2} \Bigg/ \frac{2}{\Delta\omega \dfrac{\varepsilon A}{l}}$$

$$= \frac{\Gamma_m^2 c^2 \Delta\omega \varepsilon A}{2(\omega n_0^3 \gamma_{63})^2 l} = \frac{\Gamma_m^2 \cdot \Delta\nu \cdot \varepsilon A \lambda^2}{4\pi n_0^6 \gamma_{63}^2 l}$$

渡越时间的影响（高频调制下，晶体中的电场是时变的）

渡越时间：$\tau_d = nl/c$

位相延迟：$\Gamma = \alpha El$，$\alpha = \frac{\omega}{c} n_0^3 \gamma_{63}$

高速调制时，

$$\Gamma(t) = \alpha \int_0^l E(t')\,dz = \frac{\alpha c}{n} \int_{t-\tau_d}^{t} E(t')\,dt'$$

假设：$E(t') = E_m e^{i\omega_m t'}$

$$\Gamma(t) = \frac{\alpha c}{n} E_m \int_{t-\tau_d}^{t} e^{i\omega_m t'}\,dt'$$

$$= \Gamma_0 \left(\frac{1-e^{-i\omega_m \tau_d}}{i\omega_m \tau_d}\right) e^{-i\omega_m t}$$

$$r = \frac{1-e^{-i\omega_m \tau_d}}{i\omega_m \tau_d}$$

缩减因子

$$\Gamma_0 = \frac{\alpha c}{n} \tau_d E_m = \frac{\alpha c}{n} \frac{nl}{c} E_m = \alpha l E_m$$

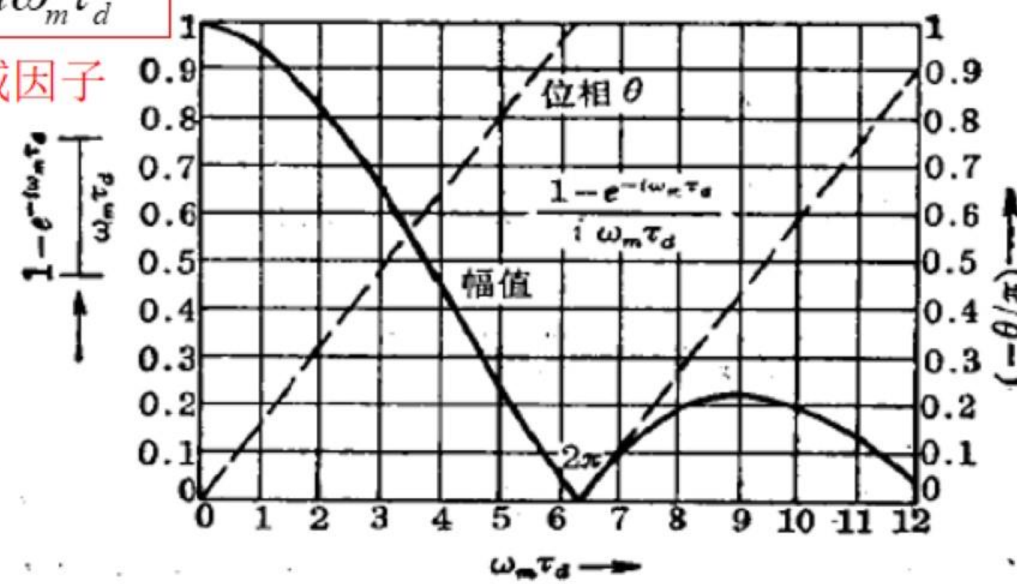
峰值相位延迟

当 $\omega_m \tau_d \ll 1$ 即 $\tau_d \ll \frac{T_m}{2\pi}$ 时，$r \approx 1$

最高调制频率判据：

$\omega_m \tau_d = \frac{\pi}{2}$ 时 $|r| = 0.9$

$$2\pi \nu_m \frac{nl}{c} = \frac{\pi}{2} \Rightarrow \nu_m = \frac{c}{4nl}$$



行波调制

思想： 调制场沿光场的传播方向传播，且相速度彼此相等。

光波在t时刻从$z=0$面进入晶体，t'时刻波面位置 $z(t')=\frac{c}{n}(t'-t)$

相位延迟：$\Gamma(t)=\frac{\alpha c}{n}\int_{t}^{t+\tau_d}E[t',z(t')]\,dt'$

调制场：$E(t',z)=E_m e^{i(\omega_m t'-k_m z)}=E_m e^{i[\omega_m t'-k_m\frac{c}{m}(t'-t)]}$

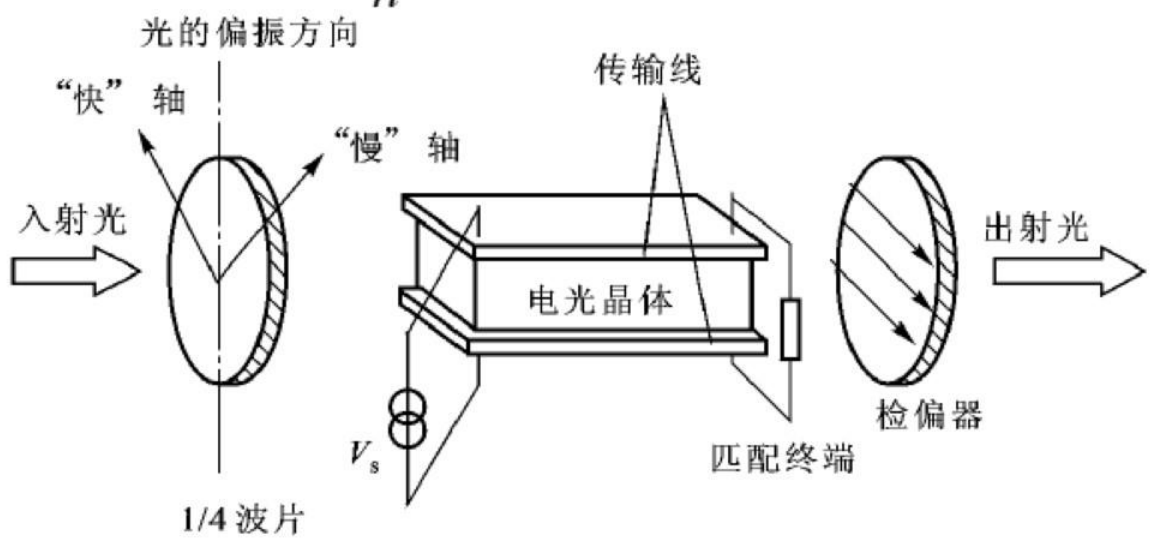
$$\Gamma(t)=\frac{\alpha c}{n}\int_{t}^{t+\tau_d}E[t',z(t')]\,dt'=\frac{\alpha c}{n}\int_{t}^{t+\tau_d}E_m e^{i[\omega_m t'-k_m\frac{c}{m}(t'-t)]}\,dt'=\Gamma_0\left(\frac{e^{-i\omega_m\tau_d(1-\frac{c}{nc_m})}-1}{i\omega_m\tau_d(1-\frac{c}{nc_m})}\right)e^{-i\omega_m t}$$

$$\Gamma_0=\frac{\alpha c}{n}\tau_d E_m=\alpha l E_m$$

$$r=\frac{e^{-i\omega_m\tau_d(1-\frac{c}{nc_m})}-1}{i\omega_m\tau_d(1-\frac{c}{nc_m})}$$

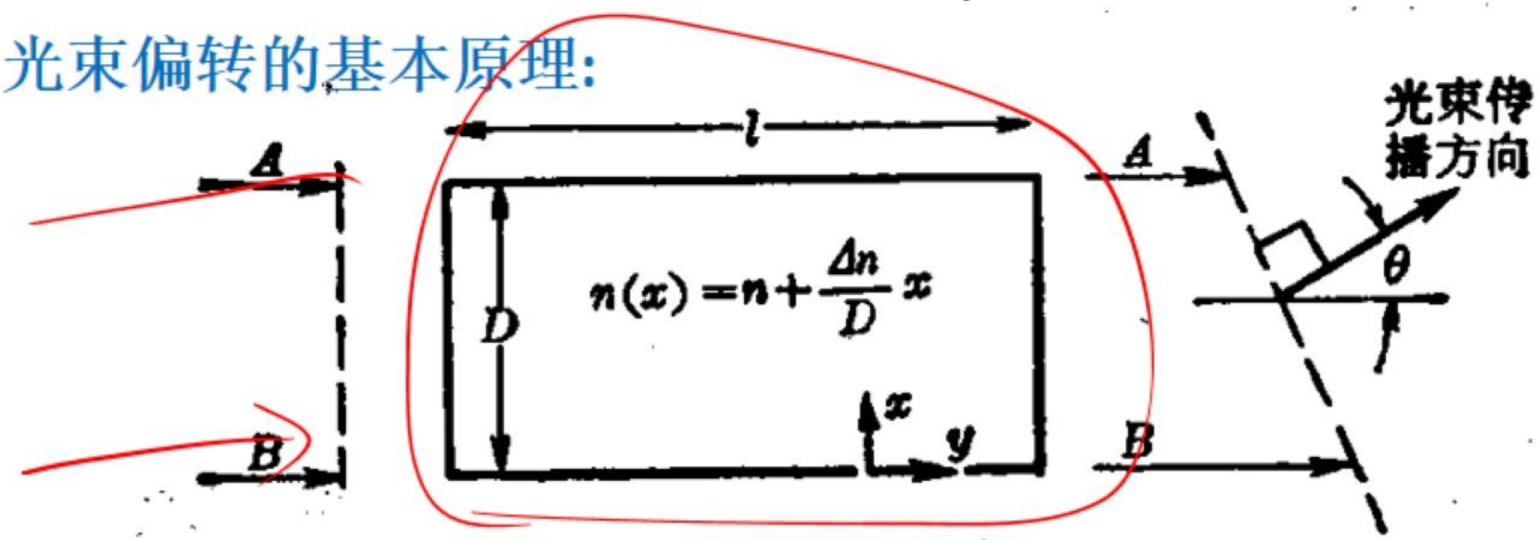
$$if\ \ c_m=\frac{c}{n},\ \ r=1$$

$c_m=\frac{\omega_m}{k_m}$ 行波速度



光束偏转

光束偏转的基本原理:



晶体折射率沿x方向线性变化: $n(\mathrm{x}) = \mathrm{n} + \frac{\Delta n}{D}x$

$T_A = \frac{l}{c}(\mathrm{n} + \Delta n) \quad T_B = \frac{l}{c}\mathrm{n}$, 光线A, B的光程差: $\Delta y = \frac{c}{n}(T_A - T_B) = l\frac{\Delta n}{n}$

偏转角: $\theta' = -\frac{\Delta y}{D} = -\frac{l\frac{\Delta n}{n}}{D} = -\frac{l}{n}\frac{\Delta n}{D} = -\frac{l}{n}\frac{dn}{dx}$

端面折射: $\frac{\sin\theta}{\sin\theta'} \approx \frac{\theta}{\theta'} = n \quad \therefore\ \theta = n\theta' = n(-\frac{l}{n}\frac{dn}{dx}) = -l\frac{dn}{dx} = -l\frac{\Delta n}{D}$

利用KDP晶体棱镜实现光束偏转:

双棱镜KDP光束偏转器

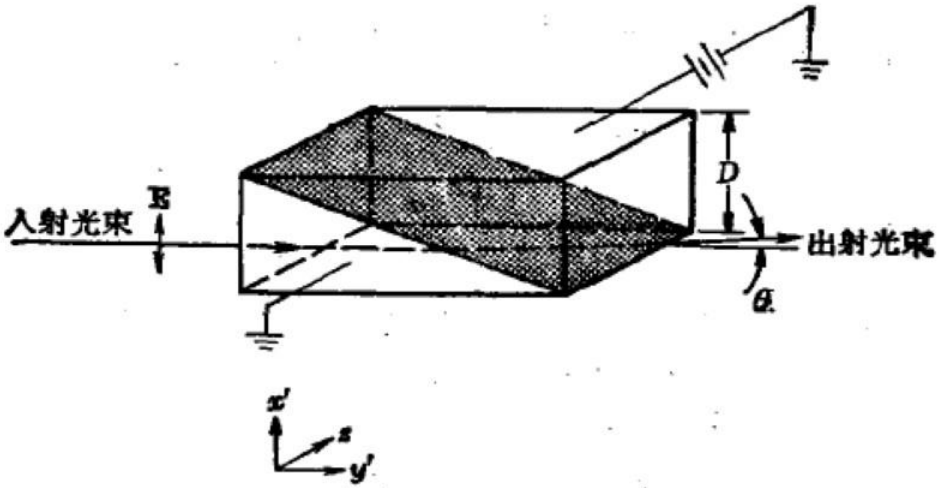


图 14.14 双棱镜 KDP 光束偏转器。上面的和下面的棱镜其 z 轴方向相反。外加偏转电场平行于 z 轴

$$n_A = n_o - \frac{1}{2} n_o^3 \gamma_{63} E_z$$

$$n_B = n_o + \frac{1}{2} n_o^3 \gamma_{63} E_z$$

$$\theta = -\frac{l}{D}(n_A - n_B) = \frac{l}{D} n_o^3 \gamma_{63} E_z$$

m对棱镜级联:

$$\theta_{tot} = m\theta = m\frac{l}{D} n_o^3 \gamma_{63} E_z$$

举例:

$l = D = h = 1cm,\ \gamma_{63} = 10.5\times10^{-12} m/V,$

$n_0 = 1.51,\ V=1000V$

$\theta = 35\times10^{-7} rad$

光束偏转角与高斯光束远场发散角比较:

远场发散角: $\theta_{束} \equiv \dfrac{\lambda}{n_0 \pi w_0}$

取 $D \ge 2w_0$

$$\frac{\theta}{\theta_{束}} = \frac{\pi n_0^4 r_{63}}{2\lambda} E_z l = \frac{n_0}{4}\cdot\frac{2\pi}{\lambda} n_0^3 r_{63} E_z l = \frac{n_0}{4}\Delta\Gamma$$

$\Delta\Gamma = \pi$ 时, $\dfrac{\theta}{\theta_{束}} \approx 1$

光弹效应

声振动⇒介质弹性形变(密度起伏)⇒折射率改变⇒光场响应

应变场

$$S_{kl}(\mathbf{r})=\frac{1}{2}\left[\frac{\partial u_k(\mathbf{r})}{\partial x_l}+\frac{\partial u_l(\mathbf{r})}{\partial x_k}\right]$$

$$\Delta\left(\frac{1}{n^2}\right)_{id}=p_{idkl}S_{kl}$$

→光弹系数张量

$$\Delta\varepsilon'_{id}=-\varepsilon'_i\varepsilon'_d\,p_{idkl}S_{kl}$$

$$\left(\frac{1}{n^2}\right)_{\substack{ij\\i\neq j}}=-\frac{\varepsilon'_{ij}}{\varepsilon'_i\varepsilon'_j}$$

$$\left(\frac{1}{n^2}\right)_{ii}=\frac{1}{\varepsilon'_i}$$

$$2\omega_e=D_iE_i=\varepsilon_{ij}E_iE_j$$

$$D_i=\varepsilon_{ij}E_j$$

$$E_i=g_{ij}D_j$$

$$g_{ii}=(\varepsilon_{ii})^{-1}$$

$$g_{\substack{ij\\i\neq j}}\simeq-\frac{\varepsilon_{ji}}{\varepsilon_{ii}\varepsilon_{jj}}=-\frac{\varepsilon_{ij}}{\varepsilon_{ii}\varepsilon_{jj}}$$

($\varepsilon_{ij}(i\neq j)\ll\varepsilon_{ii}$)

$$\varepsilon'_{ij}\equiv\varepsilon_{ij}/\varepsilon_0$$

$$2\omega_e\varepsilon_0=\frac{D_x^2}{\varepsilon'_{11}}+\frac{D_y^2}{\varepsilon'_{22}}+\frac{D_z^2}{\varepsilon'_{33}}-2\frac{\varepsilon'_{32}}{\varepsilon'_{33}\varepsilon'_{22}}D_zD_y-2\frac{\varepsilon'_{31}}{\varepsilon'_{33}\varepsilon'_{11}}D_zD_x-2\frac{\varepsilon'_{21}}{\varepsilon'_{22}\varepsilon'_{11}}D_xD_y$$

$$\mathbf{D}=\sqrt{2\omega_e\varepsilon_0}\,\mathbf{r}$$

$$\frac{x^2}{\varepsilon'_{11}}+\frac{y^2}{\varepsilon'_{22}}+\frac{z^2}{\varepsilon'_{33}}-2\frac{\varepsilon'_{32}}{\varepsilon'_{33}\varepsilon'_{22}}zy-2\frac{\varepsilon'_{31}}{\varepsilon'_{33}\varepsilon'_{11}}zx-2\frac{\varepsilon'_{21}}{\varepsilon'_{22}\varepsilon'_{11}}xy=1$$

$$\left(\frac{1}{n^2}\right)_1x^2+\left(\frac{1}{n^2}\right)_2y^2+\left(\frac{1}{n^2}\right)_3z^2+2\left(\frac{1}{n^2}\right)_4yz+2\left(\frac{1}{n^2}\right)_5xz+2\left(\frac{1}{n^2}\right)_6xy=1$$

$$D_i = \varepsilon_{id} E_d = \varepsilon_0 E_i + P_i$$

$$P_i = \varepsilon_{id} E_d - \varepsilon_0 E_i = (\varepsilon_{id} - \varepsilon_0 \delta_{id}) E_d$$

$$\Delta P_i = \Delta \varepsilon_{id} E_d = \varepsilon_0 \Delta \varepsilon'_{id} E_d$$

$$\Delta \varepsilon'_{id} = -\varepsilon'_i \varepsilon'_d p_{idkl} S_{kl}$$

$$\Delta P_i = -\frac{\varepsilon_i \varepsilon_d}{\varepsilon_0} p_{idkl} S_{kl} E_d$$

光场E_d通过应变场S_{kl}产生ΔP_i，从而与E_i光场发生耦合！！

$$\nabla^2 \mathbf{E} = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \frac{\partial^2}{\partial t^2}(\Delta \mathbf{P})$$

*应变的定义 $S_{kl}(\vec{r}) = \frac{1}{2}[\frac{\partial u_k(\vec{r})}{\partial x_l} + \frac{\partial u_l(\vec{r})}{\partial x_k}]$

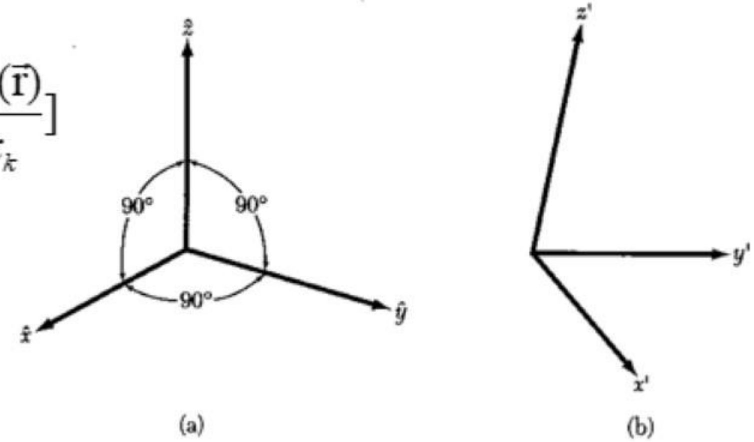


Figure 14 Coordinate axes for the description of the state of strain; the orthogonal unit axes in the unstrained state (a) are deformed in the strained state (b).

$$\mathbf{x}' = (1 + \epsilon_{xx})\hat{\mathbf{x}} + \epsilon_{xy}\hat{\mathbf{y}} + \epsilon_{xz}\hat{\mathbf{z}} \ ;$$
$$\mathbf{y}' = \epsilon_{yx}\hat{\mathbf{x}} + (1 + \epsilon_{yy})\hat{\mathbf{y}} + \epsilon_{yz}\hat{\mathbf{z}} \ ;$$
$$\mathbf{z}' = \epsilon_{zx}\hat{\mathbf{x}} + \epsilon_{zy}\hat{\mathbf{y}} + (1 + \epsilon_{zz})\hat{\mathbf{z}} \ .$$

$$\mathbf{r}' = x\mathbf{x}' + y\mathbf{y}' + z\mathbf{z}'$$
$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$
$$\mathbf{R} \equiv \mathbf{r}' - \mathbf{r} = x(\mathbf{x}' - \hat{\mathbf{x}}) + y(\mathbf{y}' - \hat{\mathbf{y}}) + z(\mathbf{z}' - \hat{\mathbf{z}})$$
$$\mathbf{R}(\mathbf{r}) \equiv (x\epsilon_{xx} + y\epsilon_{yx} + z\epsilon_{zx})\hat{\mathbf{x}} + (x\epsilon_{xy} + y\epsilon_{yy} + z\epsilon_{zy})\hat{\mathbf{y}} + (x\epsilon_{xz} + y\epsilon_{yz} + z\epsilon_{zz})\hat{\mathbf{z}} \ .$$
$$\mathbf{R}(\mathbf{r}) = u(\mathbf{r})\hat{\mathbf{x}} + v(\mathbf{r})\hat{\mathbf{y}} + w(\mathbf{r})\hat{\mathbf{z}} \ .$$

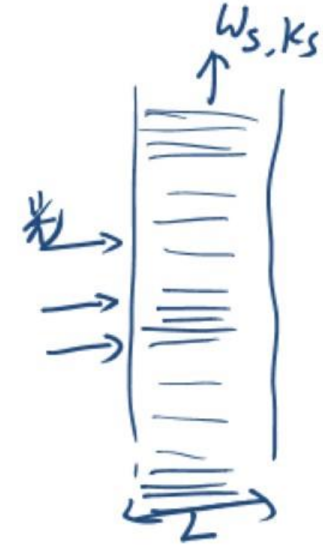
$$x\epsilon_{xx} \cong x\frac{\partial u}{\partial x} \ ; \qquad y\epsilon_{yx} = y\frac{\partial u}{\partial u} \ ; \qquad \text{etc.}$$

$$e_{xx} \equiv \epsilon_{xx} = \frac{\partial u}{\partial x} \ ; \qquad e_{yy} \equiv \epsilon_{yy} = \frac{\partial v}{\partial y} \ ; \qquad e_{zz} \equiv \epsilon_{zz} = \frac{\partial w}{\partial z} \ ,$$

$$e_{xy} \equiv \mathbf{x}' \cdot \mathbf{y}' \cong \epsilon_{yx} + \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \ ;$$
$$e_{yz} \equiv \mathbf{y}' \cdot \mathbf{z}' \cong \epsilon_{zy} + \epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \ ;$$
$$e_{zx} \equiv \mathbf{z}' \cdot \mathbf{x}' \cong \epsilon_{zx} + \epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \ .$$

Reference to "Introduction to Solid State Physics (8th Edition)" by C. Kittel, P73-75

声光调制与偏转



1、拉曼-纳斯（Raman-Nath）衍射

声调制频率低，声光相互作用距离短（薄声光栅）

$$\Delta\varepsilon'_{id} = -\varepsilon'_i \varepsilon'_d p_{idkl} S_{kl}$$

考虑简单的各向同性情形，上式则为 $\Delta\varepsilon' = -\varepsilon'^2 PS$

$$\Delta n = \Delta(\sqrt{\varepsilon'}) = \frac{1}{2}\frac{\Delta\varepsilon'}{\sqrt{\varepsilon'}} = -\frac{1}{2}\frac{\varepsilon'^2}{\sqrt{\varepsilon'}}PS = -\frac{1}{2}n^3 PS$$

声波引起的应变场 $S = S_0 \sin(\omega_s t - k_s x)$

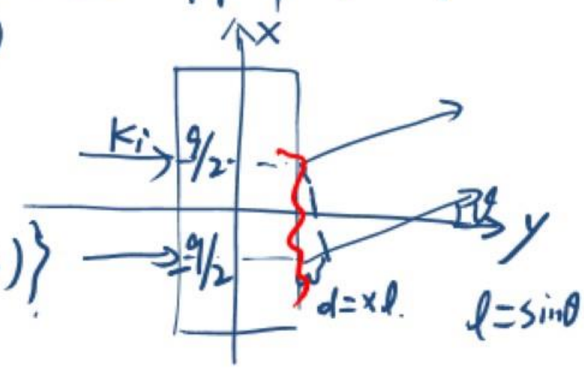
介质的折射率分布：$n_{(x,t)} = n_0 + \left(-\frac{1}{2}n_0^3\right)PS_0 \sin(\omega_s t - k_s x) = n_0 - \Delta n_0 \sin(\omega_s t - k_s x)$

一般，声频远低于光频，即 $\lambda_s \gg \lambda$，故声光作用时可忽略折射率分布的时间项，

即认为声光栅静止：$n_{(x)} = n_0 + \Delta n_0 \sin(k_s x)$

单色平面波入射：$E_{in} = A\exp(i\omega_c t)$

出射光（被调制的波面）：$E_{out} = A\exp\{i\omega_c(t - n_{(x)}L/c)\}$



远场衍射光：$E_p = \int_{-q/2}^{q/2} \exp\{ik_i[lx + L\Delta n \sin(k_s x)]\} dx$

记 $v = \Delta n k_i L = \frac{2\pi}{\lambda}\Delta n L$

$$E_p = \int_{-q/2}^{q/2} \{\cos[k_i l x + v\sin(k_s x)] + i\sin[k_i l x + v\sin(k_s x)]\} dx$$

奇函数，对称积分为0

$$= \int_{-q/2}^{q/2} \cos(k_i l x)\cos[v\sin k_s x] - \sin(k_i l x)\sin[v\sin(k_s x)]$$

又 $\cos[v\sin(k_s x)] = 2\sum_{r=0}^{\infty} J_{2r}(v)\cos(2rk_s x)$

$\sin[v\sin(k_s x)] = 2\sum_{r=0}^{\infty} J_{2r+1}(v)\sin[(2r+1)k_s x]$

$$E_p = q\sum_{r=0}^{\infty} J_{2r}(v)\left\{\frac{\sin(lk_i + 2rk_s)q/2}{(lk_i + 2rk_s)q/2} + \frac{\sin(lk_i - 2rk_s)q/2}{(lk_i - 2rk_s)q/2}\right\}$$

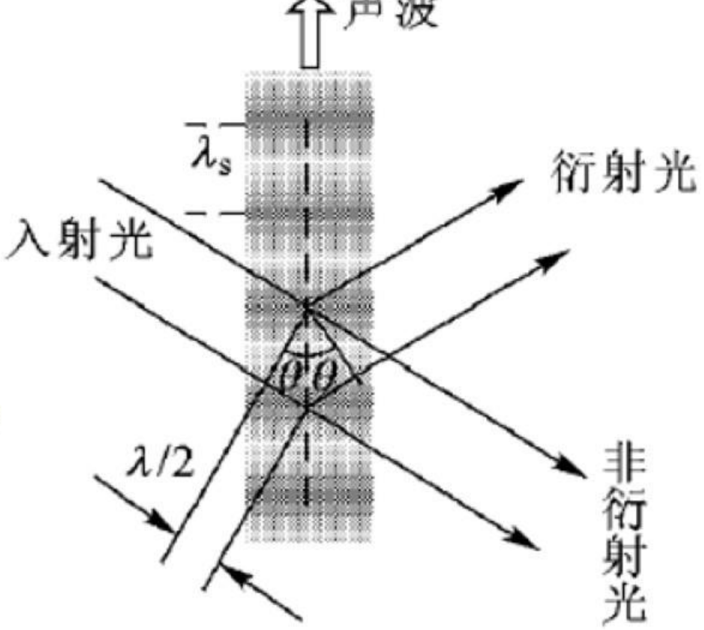
$$+ q\sum_{r=0}^{\infty} J_{2r+1}(v)\left\{\frac{\sin[lk_i + (2r+1)k_s]q/2}{[lk_i + (2r+1)k_s]q/2} - \frac{\sin[lk_i - (2r+1)k_s]q/2}{[lk_i - (2r+1)k_s]q/2}\right\}$$

衍射极大方向：$lk_i \pm mk_s = 0 \Rightarrow \sin\theta = l = \pm m\frac{k_s}{k_i} = \pm m\frac{\lambda}{\lambda_s}$

2、布拉格衍射

- 声波频率较高
- 声光相互作用长度较大
- 光束与声波波面之间形成一定的角度

故声波在介质中形成“体光栅”



布拉格衍射的耦合波理论求解

$$\nabla^2\mathbf{E}=\mu\varepsilon\frac{\partial^2 E}{\partial t^2}+\mu\frac{\partial^2}{\partial t^2}(\Delta\mathbf{P})$$

$$\Delta P_i=-\frac{\varepsilon_i\varepsilon_d}{\varepsilon_0}p_{idkl}S_{kl}E_d$$

$$E_i(\mathbf{r},t)=\frac{1}{2}E_i(r_i)e^{i(\omega_i t-\mathbf{k}_i\cdot\mathbf{r})}+\text{c.c.}$$ ：入射光，沿 $\vec{k}_i$ 方向传播，偏振为 $\vec{E}_i$

$$E_d(\mathbf{r},t)=\frac{1}{2}E_d(r_d)e^{i(\omega_d t-\mathbf{k}_d\cdot\mathbf{r})}+\text{c.c.}$$ ：衍射光，沿 $\vec{k}_d$ 方向传播，偏振为 $\vec{E}_d$

$$\nabla^2 E_i(\mathbf{r},t)\simeq-\frac{1}{2}\left(k_i^2E_i+2ik_i\frac{dE_i}{dr_i}\right)e^{i(\omega_i t-\mathbf{k}_i\cdot\mathbf{r})}+\text{c.c.}$$

慢变近似 $\nabla^2 E_i \ll k_i\frac{dE_i}{dr_i}$

$$\mu\varepsilon\frac{\partial^2}{\partial t^2}E_i=\frac{1}{2}\mu\varepsilon(i\omega_i)^2E_ie^{i(\omega_i t-\vec{k}_i\cdot\vec{r})}+c.c.=-\frac{1}{2}k_i^2E_ie^{i(\omega_i t-\vec{k}_i\cdot\vec{r})}+c.c.$$

抵消!!

$$\Delta P_i(\mathbf{r},t)=-\frac{1}{2}\varepsilon_0\varepsilon_i'\varepsilon_d'p_{idkl}S_{kl}(\mathbf{r},t)\left(E_d(r_d)e^{i(\omega_d t-\mathbf{k}_d\cdot\mathbf{r})}+c.c.\right)$$

$$S_{kl}(\mathbf{r},t)=\frac{S_{kl}}{2}e^{i(\omega_s t-\mathbf{k}_s\cdot\mathbf{r})}+\text{c.c.}$$

$$\Delta P_i(\mathbf{r},t)=-\frac{1}{4}\varepsilon_0\varepsilon_i'\varepsilon_d'p_{idkl}E_d(r_d)\left[e^{i(\omega_d t-\mathbf{k}_d\cdot\mathbf{r})}+\text{c.c.}\right]\times S_{kl}\left[e^{i(\omega_s t-\mathbf{k}_s\cdot\mathbf{r})}+\text{c.c.}\right]$$

$$=-\frac{1}{4}\varepsilon_0\varepsilon_i'\varepsilon_d'p_{idkl}E_d(r_d)S_{kl}\left\{e^{i(\omega_d+\omega_s)t-(\vec{k}_d+\vec{k}_s)\cdot\vec{r}}+e^{i(\omega_d-\omega_s)-(\vec{k}_d-\vec{k}_s)\cdot\vec{r}}+c.c.\right\}$$

为使波动方程中非线性驱动项 ΔP 在时间和空间上不均衡，要求：

$$\omega_i=\omega_d\pm\omega_s$$

$$\mathbf{k}_i\simeq\mathbf{k}_d\pm\mathbf{k}_s$$

先考虑 $\omega_i = \omega_d + \omega_s$ 的情形，

$$\mu \frac{\partial^2}{\partial t^2} \varphi_i = \mu \cdot \frac{1}{4} \varepsilon_0 \varepsilon_i' \varepsilon_d' P_{idkl} E_{d(r_d)} \cdot (\omega_d + \omega_s)^2 S_{kl} \cdot e^{i[(\omega_s+\omega_d)t - (\vec{k}_s + \vec{k}_d)\cdot\vec{r}]} + c.c.$$

($\omega_d + \omega_s \to \omega_i$)

$$\Rightarrow -ik_i \frac{dE_i}{dr_i} = \mu \cdot \frac{1}{4} \varepsilon_0 \varepsilon_i' \varepsilon_d' P_{idkl} E_{d(r_d)} \omega_i^2 e^{-i(\vec{k}_d + \vec{k}_s)\cdot\vec{r}} S_{kl} \cdot e^{i\vec{k}_i\cdot\vec{r}}$$

$$\Rightarrow \frac{dE_i}{dr_i} = \frac{i}{4} \sqrt{\mu \varepsilon_0 \varepsilon_i'}\, \varepsilon_d' \omega_i P_{idkl} S_{kl} E_d e^{i(\vec{k}_i - \vec{k}_s - \vec{k}_d)\cdot\vec{r}}$$

$$= i \eta_{id} E_d e^{i(\vec{k}_i - \vec{k}_s - \vec{k}_d)\cdot\vec{r}}$$

$$\eta_{id} = \frac{1}{4} \sqrt{\mu \varepsilon_0 \varepsilon_i'}\, \varepsilon_d' \omega_i P_{idkl} = \frac{1}{4} \frac{\omega_i}{c} \sqrt{\varepsilon_i'}\, \varepsilon_d' P_{idkl} S_{kl} \approx \frac{1}{4} \frac{2\pi}{\lambda} n^3 P_{idkl} S_{kl}$$

同样对 E_d，∂P_d 可以得到类似的方程！

$$\frac{dE_i}{dr_i} = i\eta_{id} E_d e^{i(\mathbf{k}_i - \mathbf{k}_s - \mathbf{k}_d)\cdot\mathbf{r}}$$

$$\frac{dE_d}{dr_d} = i\eta_{di} E_i e^{-i(\mathbf{k}_i - \mathbf{k}_s - \mathbf{k}_d)\cdot\mathbf{r}}$$

$$\eta_{di} = \eta_{id} \simeq \frac{\pi n^3}{2\lambda} p_{idkl} S_{kl}$$

相位匹配条件：

$$\vec{k}_i = \vec{k}_s + \vec{k}_d$$

不完全匹配时的相干距离

$$l_c = \frac{\pi}{|\vec{k}_i - \vec{k}_d - \vec{k}_s|}$$

假定 $n_i = n_d$

又 $\omega_s << \omega_i$ 故 $k_i \approx k_d = k$

由右边的矢量图形得

$$k_s = 2k\sin\theta$$

$$\Downarrow \quad k_s = 2\pi/\lambda_s, \quad k = 2\pi n/\lambda$$

$$2\lambda_s \sin\theta = \frac{\lambda}{n}$$

or $\sin\theta = \frac{\lambda/n}{2\lambda_s}$

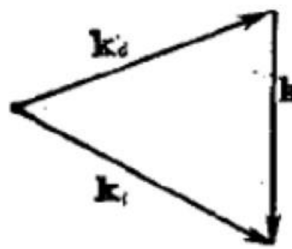
只有一阶布喇格衍射

(晶体衍射有高阶！)

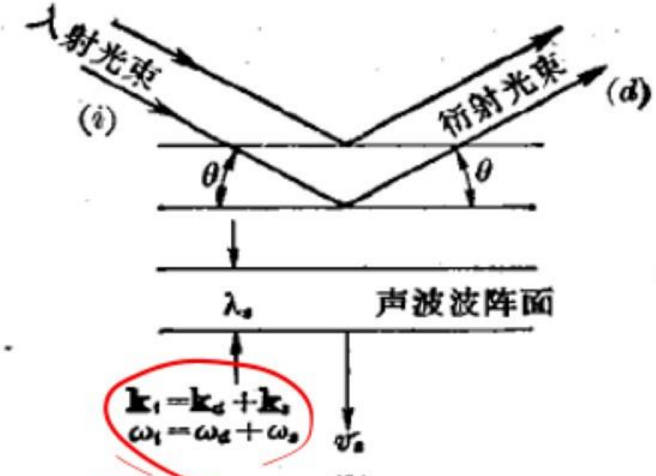
晶体衍射中 $\omega_d = \omega_i$

而声光中 $\omega_d = \omega_i \pm \omega_s$

原因：声波是运动的，晶格是固定的。

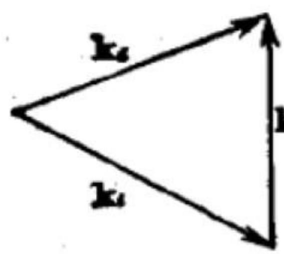


(a)

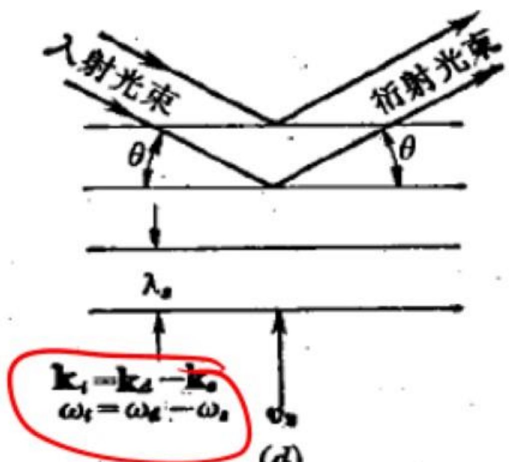


(b)

能量与动量守恒

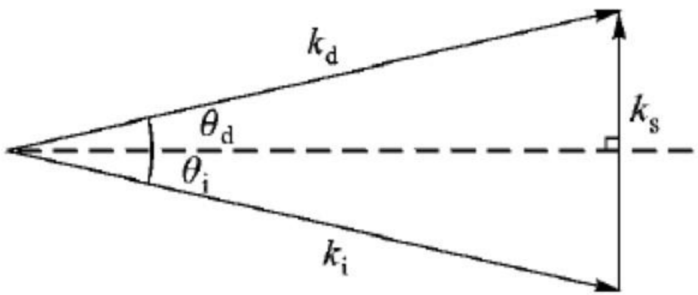


(c)



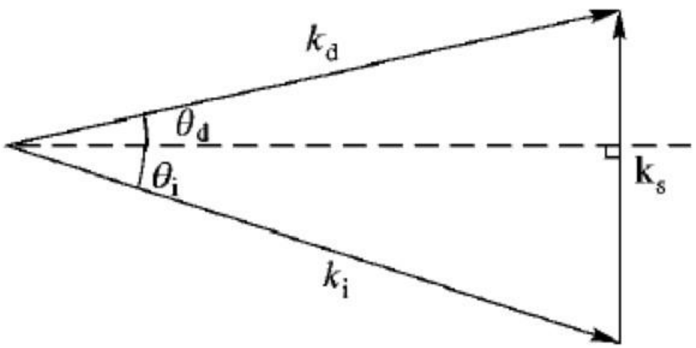
(d)

$$\vec{k}_d = \vec{k}_i + \vec{k}_s$$



正常Bragg衍射波矢图

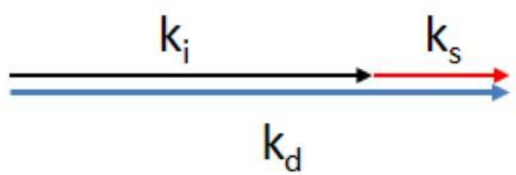
$$n_i = n_d$$



异常Bragg衍射波矢图

$$n_i \neq n_d$$

异常Bragg衍射的共线情形:



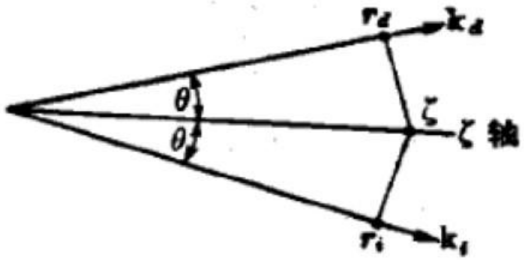
λ随λ_s的改变而改变: 声光滤波器

$$\frac{2\pi}{\lambda}(n_d - n_i) = \frac{2\pi}{\lambda_s} \Rightarrow (n_d - n_i) = \frac{\lambda}{\lambda_s}$$

完全相位匹配下($\vec{k}_i = \vec{k}_s + \vec{k}_d$)的耦合波方程的求解

$$\frac{dE_i}{dr_i} = i\eta E_d$$

$$\frac{dE_d}{dr_d} = i\eta E_i$$



$$r_i = \zeta\cos\theta, \quad r_d = \zeta\cos\theta$$

只有E_i入射，$E_d(0)$=0:

$$E_i(r_i) = E_i(0)\cos(\eta r_i)$$

$$E_d(r_d) = iE_i(0)\sin(\eta r_d)$$

$$\frac{dE_i}{d\zeta} = \frac{dE_i}{dr_i}\cos\theta = i\eta E_d\cos\theta$$

$$\frac{dE_d}{d\zeta} = i\eta E_i\cos\theta$$

能量守恒: $|E_i(r_i)|^2 + |E_d(r_d = r_i)|^2 = |E_i(0)|^2$

$$E_i(\zeta) = E_i(0)\cos(\eta\zeta\cos\theta) + iE_d(0)\sin(\eta\zeta\cos\theta)$$

$$E_d(\zeta) = E_d(0)\cos(\eta\zeta\cos\theta) + iE_i(0)\sin(\eta\zeta\cos\theta)$$

$$E_i(r_i) = E_i(0)\cos(\eta r_i) + iE_d(0)\sin(\eta r_i)$$

$$E_d(r_d) = E_d(0)\cos(\eta r_d) + iE_i(0)\sin(\eta r_d)$$

衍射效率: $\frac{I_{衍射}}{I_{入射}} = \frac{E^2{}_{衍射}}{E_1^2(0)} = \sin^2(\eta l) = \sin^2\left(\frac{\pi n^3}{2\lambda}pSl\right)$

$$S=\sqrt{\frac{2I_{声}}{\rho v_s^3}}$$

式中 v_s 为声波在介质中传播的速度，而 ρ 为质量密度(

把式(14.9–20)和式(14.9–21)合并起来可得

$$\frac{I_{衍射}}{I_{入射}}=\sin^2\left(\frac{\pi l}{\sqrt{2}\lambda}\sqrt{\frac{n^6p^2}{\rho v_s^3}I_{声}}\right)$$

利用下式定义的衍射品质因素

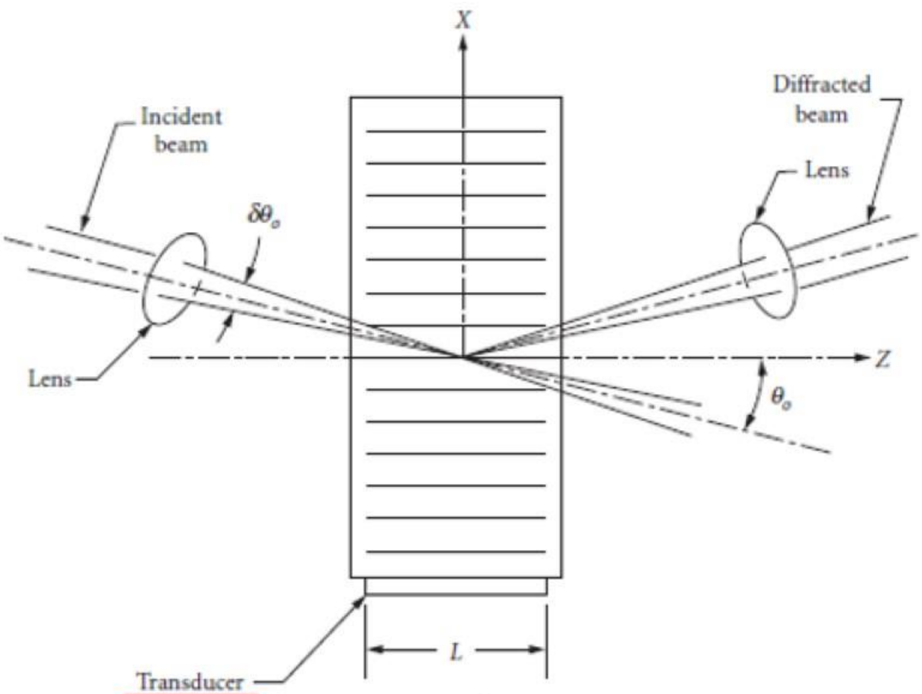
$$M\equiv\frac{n^6p^2}{\rho v_s^3}$$

于是式(14.9–22)变为

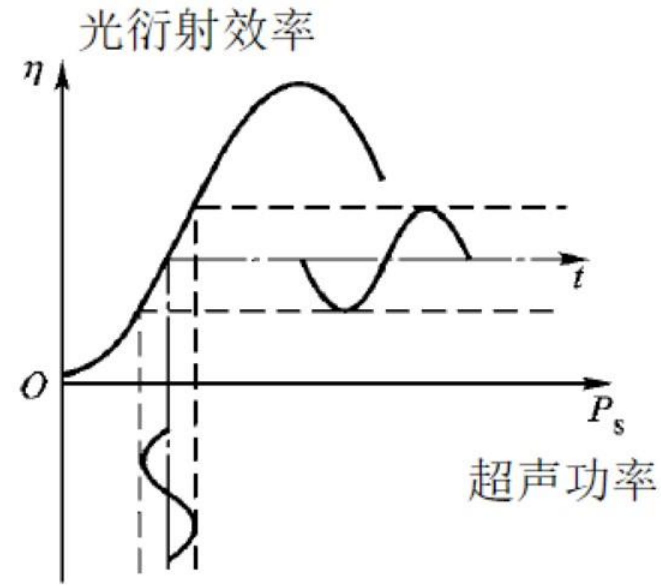
$$\frac{I_{衍射}}{I_{入射}}=\sin^2\left(\frac{\pi l}{\sqrt{2}\lambda}\sqrt{MI_{声}}\right)$$

$\propto I_{声}$ (小信号展开)

声光调制器

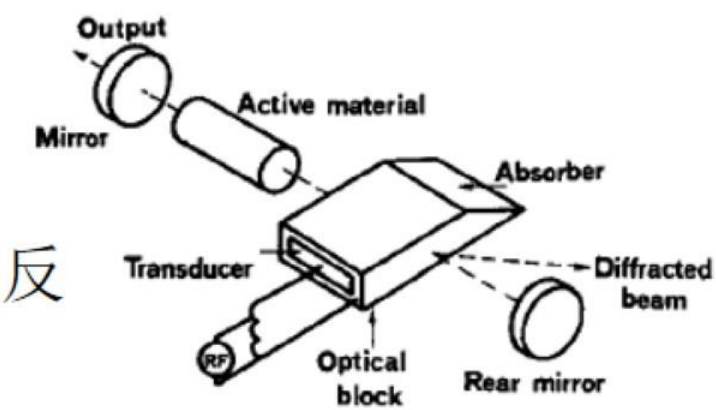


电声换能器

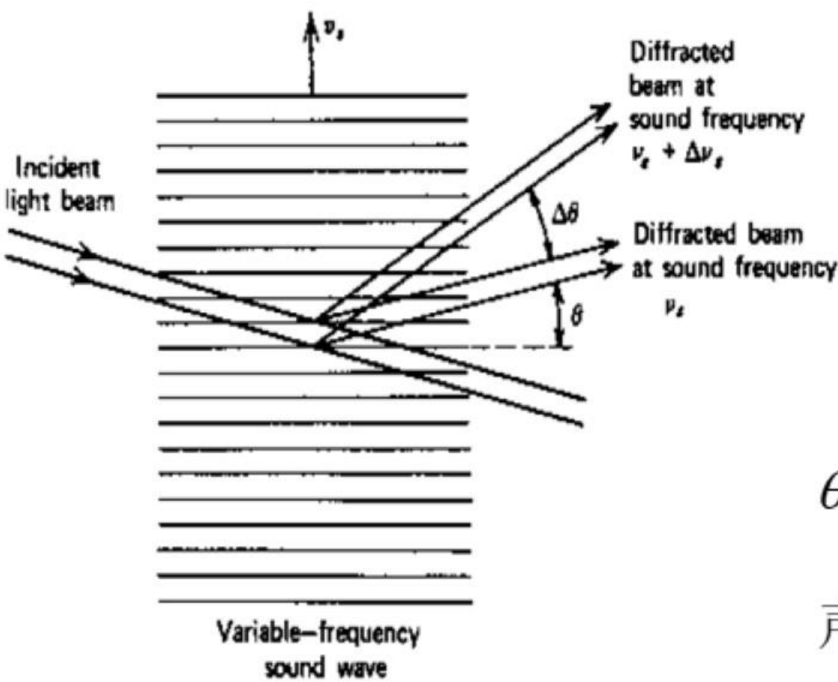


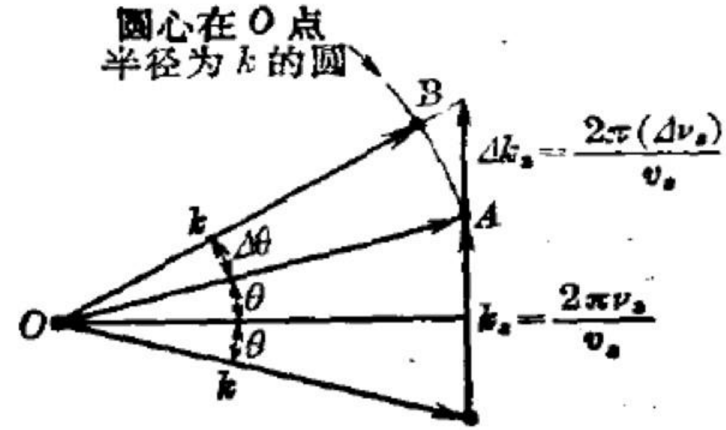
声光调制器的构成：

声光介质、电声换能器、吸声（或反射）装置、驱动电源



声光偏转





$$\theta = \frac{k_s}{2k} = \frac{\lambda}{2n\lambda_s} = \frac{\lambda}{2nV_s}\nu_s$$

声频改变$\Delta\nu_s$

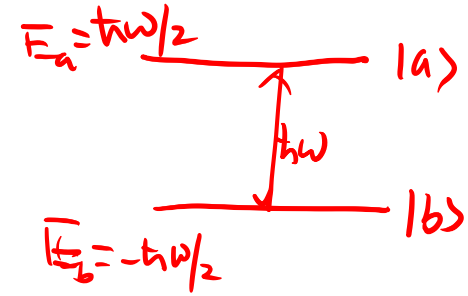
$$\Delta\theta = \frac{\Delta k_s}{k} = \frac{2\pi(\Delta\nu_s)/\mathrm{V}_s}{2\pi n/\lambda} = \frac{\lambda}{nV_s}\Delta\nu_s$$

Ch15 辐射场与原子系统的相干相互作用

问题定义：强场和长驰豫时间情况下，原子对场的响应比碰撞时间短

辐射场与二能级原子相互作用的矢量模型

二能级原子与光场作用（不计碰撞等作用）



$$\mathcal{H}\psi = i\hbar\frac{\partial \psi}{\partial t}, \quad \mathcal{H} = \mathcal{H}_0 + V(t)$$

↙ 孤立原子哈密顿量　　→ 外场作用

波函数在能量本征态上展开：

$$\psi(t) = a(t)u_a + b(t)u_b, \quad \text{其中 } \mathcal{H}_0 u_a = \frac{1}{2}\hbar\omega u_a, \quad \mathcal{H}_0 u_b = -\frac{1}{2}\hbar\omega u_b$$

$a(t), b(t)$ 的实部、虚部共4个参数就可确定体系状态。

考虑到 $\psi(t)$ 有相位任意性，故只需三个参数。

$$\rho_{nm} = \overline{c_n c_m^*}$$

Define: $\vec{r} = (r_1, r_2, r_3)$

$$r_1 = ab^* + ba^*$$
$$r_2 = i(ab^* - ba^*)$$
$$r_3 = aa^* - bb^*$$

$\Longleftrightarrow$

$$\begin{cases} r_1 = 2\mathrm{Re}(\rho_{21}) \\ r_2 = -2\mathrm{Im}(\rho_{21}) \\ r_3 = \rho_{22} - \rho_{11} \end{cases}$$

$$|r|^2 = (|a|^2 + |b|^2)^2 = \int \psi^*\psi \, dv = 1 \quad \text{归一化.}$$

$$i\hbar\frac{\partial\psi}{\partial t}=\mathcal{H}\psi \Rightarrow i\hbar(\dot{a}u_a+\dot{b}u_b)=(\mathcal{H}_0+V)(au_a+bu_b)$$

u_a^* 左乘并积分.
$$\frac{da}{dt}=-\frac{i}{\hbar}\left[a\left(\frac{\hbar\omega}{2}+V_{aa}\right)+bV_{ab}\right]$$

$V_{aa}=\langle a|V|a\rangle$

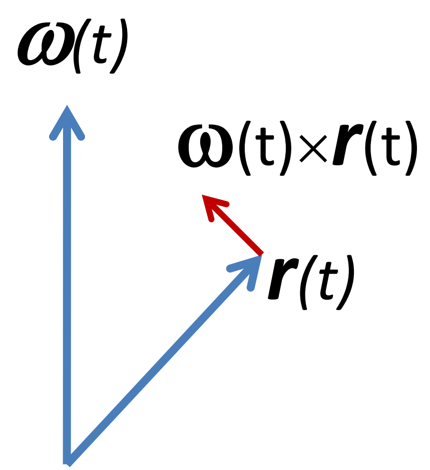
$V_{ab}=\langle a|V|b\rangle$

$V_{bb}=\langle b|V|b\rangle$

u_b^* :
$$\frac{db}{dt}=-\frac{i}{\hbar}\left[b\left(-\frac{\hbar\omega}{2}+V_{bb}\right)+aV_{ba}\right.$$

$$\omega_1\equiv(V_{ab}+V_{ba})/\hbar$$
$$\omega_2\equiv i(V_{ab}-V_{ba})/\hbar$$
$$\omega_3=\omega$$

则.
$$\frac{d\vec{r}}{dt}=\vec{\omega}_{(t)}\times\vec{r}$$



偏振跃迁： $V = -\mu_x E_x - \mu_y E_y$

定义

$$\mu^+ \equiv \mu_x + i\mu_y, \quad E^+ \equiv E_x + iE_y$$

$$\mu^- \equiv \mu_x - i\mu_y, \quad E^- \equiv E_x - iE_y$$

$\mu_x = \frac{1}{2}(\mu^+ + \mu^-)$

$\mu_y = \frac{1}{2i}(\mu^+ - \mu^-)$

$|a\rangle, |m+1\rangle$

$|b\rangle, |m\rangle$

$$V = -\frac{1}{2}(\mu^+ E^- + \mu^- E^+)$$

对于 $\Delta m = \pm 1$ 的跃迁有

$$\langle m+1|\mu^-|m\rangle = 0$$
$$\langle m|\mu^+|m+1\rangle = 0 \qquad (15.1\text{-}14)$$

因此由式(15.1-12)和(15.1-13)得

$$V_{ab} = -\frac{1}{2}\mu_{ab}^+(E_x - iE_y)$$
$$V_{ba} = -\frac{1}{2}\mu_{ba}^-(E_x + iE_y) \qquad (15.1\text{-}15)$$

$= \langle b| -\frac{1}{2}\mu^- E^+ |a\rangle$

$\mu_{ba}^- = \langle b|\mu^-|a\rangle$

$V_{ab} = \langle a|V|b\rangle$
$= \langle a| -\frac{1}{2}(\mu^+E^- + \mu^-E^+)|b\rangle$
$= \langle m+1| -\frac{1}{2}\mu^+E^- |m\rangle$
$= \langle m+1|\mu^+|m\rangle(-\frac{1}{2})E^-$
$= -\frac{1}{2}\mu_{ab}^+(E_x - iE_y)$

$\langle m+1|\mu^+|m\rangle = \langle a|\mu^+|b\rangle = \mu_{ab}^+$

μ_a 和 μ_b 的位相可自由选取。我们选取 μ_a 和 μ_b 的位相使 μ_{ab}^+ 为正实数，指定为 2μ。由于 $\mu_{ab}^+ = (\mu_{ba}^-)^*$，所以有

$$\mu_{ab}^+ = \mu_{ba}^- \equiv 2\mu, \quad \mu = \langle b|\mu_x|a\rangle \qquad (15.1\text{-}16)$$

$$\omega_1(t) = (V_{ab} + V_{ba})/\hbar = -\frac{2\mu E_x(t)}{\hbar}$$

$$\omega_2(t) = i(V_{ab} - V_{ba})/\hbar = -\frac{2\mu E_y(t)}{\hbar}$$

$\omega_3 = \omega$

$$\langle\mu_x\rangle=\frac{1}{2}\langle\mu^{+}+\mu^{-}\rangle$$

$$=\frac{1}{2}\int(a^{*}u_a^{*}+b^{*}u_b^{*})(\mu^{+}+\mu^{-})(au_a+bu_b)dv$$

作替换 $u_a\to|m+1\rangle$, $u_b\to|m\rangle$，由式(15.1–14)可得

$$\langle\mu_x\rangle=\mu r_1 \quad \text{和} \quad \langle\mu_y\rangle=\mu r_2 \qquad (15.1\text{–}18)$$

因而偶极矩算符的平均值(它相应于一个原子系统的偶极辐射)在物理平面 x–y 中的状态与矢量 $\mathbf{r}$ 在假想平面 1–2 中的状态相同。

由上述讨论显然可见，处理二能级原子系统与电磁场偶极相互作用的问题，都归结为求解 $\mathbf{r}(t)$ 的下列矢量方程

$$|\psi\rangle=a|m+1\rangle+b|m\rangle \quad (|a\rangle,\ |b\rangle)$$

$$\frac{d\mathbf{r}}{dt}=\boldsymbol{\omega}(t)\times\mathbf{r} \qquad (15.1\text{–}8)$$

$$\langle\mu_x\rangle=\frac{1}{2}\langle\mu^{+}+\mu^{-}\rangle$$
$$=\frac{1}{2}\langle\psi|\mu^{+}+\mu^{-}|\psi\rangle$$
$$=\frac{1}{2}\left\{(\langle m+1|a^{*}+\langle m|b^{*})|(\mu^{+}+\mu^{-})|(a|m+1\rangle+b|m\rangle)\right\}$$
$$=\frac{1}{2}(a^{*}b\mu^{+}_{ab}+b^{*}a\mu^{-}_{ba})$$
$$=\frac{1}{2}(a^{*}b+ba^{*})\times 2\mu$$
$$=\mu r_1$$

$$\mu^{+}_{ab}=\mu^{-}_{ba}\equiv 2\mu$$

$$\langle\mu_y\rangle=\langle\frac{1}{2i}(\mu^{+}-\mu^{-})\rangle$$
$$=\frac{1}{2i}\langle a^{*}u_a^{*}+b^{*}u_b^{*}|(\mu^{+}-\mu^{-})|au_a+bu_b\rangle$$
$$=\frac{1}{2i}(a^{*}b\mu^{+}_{ab}-b^{*}a\mu^{-}_{ba})$$
$$=\frac{1}{2i}(a^{*}b-b^{*}a)\cdot 2\mu$$
$$=i(ab^{*}-ba^{*})\cdot\mu$$
$$=\mu r_2$$

$r_3 = aa^* - bb^*$: $\langle H_0 \rangle = \langle \psi | H_0 | \psi \rangle$

$= \langle a^* u_a^* + b^* u_b^* | H_0 | a u_a + b u_b \rangle$

$= (aa^* - bb^*) \frac{1}{2}\hbar\omega$

$= r_3 \cdot \frac{1}{2}\hbar\omega$

$\mathcal{H}_0 | u_a \rangle = \frac{1}{2}\hbar\omega | u_a \rangle$

$\mathcal{H}_0 | u_b \rangle = -\frac{1}{2}\hbar | u_b \rangle$

$\mathcal{H}_0$: 非微扰哈密顿量.

小结：偶极相互作用下：

$\frac{d\vec{r}}{dt} = \vec{\omega}(t) \times \vec{r}$:

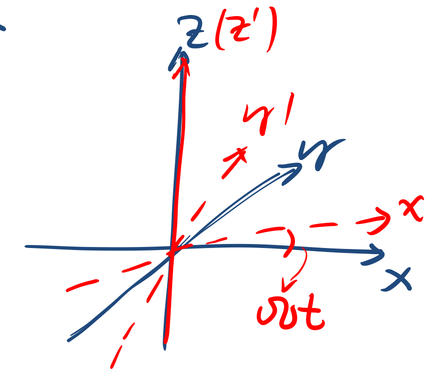
$$\begin{cases} \omega_1 \equiv (V_{ab} + V_{ba})/\hbar = -\frac{2\mu E_x(t)}{\hbar} \\ \omega_2 \equiv i(V_{ab} - V_{ba})/\hbar = -\frac{2\mu E_y(t)}{\hbar} \\ \omega_3 \equiv \omega \end{cases}$$

$\mu = \langle a | \mu_x | b \rangle$

$$\begin{cases} \mu r_1 = \langle \mu_x \rangle \\ \mu r_2 = \langle \mu_y \rangle \\ r_3 \frac{1}{2}\hbar\omega = \langle \mathcal{H}_0 \rangle \end{cases}$$

旋转坐标系变换：

$$\frac{d\vec{r}}{dt}=\vec{\omega}\times\vec{r} \Longrightarrow \frac{d\vec{r}_R}{dt}=\left(\frac{dr}{dt}\right)_R-\vec{\Omega}\times\vec{r}_R$$



坐标 (x', y', z') 相对于 (x, y, z) 作旋转：
绕 z 轴以角速度 Ω 转动。

$$\vec{\Omega}=(0,0,\Omega)$$

变换矩阵 T

$$\vec{A}_R=\begin{pmatrix}A_I\\A_{II}\\A_{III}\end{pmatrix}=\begin{pmatrix}\cos\Omega t & \sin\Omega t & 0\\-\sin\Omega t & \cos\Omega t & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix}=T\vec{A}$$

$$\vec{r}_R=T\vec{r}\Rightarrow\frac{d\vec{r}_R}{dt}=T\frac{d\vec{r}}{dt}+\frac{dT}{dt}\vec{r}$$

记为 $\left(\frac{d\vec{r}}{dt}\right)_R$

$$\frac{dT}{dt}\vec{r}=\begin{pmatrix}-\Omega\sin\Omega t & \Omega\cos\Omega t & 0\\-\Omega\cos\Omega t & -\Omega\sin\Omega t & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}=\begin{pmatrix}-\Omega r_1\sin\Omega t+\Omega r_2\cos\Omega t\\-\Omega r_1\cos\Omega t-\Omega r_2\sin\Omega t\\0\end{pmatrix}$$

$$\vec{\Omega}\times\vec{r}_R = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ r_{I} & r_{II} & r_{III} \end{vmatrix} = \hat{i}(-\Omega r_{II}) + \hat{j}(\Omega r_{I})$$

$$= \hat{i}(\Omega r_1 \sin\Omega t - \Omega r_2 \cos\Omega t) + \hat{j}(\Omega r_1 \cos\Omega t + \Omega r_2 \sin\Omega t)$$

$$= -\frac{d\tau}{dt}\vec{r}$$

故 $$\frac{d\vec{r}_R}{dt} = \left(\frac{dr}{dt}\right)_R - \vec{\Omega}\times\vec{r}_R \quad \#$$

$$= (\vec{\omega}\times\vec{r})_R - \vec{\Omega}\times\vec{r}_R$$

$$= (\vec{\omega}_R - \vec{\Omega})\times\vec{r}_R$$

* 无外场时 $\vec{r}_{(t)}$ 的演化：

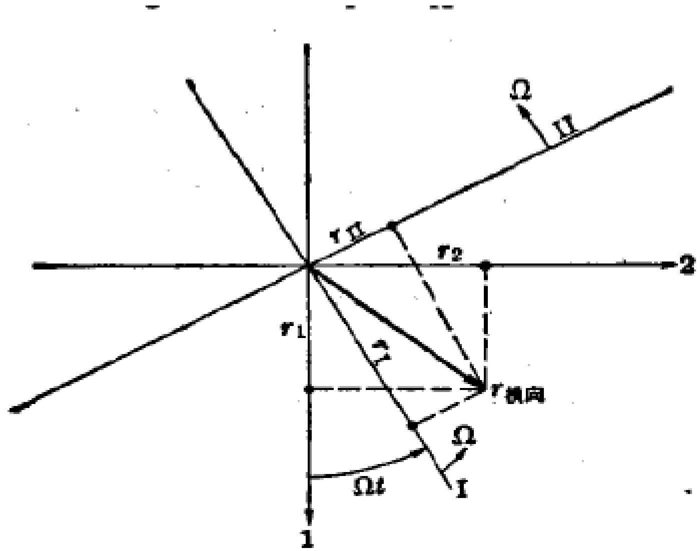
$\omega_1 = \omega_2 = 0, \quad \omega_3 = \omega \Rightarrow \vec{\omega} = \hat{a}_3\omega$

旋转坐标系：$\vec{\Omega} = \hat{a}_{III}\Omega$，且 $\hat{a}_{III} = \hat{a}_3$

$$\frac{d\vec{r}_R}{dt} = \hat{a}_{III}(\omega - \Omega)\times\vec{r}_R \Longrightarrow \vec{r}_R = (r_I, r_{II}, r_{III}) = \text{const}$$

若取 $\Omega = \omega$

$$\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = T^{-1}\vec{r}_R = \begin{pmatrix} r_I\cos\omega t - r_{II}\sin\omega t \\ r_I\sin\omega t + r_{II}\cos\omega t \\ r_{III} \end{pmatrix} \Rightarrow \begin{cases} r_1 = r_I\cos\omega t - r_{II}\sin\omega t \\ r_2 = r_I\sin\omega t + r_{II}\cos\omega t \\ r_3 = \sqrt{1 - r_I^2 - r_{II}^2} \end{cases}$$



r_I, r_{II} 与极化强度 P_x 之间的关系：

令 $R_x = U\cos\omega t - V\sin\omega t$

$$P_x = N\langle\mu_x\rangle = N\mu r_1 = N\mu(r_I\cos\omega t - r_{II}\sin\omega t)$$

$$U = N\mu r_I = (N_a - N_b)\mu r_I$$

$$V = N\mu r_{II} = (N_a - N_b)\mu r_{II}$$

$\vec{r}(t)$ 在外场作用下的演化：

设在 xy 平面中存在圆偏光：$\begin{cases} E_x = E\cos\omega_0 t \\ E_y = E\sin\omega_0 t \end{cases}$

则
$$\omega_1 = -\frac{2\mu}{\hbar}E_x = -\frac{2\mu}{\hbar}E\cos\omega_0 t$$
$$\omega_2 = -\frac{2\mu}{\hbar}E_y = -\frac{2\mu}{\hbar}E\sin\omega_0 t$$
$$\omega_3 = \omega$$

$$|\vec{\omega}| = \sqrt{\left(\frac{2\mu E}{\hbar}\right)^2 + \omega^2}$$

于转动坐标系 $\vec{\Omega} = \omega_0 \hat{a}_{III}$ 中 $\vec{\omega}_R = \left(-\frac{2\mu E}{\hbar}, 0, \omega\right)$

$$\frac{d\vec{r}_R}{dt} = (\vec{\omega}_R - \vec{\Omega}) \times \vec{r}_R = \left[\hat{a}_I\left(-\frac{2\mu E}{\hbar}\right) + \hat{a}_{III}(\omega - \omega_0)\right] \times \vec{r}_R$$
$$= \vec{\omega}_{eff} \times \vec{r}_R$$

$$\vec{\omega}_{eff} = \hat{a}_I\left(-\frac{2\mu E}{\hbar}\right) + \hat{a}_{III}(\omega - \omega_0)$$

$$\omega_e = |\vec{\omega}_{eff}| = \sqrt{\left(\frac{2\mu E}{\hbar}\right)^2 + (\omega_0 - \omega)^2}$$ 拉比频率

when $\omega = \omega_0$

$$\omega_e = \frac{2\mu E}{\hbar}$$ 共振拉比频率

$$r_{\rm I} = \frac{\omega_{\rm I}(\omega - \omega_0)}{\omega_e^2}(1 - \cos \omega_e t)$$

$$r_{\rm II} = -\frac{\omega_{\rm I}}{\omega_e} \sin \omega_e t$$

$$r_{\rm III} = 1 - 2\left(\frac{\omega_{\rm I}}{\omega_e}\right)^2 \sin^2\left(\frac{\omega_e t}{2}\right)$$

$\omega_{\rm I} = -\frac{2\mu E}{\hbar}$ 为负值

① 共振时 ($\omega = \omega_0$)，

$\vec{\omega}_{eff} = (-\frac{\mu E}{\hbar}, 0, 0)$

$\vec{r}_R = (r_{\rm II}\hat{a}_{\rm II} + r_{\rm III}\hat{a}_{\rm III})$；$r_{\rm I} = 0$

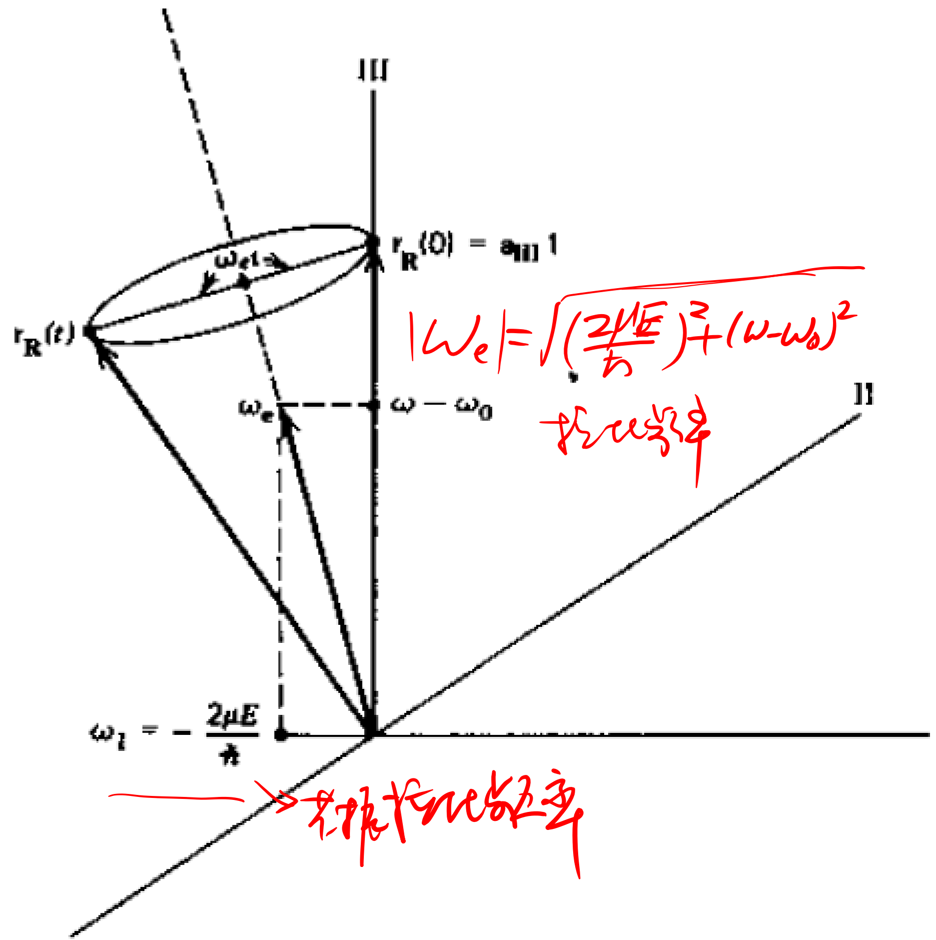
$\vec{r}_R$ 在 II III 平面内旋转

② 非共振时，如右图，

ω_{eff} 在 I III 平面内。

若 $\vec{r}_R(0) = \hat{a}_{\rm III} \cdot 1$ 则 $r_{\rm III}(t)$ 总达不到-1

反之亦然。



$$\frac{d\mathbf{r}_R(\omega)}{dt} = \left[\mathbf{a}_{\rm I}\left(-\frac{2\mu E}{\hbar}\right) + \mathbf{a}_{\rm III}(\omega - \omega_0)\right] \times \mathbf{r}_R(\omega)$$

③ 无耗散时，$\vec{r}_R$ 在旋转中保持长度，即 $|\vec{r}_R|^2 = |r_{\rm I}|^2 + |r_{\rm II}|^2 + |r_{\rm III}|^2 = 1$

$$\begin{cases} \Gamma_{III}=|a|^2-|b|^2 \\ |a|^2+|b|^2=1 \end{cases}$$

$\Downarrow$

$$|a|^2=1-\left(\frac{\omega_I}{\omega_e}\right)^2\sin^2\left(\frac{\omega_e t}{2}\right)$$

$$|b|^2=\left(\frac{\omega_I}{\omega_e}\right)^2\sin^2\left(\frac{\omega_e t}{2}\right)$$

当共振 $(\omega=\omega_0)$ $\omega_e=\omega_I$

$$|a|^2=\cos^2\left(\frac{\omega_I t}{2}\right)$$

$$|b|^2=\sin^2\left(\frac{\omega_I t}{2}\right)$$

Rabbi 振荡

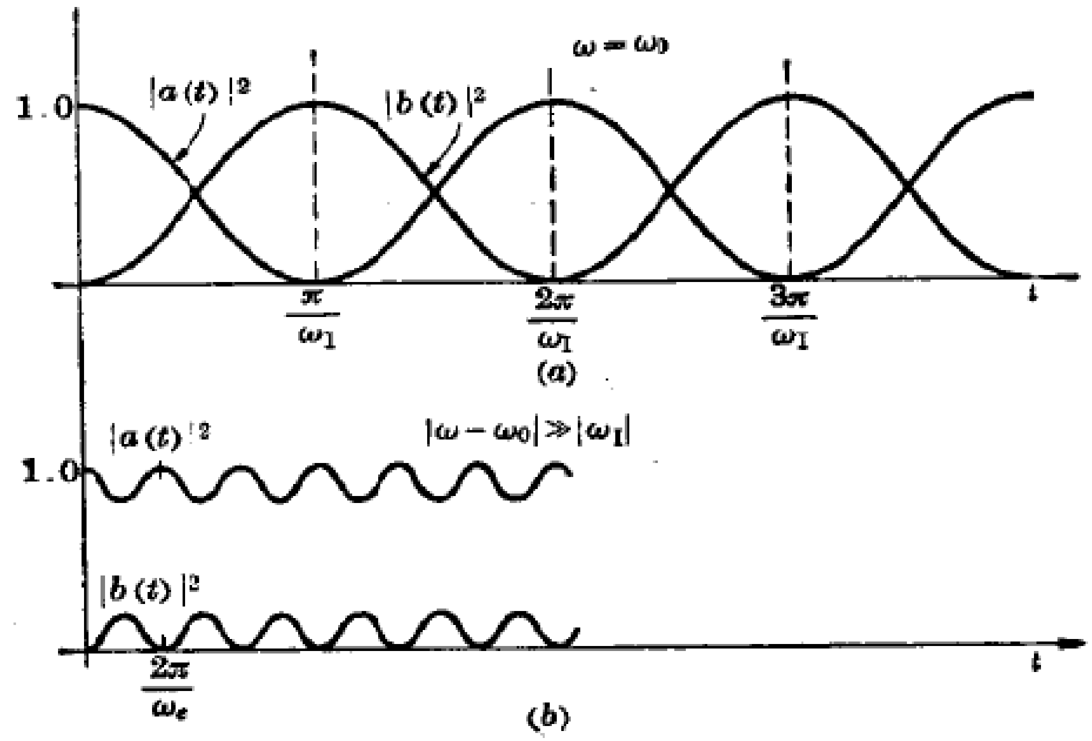


图 15.4 当有外光场时粒子数占据几率 $|a|^2$ 和 $|b|^2$ 的振荡

(a) $\omega=\omega_0$（共振）； (b) $|\omega-\omega_0|\gg|\omega_I|$

超辐射

(自由感应衰减)

初始原子处于基态，即 $\vec{r}_R(0) = -1\hat{a}_{III}$，加 $\frac{\pi}{2}$ 脉冲 ($\omega_I t_0 = \frac{\pi}{2}$) 后撤去外场。

t_0 时刻：$r_I = r_{III} = 0$

$r_{II} = -1$

$\langle \mu_x \rangle = \mu r_1$

$\langle \mu_y \rangle = \mu r_2$

无外场下，

$$\begin{cases} r_1 = r_I \cos\omega t - r_{II}\sin\omega t \\ r_2 = r_I \sin\omega t + r_{II}\cos\omega t \end{cases}$$

$$\mu_x = \mu r_1 = \mu \sin\omega(t - t_0)$$

$$\mu_y = \mu r_2 = -\mu \cos\omega(t - t_0)$$

$$P_x = (N_b - N_a)\mu \sin\omega(t - t_0)$$

$$P_y = -(N_b - N_a)\mu \cos\omega(t - t_0)$$

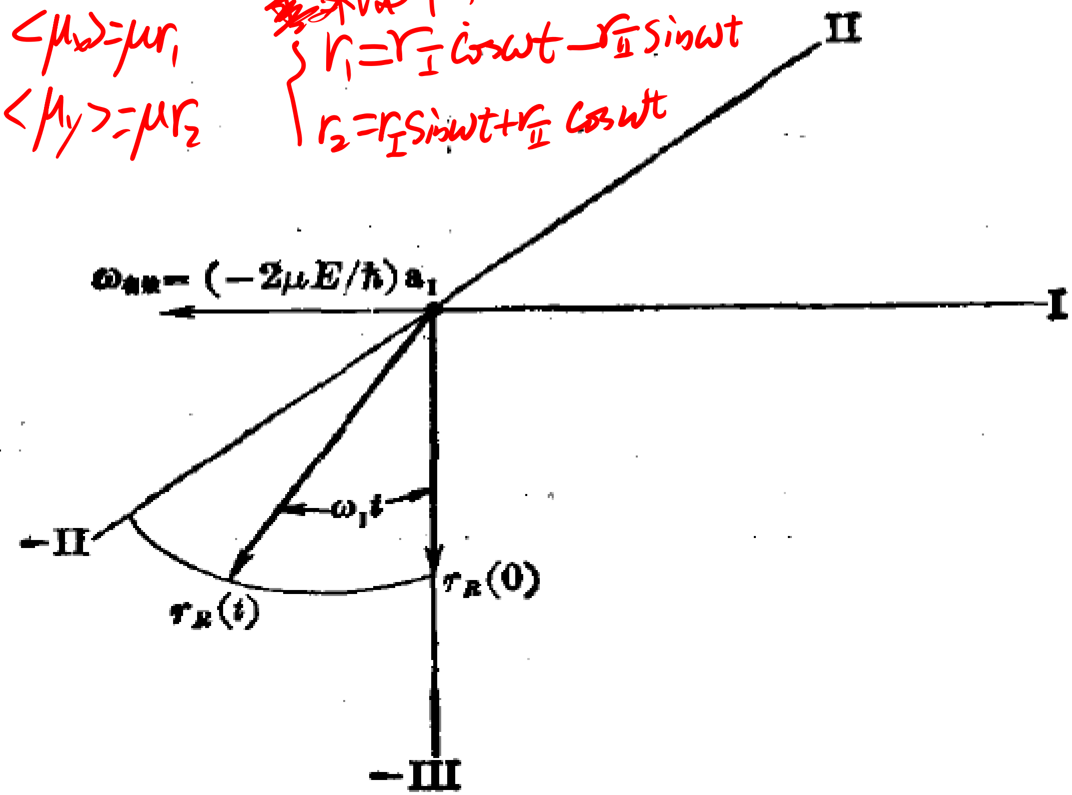
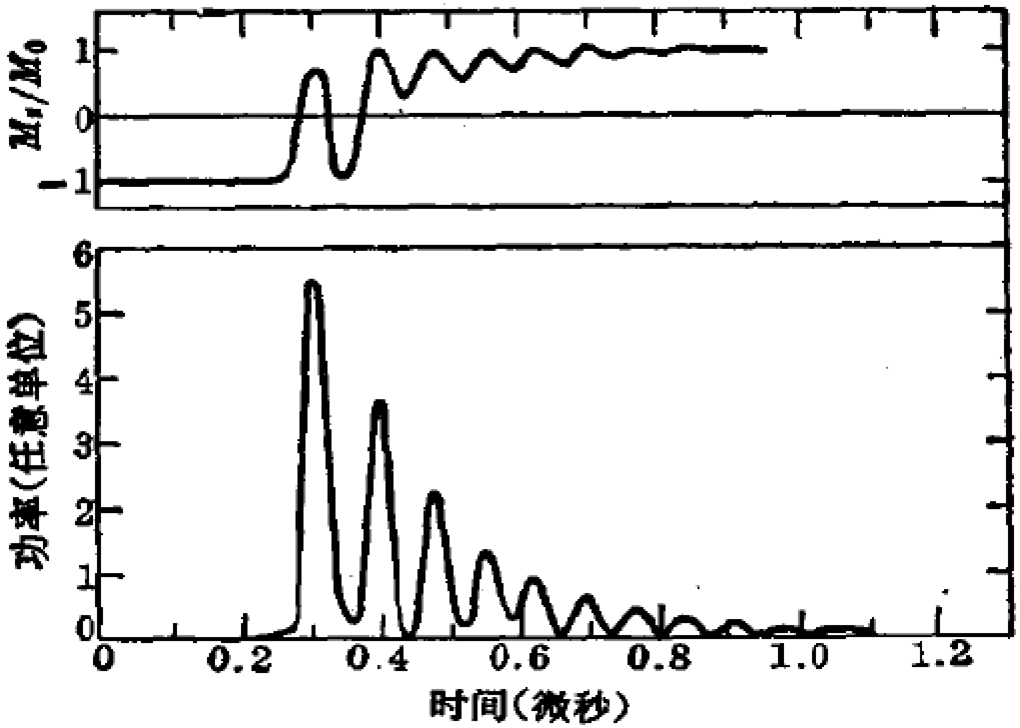


图 15.5 在(I, II, III)空间中 $\mathbf{r}_R$ 的运动，当 $t=0$ 时原子处于基态 $|b\rangle$。矢量 $\mathbf{r}_R$ 的顶端在(II, III)平面中描绘一个圆。若在时刻 $t_0=\pi/2\omega_I$ 去掉场，矢量 $\mathbf{r}_R(t>t_0)$ 沿着 −II 方向

$$功率=\frac{\omega Q(N_b-N_a)^2\mu^2V_s^2}{\varepsilon V_0} \propto (N_b-N_a)^2 \propto N_b^2$$



$$\mathbf{E} = (N_b - N_a)\hbar\omega V_s / 2$$

$$\tau = \frac{\mathscr{E}}{\text{功率}}$$

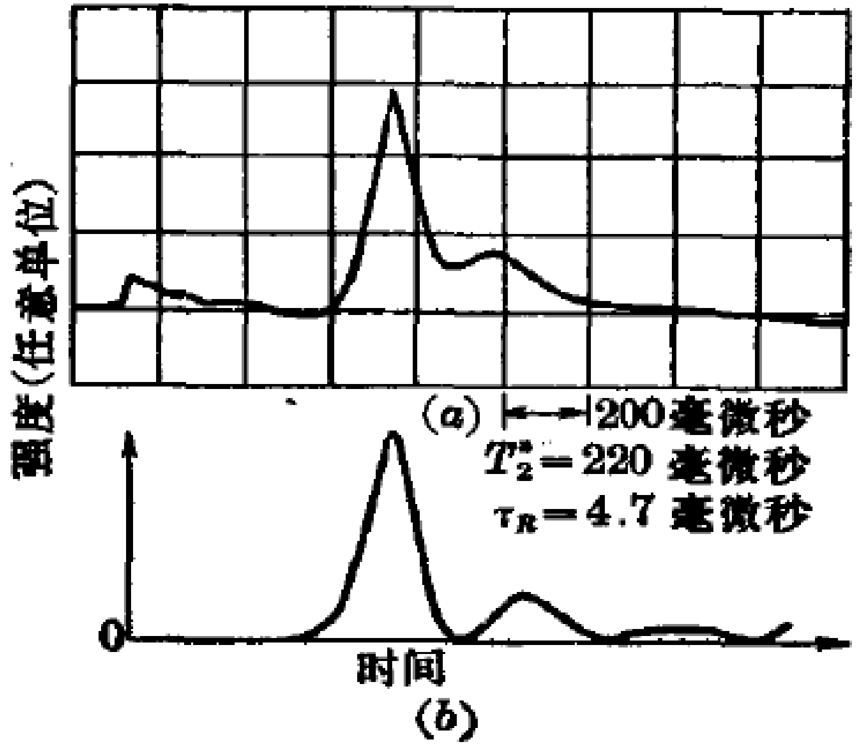
$$= \frac{\hbar\varepsilon}{2Q(N_b - N_a)\mu^2}\left(\frac{V_c}{V_s}\right)$$

初始原子处于基态

$N_b \gg N_a$

故 $\tau \propto \frac{1}{N_b}$

多原子合作辐射，寿命缩短



光子回波

(薄样品)

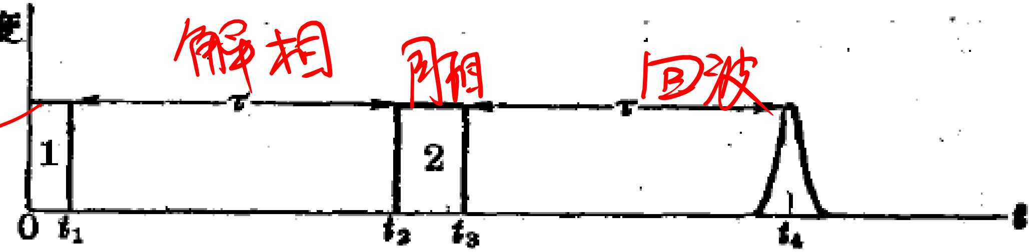
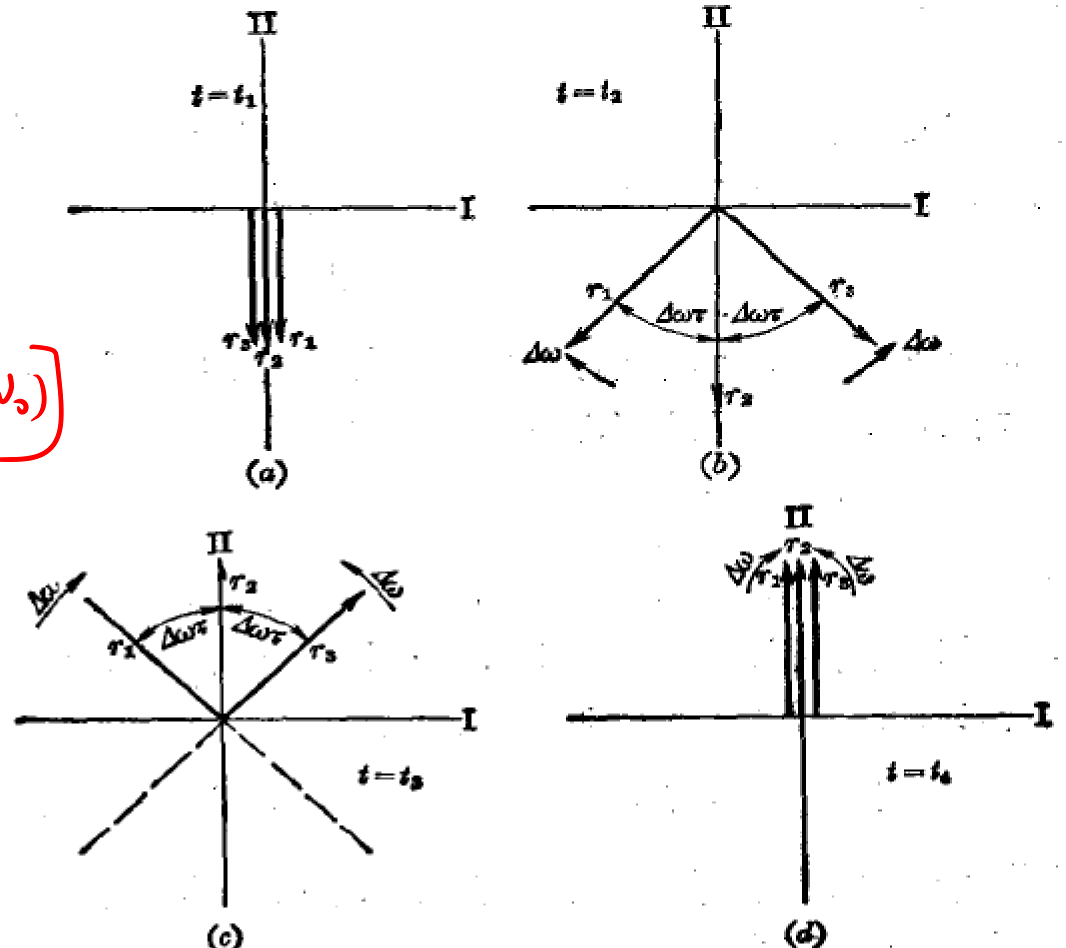


图 15.8　一个“π/2”脉冲 ① 和一个“π”脉冲 ② 作用于有吸收的原子的系综，在 t_4 时刻激发出辐射回波脉冲

ω_0：光脉冲中的中心频率

$$\frac{d\vec{r}_R}{dt} = \left[\hat{a}_I\left(-\frac{2\mu E}{\hbar}\right) + \hat{a}_{III}(\omega-\omega_0)\right] \times \vec{r}_R(\omega)$$



考虑三个原子的共振频率

分别为：$\omega_1 = \omega_0 - \Delta\omega$

$\omega_2 = \omega_0$

$\omega_3 = \omega_0 + \Delta\omega$

N个无碰撞非均匀加宽原子组成的体系，其线型函数为 $g(\delta\omega)$

$$\int_{-\infty}^{\infty} g(\delta\omega)\, d\delta\omega = 1$$

定义 $\vec{R}(t)$ 矢量为所有原子 $\vec{r}$ 的矢量和。

在 $t_1 \sim t_2$ 时间段：

$$\vec{R}(t) = -iN\int_{-\infty}^{\infty} g(\delta\omega)\, e^{i\delta\omega(t-t_1)}\, d\delta\omega$$

if $\delta\omega(t-t_1) > \pi$，~~then~~ $\vec{R}(t) \to 0$ 自由感应衰减

t_2 时刻，$(t_2 - t_1 = \tau)$：

$$\vec{R}(t_2) = -iN\int_{-\infty}^{\infty} g(\delta\omega)\, e^{i\delta\omega\tau}\, d\delta\omega$$

$t_2 \sim t_3$：π 脉冲的作用：$\delta\omega\tau \to \pi - \delta\omega\tau$

so，

$$\vec{R}(t_3) = -iN\int_{-\infty}^{\infty} g(\delta\omega)\, e^{i(\pi-\delta\omega\tau)}\, d\delta\omega$$

after t_3：

$$\vec{R}(t>t_3) = -iN\int_{-\infty}^{\infty} g(\delta\omega)\, e^{i(\pi-\delta\omega\tau)} \cdot e^{i\delta\omega(t-t_3)}\, d\delta\omega$$

$t = t_1$ 时，$\vec{R}(t_1) = -iN$

$\vec{R}$ // -II 轴：表示 $\frac{\pi}{2}$ 脉冲使原子由基态向 -II 轴方向激发

t_4 时刻，$t_4 = t_3 + \tau$

$$\vec{R}(t_4) = -iN\int_{-\infty}^{\infty} g(\delta\omega)\, e^{i(\pi-\delta\omega\tau)} \cdot e^{i\delta\omega(t_4-t_3)}\, d\delta\omega$$

$$= iN\int_{-\infty}^{\infty} g(\delta\omega)\, e^{i\pi}\, d\delta\omega$$

$$= iN$$

自感应透明

厚样品假设

现象： 强度高于一定阈值的共振短脉冲通过正常吸收介质时有异常低的衰减。且脉冲的形态、能量保持稳定。

要求： 脉冲宽度<介质弛豫时间

脉冲中心频率与介质吸收峰共振$\omega = \omega_0$

Bloch方程

$$E_x(z,t) = \frac{1}{2}\left\{\mathcal{E}(z,t)\, e^{i[k_0 z - \omega_0 t + \phi(z,t)]} + c.c.\right\}$$

$$P_x(z,t) = \frac{1}{2}\left\{[U(z,t) + iV(z,t)]\, e^{i[k_0 z - \omega_0 t + \phi(z,t)]} + c.c.\right\}$$

代入波动方程：

$$\frac{\partial^2 E}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

慢变振幅近似 $k_0 A \gg \frac{\partial A}{\partial z}$，$\omega_0 A \gg \frac{\partial A}{\partial t}$

$$\Downarrow$$

$$\begin{cases} \dfrac{\partial \mathcal{E}}{\partial z} + \dfrac{n}{c}\dfrac{\partial \mathcal{E}}{\partial t} = -\dfrac{\omega_0 c \mu_0}{2n} V \\ \mathcal{E}\left(\dfrac{\partial \phi}{\partial z} + \dfrac{n}{c}\dfrac{\partial \phi}{\partial t}\right) = \dfrac{\omega_0 c \mu_0}{2n} U \end{cases}$$

密度矩阵：$\frac{\partial}{\partial t}(\rho_{11}-\rho_{22}) = \frac{2i\mu}{\hbar}E(\rho_{21}-\rho_{12})$

$$\frac{\partial}{\partial t}\rho_{21} = -i\omega\rho_{21} + \frac{i\mu}{\hbar}E(\rho_{11}-\rho_{22})$$

$$\hbar\omega = E_2 - E_1 \quad , \quad \Delta\omega \equiv \omega - \omega_0$$

$$\begin{cases} U(z,t) = \int_{-\infty}^{\infty} u(\Delta\omega, z, t)\, g(\Delta\omega)\, d\Delta\omega \\ V(z,t) = \int_{-\infty}^{\infty} v(\Delta\omega, z, t)\, g(\Delta\omega)\, d\Delta\omega \end{cases}$$

$$P(\Delta\omega, z, t) = N\mu[\rho_{21}(\Delta\omega, z, t) + \rho_{12}(\Delta\omega, z, t)]$$

$$P(z,t) = \int_{-\infty}^{\infty} P(\Delta\omega, z, t)\, g(\Delta\omega)\, d\Delta\omega$$

$$= N\mu\int_{-\infty}^{\infty}[\rho_{21}(\Delta\omega, z, t) + c.c.]\, g(\Delta\omega)\, d\Delta\omega$$

$\frac{1}{2}(U+iV)\overline{e}+c.c.$

$$= \frac{1}{2}\{[(U+iV)e^{i[k_0 z - \omega_0 t + \phi(z,t)]} + c.c.]\}$$ 经典

$$= \frac{1}{2}\{[\int_{-\infty}^{\infty} u(\Delta\omega, z, t)\, g(\Delta\omega)\, d\Delta\omega + i\int_{-\infty}^{\infty} v(\Delta\omega, z, t)\, g(\Delta\omega)\, d\Delta\omega]$$

$$\cdot e^{i(k_0 z - \omega_0 t + \phi(z,t))} + c.c.\}$$

$$N\mu\,\rho_{21}(\delta\omega, z, t) = \frac{1}{2}\left[u(\delta\omega, z, t) + i v(\delta\omega, z, t)\right] e^{i[k_0 z - \omega_0 t + \phi(z,t)]}$$

$$\rho_{21}(\delta\omega, z, t) = \frac{1}{2N\mu}\left[u(\delta\omega, z, t) + i v(\delta\omega, z, t)\right] e^{i[k_0 z - \omega_0 t + \phi(z,t)]}$$

代入 $\frac{\partial}{\partial t}\rho_{21} = -i\omega\rho_{21} + \frac{i\mu}{\hbar}E(\rho_{11} - \rho_{22})$

分别由实部、虚部相等得

$\rho = (u + iv)e$

$$\begin{cases} \frac{\partial u}{\partial t} = v\left(\delta\omega + \frac{\partial \phi}{\partial t}\right) - \frac{u}{T_2} \\ \frac{\partial v}{\partial t} = -u\left(\delta\omega + \frac{\partial \phi}{\partial t}\right) + \frac{\mu\mathcal{E}(z,t)}{\hbar} w - \frac{v}{T_2} \end{cases}$$

$$w(\delta\omega, z, t) \equiv N\mu\left[\rho_{11}(\delta\omega, z, t) - \rho_{22}(\delta\omega, z, t)\right]$$

$$\frac{\partial w}{\partial t} = N\mu\frac{\partial}{\partial t}(\rho_{11} - \rho_{22}) = N\mu \cdot \frac{2i\mu}{\hbar}E(\rho_{21} - \rho_{12})$$

$$= -\frac{\mu}{\hbar}\mathcal{E}(z,t)\,v - \frac{w - w_0}{\tau}$$

弛豫

场方程：

$$\frac{\partial \mathcal{E}}{\partial z}+\frac{n}{c}\frac{\partial \mathcal{E}}{\partial t}=-\frac{\omega_0 c\mu_0}{2n}V=-\frac{\omega_0 c\mu_0}{2n}\int_{-\infty}^{\infty} v(\Delta\omega,z,t)\,g(\Delta\omega)\,d(\Delta\omega)$$

~~$$\mathcal{E}\left(\frac{\partial\phi}{\partial z}+\frac{n}{c}\frac{\partial\phi}{\partial t}\right)=\frac{\omega_0 c\mu_0}{2n}U=\frac{\omega_0 c\mu_0}{2n}\int_{-\infty}^{\infty} u(\Delta\omega,z,t)\,g(\Delta\omega)\,d(\Delta\omega)$$~~

若不考虑任意位相变化即：$\phi(z,t)=0$；

也不计弛豫作用，即脉冲持续时间 $\ll$ 弛豫时间 T_2, T_1.

则Bloch方程为

$$\begin{cases}\dfrac{\partial u}{\partial t}=\Delta\omega\, v\\[2mm] \dfrac{\partial v}{\partial t}=-\Delta\omega\, u+\dfrac{\mu\mathcal{E}}{\hbar}w\\[2mm] \dfrac{\partial w}{\partial t}=-\dfrac{\mu\mathcal{E}}{\hbar}v\\[2mm] \dfrac{\partial \mathcal{E}}{\partial z}+\dfrac{n}{c}\dfrac{\partial \mathcal{E}}{\partial t}=-\dfrac{\omega_0 c\mu_0}{2n}\displaystyle\int_{-\infty}^{\infty} v(\Delta\omega,z,t)\,g(\Delta\omega)\,d(\Delta\omega)\end{cases}$$

假设 $g(\Delta\omega)$ 为偶函数

$u(\Delta\omega,z,t)$ 是 $\Delta\omega$ 的奇函数

$v(\Delta\omega,z,t)$ 是 $\Delta\omega$ 的偶函数

$w(\Delta\omega,z,t)$ 是 $\Delta\omega$ 的偶函数

"Bloch矢量" ← 记 $\vec{r}=N\mu(u\hat{e}_u+v\hat{e}_v+w\hat{e}_w)$ （r_{I}, r_{II}, r_{III}）

"旋矢" ← $\vec{\Gamma}=-\hat{e}_u\left(\frac{\mu\mathcal{E}}{\hbar}\right)-\hat{e}_w\Delta\omega$

Bloch方程

$\Updownarrow$

$$\frac{\partial}{\partial t}\vec{r}=\vec{\Gamma}\times\vec{r}$$

面积定理：自感透明的理论解释

两个特例（引理）：

$T = -\hat{e}_u\left(\frac{\mu\mathcal{E}}{\hbar}\right) - \hat{e}_w \Delta\omega \underset{\Delta\omega=0\text{时}}{=} -\hat{e}_u\left(\frac{\mu\mathcal{E}}{\hbar}\right)$

引理(1)：Bloch方程的共振解：

对 $\Delta\omega=0$ 的原子，Bloch方程的解为

$$u(0,z,t)=0$$
$$v(0,z,t)=w_0\sin\theta(z,t)$$
$$w(0,z,t)=w_0\cos\theta(z,t)$$

这里，$\theta(z,t) \equiv \frac{\mu}{\hbar}\int_{-\infty}^{t}\mathcal{E}(z,t')dt'$ 表示矢量 $\vec{r}(0,z,t)$ 绕u轴的转动角。

证明：$\Delta\omega=0$ 并假设 $u(0,z,-\infty)=v(0,z,-\infty)=0$

则
$$\begin{cases} \frac{\partial u}{\partial t}=0 \\ \frac{\partial v}{\partial t}=\frac{\mu}{\hbar}\mathcal{E}w \\ \frac{\partial w}{\partial t}=-\frac{\mu}{\hbar}\mathcal{E}v \end{cases}$$

$\frac{\partial u}{\partial t}=0 \Rightarrow u(0,z,t)=u(0,z,-\infty)=0$

$$\Rightarrow \frac{\partial}{\partial t}(v^2+w^2)=2v\frac{\partial v}{\partial t}+2w\frac{\partial w}{\partial t} = 2v\frac{\mu}{\hbar}\mathcal{E}w+2w\left(-\frac{\mu}{\hbar}\mathcal{E}v\right)=0$$

$$\Rightarrow v^2+w^2=w_0^2$$

$$\Downarrow$$

$$\frac{\partial w}{\partial t}=-\frac{\mu}{\hbar}\mathcal{E}v=-\frac{\mu}{\hbar}\mathcal{E}\sqrt{w_0^2-w^2} \Rightarrow \int_{-\infty\,(w_0)}^{t\,(w)}\frac{dw}{\sqrt{w_0^2-w^2}}=\int_{-\infty}^{t}-\frac{\mu}{\hbar}\mathcal{E}\,dt$$

$$\Rightarrow -\cos^{-1}\frac{w}{W_0}\Big|_{W_0}^{w} = -\cos\frac{w}{W_0} = -\frac{\mu}{\hbar}\int_{-\infty}^{t} \mathcal{E}(z,t')\,dt'$$

$$\Rightarrow W_{(v,z,t)} = W_0\cos\left(\frac{\mu}{\hbar}\int_{-\infty}^{t} \mathcal{E}(z,t')\,dt'\right)$$

拉比振动

$$= W_0\cos\theta_{(z,t)}$$

记 $\theta_{(z,t)} = \frac{\mu}{\hbar}\int_{-\infty}^{t} \mathcal{E}(z,t')\,dt'$ 脉冲面积

$$v = \sqrt{W_0^2 - W^2} = W_0\sin\theta_{(z,t)}$$

"光学章动"

引理(2)(非共振自由衰减公式):

若 $t \geq t_0$ 时 $\tau(z,t)=0$，那么当 $t>t_0$ 时有

$$u(\omega, z, t) = u_0 \cos[\omega(t-t_0)] + v_0 \sin[\omega(t-t_0)]$$

$$v(\omega, z, t) = -u_0 \sin[\omega(t-t_0)] + v_0 \cos[\omega(t-t_0)]$$

$$w(\omega, z, t) = w(\omega, z, t_0)$$

；这里 $u_0 \equiv u(\omega, z, t_0)$，$v_0 \equiv v(\omega, z, t_0)$

证明：$\tau(z,t)=0$ 所以

$$\begin{cases} \frac{\partial u}{\partial t} = \omega \cdot v \\ \frac{\partial v}{\partial t} = -\omega \cdot u \\ \frac{\partial w}{\partial t} = 0 \end{cases}$$

前两式 $\Rightarrow \frac{\partial}{\partial t}(u^2+v^2) = 2u\frac{\partial u}{\partial t} + 2v\frac{\partial v}{\partial t} = 0$

第三式 $\Rightarrow w(\omega, z, t) = w(\omega, z, t_0)$

$\therefore u^2+v^2 = u_0^2+v_0^2 \Rightarrow v = \sqrt{(u_0^2+v_0^2)-u^2}$

$$\frac{du}{dt} = \omega v = \omega\sqrt{(u_0^2+v_0^2)-u^2} \Rightarrow \int_{t_0}^{t} \frac{du}{\sqrt{(u_0^2+v_0^2)-u^2}} = \int_{t_0}^{t} \omega\, dt$$

$$\Rightarrow -\cos^{-1}\left[\frac{u}{\sqrt{u_0^2+v_0^2}}\right]\Bigg|_{u_0}^{u} = \omega(t-t_0)$$

$$\Rightarrow \frac{u}{\sqrt{u_0^2+v_0^2}} = \cos\left[\cos^{-1}\frac{u_0}{\sqrt{u_0^2+v_0^2}} - \Delta\omega(t-t_0)\right]$$

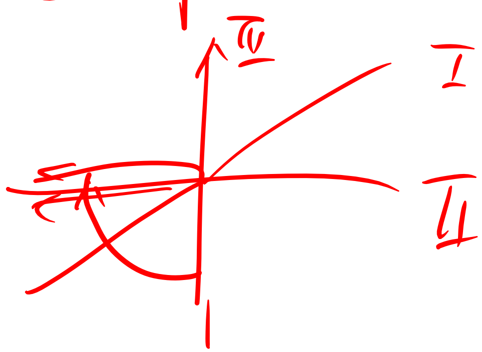
$$= \frac{u_0}{\sqrt{u_0^2+v_0^2}}\cos[\Delta\omega(t-t_0)] + \frac{v_0}{\sqrt{u_0^2+v_0^2}}\sin(\Delta\omega(t-t_0))$$

$$\Rightarrow u(\Delta\omega, z, t) = u_0\cos[\Delta\omega(t-t_0)] + v_0\sin[\Delta\omega(t-t_0)]$$

$$v(\Delta\omega, z, t) = -u_0\sin[\Delta\omega(t-t_0)] + v_0\cos[\Delta\omega(t-t_0)]$$

#

长波辐射



受脉冲后

t_0 时刻 u_0

$v_0 = 0$

$$\begin{cases} u = u_0\cos[\Delta\omega(t-t_0)] \\ v = -u_0\sin[\Delta\omega(t-t_0)] \end{cases}$$

面积定理：$\frac{dA}{dz} = -\frac{\alpha}{2}\sin A$

定义，$A(z) = \lim_{t\to\infty} \theta(z,t) = \frac{\mu}{\hbar}\int_{-\infty}^{\infty} \mathcal{E}(z,t')\,dt'$

~~用～的解释~~ $\alpha = \frac{\omega_0 \pi \mu_0 N \mu^2 c g(0)}{n\hbar}$

证明：从A(z,A)的定义出发：

$$\frac{dA}{dz} = \lim_{t\to\infty} \frac{\mu}{\hbar}\int_{-\infty}^{t} \frac{\partial}{\partial z}\mathcal{E}(z,t')\,dt'$$

又场方程 $\frac{\partial \mathcal{E}}{\partial z} + \frac{n}{c}\frac{\partial \mathcal{E}}{\partial t} = -\frac{\omega_0 c\mu}{2n}\int_{-\infty}^{\infty} v\, g(\Delta\omega)\,d(\Delta\omega)$

$$\frac{dA}{dz} = \lim_{t\to\infty} \frac{\mu}{\hbar}\int_{-\infty}^{t}\left[-\frac{\omega_0 c\mu}{2n}\int_{-\infty}^{\infty} v(\Delta\omega,z,t')\,g(\Delta\omega)\,d\Delta\omega - \frac{n}{c}\frac{\partial \mathcal{E}}{\partial t'}\right]dt'$$

$$= \lim_{t\to\infty} \frac{\mu}{\hbar}\left[-\frac{n}{c}\mathcal{E}\Big|_{-\infty}^{t} - \frac{\omega_0 c\mu}{2n}\int_{-\infty}^{\infty} d(\Delta\omega)\,g(\Delta\omega)\int_{-\infty}^{t} dt'\, v(\Delta\omega,z,t')\right]$$

$$= \frac{\mu}{\hbar}\left(-\frac{n}{c}\right)\left[\mathcal{E}(z,\infty)-\mathcal{E}(z,-\infty)\right] - \lim_{t\to\infty}\frac{\mu\omega_0 c\mu_0}{2n\hbar}\int_{-\infty}^{\infty} d(\delta\omega)\, g(\delta\omega)\int_{-\infty}^{t} dt'\, v(\delta\omega,z,t')$$

$\mathcal{E}(z,\infty)=0$, $\mathcal{E}(z,-\infty)=0$

$$= -\lim_{t\to\infty}\frac{\mu\omega_0 c\mu_0}{2n\hbar}\int_{-\infty}^{\infty} d(\delta\omega)\, g(\delta\omega)\int_{-\infty}^{t} dt'\, v(\delta\omega,z,t')$$

又 $\frac{\partial u}{\partial t} = \delta\omega\cdot v$ (Bloch方程)

$$= -\frac{\mu\omega_0 c\mu_0}{2n\hbar}\lim_{t\to\infty}\int_{-\infty}^{\infty} d(\delta\omega)\, g(\delta\omega)\int_{-\infty}^{t}\frac{1}{\delta\omega}\frac{\partial u}{\partial t'}\, dt'$$

$$= -\frac{\mu\omega_0 c\mu_0}{2n\hbar}\lim_{t\to\infty}\int_{-\infty}^{\infty} d(\delta\omega)\,\frac{g(\delta\omega)}{\delta\omega}\left[u(\delta\omega,z,t)-u(\delta\omega,z,-\infty)\right]$$

$u(\delta\omega,z,-\infty)=0$

$$= -\frac{\mu\omega_0 c\mu_0}{2n\hbar}\lim_{t\to\infty}\int_{-\infty}^{\infty} d(\delta\omega)\,\frac{g(\delta\omega)}{\delta\omega}\cdot u(\delta\omega,z,t)$$

对脉冲光场，可以取一个时刻 t_0，使得 $t>t_0$ 时 $\mathcal{E}(z,t)=0$

则可用引理(2)得：$u(\delta\omega,z,t)=u(\delta\omega,z,t_0)\cos[\delta\omega(t-t_0)]+v(\delta\omega,z,t_0)\sin[\delta\omega(t-t_0)]$

$$\text{故}\ \frac{dA}{dt} = -\frac{\mu\omega_0 c\mu}{2n\hbar}\lim_{t\to\infty}\int_{-\infty}^{\infty} d(\delta\omega)\,\frac{g(\delta\omega)}{\delta\omega}\left\{u_0\cos[\delta\omega(t-t_0)]+v_0\sin[\delta\omega(t-t_0)]\right\}$$

由于 $\cos[\Delta\omega(t-t_0)]$ 和 $\sin[\Delta\omega(t-t_0)]$ 函数的振荡特性，
所以在 $t\to\infty$ 的极限下，被积函数只在 $\Delta\omega\approx 0$ 附近很小范围的值对积分有贡献，

故 $u(\Delta\omega,z,t_0)\approx a_1\Delta\omega + a_2\Delta\omega^3+\cdots$

> u对Δω是奇函数

$g(\Delta\omega)$ 取 $g(0)$

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} d(\Delta\omega)\frac{g(\Delta\omega)}{\Delta\omega}u(\Delta\omega,z,t_0)\cos[\Delta\omega(t-t_0)]$$

$$=\lim_{t\to\infty}\int_{-\infty}^{\infty} d(\Delta\omega)\frac{g(0)}{\Delta\omega}\cdot a_1\Delta\omega\cos[\Delta\omega(t-t_0)]$$

$$=\lim_{t\to\infty} g(0)a_1\int_{-\infty}^{\infty}\cos[\Delta\omega(t-t_0)]\,d(\Delta\omega)=g(0)a_1\lim_{t\to\infty}\left.\frac{\sin[\Delta\omega(t-t_0)]}{t-t_0}\right|_{\Delta\omega=-\infty}^{\Delta\omega=\infty}=0$$

> 有界（分子 $\sin[\Delta\omega(t-t_0)]$）；分母 $t-t_0\to\infty$

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} d(\Delta\omega)\frac{g(\Delta\omega)}{\Delta\omega}v(\Delta\omega,z,t_0)\sin[\Delta\omega(t-t_0)]$$

> 偶函数 v 在 Δω=0 附近展开
> 保留零阶，则
> $v(\Delta\omega,z,t_0)=v(0,z,t_0)$

$$=\lim_{t\to\infty}\int_{-\infty}^{\infty} d(\Delta\omega)\frac{g(0)}{\Delta\omega}v(0,z,t_0)\sin[\Delta\omega(t-t_0)]$$

$$=\lim_{t\to\infty} g(0)v(0,z,t_0)\int_{-\infty}^{\infty} d(\Delta\omega)\frac{\sin[\Delta\omega(t-t_0)]}{\Delta\omega}$$

> $\int_{-\infty}^{\infty} d(\Delta\omega)\frac{\sin[\Delta\omega(t-t_0)]}{\Delta\omega}=\pi$

因此 $\frac{dA}{dz} = -\frac{w_0 c\mu_0\mu}{2n\hbar} v(0,z,t_0)\, g(0)\, \pi$

{ 引理(1)知 $v(0,z,t_0) = W_0 \sin\theta(z,t_0) = W_0 \sin A$

Because $\mathcal{E}_0(z, t > t_0) = 0 \Rightarrow \theta(z,t_0) = \theta(z,\infty)$
$= A$

这样 $\frac{dA}{dz} = -\frac{\pi W_0 c\mu_0\mu}{2n\hbar} W_0 \sin A \cdot g(0)$

$= -\frac{\pi W_0 c\mu_0 \mu W_0 g(0)}{2n\hbar} \sin A$

$= -\frac{\alpha}{2} \sin A.$

$\alpha = \frac{\pi w_0 c\mu_0 \mu W_0 g(0)}{n\hbar} \underset{W_0 = N\mu}{=} \frac{W_0 \pi \mu_0 N \mu^2 c\, g(0)}{n\hbar}$

初始原子处于基态

$W(\omega, z, -\infty) = N\mu\,[\rho_{11}(\omega, z, -\infty) - \rho_{22}(\omega, z, -\infty)]$, where $\rho_{11}(\omega, z, -\infty) = 1$ and $\rho_{22}(\omega, z, -\infty) = 0$

$= N\mu.$

讨论:

① 弱脉冲(小面积A)情况:

$$\sin A \simeq A$$

$$\frac{dA}{dz} = -\frac{\alpha}{2}\sin A = -\frac{\alpha}{2}A \Rightarrow A(z) = A(0)e^{-\frac{\alpha}{2}z}$$

~~脉冲能量~~ (光强) $\mathcal{E}^2(z) = \mathcal{E}^2(0)e^{-\alpha z}$

Beer定律. 线性吸收

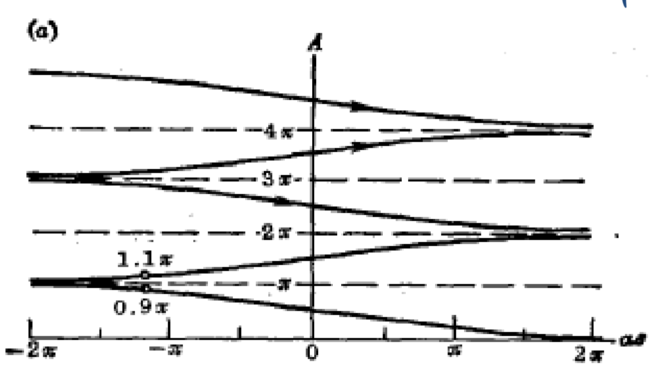
α: 线性吸收系数

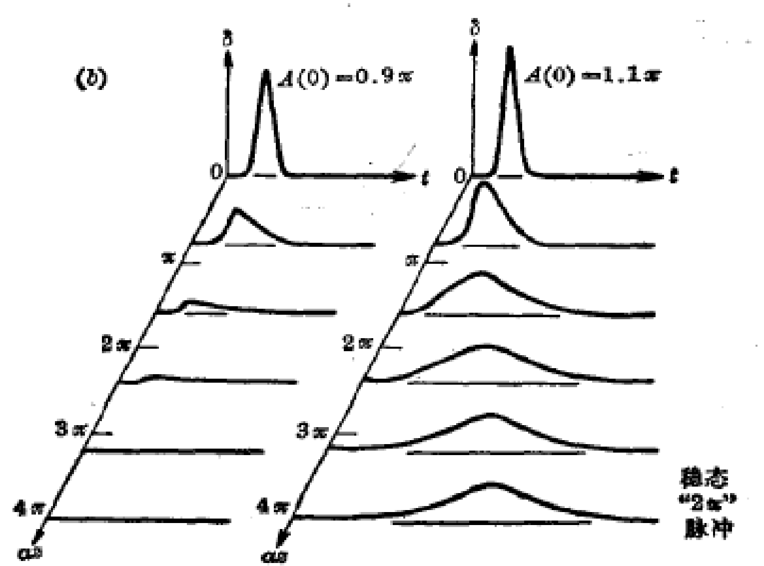
② 平衡解:

$$\frac{dA}{dz} = 0 \Rightarrow \sin A = 0 \Rightarrow A = m\pi. \quad (m=0,1,2,\dots)$$

m = 奇数: 非稳定平衡

m = 偶数: 稳定平衡





③ "2π脉冲"可以无损耗通过介质

对完全共振 $\Delta\omega=0$ 情况，假设 $W_0=-1$，那么前半脉冲使 W 从 $-1\to+1$，后半脉冲使 W 从 $+1\to-1$。总的来看脉冲的能量不变

④ 对增益介质：$\alpha\to-\alpha$

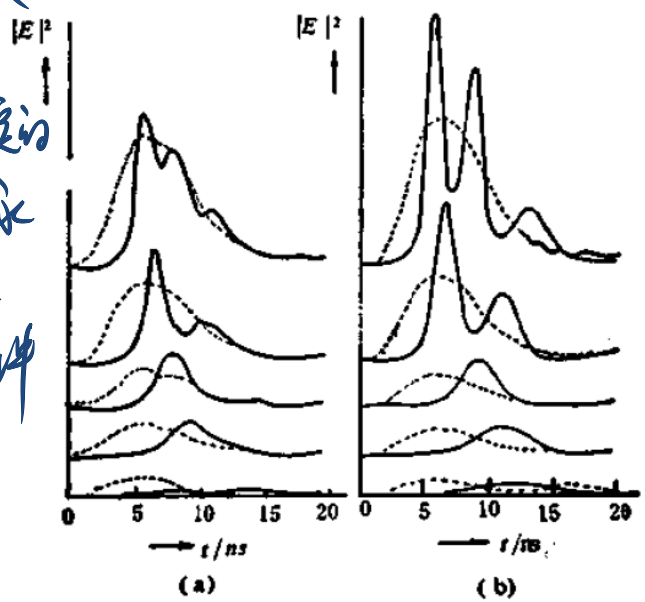
$$\frac{dA}{dz}=-\frac{\alpha}{2}\sin A \Rightarrow \frac{dA}{d(-z)}=-\frac{(-\alpha)}{2}\sin A$$

等价于吸收介质中脉冲沿 $-z$ 方向传播

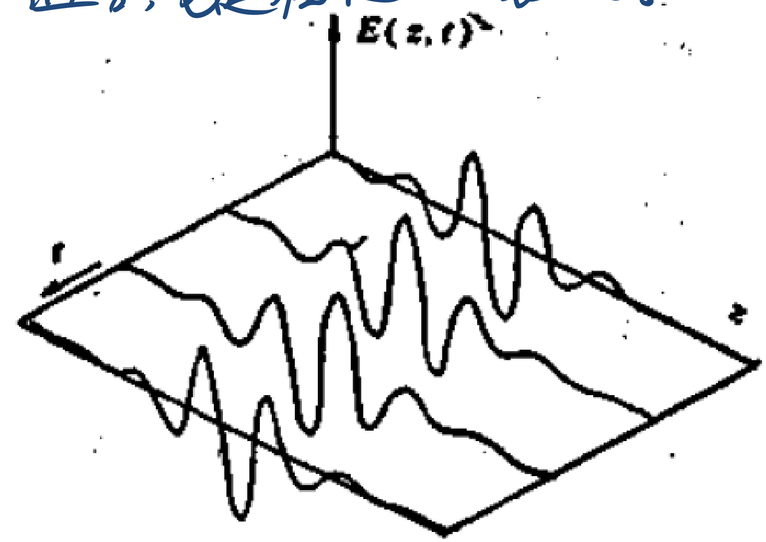
⑤ 脉冲分裂：

只有2π脉冲是真正稳定的

4π、6π …脉冲会分裂成多个2π脉冲



⑥ 0π脉冲可看作2π和−2π脉冲的组合，它是稳定不分裂的。



稳态解:自感透明脉冲的形状

稳定的自感应透明脉冲不仅具有稳定的“脉冲面积”； 还具有确定的脉冲形状和脉宽。稳定脉冲以速度V在介质中传播，其时空坐标的组合形成一个宗量$\gamma = t - \frac{1}{V}$，对任意时空波函数$f(\gamma)$有$\frac{\partial f}{\partial t} = \frac{df}{d\gamma}$； $\frac{\partial f}{\partial z} = -\frac{1}{V}\frac{df}{d\gamma}$

$$\frac{\partial \mathscr{E}}{\partial z} + \frac{n}{c}\frac{\partial \mathscr{E}}{\partial t} = -\frac{\omega_0 c \mu_0}{2n}\int_{-\infty}^{\infty} v(\Delta\omega, z, t)\, g(\Delta\omega)\, d(\Delta\omega)$$

$$\frac{\partial u}{\partial t} = \Delta\omega v$$

$$\frac{\partial v}{\partial t} = -\Delta\omega u + \frac{\mu \mathscr{E}}{\hbar} w$$

$$\frac{\partial w}{\partial t} = -\frac{\mu \mathscr{E}}{\hbar} v$$

Bloch方程

以速度V传播

$$\frac{du}{d\gamma} = (\Delta\omega) v$$

$$\frac{dv}{d\gamma} = -(\Delta\omega) u + \frac{\mu}{\hbar}\mathscr{E} w$$

$$\frac{dw}{d\gamma} = -\frac{\mu}{\hbar}\mathscr{E} v$$

$$\frac{d\mathscr{E}}{d\gamma}\left(\frac{n}{c} - \frac{1}{V}\right) = \frac{-\omega_0 c \mu_0}{2n}\int_{-\infty}^{\infty} v(\Delta\omega, \gamma)\, g(\Delta\omega)\, d(\Delta\omega)$$

稳态Bloch方程求解

对 $v(\Delta\omega, z)$ 作变量分离：$v(\Delta\omega, z) = v(z) f(\Delta\omega)$

$v(z) = v(0, z)$

$f(0) = 1$

物理意义：

不同频率($\Delta\omega$)的原子对光场的响应都是一样的 $v(0,z)$，但不同频率的原子对极化强度 $v(\Delta\omega, z)$ 贡献的振幅是不同的，其差别体现在分布函数 $f(\Delta\omega)$ 上.

$$\frac{du}{dz} = \Delta\omega\, v = v(z)\cdot[\Delta\omega f(\Delta\omega)]$$

$$u(\Delta\omega, -\infty) = 0$$

$$\Downarrow$$

$$u(\Delta\omega, z) = \left[\int_{-\infty}^{z} v(z')dz'\right]\cdot[\Delta\omega f(\Delta\omega)]$$

$$\equiv u(z)\cdot[\Delta\omega f(\Delta\omega)]$$

$$\frac{dw}{dz} = -\frac{\mu}{\hbar}\mathcal{E} v = -\frac{\mu}{\hbar}\mathcal{E}(z)\, v(z)\cdot f(\Delta\omega)$$

$$\Downarrow \quad w(\Delta\omega, -\infty) = N\mu$$

$$w = N\mu(\rho_{11} - \rho_{22})$$

$$w(\Delta\omega, z) = N\mu - \frac{\mu}{\hbar} f(\Delta\omega)\left[\int_{-\infty}^{z}\mathcal{E}(z')v(z')dz'\right]$$

$$= N\mu - f(\Delta\omega)\cdot w(z)$$

$$\begin{cases} u(\Delta\omega, z) = u(z)\cdot\Delta\omega f(\Delta\omega) \\ v(\Delta\omega, z) = v(z)\cdot f(\Delta\omega) \\ w(\Delta\omega, z) = N\mu - w(z)\cdot f(\Delta\omega) \end{cases}$$

$$\frac{d\mathcal{E}(\gamma)}{d\gamma} = \frac{\omega_0 c\mu_0}{2n\left(\frac{1}{v}-\frac{n}{c}\right)} \cdot v(\gamma)\int_{-\infty}^{\infty} f(\delta\omega)\, g(\delta\omega)\, d(\delta\omega)$$

$$\equiv \frac{\hbar}{N\mu^2\tau^2} v(\gamma)$$

$$\begin{cases} u(\delta\omega,\gamma) = u(\gamma)\,\delta\omega\, f(\delta\omega) \\ v(\delta\omega,\gamma) = v(\gamma) f(\delta\omega) \\ w(\delta\omega,\gamma) = N\mu - w(\gamma) f(\delta\omega) \end{cases}$$

Here, $\frac{1}{\tau^2} \equiv \frac{\omega_0 c\mu_0 N\mu^2}{2n\hbar\left(\frac{1}{v}-\frac{n}{c}\right)}\int_{-\infty}^{\infty} f(\delta\omega)\, g(\delta\omega)\, d(\delta\omega)$

$$\frac{dw(\delta\omega,\gamma)}{d\gamma} = 0 - \frac{dw(\gamma)}{d\gamma} \cdot f(\delta\omega)$$

$$\frac{du(\gamma)}{d\gamma} = \frac{1}{f(\delta\omega)\cdot\delta\omega}\,\frac{du(\delta\omega,\gamma)}{d\gamma} = \frac{1}{f(\delta\omega)\cdot\delta\omega}\cdot\delta\omega\cdot v(\gamma) f(\delta\omega) = v(\gamma)$$

$$\frac{dv(\gamma)}{d\gamma} = \frac{1}{f(\delta\omega)}\,\frac{dv(\delta\omega,\gamma)}{d\gamma} = \frac{1}{f(\delta\omega)}\left[-\delta\omega\, u(\delta\omega,\gamma) + \frac{\mu}{\hbar}\mathcal{E}(\gamma)\, w(\delta\omega,\gamma)\right]$$

$$= \frac{1}{f(\delta\omega)}\left[-(\delta\omega)^2 f(\delta\omega)\, u(\gamma) + \frac{\mu}{\hbar}\mathcal{E}(\gamma)\left(N\mu - w(\gamma) f(\delta\omega)\right)\right]$$

$$= \frac{1}{f(\delta\omega)}\left[-(\delta\omega)^2 f(\delta\omega)\, u(\gamma) + \frac{N\mu^2}{\hbar}\mathcal{E}(\gamma) - \frac{\mu}{\hbar}\mathcal{E}(\gamma)\, w(\gamma) f(\delta\omega)\right]$$

$$\frac{dw(\gamma)}{d\gamma} = -\frac{1}{f(\delta\omega)}\,\frac{dw(\delta\omega,\gamma)}{d\gamma} = -\frac{1}{f(\delta\omega)}\cdot\left\{-\frac{\mu}{\hbar}\mathcal{E}(\gamma)\cdot v(\gamma)\cdot f(\delta\omega)\right\}$$

$$= \frac{\mu}{\hbar}\mathcal{E}(\gamma)\, v(\gamma)$$

$$\left.\begin{aligned}\frac{du(\gamma)}{d\gamma}&=v(\gamma)\\ \frac{d\mathscr{E}(\gamma)}{d\gamma}&=\frac{\hbar}{N\mu^2\tau^2}v(\gamma)\end{aligned}\right\}\Rightarrow \frac{du(\gamma)}{d\gamma}=\frac{N\mu^2\tau^2}{\hbar}\frac{d\mathscr{E}(\gamma)}{d\gamma}\Rightarrow u(\gamma)=\frac{N\mu^2\tau^2}{\hbar}\mathscr{E}(\gamma)$$

(初值 $\mathscr{E}(-\infty)=0$, $u(-\infty)=0$)

$$\left.\begin{aligned}\frac{dw(\gamma)}{d\gamma}&=\frac{\mu}{\hbar}\mathscr{E}(\gamma)v(\gamma)\\ \frac{d\mathscr{E}(\gamma)}{d\gamma}&=\frac{\hbar}{N\mu^2\tau^2}v(\gamma)\end{aligned}\right\}\Rightarrow \frac{dw(\gamma)}{d\gamma}=\frac{\mu}{\hbar}\mathscr{E}(\gamma)\frac{N\mu^2\tau^2}{\hbar}\frac{d\mathscr{E}(\gamma)}{d\gamma}$$

$$=\frac{N\mu^3\tau^2}{2\hbar^2}\frac{d}{d\gamma}\left(\mathscr{E}^2(\gamma)\right)$$

$$\Rightarrow w(\gamma)=\frac{N\mu^3\tau^2}{2\hbar^2}\mathscr{E}^2(\gamma)$$

(初值 $w(-\infty)=0$, $\mathscr{E}(-\infty)=0$)

将上面的 $u(\gamma)$, $w(\gamma)$ 表达式代入 $\frac{dv(\gamma)}{d\gamma}$ 的方程得

$$\frac{dv(\gamma)}{d\gamma}=-\frac{N\mu^2\tau^2}{\hbar}\mathscr{E}(\gamma)(\Delta\omega)^2+\frac{N\mu^2}{\hbar f(\Delta\omega)}\mathscr{E}(\gamma)-\frac{N\mu^4\tau^2}{2\hbar^3}\mathscr{E}^3(\gamma)$$

$$=-\frac{N\mu^4\tau^2}{2\hbar^3}\mathscr{E}^3(\gamma)+\frac{N\mu^2}{\hbar}\mathscr{E}(\gamma)\left[\frac{1}{f(\Delta\omega)}-(\Delta\omega)^2\tau^2\right]$$

此项等于与ω无关的常数!

$$\left.\begin{aligned}&\frac{1}{f(\delta\omega)}-\delta\omega^2\tau^2=A\\&f(0)=1\end{aligned}\right\}\Rightarrow f(\delta\omega)=\frac{1}{1+(\delta\omega)^2\tau^2}$$

对瞬态过程，不计耗散，矢量 $u\hat{e}_u+v\hat{e}_v+w\hat{e}_w$ 在脉冲传播过程中保持长度。初始时刻原子完全处于下能级，即 $u(-\infty)=v(-\infty)=0$，$w_0=N\mu$（即 $f(0)=1$）

$$\therefore\ u^2+v^2+w^2=w_0^2=N^2\mu^2$$

即 $$N^2\mu^2=u^2(\gamma)\,\delta\omega^2 f^2(\delta\omega)+v^2(\gamma)f^2(\delta\omega)+\left(N\mu-w(\gamma)f(\delta\omega)\right)^2$$

$$=\left[u^2(\gamma)\delta\omega^2+v^2(\gamma)+w^2(\gamma)\right]f^2(\delta\omega)+N^2\mu^2-2N\mu w(\gamma)f(\delta\omega)$$

$$\Rightarrow\left[u^2(\gamma)\delta\omega^2+v^2(\gamma)+w^2(\gamma)\right]f=2N\mu w(\gamma)$$

$$\Rightarrow v^2(\gamma)=2N\mu\cdot\frac{N\mu^3\tau^2}{2\hbar}\mathcal{E}^2(1+\delta\omega^2\tau^2)-\left(\frac{N\mu^2\tau^2}{\hbar}\mathcal{E}\right)^2\cdot\delta\omega^2-\left(\frac{N\mu^3\tau^2}{2\hbar^2}\mathcal{E}^2\right)^2$$

$$=\frac{N^2\mu^4\tau^2}{\hbar^2}\mathcal{E}^2\left(1-\frac{\mu^2\tau^2}{4\hbar^2}\mathcal{E}^2\right)$$

$$\Rightarrow v(\gamma)=\frac{N\mu^2\tau}{\hbar}\mathcal{E}\sqrt{1-\left(\frac{\mu\tau}{2\hbar}\mathcal{E}\right)^2}$$

代入方程 $\frac{d\mathcal{E}(\gamma)}{d\gamma} = \frac{\hbar}{N\mu^2\tau^2}v(\gamma)$

得 $\frac{d\mathcal{E}}{d\gamma} = \frac{1}{\tau}\mathcal{E}\sqrt{1-\left(\frac{\mu\tau}{2\hbar}\mathcal{E}\right)^2} \Rightarrow \frac{e^{-\left(\frac{\gamma}{\tau}\right)}\mathcal{E}}{1+\sqrt{1-\left(\frac{\mu\tau}{2\hbar}\mathcal{E}\right)^2}} = \frac{e^{-\frac{\gamma_0}{\tau}}\mathcal{E}(\gamma_0)}{1+\sqrt{1-\left(\frac{\mu\tau}{2\hbar}\mathcal{E}(\gamma_0)\right)^2}} \equiv B$

$$\Rightarrow \mathcal{E}(\gamma) = \frac{2}{\frac{1}{B}e^{-\gamma/\tau} + \frac{\mu^2\tau^2}{4\hbar^2}Be^{\gamma/\tau}}$$

Def: $B = \frac{2\hbar}{\mu\tau}e^{-\gamma_p/\tau}$ 得 $\mathcal{E}(\gamma) = \frac{2\hbar}{\mu\tau}\operatorname{sech}\left(\frac{\gamma-\gamma_p}{\tau}\right)$

选择时空坐标原点使 $\gamma_p = 0$ 则

$$\mathcal{E}(\gamma) = \frac{2\hbar}{\mu\tau}\operatorname{sech}\left(\frac{\gamma}{\tau}\right) = \frac{2\hbar}{\mu\tau}\operatorname{sech}\left(\frac{t-z/V}{\tau}\right)$$

脉冲稳态传播的Bloch方程解。

$$u(\Delta\omega, z, t) = 2N\mu \frac{(\Delta\omega)\tau}{1+(\Delta\omega)^2\tau^2} \operatorname{sech}\left(\frac{t-\frac{z}{V}}{\tau}\right)$$

$$w(\Delta\omega, z, t) = N\mu - 2N\mu \frac{1}{1+(\Delta\omega)^2\tau^2} \operatorname{sech}^2\left(\frac{t-\frac{z}{V}}{\tau}\right)$$

$$v(\Delta\omega, z, t) = -2N\mu \frac{1}{1+(\Delta\omega)^2\tau^2}$$

$$\times \tanh\left(\frac{t-\frac{z}{V}}{\tau}\right) \operatorname{sech}\left(\frac{t-\frac{z}{V}}{\tau}\right)$$

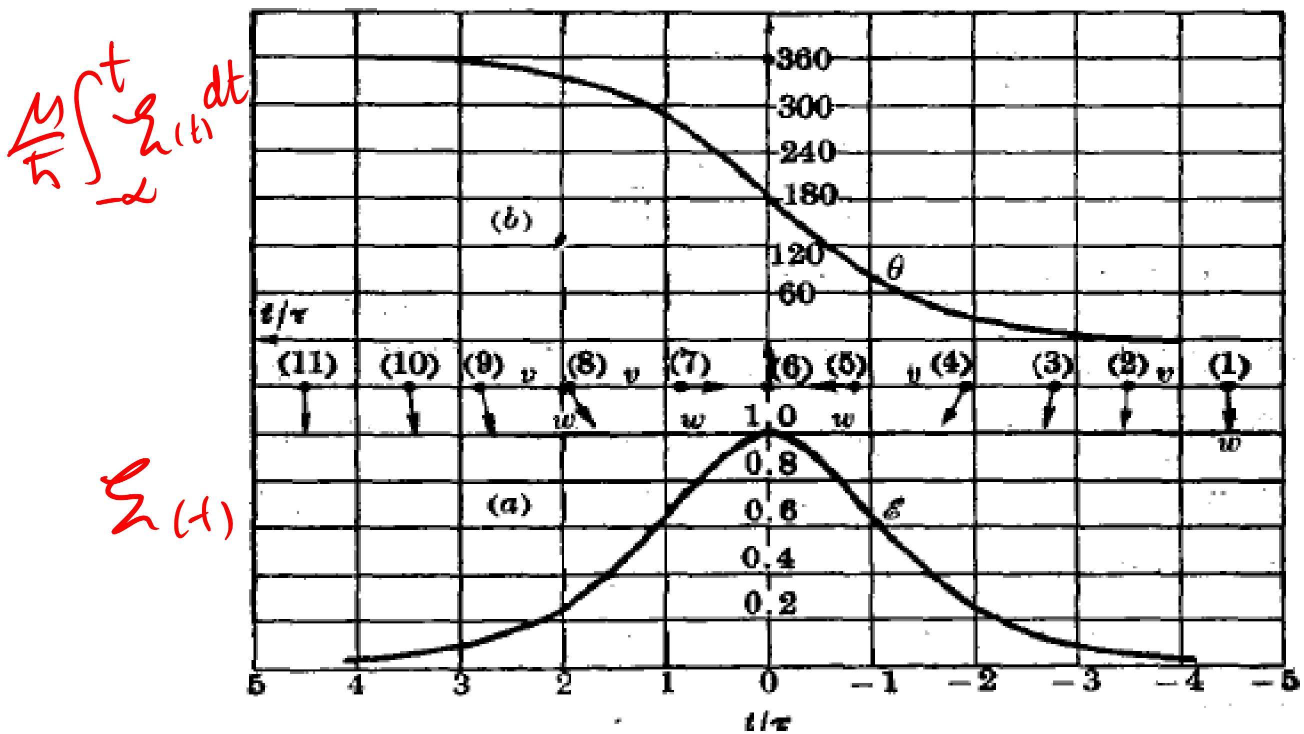
τ是脉冲宽度

V是脉冲传播速度。

$$\mathscr{E}(z, t) = \frac{2\hbar}{\mu\tau} \operatorname{sech}\left(\frac{t-\frac{z}{V}}{\tau}\right)$$

$$\frac{1}{V} = \frac{n}{c} + \frac{\omega_0 c \mu_0 N \mu^2 \tau^2}{2n\hbar} \int_{-\infty}^{\infty} \frac{g(\Delta\omega)}{1+(\Delta\omega)^2\tau^2} d(\Delta\omega)$$

$$= \frac{n}{c} + \frac{\alpha\tau^2}{2\pi g(0)} \int_{-\infty}^{\infty} \frac{g(\Delta\omega)}{1+(\Delta\omega)^2\tau^2} d(\Delta\omega)$$



360
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(b)
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v
w
1.0
0.8
0.6
0.4
0.2
(a)
ℰ
5
4
3
2
1
0
−1
−2
−3
−4
−5
t/τ

自感透明稳态脉冲的传播速度：

$$\frac{1}{V}=\frac{n}{c}+\frac{\alpha\tau^2}{2\pi g(0)}\int_{-\infty}^{\infty}\frac{g(\Delta\omega)}{1+(\Delta\omega)^2\tau^2}d(\Delta\omega)$$

$$f=\frac{1}{1+\Delta\omega^2\tau^2} \qquad \int f\cdot g\, d(\Delta\omega)=\frac{1/\tau^2}{\frac{1}{\tau^2}+(\Delta\omega)^2}$$

考虑洛伦兹线型：

$$g(\Delta\omega)=\frac{\Delta\omega_{原子}}{2\pi\left[(\Delta\omega)^2+\left(\frac{\Delta\omega_{原子}}{2}\right)^2\right]}$$

Case 1: $\Delta\omega_{原子}\tau\gg1$ "宽跃迁"

$$\int_{-\infty}^{\infty}\frac{g(\Delta\omega)}{1+\Delta\omega^2\tau^2}d\Delta\omega=\int_{-\infty}^{\infty}\frac{g(0)}{1+\Delta\omega^2\tau^2}d\Delta\omega=g(0)\int_{-\infty}^{\infty}\frac{1}{1+y^2}dy\cdot\frac{1}{\tau}=\frac{g(0)}{\tau}\pi$$

$$\therefore\ \frac{1}{V}=\frac{n}{c}+\frac{\alpha\tau^2}{2\pi g(0)}\cdot\frac{g(0)}{\tau}\cdot\pi=\frac{n}{c}+\frac{\alpha}{2}\tau$$

吸收介质 $\alpha>0$ $V<\frac{c}{n}$

增益介质 $\alpha<0$ $V>\frac{c}{n}$

Case 2: $\Delta\omega_{原子}\tau\ll1$ "窄跃迁"

$g(\Delta\omega)$ 可看作 $\delta(\Delta\omega)$ 函数，$g(0)=\frac{\Delta\omega_a}{2\pi\left(\frac{\Delta\omega_a}{2}\right)^2}=\frac{2}{\pi\Delta\omega_a}$

$$\therefore\ \frac{1}{V}=\frac{n}{c}+\frac{\alpha\tau^2}{2\pi g(0)}\cdot1=\frac{n}{c}+\frac{\alpha\tau^2}{2\pi\cdot\frac{2}{\pi\Delta\omega_a}}=\frac{n}{c}+\frac{\alpha\Delta\omega_a}{4}\tau^2$$

http://journals.aps.org/prl/international-year-of-light

See-through Materials

Careful control of atom excitations can make an opaque material transparent to certain wavelengths of light. Klaus Boller and co-workers first demonstrated this phenomenon in the early 1990s using strontium vapor. Strontium has two ground states that can both be excited to the same excited state. By carefully tuning the frequencies of two incident lasers, Boller *et al.* were able to ensure that the probabilities of the different excitation pathways destructively interfered, cancelling out any excitation. The strontium vapor, which was opaque to the separate lasers, was now transparent to both. Electromagnetically induced transparency has since been achieved in atomic gases, diamond, and superconducting qubits. As well as making materials transparent, this effect has been used to slow and stop light, to measure the velocity of cold atoms, to induce lasing, and for high-precision magnetometry.

Observation of electromagnetically induced transparency
K.-J. Boller, A. Imamoğlu, and S. E. Harris
Phys. Rev. Lett. **66**, 2593 (1991)

http://journals.aps.org/prl/international-year-of-light

Superradiant Atom Emission

The probability that an excited two-level system (e.g., an excited atom) will emit a photon decreases exponentially with time. However, this emission rate is much higher if a second atom is placed nearby—even if the second atom is in its ground state. Robert Dicke made this surprising theoretical discovery, which occurs because the quantum states of nearby atoms are correlated, in 1954. Dicke also showed that emission events in a large group of particles are not independent, leading to a significant increase in the radiative power of the system, a behavior he named superradiance. Since Dicke's exploratory study, the phenomenon of superradiance has been observed in many physical systems such as optically pumped hydrogen fluoride gas, quantum dots, and superconducting qubits. The effect has recently been used to make a superradiant laser, where the correlated atomic emission boosted photon emission by a factor of 10,000.

Coherence in Spontaneous Radiation Processes

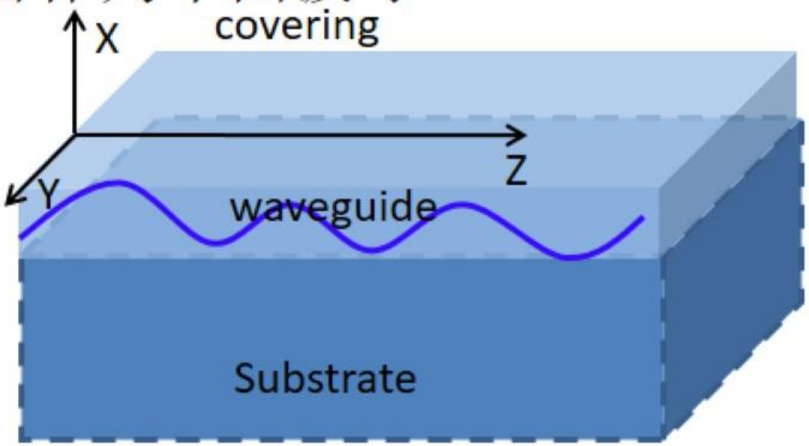
R. H. Dicke

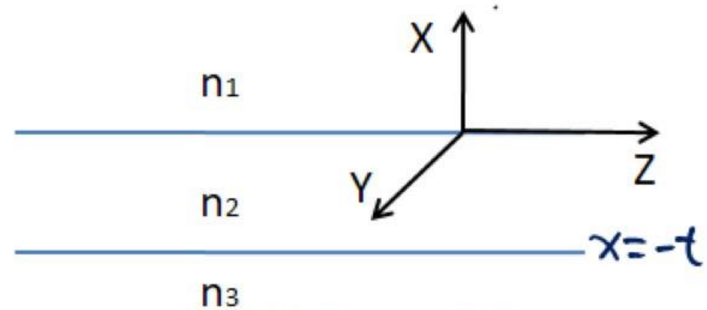
Phys. Rev. **93**, 99 (1954)

Ch19 光学电介质波导

平面波导

在Y方向上的尺度>>X方向的尺度的情况下，介质波导可以近似看作为平面波导





一般地，要求$n_2>n_3>n_1$

理论基础：Maxwell方程组+连续性边界条件

Helmhotz方程

$$\nabla^2\vec{E}(\vec{r})+k^2n^2(\mathrm{\bar{r}})\vec{E}(\vec{r})=0$$

波导的谐波电磁场：

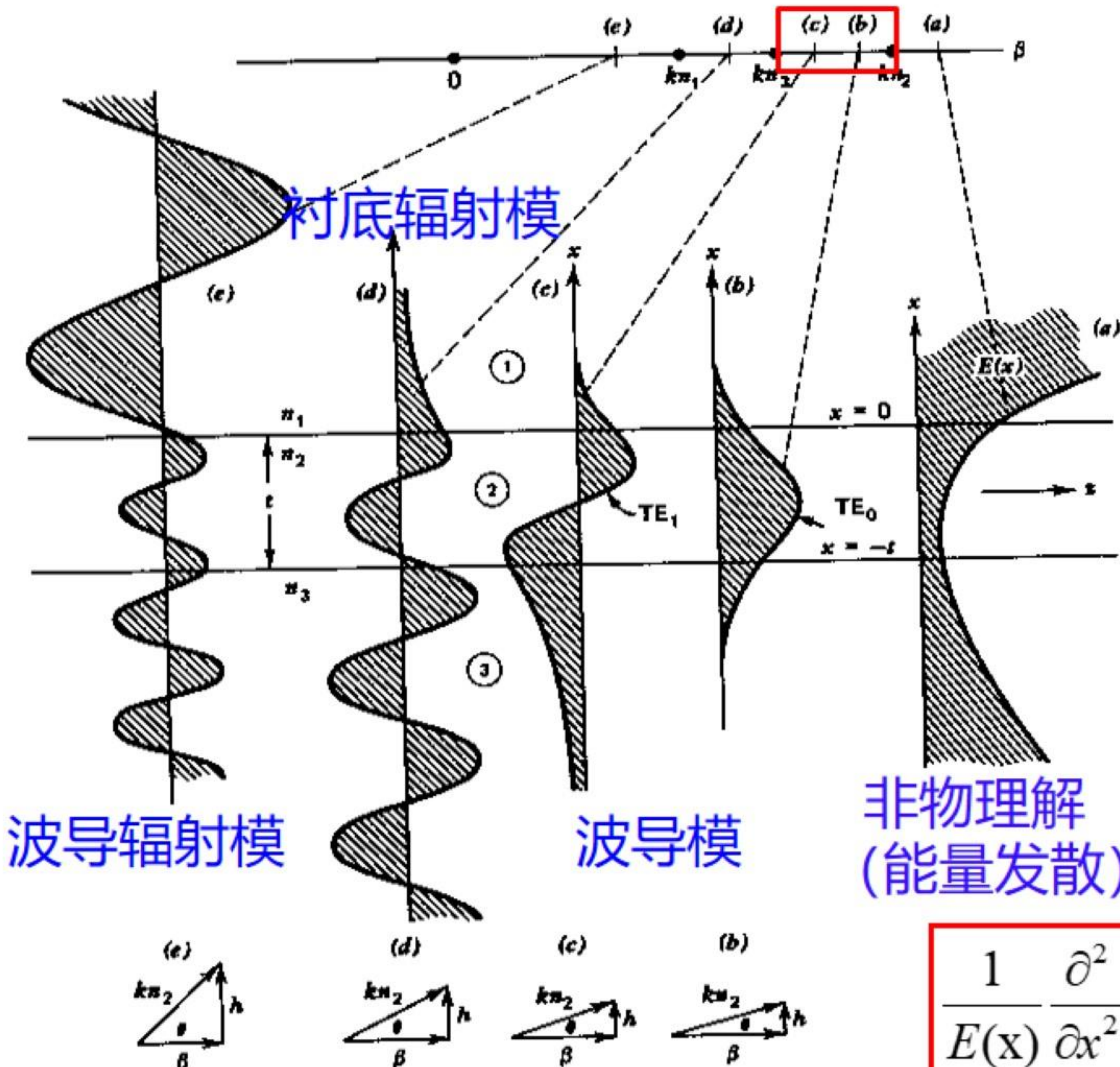
$$\vec{E}(\vec{r},t)=\vec{E}(\vec{r})e^{i\omega t}=\vec{E}(x,y)e^{i(\omega t-\beta z)}$$

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})\vec{E}(x,y)+[k^2n^2(\mathrm{\bar{r}})-\beta^2]\vec{E}(x,y)=0$$

对平面波导，$E(\mathrm{x,y})=E(\mathrm{x})$，所以，$\partial_y=0$

$$\frac{\partial^2}{\partial x^2}\vec{E}(x)+[k^2n^2(\mathrm{\bar{r}})-\beta^2]\vec{E}(x)=0$$

$$\frac{1}{E(\mathrm{x})}\frac{\partial^2}{\partial x^2}E(\mathrm{x})=\beta^2-k^2n_i^2(\mathrm{\bar{r}})$$



$$\beta^2 + h^2 = k^2 n_2^2$$

n_i选取n_1、 n_2、 n_3分别表示覆盖层、波导层和衬底层的介质折射率。

$$\frac{1}{E(\mathrm{x})}\frac{\partial^2}{\partial x^2}E(\mathrm{x}) = \beta^2 - k^2 n_i^2(\bar{\mathrm{r}})$$

平面波导的TM模和TE模

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

$$\cancel{\frac{\partial E_x}{\partial y}} - \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\cancel{\frac{\partial H_z}{\partial y}} - \frac{\partial H_y}{\partial z} = \frac{\partial D_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{\partial D_y}{\partial t}$$

$$\cancel{\frac{\partial H_x}{\partial y}} - \frac{\partial H_y}{\partial x} = \frac{\partial D_z}{\partial t}$$

Y方向上平移不变，所以， $\partial_y = 0$

$TM:(E_x, E_z, H_y)$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

$$\cancel{\frac{\partial H_z}{\partial y}} - \frac{\partial H_y}{\partial z} = \frac{\partial D_x}{\partial t}$$

$$\cancel{\frac{\partial H_x}{\partial y}} - \frac{\partial H_y}{\partial x} = \frac{\partial D_z}{\partial t}$$

$TE:(H_x, H_z, E_y)$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{\partial D_y}{\partial t}$$

$$\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

$$\cancel{\frac{\partial E_x}{\partial y}} - \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

平面波导的模式色散特性

TE模

$$\nabla^2 E_y = \frac{n_i^2}{c^2}\frac{\partial^2 E_y}{\partial t^2},\quad i=1,2,3 \qquad E_y(x,z,t)=\mathscr{E}_y(x)e^{i(\omega t-\beta z)}$$

$$\mathscr{E}_y = \begin{cases} C\exp(-qx) & 0 \le x < \infty \\ C[\cos(hx) - (q/h)\sin(hx)] & -t \le x \le 0 \\ C[\cos(ht) + (q/h)\sin(ht)]\exp[p(x+t)] & -\infty < x \le -t \end{cases}$$

$$h = (n_2^2k^2 - \beta^2)^{1/2}$$

$$q = (\beta^2 - n_1^2k^2)^{1/2}$$

$$p = (\beta^2 - n_3^2k^2)^{1/2}$$

$\mathscr{E}_y$ 在 $x=0,-t$ 边界连续

$\frac{\partial \mathscr{E}_y}{\partial x}$ 在 $x=0,-t$ 边界连续!

能量归约化

$$-\frac{1}{2}\int_{-\infty}^{\infty} E_y H_x^* \, dx = \frac{\beta_m}{2\omega\mu}\int_{-\infty}^{\infty}[\mathscr{E}_y^{(m)}(x)]^2\,dx = 1$$

$$C_m = 2h_m\left[\frac{\omega\mu}{|\beta_m|\left(t+\frac{1}{q_m}+\frac{1}{p_m}\right)(h_m^2+q_m^2)}\right]^{1/2}$$

$$h\sin(ht) - q\cos(ht) = p\left[\cos(ht) + \frac{q}{h}\sin(ht)\right]$$

模式正交条件

$$\int_{-\infty}^{\infty}\mathscr{E}_y^{(l)}\mathscr{E}_y^{(m)}\,dx = \frac{2\omega\mu}{\beta_m}\delta_{l,m}$$

模式本征方程：

$$\tan(ht) = \frac{q+p}{h\left(1-\frac{pq}{h^2}\right)}$$

TM模

$$\nabla^2 H_y = \frac{n_i^2}{c^2}\frac{\partial^2 H_y}{\partial t^2}, \quad i = 1,2,3$$

$$H_y(x, z, t) = \mathcal{H}_y(x) e^{i(\omega t - i\beta z)}$$

$$E_x(x, z, t) = \frac{i}{\omega\varepsilon}\frac{\partial H_y}{\partial z} = \frac{\beta}{\omega\varepsilon}\mathcal{H}_y(x) e^{i(\omega t - \beta z)}$$

$$E_z(x, z, t) = -\frac{i}{\omega\varepsilon}\frac{\partial H_y}{\partial x}$$

$$\mathcal{H}_y(x) = \begin{cases} -C\left[\frac{h}{\bar{q}}\cos(ht) + \sin(ht)\right] e^{p(x+t)} & x < -t \\ C\left[-\frac{h}{\bar{q}}\cos(hx) + \sin(hx)\right] & -t < x < 0 \\ -\frac{h}{\bar{q}} C e^{-qx} & x > 0 \end{cases}$$

能量归约化

$$\frac{1}{2}\int_{-\infty}^{\infty} H_y E_x^* \, dx = \frac{\beta}{2\omega}\int_{-\infty}^{\infty}\frac{\mathcal{H}_y^2(x)}{\varepsilon(x)}\, dx = 1$$

$$\int_{-\infty}^{\infty}\frac{[\mathcal{H}_y^{(m)}(x)]^2}{n^2(x)}\, dx = \frac{2\omega\varepsilon_0}{\beta_m}$$

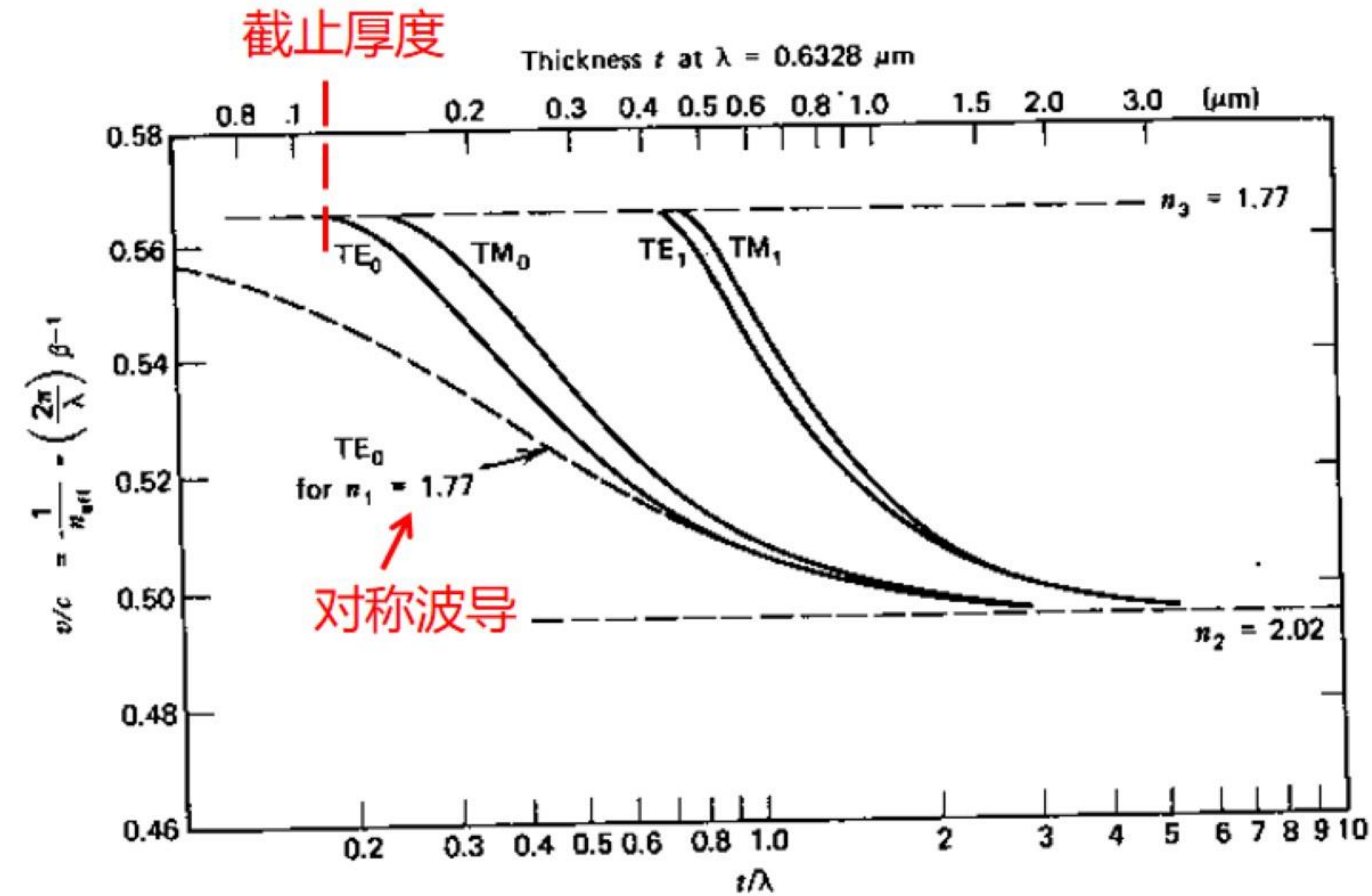
$$C_m = 2\sqrt{\frac{\omega\varepsilon_0}{\beta_m t_{\text{eff}}}}$$

$$t_{\text{eff}} = \frac{\bar{q}^2 + h^2}{\bar{q}^2}\left(\frac{t}{n_2^2} + \frac{q^2 + h^2}{\bar{q}^2 + h^2}\frac{1}{n_1^2 q} + \frac{p^2 + h^2}{\bar{p}^2 + h^2}\frac{1}{n_3^2 p}\right)$$

模式本征方程:

$$\tan(ht) = \frac{h(\bar{p} + \bar{q})}{h^2 - \bar{p}\bar{q}}$$

$$\bar{p} = \frac{n_2^2}{n_3^2} p \qquad \bar{q} = \frac{n_2^2}{n_1^2} q$$



蓝宝石衬底上ZnO波导的束缚模色散曲线

波导间的耦合

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) = \mu\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu \frac{\partial^2}{\partial t^2} \mathbf{P}(\mathbf{r}, t)$$

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0(\mathbf{r}, t) + \mathbf{P}_{\text{pert}}(\mathbf{r}, t)$$

$$\mathbf{P}_0(\mathbf{r}, t) = [\varepsilon(\mathbf{r}) - \varepsilon_0]\mathbf{E}(\mathbf{r}, t)$$

$$\nabla^2 E_y - \mu\varepsilon(\mathbf{r}) \frac{\partial^2}{\partial t^2} E_y = \mu \frac{\partial^2}{\partial t^2} [P_{\text{pert}}(\mathbf{r}, t)]_y$$

$$E_y(\mathbf{r}, t) = \frac{1}{2} \sum_m A_m(z) \mathscr{E}_y^{(m)}(x) e^{i(\omega t - \beta_m z)} + \text{c.c.}$$

只考虑该方程展开!

$$\left(\frac{\partial^2}{\partial x^2} - \beta_m^2\right) \mathscr{E}_y^{(m)}(\mathbf{r}) + \omega^2 \mu\varepsilon(\mathbf{r}) \mathscr{E}_y^{(m)}(\mathbf{r}) = 0$$

本征模方程!

$$e^{i\omega t} \sum_m \left\{ \frac{A_m}{2} \left[-\beta_m^2 \mathscr{E}_y^{(m)} + \frac{\partial^2 \mathscr{E}_y^{(m)}}{\partial x^2} + \omega^2 \mu\varepsilon(\mathbf{r}) \mathscr{E}_y^{(m)} \right] e^{-i\beta_m z} + \frac{1}{2}\left(-2i\beta_m \frac{dA_m}{dz} + \frac{d^2 A_m}{dz^2} \right) \mathscr{E}_y^{(m)} e^{-i\beta_m z} \right\} + \text{c.c.} = \mu \frac{\partial^2}{\partial t^2} [\vec{P}_{\text{pert}}(\vec{r}, t)]_y$$

$$\left| \frac{d^2 A_m}{dz^2} \right| \ll \beta_m \left| \frac{dA_m}{dz} \right|$$ 慢变假设

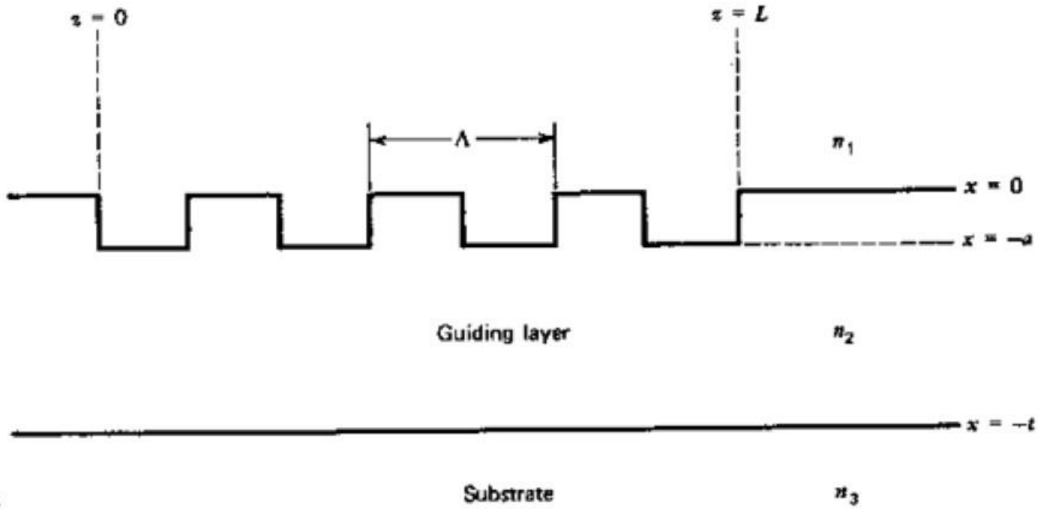
$$\sum_m -i\beta_m \frac{dA_m}{dz} \mathscr{E}_y^{(m)} e^{i(\omega t - \beta_m z)} + \text{c.c.} = \mu \frac{\partial^2}{\partial t^2} [P_{\text{pert}}(\mathbf{r}, t)]_y$$

$$\int_{-\infty}^{\infty} \mathscr{E}_{y}^{(s)}(x) \cdots dx$$

$$\int_{-\infty}^{\infty} \mathscr{E}_y^{(l)} \mathscr{E}_y^{(m)} dx = \frac{2\omega\mu}{\beta_m} \delta_{lm}$$ 模式正交性

$$\frac{dA_s^{(-)}}{dz} e^{i(\omega t + \beta_s z)} - \frac{dA_s^{(+)}}{dz} e^{i(\omega t - \beta_s z)} - \text{c.c.} = -\frac{i}{2\omega} \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} [P_{\text{pert}}(\mathbf{r}, t)]_y \mathscr{E}_y^{(s)}(x)\, dx$$

周期性波导



思路：

波导上电介质的周期性波纹当作介电常数微扰 $\Delta\varepsilon(\vec{r})=\varepsilon_0\Delta(n^2(\vec{r}))$ 来处理

则微扰极化为：$\vec{P}_{pert}=\Delta\varepsilon(\vec{r})\vec{E}(\vec{r},t)=\Delta(n^2(\vec{r}))\varepsilon_0\vec{E}(\vec{r},t)$

标量，故 $\vec{E}$ 的不同偏振分量之间不会耦合，TE, TM彼此独立

下面以TE模为例，

将上面的 $\vec{E}$ 用波导模展开，

$$[\vec{P}_{pert}(\vec{r},t)]_y=\frac{\Delta(n^2(\vec{r}))\varepsilon_0}{2}\sum_m\left[A_m\mathcal{E}_y^{(m)}(x)e^{i(\omega t-\beta_m z)}+c.c.\right]$$

将 $[\vec{P}_{pert}(\vec{r},t)]_y$ 代入上节波微扰方程得：

$$\frac{dA_s^{(-)}}{dz}e^{i(\omega t+\beta_s z)}-\frac{dA_s^{(+)}}{dz}e^{i(\omega t-\beta_s z)}-\text{c.c.}$$

$$=-\frac{i\varepsilon_0}{4\omega}\frac{\partial^2}{\partial t^2}\sum_m\left[A_m\int_{-\infty}^{\infty}\Delta n^2(x,z)\mathscr{E}_y^{(m)}(x)\mathscr{E}_y^{(s)}(x)\,dx\,e^{i(\omega t-\beta_m z)}+\text{c.c.}\right]$$

为满足位相匹配，假设求和项中第 m 个模可耦合到 s 个模中

即 $\Delta(n^2(x,z))e^{\mp i\beta_m z}$ 包含有 $e^{-i\beta_s z}$ 或 $e^{i\beta_s z}$ 项

假设 $\Delta(n^2(x,z))$ 是周期函数，其周期为 Λ 且 矩形 则：$\Delta n^2(x,z)=\Delta n^2(x)\sum_{q=-\infty}^{\infty}a_q e^{i(2q\pi/\Lambda)z}$ 代入上式

且取 q=l，m=s 则 $\frac{2l\pi}{\Lambda}-\beta_s\approx\beta_s$

$=\beta_0$

$$\frac{dA_s^{(-)}}{dz}=\frac{i\omega\varepsilon_0}{4}A_s^{(+)}\int_{-\infty}^{\infty}\Delta n^2(x)[\mathscr{E}_y^{(s)}(x)]^2\,dx\,a_l e^{i[(2l\pi/\Lambda)-2\beta_s]z}$$

$$\frac{dA_s^{(-)}}{dz}=\kappa A_s^{(+)}e^{-i2(\Delta\beta)z}$$

$$\frac{dA_s^{(+)}}{dz}=\kappa^*A_s^{(-)}e^{i2(\Delta\beta)z}$$

耦合系数 $\kappa=\frac{i\omega\varepsilon_0 a_l}{4}\int_{-\infty}^{\infty}\Delta n^2(x)[\mathscr{E}_y^{(s)}(x)]^2\,dx$

位相失配量 $\Delta\beta\equiv\beta_s-\frac{l\pi}{\Lambda}\equiv\beta_s-\beta_0$

功率守恒

$$\frac{d}{dz}[|A_s^{(-)}|^2-|A_s^{(+)}|^2]=0$$

$$\frac{dA}{dz} = \kappa_{ab} B e^{-i2(\Delta\beta)z}$$

$$\frac{dB}{dz} = \kappa_{ab}^* A e^{+i2(\Delta\beta)z}$$

$$\Delta\beta = \beta - \beta_0$$

$$\beta_0 = l\pi/\Lambda, \; l = 1, 2, 3, \; . \; . \; .$$

$$A(z)e^{i\beta z} = B(0)\frac{i\kappa_{ab}e^{i\beta_0 z}}{-\Delta\beta \sinh(SL) + iS\cosh(SL)} \sinh[S(z-L)]$$

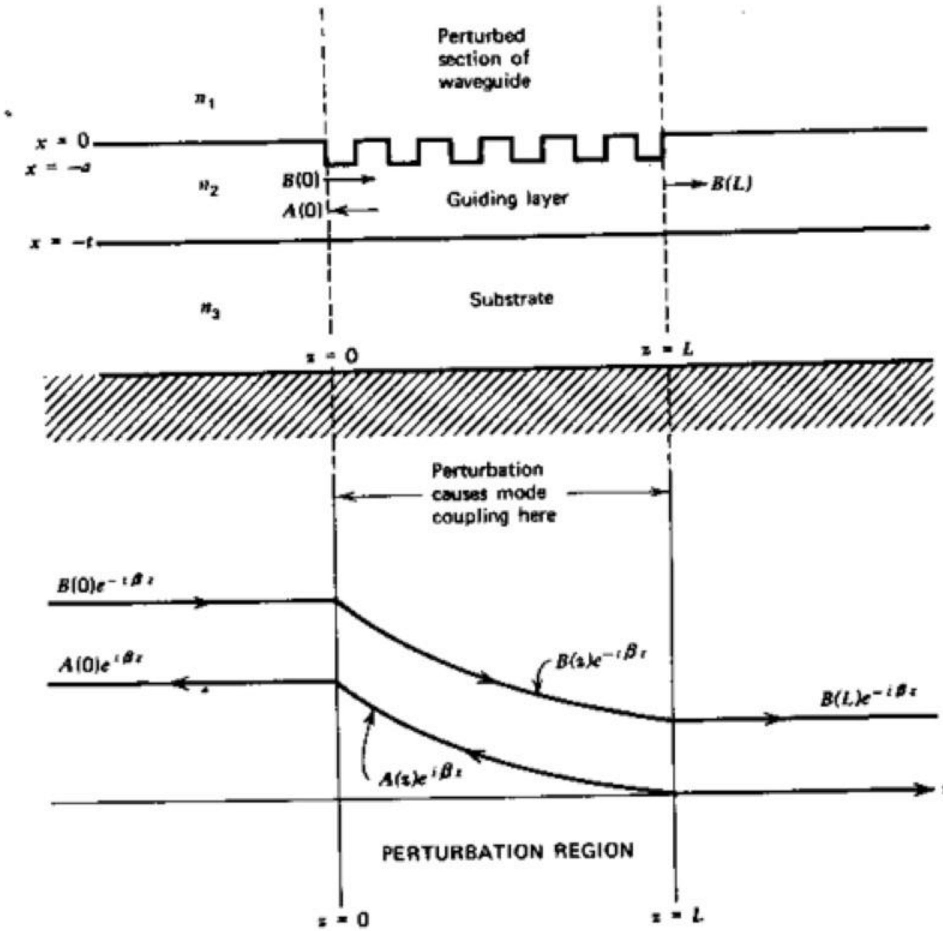
$$B(z)e^{-i\beta z} = B(0)\frac{e^{-i\beta_0 z}}{-\Delta\beta \sinh(SL) + iS\cosh(SL)}$$

$$\cdot\{\Delta\beta \sinh[S(z-L)] + iS\cosh[S(z-L)]\}$$

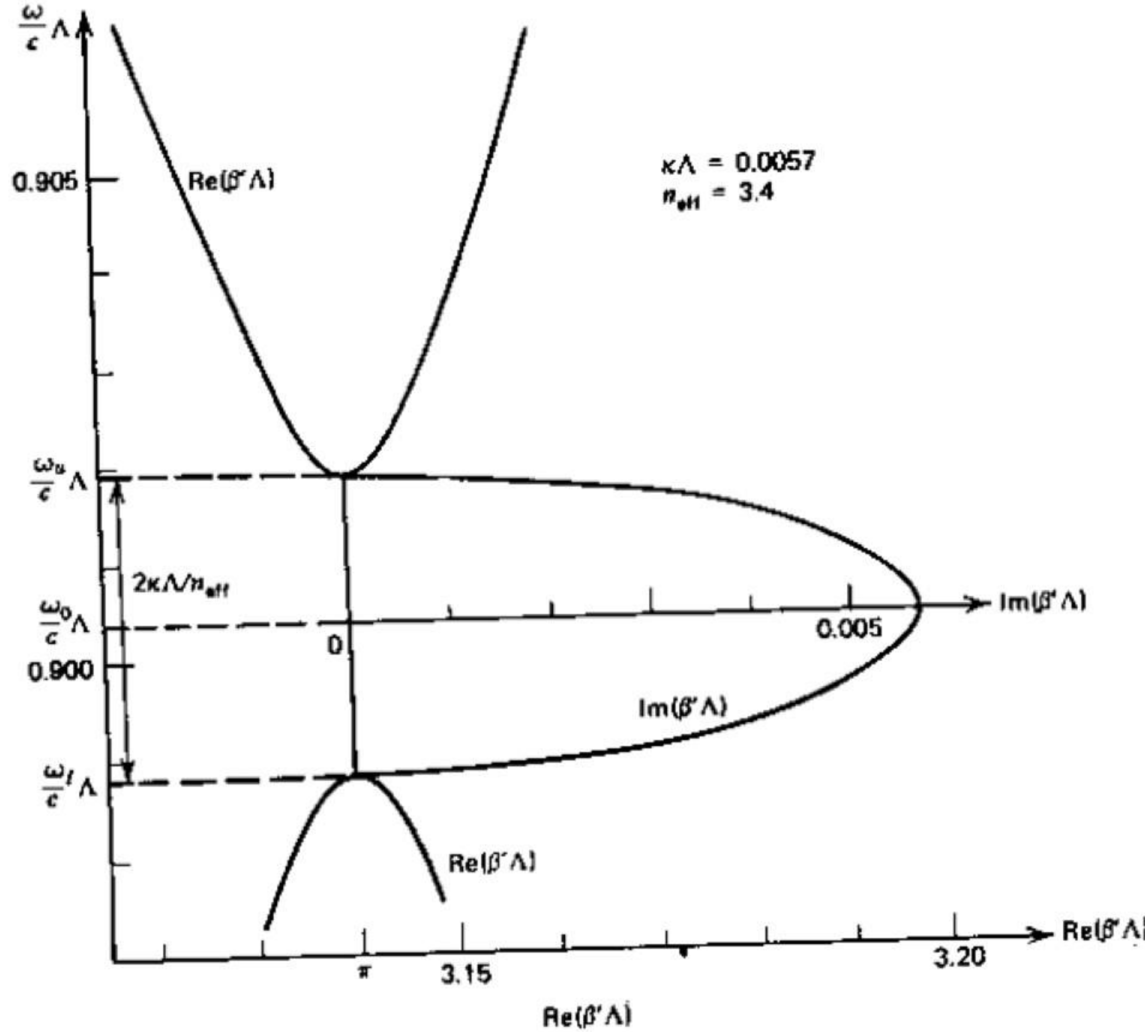
$$S = \sqrt{\kappa^2 - (\Delta\beta)^2}, \qquad \kappa \equiv |\kappa_{ab}|$$

$$A(z) = B(0)\left(\frac{\kappa_{ab}}{\kappa}\right)\frac{\sinh[\kappa(z-L)]}{\cosh(\kappa L)}$$

$$B(z) = B(0)\frac{\cosh[\kappa(z-L)]}{\cosh(\kappa L)}$$



$$(\Delta\omega)_{gap} = \frac{2\kappa c}{n_{eff}}$$



$$\beta')_{max} = \kappa$$

$$\beta(\omega) \approx (\omega/c)n_{eff}$$

$$\beta' = \beta_0 \pm iS = l\frac{\pi}{\Lambda} \pm i\sqrt{\kappa^2 - [\beta(\omega) - \beta_0]^2}$$

$$\beta' \cong \frac{l\pi}{\Lambda} \pm i\left[\kappa^2 - \left(\frac{n_{eff}}{c}\right)^2(\omega - \omega_0)^2\right]^{1/2}$$

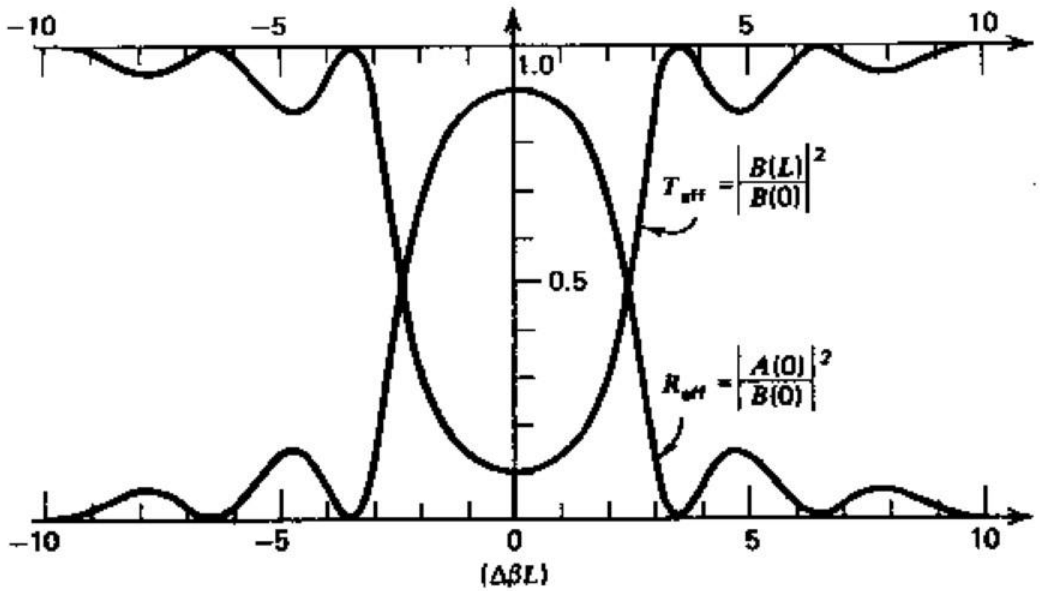


FIGURE 22.7 The transmission and reflection characteristics of a corrugated section of length L as a function of the detuning $\Delta\beta L \approx [(\omega - \omega_0)L/c]n_{eff}$. ($\kappa L = 1.84$.)

$$T_{eff} = \left|\frac{B(L)}{B(0)}\right|^2$$

$$R_{eff} = \left|\frac{A(0)}{B(0)}\right|^2$$

$$A(z)e^{i\beta z} = B(0)\,\frac{i\kappa_{ab}e^{i\beta_0 z}}{-\Delta\beta\sinh(SL) + iS\cosh(SL)}\sinh[S(z-L)]$$

$$B(z)e^{-i\beta z} = B(0)\,\frac{e^{-i\beta_0 z}}{-\Delta\beta\sinh(SL) + iS\cosh(SL)} \cdot \{\Delta\beta\sinh[S(z-L)] + iS\cosh[S(z-L)]\}$$

分布反馈激光

基本原理：接近Bragg频率处，周期介质具有足够高的增益时，无需端面反射镜也可以产生激光振荡。周期调制可以作用在整个波导层或者波导边界上。反馈由相干后向散射提供。

(1)整个波导层被周期性调制

$$n(z) = n + n_1 \cos 2\beta_0 z$$

$$\gamma(z) = \gamma + \gamma_1 \cos 2\beta_0 z$$

$$k^2 = \omega^2 \mu \varepsilon = \omega^2 \mu (\varepsilon_r + i\varepsilon_i)$$

$$= k_0^2 n^2(z) \left[1 + i \frac{2\gamma(z)}{k_0 n} \right]$$

$$k^2(z) = k_0^2 n^2 + i2k_0 n\gamma + 4k_0 n \left(\frac{\pi n_1}{\lambda} + i\frac{\gamma_1}{2} \right) \cos 2\beta_0 z$$

$$\kappa = \frac{\pi n_1}{\lambda} + i\frac{\gamma}{2}$$

$$k^2(z) = \beta^2 + i2\beta\gamma + 4\beta\kappa \cos(2\beta_0 z)$$

$$\frac{d^2E}{dz^2}+k^2(z)E=0$$

$$k^2(z)=\beta^2+i2\beta\gamma+4\beta\varkappa\cos(2\beta_0 z)$$

$$\frac{d^2E}{dz^2}+[\beta^2+i2\beta\gamma+4\beta\varkappa\cos(2\beta_0 z)]E=0$$

$$E(z)=A'(z)e^{i\beta' z}+B'(z)e^{-i\beta' z}$$

$$\beta'^2=\beta^2+i2\beta\gamma$$

$$(\beta'\simeq\beta+i\gamma,\ \gamma\ll\beta)$$

$$\frac{d^2}{dz^2}[A'(z)e^{i\beta' z}]=-\left(\beta'^2A'-2i\beta'\frac{dA'}{dz}-\frac{d^2A'}{dz^2}\right)e^{i\beta' z}$$

慢变振幅近似： $d^2A'/dz^2\ll\beta'(dA'/dz)$

$$i\beta'\frac{dA'}{dz}e^{i\beta' z}-i\beta'\frac{dB'}{dz}e^{-i\beta' z}$$

$$=-\beta\varkappa e^{i(2\beta_0-\beta')z}B'-\beta\varkappa e^{-i(2\beta_0-\beta')z}A'$$

$$-\beta\varkappa e^{i(2\beta_0+\beta')z}A'-\beta\varkappa e^{-i(2\beta_0+\beta')z}B'$$

在近Bragg条件下，$\beta\approx\beta_0$，保留同步项

$$\frac{dA'}{dz}=i\varkappa B'e^{-i2(\beta'-\beta_0)z}=i\varkappa B'e^{-i2(\Delta\beta+i\gamma)z}$$

$$\frac{dB'}{dz}=-i\varkappa A'e^{+i2(\beta'-\beta_0)z}=-i\varkappa A'e^{i2(\Delta\beta+i\gamma)z}$$

$$\Delta\beta\equiv\beta-\beta_0$$

(2)波导边界被周期调制

$$\frac{dA}{dz} = \kappa_{ab} B e^{-i2(\Delta\beta)z}$$

$$\Delta\beta = \beta - \beta_0$$

$$\frac{dB}{dz} = \kappa_{ab}^{*} A e^{+i2(\Delta\beta)z} \qquad \beta_0 = l\pi/\Lambda,\ l = 1, 2, 3, \ldots$$

引入增益

$$\frac{dA}{dz} = \kappa_{ab} B e^{-i2(\Delta\beta)z} - \gamma A$$

$$\frac{dB}{dz} = \kappa_{ab}^{*} A e^{i2(\Delta\beta)z} + \gamma B$$

$$A(z) = A'(z) e^{-\gamma z}$$

$$B(z) = B'(z) e^{\gamma z}$$

$$\frac{dA'}{dz} = \kappa_{ab} B' e^{-i2(\Delta\beta + i\gamma)z}$$

$$\frac{dB'}{dz} = \kappa_{ab}^{*} A' e^{+i2(\Delta\beta + i\gamma)z}$$

$$\Delta\beta \to \Delta\beta + i\gamma$$

$$B'(z) e^{[(-i\beta+\gamma)z]} = B(0) \frac{e^{-i\beta_0 z}\{(\gamma - i\Delta\beta)\sinh[S(L-z)] - S\cosh[S(L-z)]\}}{(\gamma - i\Delta\beta)\sinh(SL) - S\cosh(SL)}$$

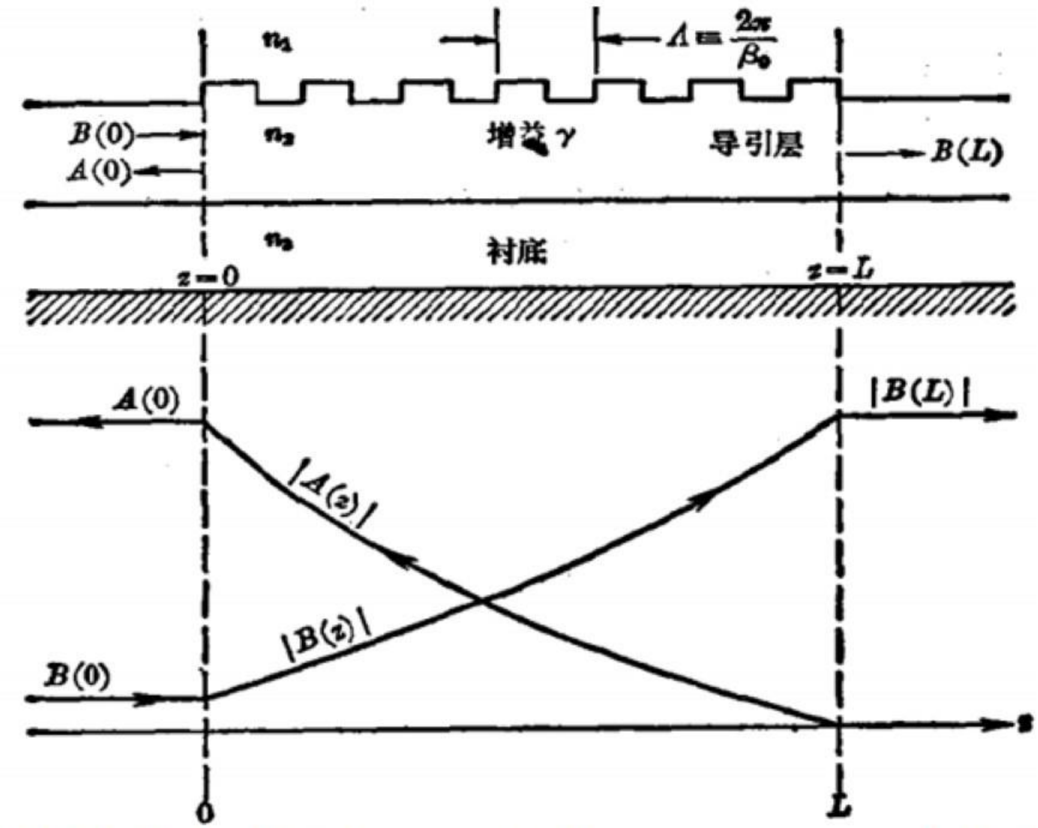
$$A'(z) e^{[(i\beta-\gamma)z]} = B(0) \frac{\kappa_{ab} e^{i\beta_0 z}\sinh[S(L-z)]}{(\gamma - i\Delta\beta)\sinh(SL) - S\cosh(SL)}$$

$$S^2 = \kappa^2 + (\gamma - i\Delta\beta)^2, \qquad \kappa^2 = |\kappa_{ab}|^2$$

反射率: $$\frac{E_r(0)}{E_i(0)}=\frac{\varkappa_{ab}\sinh(SL)}{(\gamma-i\Delta\beta)\sinh(SL)-S\cosh(SL)}$$

透射率: $$\frac{E_i(L)}{E_i(0)}=\frac{-Se^{-i\beta_0 L}}{(\gamma-i\Delta\beta)\sinh(SL)-S\cosh(SL)}$$

激射条件: $$(\gamma-i\Delta\beta)\sinh(SL)=S\cosh(SL)$$



有增益的周期性波导在接近Bragg条件$\beta \simeq l\pi / \Lambda$ 时的入射场和反射场

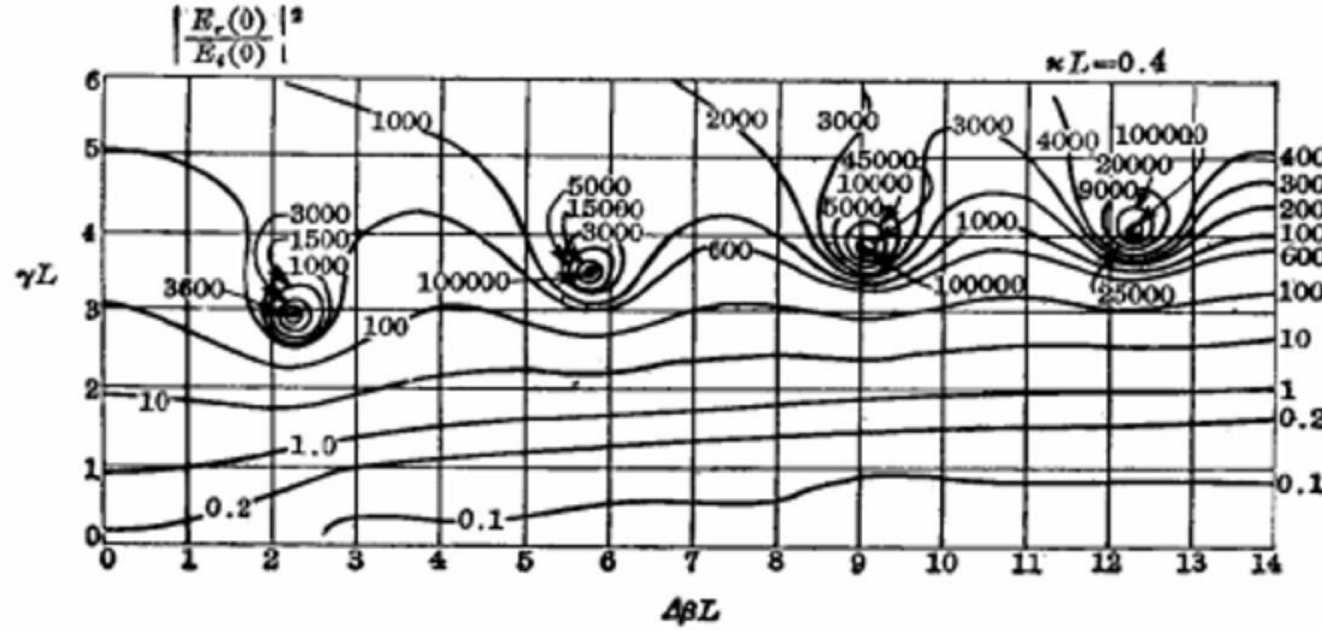


图 19.10　在 $\Delta\beta L$–γL 平面内反射增益的等高线

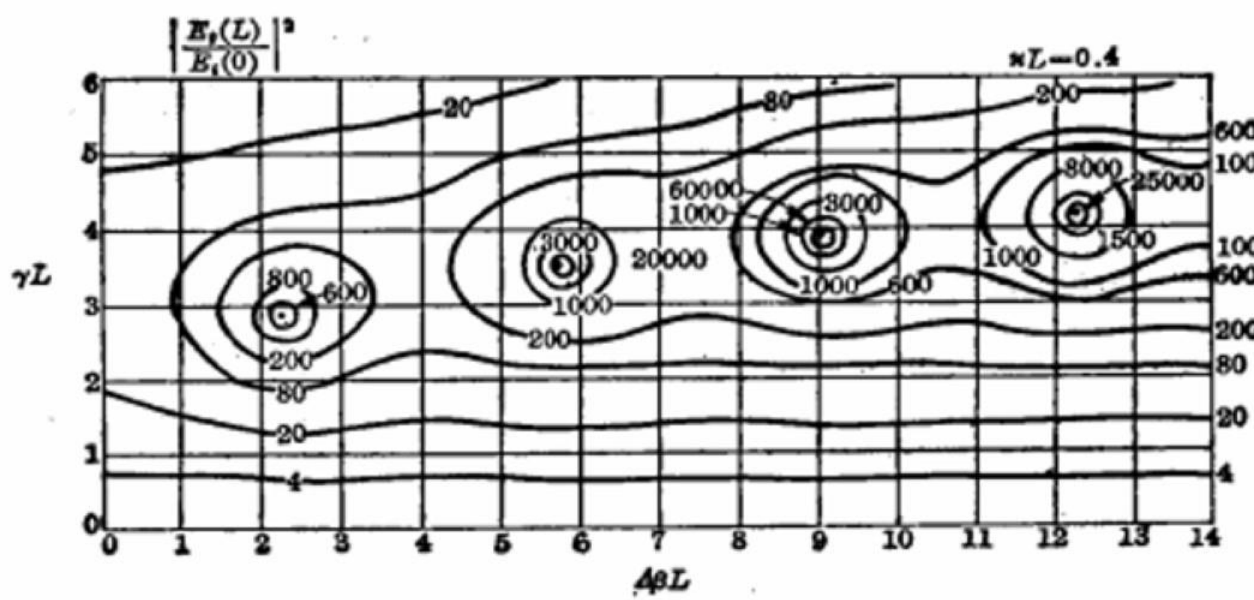


图 19.11　在 $\Delta\beta L$–γL 平面内透过增益的等高线

分布反馈激光振荡条件

$$\frac{S-(\gamma-i\Delta\beta)}{S+(\gamma-i\Delta\beta)}e^{2SL}=-1$$

高增益近似下，$\gamma>>\kappa$

$$S\simeq-(\gamma-i\Delta\beta)\left[1+\frac{\varkappa^2}{2(\gamma-i\Delta\beta)^2}\right]$$

$$S-(\gamma-i\Delta\beta)=-2(\gamma-i\Delta\beta)$$

$$S+(\gamma-i\Delta\beta)=\frac{-\varkappa^2}{2(\gamma-i\Delta\beta)}$$

$$\frac{+4(\gamma-i\Delta\beta)^2}{\varkappa^2}e^{2SL}=-1$$

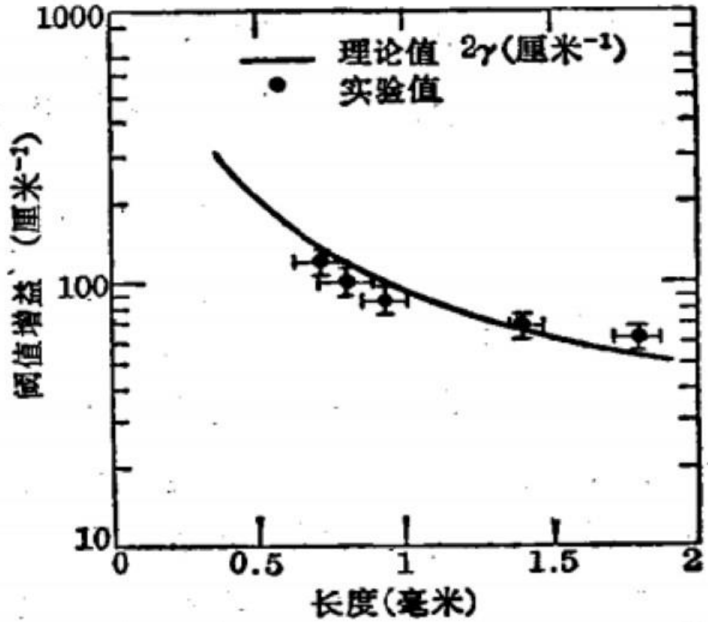


图 19.13 分布反馈激光器阈值增益特性的实验数据和理论曲线[19]

位相条件(激光频率)

$$2\tan^{-1}\frac{(\Delta\beta)_m}{\gamma_m}-2(\Delta\beta)_m L+\frac{(\Delta\beta)_m L\varkappa^2}{\gamma_m^2+(\Delta\beta)_m^2}=-(2m+1)\pi,\quad m=0,\ \pm1,\ \pm2,\ \cdots$$

$$(\Delta\beta)_m L\simeq-\left(m+\frac{1}{2}\right)\pi$$

振幅条件(阈值增益)

$$\frac{e^{2\gamma_m L}}{\gamma_m^2+(\Delta\beta)_m^2}=\frac{4}{\varkappa^2}$$

激光频率： $\omega_m=\omega_0-\left(m+\frac{1}{2}\right)\frac{\pi c}{n_{有效}L}$

模式频率间隔：$\omega_m-\omega_{m-1}\simeq\frac{\pi c}{n_{有效}L}$