现代数学物理方法

第一章, 群论基础

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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几句说在课前的话

任课教师信息:

姓名：杨焕雄
电话：18949882795
邮箱：hyang@ustc.edu.cn

教学内容:

此次教学的主体内容是物理学中出现的李群、李代数(重点).

教材与参考书推荐:

- H. Georgi, Lie algebras in particle physics, 2e, CRC, 2018
- A. M. Bincer, Lie groups and Lie algebras, a physicist's perspective, OUP, 2013
- A. Zee, Group theory in a nutshell for physicits, PUP, 2016

Why Group Theory ?

Group Theory is the study of symmetries.

Symmetries in Physics :

- Gauss law in electrostatics,

$$\oint \vec{E}\cdot d\vec{s} = Q/\epsilon_0 \qquad \rightsquigarrow \qquad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q\,\vec{r}}{r^3}$$

- The dynamical law for a charged particle in electromagnetic field,

$$\frac{d\vec{p}}{dt} = q\vec{E} + \vec{J}\times\vec{B}$$

- Lagrangian describing Strong, weak and electromagnetic interactions,

$$\mathscr{L}_{\text{int}} \sim ig\bar{\Psi}\gamma^{\mu}\Psi T^{i} A_{\mu}^{i}$$

Group:

A group G is a set of elements with a rule for assigning to every (ordered) pair of elements, satisfying

- If $f, g \in G$, then $fg \in G$.
- For $f, g, h \in G$, $f(gh) = (fg)h$.
- There is an identity element, e, such that for all $f \in G$, $ef = fe = f$.
- Every element $f \in G$ has an inverse, f^{-1}, such that $ff^{-1} = f^{-1}f = 1$.

Therefore, a group G is a multiplication table specifying $g_1 g_2$ for both g_1 and g_2 belonging to G. e.g.,

	e	g_1	g_2
e	e	g_1	g_2
g_1	g_1	$g_1 g_1$	$g_1 g_2$
g_2	g_2	$g_2 g_1$	$g_2 g_2$

Focus:

Our focus in this course will be on the Group Representation Theory.

Group Representations:

A representation $D(G)$ of group G is a mapping between the elements $g \in G$ and a set of linear operators $D(g)$ with the properties,

1. $D(e) = 1$
2. $D(g_1)D(g_2) = D(g_1 g_2)$

The representation of a group G does also form a group.

Finite group: Z_3

A group is finite if it has a finite number of elements. The number of elements in a finite group G is called the order of G.
The group Z_3 is a finite group of order 3.

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Notice that every row and column of the multiplication table contains each group elements exactly once. This is because

$$a^2 = b, \;\; b^2 = a, \;\; ab = ba = e \quad \rightsquigarrow \; e^{-1} = e, \; a^{-1} = b$$

An Abelian group is one in which the multiplication of arbitrary two elements is commutative,

$$g_1 g_2 = g_2 g_1$$

Evidently, Z_3 is Abelian.

Finite group: Z_3

A representation of Z_3:

$$D(e) = 1, \;\; D(a) = e^{2\pi i/3}, \;\; D(b) = e^{-2\pi i/3}.$$

Multiplication table reads,

	$D(e)$	$D(a)$	$D(b)$
$D(e)$	$D(e)$	$D(a)$	$D(b)$
$D(a)$	$D(a)$	$D(b)$	$D(e)$
$D(b)$	$D(b)$	$D(e)$	$D(a)$

=

	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
1	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
$e^{2\pi i/3}$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1
$e^{-2\pi i/3}$	$e^{-2\pi i/3}$	1	$e^{2\pi i/3}$

The dimension of a representation is the dimension of the linear space on which the operators in the representation act. Hence, *The above representation of Z_3 is 1-dimensional.*

Regular Representation

Here is another representation of Z_3, which is 3-dimensional,

$$D(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; D(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; D(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

This is called the regular representation of Z_3.

Definition :

The regular representation of a group is constructed by taking the group elements $\{g_1, \; g_2, \; \cdots\}$ themselves as the orthonormal base vectors $\{|g_1\rangle, \; |g_2\rangle, \; \cdots\}$ of the representation space,

$$D_{\text{reg}}(g_1)\,|g_2\rangle = |g_1 g_2\rangle$$

Hence,

$$[D_{\text{reg}}(g)]_{ij} = \langle g_i|\,D_{\text{reg}}(g)\,|g_j\rangle = \langle g_i|g g_j\rangle$$

The dimension of $D_{\text{reg}}(G)$ is the order of group G.

$D_{\text{reg}}(Z_3)$:

We now construct the regular representation of Z_3. Let $|1\rangle = |e\rangle$, $|2\rangle = |a\rangle$ and $|3\rangle = |b\rangle$ and

$$\langle i|j\rangle = \delta_{ij}, \quad \sum_{i=1}^{3} |i\rangle\langle i| = 1,$$

we get

$$[D_{\text{reg}}(a)]_{11} = \langle e|ae\rangle = \langle e|a\rangle = 0, \quad [D_{\text{reg}}(a)]_{12} = \langle e|aa\rangle = \langle e|b\rangle = 0,$$
$$[D_{\text{reg}}(a)]_{13} = \langle e|ab\rangle = \langle e|e\rangle = 1, \quad [D_{\text{reg}}(a)]_{21} = \langle a|ae\rangle = \langle a|a\rangle = 1,$$
$$[D_{\text{reg}}(a)]_{22} = \langle a|aa\rangle = \langle a|b\rangle = 0, \quad [D_{\text{reg}}(a)]_{23} = \langle a|ab\rangle = \langle a|e\rangle = 0,$$
$$[D_{\text{reg}}(a)]_{31} = \langle b|ae\rangle = \langle b|a\rangle = 0, \quad [D_{\text{reg}}(a)]_{32} = \langle b|aa\rangle = \langle b|b\rangle = 1,$$
$$[D_{\text{reg}}(a)]_{33} = \langle b|ab\rangle = \langle b|e\rangle = 0.$$

Namely,

$$D(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Similarly we can get another matrices $D_{\rm reg}(e)$ and $D_{\rm reg}(b)$ of the regular representation of group Z_3.

Trace of a matrix is defined as the sum of its diagonal elements. Therefore, for a regular representation of a group G, we have:

$$\mathrm{Tr}[D_{\rm reg}(e)] = N, \quad \mathrm{Tr}[D_{\rm reg}(g)] = 0 \; (g \neq e),$$

where N is the order of the group G.

- A general p-dimensional representation of G is spanned by p orthonormal base vectors $\{|1\rangle, \; |2\rangle, \cdots, |p\rangle\}$ satisfying the conditions $\langle i|j\rangle = \delta_{ij}$ and $\sum_i |i\rangle\langle i| = 1$. The representation matrices are defined as:

$$[D(g)]_{ij} = \langle i| D(g) |j\rangle, \quad g \in G$$

These matrices do indeed form a representation of the G, relying on the fact $D(g_1 g_2) = D(g_1)D(g_2)$.

Equivalent Representations

What makes the idea of group representations so powerful is the fact that they live in linear spaces. The powerful thing about linear spaces is that we are free to choose the base vectors (states) by making a linear transformation, $|\psi\rangle \rightsquigarrow |\psi'\rangle = S^{-1}|\psi\rangle$.

Such a transformation on the base vectors of the linear space induces a similarity transformation on the linear operators:

$$D(g) \quad \rightsquigarrow D'(g) = S^{-1}D(g)S$$

Obviously, $D'(G)$ is a representation of G if $D(G)$ is,

1. $D'(e) = 1$
2. $D'(g_1g_2) = D'(g_1)D'(g_2)$

$D'(G)$ and $D(G)$ are said to be equivalent because they differ just by a trivial choice of base vectors.

Unitary Representations:

1. A representation of group $G = \{g\}$ is unitary if and only if all the matrix elements $\{D(g)\}$ of $D(G)$ are unitary,

$$[D(g)]^{\dagger} = [D(g)]^{-1}, \quad \forall g \in G$$

2. It will turn out that all representations of finite groups are equivalent to unitary representations.

Examples:

Both given two representations of Abelian group Z_3 are unitary:

- 1-dimensional representation:

$$D_1(e) = 1, \;\; D_1(a) = e^{2\pi i/3}, \;\; D_1(b) = e^{-2\pi i/3}.$$

- 3-dimensional representation:

$$D_2(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; D_2(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$D_2(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Reducible Representations:

A representation is called reducible if it has an invariant subspace: *the action of any $D(g)$ on any vector in the subspace is still in the subspace.*

Projection operator:

Let P_1 be the projection operator of the subspace S_1 of space S, then

1. $P_1 S = S_1$
2. $P_1^2 = P_1$

Consequently, P_1 is an identity operator on S_1: $P_1 |\varphi\rangle = |\varphi\rangle$, $\forall |\varphi\rangle \in S_1$.

If $D(G)$ has an invariant subspace (so that D is reducible), we have:

$$(1 - P_1) D(g) P_1 = 0, \quad \forall g \in G$$

$$\rightsquigarrow \qquad D(g) P_1 \sim P_1, \qquad \forall g \in G$$

Examples :

- The trivial $D = \{D(g) = 1, \ \forall g \in G\}$ of every group G is a reducible representation.

- The regular representation of Z_3 is reducible, due to the fact it has an invariant subspace projected on by

$$P = P^2 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Checking : Because

$$D_{\text{reg}}(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{\text{reg}}(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$D_{\text{reg}}(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

we have:

$$D_{\text{reg}}(g)P = P, \quad \forall g \in Z_3$$

Irreducible Representations:

A representation is *irreducible* if it has no nontrivial invariant space.

Completely Reducible Representations:

A representation is *completely reducible* if it is equivalent to a representation whose matrix elements have the following block diagonal form:

$$D(g) = \begin{bmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad \forall \ g \in G$$

where $D_j(G) = \{D_j(g)\}$ are *irreducible* representations of G for all subscripts j.

1. A representation D in block diagonal form is said to be the direct sum of the sub-representations D_j,

$$D = D_1 \oplus D_2 \oplus \cdots \oplus D_M = \oplus_{j=1}^{M} D_j$$

Consequently, *A completely reducible representation can be decomposed into a direct sum of irreducible representations.*

Question:
Construct a similarity transformation so that the regular representation of Z_3 is written as the direct sum of some of its irreducible representations.

Solution:
Consider the unitary matrix S,

$$S = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{bmatrix}$$

we see:

1.

$$
\begin{aligned}
D'_{\text{reg}}(e) &= S^\dagger D_{\text{reg}}(e) S \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

2.

$$
\begin{aligned}
D'_{\text{reg}}(a) &= S^\dagger D_{\text{reg}}(a) S \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{bmatrix}
\end{aligned}
$$

3.

$$
\begin{aligned}
D'_{\text{reg}}(b) &= S^\dagger D_{\text{reg}}(b) S \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{bmatrix}\begin{bmatrix} 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{bmatrix}
\end{aligned}
$$

Hence, in $D_{\text{reg}}(Z_3)$, the involved irreducible representations of Abelian group $Z_3 = \{e, a, b\}$ are

1. $D_1(Z_3) = \{1, 1, 1\}$
2. $D_2(Z_3) = \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$
3. $D_3(Z_3) = \{1, e^{-2\pi i/3}, e^{2\pi i/3}\}$

All of these irreducible representations are 1-dimensional.

Transformation Groups:

There is a natural multiplication law for transformations of a physics system.

If the transformation group $G = \{g\}$ is the symmetry of a quantum mechanical system, then,

- For each group element g, there is a unitary operator $D(g)$ that maps the Hilbert space into itself,

$$D(g): \quad |\psi\rangle \quad \rightarrow \quad |\psi'\rangle = D(g)\,|\psi\rangle$$

- The full set of these unitary operators $\{D(g)\}$ form a unitary representation of G on the Hilbert space.

- The transformed states are subject to the same Schrödinger equation as the original states,

$$\left.\begin{aligned} i\hbar\tfrac{d}{dt}\,|\psi\rangle &= H|\psi\rangle \\ i\hbar\tfrac{d}{dt}\,[D(g)\,|\psi\rangle] &= H\,[D(g)\,|\psi\rangle] \end{aligned}\right\} \quad \rightsquigarrow \quad [D(g), H] = 0$$

$[D(g), H] = 0$ implies:

1. The transformed states have the same energy as the original states.
2. The full set of the energy eigenstates belonging to the same energy eigenvalue forms a complete set of basis vectors of an irreducible representation of the transformation group G.

Problems:

1. Find the multiplication table for a group with 3 elements and prove that it is unique.
2. Find all essentially different multiplication tables for groups with 4 elements (which can not be related by renaming elements).

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Parity:

Parity is the operation of reflection in a mirror. *Reflecting twice gets you back to where you started,*

$$p^2 = e$$

The group including parity operation is Z_2:

	e	p
e	e	p
p	p	e

Representations of Z_2:

- Z_2 has only 2 irreducible representations. The first one is trivial,

$$D_1(e) = D_1(p) = 1.$$

- The second irreducible representation of Z_2 consists of

$$D_2(e) = 1, \;\; D_2(p) = -1.$$

- *Any representation of Z_2 is completely reducible.* The Hilbert space of any parity invariant system can be decomposed into states that behave like irreducible representations, on which $D(p)$ is either 1 or -1.
  1. The energy eigensates on which $D(p) = 1$ have an even parity.
  2. The energy eigensates on which $D(p) = -1$ have an odd parity.

Definition:

S_3 is the permutation group (or symmetric group) on 3 objects,

$$a_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = (123) = (231) = (312)$$

$$a_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = (132) = (213) = (321)$$

$$a_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = (12) = (21)$$

$$a_4 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = (23) = (32)$$

$$a_5 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = (13) = (31)$$

$$e = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Properties:

Basically,

1. $(ab) = (ba)$
2. $(ab)(ba) = e$
3. $(ab)(bc) = (abc)$

In general,

1. $(123 \cdots N) = (12)(23)(34) \cdots (N-1, N)$
2. $(123 \cdots N) = (1N)(1, N-1)(1, N-2) \cdots (13)(12)$

$$\rightsquigarrow \quad a_1 a_2 = (123)(321) = e, \quad a_1 a_3 = (123)(12) = (13) = a_5$$

Generators:

S_3 has *two* generators. They can be chosen as

$$\{a_1 = (123), \;\; a_3 = (12)\}$$

From these generators, we have $a_2 = a_1 a_1$, $a_4 = a_3 a_1$, $a_5 = a_1 a_3$ and $e = a_1 a_1 a_1 = a_3 a_3$.

Non-Abelian:

S_3 is non-abelian because its multiplication law is not commutative. *e.g.*,

$$a_4 = a_3 a_1 \neq a_1 a_3 = a_5$$

It is the lack of commutativity that makes group theory very useful in *physics*.

Multiplication Table of S_3:

	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	e	a_5	a_3	a_4
a_2	a_2	e	a_1	a_4	a_5	a_3
a_3	a_3	a_4	a_5	e	a_1	a_2
a_4	a_4	a_5	a_3	a_2	e	a_1
a_5	a_5	a_3	a_4	a_1	a_2	e

Permutation group is an important transformation group in quantum mechanics, in particular in the system of identical particles.

An irreducible representation of S_3:

$$D(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D(a_1) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D(a_2) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \qquad D(a_3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D(a_4) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \qquad D(a_5) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

Discussions:

- The nontrivial representations of a non-Abelian group must be *matrices* rather than numbers. Only matrices can reproduce the non-commutative multiplication laws.
- In an irreducible representation, Not all of the matrices are diagonal.

Question:

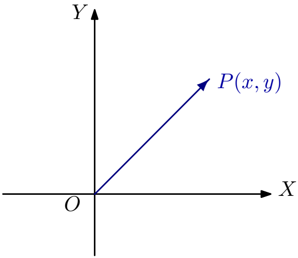
How to obtain this representation ?

My Explanation:

The two generators of S_3 obey,

$$(a_1)^3 = (a_3)^2 = 1$$

We can identify a_1 by a rotation in XY plane at an angle $2\pi/3$ with respect to X-axis, and a_3 a reflection about Y-axis. Therefore, on an arbitrary vector, $\vec{r} = x\vec{i} + y\vec{j} \sim \begin{bmatrix} x \\ y \end{bmatrix}$,



we have:

$$D(a_3)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

Hence,

$$D(a_3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly,

$$D(a_1) = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

Based on these two generators, we get:

$$
\begin{aligned}
D(a_2) &= [D(a_1)]^2 \\
&= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \\
&= \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
D(a_4) &= D(a_3)D(a_1) \\
&= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
D(a_5) &= D(a_1)D(a_3) \\
&= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}
\end{aligned}
$$

Of course,

$$
\begin{aligned}
D(e) &= [D(a_3)]^2 \\
&= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Addition of integers:

The integers form an infinite group $\mathbb{Z}$ under addition:

$$x \circ y := x + y$$

Checking:

1. If x and y are integers, $x + y$ is also an integer.
2. For three integers x, y and z, $(x + y) + z = x + (y + z)$.
3. Identity element exists, $e = 0$.
4. Inverse elements exist, $x^{-1} = -x$.

Multiplication table:

Since this group is infinite, the explicit *multiplication table* for it is impossible.

The additive group $\mathbb{Z}$ has a representation as follows:

$$D(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad \forall x \in \mathbb{Z}$$

Checking:

$$D(e) = \begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D(x)D(y) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} = D(x+y)$$

This representation is reducible but it is not completely reducible.

Reducibility:

Construct the projection operator P for subspace spanned by the base vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Because

$$D(x)P_1 = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P_1$$

this representation is reducible.

However,

$$D(x)P_2 = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix} \neq P_2$$

Therefore, it is not completely reducible.

Theorem 1:

Every representation of a finite group is equivalent to a unitary representation.

Proof:

Suppose $D(G)$ is a representation of a finite group $G = \{g\}$, from which we can construct a hermitian matrix S,

$$S = \sum_{g \in G} \left[D(g)\right]^{\dagger} D(g)$$

Consider the eigenvalue equation of this hermitian matrix,

$$S\,|\lambda_n\rangle = \lambda_n\,|\lambda_n\rangle, \quad n = 1,\ 2,\ 3,\ \cdots$$

Hence,

$$\lambda_n = \langle\lambda_n|\,S\,|\lambda_n\rangle = \langle\lambda_n|\sum_{g \in G}\left[D(g)\right]^{\dagger} D(g)\,|\lambda_n\rangle = \sum_{g \in G}\left\|D(g)\,|\lambda_n\rangle\right\|^2$$

Proof (continued):

i.e.,

$$\lambda_n = \|D(e)\,|\lambda_n\rangle\|^2 + \cdots \geqslant \|D(e)\,|\lambda_n\rangle\|^2 = \||\lambda_n\rangle\|^2 > 0$$

All of the eigenvalues of the hermitian matrix S are not only *real* but also *positive*.

As is well known, a hermitian matrix can be diagonalized via a unitary transformation,

$$S = U^\dagger \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} U$$

Relying on the fact that $\lambda_n > 0$, the square root of S is also a hermitian matrix

$$X = \sqrt{S} = U^\dagger \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots \\ 0 & \sqrt{\lambda_2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} U$$

Proof (continued):

This hermitian matrix is invertible,

$$X^{-1} = \frac{1}{\sqrt{S}} = U^{\dagger} \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & \cdots \\ 0 & \frac{1}{\sqrt{\lambda_2}} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} U$$

Construct a similarity transformation with this invertible X, we have:

$$D'(g) = X\, D(g)\, X^{-1}, \quad \forall g \in G$$

The new representation $D'(G)$ is equivalent to the old representation $D(G)$. Moreover, *it is unitary*.

Proof (continued):

$$
\begin{aligned}
\left[D'(g)\right]^\dagger D'(g) &= \left[XD(g)X^{-1}\right]^\dagger XD(g)X^{-1} \\
&= (X^{-1})^\dagger \left[D(g)\right]^\dagger X^\dagger XD(g)X^{-1} \\
&= X^{-1}\left[D(g)\right]^\dagger X^2 D(g)X^{-1} \\
&= X^{-1}\left[D(g)\right]^\dagger SD(g)X^{-1} \\
&= X^{-1}\left[D(g)\right]^\dagger \left\{ \sum_{h\in G} \left[D(h)\right]^\dagger D(h) \right\} D(g)X^{-1} \\
&= X^{-1}\left\{ \sum_{h\in G} \left[D(hg)\right]^\dagger D(hg) \right\} X^{-1} \\
&= X^{-1}SX^{-1} = 1
\end{aligned}
$$

Theorem 2:

Every representation of a finite group is completely reducible.

Proof:

- It is sufficient to consider unitary representations.
- If the representation is irreducible, the required proof is achieved because it is already in block diagonal form.
- If the representation $D(G) = \{D(g)\}$ is reducible, there exists a projection operator P_1 such that

$$(1 - P_1)D(g)P_1 = 0, \quad \forall\, g \in G$$

Taking its hermitian conjugation gives,

$$\begin{aligned} 0 &= P_1\,[D(g)]^\dagger\,(1 - P_1) = P_1\,[D(g)]^{-1}\,(1 - P_1) \\ &= P_1 D(g^{-1})(1 - P_1), \quad \forall g \in G \end{aligned}$$

Proof (continued):

- Equivalently,

$$P_1 D(g)(1 - P_1) = 0, \quad \forall g \in G$$

This equation demonstrates that the subspace of the complementary projection operator $P_2 = (1 - P_1)$ is also invariant under $D(G)$:

$$(1 - P_2)D(g)P_2 = 0, \quad \forall\, g \in G$$

- By induction, we eventually completely reduce the representation $D(G)$.

Subgroups:

Subgroup :

A group H whose elements are all elements of a group G is called a subgroup of G.

Examples :

1. The identity e. (trivial)
2. The group G itself. (trivial)
3. $S_3 = \{e, a_1, a_2, a_3, a_4, a_5\}$ has the following nontrivial subgroups:

$$G_1 = \{e, a_1, a_2\}$$
$$G_2 = \{e, a_3\}$$
$$G_3 = \{e, a_4\}$$
$$G_4 = \{e, a_5\}$$

Cosets:

Right Coset of subgroup H:

The right coset of subgroup H in G is the set of elements of the form Hg for some *fixed* element $g \in G$.

Examples:
The cosets of subgroup $Z_3 = \{e, a_1, a_2\}$ of the permutation group S_3 consist of the following elements,

$$Z_3 a_1 = \{e, a_1, a_2\} a_1 = \{a_1, a_2, e\} = Z_3$$
$$Z_3 a_4 = \{e, a_1, a_2\} a_4 = \{a_4, a_3, a_5\}$$

Properties:

- The number of elements in each coset is the order of subgroup H.
- Every element of G must belong to one and only one coset.
- For a finite group G, the order of its subgroup H must be a factor of the order of G.

Coset space G/H:

It is the linear space in which each coset of subgroup H is taken as a single element.

Normal Subgroup:

A subgroup H of G is called an invariant or normal subgroup if for every $g \in G$,

$$gH = Hg$$

- The trivial subgroups e and G are normal for any group G.
- If H is normal, $gH = Hg$, the coset space G/H forms a group under the same multiplication law in G:

$$(Hg_1)(Hg_2) = H(g_1 H)g_2 = H(Hg_1)g_2 = H(g_1 g_2) \in G/H$$

In this case, the coset space G/H is called Factor group of G by H.

Normal subgroup of S_3:

1. Among the nontrivial subgroups of S_3, only is Z_3 the normal subgroup:

$$\begin{aligned}
eZ_3 &= e\{e, a_1, a_2\} = \{e, a_1, a_2\} = \{e, a_1, a_2\}e = Z_3e \\
a_1Z_3 &= a_1\{e, a_1, a_2\} = \{a_1, a_2, e\} = \{e, a_1, a_2\}a_1 = Z_3a_1 \\
a_2Z_3 &= a_2\{e, a_1, a_2\} = \{a_2, e, a_1\} = \{e, a_1, a_2\}a_2 = Z_3a_2 \\
a_3Z_3 &= a_3\{e, a_1, a_2\} = \{a_3, a_4, a_5\} = \{e, a_2, a_1\}a_3 = Z_3a_3 \\
a_4Z_3 &= a_4\{e, a_1, a_2\} = \{a_4, a_5, a_3\} = \{e, a_2, a_1\}a_4 = Z_3a_4 \\
a_5Z_3 &= a_5\{e, a_1, a_2\} = \{a_5, a_3, a_4\} = \{e, a_2, a_1\}a_5 = Z_3a_5
\end{aligned}$$

2. The other subgroups of S_3 are not normal subgroups. e.g.,

$$a_5\{e, a_4\} = \{a_5, a_2\} \neq \{a_5, a_1\} = \{e, a_4\}a_5$$

3. The factor group S_3/Z_3 is,

$$S_3/Z_3 = Z_2 \quad \rightsquigarrow \quad Z_2 \text{ is parity group.}$$

Center of a group:

The center of a group G is the set of all elements of G that commute with all elements of G.

Discussions:

1. The center is always an Abelian, normal subgroup of G.
2. It may be trivial, consisting only of the identity, or of the whole group G.

Homework:

1. There is a simple n-dimensional representation D of S_n called the defining representation, where the objects being permuted are just the basis vectors of an n-dimensional vector space:

$$|1\rangle, \ |2\rangle, \ \cdots, \ |n\rangle$$

The representation D is defined as $D[(\xi_j \xi_k)]\,|j\rangle = |k\rangle$. Show that this representation is reducible.

现代数学物理方法

第一章, 群论基础

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Conjugate elements:

Given two elements f and g of a group G, one can define the third element $g^{-1}fg \in G$. Let

$$g^{-1}fg = h$$

- Two elements f and h of G connected this way are called **conjugate**.
- If the element f is conjugate to h and h is conjugate to p, then f is conjugate of p.
- The set of all elements in G that are conjugate one another is called to form a **conjugacy class**. The element f is in the conjugacy class C_f, given by

$$C_f = \left\{g^{-1}fg, \ \forall g \in G\right\}$$

Conjugacy classes:

In a group G, the conjugacy class $S = \{g_1, g_2, \cdots\}$ consisting of some elements of G has the property

$$g^{-1}Sg = S, \quad \forall g \in G$$

Corollaries:

1. *A subgroup that is a union of conjugacy classes is a normal subgroup.*
2. *In an Abelean group, each group element forms an independent conjugacy class.*

Example:
Group S_3 has 3 conjugacy classes:

1. $C_1 = \{e\}$
2. $C_2 = \{a_1, a_2\}$
3. $C_3 = \{a_3, a_4, a_5\}$

Checking:

- The identity $\{e\}$ forms a conjugacy class itself, due to the fact that

$$g^{-1}eg = e, \quad \forall\, g \in S_3$$

- Moreover,

$$\begin{aligned}
(a_3)^{-1}a_1a_3 &= a_3a_1a_3 = a_4a_3 = a_2 \\
(a_4)^{-1}a_1a_4 &= a_4a_1a_4 = a_5a_4 = a_2 \\
(a_5)^{-1}a_1a_5 &= a_5a_1a_5 = a_3a_5 = a_2
\end{aligned}$$

  The set $C_2 = \{a_1, a_2\}$ forms another conjugacy class of S_3.

- Similar calculations yield,

$$\begin{aligned}
(a_1)^{-1}a_3a_1 &= a_2a_3a_1 = a_4a_1 = a_5 \\
(a_2)^{-1}a_3a_2 &= a_1a_3a_2 = a_5a_2 = a_4 \\
(a_4)^{-1}a_3a_4 &= a_4a_3a_4 = a_2a_4 = a_5 \\
(a_5)^{-1}a_3a_5 &= a_5a_3a_5 = a_1a_5 = a_4
\end{aligned}$$

  Namely, $C_3 = \{a_3, a_4, a_5\}$ forms the 3rd conjugacy class of S_3.

Other concepts in group theory:

1. An isomorphism is a *one-to-one* mapping of group onto another group that preserves the multiplication law.

2. An automorphism is a *one-to-one* mapping of a group onto itself that preserve the multiplication law.

3. An inner automorphism is an automorphism that can be cast as the mapping

$$G \rightarrow G' = gGg^{-1}$$

for a fixed group element $g \in G$.

4. An outer automorphism is an automorphism that can not be written as gGg^{-1} for any group element $g \in G$.

Schur's second lemma:

If

$$D_1(g)A = AD_2(g), \quad \forall g \in G$$

where D_1 and D_2 are inequivalent, irreducible representations of group G, then $A = 0$.

Proof:

The spaces and their dimensions of these two nonequivalent irreducible representations are denoted as $\mathcal{S}_1(d_1)$ and $\mathcal{S}_2(d_2)$ respectively, with $d_1 \geqslant d_2$.

Let A be an operator which maps from $\mathcal{S}_2$ into $\mathcal{S}_1$. When applied to $\mathcal{S}_2$, this A generates a subspace $\mathcal{S}_3$ of $\mathcal{S}_1$:

$$\mathcal{S}_3 = \{A|\Psi\rangle \in \mathcal{S}_1, \quad \forall |\Psi\rangle \in \mathcal{S}_2\}$$

with dimension $d_3 \leqslant d_2 \leqslant d_1$.

It follows from the proposed assumption that,

$$D_1(g)A\,|\Psi\rangle = AD_2(g)\,|\Psi\rangle = A\left[D_2(g)\,|\Psi\rangle\right] \equiv A|\Psi_g\rangle \quad \in \mathcal{S}_3$$

Because $|\Psi_g\rangle \equiv D_2(g)\,|\Psi\rangle \in \mathcal{S}_2$. Thus, $D_1(g)\mathcal{S}_3 = \mathcal{S}_3$. $\rightsquigarrow$ $\mathcal{S}_3$ is an invariant subspace of $\mathcal{S}_1$.

That $D_1(G)$ is an irreducible representation of G implies $\mathcal{S}_1$ *has no true invariant subspace.*

- Because $\mathcal{S}_3$ is an invariant subspace of $\mathcal{S}_1$, there is a contradiction unless $\mathcal{S}_3$ is either a null space ($A = 0$) or the full $\mathcal{S}_1$.
- The second possibility is excluded by the assumption that $D_1(G)$ and $D_2(G)$ are different (nonequivalent) representations[1].

Therefore, the single possibility $A = 0$ remains.

[1] The second possibility happens when $d_3 = d_1 = d_2$. However, if $d_2 = d_1$, we could invert A so that the two representations would be equivalent,

$$D_1(g) = AD_2(g)A^{-1}, \quad \forall g \in G.$$

Schur's first lemma:

If

$$D(g)A = AD(g), \quad \forall g \in G$$

where D is a finite dimensional irreducible representation of group G, then[a], $A \propto I$.

[a] In other words, if a matrix A commutes with all elements of a finite dimensional irreducible representation, it must be proportional to the unit matrix I.

Proof:

The condition of a finite dimensional representation is important. Any finite dimensional matrix A has at least one eigenvalue,

$$A|\lambda\rangle = \lambda|\lambda\rangle \rightsquigarrow (A - \lambda I)|\lambda\rangle = 0.$$

This is because the characteristic equation

$$\det(A - \lambda I) = 0$$

has at least one root for finite dimensional A.

Proof (continued):

Let P be the projection operator of the corresponding eigenstate $|\lambda\rangle$,

$$(A - \lambda I)P = 0$$

The assumption $D(g)A = AD(g)$ for all $g \in G$ does then imply,

$$(A - \lambda I)D(g)P = D(g)(A - \lambda I)P = 0$$

This equation has two possible solutions:

1. either $D(g)P \propto P$
2. or $A = \lambda I$

The first possibility is excluded because $D(G)$ is assumed to be an irreducible representation of G.

Consequently,

$$A = \lambda I \propto I$$

Remark:

Schur's first lemma can be alternatively written as,

$$A^{-1}D(g)A = D(g),\ \forall g \in G \quad \rightsquigarrow \quad A \propto I$$

for any irreducible representation $D(G)$.

Appendix:

In Schur's second lemma, the nonsingularity of matrix A if $A \neq 0$ can be justified as follows. By assumption, A satisfies the equality

$$D_1(g)A = AD_2(g), \quad \forall\ g \in G$$

where $D_1(G)$ and $D_2(G)$ could reasonably be assumed to be two unitary representations of G. Taking its Hermitian conjugate,

$$A^\dagger\Big[D_1(g)\Big]^\dagger = \Big[D_2(g)\Big]^\dagger A^\dagger \quad \rightsquigarrow \quad A^\dagger\Big[D_1(g)\Big]^{-1} = \Big[D_2(g)\Big]^{-1} A^\dagger$$

Since $\big[D(g)\big]^{-1} = D(g^{-1})$, the above equation can be recast as

$$A^\dagger D_1(g^{-1}) = D_2(g^{-1})A^\dagger, \quad \forall\, g \in G$$

Equivalently,

$$\boxed{A^\dagger D_1(g) = D_2(g)A^\dagger, \quad \forall\, g \in G}$$

By Combining this with $AD_2(g) = D_1(g)A$, which is the assumed equality in Schur's second lemma, we have:

$$\begin{aligned}(AA^\dagger)D_1(g) &= A\left[A^\dagger D_1(g)\right] = A\left[D_2(g)A^\dagger\right] \\ &= \left[AD_2(g)\right]A^\dagger = \left[D_1(g)A\right]A^\dagger = D_1(g)(AA^\dagger)\end{aligned}$$

Because $D_1(G)$ is assumed to be an irreducible representation of G,

$$AA^\dagger \propto I$$

according to Schur's first lemma.Therefore, A is nonsingular, $\det A \neq 0$.

Schur's lemma in QM:

Hilbert Space:

The orthonormal basis states of an QM object are of the form,

$$|a,\ j,\ x\rangle, \quad (1 \leqslant j \leqslant n_a)$$

where a labels the irreducible representation $D_a(G)$, j lables the states within $D_a(G)$ and x lables the other physical parameters. These states satisfy the relations:

$$\langle b,\ k,\ y|a,\ j,\ x\rangle = \delta_{ba}\delta_{kj}\delta_{yx}, \quad \sum_{a,j,x} |a,\ j,\ x\rangle\langle a,\ j,\ x| = I$$

Symmetry:

In QM, the symmetry is expressed as

$$\big[H, D(g)\big] = 0, \quad \forall g \in G$$

- Under the symmetry transformation, the states in Hilbert space transform like,

$$|\psi\rangle \to |\psi'\rangle = D(g)\,|\psi\rangle$$
$$\langle\psi| \to \langle\psi'| = \langle\psi|\,[D(g)]^\dagger$$

- The operators transform like

$$\mathcal{O} \to \mathcal{O}' = D(g)\mathcal{O}\,[D(g)]^\dagger$$

so that all matrix elements $\langle\phi|\,\mathcal{O}\,|\psi\rangle$ kept unchanged.

- An invariant observable satisfies,

$$\mathcal{O} \to \mathcal{O}' = D(g)\mathcal{O}\,[D(g)]^\dagger = \mathcal{O}$$

i.e.,

$$[\mathcal{O}, D(g)] = 0, \quad \forall\, g \in G$$

We have supposed that $D(G)$ forms a finite dimensional representation of group G.

Hence, $D(G)$ can be equivalent to a unitary and completely reducible representation:

$$\langle a,\ j,\ x|\, D(g)\, |b,\ k,\ y\rangle = \delta_{ab}\delta_{xy} \left[D_a(g)\right]_{jk}$$

Consequently,

$$D(g) = \sum_{a,j,k,x} |a,\ j,\ x\rangle \left[D_a(g)\right]_{jk} \langle a,\ k,\ x|$$

In detail,

$$
\begin{aligned}
D(g) &= \left[\sum_{a,j,x} |a,j,x\rangle\langle a,j,x| \right] D(g) \left[\sum_{b,k,y} |b,k,y\rangle\langle b,k,y| \right] \\
&= \sum_{a,j,x}\sum_{b,k,y} |a,j,x\rangle \left[\langle a,j,x| \, D(g) \, |b,k,y\rangle \right] \langle b,k,y| \\
&= \sum_{a,j,x}\sum_{b,k,y} |a,j,x\rangle \left\{ \delta_{ab}\delta_{xy} \left[D_a(g)\right]_{jk} \right\} \langle b,k,y| \\
&= \sum_{a,j,k,x} |a,\ j,\ x\rangle \left[D_a(g)\right]_{jk} \langle a,\ k,\ x|
\end{aligned}
$$

Wigner-Eckart Theorem:

For an invariant observable operator $\mathscr{O}$,

$$\left[\mathscr{O}, D(g)\right] = 0, \quad \forall\, g \in G$$

we get:

$$\begin{aligned} 0 &= \langle a, j, x|\left[\mathscr{O}, D(g)\right]|b, k, y\rangle \\ &= \sum_i \left\{ \langle a, j, x|\, \mathscr{O}\, |b, i, y\rangle \left[D_b(g)\right]_{ik} - \left[D_a(g)\right]_{ji} \langle a, i, x|\, \mathscr{O}\, |b, k, y\rangle \right\} \end{aligned}$$

The matrix element $\langle a, j, x|\, \mathscr{O}\, |b, k, y\rangle$ satisfies the hypotheses of Schur's Lemmas. Therefore, it either vanishes when $a \neq b$ or is proportional to identity δ_{jk} for $a = b$,

$$\langle a, j, x|\, \mathscr{O}\, |b, k, y\rangle = f_a(x, y)\delta_{ab}\delta_{jk}$$

This conclusion is called the Wigner-Eckart theorem.

Orthogonality relations:

Suppose that $D_a(G)$ and $D_b(G)$ are two finite dimensional irreducible representations of G. We define a linear operator:

$$A^{ab}_{jl} \equiv \sum_{g \in G} D_a(g^{-1}) \, |a, j\rangle\langle b, l| \, D_b(g)$$

Then,

$$\begin{aligned}
D_a(g_1) A^{ab}_{jl} &= \sum_{g \in G} D_a(g_1) D_a(g^{-1}) \, |a, j\rangle\langle b, l| \, D_b(g) \\
&= \sum_{g \in G} D_a(g_1 g^{-1}) \, |a, j\rangle\langle b, l| \, D_b(g) \\
&= \sum_{h \in G} D_a(h^{-1}) \, |a, j\rangle\langle b, l| \, D_b(h g_1) \\
&= \sum_{h \in G} D_a(h^{-1}) \, |a, j\rangle\langle b, l| \, D_b(h) D_b(g_1) \\
&= \left[\sum_{h \in G} D_a(h^{-1}) \, |a, j\rangle\langle b, l| \, D_b(h) \right] D_b(g_1) = A^{ab}_{jl} D_b(g_1)
\end{aligned}$$

Schur's lemmas indicate that,

$$A^{ab}_{jl} = \sum_{g\in G} D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g) = \delta_{ab}\lambda^a_{jl} I$$

By computing the trace of the above equation in the sub-Hilbert space of dimension n_a,

$$\begin{aligned}
\delta_{ab}\lambda^a_{jl}\, n_a &= \delta_{ab}\lambda^a_{jl}\,\mathrm{Tr} I = \mathrm{Tr} A^{ab}_{jl} \\
&= \mathrm{Tr}\left[\sum_{h\in G} D_a(h^{-1})\,|a,j\rangle\langle b,l|\,D_b(h)\right] \\
&= \delta_{ab}\left[\sum_{h\in G} \langle a,l|\,D_a(h)D_a(h^{-1})\,|a,j\rangle\right] \\
&= \delta_{ab}\left[\sum_{h\in G} \langle a,l|\,D_a(hh^{-1})\,|a,j\rangle\right] \\
&= \delta_{ab}\sum_{h\in G} \langle a,l|a,j\rangle = N\delta_{ab}\delta_{jl} \quad \rightsquigarrow \quad \lambda^a_{jl} = \frac{N}{n_a}\delta_{jl}
\end{aligned}$$

Therefore,

$$\sum_{g\in G} D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g) = \frac{N}{n_a}\delta_{ab}\delta_{jl}I$$

Orthogonality relations:

The matrix element of the above equation between the states $|a,k\rangle$ and $|b,m\rangle$ reads,

$$\begin{aligned}\frac{N}{n_a}\delta_{ab}\delta_{jl}\delta_{km} &= \frac{N}{n_a}\delta_{ab}\delta_{jl}\,\langle a,k|a,m\rangle \\ &= \langle a,k|\left[\frac{N}{n_a}\delta_{ab}\delta_{jl}I\right]|b,m\rangle \\ &= \langle a,k|\left[\sum_{g\in G} D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g)\right]|b,m\rangle \\ &= \sum_{g\in G}\langle a,k|\,D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g)\,|b,m\rangle\end{aligned}$$

These equations are known as the *orthogonality relations* for the matrix elements of irreducible representations. They can be rewritten as:

$$\sum_{g\in G} \frac{n_a}{N} \left[D_a(g^{-1})\right]_{kj} \left[D_b(g)\right]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}$$

Notice:

- The matrix elements $[D_a(g)]_{jk}$ are linearly independent of one another.
- The whole set of $[D_a(g)]_{jk}$ are complete. An arbitrary function of g can be expanded in them.

For the unitary irreducible representations, the orthogonality can be recast as,

$$\sum_{g\in G} \frac{n_a}{N} [D_a(g)]^*_{jk} [D_b(g)]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}$$

With proper normalization,

$$\Phi^a_{jk}(g) \equiv \sqrt{\frac{n_a}{N}}\ [D_a(g)]_{jk}$$

the matrix elements of unitary irreducible representations become the orthonormal functions of the group elements $\{g\}$:

$$\sum_{g\in G} \left[\Phi^a_{jk}(g)\right]^*\ \Phi^b_{lm}(g) = \delta_{ab}\delta_{jl}\delta_{km}$$

Definition:

The characters $\chi_D(g)$ of a group representation $D(G)$ are the traces of the matrices $\{D(g)\}$ in the representation,

$$\chi_D(g) = \mathrm{Tr}\,[D(g)] \;= \sum_i [D(g)]_{ii}$$

Orthogonality:

The characters of non-equivalent irreducible representations are different from each other. In fact, they satisfy the so-called orthogonality relations,

$$\frac{1}{N}\sum_{g\in G}\chi^*_{D_a}(g)\chi_{D_b}(g) = \delta_{ab}$$

Therefore, the characters of different irreducible representations are different.

Proof:

Notice that $n_a = \sum_i \delta_{ii}$ is the dimension of $D_a(G)$. It follows from the orthogonality relations

$$\sum_{g\in G} \frac{n_a}{N} \left[D_a(g^{-1})\right]_{kj} \left[D_b(g)\right]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}$$

that

$$\begin{aligned}
\delta_{ab} n_a &= \delta_{ab} \sum_j \delta_{jj} = \sum_j \sum_l \delta_{ab}\delta_{jl}\delta_{jl} \\
&= \sum_j \sum_l \left\{ \sum_{g\in G} \frac{n_a}{N} \left[D_a(g^{-1})\right]_{jj} \left[D_b(g)\right]_{ll} \right\} \\
&= \sum_{g\in G} \frac{n_a}{N} \left\{ \sum_j \left[D_a(g)\right]^*_{jj} \right\} \left\{ \sum_l \left[D_b(g)\right]_{ll} \right\} \\
&= \frac{n_a}{N} \sum_{g\in G} \chi^*_{D_a}(g) \chi_{D_b}(g) \quad \rightsquigarrow \quad \frac{1}{N} \sum_{g\in G} \chi^*_{D_a}(g) \chi_{D_b}(g) = \delta_{ab}
\end{aligned}$$

Properties of $\chi_D(G)$:

- The characters are constants on conjugacy classes.

$$\begin{aligned} \chi_D(g) &= \mathrm{Tr} D(g) = \mathrm{Tr}\left[D(h)^{-1} D(g) D(h)\right] \\ &= \mathrm{Tr}\left[D(h^{-1}) D(g) D(h)\right] \\ &= \mathrm{Tr} D(h^{-1} g h) \\ &= \chi_D(h^{-1} g h) \end{aligned}$$

- By labeling the conjugacy classes in integers α and letting κ_α be the number of elements in C_α, we can rewrite the previous orthogonality relations of the characters as,

$$\frac{1}{N} \sum_\alpha \kappa_\alpha \chi^*_{D_a}(g_\alpha) \chi_{D_b}(g_\alpha) = \delta_{ab}$$

From this we get,

$$\begin{aligned}
\chi_{D_b}(g_\beta) &= \sum_a \left[\delta_{ab}\chi_{D_a}(g_\beta)\right] \\
&= \sum_a \left[\chi_{D_a}(g_\beta)\frac{1}{N}\sum_\alpha \kappa_\alpha \chi^*_{D_a}(g_\alpha)\chi_{D_b}(g_\alpha)\right] \\
&= \frac{1}{N}\sum_\alpha \kappa_\alpha \left[\sum_a \chi^*_{D_a}(g_\alpha)\chi_{D_a}(g_\beta)\right]\chi_{D_b}(g_\alpha)
\end{aligned}$$

Therefore,

$$\sum_a \chi^*_{D_a}(g_\alpha)\chi_{D_a}(g_\beta) = \frac{N}{\kappa_\alpha}\delta_{\alpha\beta}$$

Corollaries:

- The finite dimensional representation $D(G)$ of group G is irreducible iff

$$\frac{1}{N}\sum_{\alpha}\kappa_{\alpha}|\chi_{D}(g_{\alpha})|^{2}=1$$

- There is a relation between the order of group G and the dimensions of its irreducible representations

$$N=\sum_{a}n_{a}^{2}$$

Remark:

The formula $N = \sum_a n_a^2$ is shown below.

Suppose that G has a finite dimensional reducible representation $D(G)$, which can be expressed as the direct sum of a set of irreducible representations,

$$D(g) \sim \oplus_{a=1}^{M} c_a D_a(g), \quad \forall\, g \in G$$

This implies $\chi_D(g) = \sum_{a=1}^{M} c_a \chi_{D_a}(g)$. Therefore,

$$\frac{1}{N} \sum_{g \in G} \chi_{D_a}^*(g) \chi_D(g) = \sum_{b=1}^{M} c_b \left[\frac{1}{N} \sum_{g \in G} \chi_{D_a}^*(g) \chi_{D_b}(g) \right]$$

$$= \sum_{b=1}^{M} c_b \delta_{ab}$$

$$= c_a \quad \rightsquigarrow \quad c_a = \frac{1}{N} \sum_{g \in G} \chi_{D_a}^*(g) \chi_D(g)$$

Consider the regular representation $D_{\text{reg}}(G)$, where

$$\chi_{\text{reg}}(e) = \text{Tr} D_{\text{reg}}(e) = N,$$
$$\chi_{\text{reg}}(g) = \text{Tr} D_{\text{reg}}(g) = 0, \quad \forall\, g \neq e$$

Hence,

$$c_a = \frac{1}{N} \sum_{g \in G} \chi^*_{D_a}(g) \chi_{\text{reg}}(g) = \chi^*_{D_a}(e) = n_a$$

and

$$N = \chi_{\text{reg}}(e) = \sum_{a=1}^{M} c_a \chi_{D_a}(e) = \sum_{a=1}^{M} n_a^2$$

Corollary:

The number of non-equivalent irreducible representations of a finite group is equal to the number of its conjugacy classes.

Explanation:

Let $F(g_1)$ be a function of group element g_1 that is some constant on each conjugacy class,

$$F(g_1) = F(h^{-1} g_1 h)$$

The full set of $[D_a(g)]_{jk}$ of the irreducible representations are complete. Thereby, $F(g_1)$ can be expanded in terms of these matrix elements,

$$F(g_1) = \sum_{a,j,k} c^a_{jk} [D_a(g_1)]_{jk}$$

That $F(g_1)$ is some constant on each conjugacy class further suggests:

$$F(g_1) = \sum_a \left[\sum_j \left(\frac{c^a_{jj}}{n_a} \right) \right] \chi_{D_a}(g_1)$$

In detail,

$$
\begin{aligned}
F(g_1) &= \frac{1}{N}\sum_{g\in G} F(g^{-1}g_1 g) = \frac{1}{N}\sum_{g\in G}\sum_{a,j,k} c^a_{jk}\left[D_a(g^{-1}g_1 g)\right]_{jk} \\
&= \frac{1}{N}\sum_{g\in G}\sum_{a,j,k} c^a_{jk}\left\{\left[D_a(g^{-1})\right]_{jl}\left[D_a(g_1)\right]_{lm}\left[D_a(g)\right]_{mk}\right\} \\
&= \frac{1}{N}\sum_{a,j,k} c^a_{jk}\left\{\sum_{g\in G}\left[D_a(g^{-1})\right]_{jl}\left[D_a(g)\right]_{mk}\right\}\cdot\left[D_a(g_1)\right]_{lm} \\
&= \frac{1}{N}\sum_{a,j,k} c^a_{jk}\left\{\frac{N}{n_a}\delta_{lm}\delta_{jk}\right\}\cdot\left[D_a(g_1)\right]_{lm} \\
&= \sum_a\left[\sum_j\left(\frac{c^a_{jj}}{n_a}\right)\right]\left[D_a(g_1)\right]_{ll} \\
&= \sum_a\left[\sum_j\left(\frac{c^a_{jj}}{n_a}\right)\right]\chi_{D_a}(g_1)
\end{aligned}
$$

This formula

$$F(g_1) = \sum_a \left[\sum_j \left(\frac{c^a_{jj}}{n_a} \right) \right] \chi_{D_a}(g_1)$$

for functions that are constants on the conjugacy classes implies that the characters of the independent irreducible representations form a complete, orthonormal set of basis vectors in "Class Space".

Therefore,

the number of irreducible representations of a group G equals to the number of its conjugacy classes.

Recall that $N = \sum_a n_a^2$.

- All of the irreducible representations of a finite Abelian group are 1-dimensional.

An example:

Question:

Determine the characters of all independent irreducible representations of permutation group S_3.

Solution:

There are 3 independent conjugacy classes in S_3. Hence S_3 has 3 non-equivalent irreducible representations D_0, D_1 and D_2 in total.

D_0 is the trivial 1-dimensional irreducible representation,

$$D_0(g) = 1, \quad \forall\, g \in S_3$$

It means $\chi_0(g) = 1, \ \forall g \in S_3$. The constraint $N = \sum_a n_a^2$ further indicates:

$$6 = 1 + n_1^2 + n_2^2$$

Hence, $n_1 = 1$ and $n_2 = 2$. $\rightsquigarrow$ Besides D_0, S_3 has a 1d irreducible representation D_1 and a 2d irreducible representation D_2.

The elements of the Factor Group $S_3/Z_3 = Z_2$ form the cosets of subgroup Z_3,

$$Z_3 = \{e,\ a_1,\ a_2\}, \quad Z_3 a_3 = \{a_3,\ a_4,\ a_5\}$$

We can identify D_1 as this $Z_2 = \{1,\ -1\}$:

$$\begin{cases} D_1(e) = D_1(a_1) = D_1(a_2) = 1, \\ D_1(a_3) = D_1(a_4) = D_1(a_5) = -1. \end{cases}$$

The corresponding characters read,

$$\begin{cases} \chi_1(e) = \chi_1(a_1) = \chi_1(a_2) = 1, \\ \chi_1(a_3) = \chi_1(a_4) = \chi_1(a_5) = -1. \end{cases}$$

So far we have got an unfinished Characters table for S_3:

	$\{e\}$	$\{a_1, a_2\}$	$\{a_3, a_4, a_5\}$
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	?	?

We can fill the remaining 2 entries by using orthogonality relations of the characters,

$$\sum_\alpha \kappa_\alpha \chi^*_{D_a}(g_\alpha)\chi_{D_b}(g_\alpha) = N\,\delta_{ab}$$

Concretely,

$$\begin{aligned}
6 &= |\chi_2(e)|^2 + 2|\chi_2(a_1)|^2 + 3|\chi_2(a_3)|^2 \\
&= 4 + 2|\chi_2(a_1)|^2 + 3|\chi_2(a_3)|^2 \\
0 &= \chi_1^*(e)\chi_2(e) + 2\chi_1^*(a_1)\chi_2(a_1) + 3\chi_1^*(a_3)\chi_2(a_3) \\
&= 2 + 2\chi_2(a_1) - 3\chi_2(a_3) \\
0 &= \chi_0^*(e)\chi_2(e) + 2\chi_0^*(a_1)\chi_2(a_1) + 3\chi_0^*(a_3)\chi_2(a_3) \\
&= 2 + 2\chi_2(a_1) + 3\chi_2(a_3)
\end{aligned}$$

Therefore,

$$\chi_2(a_1) = -1, \ \ \chi_2(a_3) = 0.$$

Exercise (optional):

Show these results by checking the alternative orthogonality relations

$$\sum_a \chi^*_{D_a}(g_\alpha)\chi_{D_a}(g_\beta) = \frac{N}{\kappa_\alpha}\delta_{\alpha\beta}$$

The *finished* Characters Table of S_3 is,

	$\{e\}$	$\{a_1, a_2\}$	$\{a_3, a_4, a_5\}$
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	-1	0

Homework:

1. Suppose that D_1 and D_2 are equivalent, irreducible representations of a finite group G such that

$$D_2(g) = SD_1(g)S^{-1}, \quad \forall g \in G.$$

What can you say about an operator A that satisfies

$$AD_1(g) = D_2(g)A, \quad \forall g \in G \ ?$$

现代数学物理方法

第一章, 群论基础

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Projection Operator:

- Characters can be used to decompose an reducible representation into its irreducible ingredients. The key bridge to this end is the **Projection Operator** of an irreducible component representation.

Let $D(G)$ be an arbitrary representation of finite group $G = \{g\}$ (of order N) that contains an n_a-dimensional irreducible representation $D_a(G)$ with characters $\{\chi_a(g)\}$. Then

$$P_a = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g) D(g)$$

is the projection operator onto the subspace of $D_a(G)$.

The matrix elements of P_a in a given representation space of $D(G)$ read

$$[P_a]_{ij} = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g) \, [D(g)]_{ij}$$

Explanation:

Recall that every representation of a finite group is equivalently unitary and completely reducible,

$$D(g) \sim \oplus_{a=1}^{s} c_a D_a(g), \quad \forall\, g \in G$$

we see,

$$[P_a]_{ij} = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g)\, [D(g)]_{ij} \sim \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g)\, [\oplus_{b=1}^{s} c_b D_b(g)]_{ij}$$

Recall the orthogonality relations between irreducible representations:

$$\frac{n_a}{N} \sum_{g \in G} [D_a(g)]^*_{jk}\, [D_b(g)]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}$$

We have

$$\frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g)\, [D_b(g)]_{lm} = \delta_{ab}\delta_{lm}$$

Hence, P_a gives 1 on the subspaces that transform like $D_a(G)$ and 0 on all the other subspaces.

An example:

Question:

Here is a 3-dimensional representation of S_3,

$$D(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D(a_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad D(a_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$D(a_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D(a_4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad D(a_5) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

1. Is it irreducible ?
2. Is it the regular representation of S_3 ?
3. Evaluate the projection operators of the irreducible representations of S_3 in this 3-dimensional reducible representation.

Solution:

1. No. It is not an irreducible because its dimension is $n = 3$, violating the required relation $\sum_a n_a^2 = 6$.

2. No. The regular representation of S_3 should be 6-dimensional.

3. The projection operators of 3 irreducible representations of S_3 are evaluated from $P_a = \frac{n_a}{N} \sum_{g \in G} \chi^*_{D_a}(g) D(g)$. The results are as follows:

$$P_0 = \frac{1}{6} \sum_{g \in S_3} D(g) = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_1 = \frac{1}{6} \left[D(e) + \sum_{j=1}^{2} D(a_j) - \sum_{j=3}^{5} D(a_j) \right] = 0$$

$$P_2 = \frac{2}{6} \left[2D(e) - \sum_{j=1}^{2} D(a_j) \right] = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Simple calculations lead to $(P_j)^2 = P_j = (P_j)^\dagger$ for $j = 0, 1, 2$. Hence, $D = D_0 \oplus D_2$.

QM Background:

In QM, we are interested in the eigenstates of an invariant hermitian operator, in particular the invariant hamiltonian under group G,

$$[D(g), H] = 0$$

where

$$H|n\rangle = \lambda_n |n\rangle, \quad n = 0, 1, 2, \cdots$$

Theorem:

1. If H commutes with all the elements $\{D(g)\}$ of a representation of group G, then you can choose the eigenstates of H to transform according to irreducible representations of G.
2. If an irreducible representation appears only once in the Hilbert space, every state in the irreducible representation is an eigenstate of H with the same eigenvalue.

Proof:

- Due to the assumption that $[D(g), H] = 0$, the transformations in the representation $D(G)$ do not change the eigenvalues of operator H,

$$H|n\rangle = \lambda_n |n\rangle,$$
$$H[D(g)|n\rangle] = D(g)H|n\rangle = \lambda_n [D(g)|n\rangle]$$

- If G is finite, $D(G)$ can be decomposed into a direct sum of some irreducible representations $D_i(G)$:

$$D(G) = \oplus_i D_i(G)$$

Thus we can divide up the Hilbert space into some subspaces:

1. The i-th subspace is labelled by the eigenvalue λ_i of H.
2. The i-th subspace furnishes an irreducible representation $D_i(G)$ of group G.

- Eigenvectors $\{|i, \alpha\rangle; \alpha = 1, 2, \cdots, n_i\}$ of H belonging to λ_i

$$H|i, \alpha\rangle = \lambda_i |i, \alpha\rangle$$

can be chosen in terms of the irreducible representation $D_i(G)$:

$$g: \quad D_i(g)|i, \alpha\rangle = |i, \beta\rangle, \quad \forall g \in G$$

where $\alpha, \beta = 1, 2, \cdots, n_i$ and $i = 1, 2, 3, \cdots$.

- Consider an arbitrary vector in the whole Hilbert space,

$$|a, j, x\rangle, \quad 1 \leqslant j \leqslant n_a,$$

where x stands for the times the $D_i(G)$ appearing in Hilbert space. Then,

$$H|a, j, x\rangle = \sum_y c_y |a, j, y\rangle$$

If x and y take only one value, $|a, j, x\rangle$ becomes an eigenvector of H.

Tensor product representation:

Question:

How to put known representations together to form a new representation (with higher dimensions) ?

Suppose that D_1 is an m-dimensional representation acting on a space with basis vectors

$$|i\rangle, \quad (i = 1, 2, \cdots, m)$$

D_2 is an n-dimensional representation acting on a space with basis vectors

$$|\alpha\rangle, \quad (\alpha = 1, 2, \cdots, n)$$

We can make an mn-dimensional representation space, called the tensor product space, by defining its basis vectors as,

$$|i, \alpha\rangle = |i\rangle \otimes |\alpha\rangle, \qquad (i = 1, 2, \cdots, m; \;\; \alpha = 1, 2, \cdots, n)$$

In this space we define the so-called tensor product representation $D_{1\times 2} = D_1 \otimes D_2$,

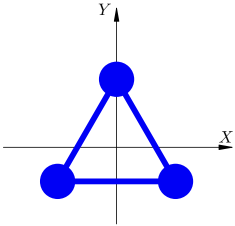
$$\langle i, \alpha| \, D_{1\times 2}(g) \, |j, \beta\rangle \equiv \langle i| \, D_1(g) \, |j\rangle \cdot \langle \alpha| \, D_2(g) \, |\beta\rangle$$

Remarks:

1. The tensor product representation is indeed a representation of group G [Homework (optional)].
2. In general, the tensor product representation is not an irreducible representation.
3. One of our favorite pastimes is to decompose a reducible tensor representation into the direct sum of irreducible representations of the group G.

Example:

Three blocks are connected by springs in a triangle. The system is suposed to be free to slide on a frictionless surface.

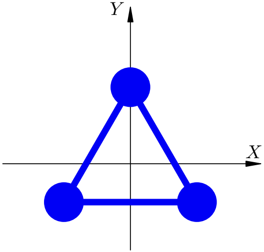


Properties of the model:

- The system has an S_3 symmetry.
- The system has 6 degrees of freedom, described by the x and y coordinates of the 3 blocks:

$$|\vec{r}\rangle = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{12} \\ r_{21} \\ r_{22} \\ r_{31} \\ r_{32} \end{bmatrix} = |r_{i\alpha}\rangle$$

where α labels coordinate x or y, and i labels the blocks.



- This 6-dimensional configuration space can be viewed as a tensor product space of a 3-dimensional space of the blocks

$$|\xi\rangle = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

and a 2-dimensional space of coordinates x and y,

$$|\zeta\rangle = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

That is:

$$|r_{i\alpha}\rangle = |\xi\rangle \otimes |\zeta\rangle$$

Namely,

$$r_{i\alpha} = \xi_i \zeta_\alpha, \quad (i = 1, 2, 3; \;\; \alpha = 1, 2.)$$

- Suppose that the representations of S_3 on 3-dimensional space $\{|\xi\rangle\}$ and 2-dimensional space $\{|\zeta\rangle\}$ could *respectively* be given by the previous D_3,

$$D_3(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D_3(a_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D_3(a_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad D_3(a_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_3(a_4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad D_3(a_5) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and D_2,

$$D_2(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D_2(a_1) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D_2(a_2) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \quad D_2(a_3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_2(a_4) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \qquad D_2(a_5) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

we have a 6-dimensional representation $D_6(S_3)$ whose elements read,

$$[D_6(S_3)]_{i\alpha j\beta} = [D_3(S_3)]_{ij} \cdot [D_2(S_3)]_{\alpha\beta}$$

The characters of $D_6(S_3)$ are:

$$\begin{aligned} \chi_6(S_3) &= \sum_{i\alpha} [D_6(S_3)]_{i\alpha i\alpha} = \left\{ \sum_i [D_3(S_3)]_{ii} \right\} \cdot \left\{ \sum_\alpha [D_2(S_3)]_{\alpha\alpha} \right\} \\ &= \chi_3(S_3)\chi_2(S_3) \end{aligned}$$

Theorem:

The characters of a tensor product representation are the products of the characters of the factor representations,

$$\chi_{D_1 \times D_2} = \chi_{D_1} \chi_{D_2}$$

The characters of $D_6(S_3)$ are then given by,

	$\{e\}$	$\{a_1, a_2\}$	$\{a_3, a_4, a_5\}$
χ_3	3	0	1
χ_2	2	-1	0
χ_6	6	0	0

$D_6(S_3)$ has the same characters as the regular representation $D_{\text{reg}}(S_3)$. Consequently,

1. $D_6(S_3)$ and $D_{\text{reg}}(S_3)$ are equivalent to each other (because the similarity transformations do not change the characters).
2. $D_6(S_3)$ contains D_0 and D_1 once but D_2 twice.

For completeness, we write down explicitly an element of $D_6(S_3)$:

$$D_6(a_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 & 0 & \sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 & 0 & 0 & 0 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 \\ 0 & 0 & \sqrt{3}/2 & -1/2 & 0 & 0 \end{bmatrix}$$

Permutation group S_n:

1. Any element of the permutation group S_n on n-objects can be expressed in terms of cycles. e.g.,

$$\begin{cases} e = (1)(2)\cdots(n) \\ a_1 = (12)(3)(4)\cdots(n) \\ a_j = (1243)(5)(6)(79)(8)\cdots(n) \end{cases}$$

2. Each cycle is written as a set of numbers in parentheses, indicating the set of objects that are cyclically permuted.
3. Each element of S_n involves each integer from 1 to n in exactly one cycle.

Illustration:

- (1) means $x_1 \to x_1$.
- (1372) means $x_1 \to x_3 \to x_7 \to x_2 \to x_1$.

Definition of j-cycle in S_n:

In S_n, a j-cycle is defined as

$$(\xi_1 \xi_2 \xi_3 \cdots \xi_j), \quad 1 \leqslant j \leqslant n.$$

If an element of S_n has k_j j-cycles, then

$$\sum_{j=1}^{n} j k_j = n$$

An Example in S_9:

$$(123)(456)(78)(9) \rightsquigarrow \begin{cases} k_1 = k_2 = 1 \\ k_3 = 2 \\ k_4 = k_5 = \cdots = k_9 = 0 \end{cases}$$

Interchange:

An interchange is a 2-cycle, the permutation between two objects,

$$(\xi_i \xi_j), \quad 1 \leqslant i, j \leqslant n, \ \ (i \neq j)$$

Remarks:

- Except the trivial 1-cycle, each group element in S_n can be written out in terms of the ordered product of interchanges. *e.g.* in S_9,

$$(123)(456)(78)(9) = (12)(23)(45)(56)(78)(9)$$

- The inner automorphism built from "interchanges" does not change the *cycle structure* $\{k_1 k_2 \cdots k_n\}$ of any element in S_n.

$$
\begin{aligned}
(\xi_j \xi_i)(\cdots \xi_1 \xi_i \xi_2 \cdots)(\cdots \xi_3 \xi_j \xi_4 \cdots)(\xi_i \xi_j) \\
= (\cdots \xi_1 \xi_j \xi_2 \cdots)(\cdots \xi_3 \xi_i \xi_4 \cdots) \\
(\xi_j \xi_i)(\cdots \xi_1 \xi_i \xi_2 \cdots \xi_3 \xi_j \xi_4 \cdots)(\xi_i \xi_j) \\
= (\cdots \xi_1 \xi_j \xi_2 \cdots \xi_3 \xi_i \xi_4 \cdots)
\end{aligned}
$$

Therefore, the inner automorphism $g g_1 g^{-1}$ built from an arbitrary permutation $g \in S_n$ does not change the cycle structure of element $g_1 \in S_n$.

Examples in S_4:

1. $(12) \cdot (1234) \cdot (12) = (1342)$
2. $(12) \cdot (23) \cdot (12) = (13)$
3. $(12) \cdot (13)(24) \cdot (12) = (14)(23)$

Conjugacy classes in S_n:

1. In S_n, the conjugacy classes consist of all possible permutations with a particular cycle structure.

2. The conjugacy classes can be labeled by the set of integers $\{k_1, k_2, \cdots, k_n\}$, where k_i is the number of i-cycle but i the *length* of i-cycle[1].

3. The number of group elements in each conjugacy class $\{k_1, k_2, \cdots, k_n\}$ of S_n is,

$$\# = \frac{n!}{\prod_{j=1}^{n} j^{k_j} (k_j)!}$$

[1] For example, the group elements $(1)(234)$, $(2)(341)$, $(3)(412)$ and $(4)(123)$ in S_4 are in the same conjugacy class.

Proof:

Each permutation of objects (from 1 to n) gives a permutation in the class, the total number is $n!$. Hence, the number of group elements in class $\{k_1, k_2, \cdots, k_n\}$ should be proportional to $n!$,

$$\# \propto n!$$

But cyclic order doesn't matter within a cycle, e.g., (1234) is the same as (2341), (3412) and (4123),

$$\# \propto \frac{n!}{\prod_{j=1}^{n} j^{k_j}}$$

Furthermore, the order does not matter also at all between cycles of the same length, e.g., $(12)(34)$ is the same as $(34)(12)$,

$$\rightsquigarrow \quad \# = \frac{n!}{\prod_{j=1}^{n} j^{k_j}} \cdot \frac{1}{\prod_{j=1}^{n} (k_j)!} = \frac{n!}{\prod_{j=1}^{n} j^{k_j} (k_j)!}$$

Example: S_3

In S_3, there are totally 3 conjugacy classes[2]:

$$\mathcal{C}_1 = \{e\}, \;\; \mathcal{C}_2 = \{(12), (23), (31)\}, \;\; \mathcal{C}_3 = \{(123), (321)\}$$

The number of group elements in each class is calculated as,

$$\#\mathcal{C}_1 = \frac{3!}{(1^3 \cdot 3!)(2^0 \cdot 0!)(3^0 \cdot 0!)} = 1$$
$$\#\mathcal{C}_2 = \frac{3!}{(1^1 \cdot 1!)(2^1 \cdot 1!)(3^0 \cdot 0!)} = 3$$
$$\#\mathcal{C}_3 = \frac{3!}{(1^0 \cdot 0!)(2^0 \cdot 0!)(3^1 \cdot 1!)} = 2$$

[2] In S_3, $e = (1)(2)(3)$ and the group element (12) stands for $(12)(3)$, and so on.

Example: S_4

There are totally 5 conjugacy classes in S_4,

$$
\begin{aligned}
\mathcal{C}_1 &= \{e\} \\
\mathcal{C}_2 &= \{(12), (13), (14), (23), (24), (34)\} \\
\mathcal{C}_3 &= \{(123), (124), (134), (234), (321), (421), (431), (432)\} \\
\mathcal{C}_4 &= \{(12)(34), (13)(24), (14)(23)\} \\
\mathcal{C}_5 &= \{(1234), (1243), (1324), (1342), (1423), (1432)\}
\end{aligned}
$$

The number of group elements in each class is calculated as follows:

$$
\begin{aligned}
\#\mathcal{C}_1 &= \frac{4!}{(1^4 \cdot 4!)(2^0 \cdot 0!)(3^0 \cdot 0!)(4^0 \cdot 0!)} = 1 \\
\#\mathcal{C}_2 &= \frac{4!}{(1^2 \cdot 2!)(2^1 \cdot 1!)(3^0 \cdot 0!)(4^0 \cdot 0!)} = 6 \\
\#\mathcal{C}_3 &= \frac{4!}{(1^1 \cdot 1!)(2^0 \cdot 0!)(3^1 \cdot 1!)(4^0 \cdot 0!)} = 8
\end{aligned}
$$

Homework:

$$\#\mathcal{C}_4 = \frac{4!}{(1^0 \cdot 0!)(2^2 \cdot 2!)(3^0 \cdot 0!)(4^0 \cdot 0!)} = 3$$

$$\#\mathcal{C}_5 = \frac{4!}{(1^0 \cdot 0!)(2^0 \cdot 0!)(3^0 \cdot 0!)(4^1 \cdot 1!)} = 6$$

Problems:

1. How many conjugacy classes are there in symmetric group S_6 ?
   How many group elements are there in each of these classes ?

现代数学物理方法

第一章, 群论基础

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Young Tableaux:

Definition of Young Tableaux:

It is convevient (and then useful) to represent each j-cycle by a column of boxes of length j, top-justified and arranged in order of decreasing j as you go to the right. *In S_n, the total number of boxes is n.*

These collections of boxes are called Young Tableaux.

Importance of Young tableaux:

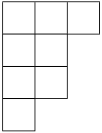
1. Each different tableaux of n-boxes represents a different conjugacy class of S_n.
2. The Young tableaux are in *one-to-one* correspondence with the irreducible representations of S_n.

Illustration:

1. The identity element in S_4 consists of four 1-cycles. It is represented as

2. The elements $(1324)(658)(7)$ and $(1)(362)(5478)$ in S_8 contain a 4-cycle, a 3-cycle and a 1-cycle.

   Both elements are represented as

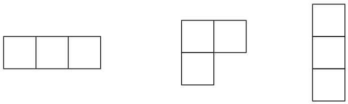


Example:

S_3 has 3 conjugacy classes, *i.e.*,

$$\{e\}, \quad \{(12),(23),(31)\}, \quad \{(123),(321)\}$$

With Young tableaux they could be represented as,



respectively.

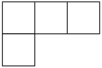
The numbers of group elements in these conjugacy classes are:

$$\frac{3!}{3!} = 1, \qquad \frac{3!}{2} = 3, \qquad \frac{3!}{3} = 2.$$

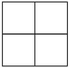
Example:

The classes and the corresponding numbers of group elements of S_4 are,

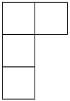
$$\frac{4!}{4!} = 1$$



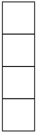
$$\frac{4!}{2 \times 2} = 6$$



$$\frac{4!}{2^2 \times 2!} = 3$$



$$\frac{4!}{3} = 8$$



$$\frac{4!}{4} = 6$$

Representation of S_n:

Young tableaux can be used to construct the irreducible representations of S_n.

Steps:

- We begin by putting the integers from 1 to n in the boxes of the tableaux in all possible ways. There are $n!$ ways to do this.
- We identify each assignment of integers 1 to n to the boxes with a state in the regular representation of S_n.

Concretely,

> *by defining a standard ordering, saying from left to right and then top to down, we translate from the integers in the boxes of the Young tableaux to a state associated with a particular permutation.*

An example in S_7:

$$\begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 2 \\ \hline 1 & 7 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \rightsquigarrow \quad |6532174\rangle$$

This state is associated with the permutation:

$$|1234567\rangle \quad \rightsquigarrow \quad |6532174\rangle$$

Obviously, it is $(167425)(3)$.

- For a particular tableaux, we first symmetrize the corresponding state in the numbers in each row, and then anti-symmetrize it in the numbers in each column.

  e.g.,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad \rightsquigarrow \quad [e + (12)]\,|12\rangle = |12\rangle + |21\rangle$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} \quad \rightsquigarrow$$

$$[e - (13)][e + (12)]\,|123\rangle = |123\rangle + |213\rangle - |321\rangle - |231\rangle$$

$$\begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 2 \\ \hline 1 & 7 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad \rightsquigarrow \; ?$$

- The set of states constructed in this way spans some subspaces of the regular representation. Such a subspace defines actually an irreducible representation of S_n.

Question:

Find all of the irreducible representations of S_3 by using Young tableaux.

Solution:

- The Young tableau $\boxed{}\boxed{}\boxed{}$ gives a completely symmetrized state:

$$\boxed{1}\boxed{2}\boxed{3}$$

$$\rightsquigarrow \quad |\Psi_0\rangle = |123\rangle + |231\rangle + |312\rangle + |132\rangle + |213\rangle + |321\rangle$$

Because

$$D_0[g]\,|\Psi_0\rangle = |\Psi_0\rangle, \quad \forall g \in S_3$$

$\boxed{}\boxed{}\boxed{}$ is associated with a 1-dimensional subspace which defines the trivial (irreducible) representation of S_3:

$$\begin{aligned} D_0[e] &= D_0[(12)] = D_0[(13)] \\ &= D_0[(23)] = D_0[(123)] = D_0[(132)] = 1 \end{aligned}$$

- The Young tableau $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}$ gives a completely antisymmetric state,

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \rightsquigarrow |\Psi_1\rangle = |123\rangle - |213\rangle - |321\rangle - |132\rangle + |231\rangle + |312\rangle$$

This state spans another 1-dimensional irreducible subspace which defines the so-called alternate representation D_1 of S_3:

$$D_1[e]\,|\Psi_1\rangle = D_1[(123)]\,|\Psi_1\rangle = D_1[(132)]\,|\Psi_1\rangle = |\Psi_1\rangle$$
$$D_1[(12)]\,|\Psi_1\rangle = D_1[(23)]\,|\Psi_1\rangle = D_1[(13)]\,|\Psi_1\rangle = -\,|\Psi_1\rangle$$

Therefore,

$$D_1[e] = D_1[(123)] = D_1[(132)] = 1$$
$$D_1[(12)] = D_1[(23)] = D_1[(13)] = -1$$

- The Young tableau $\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ gives the following states:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \rightsquigarrow \quad |\psi_{21}\rangle = |123\rangle + |213\rangle - |321\rangle - |231\rangle$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \rightsquigarrow \quad |\psi_{22}\rangle = |132\rangle + |312\rangle - |231\rangle - |321\rangle$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \rightsquigarrow \quad |213\rangle + |123\rangle - |312\rangle - |132\rangle = |\psi_{21}\rangle - |\psi_{22}\rangle$$

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \rightsquigarrow \quad |231\rangle + |321\rangle - |132\rangle - |312\rangle = -|\psi_{22}\rangle$$

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \rightsquigarrow \quad |312\rangle + |132\rangle - |213\rangle - |123\rangle = -|\psi_{21}\rangle + |\psi_{22}\rangle$$

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \rightsquigarrow \quad |321\rangle + |231\rangle - |123\rangle - |213\rangle = -|\psi_{21}\rangle$$

Therefore, $\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ is associated with a 2-d irreducible representation of S_3.

Explanation:

The state related to the Young tableau

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

is determined as follows:

$$\begin{aligned} |\psi_{213}\rangle &= [e-(23)][e+(12)]\,|213\rangle \\ &= [e-(23)+(12)-(132)]\,|213\rangle \\ &= |213\rangle - |312\rangle + |123\rangle - |132\rangle \end{aligned}$$

Recall that,

$$\begin{aligned} |\psi_{21}\rangle &= |123\rangle + |213\rangle - |321\rangle - |231\rangle \\ |\psi_{22}\rangle &= |132\rangle + |312\rangle - |231\rangle - |321\rangle \end{aligned}$$

Hence,

$$|\psi_{213}\rangle = |\psi_{21}\rangle - |\psi_{22}\rangle$$

- To find this 2-dimensional representation, we need only consider the so-called standard Young tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \rightsquigarrow |\psi_{21}\rangle = |123\rangle + |213\rangle - |321\rangle - |231\rangle$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \rightsquigarrow |\psi_{22}\rangle = |132\rangle + |312\rangle - |231\rangle - |321\rangle$$

Standard Young tableaux:

1. In a standard Young tableau, the filled numbers increase within a row from left to right and within a column from top to down.
2. For a given Young tableau, the number of the standard Young tableaux is the same as the dimensions of the corresponding irreducible representation.

Remark:

The standard Young tableaux of S_3 are as follows:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}\;; \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}\;; \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\;, \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}\;.$$

- Go back to the construction of the 2-d irreducible representation of S_3. On the states $|\psi_{21}\rangle$ and $|\psi_{22}\rangle$ that correspond to the standard Young tableaux,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

we have,

$$\begin{aligned} D_2[(12)]\,|\psi_{21}\rangle &= D_2[(12)]\Big\{\,|123\rangle + |213\rangle - |321\rangle - |231\rangle\,\Big\} \\ &= \Big\{\,|213\rangle + |123\rangle - |312\rangle - |132\rangle\,\Big\} \\ &= |\psi_{21}\rangle - |\psi_{22}\rangle \end{aligned}$$

$$D_2[(12)]\,|\Psi_{22}\rangle = D_2[(12)]\Big\{ |132\rangle + |312\rangle - |231\rangle - |321\rangle \Big\}$$
$$= \Big\{ |231\rangle + |321\rangle - |132\rangle - |312\rangle \Big\}$$
$$= -\,|\Psi_{22}\rangle$$

By setting $|\psi_{21}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|\psi_{22}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we get:

$$D_2[(12)] = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

- Besides,

$$D_2[(23)]\,|\psi_{21}\rangle = D_2[(23)]\left\{ |123\rangle + |213\rangle - |321\rangle - |231\rangle \right\}$$
$$= \left\{ |132\rangle + |312\rangle - |231\rangle - |321\rangle \right\}$$
$$= |\Psi_{22}\rangle$$

$$D_2[(23)]\,|\psi_{22}\rangle = D_2[(23)]\left\{ |132\rangle + |312\rangle - |231\rangle - |321\rangle \right\}$$
$$= \left\{ |123\rangle + |213\rangle - |321\rangle - |231\rangle \right\}$$
$$= |\psi_{21}\rangle$$

Hence,

$$D_2[(23)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- The remaining representation matrices are calculated in terms of the above two. For example,

$$D_2[(123)] = D_2[(12)(23)] = D_2[(12)]D_2[(23)]$$
$$= \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

- In conclusion, the 2-d irreducible Rep. $D_2(S_3)$ is realized by,

$$D_2[e] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_2[(12)] = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$D_2[(13)] = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \quad D_2[(23)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D_2[(123)] = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad D_2[(132)] = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

Discussions:

- The obtained 2-d representation D_2 is indeed irreducible, because it leads to the expected characters,

$$\begin{aligned}
&\chi_2[e] = 2 \\
&\chi_2[(123)] = \chi_2[(132)] = -1 \\
&\chi_2[(12)] = \chi_2[(13)] = \chi_2[(23)] = 0
\end{aligned}$$

- Obviously, D_2 is not a unitary representation.

To get the equivalent unitary representation, we introduce an auxiliary hermitian matrix H,

$$H = \sum_{g \in S_3} \left[D_2(g)\right]^\dagger D_2(g) = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$$

The eigenvalue equation of matrix H reads,

$$\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

$\rightsquigarrow \quad 0 = \begin{vmatrix} 8-\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix} = (8-\lambda)^2 - 16.$ As expected, *both eigenvalues are positive:*

$$\lambda = \begin{cases} 12 \\ 4 \end{cases}$$

The corresponding eigenvectors of H read,

$$|\lambda = 12\rangle = \frac{1}{\sqrt{2}} e^{i\phi_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad |\lambda = 4\rangle = \frac{1}{\sqrt{2}} e^{i\phi_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where ϕ_1 and ϕ_2 are two arbitrary real parameters (phases). These two eigenvectors can be used to define a unitary matrix

$$u = \begin{bmatrix} \frac{e^{i\phi_1}}{\sqrt{2}} & \frac{e^{i\phi_2}}{\sqrt{2}} \\ \frac{e^{i\phi_1}}{\sqrt{2}} & -\frac{e^{i\phi_2}}{\sqrt{2}} \end{bmatrix}$$

With u we can diagonalize H,

$$H = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} = u \begin{bmatrix} 12 & 0 \\ 0 & 4 \end{bmatrix} u^\dagger$$

$$= \begin{bmatrix} \frac{e^{i\phi_1}}{\sqrt{2}} & \frac{e^{i\phi_2}}{\sqrt{2}} \\ \frac{e^{i\phi_1}}{\sqrt{2}} & -\frac{e^{i\phi_2}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{e^{-i\phi_1}}{\sqrt{2}} & \frac{e^{-i\phi_1}}{\sqrt{2}} \\ \frac{e^{-i\phi_2}}{\sqrt{2}} & -\frac{e^{-i\phi_2}}{\sqrt{2}} \end{bmatrix}$$

We define the square root matrix $\Omega = \sqrt{H}$,

$$\Omega = u \begin{bmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{4} \end{bmatrix} u^\dagger$$

$$= \begin{bmatrix} \frac{e^{i\phi_1}}{\sqrt{2}} & \frac{e^{i\phi_2}}{\sqrt{2}} \\ \frac{e^{i\phi_1}}{\sqrt{2}} & -\frac{e^{i\phi_2}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{e^{-i\phi_1}}{\sqrt{2}} & \frac{e^{-i\phi_1}}{\sqrt{2}} \\ \frac{e^{-i\phi_2}}{\sqrt{2}} & -\frac{e^{-i\phi_2}}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{bmatrix}$$

Matrix Ω:

- Ω is a hermitian matrix.
- Since $\det\Omega = 4\sqrt{3} \neq 0$, Ω has an inverse. The inverse matrix is also a hermitian.
- The inverse of Ω reads,

$$\Omega^{-1} = \frac{1}{4\sqrt{3}} \begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} \end{bmatrix}$$

The 2-dimensional unitary irreducible representation of S_3 is then constructed as,

$$D_2^{\text{unitary}}(g) = \Omega D_2(g)\Omega^{-1}, \quad \forall\, g \in S_3$$

Explicitly,

$$D_2^{\text{unitary}}[e] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_2^{\text{unitary}}[(12)] = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$$

$$D_2^{\text{unitary}}[(13)] = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$D_2^{\text{unitary}}[(23)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D_2^{\text{unitary}}[(123)] = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D_2^{\text{unitary}}[(132)] = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

Warning:

The matrix forms of the 2-dimensional unitary irreducible representation of S_3 are still not unique, although they are equivalent to each other.

An alternative realization of this 2-d irreducible unitary representation for S_3 is,

$$D_2(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D_2[(123)] = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D_2[(132)] = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D_2[(12)] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_2[(23)] = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$$D_2[(13)] = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

Homework:

Question (optional):

1. Please find a similarity transformation to relate these two equivalent unitary representations of S_3.

   Hint: Try to diagonize the matrix $D_2^{\text{unitary}}[(12)]$. We conclude that the two unitary representations are equivalent to each other by a similarity transformation,

$$u = u^\dagger = u^{-1} = \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3}-1 & \sqrt{3}+1 \\ \sqrt{3}+1 & 1-\sqrt{3} \end{bmatrix}$$

Problem:

1. Find the group of all the discrete rotations that leave a regular tetrahedron invariant by labeling the four vertices and considering the rotations as permutations on the four vertices. This defines a four dimensional representation of a group. Find the conjugacy classes and the characters of the irreducible representations of this group.

现代数学物理方法

第二章, 李群

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Lie Groups:

Lie groups G are groups where the group elements $g \in G$ depends smoothly on a set of continuous real parameters,

$$g = g(\alpha)$$

where

$$\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_N\} = \{\alpha_a \,|1 \leqslant a \leqslant N\}$$

In general, we choose parameters $\{\alpha_a\}$ so that the identity can be expressed as

$$e = g(\alpha)\,|_{\alpha=0} = g(0)$$

If we find a representation $D(G)$, we have similarly,

$$1 = D(\alpha)\,|_{\alpha=0} = D(0)$$

Generators:

In some neighborhood of the *identity*, the elements of a Lie group G or its representation $D(G)$ can be Taylor expanded as,

$$
\begin{aligned}
D(d\alpha) &= 1 + \sum_{a=1}^{N} d\alpha_a \left[\frac{\partial D(\alpha)}{\partial \alpha_a}\right]_{\alpha=0} + \cdots \\
&= 1 + i \sum_{a=1}^{N} d\alpha_a X_a + \cdots \\
&\approx 1 + i d\alpha_a X_a
\end{aligned}
$$

where

$$X_a = -i \frac{\partial D(\alpha)}{\partial \alpha_a} \Big|_{\alpha=0}, \quad (a = 1,\ 2,\ \cdots,\ N)$$

are called the generators of group G in its representation $D(G)$.

Discussions:

1. X_a are independent of one another.
2. The factor i is included in the definition of generators X_a so that if the representation is unitary, X_a will be hermitian matrices.
3. The representation of the group elements for finite parameters $\alpha = \{\alpha_a\}$ can be defined as,

$$D(\alpha) = \lim_{k\to\infty}\left[1 + i\left(\frac{\alpha_a}{k}\right)X_a\right]^k = \exp(i\alpha_a X_a) = e^{i\alpha_a X_a}$$

   This procedure is called *exponential mapping*. It implies that, *at least in some neighborhood of identity*, the group elements can be written out in terms of the generators.

4. The exponential of a matrix is always defined as a power series,

$$e^{i\alpha_a X_a} = \sum_{n=0}^{\infty}\frac{i^n}{n!}(\alpha_a X_a)^n$$

We now consider the multiplication of two group elements of a Lie group G,

$$g_\alpha = e^{i\alpha_a X_a}, \qquad g_\beta = e^{i\beta_a X_a}.$$

That the generators X_a are matrices indicates,

$$g_\alpha g_\beta = e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i(\alpha+\beta_a)X_a}$$

- Because the exponentials form a representation of the group G, it must be true that the product of two exponentials is also an exponential of the generators,

$$\begin{aligned} g_\alpha g_\beta &= e^{i\alpha_a X_a} e^{i\beta_b X_b} \\ &= e^{i\gamma_a X_a} \\ &= g_\gamma \end{aligned}$$

The parameters γ_a are determined by,

$$\begin{aligned} i\gamma_a X_a &= \ln\left(e^{i\alpha_a X_a} e^{i\beta_b X_b}\right) = \ln[1 + (e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1)] \\ &= \ln(1+K) \\ &= K - \frac{K^2}{2} + \frac{K^3}{3} - \cdots \end{aligned}$$

where $K = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$. Explicitly,

$$\begin{aligned} K &= \left[1 + i(\alpha_a X_a) - \frac{1}{2}(\alpha_a X_a)^2 + \cdots\right] \\ &\qquad \cdot \left[1 + i(\beta_b X_b) - \frac{1}{2}(\beta_b X_b)^2 + \cdots\right] - 1 \\ &= i(\alpha_a + \beta_a) X_a - \alpha_a \beta_b X_a X_b \\ &\qquad - \frac{1}{2}\left[(\alpha_a X_a)^2 + (\beta_a X_a)^2\right] + \cdots \end{aligned}$$

and

$$K^2 \approx \left[i(\alpha_a + \beta_a) X_a\right]^2 = -\alpha_a \beta_b (X_a X_b + X_b X_a) - \left[(\alpha_a X_a)^2 + (\beta_a X_a)^2\right]$$

Therefore,

$$\begin{aligned} i\gamma_a X_a &= K - K^2/2 + \cdots \\ &= i(\alpha_a + \beta_a)X_a - \frac{1}{2}\alpha_a\beta_b\Big(X_aX_b - X_bX_a\Big) \\ &= i(\alpha_a + \beta_a)X_a - \frac{1}{2}\alpha_a\beta_b\,[X_a,\ X_b] \end{aligned}$$

where

$$[A,\ B] = AB - BA$$

is called the *Lie bracket* between two generators A and B.

- We conclude that,

$$(\alpha_a\beta_b)[X_a,\ X_b] = -2i(\gamma_c - \alpha_c - \beta_c)X_c$$

That is to say: *the generators of the Lie group G form an closed algebra under Lie brackets.* It is called the Lie algebra.

Lie algebras are generally written as,

$$[X_a,\ X_b] = i f_{abc} X_c$$

The coefficients f_{abc} are known as the structure constants of the Lie group G.

Properties of f_{abc}:

1. $f_{abc} = -f_{bac}$
2. *The generators of a unitary representation of Lie group G are hermitian matrices*. Consequently, all of the structure constants are real,

$$f^*_{abc} = f_{abc}$$

3. The structure constants satisfy the so-called Jacobi identity,

$$f_{abd} f_{dce} + f_{bcd} f_{dae} + f_{cad} f_{dbe} = 0.$$

Proof:
The *reality* of f_{abc} is proved as follows,

$$\begin{aligned} -if^*_{abc}X_c &= (if_{abc}X_c)^\dagger = \{[X_a,\ X_b]\}^\dagger = (X_aX_b - X_bX_a)^\dagger \\ &= (X_b)^\dagger(X_a)^\dagger - (X_a)^\dagger(X_b)^\dagger \\ &= X_bX_a - X_aX_b = -[X_a,\ X_b] = -if_{abc}X_c \end{aligned}$$

Hence, $f^*_{abc} = f_{abc}$.

Similar to the Poisson brackets in classical mechanics, the Lie brackets obey the so-called Jacobi identity,

$$[[X_a,\ X_b],\ X_c] + \text{Cyclic Permutations} = 0.$$

Explicitly,

$$[[X_a,\ X_b],\ X_c] + [[X_b,\ X_c],\ X_a] + [[X_c,\ X_a],\ X_b] = 0.$$

Here we check this formula. By definition of the Lie brackets

$$\begin{aligned}[[X_a,\ X_b],\ X_c] &= [X_aX_b - X_bX_a,\ X_c] \\ &= (X_aX_b - X_bX_a)X_c - X_c(X_aX_b - X_bX_a) \\ &= X_aX_bX_c - X_bX_aX_c - X_cX_aX_b + X_cX_bX_a\end{aligned}$$

Cyclic permutations of above equation lead to

$$\begin{aligned}[[X_b,\ X_c],\ X_a] &= X_bX_cX_a - X_cX_bX_a - X_aX_bX_c + X_aX_cX_b \\ [[X_c,\ X_a],\ X_b] &= X_cX_aX_b - X_aX_cX_b - X_bX_cX_a + X_bX_aX_c\end{aligned}$$

Obviously, the sum of these three terms vanishes:

$$[[X_a,\ X_b],\ X_c] + [[X_b,\ X_c],\ X_a] + [[X_c,\ X_a],\ X_b] = 0.$$

Because

$$[[X_a,\ X_b],\ X_c] = [if_{abd}X_d,\ X_c] = -f_{abd}f_{dce}X_e$$

The Jacobi identities put some stringent constraints on the structure constants:

$$f_{abd}f_{dce} + f_{bcd}f_{dae} + f_{cad}f_{dbe} = 0.$$

Adjoint Representation:

Define a set of hermitian matrices T_a from the structure constants,

$$(T_a)_{bc} = -i f_{abc} \,, \quad (T_a)_{bc} = (T_a)^*_{cb}.$$

We can rewrite the above Jacobi identities as,

$$\begin{aligned} 0 &= f_{abd} f_{dce} + f_{bcd} f_{dae} + f_{cad} f_{dbe} \\ &= -f_{abd} f_{cde} + f_{cbd} f_{ade} - f_{acd} f_{dbe} \\ &= (T_a)_{bd} (T_c)_{de} - (T_c)_{bd} (T_a)_{de} - i f_{acd} (T_d)_{be} \\ &= \left([T_a,\ T_c] \right)_{be} - i f_{acd} (T_d)_{be} \end{aligned}$$

Therefore, the structure constants themselves generate a representation of the Lie algebra:

$$[T_a,\ T_c] = i f_{acd}\, T_d$$

It is called the adjoint representation.

Discussions:

- For a unitary adjoint representation of a Lie group G, because

$$(T_a)_{bc} = -i f_{abc}$$

  its hermitian generators are pure imaginary and then antisymmetric matrices. Hence, f_{abc} becomes totally antisymmetric about its indices. In particular,

$$f_{abc} = -f_{acb}.$$

- The dimension of the adjoint representation is just the number of independent generators, which is also the number of real parameters required to describe a group element.

- The scalar product in the linear space of the generators is defined as the following trace,

$$\mathrm{Tr}(X_a X_b)$$

which is symmetric for interchanging indices a and b.
In the adjoint representation,

$$\begin{aligned}\mathrm{Tr}(T_a T_b) &= (T_a)_{cd}(T_b)_{dc} \\ &= (-i f_{acd})(-i f_{bdc}) \\ &= -f_{acd} f_{bdc} \\ &= f_{acd} f_{bcd}\end{aligned}$$

Since the basic symmetric quantity is δ_{ab}, this scalar product can be cast as a simple canonical form,[1]

$$\mathrm{Tr}(T_a T_b) = \lambda^a \delta_{ab}$$

Therefore,

$$f_{acd} f_{bcd} \propto \delta_{ab}$$

[1] There is no sum over index a.

Explanation:

If $\mathrm{Tr}(T_a T_b) \neq \lambda^a \delta_{ab}$, we can give $(T_a)_{bc} = -i f_{abc}$ up and redefine a set of new generators for the adjoint representation. Firstly, let us do a linear transformation on generators X_a,

$$X_a \rightsquigarrow X'_a = L_{ab} X_b$$

L must be invertible, $X_b = (L^{-1})_{bc} X'_c$. The Lie bracket between new generators X'_a and X'_b is either

$$[X'_a,\ X'_b] = i f'_{abc} X'_c$$

or

$$\begin{aligned}[X'_a,\ X'_b] &= L_{ai} L_{bj} [X_i,\ X_j] = L_{ai} L_{bj}\ (i f_{ijk} X_k) \\ &= i L_{ai} L_{bj} (L^{-1})_{kc} f_{ijk}\ X'_c\end{aligned}$$

Therefore,

$$f_{abc} \rightsquigarrow f'_{abc} = L_{ai} L_{bj} (L^{-1})_{kc} f_{ijk}$$

The new generators of adjoint representation are then defined as:

$$\begin{aligned}(T'_a)_{bc} &= -if'_{abc} \\ &= L_{ai}L_{bj}(L^{-1})_{kc}(T_i)_{jk}\end{aligned}$$

The trace of the product of two new generators T'_a and T'_b reads,

$$\begin{aligned}\mathrm{Tr}(T'_aT'_b) &= (T'_a)_{cd}(T'_b)_{dc} \\ &= L_{ai}L_{cj}(L^{-1})_{kd}(T_i)_{jk}\ L_{bm}L_{dn}(L^{-1})_{lc}(T_m)_{nl} \\ &= \delta_{jl}\delta_{kn}\ L_{ai}L_{bm}(T_i)_{jk}(T_m)_{nl} \\ &= L_{ai}L_{bm}(T_i)_{jk}(T_m)_{kj} \\ &= L_{ai}\mathrm{Tr}(T_iT_m)(L^T)_{mb}\end{aligned}$$

The matrix consisting of $\mathrm{Tr}(T_iT_m)$ is real symmetrical which turns out to be a Hermitian matrix. Hence, we can diagonalize it with an appropriate orthogonal matrix L ($L^T = L^{-1}$). Suppose we have done this, so that

$$\mathrm{Tr}(T'_aT'_b) = k^a\delta_{ab} \text{ (no summation over index } a)$$

Compact Lie algebras:

From now on we shall assume that all of the coefficients in $\{\lambda^a\}$ are positive and equal to each other. This defines a class of algebras called compact Lie algebras:

$$\mathrm{Tr}(T_a T_b) = \lambda\, \delta_{ab}$$

The structure constants of a compact Lie algebra are completely antisymmetric,

$$\begin{aligned} f_{abc} &= -i\lambda^{-1}(i f_{abd})\lambda\delta_{dc} \\ &= -i\lambda^{-1}(i f_{abd})\mathrm{Tr}(T_d T_c) \\ &= -i\lambda^{-1}\mathrm{Tr}\big[(i f_{abd} T_d) T_c\big] \\ &= -i\lambda^{-1}\mathrm{Tr}\big\{[T_a,\, T_b] T_c\big\} \\ &= -i\lambda^{-1}\mathrm{Tr}(T_a T_b T_c - T_b T_a T_c) \end{aligned}$$

Namely,

$$f_{abc} = -f_{bac} = f_{bca} = -f_{cba} = f_{cab} = -f_{acb}$$

Theorem:

The adjoint representation of a compact Lie algebra is *unitary*.

In fact, the reality of f_{abc} and its symmetry guarantee that the generators $(T_a)_{bc} = -if_{abc}$ are not only pure imaginary but anti-symmetric also.

Therefore,

$$\begin{aligned}
[(T_a)^\dagger]_{bc} &= [(T_a)^*]_{cb} \\
&= [(T_a)_{cb}]^* \\
&= (-if_{acb})^* \\
&= if_{acb} \\
&= -if_{abc} \\
&= (T_a)_{bc}
\end{aligned}$$

Namely,

$$(T_a)^\dagger = T_a$$

This is very the expected hermitility.

Invariant subalgebra:

An invariant subalgebra is some set of generators $\mathcal{H} = \{X_a\}$ which goes into itself under Lie brackets with any element Y_b of the whole algebra,

$$[X_a, Y_b] = i f_{abc} X_c$$

for an arbitrary generator Y_b of group G.

When exponentiated, an invariant subalgebra generates an subgroup $H = \{h\}$ of G,

$$h = e^{i\alpha_a X_a}, \ \forall X_a \in \mathcal{H}.$$

For an arbitrary group element $g = e^{i\beta_b Y_b}$ in G, we see,

$$\begin{aligned} g^{-1}hg &= e^{-i\beta_b Y_b} e^{i\alpha_a X_a} e^{i\beta_c Y_c} = e^{-i\beta_b Y_b}\left[\sum_{n=0}^{\infty}\frac{i^n}{n!}(\alpha_a X_a)^n\right] e^{i\beta_c Y_c} \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\left[e^{-i\beta_b Y_b}(\alpha_a X_a) e^{i\beta_c Y_c}\right]^n \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}(\alpha_a X'_a)^n = e^{i\alpha_a X'_a} \end{aligned}$$

where

$$
\begin{aligned}
X'_a &= e^{-i\beta_b Y_b} X_a e^{i\beta_c Y_c} \\
&= X_a - i\beta_b [Y_b, X_a] - \frac{1}{2!}\beta_b\beta_c [Y_b,\ [Y_c,\ X_a]] + \cdots
\end{aligned}
$$

does still belong to the subalgebra $\mathcal{H}$. As a result, the considered exponentials form an invariant subgroup of G.

Remark:

The whole algebra and the null set ϕ are two trivial invariant subalgebras.

Simple Lie Algebras:

Definition:

A Lie algebra which has no nontrivial invariant subalgebras is called *simple Lie algebra*.
A simple Lie algebra generates a *simple Lie group*.

Theorem:
The adjoint representation of a simple Lie group G with generators $(T_a)_{bc} = -i f_{abc}$ satisfying

$$\mathrm{Tr}(T_a T_b) = \lambda \, \delta_{ab}$$

is irreducible.

Proof:

If the adjoint representation were reducible, there were an invariant subspace in the adjoint representation sapnned by some subset of generators,

$$T_j, \quad 1 \leqslant j \leqslant K$$

The rest of the generators are labeled as,

$$T_\alpha, \quad K+1 \leqslant \alpha \leqslant N$$

Because the indices j $(j = 1, 2, \cdots, K)$ label an invariant subspace, we must have

$$-if_{aj\beta} = (T_a)_{j\beta} = 0, \qquad \begin{cases} 1 \leqslant a \leqslant N \\ 1 \leqslant j \leqslant K \\ K+1 \leqslant \beta \leqslant N \end{cases}$$

If $Tr(T_a T_b) = \lambda\delta_{ab}$, *the structure constants are completely antisymmetric about their three indices.* Consequently, $f_{aj\beta} = 0$ means:

$$f_{ij\beta} = f_{j\beta i} = f_{\beta ij} = 0, \quad (1 \leqslant i, j \leqslant K, \ K+1 \leqslant \beta \leqslant N)$$

and

$$f_{\alpha j\beta} = f_{j\beta\alpha} = f_{\beta\alpha j} = 0, \quad (1 \leqslant j \leqslant K, \ K+1 \leqslant \alpha, \beta \leqslant N)$$

The nonzero structure constants would be:

$$f_{ijk}, \quad (1 \leqslant i, j, k \leqslant K)$$
$$f_{\alpha\beta\gamma}, \quad (K+1 \leqslant \alpha, \beta, \gamma \leqslant N)$$

The algebra contained two nontrivial invariant subalgebras, and not simple. *Contrary to the initial assumption !* Q.E.D.

Abelian invariant subalgebras:

An abelian invariant sub-algebra consists of a single generator which commutes with all of the generators of the Lie group G.

1. We call such a sub-algebra a $U(1)$ factor of the group.
2. If X_a is a $U(1)$ generator, $f_{abc} = 0$ for all possible b and c.

Semi-simple Lie algebras:

The Lie algebras without Abelian invariant sub-algebras are called semi-simple Lie algebras.

Cartan subalgebra:

In any Lie group, the maximum set of mutually commuting generators $H_a \ (a = 1, 2, \cdots, r)$ generates an abelian subalgebra $\mathbb{h}$,

$$[H_a, \ H_b] = 0$$

which is called the **Cartan subalgebra.**

1. The number of generators in $\mathbb{h}$ is the **rank** of the corresponding Lie algebra $\mathbb{g}$.
2. The Cartan generators H_a can be simultaneously diagonalized, and their eigenvalues or diagonal elements are the **weights**

$$H_a \left|\mu, x, D\right\rangle = \mu_a \left|\mu, x, D\right\rangle$$

   in which D labels the representation and x whatever other variables are needed to specify the state.
3. The vector $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_r)$ is called the **weight vector**.
4. The weights of the *adjoint representation* is called the **roots**.

Consider a Lie group G and its representation spanned by the states or column vectors

$$|i\rangle, \quad i = 1, 2, 3, \cdots$$

Generators:

The generators $\{X_a\}$ of this representation can be thought of as either linear operators acting on the representation space,

$$X_a |i\rangle = \sum_j |j\rangle\langle j| X_a |i\rangle = \sum_j |j\rangle (X_a)_{ji}$$

Group elements:

The group elements $e^{i\alpha_a X_a}$ can be thought of as transformations of the states,

$$e^{i\alpha_a X_a}: \quad |i\rangle \rightsquigarrow |i'\rangle = e^{i\alpha_a X_a} |i\rangle, \quad \langle i| \rightsquigarrow \langle i'| = \langle i| e^{-i\alpha_a X_a}.$$

For a state generated from $|i\rangle$ by acting an operator $\mathcal{O}$: $\mathcal{O}\,|i\rangle$, we see,

$$e^{i\alpha_a X_a}:\quad \mathcal{O}\,|i\rangle \rightsquigarrow \mathcal{O}'\,|i'\rangle \;= e^{i\alpha_a X_a}\,\mathcal{O}\,|i\rangle$$
$$= e^{i\alpha_a X_a}\,\mathcal{O} e^{-i\alpha_b X_b} e^{i\alpha_c X_c}\,|i\rangle$$
$$= e^{i\alpha_a X_a}\,\mathcal{O} e^{-i\alpha_b X_b}\,|i'\rangle$$

Hence,

$$e^{i\alpha_a X_a}:\quad \mathcal{O} \rightsquigarrow \mathcal{O}' = e^{i\alpha_a X_a}\,\mathcal{O} e^{-i\alpha_b X_b}$$

Invarant operators:

If $\mathcal{O}$ is an invariant operator under $G = \{e^{i\alpha_a X_a}\}$, then

$$[e^{i\alpha_a X_a},\ \mathcal{O}] = 0$$

Equivalently,

$$[X_a,\ \mathcal{O}] = 0, \quad \forall\ a$$

This conclusion can alternativley be obtained in the following manner. Under an infinitesimal transformation of Lie group G,

$$e^{i\alpha_a X_a} \approx 1 + i\alpha_a X_a$$

the variation of the operator $\mathscr{O}$ can be expressed as,

$$\begin{aligned} \delta\mathscr{O} &= \mathscr{O}' - \mathscr{O} \\ &= e^{i\alpha_a X_a}\mathscr{O}e^{-i\alpha_b X_b} - \mathscr{O} \\ &= (1 + i\alpha_a X_a)\mathscr{O}(1 - i\alpha_b X_b) - \mathscr{O} \end{aligned}$$

Namely,

$$\delta\mathscr{O} \approx i\alpha_a[X_a,\ \mathscr{O}]$$

- The invariance of $\mathscr{O}$ under this Lie group transformation is then recast as:

$$[X_a,\ \mathscr{O}] = 0, \quad \forall\ a.$$

Fun with exponentials:

As remarked previously, the exponential is alternatively defined as a power series expansion,

$$\exp(i\alpha_a X_a) = \sum_{n=0}^{\infty} \frac{i^n}{n!} (\alpha_a X_a)^n$$

In general, the generators do not commute mutually, $[X_a,\ X_b] \neq 0$. However,

$$\begin{aligned}
[\alpha_a X_a,\ \alpha_b X_b] &= (\alpha_a \alpha_b)[X_a,\ X_b] = i(\alpha_a \alpha_b) f_{abc} X_c \\
&= \frac{i}{2}(\alpha_a \alpha_b) f_{abc} X_c + \frac{i}{2}(\alpha_a \alpha_b) f_{abc} X_c \\
&= \frac{i}{2}\left[(\alpha_a \alpha_b) f_{abc} X_c + (\alpha_b \alpha_a) f_{bac} X_c\right] \\
&= \frac{i}{2}\left[(\alpha_a \alpha_b) f_{abc} X_c - (\alpha_a \alpha_b) f_{abc} X_c\right] \\
&= 0
\end{aligned}$$

As a result, for an arbitrary real parameter ξ,

$$\begin{aligned}\frac{\partial}{\partial \xi}\exp(i\xi\alpha_a X_a) &= i(\alpha_b X_b)\exp(i\xi\alpha_a X_a) \\ &= i\exp(i\xi\alpha_a X_a)(\alpha_b X_b)\end{aligned}$$

Question:

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = ?$$

It follows from the above definition that,

$$\begin{aligned}\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} &= \sum_{n=0}^{\infty} \frac{i^n}{n!}\partial_{\alpha_b}(\alpha_a X_a)^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n!}\left[\sum_{m=0}^{n-1}(i\alpha_a X_a)^m iX_b(i\alpha_c X_c)^{n-1-m}\right]\end{aligned}$$

Using the famous mathematical identity,

$$\frac{(n-1-m)!m!}{n!} = \frac{\Gamma(n-m)\Gamma(m+1)}{\Gamma(n+1)}$$
$$= B(n-m, m+1)$$
$$= \int_0^1 d\zeta \zeta^m (1-\zeta)^{(n-1-m)}$$

i.e,

$$1 = \frac{n!}{m!(n-1-m)!} \int_0^1 d\zeta \zeta^m (1-\zeta)^{(n-1-m)}$$

we reexpress the above derivative as,

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{m=0}^{n-1} \frac{n!}{m!(n-1-m)!} \int_0^1 d\zeta \zeta^m (1-\zeta)^{(n-1-m)} \right.$$
$$\left. (i\alpha_a X_a)^m i X_b (i\alpha_c X_c)^{(n-1-m)} \right]$$

i.e.,

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \int_0^1 d\zeta \left[\frac{(i\zeta \alpha_a X_a)^m}{m!} \right] (iX_b) \cdot \left\{ \frac{[i(1-\zeta)\alpha_c X_c]^{(n-m)}}{(n-m)!} \right\}$$

Because the factorial of an arbitrary negative integer is infinity, e.g.,

$$(-3)! = \infty$$

we can recast the above equation as

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^1 d\zeta \left[\frac{(i\zeta \alpha_a X_a)^m}{m!} \right] (iX_b) \cdot \left\{ \frac{[i(1-\zeta)\alpha_c X_c]^{(n-m)}}{(n-m)!} \right\}$$

By order exchange of summations, we have:

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_0^1 d\zeta \left[\frac{(i\zeta \alpha_a X_a)^m}{m!} \right] (iX_b) \cdot \left\{ \frac{[i(1-\zeta)\alpha_c X_c]^{(n-m)}}{(n-m)!} \right\}$$

$$= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \int_0^1 d\zeta \left[\frac{(i\zeta \alpha_a X_a)^m}{m!} \right] (iX_b) \cdot \left\{ \frac{[i(1-\zeta)\alpha_c X_c]^{(n-m)}}{(n-m)!} \right\}$$

Equivalently,

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \int_0^1 d\zeta \left[\sum_{m=0}^{\infty} \frac{(i\zeta \alpha_a X_a)^m}{m!} \right] (iX_b) \left\{ \sum_{k=0}^{\infty} \frac{[i(1-\zeta)\alpha_c X_c]^k}{k!} \right\}$$

That is,

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \int_0^1 d\zeta \; e^{i\zeta\alpha_a X_a} \; iX_b \; e^{i(1-\zeta)\alpha_c X_c}$$

Homework:

1. Find the explicit expression of the matrix $e^{i\alpha A}$ with
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$
2. If $[A, \; B] = B$, calculate $e^{i\alpha A} B e^{-i\alpha A}$.
3. Carry out the expansion of γ_c in
$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\gamma_c X_c}$$
to third order of α_a and β_b.

现代数学物理方法

第二章, 李群

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Rotation group $SO(3)$:

Consider a vector $\vec{r}$ in 3-dimensional space,

$$\vec{r} = \sum_{a=1}^{3} \vec{e}_a x_a \sim \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Rotation:

A linear transformation g

$$g : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = g \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

that leaves the bilinear form $\sum_{a=1}^{3} x_a x_a = x_1^2 + x_2^2 + x_3^2$ invariant is called a 3-dimensional rotation.

Because

$$x_1^2 + x_2^2 + x_3^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1'^2 + x_2'^2 + x_3'^2 = \begin{bmatrix} x_1' & x_2' & x_3' \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} g^T g \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

the 3-dimensional rotation transformations should be expressed as a set of 3×3 real orthogonal matrices,

$$g^T g = 1$$

Therefore,

$$1 = \det\left(g^T g\right) = \left[\det(g)\right]^2 \quad \rightsquigarrow \quad \det(g) = \pm 1$$

The determinant of every orthogonal matrix is either

$$\det(g) = +1$$

in which case the transformation describes pure rotation, or

$$\det(g) = -1$$

in which case it describes a rotation-reflection.

Orthogonal group $O(3)$:

The aggregate of all real orthogonal 3-dimensional matrices

$$g^T g = 1, \quad \det\{g\} = \pm 1$$

forms a Lie group, $O(3)$, the so-called 3-dimensional orthogonal group.

$SO(3)$:

Special orthogonal group $SO(3)$:

The aggregate of all pure 3-dimensional rotations

$$g^T g = 1, \quad \det(g) = 1$$

forms a Lie group, $SO(3)$, the 3-dimensional special orthogonal group.

Question:

What is the orthogonal matrix describing a pure rotation with an angle ψ about some direction

$$\vec{n} = \sin\theta\cos\phi\vec{e}_1 + \sin\theta\sin\phi\vec{e}_2 + \cos\theta\vec{e}_3 \sim \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} ?$$

Solution:

In 3-dimensional Cartesian space, the other two *independent* unit vectors orthogonal to $\vec{n}$ read

$$\begin{aligned}\vec{t}_1 &= \cos\theta\cos\phi\vec{e}_1 + \cos\theta\sin\phi\vec{e}_2 - \sin\theta\vec{e}_3,\\ \vec{t}_2 &= -\sin\phi\vec{e}_1 + \cos\phi\vec{e}_2.\end{aligned}$$

From these three unit vectors we find the following *pure rotation* from $\vec{e}_3$ to $\vec{n}$:

$$h = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Evidently,

$$h: \quad \vec{e}_3 \sim \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \rightsquigarrow \quad h\vec{e}_3 \sim h\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \sim \vec{n}$$

The expected orthogonal matrix describing the pure rotation with an angle ψ about the direction $\vec{n}$ is,

$$
\begin{aligned}
g &= h \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} h^T \\
&= \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}
\end{aligned}
$$

The explicit expressions for matrix elements, for example, read

$$
g_{11} = c_\psi + s_\theta^2 c_\phi^2 (1 - c_\psi), \qquad g_{12} = s_\theta^2 c_\phi s_\phi (1 - c_\psi) - c_\theta s_\psi,
$$
$$
g_{13} = s_\theta c_\theta c_\phi (1 - c_\psi) + s_\theta s_\phi s_\psi, \quad \cdots
$$

where $c_\theta = \cos\theta$ and $s_\psi = \sin\psi$, eta.

In general,

$$\big[g(\theta,\phi,\psi)\big]_{ab} = \delta_{ab}c_\psi + n_a n_b(1 - c_\psi) - \epsilon_{abc}n_c s_\psi$$

where indices a, b and c take their values from 1 to 3, and $n_1 = s_\theta c_\phi$, $n_2 = s_\theta s_\phi$ and $n_3 = c_\theta$.

Generators of $SO(3)$:

In this definition representation, the generators of $SO(3)$ are defined by,

$$\big[X(\theta,\phi)\big]_{ab} = -i\partial_\psi\big[g(\theta,\phi,\psi)\big]_{ab}\,|_{\psi=0} = i\epsilon_{abc}n_c$$

Along the 3 axes of the Cartisian coordinate frame, we have:

$$(X_1)_{ab} = i\epsilon_{ab1} = i(\delta_{a2}\delta_{b3} - \delta_{a3}\delta_{b2}), \quad \rightsquigarrow \quad X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$$

$$(X_2)_{ab} = i\epsilon_{ab2} = i(\delta_{a3}\delta_{b1} - \delta_{a1}\delta_{b3}), \quad \rightsquigarrow \quad X_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$$

$$(X_3)_{ab} = i\epsilon_{ab3} = i(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1}), \quad \rightsquigarrow \quad X_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In short, in Cartisian coordinates, the generators of $SO(3)$ are as follows:

$$(X_a)_{mn} = i\epsilon_{mna}$$

Based on the famous mathematical identity

$$\epsilon_{ijk}\epsilon_{mnk} = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})$$

we get:

$$\begin{aligned} [X_a,\ X_b]_{mn} &= (X_a)_{mk}(X_b)_{kn} - (X_b)_{mk}(X_a)_{kn} \\ &= -\epsilon_{mka}\epsilon_{knb} + \epsilon_{mkb}\epsilon_{kna} = \epsilon_{amk}\epsilon_{bnk} - \epsilon_{bmk}\epsilon_{ank} \\ &= \delta_{ab}\delta_{mn} - \delta_{an}\delta_{mb} - \delta_{ba}\delta_{mn} + \delta_{bn}\delta_{ma} \\ &= \delta_{am}\delta_{bn} - \delta_{an}\delta_{bm} = \epsilon_{abc}\epsilon_{mnc} \\ &= -i\epsilon_{abc}(i\epsilon_{mnc}) = -i\epsilon_{abc}(X_c)_{mn} \end{aligned}$$

That is,

$$[X_a,\ X_b] = -i\epsilon_{abc}X_c$$

The structure constants of $SO(3)$ are components ϵ_{ijk} of the Levi-Civita antisymmetric tensor.

Relying on the fact,

$$-(X_a)_{bc} = -i\epsilon_{abc}$$

the definition representation of $SO(3)$ is just its adjoint representation.

Casimir operators:

Casimir operators of a Lie group are such operators that commute with all generators of the group.

- $SO(3)$ has one Casimir operator:

$$X^2 = \sum_{a=1}^{3} X_a X_a$$

Racah Theorem :

Here is a simple check:

$$
\begin{aligned}
[X^2,\ X_a] &= \sum_{b=1}^{3}[X_b X_b,\ X_a] = \sum_{b=1}^{3}\Big\{[X_b,\ X_a]X_b + X_b[X_b,\ X_a]\Big\} \\
&= \sum_{b,c=1}^{3}\left(-i\epsilon_{bac}X_c X_b - i\epsilon_{bac}X_b X_c\right) \\
&= i\sum_{b,c=1}^{3}\epsilon_{abc}(X_b X_c + X_c X_b) \ = 0.
\end{aligned}
$$

Racah theorem:

For any semi-simple Lie group G *of rank* l*, there exists a set of* l *Casimir operators,*

$$
C_\lambda = C_\lambda(X_1,\ X_2,\ \cdots,X_N), \quad (1 \leqslant \lambda \leqslant l)
$$

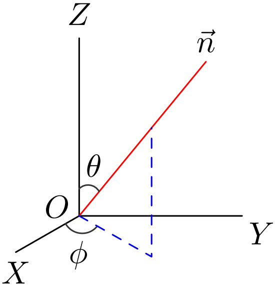
that commute with every generator of the group and therefore also amongst themselves, $[C_\lambda,\ C_\sigma] = 0$*.*

Group elements of $SO(3)$:

The general group elements of $SO(3)$, which describe the pure rotation with an angle ψ about the direction $\vec{n} = (s_\theta c_\phi, s_\theta s_\phi, c_\theta)$, read:[1]

$$[g(\theta, \phi, \psi)]_{ab} = \delta_{ab}\, c_\psi + n_a n_b\, (1 - c_\psi) - \epsilon_{abc} n_c\, s_\psi$$

where $n_1 = s_\theta c_\phi$, $n_2 = s_\theta s_\phi$ and $n_3 = c_\theta$.



[1] The ranges for the parameters take their values are $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi, \psi \leqslant 2\pi$.

In particular,

$$g\left(\frac{\pi}{2}, 0, \psi\right) \equiv R_x(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

Similarly,

$$g\left(\frac{\pi}{2}, \frac{\pi}{2}, \psi\right) \equiv R_y(\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}$$

and

$$g(0, 0, \psi) \equiv R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With the previously defined generators,

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

these special group elements of $SO(3)$ can be expressed as

$$R_x(\psi) = e^{i\psi X_1}, \quad R_y(\psi) = e^{i\psi X_2}, \quad R_z(\psi) = e^{i\psi X_3}$$

In general,

$$g(\theta, \phi, \psi) \equiv R_{\vec{n}}(\psi) = e^{i\psi \vec{n}\cdot\vec{X}} = e^{i\psi(s_\theta c_\phi X_1 + s_\theta s_\phi X_2 + c_\theta X_3)}$$

Our check is as follows:

$$(\vec{n} \cdot \vec{X})_{ij} = n_a (X_a)_{ij} = i\epsilon_{ija} n_a$$

$$
\begin{aligned}
\left[(\vec{n}\cdot\vec{X})^2\right]_{ij} &= (\vec{n}\cdot\vec{X})_{ik}(\vec{n}\cdot\vec{X})_{kj} \\
&= (i\epsilon_{ika}n_a)(i\epsilon_{kjb}n_b) \\
&= -\epsilon_{ika}\epsilon_{kjb}n_a n_b \\
&= \epsilon_{iak}\epsilon_{jbk}n_a n_b \\
&= (\delta_{ij}\delta_{ab} - \delta_{ib}\delta_{ja})n_a n_b \\
&= \delta_{ij}n_a n_a - n_i n_j \\
&= \delta_{ij} - n_i n_j
\end{aligned}
$$

In the last step, we have used the the condition $n_a n_a = 1$ for unit vector $\vec{n}$. Moreover,

$$
\begin{aligned}
\left[(\vec{n}\cdot\vec{X})^3\right]_{ij} &= \left[(\vec{n}\cdot\vec{X})^2\right]_{ik}(\vec{n}\cdot\vec{X})_{kj} \\
&= (\delta_{ik} - n_i n_k)(-i\epsilon_{kja}n_a) \\
&= -i\epsilon_{ija}n_a + i\epsilon_{akj}n_a n_k n_i \\
&= -i\epsilon_{ija}n_a = (\vec{n}\cdot\vec{X})_{ij}
\end{aligned}
$$

$$\left[(\vec{n}\cdot\vec{X})^4\right]_{ij} = [(\vec{n}\cdot\vec{X})^3]_{ik}(\vec{n}\cdot\vec{X})_{kj} = (\vec{n}\cdot\vec{X})_{ik}(\vec{n}\cdot\vec{X})_{kj} = [(\vec{n}\cdot\vec{X})^2]_{ij}$$

In general, for an arbitrary positive integer $m \in \mathbb{Z}^+$,

$$[(\vec{n}\cdot\vec{X})^{2m-1}]_{ij} = i\epsilon_{ija}n_a, \quad [(\vec{n}\cdot\vec{X})^{2m}]_{ij} = \delta_{ij} - n_i n_j \ .$$

Hence,

$$\begin{aligned}
\left[e^{i\psi(\vec{n}\cdot\vec{X})}\right]_{ij} &= \left[1 + i\psi(\vec{n}\cdot\vec{X}) + \frac{i^2\psi^2}{2!}(\vec{n}\cdot\vec{X})^2 + \frac{i^3\psi^3}{3!}(\vec{n}\cdot\vec{X})^3 + \cdots\right] \\
&= \delta_{ij} + i(\vec{n}\cdot\vec{X})_{ij}\left[\psi - \frac{\psi^3}{3!} + \cdots\right] \\
&\quad + [(\vec{n}\cdot\vec{X})^2]_{ij}\left[-\frac{\psi^2}{2!} + \frac{\psi^4}{4!} - \cdots\right] \\
&= \delta_{ij} + i(\vec{n}\cdot\vec{X})_{ij}s_\psi + [(\vec{n}\cdot\vec{X})^2]_{ij}(c_\psi - 1) \\
&= \delta_{ij} - \epsilon_{ija}n_a s_\psi + (\delta_{ij} - n_i n_j)(c_\psi - 1)
\end{aligned}$$

As expected,

$$\left[e^{i\psi(\vec{n}\cdot\vec{X})}\right]_{ij} = c_\psi \delta_{ij} + n_i n_j (1 - c_\psi) - \epsilon_{ijk} n_k s_\psi = \left[g(\theta, \phi, \psi)\right]_{ij}$$

In matrix form,
the group elements of $SO(3)$ in its adjoint representation are expressed as:

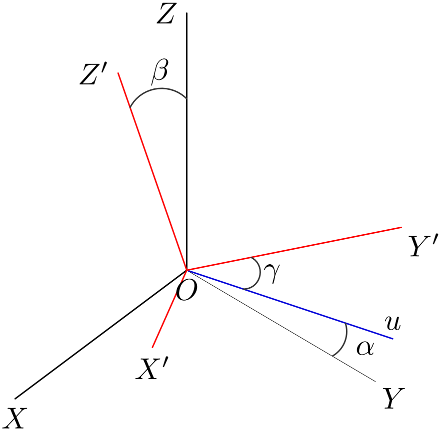
$$g(\theta, \phi, \psi) = e^{i\psi(\vec{n}\cdot\vec{X})} = e^{i\psi(s_\theta c_\phi X_1 + s_\theta s_\phi X_2 + c_\theta X_3)}$$

where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi,\ \psi \leqslant 2\pi$.

Evidently,
3 parameters are required to describe an arbitrary 3-dimensional rotation. *They may be related to the rotation axis*[2] *and the angle* ψ *of rotation.*

[2]The axis $\vec{n}$ is described by 2 parameters θ and ϕ. Since $g(\vec{n}, \psi) = g(-\vec{n}, 2\pi - \psi)$, the space of $SO(3)$ group parameters is a sphere of radius π, i.e., $0 \leqslant \phi \leqslant 2\pi$ and $0 \leqslant \theta,\ \psi \leqslant \pi$, if the one-to-one correspondence exists between the parameters and the $SO(3)$ group elements.

Alternatively, the 3 parameters may be chosen as Euler angles, defined as the *three successive angles of rotation* by the sequent rotations from the fixed system of axes $Oxyz$:



1. Rotate through angle α about axis Oz, carrying Oy into Ou;
2. Rotate through angle β about axis Ou, carrying Oz into Oz';
3. Rotate through angle γ about axis Oz', carrying Ou into Oy';

At the end of this process Ox will have been carried into Ox'. The range of these Euler angles is $0 \leqslant \alpha, \gamma \leqslant 2\pi$ and $0 \leqslant \beta \leqslant \pi$.

Euler angle representation:

The net rotation is described by the orthogonal matrix,

$$R(\alpha,\ \beta,\ \gamma) = e^{i\gamma X_{z'}} e^{i\beta X_u} e^{i\alpha X_z} = R_{z'}(\gamma) R_u(\beta) R_z(\alpha)$$

Because the factor rotation $R_z(\alpha) = e^{i\alpha X_z}$ carries axis Oy into ou,

$$X_u = R_z(\alpha) X_y R_z(-\alpha) = e^{i\alpha X_z} X_y e^{-i\alpha X_z}$$

Hence,

$$R_u(\beta) = e^{i\beta X_u} = e^{i\alpha X_z} e^{i\beta X_y} e^{-i\alpha X_z}$$

Similarly, because $R_u(\beta)$ carries axis Oz into Oz', we have,

$$R_{z'}(\gamma) = e^{i\gamma X_{z'}} = e^{i\beta X_u} e^{i\gamma X_z} e^{-i\beta X_u}$$

Consequently,

$$\begin{aligned}
R(\alpha,\ \beta,\ \gamma) &= R_{z'}(\gamma) R_u(\beta) R_z(\alpha) \\
&= \left[e^{i\beta X_u} e^{i\gamma X_z} e^{-i\beta X_u} \right] e^{i\beta X_u}\ R_z(\alpha) \\
&= e^{i\beta X_u}\ e^{i\gamma X_z}\ R_z(\alpha) \\
&= \left[e^{i\alpha X_z}\ e^{i\beta X_y}\ e^{-i\alpha X_z} \right]\ e^{i\gamma X_z}\ e^{i\alpha X_z} \\
&= e^{i\alpha X_z} e^{i\beta X_y} e^{i\gamma X_z}
\end{aligned}$$

In conclusion, an arbitrary pure rotation in 3-dimensional Cartesian space can be recast as

$$R(\alpha,\ \beta,\ \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma) = e^{i\alpha X_z} e^{i\beta X_y} e^{i\gamma X_z}$$

in terms of Euler angles α, β and γ in the original fixed coordinate system $Oxyz$.

The range of Euler angles:

It follows from the explicit orthogonal matrices $R_y(\beta)$ and $R_z(\alpha)$ that,

$$R_z(\gamma)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}c_\gamma & -s_\gamma & 0\\ s_\gamma & c_\gamma & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$R_y(\beta)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}c_\beta & 0 & s_\beta\\ 0 & 1 & 0\\ -s_\beta & 0 & c_\beta\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}s_\beta\\0\\c_\beta\end{bmatrix}$$

$$R_z(\alpha)\begin{bmatrix}s_\beta\\0\\c_\beta\end{bmatrix} = \begin{bmatrix}c_\alpha & -s_\alpha & 0\\ s_\alpha & c_\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}s_\beta\\0\\c_\beta\end{bmatrix} = \begin{bmatrix}s_\beta c_\alpha\\ s_\beta s_\alpha\\ c_\beta\end{bmatrix}$$

It implies,

$$R(\alpha,\ \beta,\ \gamma)\begin{bmatrix}0\\0\\1\end{bmatrix} = R_z(\alpha)R_y(\beta)R_z(\gamma)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}s_\beta c_\alpha\\ s_\beta s_\alpha\\ c_\beta\end{bmatrix}$$

Namely,

$$R(\alpha,\ \beta,\ \gamma)\vec{e}_3 = \vec{n} = s_\beta c_\alpha \vec{e}_1 + s_\beta s_\alpha \vec{e}_2 + c_\beta \vec{e}_3$$

Hence $0 \leqslant \alpha \leqslant 2\pi$ and $0 \leqslant \beta \leqslant \pi$.

Similarly,

$$[R_z(\alpha)]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_\alpha & s_\alpha & 0 \\ -s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[R_y(\beta)]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_\beta \\ 0 \\ c_\beta \end{bmatrix}$$

$$[R_z(\gamma)]^T \begin{bmatrix} -s_\beta \\ 0 \\ c_\beta \end{bmatrix} = \begin{bmatrix} c_\gamma & c_\gamma & 0 \\ -s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_\beta \\ 0 \\ c_\beta \end{bmatrix} = \begin{bmatrix} -s_\beta c_\gamma \\ s_\beta s_\gamma \\ c_\beta \end{bmatrix}$$

These formulae yield,

$$[R(\alpha,\ \beta,\ \gamma)]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [R_z(\gamma)]^T [R_y(\beta)]^T [R_z(\alpha)]^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_\beta c_\gamma \\ s_\beta s_\gamma \\ c_\beta \end{bmatrix}$$

That is to say,

$$\begin{aligned} [R(\alpha,\ \beta,\ \gamma)]^T \vec{e}_3 &= \vec{n}' \\ &= -s_\beta c_\gamma \vec{e}_1 + s_\beta s_\gamma \vec{e}_2 + c_\beta \vec{e}_3 \\ &= s_\beta c_{(\pi-\gamma)} \vec{e}_1 + s_\beta s_{(\pi-\gamma)} \vec{e}_2 + c_\beta \vec{e}_3 \end{aligned}$$

Hence $0 \leqslant (\pi - \gamma) \leqslant 2\pi$ or equivalently $-\pi \leqslant \gamma \leqslant \pi$.

We conclude that the range of Euler angles in $R(\alpha, \beta, \gamma)$ are:

$$0 \leqslant \alpha,\ \gamma \leqslant 2\pi, \quad 0 \leqslant \beta \leqslant \pi.$$

$SO(3)$ rotation in Hilbert space:

Scalar wave function :

Scalar wave-function has one-component $\psi(\vec{x})$. Under a rotation of position coordinates, $\vec{x} \rightsquigarrow \vec{x}' = R\vec{x}$, it remains invariant,

$$\psi(\vec{x}) \rightsquigarrow \psi'(\vec{x}') = \psi(\vec{x})$$

As a result,

$$\psi'(\vec{x}) = \psi(R^{-1}\vec{x})$$

Here R^{-1} is the inverse of a 3×3 coordinate rotation matrix R.

Let us introduce the operator $\mathcal{R}$ in Hilbert space to describe *the rotation of the wave functions themselves*,

$$\begin{aligned} \vec{x} &\rightsquigarrow \vec{x}' = R\vec{x}, \\ \psi(\vec{x}) &\rightsquigarrow \psi'(\vec{x}) = \mathcal{R}\psi(\vec{x}) \end{aligned}$$

Therefore,

$$\mathcal{R}\psi(\vec{x}) = \psi(R^{-1}\vec{x})$$

The complete set of operators $\{\mathcal{R}\}$ defines a representation of $SO(3)$, called *the rotation group in Hilbert space.*

Proof:

The unit element in $\{\mathcal{R}\}$ does trivially exist. Moreover, under two successive coordinate rotations,

$$\vec{x} \rightsquigarrow \vec{x}' = R_1\vec{x} \rightsquigarrow \vec{x}'' = R_2\vec{x}' = R_2R_1\vec{x}$$

the scalar wave function $\psi(\vec{x})$ transforms into:

$$\psi(\vec{x}) \rightsquigarrow \psi'(\vec{x}') = \psi(\vec{x}) \rightsquigarrow \psi''(\vec{x}'') = \psi'(\vec{x}') = \psi(\vec{x})$$

Namely,

$$\psi''(\vec{x}) = \psi((R_2R_1)^{-1}\vec{x})$$

On the other hand, $\mathcal{R}_1\psi(\vec{x}) = \psi'(\vec{x})$ and $\mathcal{R}_2\psi'(\vec{x}) = \psi''(\vec{x})$. Hence,

$$\psi''(\vec{x}) = \mathcal{R}_2\psi'(\vec{x}) = \mathcal{R}_2\mathcal{R}_1\psi(\vec{x})$$

By comparison, we get

$$\mathcal{R}_2\mathcal{R}_1\psi(\vec{x}) = \psi((R_2R_1)^{-1}\vec{x})$$

This justifies that the rule

$$\mathcal{R}\psi(\vec{x}) = \psi(R^{-1}\vec{x})$$

is kept by the successive transformations, as expected. So $\{\mathcal{R}\}$ forms a representation of $SO(3)$ in Hilbert space.

- Recall that the rotation matrices in coordinate space are expressed as $R_{\vec{n}}(\psi) = e^{i\psi(\vec{n}\cdot\vec{X})}$, whose infinitesimal form reads,

$$[R_{\vec{n}}(\varphi)]_{ij} \approx \delta_{ij} + i\varphi(\vec{n}\cdot\vec{X})_{ij} = \delta_{ij} - \varphi\epsilon_{ijk}n_k$$

Hence, the infinitesimal rotation in Hilbert space should satisfy,

$$\begin{aligned}\mathcal{R}_{\vec{n}}(\varphi)\psi(\vec{x}) &= \psi(R_{\vec{n}}^{-1}(\varphi)\vec{x}) = \psi([R_{\vec{n}}^{-1}(\varphi)]_{ij}x_j) \\ &= \psi(x_i + \varphi\epsilon_{ijk}x_jn_k) \\ &= \psi(\vec{x}) + \varphi\epsilon_{ijk}x_jn_k\partial_{x_i}\psi(\vec{x}) + \cdots\end{aligned}$$

Namely,

$$\mathcal{R}_{\vec{n}}(\varphi)\psi(\vec{x}) \approx \psi(\vec{x}) - \varphi n_i \epsilon_{ijk} x_j \partial_k \psi(\vec{x})$$

Generators:

Define the generators L_i $(i = 1, 2, 3)$ of $SO(3)$ in Hilbert space by

$$\mathcal{R}_{\vec{n}}(\varphi) \approx 1 - i\varphi(\vec{n} \cdot \vec{L})$$

- These generators turn out to be the orbital angular momentum operators:

$$L_i = -i\epsilon_{ijk} x_j \partial_k$$

- It is easy to check that

$$[L_i,\ L_j] = i\epsilon_{ijk} L_k$$

Multicomponent wave functions :

Under a 3-dimensional rotation $\vec{x} \rightsquigarrow \vec{x}' = R\vec{x}$ in coordinate space, the components of a multicomponent wave function

$$\begin{bmatrix} \psi_1(\vec{x}) \\ \psi_2(\vec{x}) \\ \vdots \\ \psi_N(\vec{x}) \end{bmatrix}$$

transform as,

$$\mathcal{R}\psi_a(\vec{x}) = D_{ab}\psi_b(R^{-1}\vec{x}), \qquad (a, b = 1, 2, \cdots, N)$$

In addition to the coordinate transformation $R^{-1}\vec{x}$, *a $N \times N$ matrix D has to act on the internal degrees of freedom so that a linear combination of the wave function components forms.*

Hence,

$$\mathcal{R}_{\vec{n}}(\varphi) = e^{-i\varphi(\vec{n}\cdot\vec{L})} D_{\vec{n}}(\varphi)$$

The matrix D must be unitary and so it can be written as:

$$D_{\vec{n}}(\varphi) = e^{-i\varphi(\vec{n}\cdot\vec{S})}$$

with the $N \times N$ hermitian matrices $\vec{S}$ obeying Lie brackets

$$[S_i,\ S_j] = i\epsilon_{ijk}S_k$$

and

$$[S_i,\ L_j] = 0$$

Such a $\vec{S}$ is called the spin angular momentum of the particle described by the given multi-component wave function. *e.g.*,

1. $N = 1$, scalar.
2. $N = 2$, spinor.
3. $N = 3$, vector.
4. $N = 4$, double-spinor ?

$SO(N)$:

$O(N)$:

The orthogonal group $O(N)$ is formed by the set of all $N \times N$ real orthogonal matrices

$$R^T R = 1, \quad R^* = R$$

under the matrix multiplications.

- Obviously,

$$\det R = \pm 1$$

- The condition $R^T R = 1$ stands for $N(N+1)/2$ independent constraints

$$R_{ij} R_{ik} = \delta_{jk}$$

  Hence, the number of independent real parameters for describing an $O(N)$ group element is:

$$g = N^2 - \frac{1}{2} N(N+1) = \frac{1}{2} N(N-1)$$

$SO(N)$:

$SO(N)$ is the normal subgroup of $O(N)$ consisting of the $N \times N$ real orthogonal matrices with unit determinant,

$$\det R = 1$$

Remarks:

- The total number of real independent parameters for describing a $SO(N)$ group element is $N(N-1)/2$.
- These real parameters can be written as

$$\omega_{ab}, \qquad (a, b = 1, 2, \cdots, N)$$

with antisymmetry,

$$\omega_{ab} = -\omega_{ba}$$

Consequently, an arbitrary $SO(N)$ group element is expressed as,

$$R = \exp\left[-i \sum_{b>a} \sum_{a=1}^{N-1} \omega_{ab}\, T_{ab}\right]$$

where T_{ab} with symmetry $T_{ab} = -T_{ba}$ are $N(N-1)/2$ generators of $SO(N)$.

Discussions:

- Because R is real and unitary, *each generator T_{ab} is purely imaginary and antisymmetric hermitian matrix.*
- $\det R = 1$ requires that *all T_{ab} are traceless.*

We choose the generators of $SO(N)$ in its definition representation as

$$(T_{ab})_{jk} = -i(\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj})$$

where indices a, b label the name of the generator T_{ab}, while indices j, k specify the matrix element of T_{ab}.

Obviously,

1. T_{ab} are purely imaginary.
2. $(T_{ab})_{jk} = -(T_{ab})_{kj}$
3. $\text{Tr}(T_{ab}) = (T_{ab})_{jj} = -i(\delta_{aj}\delta_{bj} - \delta_{aj}\delta_{bj}) = -i(\delta_{ab} - \delta_{ab}) = 0$

$so(N)$ algebra is,

$$\begin{aligned}
[T_{ab},\ T_{cd}]_{ij} &= (T_{ab})_{ik}(T_{cd})_{kj} - (T_{cd})_{ik}(T_{ab})_{kj} \\
&= -(\delta_{ai}\delta_{bk} - \delta_{ak}\delta_{bi})(\delta_{ck}\delta_{dj} - \delta_{cj}\delta_{dk}) \\
&\qquad +(\delta_{ci}\delta_{dk} - \delta_{ck}\delta_{di})(\delta_{ak}\delta_{bj} - \delta_{aj}\delta_{bk}) \\
&= -i\delta_{bc}(T_{ad})_{ij} + i\delta_{bd}(T_{ac})_{ij} + i\delta_{ac}(T_{bd})_{ij} - i\delta_{ad}(T_{bc})_{ij}
\end{aligned}$$

Namely,

$$[T_{ab},\ T_{cd}] = -i(\delta_{ad}T_{bc} + \delta_{bc}T_{ad} - \delta_{ac}T_{bd} - \delta_{bd}T_{ac})$$

Equivalently,

$$[T_{ab},\ T_{cd}] = i f_{ab,cd,ij} T_{ij}$$

where the structure constants

$$\begin{aligned}
f_{ab,cd,ij} &= \frac{1}{2}\Big[\delta_{ad}\delta_{ci}\delta_{bj} - \delta_{ad}\delta_{bi}\delta_{cj} + \delta_{bc}\delta_{di}\delta_{aj} - \delta_{bc}\delta_{ai}\delta_{dj} \\
&\qquad - \delta_{ac}\delta_{di}\delta_{bj} + \delta_{ac}\delta_{bi}\delta_{dj} - \delta_{bd}\delta_{ci}\delta_{aj} + \delta_{bd}\delta_{ai}\delta_{cj}\Big]
\end{aligned}$$

are completely antisymmetric for exchanging any two groups of indices.

Note:

- The definition representation of $SO(N)$ is not its *adjoint* representation for $N \neq 3$. The former is N-dimensional, but the latter has dimension $N(N-1)/2$.
- Due to the complete antisymmetry of the structure constants, the adjoint representation of $SO(N)$ is unitary.
- For $SO(2M)$ and $SO(2M+1)$, the mutually commuting generators are:

$$H_a = T_{(2a-1)(2a)}, \quad (1 \leqslant a \leqslant M)$$

The normalization conditions of the $SO(N)$ generators in its definition representation read,

$$\begin{aligned}
\mathrm{Tr}(T_{ab}T_{cd}) &= (T_{ab})_{ij}(T_{cd})_{ji} \\
&= -(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{di} - \delta_{ci}\delta_{dj}) \\
&= 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})
\end{aligned}$$

Definition Rep. of $SU(N)$:

The aggregate of all $N \times N$ unitary matrices $\{u\}$ with unit determinant provides the group $SU(N)$,

$$u^{\dagger}u = uu^{\dagger} = 1, \quad \det u = 1$$

Number of the real parameters :

- The unitary condition can be written as

$$\delta_{ij} = (u^{\dagger})_{ik}u_{kj} = u^{*}_{ki}u_{kj}$$

  It gives N real constraints when $i = j$ while $N(N-1)/2$ complex constraints or equivalently $N(N-1)$ real constraints when $i \neq j$.

- $\det u = 1$ gives an additional constraint.

Totally, the number of real independent parameters for describing an arbitrary $SU(N)$ group element should be,

$$g = 2N^2 - N - N(N-1) - 1 = N^2 - 1$$

These $N^2 - 1$ real parameters could be chosen to be

$$\begin{cases} \omega_{ab}^{(1)} \\ \omega_{ab}^{(2)} \\ \omega_{c}^{(3)} \qquad a = 1, 2, \cdots, N-1; \quad a < b; \quad b, c = 2, 3, \cdots, N \end{cases}$$

with properties

$$\omega_{ab}^{(1)} = \omega_{ba}^{(1)}, \;\; \omega_{ab}^{(2)} = -\omega_{ba}^{(2)}.$$

Generators:

The (N^2-1) *traceless hermitian* generators of the definition Rep. of unitary group $SU(N)$ could be chosen as follows:

1. $N(N-1)/2$ *hermitian* $T^{(1)}_{ab}$ $(a<b)$ *with* $T^{(1)}_{ab}=T^{(1)}_{ba}$
2. $N(N-1)/2$ *hermitian* $T^{(2)}_{ab}$ $(a<b)$ *with* $T^{(2)}_{ab}=-T^{(2)}_{ba}$
3. $(N-1)$ *diagonal hermitian* $T^{(3)}_{c}$

so that

$$u=\exp\left[\sum_{a<b}\sum_{b=2}^{N}(\omega^{(1)}_{ab}T^{(1)}_{ab}+\omega^{(2)}_{ab}T^{(2)}_{ab})+\sum_{c=2}^{N}\omega^{(3)}_{c}T^{(3)}_{c}\right]$$

The matrix elements of these traceless hermitian generators can explicitly be defined as,

$$(T^{(1)}_{ab})_{ij}=\frac{1}{2}\left(\delta_{ai}\delta_{bj}+\delta_{aj}\delta_{bi}\right)$$

$$(T_{ab}^{(2)})_{ij} = -\frac{i}{2}\Big(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\Big)$$

and

$$(T_c^{(3)})_{ij} = \begin{cases} \delta_{ij}\frac{1}{\sqrt{2c(c-1)}}, & \text{if} \quad i < c\,; \\ -\delta_{ij}\sqrt{\frac{(c-1)}{2c}}, & \text{if} \quad i = c\,; \\ 0, & \text{if} \quad i > c. \end{cases}$$

For $SU(2)$, they are simply related to the famous Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Obviously,

$$T_{12}^{(1)} = \sigma_1/2, \quad T_{12}^{(2)} = \sigma_2/2, \quad T_2^{(3)} = \sigma_3/2.$$

$SU(2)$:

Remainder:

The aggregate of all unitary matrices of order 2 and determinant unity forms the group $SU(2)$.

An arbitrary $SU(2)$ group element has the form,

$$u(\omega) = e^{i\left[\omega_{12}^{(1)}T_{12}^{(1)} + \omega_{12}^{(2)}T_{12}^{(2)} + \omega_{2}^{(3)}T_{2}^{(3)}\right]}$$

Equivalently,

$$u(\vec{n}, \psi) = e^{i\psi(\vec{n}\cdot\vec{\sigma})/2}$$

where

$$\vec{n} = c_\theta \vec{e}_3 + s_\theta c_\phi \vec{e}_1 + s_\theta s_\phi \vec{e}_2$$

is a two-parameter unit vector in the 3-dimensional parameter space. So, $-\vec{n} = c_{(\pi-\theta)}\vec{e}_3 + s_{(\pi-\theta)}c_{(\pi+\phi)}\vec{e}_1 + s_{(\pi-\theta)}s_{(\pi+\phi)}\vec{e}_2$.

The Pauli matrices satisfy relation

$$\sigma_a \sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c.$$

Hence,

$$(\vec{n}\cdot\vec{\sigma})^2 = n_a n_b \sigma_a \sigma_b = n_a n_b(\delta_{ab} + i\epsilon_{abc}\sigma_c) = n_a n_a = 1$$

The $SU(2)$ group element becomes,

$$\begin{aligned}
u(\vec{n},\psi) &= e^{i\psi(\vec{n}\cdot\vec{\sigma})/2} \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!}(\psi/2)^n(\vec{n}\cdot\vec{\sigma})^n \\
&= \cos(\psi/2) + i\sin(\psi/2)(\vec{n}\cdot\vec{\sigma}) \\
&= \cos(\psi/2) + i\sin(\psi/2)\begin{bmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\psi/2) + i\sin(\psi/2)c_\theta & i\sin(\psi/2)s_\theta e^{-i\phi} \\ i\sin(\psi/2)s_\theta e^{i\phi} & \cos(\psi/2) - i\sin(\psi/2)c_\theta \end{bmatrix}
\end{aligned}$$

It follows from

$$u(\vec{n},\psi) = \begin{bmatrix} \cos(\psi/2) + i\sin(\psi/2)c_\theta & i\sin(\psi/2)s_\theta e^{-i\phi} \\ i\sin(\psi/2)s_\theta e^{i\phi} & \cos(\psi/2) - i\sin(\psi/2)c_\theta \end{bmatrix}$$

that:

1. $\det u = \cos^2(\psi/2) + \sin^2(\psi/2)c_\theta^2 + \sin^2(\psi/2)s_\theta^2 = 1.$
2. $u(\vec{n},\psi)$ is indeed unitary, $u^\dagger(\vec{n},\psi) = u^{-1}(\vec{n},\psi)$, with

$$u^\dagger(\vec{n},\psi) = \begin{bmatrix} \cos(\psi/2) - i\sin(\psi/2)c_\theta & -i\sin(\psi/2)s_\theta e^{-i\phi} \\ -i\sin(\psi/2)s_\theta e^{i\phi} & \cos(\psi/2) + i\sin(\psi/2)c_\theta \end{bmatrix}$$

3. $u(\vec{n},2\pi) = -1$ while $u(\vec{n},\psi) = -u(-\vec{n},2\pi-\psi)$. Therefore, the range for these 3 real parameters taking their values could be,

$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi \leqslant 2\pi, \quad 0 \leqslant \psi \leqslant 2\pi.$$

4. There is a Homomorphism between the groups $SO(3)$ and $SU(2)$,

$$u^\dagger(\vec{n},\psi)\sigma_b u(\vec{n},\psi) = \sum_{a=1}^{3} \sigma_a \left[R(\vec{n},\psi)\right]_{ab}$$

Homomorphism between $SO(3)$ and $SU(2)$:

So, two $SU(2)$ matrices, $u(\vec{n}, \psi)$ and $u(-\vec{n}, 2\pi - \psi)$, correspond to the same $SO(3)$ rotation $R(\vec{n}, \psi)$.

Proof:
Consider an arbitrary vector $\vec{r}$ in the $SU(2)$ parameter space,

$$\vec{r} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Because

$$u(\vec{n}, \psi) = e^{i\psi(\vec{n}\cdot\vec{\sigma})/2} = \cos(\psi/2) + i\sin(\psi/2)(\vec{n}\cdot\vec{\sigma})$$

we have

$$\begin{aligned} & u^\dagger(\vec{n}, \psi)(\vec{r}\cdot\vec{\sigma})u(\vec{n}, \psi) \\ & \quad = \Big[\cos(\psi/2) - i\sin(\psi/2)(\vec{n}\cdot\vec{\sigma})\Big](\vec{r}\cdot\vec{\sigma}) \\ & \qquad \cdot\Big[\cos(\psi/2) + i\sin(\psi/2)(\vec{n}\cdot\vec{\sigma})\Big] \end{aligned}$$

$$= \cos^2(\psi/2)(\vec{r}\cdot\vec{\sigma}) - i\sin(\psi/2)\cos(\psi/2)\left[(\vec{n}\cdot\vec{\sigma}),\ (\vec{r}\cdot\vec{\sigma})\right] + \sin^2(\psi/2)(\vec{n}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma})$$

Employment of identity $\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$ yields,

$$[(\vec{n}\cdot\vec{\sigma}),\ (\vec{r}\cdot\vec{\sigma})] = n_a x_b[\sigma_a, \sigma_b] = 2i n_a x_b \epsilon_{abc}\sigma_c = 2i(\vec{n}\times\vec{r}\,)\cdot\vec{\sigma}$$

and

$$\begin{aligned}
(\vec{n}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma}) &= n_a n_b x_c \sigma_a\sigma_c\sigma_b \\
&= n_a n_b x_c(\delta_{ac} + i\epsilon_{acd}\sigma_d)\sigma_b \\
&= (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) + i n_a n_b x_c \epsilon_{acd}(\delta_{db} + i\epsilon_{dbe}\sigma_e) \\
&= (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - i n_a n_b x_c \epsilon_{abc} - n_a n_b x_c(\epsilon_{acd}\epsilon_{bed})\sigma_e \\
&= (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - n_a n_b x_c(\delta_{ab}\delta_{ce} - \delta_{ae}\delta_{cb})\sigma_e \\
&= (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - (\vec{r}\cdot\vec{\sigma}) + (\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) \\
&= 2(\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - (\vec{r}\cdot\vec{\sigma})
\end{aligned}$$

Therefore,

$$\begin{aligned}
u^\dagger(\vec{n},\psi)(\vec{r}\cdot\vec{\sigma})u(\vec{n},\psi) &= \Big[\cos^2(\psi/2) - \sin^2(\psi/2)\Big](\vec{r}\cdot\vec{\sigma}) \\
&\quad + 2\sin(\psi/2)\cos(\psi/2)(\vec{n}\times\vec{r}\,)\cdot\vec{\sigma} \\
&\quad + 2\sin^2(\psi/2)(\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) \\
&= \cos\psi(\vec{r}\cdot\vec{\sigma}) + \sin\psi(\vec{n}\times\vec{r}\,)\cdot\vec{\sigma} + (1-\cos\psi)(\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) \\
&= \cos\psi\,\sigma_a x_a + \sin\psi\,\sigma_a\epsilon_{acb}n_c x_b + (1-\cos\psi)n_b x_b n_a\sigma_a \\
&= \sigma_a\Big[\delta_{ab}\cos\psi + n_a n_b(1-\cos\psi) - \epsilon_{abc}n_c\sin\psi\Big]x_b
\end{aligned}$$

Recall that the $SO(3)$ group element

$$R(\vec{n},\psi) \equiv g(\theta,\phi,\psi) = e^{i\psi(\vec{n}\cdot\vec{X})}$$

can explicitly be expressed as

$$\big[R(\vec{n},\psi)\big]_{ab} = \delta_{ab}\cos\psi + n_a n_b(1-\cos\psi) - \epsilon_{abc}n_c\sin\psi$$

Therefore,

$$u^\dagger(\vec{n},\psi)(\vec{r}\cdot\vec{\sigma})u(\vec{n},\psi) = \sigma_a\big[R(\vec{n},\psi)\big]_{ab} x_b$$

It implies that the unitary group $SU(2)$ is homomorphic to the orthogonal group $SO(3)$,

$$u^\dagger(\vec{n},\psi)\,\sigma_b\,u(\vec{n},\psi) = \sigma_a\big[R(\vec{n},\psi)\big]_{ab}$$

Recall that

$$R(-\vec{n},\ 2\pi-\psi) = R(\vec{n},\psi)$$

we have also,

$$\begin{aligned} u^\dagger(-\vec{n},2\pi-\psi)\,\sigma_b\,u(-\vec{n},2\pi-\psi) &= \sigma_a\big[R(-\vec{n},2\pi-\psi)\big]_{ab} \\ &= \sigma_a\big[R(\vec{n},\psi)\big]_{ab} \end{aligned}$$

Therefore, two unitary matrices of $SU(2)$:

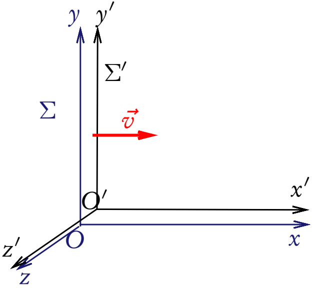
$$u(\vec{n},\psi),\quad u(-\vec{n},2\pi-\psi) = -u(\vec{n},\psi)$$

are mapped to the same rotation matrix $R(\vec{n},\psi)$ in $SO(3)$.

The genuine Lorentz transformations (LTs), called **boost**, are those connecting two inertial frames moving with a relative speed v.

If the relative motion ia along the common x_1-direction, boost is:

$$\begin{aligned} x_1' &= \gamma(x_1 - \beta ct) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ ct' &= \gamma(ct - \beta x_1) \end{aligned}$$



where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

Introduce the so-called boost parameter ζ by setting,

$$\gamma = \cosh\zeta\ , \qquad \gamma\beta = -\sinh\zeta\ .$$

Genuine LTs can be viewed as pseudo-orthogonal transformations in 4-dimensional Minkowski space $\mathbb{M}_4$,

$$\begin{bmatrix} ct' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

As expected,

$$\cosh^2\zeta - \sinh^2\zeta = \gamma^2 - \gamma^2\beta^2 = \left[\frac{1}{\sqrt{1-\beta^2}}\right]^2 (1-\beta^2) = 1$$

- The characteristic of Lorentz transformations is that they preserve the invariance of the **interval:**

$$S^2 = x_1^2 + x_2^2 + x_3^2 - c^2t^2 = x_1'^2 + x_2'^2 + x_3'^2 - c^2t'^2$$

The boost matrix

$$B = \begin{bmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are not orthogonal matrices, $BB^T \neq 1$. However, by introducing the metric matrix η in $\mathbb{M}_4$,

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we have:

$$B^{-1} = \eta B^T \eta = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let

$$X = \begin{bmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

the boosts and the **interval** can be expressed as

$$X' = BX, \quad S^2 = X^T \eta X$$

The interval invariance under the boosts is then manifest,

$$\begin{aligned} S'^2 &= X'^T \eta X' = X^T B^T \eta B X \\ &= X^T \eta (\eta B^T \eta) B X = X^T \eta B^{-1} B X = X^T \eta X = S^2 \end{aligned}$$

The general form of boosts reads,

$$\begin{cases} ct' &= \gamma(ct - \vec{\beta}\cdot\vec{x}) \\ \vec{x}' &= -\gamma\vec{\beta}ct + \vec{x} + \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}) \end{cases}$$

Thereby,

$$B = \begin{bmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\frac{\gamma^2\beta_1^2}{\gamma+1} & \frac{\gamma^2\beta_1\beta_2}{\gamma+1} & \frac{\gamma^2\beta_1\beta_3}{\gamma+1} \\ -\gamma\beta_2 & \frac{\gamma^2\beta_2\beta_1}{\gamma+1} & 1+\frac{\gamma^2\beta_2^2}{\gamma+1} & \frac{\gamma^2\beta_2\beta_3}{\gamma+1} \\ -\gamma\beta_3 & \frac{\gamma^2\beta_3\beta_1}{\gamma+1} & \frac{\gamma^2\beta_3\beta_2}{\gamma+1} & 1+\frac{\gamma^2\beta_3^2}{\gamma+1} \end{bmatrix}$$

- Describing an arbitrary boost requires 3 real independent parameters.
- These parameters can be chosen as β_a $(a = 1, 2, 3)$.

Using these parameters, the infinitesimal Lorentz boosts can be cast as,

$$B \approx 1 + \beta_a \frac{\partial B}{\partial \beta_a}|_{\vec{\beta}=0} = 1 + i\beta_a K_a$$

The generators for Lorentz boost are then:

$$K_a = -i \frac{\partial B}{\partial \beta_a}|_{\vec{\beta}=0}, \qquad (a = 1, 2, 3).$$

Recall $\gamma = 1/\sqrt{1-\beta^2}$. We have,

$$\frac{\partial \gamma}{\partial \beta_a} = -\gamma^3 \beta_a$$

This formula enables us to find out the explicit matrices of the boost generators:

$$K_1 = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_2 = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_3 = -i \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Obviously, these generators are not hermitian matrices:

$$K_a^\dagger = -K_a.$$

In terms of matrix elements, these boost generators have the form:

$$(K_a)_{\mu\nu} = -i(\delta_{\mu 0}\delta_{\nu a} + \delta_{\mu a}\delta_{\nu 0}), \qquad (a = 1, 2, 3).$$

Therefore,

$$\begin{aligned}
[K_a,\ K_b]_{\mu\nu} &= (K_a)_{\mu\rho}(K_b)_{\rho\nu} - (K_b)_{\mu\rho}(K_a)_{\rho\nu} \\
&= -(\delta_{\mu 0}\delta_{\rho a} + \delta_{\mu a}\delta_{\rho 0})(\delta_{\rho 0}\delta_{\nu b} + \delta_{\rho b}\delta_{\nu 0}) \\
&\qquad\qquad + (\delta_{\mu 0}\delta_{\rho b} + \delta_{\mu b}\delta_{\rho 0})(\delta_{\rho 0}\delta_{\nu a} + \delta_{\rho a}\delta_{\nu 0}) \\
&= -(\delta_{a\mu}\delta_{b\nu} - \delta_{a\nu}\delta_{b\mu})
\end{aligned}$$

Namely,

$$\begin{aligned}
[K_a,\ K_b]_{\mu 0} &= 0, \\
[K_a,\ K_b]_{0\nu} &= 0, \\
[K_a,\ K_b]_{de} &= -(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}) = -\epsilon_{abc}\epsilon_{cde}
\end{aligned}$$

Introducing 4×4 matrices $(J_a)_{\mu\nu}$ $(a = 1, 2, 3)$ by,

$$(J_a)_{\mu 0} = (J_a)_{0\nu} = 0, \qquad (J_a)_{bc} = -i\epsilon_{abc}$$

then,

$$[K_a,\ K_b]_{\mu\nu} = -i\epsilon_{abc}(J_c)_{\mu\nu} \quad \rightsquigarrow \quad [K_a,\ K_b] = -i\epsilon_{abc}J_c$$

We see that the genuine Lorentz boosts do not form a group.

The above matrix J_a $(a = 1, 2, 3)$ can be written into compact forms,

$$(J_a)_{\mu\nu} = -\frac{i}{2}\epsilon_{abc}\Big[\delta_{b\mu}\delta_{c\nu} - \delta_{b\nu}\delta_{c\mu}\Big]$$

- Each J_a is purely imaginary and antisymmetric. So, all three J_a's are hermitian matrices.
- In fact, J_a are generators of 3-d rotations in 4-dimensional Minkowski space.

Together with the boost generators K_a $(a = 1, 2, 3)$, these six traceless matrices form a closed algebra under Lie brackets,

$$\begin{cases} [K_a,\ K_b] = -i\epsilon_{abc}J_c \\ [K_a,\ J_b] = i\epsilon_{abc}K_c \\ [J_a,\ K_b] = i\epsilon_{abc}K_c \\ [J_a,\ J_b] = i\epsilon_{abc}J_c \end{cases}$$

It is called Lorentz algebra or $so(3,1)$ algebra.

We can redefine the hermitian generators of Lorentz group $SO(3,1)$ as follows:

$$J_a^{\pm} = \frac{1}{2}\Big[J_a \pm iK_a\Big] \qquad (a = 1, 2, 3).$$

Evidently,

$$(J_a^{\pm})^{\dagger} = \frac{1}{2}\Big[J_a^{\dagger} \mp iK_a^{\dagger}\Big] = \frac{1}{2}\Big[J_a \pm iK_a\Big] = J_a^{\pm}$$

With these hermitian generators, $so(3,1)$ algebra becomes,

$$[J_a^+,\ J_b^+] = i\epsilon_{abc}J_c^+$$
$$[J_a^-,\ J_b^-] = i\epsilon_{abc}J_c^-$$
$$[J_a^+,\ J_b^-] = 0$$

This shows that $\{J_a^+\}$ and $\{J_a^-\}$ each generate a group $SU(2)$, and the two groups commute.

Hence the Lorentz algebra $so(3,1)$ is equivalent to two copies of $su(2)$,

$$so(3,1) \sim su(2) \times su(2)$$

$SO(3,1)$ group elements:

In terms of the *exponential* parameterization, the group elements of Lorentz group $SO(3,1)$ are expressed as:

$$D(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \exp\left[-i \sum_{a=1}^{3} (\theta_a J_a + \lambda_a K_a)\right]$$

in some finite-dimensional representations. Surprisingly, each of them is a direct product of two $SU(2)$ group elements in their non-unitary representations:

$$D(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{-i(\theta_a - i\lambda_a)J_a^+} \, e^{-i(\theta_a + i\lambda_a)J_a^-}$$

Homework:

1. The generators of Lorentz group $SO(3,1)$ are

$$(K_a)_{\mu\nu} = -i\Big[\delta_{\mu 0}\delta_{\nu a} + \delta_{\mu a}\delta_{\nu 0}\Big]$$

$$(J_a)_{\mu\nu} = -\frac{i}{2}\epsilon_{abc}\Big[\delta_{b\mu}\delta_{c\nu} - \delta_{b\nu}\delta_{c\mu}\Big]$$

where $a, b, c = 1, 2, 3$ but $\mu, \nu = 0, 1, 2, 3$.
Please check the $so(3,1)$ algebra by computing all possible Lie brackets.

现代数学物理方法

第二章, 李群

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Unitary group $SU(2)$ has 3 independent generators

$$J_a, \qquad a = 1, 2, 3$$

which satisfy the Lie brackets,

$$[J_a,\ J_b] = i\epsilon_{abc} J_c\ , \qquad (1 \leqslant a, b, c \leqslant 3)$$

This is known as $su(2)$ algebra.

Remark:

- The $SU(2)$ structure constants ϵ_{abc} is completely anti symmetric for exchanging any two indices. Therefore,

the adjoint representation of $SU(2)$ is unitary.

Question :

What is the *adjoint* representation of $su(2)$ algebra ?

Answer :

The adjoint representation of $SU(2)$ is generated by the following traceless hermitian matrices,

$$(T_a)_{bc} = -i\epsilon_{abc}, \qquad (1 \leqslant a, b, c \leqslant 3)$$

It is 3-dimensional.

Obviously,

$$\begin{aligned}
[T_a,\ T_b]_{ij} &= (T_a)_{ik}(T_b)_{kj} - (T_b)_{ik}(T_a)_{kj} \\
&= -\epsilon_{aik}\,\epsilon_{bkj} + \epsilon_{bik}\,\epsilon_{akj} \\
&= -\delta_{aj}\,\delta_{bi} + \delta_{ai}\,\delta_{bj} \\
&= \epsilon_{abc}\,\epsilon_{ijc} \\
&= i\epsilon_{abc}\left[-i\epsilon_{cij}\right] = i\epsilon_{abc}\,(T_c)_{ij}
\end{aligned}$$

$\rightsquigarrow \quad [T_a,\ T_b] = i\epsilon_{abc}\,T_c$

The explicit matrices of the $SU(2)$ adjoint representation generators read,

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

1. Relying on the fact that $(T_a)_{jk} = -i\epsilon_{ajk}$, we have:

$$\mathrm{Tr}(T_a T_b) = (T_a)_{jk}(T_b)_{kj} = (-i)^2 \epsilon_{ajk}\epsilon_{bkj} = \epsilon_{ajk}\epsilon_{bjk} = 2\delta_{ab}$$

Therefore, the adjoint representation of $SU(2)$ is irreducible.

Our Goal here is to find out all of the finite dimensional irreducible representations of $SU(2)$.

J_3 eigenstates:

To conveniently find a finite-dimensional irreducible representations of a Lie algebra, we have to diagonalize as many of the generators in the algebra as possible.

$su(2)$ is a simple Lie algebra, in which the 3 generators don't commute with one another.

Consequently, we can only diagonalize one generator, say J_3,

$$J_3 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

where m_i is the eigenvalues of J_3,

$$J_3 \left|m_i\right\rangle = m_i \left|m_i\right\rangle$$

and $i = 1, 2, \cdots, N$.

Discussions:

1. In an irreducible representation with finite dimensions, the number of J_3's eigenvalues is obviously finite, i.e.,

   $$N \text{ takes a finite value,}$$

   among which exists the highest eigenvalue.

2. Call the highest eigenvalue of J_3 as j,

   $$J_3 |j, \alpha\rangle = j |j, \alpha\rangle$$

   where α is another label necessary if there is more than one state of highest J_3.

3. The states of the representation space can be normalized so that

   $$\langle j, \alpha | j, \beta\rangle = \delta_{\alpha\beta}$$

$su(2)$'s adjoint representation :

Consider the adjoint representation of $su(2)$.

Let the eigenvalue equation of T_3 be

$$T_3 \, |\lambda\rangle = \lambda \, |\lambda\rangle$$

Recall that

$$T_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that the eigenvalues of T_3 obey an algebraic equation,

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \qquad \rightsquigarrow \; -\lambda^3 + \lambda = 0,$$

Its solutions are:

$$\lambda = 0, \; \pm 1.$$

- The highest eigenvalue of T_3 is 1.
- Complete list of solutions to the eigenvalue problem of T_3 is:

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad |\lambda_2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad |\lambda_3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

$$\lambda_1 = 1 \qquad \lambda_2 = 0 \qquad \lambda_3 = -1$$

From these eigenvectors we can define a unitary matrix U:

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

Its inverse reads,

$$U^{-1} = U^{\dagger} = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \end{bmatrix}$$

The matrix U enables us to diagonalize the $SU(2)$ adjoint representation generator T_3,

$$\begin{aligned}
T_3^1 &= U^\dagger T_3 U \\
&= \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 0 \\ -1/\sqrt{2} & -i/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{aligned}$$

The other two generators of $SU(2)$ in its adjoint representation become,

$$T_1^1 = U^\dagger T_1 U = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$T_2^1 = U^\dagger T_2 U = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Remark:

- Among the 3 independent generators T_a^1 of $SU(2)$ adjoint representation, only is T_3^1 a diagonal matrix.

  Consequently,

  The adjoint representation of $\mathfrak{su}(2)$ algebra is irreducible.

$J_{\pm}$:

The $su(2)$ algebra can alternatively be formulated as:

$$[J_3,\ J_{\pm}] = \pm J_{\pm}, \quad [J_+,\ J_-] = J_3$$

if we introduce the so-called *raising* and *lowering* operators

$$J_{\pm} = \frac{1}{\sqrt{2}}\Big[J_1 \pm iJ_2\Big]$$

- $J_{\pm}$ are not hermitian. The meaning of $J_{\pm}$ can be revealed by the comparison of eigenvalue equation

$$J_3 \left|m\right\rangle = m \left|m\right\rangle$$

and its inference,

$$\begin{aligned} J_3 J_{\pm} \left|m\right\rangle &= \Big\{[J_3,\ J_{\pm}] + J_{\pm} J_3\Big\} \left|m\right\rangle \\ &= \Big\{ \pm J_{\pm} + J_{\pm} m\Big\} \left|m\right\rangle = (m \pm 1) J_{\pm} \left|m\right\rangle \end{aligned}$$

We now try to build the finite dimensional irreducible representations of $su(2)$. The key idea is to use the *raising* and *lowering* operators $J_\pm$.

Step 1.

Because we have assumed that j is the highest value of J_3, there is no state with $J_3 = j + 1$. Therefore,

$$J_+ |j, \alpha\rangle = 0, \ \ \forall \alpha$$

Of course, the states $|j, \alpha\rangle$ with different α are orthogonal

$$\langle j, \alpha | j, \beta \rangle = \delta_{\alpha\beta}$$

On the other hand,

$$J_- |j, \alpha\rangle = N_j(\alpha) |j - 1, \alpha\rangle$$

with $N_j(\alpha)$ the normalization coefficient.

Notice that

$$\left(J_{\pm}\right)^{\dagger} = J_{\mp}, \quad \left(|\psi\rangle\right)^{\dagger} = \langle\psi|$$

and

$$\langle j-1, \alpha | j-1, \beta \rangle = \delta_{\alpha\beta}$$

we have:

$$\begin{aligned}
N_j(\beta)^* N_j(\alpha)\delta_{\alpha\beta} &= N_j(\beta)^* N_j(\alpha) \langle j-1, \beta | j-1, \alpha \rangle \\
&= \langle j, \beta | J_+ J_- | j, \alpha \rangle \\
&= \langle j, \beta | [J_+, \ J_-] | j, \alpha \rangle \\
&= \langle j, \beta | J_3 | j, \alpha \rangle = \langle j, \beta | j | j, \alpha \rangle \\
&= j \langle j, \beta | j, \alpha \rangle \\
&= j \delta_{\alpha\beta} \qquad \rightsquigarrow \quad N_j(\alpha) = \sqrt{j} \equiv N_j
\end{aligned}$$

Hence,

$$J_- |j, \alpha\rangle = N_j |j-1, \alpha\rangle, \qquad \rightsquigarrow \quad |j-1, \alpha\rangle = \frac{1}{N_j} J_- |j, \alpha\rangle$$

The last equation further implies that,

$$\begin{aligned} J_+ |j-1,\alpha\rangle &= \frac{1}{N_j} J_+ J_- |j,\alpha\rangle \qquad \left\{ \text{Reminder: } N_j = \sqrt{j}\ . \right\} \\ &= \frac{1}{N_j} [J_+,\ J_-] |j,\alpha\rangle \\ &= \frac{1}{N_j} J_3 |j,\alpha\rangle \\ &= \frac{j}{N_j} |j,\alpha\rangle = N_j |j,\alpha\rangle \end{aligned}$$

So far we have achieved the following conclusion:

$$J_- |j,\alpha\rangle = N_j |j-1,\alpha\rangle, \qquad J_+ |j-1,\alpha\rangle = N_j |j,\alpha\rangle.$$

Step 2:

Focus on the states $J_- |j-1,\alpha\rangle$.

By an similar procedure, we can find out a set of orthonormal states $|j-2,\alpha\rangle$ which satisfy,

$$\langle j-2,\alpha|j-2,\beta\rangle = \delta_{\alpha\beta}$$

and

$$J_-\,|j-1,\alpha\rangle = N_{j-1}\,|j-2,\alpha\rangle, \qquad J_+\,|j-2,\alpha\rangle = N_{j-1}\,|j-1,\alpha\rangle.$$

Question :

What is the coefficient N_{j-1} equal to ? $\quad N_{j-1} \stackrel{?}{=} \sqrt{j-1}$

Step 3:

By continuing the procedure, we can easily build a series of orthonormal states $|j-k,\alpha\rangle$,

$$\langle j-k,\alpha|j-k,\beta\rangle = \delta_{\alpha\beta}, \qquad k = 0,\ 1,\ 2,\ \cdots$$

such that

$$\begin{cases} J_-\,|j-k,\alpha\rangle = N_{j-k}\,|j-k-1,\alpha\rangle, \\ J_+\,|j-k-1,\alpha\rangle = N_{j-k}\,|j-k,\alpha\rangle. \end{cases}$$

Explanation :

In general, we should express the action of $J_\pm$ as follows:

$$\begin{cases} J_- \left|j-k,\alpha\right\rangle = N_{j-k} \left|j-k-1,\alpha\right\rangle, \\ J_+ \left|j-k-1,\alpha\right\rangle = \widetilde{N}_{j-k} \left|j-k,\alpha\right\rangle. \end{cases}$$

Notice that,

$$\begin{aligned} N_{j-k} &= N_{j-k} \langle j-k-1,\alpha | j-k-1,\alpha\rangle \\ &= \langle j-k-1,\alpha | J_- | j-k,\alpha\rangle \end{aligned}$$

Because we have assumed that N_{j-k} is real, we have:

$$\begin{aligned} N_{j-k} &= N^*_{j-k} \\ &= \langle j-k,\alpha | J_+ | j-k-1,\alpha\rangle \\ &= \widetilde{N}_{j-k} \langle j-k,\alpha | j-k,\alpha\rangle \end{aligned}$$

That is,

$$N_{j-k} = \widetilde{N}_{j-k}$$

Hence, it is not necessary to distinguish N_{j-k} and $\widetilde{N}_{j-k}$.

The normalization coefficients N_{j-k} are generally chosen to be real, and determined by a *recursion* relation. Because,

$$\begin{aligned}
\left(N_{j-k}\right)^2 &= \left(N_{j-k}\right)^2 \langle j-k-1,\alpha|j-k-1,\alpha\rangle \\
&= \langle j-k,\alpha|\,J_+J_-\,|j-k,\alpha\rangle \\
&= \langle j-k,\alpha|\left\{[J_+,\ J_-]+J_-J_+\right\}|j-k,\alpha\rangle \\
&= \langle j-k,\alpha|\,J_3\,|j-k,\alpha\rangle + \langle j-k,\alpha|\,J_-J_+\,|j-k,\alpha\rangle \\
&= (j-k) + \left(N_{j-k+1}\right)^2
\end{aligned}$$

the expected recursion relation is,

$$\left(N_{j-k}\right)^2 - \left(N_{j-k+1}\right)^2 = j-k, \qquad k = 0,\ 1,\ 2,\ \cdots$$

- Setting $k = 1$ in the recursion relation gives,

$$\left(N_{j-1}\right)^2 = \left(N_j\right)^2 + (j-1) \ = j + (j-1) \ = 2j-1$$

$$\rightsquigarrow \quad N_{j-1} = \sqrt{2j-1} \neq \sqrt{j-1}\,.$$

It follows from the above recursion relation that,

$$\begin{aligned}
(N_j)^2 &= j \\
(N_{j-1})^2 - (N_j)^2 &= j-1 \\
(N_{j-2})^2 - (N_{j-1})^2 &= j-2 \\
(N_{j-3})^2 - (N_{j-2})^2 &= j-3 \\
\cdots & \quad \cdots \\
(N_{j-k})^2 - (N_{j-k+1})^2 &= j-k
\end{aligned}$$

The summation of these equations yields:

$$\left(N_{j-k}\right)^2 = \sum_{n=0}^{k}(j-n) = j(k+1) - \frac{k(k+1)}{2} = \frac{1}{2}(k+1)(2j-k)$$

i.e.,

$$N_m = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$$

Consequently,

$$\begin{aligned}
J_- \left|m, \alpha\right\rangle &= \tfrac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\left|m-1, \alpha\right\rangle \\
J_+ \left|m-1, \alpha\right\rangle &= \tfrac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\left|m, \alpha\right\rangle \qquad \forall \ m \leqslant j
\end{aligned}$$

Step 4:

The representations under consideration are assumed to have finite dimensions. Therefore, *there must be some maximum number of the lowering operators, p,* that we can apply to $|j,\alpha\rangle$

$$(J_-)^p\,|j,\alpha\rangle \;\propto\; |j-p,\alpha\rangle$$

so that

$$J_-\,|j-p,\alpha\rangle = 0\,.$$

Since,

$$J_-\,|j-k,\alpha\rangle = N_{j-k}\,|j-k-1,\alpha\rangle \;=\sqrt{\frac{(2j-k)(k+1)}{2}}\,|j-k-1,\alpha\rangle$$

we have:

$$N_{j-p} = \sqrt{\frac{(2j-p)(p+1)}{2}} = 0, \qquad \rightsquigarrow \quad j = \frac{p}{2}$$

p is obviously a non-negative integer. As a result,

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$$

Discussions:

1. The lowest value of m (the eigenvalue of J_3) is,

$$m_{\text{min}} = j - p = j - 2j = -j$$

2. The operator J_3 has $(2j+1)$ possible eigenvalues in total,

$$J_3 \left|m, \alpha\right\rangle = m \left|m, \alpha\right\rangle, \qquad -j \leqslant m \leqslant j \,.$$

Remark :

The parameter α for denoting the states $\left|m, \alpha\right\rangle$ is in fact unwanted.

- All of the $SU(2)$ generators do not change α. *The representation space breaks into subspaces that are invariant under $su(2)$, one for each value of α.*
- Due to the assumption of *irreducibility*, there must be only one α value. So we can drop the parameter α entirely.

In standard notation, we label the states of the *irreducible representations of* $su(2)$ by 2 parameters

$$|jm\rangle$$

where,

1. j is the highest eigenvalue of J_3 in the considered representation.
2. m is the eigenvalue of J_3 in a concrete state in the representation.

In short, the spin-j representation of $su(2)$ is defined by

$$\begin{cases} J_3\,|jm\rangle = m\,|jm\rangle \\ J_\pm\,|jm\rangle = \frac{1}{\sqrt{2}}\sqrt{(j \mp m)(j \pm m + 1)}\,|j, m \pm 1\rangle \end{cases}$$

where

$$j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2, \cdots$$

and

$$-j \leqslant m \leqslant j$$

The spin-j representation of $su(2)$ has dimensions of $(2j + 1)$.

In spin-j representation, the matrix elements of the $SU(2)$ generators are given by,

$$
\begin{aligned}
(J_3^j)_{m'm} &= \langle jm'|\, J_3 \,|jm\rangle &= m\,\delta_{m'm} \\
(J_+^j)_{m'm} &= \langle jm'|\, J_+ \,|jm\rangle &= \sqrt{(j-m)(j+m+1)/2}\,\delta_{m',m+1} \\
(J_-^j)_{m'm} &= \langle jm'|\, J_- \,|jm\rangle &= \sqrt{(j+m)(j-m+1)/2}\,\delta_{m',m-1}
\end{aligned}
$$

The last two equations can be recast as

$$
\left(J_1^j\right)_{m'm} = \frac{1}{2}\Bigg[\sqrt{(j-m)(j+m+1)}\,\delta_{m',m+1} + \sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1}\Bigg]
$$

$$
\left(J_2^j\right)_{m'm} = \frac{1}{2i}\Bigg[\sqrt{(j-m)(j+m+1)}\,\delta_{m',m+1} - \sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1}\Bigg]
$$

Examples :

- Spin-1/2 Representation of $su(2)$.

$$j = 1/2 \qquad \Rightarrow \qquad m = \pm 1/2$$

Hence,

$$J_3^{1/2} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3/2, \qquad J_1^{1/2} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1/2,$$

$$J_2^{1/2} = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \sigma_2/2.$$

Exponentiating the above generators yields the general elements of group $SU(2)$ in spin-1/2 representation:

$$g = e^{\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}} = \sum_{n=0}^{\infty} \frac{(i/2)^n}{n!}(\vec{\alpha}\cdot\vec{\sigma})^n$$

Since,

$$\begin{aligned}(\vec{\alpha}\cdot\vec{\sigma})^2 &= \alpha_a\alpha_b(\sigma_a\sigma_b) = \alpha_a\alpha_b(\delta_{ab}+i\epsilon_{abc}\sigma_c)\\ &= \alpha_a\alpha_b\delta_{ab} = \alpha_a\alpha_a \equiv \alpha^2\end{aligned}$$

we have:

$$\begin{cases}(\vec{\alpha}\cdot\vec{\sigma})^{2n} = \alpha^{2n}\\ (\vec{\alpha}\cdot\vec{\sigma})^{2n+1} = \alpha^{2n}(\vec{\alpha}\cdot\vec{\sigma})\end{cases}$$

where n is an arbitrary *non-negative* integer. Therefore,

$$\begin{aligned}e^{\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}} &= \cos(\alpha/2) + i(\vec{n}\cdot\vec{\sigma})\sin(\alpha/2)\\ &= \begin{bmatrix}\cos(\alpha/2)+in_3\sin(\alpha/2) & (in_1+n_2)\sin(\alpha/2)\\ (in_1-n_2)\sin(\alpha/2) & \cos(\alpha/2)-in_3\sin(\alpha/2)\end{bmatrix}\end{aligned}$$

where $\alpha = \sqrt{\alpha_a\alpha_a}$ and n_a are the Cartesian components of the unit vector

$$\vec{n} = \vec{\alpha}/\alpha = \vec{e}_3 c_\theta + \vec{e}_1 s_\theta c_\phi + \vec{e}_2 s_\theta s_\phi$$

This is obviously a unitary matrix with unity determinant.

- Spin-1 Representation of $su(2)$.

$$j = 1 \qquad \Rightarrow \quad m = 0, \ \pm 1.$$

Hence,

$$J_3^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad J_1^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$J_2^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}.$$

The corresponding 3-d irreducible representation of group $SU(2)$ is given by,

$$e^{i\vec{\alpha}\cdot\vec{J}^1} = e^{i(\alpha_1 J_1^1 + \alpha_2 J_2^1 + \alpha_3 J_3^1)}$$

- Spin-$3/2$ Representation of $su(2)$.

$$j = 3/2 \quad \Rightarrow \quad m = \pm 3/2,\ \pm 1/2.$$

Hence,

$$J_3^{3/2} = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix},$$

$$J_1^{3/2} = \begin{bmatrix} 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 \end{bmatrix},$$

and

$$J_2^{3/2} = \begin{bmatrix} 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 \\ i\sqrt{\frac{3}{2}} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{\frac{3}{2}} \\ 0 & 0 & i\sqrt{\frac{3}{2}} & 0 \end{bmatrix}.$$

The corresponding 4-d irreducible representation of group $SU(2)$ is given by,

$$e^{i\vec{\alpha}\cdot\vec{J}^{3/2}} = e^{i(\alpha_1 J_1^{3/2} + \alpha_2 J_2^{3/2} + \alpha_3 J_3^{3/2})}$$

Let us now consider the homomorphism between $SU(2)$ and $SO(3)$.

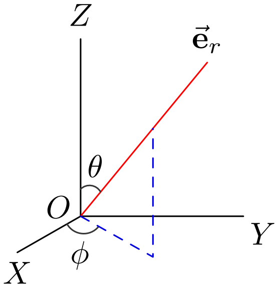
Question:

Why the magnetic quantum number m of orbital angular momentum $\vec{L}$ of an object must be an integer ?

The angular momentum operator is defined as $\vec{L} = \vec{r} \times \vec{p}$. In coordinate representation,

$$\vec{L} = -i\hbar\vec{r} \times \vec{\nabla}$$

To solve the eigenvalue problem of $\vec{L}$, we generally employ the spherical coordinates (r, θ, ϕ).



So $\vec{r} = r\vec{e}_r$,

$$\vec{e}_r = \vec{e}_3 c_\theta + \vec{e}_1 s_\theta c_\phi + \vec{e}_2 s_\theta s_\phi,$$

and

$$\begin{aligned}\vec{e}_\theta &= \partial_\theta \vec{e}_r \\ &= -\vec{e}_3 s_\theta + \vec{e}_1 c_\theta c_\phi + \vec{e}_2 c_\theta s_\phi,\end{aligned}$$

$$\begin{aligned}\vec{e}_\phi &= \frac{1}{s_\theta}\partial_\phi \vec{e}_r \\ &= -\vec{e}_1 s_\phi + \vec{e}_2 c_\phi.\end{aligned}$$

In spherical coordinates, the gradient operator $\vec{\nabla}$ becomes:

$$\vec{\nabla} = \vec{e}_r\ \partial_r + \frac{1}{r}\vec{e}_\theta \partial_\theta + \frac{1}{r s_\theta}\vec{e}_\phi \partial_\phi$$

Hence,

$$\vec{L} = -i\hbar(r\vec{e}_r) \times \vec{\nabla} = -i\hbar\left[\vec{e}_\phi \partial_\theta - \vec{e}_\theta \frac{1}{s_\theta}\partial_\phi\right]$$

Equivalently,

$$\vec{L} = -i\left[(-\vec{e}_1 s_\phi + \vec{e}_2 c_\phi)\partial_\theta - (-\vec{e}_3 s_\theta + \vec{e}_1 c_\theta c_\phi + \vec{e}_2 c_\theta s_\phi)\frac{1}{s_\theta}\partial_\phi\right]$$

Consequently, the Cartesian components of orbital angular momentum $\vec{L}$ can be expressed as

$$\begin{aligned} L_1 &= i\big[s_\phi \partial_\theta + \cot\theta c_\phi \partial_\phi\big] \\ L_2 &= -i\big[c_\phi \partial_\theta - \cot\theta s_\phi \partial_\phi\big] \\ L_3 &= -i\partial_\phi \end{aligned}$$

in terms of the spherical coordinates $(\theta,\ \phi)$.

Casimir operator L^2 of $SO(3)$:

Notice that $\vec{e}_\phi \cdot \vec{e}_\phi = \vec{e}_\theta \cdot \vec{e}_\theta = 1$ and $\vec{e}_\phi \cdot \vec{e}_\theta = 0$. The derivatives of the first two orthonormal conditions with respect to the angles θ and ϕ give,

$$\vec{e}_\phi \cdot \partial_\theta \vec{e}_\phi = \vec{e}_\phi \cdot \partial_\phi \vec{e}_\phi = 0, \qquad \vec{e}_\theta \cdot \partial_\theta \vec{e}_\theta = \vec{e}_\theta \cdot \partial_\phi \vec{e}_\theta = 0.$$

Therefore,

$$
\begin{aligned}
L^2 &= \vec{L} \cdot \vec{L} \\
&= -\left[\vec{e}_\phi \partial_\theta - \vec{e}_\theta \frac{1}{s_\theta} \partial_\phi\right] \cdot \left[\vec{e}_\phi \partial_\theta - \vec{e}_\theta \frac{1}{s_\theta} \partial_\phi\right] \\
&= -\partial_\theta^2 + (\vec{e}_\phi \cdot \partial_\theta \vec{e}_\theta) \frac{1}{s_\theta} \partial_\phi + (\vec{e}_\theta \cdot \partial_\phi \vec{e}_\phi) \frac{1}{s_\theta} \partial_\theta - \frac{1}{s_\theta^2} \partial_\phi^2
\end{aligned}
$$

Recall the transformation of basis vectors between the Cartesian and spherical coordinate systems

$$
\begin{aligned}
\vec{e}_r &= \vec{e}_3 c_\theta + \vec{e}_1 s_\theta c_\phi + \vec{e}_2 s_\theta s_\phi \\
\vec{e}_\theta &= -\vec{e}_3 s_\theta + \vec{e}_1 c_\theta c_\phi + \vec{e}_2 c_\theta s_\phi \\
\vec{e}_\phi &= -\vec{e}_1 s_\phi + \vec{e}_2 c_\phi
\end{aligned}
$$

we see that: $\vec{e}_r s_\theta + \vec{e}_\theta c_\theta = \vec{e}_1 c_\phi + \vec{e}_2 s_\phi$. Therefore,

$$
\begin{aligned}
\partial_\theta \vec{e}_\theta &= -\vec{e}_3 c_\theta - \vec{e}_1 s_\theta c_\phi - \vec{e}_2 s_\theta s_\phi \;\; = -\vec{e}_r \\
\partial_\phi \vec{e}_\phi &= -\vec{e}_1 c_\phi - \vec{e}_2 s_\phi \;\; = -\vec{e}_r s_\theta - \vec{e}_\theta c_\theta
\end{aligned}
$$

Hence,

$$(\vec{e}_\phi \cdot \partial_\theta \vec{e}_\theta) = 0, \qquad (\vec{e}_\theta \cdot \partial_\phi \vec{e}_\phi) = -c_\theta.$$

Substitution of these results into the previous formula yields,

$$L^2 = -\partial_\theta^2 - \cot\theta\,\partial_\theta - \frac{1}{s_\theta^2}\partial_\phi^2$$

In QM textbooks, L^2 is commonly recast as:

$$L^2 == -\left[\frac{1}{s_\theta}\partial_\theta(s_\theta\partial_\theta) + \frac{1}{s_\theta^2}\partial_\phi^2\right]$$

- L^2 is called the **Casimir** operator of $so(3)$. Its crucial property is,

$$[L^2,\ L_a] = 0, \quad a = 1,\ 2,\ 3.$$

Thereby, L^2 and L_3 can have common eigenvectors.

- The eigenvalue problem

$$L_3 \left| lm \right\rangle = m \left| lm \right\rangle, \quad L^2 \left| lm \right\rangle = l(l+1) \left| lm \right\rangle$$

in spherical coordinates becomes,

$$\begin{cases} \partial_\phi Y = imY, \\ s_\theta \partial_\theta (s_\theta \partial_\theta) Y + \left[s_\theta^2 l(l+1) - m^2 \right] Y = 0. \end{cases}$$

- The common eigenfunction $Y(\theta, \phi)$ of L_3 and L^2 can be factorized into

$$Y(\theta, \phi) = \Theta(\theta) e^{im\phi}$$

Insight:

If $Y(\theta, \phi)$ is single-valued under rotation: $Y(\theta, \phi + 2\pi) = Y(\theta, \phi)$, the magnetic quantum number m has to be some integers: $m \in Z$.

Question :

Why should $Y(\theta, \phi)$ be single-valued under rotation ?

Remarks :

- In QM, physical significance is attached, not to wavefunction Y itself, but to its bilinear functions, e.g., $|Y|^2$.
- These bilinear functions are unchanged by a 2π rotation *even if m is a half-integer* and Y changes sign.

For $l = m = 1/2$, the common eigenfunction of Casimir operator L^2 and L_3 becomes:

$$Y = \Theta(\theta) e^{\frac{i}{2}\phi}$$

where the factor function Θ obeys,

$$s_\theta \partial_\theta (s_\theta \partial_\theta)\Theta + \frac{1}{4}\left[3 s_\theta^2 - 1\right]\Theta = 0$$

A special solution to this equation reads,

$$\Theta(\theta) = \sqrt{s_\theta}$$

Checking:

If $\Theta(\theta) = \sqrt{s_\theta}$, we see that

$$(s_\theta \partial_\theta)\Theta = \frac{1}{2}\sqrt{s_\theta}\, c_\theta$$

$$\begin{aligned} s_\theta \partial_\theta (s_\theta \partial_\theta)\Theta &= \frac{1}{2} s_\theta \partial_\theta (\sqrt{s_\theta}\, c_\theta) \;= \frac{1}{4}\sqrt{s_\theta}\, (c_\theta^2 - 2 s_\theta^2) \\ &= \frac{1}{4}\sqrt{s_\theta}\, (1 - 3 s_\theta^2) \\ &= -\frac{1}{4}\left[3 s_\theta^2 - 1\right] \Theta \end{aligned}$$

This is just what we have expected.

$Y(\theta, \phi) = \sqrt{s_\theta}\, e^{i\phi/2}$ appears to be an acceptable wave function in QM because $|Y|^2 = |s_\theta|$ is well defined in the unit spherical surface,

$$0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \phi \leqslant 2\pi.$$

Puzzle :

What is wrong in the above argument ?

Go back to the primary definition of orbital angular momentum:[1]

$$\vec{L} = -i\vec{r} \times \vec{\nabla}$$

In Cartesian coordinates,

$$L_a = -i\epsilon_{abc}x_b\partial_{x_c}, \qquad (a = 1,\ 2,\ 3.)$$

Particularly, L_3 consists of four linear operators $\left\{x_1, x_2, \partial_{x_1}, \partial_{x_2}\right\}$:

$$L_3 = -i\left[x_1\partial_{x_2} - x_2\partial_{x_1}\right]$$

[1] It holds only for the orbital angular momentum operator of a quantum particle.

To expose L_3's interesting intrinsic structure, we now introduce four new linear operators:

$$q_1 = \tfrac{1}{\sqrt{2}}\left(x_1 - i\partial_{x_2}\right), \qquad q_2 = \frac{1}{\sqrt{2}}\left(x_1 + i\partial_{x_2}\right),$$

$$p_1 = -\tfrac{1}{\sqrt{2}}\left(x_2 + i\partial_{x_1}\right), \quad p_2 = \frac{1}{\sqrt{2}}\left(x_2 - i\partial_{x_1}\right).$$

Notice that $[\partial_{x_a},\ x_b] = \delta_{ab}$. The Lie brackets between these operators are

$$[q_a,\ q_b] = [p_a,\ p_b] = 0, \qquad [q_a,\ p_b] = i\delta_{ab}.$$

In terms of these *new* operators,

$$x_1 = \tfrac{1}{\sqrt{2}}\left(q_1 + q_2\right), \qquad x_2 = -\frac{1}{\sqrt{2}}\left(p_1 - p_2\right),$$

$$\partial_{x_1} = \tfrac{i}{\sqrt{2}}\left(p_1 + p_2\right), \qquad \partial_{x_2} = \frac{i}{\sqrt{2}}\left(q_1 - q_2\right).$$

and L_3 is recast as:

$$\begin{aligned} L_3 &= -i\left(x_1\partial_{x_2} - x_2\partial_{x_1}\right) \\ &= \frac{1}{2}\Big[\left(q_1+q_2\right)\left(q_1-q_2\right) + \left(p_1-p_2\right)\left(p_1+p_2\right)\Big] \\ &= \frac{1}{2}\Big[(q_1^2+p_1^2) - (q_2^2+p_2^2)\Big] \\ &= H_1 - H_2 \end{aligned}$$

where

$$H_a = \frac{1}{2}(q_a^2 + p_a^2), \qquad (a = 1,\ 2.)$$

are hamiltonian operators of two independent oscillators, each having mass $M = 1$ and angular frequency $\omega = 1$.

Insight :

The eigenvalues of L_3 should be the difference of eigenvalues of two independent (but with identical parameters $M = \omega = 1$) harmonic oscillator Hamiltonians.

The eigenvalues of a harmonic oscillator Hamiltonian $H_a = \frac{1}{2}(q_a^2 + p_a^2)$ are well-known,

$$E_{n_a} = n_a + \frac{1}{2}$$

with n_a some nonnegative integers.

Consequently, the eigenvalues of orbital angular momentum L_3 are equal to,

$$m = \left(n_1 + \frac{1}{2}\right) - \left(n_2 + \frac{1}{2}\right) = n_1 - n_2 \quad \in Z$$

Namely, the orbital angular momentum eigenvalues must be some integers. *The possibility for m being a half-integer is forbidden.* [2]

[2] This demonstration can be regarded as an indirect justification for the conventional boundary condition $Y(\theta, \phi + 2\pi) = Y(\theta, \phi)$ that leads to the same result.

Tensor product representations:

Consider the tensor product representations of a Lie group G.

Suppose

$$D(g)\,|i\rangle = \sum_{j=1}^{N} \big[D_1(g)\big]_{ji}\,|j\rangle\,, \quad D(g)\,|\alpha\rangle = \sum_{\beta=1}^{M} \big[D_2(g)\big]_{\beta\alpha}\,|\beta\rangle$$

On states of tensor product $|i\rangle\,|\alpha\rangle$, we have:

$$\begin{aligned}
D_{1\times 2}(g)\,|i\rangle\,|\alpha\rangle &= \sum_{j=1}^{N}\sum_{\beta=1}^{M}\Big[D_1(g)\;D_2(g)\Big]_{j\beta,i\alpha}\,|j\rangle\,|\beta\rangle \\
&= \sum_{j=1}^{N}\sum_{\beta=1}^{M}\Big[D_1(g)\Big]_{ji}\Big[D_2(g)\Big]_{\beta\alpha}\,|j\rangle\,|\beta\rangle \\
&= \left\{\sum_{j=1}^{N}[D_1(g)]_{ji}\,|j\rangle\right\}\cdot\left\{\sum_{\beta=1}^{M}[D_2(g)]_{\beta\alpha}\,|\beta\rangle\right\}
\end{aligned}$$

i.e.,

$$\left[D_{1\times 2}(g)\right]_{j\beta,i\alpha} = \left[D_1(g)\right]_{ji}\left[D_2(g)\right]_{\beta\alpha}$$

Consider the infinitesimal group elements of the relevant representations,

$D_1(g) \approx 1+i\xi_a J_a^1, \quad D_2(g) \approx 1+i\xi_a J_a^2, \quad D_{1\times 2}(g) \approx 1+i\xi_a J_a^{1\times 2}.$

The above relation can be recast as:

$$[1 + i\xi_a J_a^{1\times 2}]_{j\beta,i\alpha} = [1 + i\xi_b J_b^1]_{ji}[1 + i\xi_c J_c^2]_{\beta\alpha}$$

$$\rightsquigarrow \quad (J_a^{1\times 2})_{j\beta,i\alpha} = (J_a^1)_{ji}\delta_{\beta\alpha} + \delta_{ji}(J_a^2)_{\beta\alpha}$$

i.e.,

$$J_a^{1\times 2} = J_a^1 \times 1 + 1 \times J_a^2$$

The action of generators on the tensor product of states is as follows:

$$J_a^{1\times 2}\left\{ |i\rangle\,|\alpha\rangle \right\} = \left\{ J_a^1\,|i\rangle \right\}\cdot|\alpha\rangle + |i\rangle\cdot\left\{ J_a^2\,|\alpha\rangle \right\}$$

J_3's value add :

Because we work in a basis $|jm\rangle$ in which J_3 ia diagonal, the J_3 values of tensor product states are just the sums of the J_3 values of the factors.

Explanation :

$$J_3\Big\{\,|j_1m_1\rangle|j_2m_2\rangle\,\Big\} = \Big\{J_3\,|j_1m_1\rangle\Big\}\,|j_2m_2\rangle + |j_1m_1\rangle\Big\{J_3\,|j_2m_2\rangle\Big\}$$
$$= (m_1+m_2)\Big\{\,|j_1m_1\rangle|j_2m_2\rangle\,\Big\}$$

The irreducible representation $\Big\{\,|jm\rangle\,\Big\}$ of $SU(2)$ is related to its tensor product representation $\Big\{\,|j_1m_1\rangle|j_2m_2\rangle\,\Big\}$ through,

$$|jm\rangle = \sum_{m_1=-j_1}^{j_1} c_{j_1j_2j,m_1(m-m_1)m}\Big\{\,|j_1m_1\rangle|j_2, m-m_1\rangle\,\Big\}$$

Remarks :

1. The coefficients $c_{j_1 j_2 j, m_1 (m-m_1) m}$ are called Clebsch-Gordon coefficients of $SU(2)$.
2. In particular, we define:

$$c_{j_1 j_2 (j_1+j_2), j_1 j_2 (j_1+j_2)} = 1.$$

Question :

How to systematically determine the Clebsch-Gordon coefficients ?

Answer :

The highest weight procedure.

Example :

Consider the spin-$1/2$ representation and spin-1 representation of $su(2)$,

$$j_1 = \frac{1}{2}, \quad j_2 = 1 \quad \rightsquigarrow \quad j_1 + j_2 = \frac{3}{2}.$$

The assumption $c_{j_1 j_2 (j_1+j_2), j_1 j_2 (j_1+j_2)} = 1$ means,

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \cdot |1, 1\rangle$$

Therefore,

$$\begin{aligned}
\sqrt{\tfrac{3}{2}}\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle &= J_-\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle \\
&= J_-\left\{\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \cdot |1, 1\rangle\right\} \\
&= \left\{J_-^{1/2}\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\right\} \cdot |1, 1\rangle + \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \cdot \left\{J_-^{1}|1, 1\rangle\right\} \\
&= \tfrac{1}{\sqrt{2}}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle \cdot |1, 1\rangle + \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \cdot |1, 0\rangle
\end{aligned}$$

Equivalently,

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \cdot |1, 1\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle \cdot |1, 0\rangle$$

Continuing this procedure yields:

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,0\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \cdot |1,-1\rangle$$

$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,-1\rangle$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,1\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \cdot |1,0\rangle$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \cdot |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \cdot |1,-1\rangle$$

Hence, the decomposition of tensor product of spin-$1/2$ and spin-1 representations of $SU(2)$

$$D_{1/2} \times D_1 \sim \bigoplus_{j=1/2}^{3/2} D_j$$

is determined by the following non-vanishing Clebsch-Gordon coefficients $c_{j_1 j_2 j, m_1 (m-m_1) m}$:

$c_{\frac{1}{2} 1 \frac{3}{2}, \frac{1}{2} 1 \frac{3}{2}} = 1$	$c_{\frac{1}{2} 1 \frac{3}{2}, -\frac{1}{2} 1 \frac{1}{2}} = 1/\sqrt{3}$
$c_{\frac{1}{2} 1 \frac{3}{2}, \frac{1}{2} 0 \frac{1}{2}} = \sqrt{2/3}$	$c_{\frac{1}{2} 1 \frac{3}{2}, -\frac{1}{2} -1 -\frac{3}{2}} = 1$
$c_{\frac{1}{2} 1 \frac{3}{2}, -\frac{1}{2} -1 -\frac{1}{2}} = 1/\sqrt{3}$	$c_{\frac{1}{2} 1 \frac{3}{2}, -\frac{1}{2} 0 -\frac{1}{2}} = \sqrt{2/3}$
$c_{\frac{1}{2} 1 \frac{1}{2}, -\frac{1}{2} 1 \frac{1}{2}} = \sqrt{2/3}$	$c_{\frac{1}{2} 1 \frac{1}{2}, \frac{1}{2} 0 \frac{1}{2}} = -1/\sqrt{3}$
$c_{\frac{1}{2} 1 \frac{1}{2}, -\frac{1}{2} 0 -\frac{1}{2}} = \sqrt{1/3}$	$c_{\frac{1}{2} 1 \frac{1}{2}, \frac{1}{2} -1 -\frac{1}{2}} = -\sqrt{2/3}$

1. Let $\left\{k\right\}$ be the spin-k representation of $su(2)$. Show that

$$\left\{j\right\} \times \left\{s\right\} = \oplus_{l=|j-s|}^{j+s} \left\{l\right\}$$

2. Calculate

$$\exp\left[i\vec{\xi} \cdot \vec{\sigma}\right]$$

where $\vec{\sigma} = \left\{\sigma_1, \sigma_2, \sigma_3\right\}$ are the pauli matrices and $\vec{\xi}$ a common 3-dimensional vector.

3. Show explicitly that the spin-1 representation of $su(2)$ obtained by the highest weight procedure with $j = 1$ is equivalent to the adjoint representation with $f_{abc} = \epsilon_{abc}$ by finding the similarity transformation that implements the equivalence.

4. Suppose that $\left(\sigma_a\right)_{ij}$ and $\left(\eta_a\right)_{xy}$ are pauli matrices in two different 2-dimensional spaces. In the 4-dimensional tensor product space, define the basis vectors as

$$\begin{aligned}
|1\rangle &= |i=1\rangle|x=1\rangle \\
|2\rangle &= |i=1\rangle|x=2\rangle \\
|3\rangle &= |i=2\rangle|x=1\rangle \\
|4\rangle &= |i=2\rangle|x=2\rangle
\end{aligned}$$

Write out the matrix elements of $\sigma_2 \times \eta_1$ in this basis.

现代数学物理方法

第二章, 李群

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Outline

1. Tensor Operators
   - Operator Basis
   - Wigner-Eckart Theorem
   - Products of Tensor Operators
2. Roots and Weights
   - Weights
   - Adjoint Representation
   - Roots
   - Lots of $su(2)$s
   - The angle between two roots
3. $su(3)$ Algebra
   - Generators
   - Root vectors of $su(3)$

Tensor operators

Goal :

In this lecture, we will define and discuss the tensor operators of the $su(2)$ [or equivalently $so(3)$] algebra.

A tensor operator transforming under the spin-s representation of $su(2)$ consists of a set of operators

$$\mathscr{O}_l^s, \quad (-s \leqslant l \leqslant s)$$

such that

$$[J_a,\ \mathscr{O}_l^s] = \mathscr{O}_m^s (J_a^s)_{ml}, \quad (a = 1,\ 2,\ 3.)$$

Orbital angular momentum :

The $su(2)$ algebra can be realized by the orbital angular momentum operators of a quantum mechanics particle, $J_a = L_a = \epsilon_{abc} x_b p_c$.

Because $[x_a,\ p_b] = i\delta_{ab}$,

$$[J_a,\ x_b] = \epsilon_{acd} x_c [p_d,\ x_b] = \epsilon_{acd} x_c (-i\delta_{db}) = -i\epsilon_{acb} x_c$$

Tensor operator examples

Recalling

$$(J_a^{\text{adj}})_{cb} = -i\epsilon_{acb} \ ,$$

we get:

$$\begin{aligned}[J_a,\ x_b] &= -i\epsilon_{acb}x_c \\ &= x_c(J_a^{\text{adj}})_{cb} \quad \rightsquigarrow \quad x_c(J_a^{1})_{cb}\end{aligned}$$

We conclude that :

1. The position vector $\vec{r} = \sum_{a=1}^{3} x_a \vec{e}_a$ is a tensor operator of $su(2)$ that transforms under the spin-1 representation.

Similarly,

$$
\begin{aligned}
[J_a,\ p_b] &= \epsilon_{acd}[x_c,\ p_b]p_d = \epsilon_{acd}(i\delta_{cb})p_d = i\epsilon_{abd}p_d \\
&= -i\epsilon_{acb}p_c = p_c(J_a^{\text{adj}})_{cb}
\end{aligned}
$$

$$
[J_a,\ J_b] = i\epsilon_{abc}J_c = -i\epsilon_{acb}J_c = J_c(J_a^{\text{adj}})_{cb}
$$

1. The momentum $\vec{p} = \sum_{a=1}^{3} p_a \vec{e}_a$ and the orbital angular momentum itself are also the tensor operators of $su(2)$ under the spin-1 representation.

Operator basis

we now consider the question about choosing an operator basis so that the standard spin-s representation generators J_a^s appears in the Lie brackets,

$$[J_a,\ \mathscr{O}_l^s] = \mathscr{O}_m^s (J_a^s)_{ml}, \quad (a = 1, 2, 3.)$$

Suppose

1. we are given a tensor operator $\mathscr{O}$ that transforms under a representation D of $su(2)$ algebra,

$$[J_a,\ \mathscr{O}_\alpha] = \mathscr{O}_\beta (J_a^D)_{\beta\alpha}\ , \quad (-s \leqslant \alpha, \beta \leqslant s)\ .$$

2. D is equivalent to the spin-s irreducible representation of $su(2)$. Namely, there is a nonsingular matrix S ($\det S \neq 0$) such that:

$$J_a^D = S^{-1} J_a^s S \quad \rightsquigarrow \quad (J_a^D)_{\beta\alpha} = (S^{-1})_{\beta j} (J_a^s)_{ji} S_{i\alpha}$$

we get,

$$[J_a,\ \mathscr{O}_\alpha] = \mathscr{O}_\beta (S^{-1})_{\beta j} (J_a^s)_{ji} S_{i\alpha}$$

It leads to:

$$[J_a,\ \mathscr{O}_\alpha](S^{-1})_{\alpha k} = \mathscr{O}_\beta (S^{-1})_{\beta j} (J_a^s)_{jk}$$

Definition :

$$\mathscr{O}_i^s \equiv \mathscr{O}_\beta (S^{-1})_{\beta i}$$

The above commutator is rewritten as:

$$[J_a,\ \mathscr{O}_i^s] = \mathscr{O}_j^s (J_a^s)_{ji}, \qquad -s \leqslant i, j \leqslant s.$$

In the standard basis, the $SU(2)$'s generator J_3 is a diagonal matrix: $(J_3^s)_{jk} = j\delta_{jk}$, $(j, k = -s, -s+1, \cdots, s-1, s)$.

Namely,

$$J_3^s = \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & s-2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix}$$

Therefore,

$$[J_3,\ \mathscr{O}_k^s] = \mathscr{O}_j^s (J_3^s)_{jk} = \mathscr{O}_j^s j\delta_{jk} = k\mathscr{O}_k^s$$

Remark :

What does the commutator $[J_3,\ \mathscr{O}^s_k] = k\mathscr{O}^s_k$ mean ?

- If we find a linear combination of the operators $\{\mathscr{O}^s_\alpha\}$ which has a definite value k of J_3 (with $|k| \leqslant s$),

$$[J_3,\ \mathscr{O}^s_\alpha] = k\sum_\beta c_{\alpha\beta}\mathscr{O}^s_\beta$$

  we can take that combination to be the tensor component $\mathscr{O}^s_k$,

$$\mathscr{O}^s_k = \sum_\alpha f_{k\alpha}\mathscr{O}^s_\alpha$$

- The other components $\{\mathscr{O}^s_i,\ i \neq k\}$ of the tensor operator $\mathscr{O}^s$ can be built up by applying raising and lowering operators.

Example:

Let

$$V^1 = \left\{V_1^1, V_0^1, V_{-1}^1\right\}$$

be the position vector operator [the tensor operator in spin-1 representation of $su(2)$] in standard basis.

1. Since $[J_3,\ V_k^1] = kV_k^1$, we see

$$[J_3,\ V_0^1] = 0.$$

   On the other hand, we have $[J_a,\ x_b] = -i\epsilon_{acb}x_c$ that implies

$$[J_3,\ x_3] = -i\epsilon_{3c3}x_c = 0.$$

   Therefore, we can identify V_0^1 as x_3,

$$V_0^1 \equiv x_3$$

2. Since $[J_a,\ \mathscr{O}_i^s] = \mathscr{O}_j^s (J_a^s)_{ji}$, we have

$$[J_\pm,\ V_0^1] = V_j^1 (J_\pm^1)_{j0} = V_j^1 \delta_{j,\pm 1} = V_{\pm 1}^1,$$

i.e.,

$$\begin{aligned}
V^1_{\pm 1} &= [J_\pm, V^1_0] \\
&= \tfrac{1}{\sqrt{2}}[J_1 \pm iJ_2,\ x_3] \\
&= \tfrac{1}{\sqrt{2}}(i\epsilon_{132}x_2 \pm i^2\epsilon_{231}x_1) \\
&= \tfrac{1}{\sqrt{2}}(-ix_2 \mp x_1) \\
&= \mp\tfrac{1}{\sqrt{2}}(x_1 \pm ix_2)
\end{aligned}$$

In conclusion, we have:

$$V^1_1 = -\frac{1}{\sqrt{2}}(x_1 + ix_2)$$

$$V^1_0 = x_3$$

$$V^1_{-1} = \frac{1}{\sqrt{2}}(x_1 - ix_2)$$

Wigner-Eckart theorem

Consider the $su(2)$ transformation of the state

$$\mathscr{O}_l^s \left| jm, \alpha \right\rangle$$

Straightforwardly,

$$\begin{aligned} J_a \mathscr{O}_l^s \left| jm, \alpha \right\rangle &= [J_a,\ \mathscr{O}_l^s] \left| jm, \alpha \right\rangle + \mathscr{O}_l^s J_a \left| jm, \alpha \right\rangle \\ &= \textstyle\sum_{k=-s}^{s} \mathscr{O}_k^s (J_a^s)_{kl} \left| jm, \alpha \right\rangle \\ &\quad + \mathscr{O}_l^s \textstyle\sum_{k=-j}^{j} \left| jk, \alpha \right\rangle \left\langle jk, \alpha \right| J_a \left| jm, \alpha \right\rangle \\ &= \textstyle\sum_{k=-s}^{s} \mathscr{O}_k^s (J_a^s)_{kl} \left| jm, \alpha \right\rangle \\ &\quad + \mathscr{O}_l^s \textstyle\sum_{k=-j}^{j} (J_a^j)_{km} \left| jk, \alpha \right\rangle \end{aligned}$$

In particular,

- J_3's value of the product of a tensor operator with a state is just the sum of the J_3's values of the operator and the state,

$$
\begin{aligned}
J_3 \mathscr{O}^s_l \left|jm,\alpha\right\rangle &= \sum_{k=-s}^{s} \mathscr{O}^s_k (J^s_3)_{kl} \left|jm,\alpha\right\rangle \\
&\qquad + \sum_{k=-j}^{j} \mathscr{O}^s_l (J^j_3)_{km} \left|jk,\alpha\right\rangle \\
&= \sum_{k=-s}^{s} \mathscr{O}^s_k (k\delta_{kl}) \left|jm,\alpha\right\rangle \\
&\qquad + \sum_{k=-j}^{j} \mathscr{O}^s_l (k\delta_{km}) \left|jk,\alpha\right\rangle \\
&= (l+m) \mathscr{O}^s_l \left|jm,\alpha\right\rangle
\end{aligned}
$$

The product of a tensor operator and a state behaves under $su(2)$ just like the tensor products of two states. Therefore, it can be decomposed into the direct sum of irreducible representations of $su(2)$.

Notice that,

1. $\mathcal{O}_s^s |jj, \alpha\rangle$ is the highest weight state in spin-$(j+s)$ Rep. of $su(2)$, with J_3 eigenvalue being $J_3 = j + s$. We can lower it to construct the rest states of the spin-$(j+s)$ representation.
2. We can find a linear combination of $J_3 = j + s - 1$ states that is the highest weight state of the spin-$(j+s-1)$ representation. By lowering it we can get the entire states of the representation.
3. The explicit states of the irreducible representations of $su(2)$ algebra can be constructed in terms of linear combinations of the states $\{\mathcal{O}_l^s |jm, \alpha\rangle\}$,

$$|JM\rangle = \sum_{l=-s}^{s} d_{sjl,JM} \mathcal{O}_l^s |j, M-l, \alpha\rangle$$

where $|j - s| \leqslant J \leqslant j + s$ and $-J \leqslant M \leqslant J$.

Recalling,

$$|JM\rangle = \sum_{l=-s}^{s} c_{sjJ,l(M-l)M}\Big[\,|sl\rangle \times |j, M-l\rangle\Big]$$

with $c_{sjJ,l(M-l)M}$ C.G. coefficients. The $su(2)$ transformation properties of states

$$\mathcal{O}_l^s\,|j, M-l, \alpha\rangle, \qquad \Big[\,|sl\rangle \times |j, M-l\rangle\Big]$$

are identical for a given J. Hence, the coefficients must be proportional:

$$d_{sjl,JM} = \frac{1}{k_J^\alpha} c_{sjJ,l(M-l)M}$$

i.e.,

$$k_J^\alpha\,|JM\rangle = \sum_{l=-s}^{s} c_{sjJ,l(M-l)M}\,\mathcal{O}_l^s\,|j, M-l, \alpha\rangle$$

Question :

What is the inverse relation ?

The C.G.coefficients are defined as:

$$c_{j_1 j_2 j, m_1 (m-m_1) m} = \Big[\langle j_1 m_1 | \times \langle j_2, m - m_1 | \Big] |jm\rangle$$

their complex conjugates read,

$$c^*_{j_1 j_2 j, m_1 (m-m_1) m} = \langle jm | \Big[|j_1 m_1\rangle \times |j_2, m - m_1\rangle \Big]$$

The completeness relation $\sum_{j,m} |jm\rangle\langle jm| = \hat{I}$ then implies that,

$$\sum_{j,m} c_{j_1 j_2 j, m_1 (m-m_1) m} c^*_{j_1' j_2' j, m_1' (m-m_1') m} = \delta_{j_1 j_1'} \delta_{j_2 j_2'} \delta_{m_1 m_1'}$$

Consequently,

$$\mathscr{O}_l^s |jm, \alpha\rangle = \sum_{J=|j-s|}^{j+s} c^*_{sjJ, lm(m+l)} k_J^\alpha |J, m+l\rangle$$

Wigner-Eckart Theorem :

The physics comes in when we express the state $k_J^\alpha |J, m+l\rangle$ in terms of the Hilbert space basis states $|J, m+l, \alpha\rangle$,

$$k_J^\alpha |J, m+l\rangle = \sum_\beta k_{\alpha\beta} |J, m+l, \beta\rangle$$

where,

- $k_{\alpha\beta}$ are known as the reduced matrix elements which depend only on α, j and $\mathscr{O}^s$.
- $k_{\alpha\beta}$ are generically denoted as,

$$k_{\alpha\beta} = \langle\langle J, \beta | \mathscr{O}^s | j, \alpha \rangle\rangle$$

If we know any non-vanishing reduced matrix element of a tensor operator between states of some given (J, β) and (j, α), we can compute all other matrix elements using the algebra.

That is to say,

$$
\begin{aligned}
&\langle J'm', \beta | \mathscr{O}_l^s | jm, \alpha \rangle \\
&= \sum_\gamma k_{\alpha\gamma} \sum_{J=|j-s|}^{j+s} c^*_{sj\,J,lm(m+l)} \langle J'm', \beta | J, m+l, \gamma \rangle \\
&= \sum_\gamma k_{\alpha\gamma} \sum_{J=|j-s|}^{j+s} c^*_{sj\,J,lm(m+l)} \delta_{J'J} \delta_{m',m+l} \delta_{\beta\gamma} \\
&= k_{\alpha\beta} \delta_{m',m+l} c^*_{sj\,J',lm(m+l)}
\end{aligned}
$$

Namely,

$$
\boxed{\langle J'm', \beta | \mathscr{O}_l^s | jm, \alpha \rangle = \delta_{m',m+l} c^*_{sj\,J',\,lm(m+l)} \cdot \langle\langle J', \beta | \mathscr{O}^s | j, \alpha \rangle\rangle}
$$

This conclusion is called Wigner-Eckart Theorem.

1. Wigner-Eckart theorem has founded wide applications in quantum mechanics.

Problem :

Suppose $\langle 1/2, 1/2, \alpha | \, X_3 \, | 1/2, 1/2, \beta \rangle = \mathscr{A}$.
Find $\langle 1/2, 1/2, \alpha | \, X_1 \, | 1/2, -1/2, \beta \rangle = ?$

Solution :

The tensor operator related to the position vector $\vec{r}$ has the standard components as follows,

$$V_1^1 = -\frac{1}{\sqrt{2}}(X_1 + iX_2), \quad V_0^1 = X_3, \quad V_{-1}^1 = \frac{1}{\sqrt{2}}(X_1 - iX_2).$$

Equivalently,

$$X_1 = \frac{1}{\sqrt{2}}(V_{-1}^1 - V_1^1), \quad X_2 = \frac{i}{\sqrt{2}}(V_{-1}^1 + V_1^1), \quad X_3 = V_0^1.$$

It follows from the Wigner-Eckart theorem that

$$\mathscr{A} = \langle 1/2, 1/2, \alpha | \, V_0^1 \, | 1/2, 1/2, \beta \rangle = c^*_{1\,\frac{1}{2}\,\frac{1}{2},0\,\frac{1}{2}\,\frac{1}{2}} \, \langle\langle 1/2, \beta | V^1 | 1/2, \alpha \rangle\rangle$$

Similarly,

$$\langle 1/2, 1/2, \alpha|\, V^1_1 \,|1/2, -1/2, \beta\rangle = c^*_{1\frac{1}{2}\frac{1}{2},1-\frac{1}{2}\frac{1}{2}} \left\langle\!\left\langle 1/2, \beta|V^1|1/2, \alpha\right\rangle\!\right\rangle$$
$$\langle 1/2, 1/2, \alpha|\, V^1_{-1} \,|1/2, -1/2, \beta\rangle = 0$$

These equations imply,

$$\begin{aligned}
&\langle 1/2, 1/2, \alpha|\, X_1 \,|1/2, -1/2, \beta\rangle \\
&\quad = \frac{1}{\sqrt{2}} \langle 1/2, 1/2, \alpha|\, (V^1_{-1} - V^1_1)\, |1/2, -1/2, \beta\rangle \\
&\quad = -\frac{1}{\sqrt{2}} \langle 1/2, 1/2, \alpha|\, V^1_1 \,|1/2, -1/2, \beta\rangle \\
&\quad = -\frac{1}{\sqrt{2}} c^*_{1\frac{1}{2}\frac{1}{2},1-\frac{1}{2}\frac{1}{2}} \left\langle\!\left\langle 1/2, \beta|V^1|1/2, \alpha\right\rangle\!\right\rangle \\
&\quad = -\frac{1}{\sqrt{2}} c^*_{1\frac{1}{2}\frac{1}{2},1-\frac{1}{2}\frac{1}{2}} \frac{\mathscr{A}}{c^*_{1\frac{1}{2}\frac{1}{2},0\frac{1}{2}\frac{1}{2}}}
\end{aligned}$$

We knew from the last lecture that

$$c_{1\,\frac{1}{2}\,\frac{1}{2},1-\frac{1}{2}\,\frac{1}{2}} = \sqrt{2/3}, \quad c_{1\,\frac{1}{2}\,\frac{1}{2},0\,\frac{1}{2}\,\frac{1}{2}} = -\sqrt{1/3}.$$

Hence,

$$\langle 1/2, 1/2, \alpha|\, X_1 \,|1/2, -1/2, \beta\rangle = \mathscr{A}$$

Discussions :

- The similar applications of Wigner-Eckart theorem will yield,

$$\langle 1/2, 1/2, \alpha|\, X_2\, |1/2, -1/2, \beta\rangle = -i\mathscr{A}$$
$$\langle 1/2, -1/2, \alpha|\, X_3\, |1/2, -1/2, \beta\rangle = -\mathscr{A}$$
$$\langle 1/2, 1/2, \alpha|\, X_3\, |1/2, -1/2, \beta\rangle = 0$$
$$\langle 1/2, -1/2, \alpha|\, X_3\, |1/2, 1/2, \beta\rangle = 0$$

- However, the Wigner-Eckart theorem is not enough for us to evaluate the matrix elements such as

$$\langle 3/2, 1/2, \alpha|\, X_3\, |1/2, 1/2, \beta\rangle$$

because we are not told the relevant reduced matrix element $\langle\langle 3/2, \beta|V^1|1/2, \alpha\rangle\rangle$.

Products of tensor operators

One of the reason that tensor operators are important is that a product of two tensor operators, $\mathscr{O}^{s_1}_{m_1}$ and $\mathscr{O}^{s_2}_{m_2}$ in the spin-s_1 and spin-s_2 representations, transforms under the tensor product representation $s_1 \times s_2$:

$$\begin{aligned}
[J_a,\ \mathscr{O}^{s_1}_{m_1}\mathscr{O}^{s_2}_{m_2}] &= [J_a,\ \mathscr{O}^{s_1}_{m_1}]\mathscr{O}^{s_2}_{m_2} + \mathscr{O}^{s_1}_{m_1}[J_a,\ \mathscr{O}^{s_2}_{m_2}] \\
&= \mathscr{O}^{s_1}_{m_1'}\mathscr{O}^{s_2}_{m_2}(J^{s_1}_a)_{m_1' m_1} + \mathscr{O}^{s_1}_{m_1}\mathscr{O}^{s_2}_{m_2'}(J^{s_2}_a)_{m_2' m_2} \\
&= \mathscr{O}^{s_1}_{m_1'}\mathscr{O}^{s_2}_{m_2'}\left[(J^{s_1}_a)_{m_1' m_1}\delta_{m_2' m_2} + \delta_{m_1' m_1}(J^{s_2}_a)_{m_2' m_2}\right] \\
&= \mathscr{O}^{s_1}_{m_1'}\mathscr{O}^{s_2}_{m_2'}\left[J^{s_1}_a \times 1 + 1 \times J^{s_2}_a\right]_{m_1' m_2', m_1 m_2}
\end{aligned}$$

In particular,

$$[J_3,\ \mathscr{O}^{s_1}_{m_1}\mathscr{O}^{s_2}_{m_2}] = (m_1 + m_2)\mathscr{O}^{s_1}_{m_1}\mathscr{O}^{s_2}_{m_2}$$

Homework :

1. Consider an operator $\mathcal{O}_x$ for $x = 1$ to 2, transforming according to the spin-$1/2$ representation of $su(2)$ as follows,

$$[J_a,\ \mathcal{O}_x] = \mathcal{O}_y(\sigma_a/2)_{yx},$$

where σ_a are Pauli matrices. Given

$$\langle 3/2, -1/2, \alpha|\, \mathcal{O}_1\, |1, -1, \beta\rangle = \mathcal{A},$$

Please find $\langle 3/2, -3/2, \alpha|\, \mathcal{O}_2\, |1, -1, \beta\rangle$.

Outline

Goal :

We are going to generalize the analysis of the irreducible representations of $su(2)$ to those of an arbitrary simple Lie algebra.

1. Firstly, we are necessary to find the largest possible set of commuting hermitian generators and use their eigenvalues to label the states. These generators are the analog of J_3 in $su(2)$.

2. The rest of the generators will be analogous to the raising and lowering operators $J_\pm$.

Cartan generators

Cartan subalgebra :

A subset of commuting Hermitian generators which is as large as possible is called a Cartan subalgebra.

1. These commuting generators are called the Cartan generators.
2. The total number m of the independent Cartan generators is called the rank of the Lie algebra.
3. In a particular irreducible representation D, the Cartan generators are formulated as m Hermitian matrices H_i $(i = 1, 2, \cdots, m)$,

$$H_i = H_i^\dagger, \qquad [H_i,\ H_j] = 0.$$

Weights

- For compact Lie algebra, we can choose a basis in which

$$\mathrm{Tr}(H_i H_j) = k_D \delta_{ij}$$

  with k_D some constant that depends on the representation and on the normalization of the generators.

After simultaneously diagonalization of the Cartan generators, the basis vectors (states) of the representaton space (of Rep. D) can be cast as,

$$|\mu, \xi, D\rangle$$

such that

$$H_i\,|\mu, \xi, D\rangle = \mu_i\,|\mu, \xi, D\rangle\,, \qquad (i = 1, 2, \cdots, m.)$$

where ξ stands for any other parameters necessary for specifying the state.

Weights :

- The eigenvalues μ_i $(i = 1, 2, \cdots, m)$ of the Cartan generators $\{H_i\}$ are called weights.

- Weights are real.

- The whole set of weights $\{\mu_i\}$ forms a m-component vector $\vec{\mu}$,
$$\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_m)$$
in weight space, called weight vector.

Adjoint representation

The adjoint representation of a Lie algebra $[X_a,\ X_b] = if_{abc}X_c$ is defined as,

$$(T_a)_{bc} = -if_{abc}$$

- Due to the Jacobi identity, this definition leads to

$$[T_a,\ T_b] = if_{abc}T_c$$

- The rows and columns of the generators $\{T_a\}$ are labeled by the same indices as that labels the generators themselves.

Thus,

- The basis vectors (states) of the adjoint representation space have a one-to-one correspondence with the generators,

$$T_a \quad \Leftrightarrow \quad |T_a\rangle$$

which implies,

$$\alpha\,|T_a\rangle + \beta\,|T_b\rangle = |\alpha T_a + \beta T_b\rangle$$

- The action of a generator on the basis states of the adjoint representation gives,

$$\begin{aligned} T_a\,|T_b\rangle &= \sum_c |T_c\rangle\langle T_c|\,T_a\,|T_b\rangle = \sum_c |T_c\rangle\,(T_a)_{cb} \\ &= \sum_c (if_{abc})\,|T_c\rangle = |\sum_c if_{abc}T_c\rangle \\ &= |[T_a,\ T_b]\rangle \end{aligned}$$

Its hermitian conjugate leads to:

$$\langle T_b| T_a^\dagger = \langle [T_a,\ T_b]|$$

- In adjoint representation, the scalar product between any two basis states $|T_a\rangle$ and $|T_b\rangle$ is defined by[1],

$$\langle T_a|T_b\rangle = \lambda^{-1}\mathrm{Tr}(T_a^\dagger T_b)$$

- In adjoint representation, the states $|H_i\rangle$ corresponding to Cartan generators are called the Cartan states. Obviously,

$$H_i\,|H_j\rangle = |[H_i,\ H_j]\rangle = |0\rangle = |0\cdot H_j\rangle = 0\,|H_j\rangle = 0$$

Besides, the Cartan states are orthonormal,

$$\langle H_i|H_j\rangle = \lambda^{-1}\mathrm{Tr}(H_i H_j) = \lambda^{-1}\cdot\lambda\delta_{ij} = \delta_{ij}.$$

---

[1] This formula is valid only for a compact Lie algebra.

Roots

Roots :

Weights of a Lie algebra in its adjoint representation are called roots.

Notice that,

- In the adjoint representation, $H_i |H_j\rangle = 0$, the Cartan states $\{|H_j\rangle\}$ have zero weights.
- The other states $\{|E_\alpha\rangle\}$ in the adjoint representation, which correspond to non-Cartan generators E_α, have non-zero weights:

$$H_i |E_\alpha\rangle = \alpha_i |E_\alpha\rangle, \qquad (i = 1, 2, \cdots, m.)$$

  i.e., $|[H_i,\ E_\alpha]\rangle = |\alpha_i E_\alpha\rangle$.

- This indicates:

$$[H_i,\ E_\alpha] = \alpha_i E_\alpha \ , \qquad (i = 1, 2, \cdots, m.)$$

Definition :

- The weights

$$\left\{ \alpha_i \middle| \, i = 1, 2, \cdots, m \right\}$$

of the adjoint representation are called roots. The weight vector

$$\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m)$$

is called a root vector of the Lie algebra.

Remarks :

- Like the $su(2)$ raising and lowering operators, the generators $\{E_\alpha\}$ related to the non-zero root vectors are not hermitian. The reason is as follows. Since $[H_i,\ E_\alpha] = \alpha_i E_\alpha$,

$$\alpha_i E_\alpha^\dagger = (\alpha_i E_\alpha)^\dagger = ([H_i,\ E_\alpha])^\dagger = -[H_i,\ E_\alpha^\dagger]$$

i.e.,

$$[H_i,\ E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger$$

By comparison we see that $E_\alpha \neq E_\alpha^\dagger$. Instead, $E_\alpha^\dagger = E_{-\alpha}$.

- In adjoint representation, states corresponding to different roots must be orthogonal.

This is because they have different eigenvalues of at least one of the Cartan generators,

$$\langle E_\alpha | E_\beta \rangle = \delta_{\alpha\beta}$$

It gives moreover,

$$\mathrm{Tr}(E_\alpha^\dagger E_\beta) = \lambda \langle E_\alpha | E_\beta \rangle = \lambda \delta_{\alpha\beta}$$

- The generators $\{E_{\pm\alpha}\}$ are raising and lowering operators for the weights.

Proof :

Consider a representation D of Lie algebra in which

$$H_i\,|\mu, D\rangle = \mu_i\,|\mu, D\rangle, \qquad (i = 1, 2, \cdots, m.)$$

Then,

$$\begin{aligned} H_i E_{\pm\alpha}\,|\mu, D\rangle &= [H_i,\ E_{\pm\alpha}]\,|\mu, D\rangle + E_{\pm\alpha} H_i\,|\mu, D\rangle \\ &= \pm\alpha_i E_{\pm\alpha}\,|\mu, D\rangle + E_{\pm\alpha}\mu_i\,|\mu, D\rangle \\ &= (\vec{\mu} \pm \vec{\alpha})_i E_{\pm\alpha}\,|\mu, D\rangle \end{aligned}$$

This result is valid for any representation, particularly true for the adjoint representation.

- Go back to the adjoint representation. We consider the state,

$$E_\alpha \left| E_{-\alpha} \right\rangle$$

This is an eigenstate of Cartan generators belonging to vanishing eigenvalue:

$$H_i E_\alpha \left| E_{-\alpha} \right\rangle = (\vec{\alpha} - \vec{\alpha})_i E_\alpha \left| E_{-\alpha} \right\rangle = 0$$

Therefore,

$$E_\alpha \left| E_{-\alpha} \right\rangle = c_i \left| H_i \right\rangle \qquad \rightsquigarrow \quad \left| [E_\alpha,\ E_{-\alpha}] \right\rangle = \left| c_i H_i \right\rangle$$

and from this we get the commutators:

$$[E_\alpha,\ E_{-\alpha}] = c_i H_i$$

We now determine the coefficients c_i:

$$\begin{aligned} c_i &= c_j \delta_{ij} = c_j \langle H_i | H_j \rangle = \langle H_i | c_j H_j \rangle = \langle H_i | [E_\alpha,\ E_{-\alpha}] \rangle \\ &= \frac{1}{\lambda} \mathrm{Tr}(H_i [E_\alpha,\ E_{-\alpha}]) \end{aligned}$$

where $\lambda \neq 0$. Equivalently,

$$\begin{aligned} \lambda c_i &= \mathrm{Tr}(H_i E_\alpha E_{-\alpha} - H_i E_{-\alpha} E_\alpha) \\ &= \mathrm{Tr}(E_{-\alpha} H_i E_\alpha - E_{-\alpha} E_\alpha H_i) \\ &= \mathrm{Tr}(E_{-\alpha} [H_i,\ E_\alpha]) \\ &= \mathrm{Tr}(E_\alpha^\dagger \alpha_i E_\alpha) \\ &= \alpha_i \mathrm{Tr}(E_\alpha^\dagger E_\alpha) \\ &= \alpha_i\ \lambda \qquad \rightsquigarrow \quad c_i = \alpha_i \end{aligned}$$

In other words,

$$[E_\alpha,\ E_{-\alpha}] = \alpha_i H_i = \vec{\alpha} \cdot \vec{H}$$

This is the analog of $[J_+,\ J_-] = J_3$ of $su(2)$ algebra.

- In adjoint representation, we now focus on the state,

$$E_\alpha \left| E_\beta \right\rangle$$

for $\vec{\alpha} + \vec{\beta} \neq 0$. This is an eigenstate of Cartan generators belonging to root vector $\vec{\alpha} + \vec{\beta}$,

$$H_i E_\alpha \left| E_\beta \right\rangle = (\vec{\alpha} + \vec{\beta})_i E_\alpha \left| E_\beta \right\rangle$$

Therefore,

$$E_\alpha \left| E_\beta \right\rangle = \mathcal{N}_{\alpha\beta} \left| E_{\alpha+\beta} \right\rangle \quad \rightsquigarrow \quad \left| [E_\alpha,\ E_\beta] \right\rangle = \left| \mathcal{N}_{\alpha\beta} E_{\alpha+\beta} \right\rangle$$

The relevant Lie brackets read,

$$[E_\alpha,\ E_\beta] = \mathcal{N}_{\alpha\beta} E_{\alpha+\beta}$$

$\mathcal{Q}$: $\mathcal{N}_{\alpha\beta} = ?$

Cartan-Weyl formalism

We have reformulated the Lie algebra $[X_i,\ X_j, =]if_{ijk}X_k$ into the so-called Cartan-Weyl basis,

$$
\begin{aligned}
[H_i,\ H_j] &= 0\ , \\
[H_i,\ E_\alpha] &= \alpha_i E_\alpha\ , \\
[E_\alpha,\ E_{-\alpha}] &= \alpha_i H_i\ , \\
[E_\alpha,\ E_\beta] &= \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta}\ , \quad (\text{ for }\ \vec{\alpha} + \vec{\beta} \neq 0.\ )
\end{aligned}
$$

The structure constants $\mathcal{N}_{\alpha,\beta}$ will be determined systematically.

Lots of $su(2)$s :

1. For each pair of non-zero root vectors $\pm\vec{\alpha}$, there is an $su(2)$ algebra of the Lie algebra $\mathfrak{g}$, with generators,

$$E_\pm = \frac{E_{\pm\alpha}}{\alpha}, \quad E_3 = \frac{\vec{\alpha}\cdot\vec{H}}{\alpha^2}$$

where $\alpha = |\vec{\alpha}|$.

Checking:

$$
\begin{aligned}
[E_3,\ E_\pm] &= \alpha^{-3}\alpha_i[H_i,\ E_{\pm\alpha}] \\
&= \pm\alpha^{-3}\alpha_i\alpha_i E_{\pm\alpha} = \pm\alpha^{-1}E_{\pm\alpha} = \pm E_\pm\ , \\
[E_+,\ E_-] &= \alpha^{-2}[E_\alpha,\ E_{-\alpha}] = \alpha^{-2}\alpha_i H_i = E_3\ .
\end{aligned}
$$

Corollaries :

- The 3 states $\{|E_3\rangle,\ |E_\pm\rangle\}$ in adjoint representation form a spin-1 representation of the associated $su(2)$ subalgebra $\{E_3,\ E_\pm\}$.

The nontrivial scalar products in subspace $\{|E_3\rangle,\ |E_\pm\rangle\}$ are,

$$
\begin{aligned}
\langle E_3|E_3\rangle &= \alpha^{-4}\alpha_i\alpha_j\langle H_i|H_j\rangle = \alpha^{-2}\ , \\
\langle E_\pm|E_\pm\rangle &= \alpha^{-2}\langle E_{\pm\alpha}|E_{\pm\alpha}\rangle = \alpha^{-2}\ .
\end{aligned}
$$

On these states, the action of generators $\{E_3,\ E_\pm\}$ is calculated below:

$$E_3 \left|E_\pm\right\rangle = \left|[E_3,\ E_\pm]\right\rangle = \left|\pm E_\pm\right\rangle = \pm \left|E_\pm\right\rangle ,$$
$$E_3 \left|E_3\right\rangle = \left|[E_3,\ E_3]\right\rangle = \left|0\right\rangle = 0 \left|E_3\right\rangle = 0 .$$

and

$$E_+ \left|E_+\right\rangle = \left|[E_+,\ E_+]\right\rangle = \left|0\right\rangle = 0 ,$$
$$E_+ \left|E_3\right\rangle = \left|[E_+,\ E_3]\right\rangle = \left|-E_+\right\rangle = - \left|E_+\right\rangle ,$$
$$E_+ \left|E_-\right\rangle = \left|[E_+,\ E_-]\right\rangle = \left|E_3\right\rangle .$$

By introducing the normalized basis states,

$$\left|1\right\rangle = \alpha \left|E_+\right\rangle = \left|E_\alpha\right\rangle$$
$$\left|2\right\rangle = \alpha \left|E_3\right\rangle = \alpha^{-1}\alpha_i \left|H_i\right\rangle$$
$$\left|3\right\rangle = \alpha \left|E_-\right\rangle = \left|E_{-\alpha}\right\rangle$$

we get:

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad E_+ = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_- = (E_+)^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This is the very spin-1 representation of $su(2)$ algebra.

- If $\vec{\alpha}$ is a root vector, no non-zero multiple of $\vec{\alpha}$ (except $-\vec{\alpha}$) is a root vector.

Proof :

Suppose $k\vec{\alpha}$ were a root vector for $k \neq \pm 1$. The corresponding generator and the state in adjoint representation read,

$$E_{k\alpha}, \quad |E_{k\alpha}\rangle .$$

Then,

$$\begin{aligned} E_3 |E_{k\alpha}\rangle &= |[E_3,\ E_{k\alpha}]\rangle = \alpha^{-2}\alpha_i |[H_i,\ E_{k\alpha}]\rangle \\ &= \alpha^{-2}\alpha_i |k\alpha_i E_{k\alpha}\rangle \\ &= k |E_{k\alpha}\rangle \end{aligned}$$

$|E_{k\alpha}\rangle$ becomes an eigenstate of E_3 belonging to eigenvalue k. Because E_3 could be recast as a generator of $su(2)$-algebra, its eigenvalue k must be a half-integer.

There are two possibilities:

- k is an integer.

  When k is an integer, $|E_{k\alpha}\rangle$ will be in such a $su(2)$ representation that contains another state $|E'_\alpha\rangle$ related to root vector $\vec{\alpha}$.

  We will show that a root vector corresponds uniquely to a generator.

  Hence,

$$|E'_\alpha\rangle = |E_\alpha\rangle \;\Leftrightarrow\; E_\alpha$$

  Recall that $|E_\alpha\rangle$ is in the spin-1 representation of $su(2)$ algebra generated by $E_3 = \alpha^{-2}\vec{\alpha}\cdot\vec{H}$ and $E_\pm = \alpha^{-1}E_{\pm\alpha}$, $-1 \leqslant k \leqslant 1$. We conclude that, $|E_{k\alpha}\rangle$'s existence is impossible unless $k \neq \pm 1$.

- k is half an odd integer.

  In this case, there were a state (and then a generator $E_{\alpha/2}$) with root vector $\vec{\alpha}/2$.

  We have seen that if $\vec{\alpha}$ is a root vector, $2\vec{\alpha}$ is not a root vector.

  Thus, if $\vec{\alpha}/2$ were a root vector, $\vec{\alpha} = 2(\vec{\alpha}/2)$ would not be a root vector $\rightsquigarrow$ absurd.

  We conclude that k cannot be half an odd integer.

1. There is a one-to-one correspondence between root vectors and the generators.

Proof:

Suppose the contrary: there were 2 independent generators E_α and E'_α corresponding to the same root vector $\vec{\alpha}$.

Choosing appropriate linear combination of E_α and E'_α, we could have:

$$0 = \langle E_\alpha | E'_\alpha \rangle = \lambda^{-1} \mathrm{Tr}(E^\dagger_\alpha E'_\alpha) = \lambda^{-1} \mathrm{Tr}(E_{-\alpha} E'_\alpha)$$

Consider the action of $su(2)$ algebra (related to root $\vec{\alpha}$) on the state $|E'_\alpha\rangle$. Because,

$$[H_i,\ E_\alpha] = \alpha_i E_\alpha, \quad [H_i,\ E'_\alpha] = \alpha_i E'_\alpha, \qquad i = 1, 2, \cdots, m$$

In adjoint representation, we have:

$$\begin{aligned} H_i E_- |E'_\alpha\rangle &= \alpha^{-1} H_i E_{-\alpha} |E'_\alpha\rangle \\ &= \alpha^{-1} [H_i,\ E_{-\alpha}] |E'_\alpha\rangle + \alpha^{-1} E_{-\alpha} H_i |E'_\alpha\rangle \\ &= -\alpha^{-1} \alpha_i E_{-\alpha} |E'_\alpha\rangle + \alpha^{-1} E_{-\alpha} |[H_i,\ E'_\alpha]\rangle \\ &= -\alpha^{-1} \alpha_i E_{-\alpha} |E'_\alpha\rangle + \alpha^{-1} E_{-\alpha} |\alpha_i E'_\alpha\rangle = 0 \end{aligned}$$

It implies,

$$E_- |E'_\alpha\rangle = c_j |H_j\rangle$$

The coefficient c_j turns out to be vanishing:

$$\begin{aligned} c_j &= \langle H_j | E_- | E'_\alpha \rangle = \langle H_j | [E_-, \ E'_\alpha] \rangle = \lambda^{-1} \mathrm{Tr}(H_j [E_-, \ E'_\alpha]) \\ &= -\lambda^{-1} \mathrm{Tr}(E_- [H_j, \ E'_\alpha]) \\ &= -\lambda^{-1} \alpha^{-1} \alpha_j \mathrm{Tr}(E_{-\alpha} E'_\alpha) = 0 \end{aligned}$$

Therefore $E_- |E'_\alpha\rangle = 0$. It implies,

- $|E'_\alpha\rangle$ is the lowest E_3 eigenstate in $su(2)$ representation.

However,

$$\begin{aligned} E_3 |E'_\alpha\rangle &= \alpha^{-2} \alpha_j H_j |E'_\alpha\rangle = \alpha^{-2} \alpha_j |[H_j, \ E'_\alpha]\rangle \\ &= \alpha^{-2} \alpha_j |\alpha_j E'_\alpha\rangle = |E'_\alpha\rangle \end{aligned}$$

This alternatively indicates that $|E'_\alpha\rangle$ is an eigenstate of E_3 belonging to eigenvalue $E_3 = 1$. A contradiction emerges:

- $|E'_\alpha\rangle$ cannot be the lowest value of E_3.

The above contradiction shows that the generator E'_α cannot exist. E_α is the unique generator related to the root vector $\vec{\alpha}$.

More generaically, for any weight $\vec{\mu}$ of a representation D of Lie algebra $\mathfrak{g}$, the E_3 value is determined by,

$$E_3 \left|\mu, \xi, D\right\rangle = \frac{\vec{\alpha} \cdot \vec{H}}{\alpha^2} \left|\mu, \xi, D\right\rangle$$
$$= \frac{\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} \left|\mu, \xi, D\right\rangle$$

Because the E_3's value must be integers or half odd integers,

$$\frac{2\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} = \text{integer}$$

From the perspective of E_3 related $su(2)$ algebra, this eigenvalue equation suggests that the state $\left|\mu, \xi, D\right\rangle$ is among the spin-j representation of this $su(2)$ for some non-negative half integer j.

Accurately, there is some non-negative integer p such that,

$$\left|jj\right\rangle_{su(2)} = (E_+)^p \left|\mu, \xi, D\right\rangle \neq 0$$

on which

$$\begin{aligned} E_3 \left|jj\right\rangle_{su(2)} &= j \left|jj\right\rangle_{su(2)} \\ E_+ \left|jj\right\rangle_{su(2)} &= (E_+)^{p+1} \left|\mu, \xi, D\right\rangle = 0 \; . \end{aligned}$$

Notice that

$$\begin{aligned} [E_3,\ E_\pm] &= \pm E_\pm \\ [E_3,\ (E_\pm)^2] &= E_\pm[E_3,\ E_\pm] + [E_3,\ E_\pm]E_\pm = \pm 2\ (E_\pm)^2 \\ [E_3,\ (E_\pm)^3] &= E_\pm[E_3,\ (E_\pm)^2] + [E_3,\ E_\pm]\ (E_\pm)^2 = \pm 3\ (E_\pm)^3 \\ &\cdots \\ [E_3,\ (E_\pm)^p] &= \pm p\ (E_\pm)^p \end{aligned}$$

we get,

$$\begin{aligned} j\,|jj\rangle_{su(2)} &= E_3(E_+)^p\,|\mu,\xi,D\rangle \\ &= [E_3,\ (E_+)^p]\,|\mu,\xi,D\rangle + (E_+)^p E_3\,|\mu,\xi,D\rangle \\ &= p(E_+)^p\,|\mu,\xi,D\rangle + (E_+)^p\left(\alpha^{-2}\vec{\mu}\cdot\vec{\alpha}\right)|\mu,\xi,D\rangle \\ &= (p+\alpha^{-2}\vec{\mu}\cdot\vec{\alpha})(E_+)^p\,|\mu,\xi,D\rangle \\ &= (p+\alpha^{-2}\vec{\mu}\cdot\vec{\alpha})\,|jj\rangle_{su(2)} \end{aligned}$$

i.e.,

$$j = p + \frac{\vec{\mu}\cdot\vec{\alpha}}{\alpha^2}$$

Likewise, there is some non-negative integer q such that,

$$|j,-j\rangle_{su(2)} = (E_-)^q\,|\mu,\xi,D\rangle \neq 0$$

on which

$$\begin{aligned} E_3\,|j,-j\rangle_{su(2)} &= -j\,|j,-j\rangle_{su(2)}\,, \\ E_-\,|j,-j\rangle_{su(2)} &= (E_-)^{q+1}\,|\mu,\xi,D\rangle = 0\,. \end{aligned}$$

From these equations we see that there is another expression for the highest eigenvalue j of E_3,

$$\begin{aligned}
-j\,|j,-j\rangle_{su(2)} &= E_3(E_-)^q\,|\mu,\xi,D\rangle \\
&= [E_3,\ (E_-)^q]\,|\mu,\xi,D\rangle + (E_-)^q E_3\,|\mu,\xi,D\rangle \\
&= -q(E_-)^q\,|\mu,\xi,D\rangle + (E_-)^q\left(\alpha^{-2}\vec{\mu}\cdot\vec{\alpha}\right)|\mu,\xi,D\rangle \\
&= (-q+\alpha^{-2}\vec{\mu}\cdot\vec{\alpha})(E_-)^q\,|\mu,\xi,D\rangle \\
&= (-q+\alpha^{-2}\vec{\mu}\cdot\vec{\alpha})\,|j,-j\rangle_{su(2)}
\end{aligned}$$

i.e.,

$$j = q - \frac{\vec{\mu}\cdot\vec{\alpha}}{\alpha^2}$$

Comparison of the above two expressions of j yields

$$j = (p+q)/2$$

and the so-called Master formula :

$$\frac{2\vec{\mu}\cdot\vec{\alpha}}{\alpha^2} = q - p$$

1. In master formula, p and q are two non-negative integers.
2. For a given weight $\vec{\mu}$ and root $\vec{\alpha}$, p and q are determined by

$$(E_\alpha)^{p+1}\,|\mu,\xi,D\rangle = 0, \quad (E_{-\alpha})^{q+1}\,|\mu,\xi,D\rangle = 0$$

respectively.

For each weight vector $\vec{\mu}$ of the representation D of Lie algebra $\mathfrak{g}$, there is a spin-j representation $\left[j = (p+q)/2\right]$ of $su(2)$ subalgebra $\{E_3,\ E_\pm\}$ related to the root vector $\vec{\alpha}$,

- Its $(2j+1)$ basis states are as follows:

$$(E_{-\alpha})^q \left|\mu, \xi, D\right\rangle, \ (E_{-\alpha})^{q-1} \left|\mu, \xi, D\right\rangle, \ \cdots, \\ E_{-\alpha} \left|\mu, \xi, D\right\rangle, \ \left|\mu, \xi, D\right\rangle, \ E_\alpha \left|\mu, \xi, D\right\rangle, \\ (E_\alpha)^2 \left|\mu, \xi, D\right\rangle, \ \cdots, (E_\alpha)^{p-1} \left|\mu, \xi, D\right\rangle, \\ (E_\alpha)^p \left|\mu, \xi, D\right\rangle.$$

with

$$E_3 (E_{-\alpha})^q \left|\mu, \xi, D\right\rangle = -\frac{(p+q)}{2} (E_{-\alpha})^q \left|\mu, \xi, D\right\rangle$$
$$E_3 (E_\alpha)^p \left|\mu, \xi, D\right\rangle = \frac{(p+q)}{2} (E_\alpha)^p \left|\mu, \xi, D\right\rangle$$

- In view of the mother algebra $\mathfrak{g}$, the weights of these states are given by,

$$\vec{\mu} + n\vec{\alpha}, \quad (-q \leqslant n \leqslant p).$$

- The roots of $\mathfrak{g}$ are weights of its adjoint representation. For each root vector $\vec{\beta}$, there is a root vector chain as follows:

$$\vec{\beta} + n\vec{\alpha}, \quad (-q \leqslant n \leqslant p).$$

where the non-negative integers p and q are determined by conditions that both $\vec{\beta} + (p+1)\vec{\alpha}$ and $\vec{\beta} - (q+1)\vec{\alpha}$ are not roots.

Properties of $\mathcal{N}_{\alpha,\beta}$

The structure constants $\mathcal{N}_{\alpha,\beta}$ appear in the Lie brackets,

$$[E_\alpha,\ E_\beta] = \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta}$$

Properties of $\mathcal{N}_{\alpha,\beta}$:

- Evidently, $\mathcal{N}_{\alpha,\beta} = -\mathcal{N}_{\beta,\alpha}$.
- There is a one-to-one correspondence between the generators and the root vectors.

  Therefore, only when all of $\vec{\alpha},\ \vec{\beta}$ and $\vec{\alpha}+\vec{\beta}$ are root vectors of $\mathfrak{g}$, $\mathcal{N}_{\alpha,\beta} \neq 0$. Otherwise, $\mathcal{N}_{\alpha,\beta} = 0$.
- For root vector chain $\{\ \vec{\beta} + n\vec{\alpha} \mid -q \leqslant n \leqslant p\ \}$,

$$\mathcal{N}_{\alpha,(\beta+p\alpha)} = \mathcal{N}_{-\alpha,(\beta-q\alpha)} = 0$$

- In adjoint representation, $\langle E_\alpha | E_\beta \rangle = \delta_{\alpha\beta}$. So, for three non-zero root vectors $\alpha,\ \beta$ and $\alpha + \beta$,

$$\begin{aligned}\langle E_\alpha |\, E_{-\beta}\, | E_{\alpha+\beta} \rangle &= \langle E_\alpha | [E_{-\beta},\ E_{\alpha+\beta}] \rangle \\ &= \langle E_\alpha | \mathcal{N}_{-\beta,\alpha+\beta} E_\alpha \rangle \\ &= \mathcal{N}_{-\beta,\alpha+\beta} \langle E_\alpha | E_\alpha \rangle \ = -\mathcal{N}_{\alpha+\beta,-\beta}\end{aligned}$$

Alternatively,

$$\langle E_\beta |\, E_{-\alpha} = \langle E_\beta |\, E_\alpha^\dagger = \langle [E_\alpha,\ E_\beta] |$$

leads to:

$$\begin{aligned}\langle E_\alpha |\, E_{-\beta}\, | E_{\alpha+\beta} \rangle &= \langle [E_\beta,\ E_\alpha] | E_{\alpha+\beta}] \rangle \\ &= \langle \mathcal{N}_{\beta,\alpha} E_{\alpha+\beta} | E_{\alpha+\beta} \rangle \\ &= \mathcal{N}_{\beta,\alpha} \langle E_{\alpha+\beta} | E_{\alpha+\beta} \rangle \ = -\mathcal{N}_{\alpha,\beta}\end{aligned}$$

Therefore,

$$\mathcal{N}_{\alpha+\beta,-\beta} = \mathcal{N}_{\alpha,\beta}\ .$$

- Consider the generators related to the root vector chain $\{\ \vec{\beta} + n\vec{\alpha}\ \}$ with $-q \leqslant n \leqslant p$. Let

$$F_n = -\mathcal{N}_{\beta+n\alpha,\alpha}\mathcal{N}_{\beta+(n+1)\alpha,-\alpha}$$

we see $F_p = F_{-q-1} = 0$. Moreover,

$$\begin{aligned}
0 &= [E_{\beta+n\alpha},\ [E_\alpha,\ E_{-\alpha}]] + [E_\alpha,\ [E_{-\alpha},\ E_{\beta+n\alpha}]] \\
&\qquad +[E_{-\alpha},\ [E_{\beta+n\alpha},\ E_\alpha]] \\
&= \alpha_j[E_{\beta+n\alpha},\ H_j] + \mathcal{N}_{-\alpha,\beta+n\alpha}[E_\alpha,\ E_{\beta+(n-1)\alpha}] \\
&\qquad +\mathcal{N}_{\beta+n\alpha,\alpha}[E_{-\alpha},\ E_{\beta+(n+1)\alpha}] \\
&= -\alpha_j(\beta_j + n\alpha_j)E_{\beta+n\alpha} + \mathcal{N}_{-\alpha,\beta+n\alpha}\mathcal{N}_{\alpha,\beta+(n-1)\alpha}E_{\beta+n\alpha} \\
&\qquad +\mathcal{N}_{\beta+n\alpha,\alpha}\mathcal{N}_{-\alpha,\beta+(n+1)\alpha}E_{\beta+n\alpha} \\
&= \Big[-\vec{\alpha}\cdot(\vec{\beta}+n\vec{\alpha}) - F_{n-1} + F_n\Big]E_{\beta+n\alpha}
\end{aligned}$$

This yields a recursion relation :

$$F_n = F_{n-1} + \vec{\alpha}\cdot(\vec{\beta}+n\vec{\alpha})$$

Therefore,

$$\begin{aligned}
F_n &= F_{n-1} + \vec{\alpha} \cdot (\vec{\beta} + n\vec{\alpha}) \\
&= F_{n-2} + \vec{\alpha} \cdot (\vec{\beta} + n\vec{\alpha}) + \vec{\alpha} \cdot \left[\vec{\beta} + (n-1)\vec{\alpha}\right] \\
&= F_{n-3} + \vec{\alpha} \cdot (\vec{\beta} + n\vec{\alpha}) + \vec{\alpha} \cdot \left[\vec{\beta} + (n-1)\vec{\alpha}\right] \\
&\qquad + \vec{\alpha} \cdot \left[\vec{\beta} + (n-2)\vec{\alpha}\right] \\
&= \cdots \\
&= F_{n-(n+q+1)} + \textstyle\sum_{i=0}^{n+q} \vec{\alpha} \cdot \left[\vec{\beta} + (n-i)\vec{\alpha}\right] \\
&= F_{-q-1} + (n+q+1)(\vec{\alpha} \cdot \vec{\beta}) \\
&\qquad + \left[n(n+q+1) - \tfrac{1}{2}(n+q+1)(n+q)\right](\vec{\alpha} \cdot \vec{\alpha}) \\
&= \tfrac{1}{2}(n+q+1)\left[2(\vec{\alpha} \cdot \vec{\beta}) + (n-q)\alpha^2\right]
\end{aligned}$$

When $n = p$, this equation is reduced to the expected master formula,

$$\frac{2(\vec{\alpha} \cdot \vec{\beta})}{\alpha^2} = q - p$$

When $n = 0$, it gives

$$F_0 = \frac{1}{2}(q+1)\left[2(\vec{\alpha}\cdot\vec{\beta}) - q\alpha^2\right] = -\frac{1}{2}p(q+1)\alpha^2$$

Notice that $F_0 = -\mathcal{N}_{\beta,\alpha}\mathcal{N}_{\beta+\alpha,-\alpha} = -\mathcal{N}_{\beta,\alpha}\mathcal{N}_{\beta,\alpha}$, we finally get:

$$(\mathcal{N}_{\alpha,\beta})^2 = \frac{1}{2}p(q+1)\alpha^2$$

Consider the scalar product of root vectors $\vec{\alpha}$ and $\vec{\beta}$,

$$\frac{2(\vec{\alpha}\cdot\vec{\beta})}{\alpha^2} = q - p$$

or

$$\frac{2(\vec{\alpha}\cdot\vec{\beta})}{\beta^2} = q' - p'$$

The first master formula implies the existence of root vector chain $\{ \vec{\beta} + n\vec{\alpha} \}$ with $-q \leqslant n \leqslant p$,while the second formula implies the existence of another root vector chain $\{ \vec{\alpha} + n'\vec{\beta} \}$ with $-q' \leqslant n' \leqslant p'$. Hence,

$$\left(\cos\theta_{\alpha\beta}\right)^2 = \frac{(\vec{\alpha}\cdot\vec{\beta})^2}{\alpha^2\beta^2} = \frac{(q-p)(q'-p')}{4}$$

What is remarkable is that $(q-p)(q'-p')$ must be a non-negative integer.

Relying on the fact that

$$-1 \leqslant \cos\theta_{\alpha\beta} \leqslant 1$$

there are only 4 choices for the angle between two distinct root vectors:

Table: The possible angles between two distinct root vectors

$(q-p)(q'-p')$	$\theta_{\alpha\beta}$
0	$\pi/2$
1	$\pi/3$ or $2\pi/3$
2	$\pi/4$ or $3\pi/4$
3	$\pi/6$ or $5\pi/6$

The basic formula for such an angle is,

$$\cos\theta_{\alpha\beta} = \pm\frac{1}{2}\sqrt{(q-p)(q'-p')}$$

The possibility $(q-p)(q'-p') = 4$, which corresponds to $\theta_{\alpha\beta} = 0$ or $\theta_{\alpha\beta} = \pi$, is not interesting.

Problems :

1. Show that $[E_\alpha,\ E_\beta]$ must be proportional to $E_{\alpha+\beta}$. What happens if $\vec{\alpha}+\vec{\beta}$ is not a root vector ?

2. Suppose that the raising operators of some Lie algebra $\mathfrak{g}$ satisfy $[E_\alpha,\ E_\beta] = \mathcal{N} E_{\alpha+\beta}$ for some nonzero $\mathcal{N}$. Calculate $[E_\alpha,\ E_{-\alpha-\beta}]$.

3. Consider the simple Lie algebra $\mathfrak{g}$ formed by the 10 matrices

$$\{\sigma_a, \sigma_a\tau_1, \sigma_a\tau_3, \tau_2\}$$

for $a = 1$ to 3, where σ_a and τ_a are Pauli matrices in orthogonal spaces. Take $H_1 = \sigma_3$ and $H_2 = \sigma_3\tau_3$ as the Cartan generators. Find: (1) the weights of the 4-dimensional Rep. generated by these matrices; (2) the weights of the adjoint representation.

$SU(3)$ Definition Rep.

In its definition representation, $SU(3)$ is the group of 3×3 unitary matrices $\{u \mid uu^\dagger = u^\dagger u = 1\}$ with unity determinant ($\det u = 1$).

The group elements of $SU(3)$ have the form

$$u = e^{i \sum_{a=1}^{8} \alpha_a X_a}$$

with X_a a set of linearly independent 3×3 traceless hermitian generators:

$$\begin{aligned} X_1 &= T_{12}^{(1)}, \quad X_2 = T_{12}^{(2)}, \quad X_3 = T_{2}^{(3)}, \\ X_4 &= T_{13}^{(1)}, \quad X_5 = T_{13}^{(2)}, \quad X_6 = T_{23}^{(1)}, \\ X_7 &= T_{23}^{(2)}, \quad X_8 = T_{3}^{(3)}. \end{aligned}$$

where

$$(T_{ab}^{(1)})_{ij} = \frac{1}{2}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi}),$$
$$(T_{ab}^{(2)})_{ij} = \frac{1}{2i}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})$$

for $a \neq b$, and

$$(T_a^{(3)})_{ij} = \begin{cases} \delta_{ij}\frac{1}{\sqrt{2a(a-1)}}, & \text{if} \quad i < a\,; \\ -\delta_{ij}\sqrt{\frac{a-1}{2a}}, & \text{if} \quad i = a\,; \\ 0, & \text{if} \quad i > a. \end{cases}$$

We can recast the generators as

$$X_a = \lambda_a/2$$

Such λ_a $(a = 1, 2, \cdots, 8)$ are called Gell-Mann matrices.

Gell-Mann matrices are explicitly written out as follows,

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The $SU(3)$ group is a compact Lie group, because its generators

$$X_a = \lambda_a/2 \qquad (a = 1, 2, \cdots, 8)$$

satisfy the uniform orthonormal conditions:

$$\mathrm{Tr}(X_a X_b) = \frac{1}{2}\delta_{ab}$$

Consequently, the structure constants $\{f_{abc}\}$ appearing in the Lie brackets $[X_a,\ X_b] = if_{abc}X_c$ are completely antisymmetric.

With Gell-Mann matrices, the $su(3)$ algebra could be recast as:

$$[\lambda_a,\ \lambda_b] = 2if_{abc}\lambda_c$$

where f_{abc} are completely antisymmetric in the indices.
The nonzero f_{abc} are

$$f_{123} = 1$$
$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = 1/2$$
$$f_{458} = f_{678} = \sqrt{3}/2$$

Besides, the Gell-Mann matrices have the following additional properties:

1. $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$
2. Completeness relation reads,

$$(\lambda_a)_{ij}(\lambda_a)_{kl} = -\frac{2}{3}\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk}$$

   where $i, j, k, l = 1,\ 2,\ 3.$
3. There exists a group of completely symmetric constants d_{abc} such that,

$$\{\lambda_a,\ \lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c$$

For completeness, we list the nonzero components of d_{abc} below:

$$\begin{cases} d_{118} = d_{228} = d_{338} = 1/\sqrt{3} \\ d_{146} = d_{157} = d_{256} = d_{344} = d_{355} = 1/2 \\ d_{247} = d_{366} = d_{377} = -1/2 \\ d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} \\ d_{888} = -1/\sqrt{3} \end{cases}$$

Casimir operators :

$SU(3)$ has two independent Casimir operators

$$C_2 = \sum_{a=1}^{8} X_a X_a, \qquad C_3 = \sum_{a,b,c=1}^{8} d_{abc} X_a X_b X_c$$

In definition representation, we have:

$$C_2 = 4/3, \qquad C_3 = 10/9.$$

Checking $\mathrm{Tr}(X_a X_b) = \frac{1}{2}\delta_{ab}$

Notice that in $T^{(1)}_{ab}$ and $T^{(2)}_{ab}$, $a \neq b$. $T^{(3)}_a$ are diagonal matrices. Thus,

$$\begin{aligned}
(T^{(1)}_{ab})_{ij}(T^{(1)}_{cd})_{ji} &= \frac{1}{4}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{di} + \delta_{ci}\delta_{dj}) \\
&= \frac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}), \\
(T^{(1)}_{ab})_{ij}(T^{(2)}_{cd})_{ji} &= \frac{1}{4i}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{di} - \delta_{ci}\delta_{dj}) = 0, \\
(T^{(1)}_{ab})_{ij}(T^{(3)}_{c})_{ji} &= \frac{1}{2}(\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi})(T^{(3)}_{c})_{ji} \\
&= \frac{1}{2}\left[(T^{(3)}_{c})_{ab} + (T^{(3)}_{c})_{ba}\right] = 0, \\
(T^{(2)}_{ab})_{ij}(T^{(2)}_{cd})_{ji} &= -\frac{1}{4}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(\delta_{cj}\delta_{di} - \delta_{ci}\delta_{dj}) \\
&= \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})
\end{aligned}$$

$$\begin{aligned}(T^{(2)}_{ab})_{ij}(T^{(3)}_c)_{ji} &= \frac{1}{2i}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi})(T^{(3)}_c)_{ji} \\ &= \frac{1}{2i}\left[(T^{(3)}_c)_{ba} - (T^{(3)}_c)_{ab}\right] \\ &= 0\end{aligned}$$

Besides, when $a < b$,

$$\begin{aligned}(T^{(3)}_a)_{ij}(T^{(3)}_b)_{ji} &= (a-1)\left[\frac{1}{\sqrt{2a(a-1)}} \cdot \frac{1}{\sqrt{2b(b-1)}}\right] \\ &\quad -\sqrt{\frac{a-1}{2a}}\frac{1}{\sqrt{2b(b-1)}} = 0\end{aligned}$$

while when $a = b$,

$$\begin{aligned}(T^{(3)}_a)_{ij}(T^{(3)}_a)_{ji} &= (a-1)\left[\frac{1}{2a(a-1)}\right] + \frac{a-1}{2a} \\ &= \frac{1}{2}\end{aligned}$$

Checking is finished.

Cartan generators

Among these generators, there are two commute mutually and they form the Cartan generators of group $SU(3)$:

$$H_1 = X_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_2 = X_8 = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Because H_1 and H_2 are already diagonal, the weights of $su(3)$ definition representation can be read off through

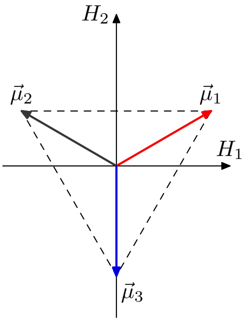
$$H_i \left|\vec{\mu}_a\right\rangle = (\vec{\mu}_a)_i \left|\vec{\mu}_a\right\rangle$$

with $i = 1, 2$ but $a = 1, 2, 3$. The result is as follows:

$\vec{\mu}_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$	$\vec{\mu}_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$	$\vec{\mu}_3 = \left(0, -\frac{1}{\sqrt{3}}\right)$
$\left\lvert\vec{\mu}_1\right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\left\lvert\vec{\mu}_2\right\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\left\lvert\vec{\mu}_3\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Weight diagram

In weight diagram, these weight vectors form an equilateral triangle:



Here,

$$\vec{\mu}_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right), \quad \vec{\mu}_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right), \quad \vec{\mu}_3 = \left(0, -\frac{1}{\sqrt{3}}\right).$$

Among them, $\vec{\mu}_1$ is the highest weight vector.

Roots of $su(3)$:

Question :

How many root vectors does $su(3)$ algebra have ?

Because

- $su(3)$ has 6 non-Cartan generators.
- There is a one-to-one correspondence between the root vectors and the non-Cartan generators.

$su(3)$ has 6 distinct root vectors: half of which are positive, another half are negative.

The 3 distinct positive root vectors can be read off from the difference of weight vectors of the above definition representation:

$$\vec{\alpha}_1 = \vec{\mu}_1 - \vec{\mu}_2 = (1, 0), \quad \vec{\alpha}_2 = \vec{\mu}_1 - \vec{\mu}_3 = (1/2, \sqrt{3}/2)$$
$$\vec{\alpha}_3 = \vec{\mu}_3 - \vec{\mu}_2 = (1/2, -\sqrt{3}/2)$$

Their negative counterparts are,

$$-\vec{\alpha}_1 = (-1, 0), \quad -\vec{\alpha}_2 = (-1/2, -\sqrt{3}/2), \quad -\vec{\alpha}_3 = (-1/2, \sqrt{3}/2).$$

The corresponding generators are those that have only one offidiagonal entry,

$$\begin{aligned} E_{\pm\alpha_1} &= \tfrac{1}{\sqrt{2}}(X_1 \pm iX_2), \qquad & E_{\pm\alpha_2} &= \tfrac{1}{\sqrt{2}}(X_4 \pm iX_5), \\ E_{\pm\alpha_3} &= \tfrac{1}{\sqrt{2}}(X_6 \mp iX_7). \end{aligned}$$

Explicitly,

$$E_{\alpha_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{\alpha_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{\alpha_3} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and

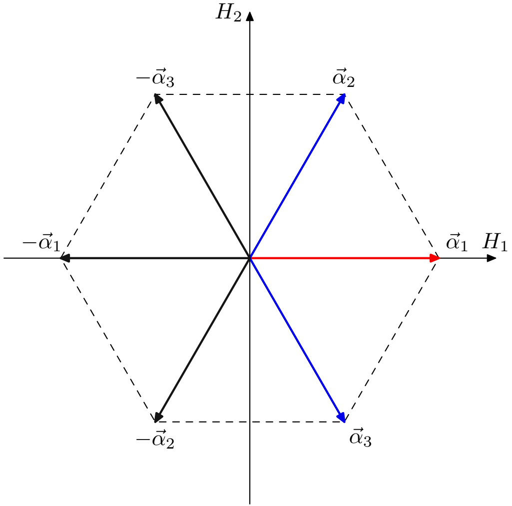
$$E_{-\alpha_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{-\alpha_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$E_{-\alpha_3} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In weight diagram, the 6 non-zero root vectors of $su(3)$

$$\pm\vec{\alpha}_1 = (\pm 1, 0), \quad \pm\vec{\alpha}_2 = (\pm 1/2, \pm\sqrt{3}/2), \quad \pm\vec{\alpha}_3 = (\pm 1/2, \mp\sqrt{3}/2),$$

form a regular hexagon:



Homework

Problems :

1. Calculate f_{147} and f_{458} in the $su(3)$ definition representation.
2. The $SU(3)$ structure constants have the property $f_{acd}f_{bcd} = 3\delta_{ab}$. Please show
$$f_{abc}\lambda_b\lambda_c = 3i\lambda_a$$
and
$$\lambda_b\lambda_a\lambda_b = -2\lambda_a/3$$
by making use of this relation.
3. Show that $X_1,\ X_2$ and X_3 generate an $su(2)$ subalgebra of $su(3)$. How does the representation generated by the Gell-Mann matrices transform under this subalgebra ?

现代数学物理方法

第二章, 李群

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Outline

Simple Roots :

Definition :

Simple roots are those positive root vectors that cannot be written as a sum of other positive root vectors.

Properties of Simple Roots :

- If $\vec{\alpha}$ and $\vec{\beta}$ are different simple roots, then $(\vec{\alpha} - \vec{\beta})$ is not a root vector.

  Proof : Let $\vec{\beta}$ be the larger so that $(\vec{\beta} - \vec{\alpha}) > 0$. The assumption that $\vec{\alpha}$ and $\vec{\beta}$ are simple roots and the fact

  $$\vec{\beta} = \vec{\alpha} + (\vec{\beta} - \vec{\alpha})$$

  indicate that $(\vec{\beta} - \vec{\alpha})$ is not a positive root vector.

- The angle $\theta_{\alpha\beta}$ between any pair of simple roots $\vec{\alpha}$ and $\vec{\beta}$ satisfies the constraint,

  $$\frac{\pi}{2} \leqslant \theta_{\alpha\beta} < \pi \ .$$

Proof : Consider two distinct simple roots $\vec{\alpha}$ and $\vec{\beta}$. Because $(\vec{\alpha}-\vec{\beta})$ is not a root vector, in the adjoint representation, we have:

$$E_{-\alpha}\left|E_{\beta}\right\rangle = E_{-\beta}\left|E_{\alpha}\right\rangle = 0.$$

Then, in the root vector chains $\{\vec{\beta}+n\vec{\alpha} \mid -q \leqslant n \leqslant p\}$ and $\{\vec{\alpha}+n'\vec{\beta} \mid -q' \leqslant n' \leqslant p'\}$, $q = q' = 0$. The master formula between these two simple roots gives,

$$\frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2} = -p \leqslant 0, \quad \frac{2\vec{\beta}\cdot\vec{\alpha}}{\beta^2} = -p' \leqslant 0,$$

where $p,\ p'$ are two nonnegative integers. Hence, $\cos\theta_{\alpha\beta} \leqslant 0$. Accurately, by combining the above two equations we get:

$$\cos\theta_{\alpha\beta} = -\sqrt{\frac{\vec{\alpha}\cdot\vec{\beta}}{\alpha^2}\cdot\frac{\vec{\beta}\cdot\vec{\alpha}}{\beta^2}} = -\frac{1}{2}\sqrt{pp'} \leqslant 0$$

Besides, the largest angle between any two positive root vectors cannot take values beyond π. As a result,

$$\frac{\pi}{2} \leqslant \theta_{\alpha\beta} < \pi\ .$$

- The simple roots are linearly independent from one another.

Proof : Consider a linear combination of the simple roots,

$$\vec{\gamma} = \sum_{\alpha} x_\alpha \vec{\alpha}$$

If all of the non-vanishing coefficients x_i have the same sign, $\vec{\gamma} \neq 0$. If there are some coefficients of each sign, we can write,

$$\vec{\gamma} = \vec{\mu} + \vec{\nu}$$

where $\vec{\mu} = \sum_\alpha x_\alpha \vec{\alpha}$ with all $x_\alpha > 0$, and $\vec{\nu} = \sum_\beta x_\beta \vec{\beta}$ with all $x_\beta < 0$. Relying on the fact $\frac{\pi}{2} \leqslant \theta_{\alpha\beta} < \pi$, $\vec{\alpha} \cdot \vec{\beta} \leqslant 0$.So,

$$\vec{\mu} \cdot \vec{\nu} = \sum_{\alpha, x_\alpha > 0} \sum_{\beta, x_\beta < 0} x_\alpha x_\beta \vec{\alpha} \cdot \vec{\beta} \geqslant 0.$$

From this we see,

$$\vec{\gamma}^2 = (\vec{\mu} + \vec{\nu})^2 = \vec{\mu}^2 + \vec{\nu}^2 + 2\vec{\mu} \cdot \vec{\nu} > 0\ .$$

$\vec{\gamma} = 0$ is possible iff all coefficients x_α vanish. In conclusion, the simple roots are linearly independent of one another.

- Any positive root vector $\vec{\phi}$ can be written as a linear combination of all simple roots with non-negative integer coefficients k_α,

$$\vec{\phi} = \sum_{\alpha, k_\alpha \geqslant 0} k_\alpha \vec{\alpha}$$

Corolleries :

1. The simple roots are not only linearly independent of each other only, they are also complete.
2. Because the root vector space has dimension m, *the rank of the Lie algebra* $\mathfrak{g}$, the number of simple roots is equal to m (the rank of the algebra), which is also the number of Cartan generators.

Question :

How to determine all the root vectors of an algebra $\mathfrak{g}$?

- It is only necessary to find out all positive root vectors,

$$\vec{\phi}_k = \sum_{\alpha, k_\alpha \geqslant 0} k_\alpha \vec{\alpha}$$

  where $\vec{\alpha}$ stands for simple roots and $k = \sum_\alpha k_\alpha$.

- All of the $\vec{\phi}_1$'s are roots because they are just the simple roots.

- Suppose we have determined the positive roots $\vec{\phi}_k$ for $k \leqslant n$. To find out $\{\vec{\phi}_{n+1}\}$, for all simple roots $\{\vec{\alpha}\}$, we consider the states

$$E_\alpha \left| E_{\phi_n} \right\rangle$$

  in $\mathfrak{g}$'s adjoint representation. These states are related to the possible roots $\{\vec{\phi}_{n+1}\}$ of the form

$$\{\vec{\phi}_{n+1}\} = \{\vec{\phi}_n\} + \vec{\alpha}$$

Question :

Is $\{\vec{\phi}_{n+1}\}$ really a root ?

- $\{\vec{\phi}_{n+1}\}$ being a root means that $E_\alpha \left|E_{\phi_n}\right\rangle$ is a true state in the adjoint representation of the Lie algebra $\mathfrak{g}$.
- From the perspective of accessory $su(2)$ (related to the simple root $\vec{\alpha}$),

$$E_3 = \alpha^{-2}\vec{\alpha}\cdot\vec{H}, \quad E_\pm = \alpha^{-1}E_{\pm\alpha},$$

  this means that there must be a positive integer p such that,

$$(E_\alpha)^p \left|E_{\phi_n}\right\rangle \neq 0, \quad (E_\alpha)^{p+1}\left|E_{\phi_n}\right\rangle = 0.$$

- Similarly, there must exist another non-negative integer q such that,

$$(E_{-\alpha})^q \left|E_{\phi_n}\right\rangle \neq 0, \quad (E_{-\alpha})^{q+1}\left|E_{\phi_n}\right\rangle = 0.$$

Claiming that these states form the spin-j representation of the above accessory $su(2)$, we have in g's adjoint representation,

$$(E_{-\alpha})^q \left|E_{\phi_n}\right\rangle = \left|j,-j\right\rangle_{su(2)}, \quad (E_\alpha)^p \left|E_{\phi_n}\right\rangle = \left|jj\right\rangle_{su(2)}.$$

So,

$$\begin{aligned} -j(E_{-\alpha})^q \left|E_{\phi_n}\right\rangle &= E_3(E_{-\alpha})^q \left|E_{\phi_n}\right\rangle \\ &= \alpha^{-2}\alpha_i H_i (E_{-\alpha})^q \left|E_{\phi_n}\right\rangle \\ &= \alpha^{-2}(\vec{\alpha}\cdot\vec{\phi_n} - q\alpha^2)(E_{-\alpha})^q \left|E_{\phi_n}\right\rangle \end{aligned}$$

and

$$\begin{aligned} j(E_\alpha)^p \left|E_{\phi_n}\right\rangle &= E_3(E_\alpha)^p \left|E_{\phi_n}\right\rangle \\ &= \alpha^{-2}\alpha_i H_i (E_\alpha)^p \left|E_{\phi_n}\right\rangle \\ &= \alpha^{-2}(\vec{\alpha}\cdot\vec{\phi_n} + p\alpha^2)(E_\alpha)^p \left|E_{\phi_n}\right\rangle \end{aligned}$$

Hence,

$$\frac{\vec{\alpha}\cdot\vec{\phi_n}}{\alpha^2} + p = j, \quad \frac{\vec{\alpha}\cdot\vec{\phi_n}}{\alpha^2} - q = -j.$$

Summation of these two equations gives,

$$\frac{2\vec{\alpha}\cdot\vec{\phi_n}}{\alpha^2} = q - p$$

Warning !

The significance of equation $\frac{2\vec{\alpha}\cdot\vec{\phi_n}}{\alpha^2} = q - p$:

- The equation is used to determine the integer p.

  We always know q, because we know the history of how $\vec{\phi_n}$ got built up by the action of the raising operators from $\vec{\phi_k}$ with the smaller k.

- If $p > 0$, $\vec{\phi_n} + \vec{\alpha}$ is a (positive) root vector.

Example 1 :

Suppose $\vec{\alpha}$ and $\vec{\beta}$ are two simple roots of a Lie algebra. Is $\vec{\alpha}+\vec{\beta}$ a root vector ?

Solution :

Take $\vec{\phi_1}=\vec{\beta}$. Because $\vec{\alpha}$ and $\vec{\beta}$ are simple roots,

$$E_{-\alpha}\left|E_{\phi_1}\right\rangle = 0.$$

Comparing with $(E_{-\alpha})^{q+1}\left|E_{\phi_1}\right\rangle = 0$, we see that $q=0$. So,

$$\frac{2\vec{\alpha}\cdot\vec{\phi_1}}{\alpha^2} = \frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2} = -p$$

If $\frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2}=0$, $\theta_{\alpha\beta}=\pi/2$, $p=0$, $\vec{\beta}+\vec{\alpha}$ is not a root vector. If $\frac{2\vec{\alpha}\cdot\vec{\beta}}{\alpha^2}<0$, $\pi/2<\theta_{\alpha\beta}<\pi$, $p>0$, $\vec{\beta}+\vec{\alpha}$ is a positive root.

Example 2 :

The $su(3)$ algebra has rank 2. Among its 3 positive roots of

$$\vec{\alpha_1} = (1/2, \sqrt{3}/2), \quad \vec{\alpha_2} = (1/2, -\sqrt{3}/2), \quad \vec{\alpha_3} = (1, 0)$$

there are only 2 simple roots. Because

$$\vec{\alpha_3} = \vec{\alpha_1} + \vec{\alpha_2}$$

$\vec{\alpha_1}$ and $\vec{\alpha_2}$ are the expected simple roots of $su(3)$ algebra.

Question :

Is $(\vec{\alpha_2} + 2\vec{\alpha_1})$ a root vector of $su(3)$?

Solution :

Construct $SU(2)$ generators from the generators related to the simple root $\vec{\alpha_1}$,

$$E_\pm = \alpha_1^{-1} E_{\pm\alpha_1} = E_{\pm\alpha_1}, \quad E_3 = \alpha_1^{-2} \vec{\alpha_1} \cdot \vec{H} = \vec{\alpha_1} \cdot \vec{H},$$

where we have noticed that

$$\alpha_1^2 = \alpha_2^2 = 1, \quad \vec{\alpha_1} \cdot \vec{\alpha_2} = -1/2.$$

Now focus on $(\vec{\alpha_2} + 2\vec{\alpha_1}) = \vec{\alpha_3} + \vec{\alpha_1}$:

$$\frac{2\vec{\alpha_3} \cdot \vec{\alpha_1}}{\alpha_1^2} = 2\vec{\alpha_3} \cdot \vec{\alpha_1} = 1 = q - p, \qquad \rightsquigarrow \quad q - p = 1.$$

On the other hand,
$\vec{\alpha_3} - \vec{\alpha_1} = \vec{\alpha_2}$ is a root but $\vec{\alpha_3} - 2\vec{\alpha_1} = \vec{\alpha_2} - \vec{\alpha_1}$ is not. This implies $q = 1$.
So, $p = 0$. $\vec{\alpha_3} + \vec{\alpha_1} = 2\vec{\alpha_2} + \vec{\alpha_1}$ is not a $su(3)$ root vector.

Constructing Lie algebra :

1. The basis states of the adjoint representation space have a one-to-one correspondence with the generator,

$$T_a \Leftrightarrow |T_a\rangle, \quad T_a\,|T_b\rangle = |[T_a,\, T_b]\rangle$$

   Thus, knowing the states in adjoint representation enable us to obtain the Lie algebra itself

$$[T_a,\, T_b] = i f_{abc} T_c$$

2. There is also a one-to-one correspondence between root vectors and the non-Cartan generators. Therefore, in adjoint representation, each root vector $\vec{\beta}$ corresponds uniquely to a basis state $|E_\beta\rangle$.

3. Associated with a simple root $\vec{\alpha}$, we can define an accessory $su(2)_\alpha$ subalgebra,

$$E_\pm = \alpha^{-1} E_{\pm\alpha}, \quad E_3 = \alpha^{-2}\vec{\alpha}\cdot\vec{H}.$$

   Some of the states $\{|E_\beta\rangle\}$ will form a spin-j representation of this $su(2)_\alpha$,

$$j = \frac{1}{2}(p+q)$$

where $p,\ q$ are two integers, determined by

$$(E_-)^{q+1}|E_\beta\rangle = 0, \quad \frac{2\vec{\beta}\cdot\vec{\alpha}}{\alpha^2} = q - p.$$

Notice that,

$$E_3|E_\beta\rangle = \frac{\vec{\beta}\cdot\vec{\alpha}}{\alpha^2}|E_\beta\rangle$$

The state $|E_\beta\rangle$ can be recast as a standard $su(2)_\alpha$ form $|jm\rangle$,

$$|E_\beta\rangle = \left|j, \frac{\vec{\beta}\cdot\vec{\alpha}}{\alpha^2}\right\rangle$$

In this way, the knowledge of $su(2)$ enable us to know exactly how $E_\pm$ act (up to a phase).

Remark :

This procedure will enable us to determine $[E_\alpha,\ E_\beta] = \mathcal{N}_{\alpha\beta}E_{\alpha+\beta}$ and then the whole algebra.

Now we illustrate the above procedure by constructing the $su(3)$ algebra from the knowledge of its simple roots.

Starting point : The algebra $su(3)$ has 2 simple roots $\vec{\alpha_1}$ and $\vec{\alpha_2}$,

$$\vec{\alpha_1} = (1/2, \sqrt{3}/2), \quad \vec{\alpha_2} = (1/2, -\sqrt{3}/2).$$

Evidently, $\alpha_1^2 = \alpha_2^2 = 1$, $\vec{\alpha_1} \cdot \vec{\alpha_2} = -1/2$.

$su(2)_{\alpha_1}$: We construct a $su(2)_{\alpha_1}$ algebra $\{E_\pm = E_{\pm\alpha_1}, E_3 = \vec{\alpha_1} \cdot \vec{H}\}$ based on simple root $\vec{\alpha_1}$. Since $[E_{-\alpha_1}, \ E_{\alpha_2}] = 0$, in adjoint representation, we have:

$$0 = |[E_{-\alpha_1}, \ E_{\alpha_2}]\rangle = E_{-\alpha_1} |E_{\alpha_2}\rangle = E_- |E_{\alpha_2}\rangle$$

i.e., $q = 0$. Together with $(q - p) = 2\vec{\alpha_2} \cdot \vec{\alpha_1}/\alpha_1^2 = -1$ we see $p = 1$, $j = (p + q)/2 = 1/2$. So, in $su(2)_{\alpha_1}$ language, $|E_{\alpha_2}\rangle$ can be written as

$$|E_{\alpha_2}\rangle = \left| j, \frac{\vec{\alpha_2} \cdot \vec{\alpha_1}}{\alpha_1^2} \right\rangle_{\alpha_1} = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_{\alpha_1}$$

Consequently,

$$|[E_{\alpha_1},\ E_{\alpha_2}]\rangle = E_{\alpha_1}|E_{\alpha_2}\rangle = E_+\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{\alpha_1} = \frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle_{\alpha_1}$$

On the other hand, in adjoint representation, the state $|E_{\alpha_3}\rangle$ related to the positive root vector $\vec{\alpha_3} = \vec{\alpha_1} + \vec{\alpha_2}$ satisfies,

$$E_3|E_{\alpha_3}\rangle = \vec{\alpha_1}\cdot\vec{\alpha_3}|E_{\alpha_3}\rangle = \frac{1}{2}|E_{\alpha_3}\rangle$$

i.e.,

$$|E_{\alpha_3}\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle_{\alpha_1}$$

The consistency between the above results implies that,

$$|[E_{\alpha_1},\ E_{\alpha_2}]\rangle = \frac{1}{\sqrt{2}}|E_{\alpha_3}\rangle$$

i.e.,

$$[E_{\alpha_1},\ E_{\alpha_2}] = \frac{1}{\sqrt{2}}E_{\alpha_3}$$

For $su(3)$, the other Lie brackets can be calculated by using Jacobi identities. e.g,

$$\begin{aligned}
[E_{-\alpha_1},\ E_{\alpha_3}] &= \sqrt{2}[E_{-\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]] \\
&= -\sqrt{2}[E_{\alpha_1},\ [E_{\alpha_2},\ E_{-\alpha_1}]] - \sqrt{2}[E_{\alpha_2},\ [E_{-\alpha_1},\ E_{\alpha_1}]] \\
&= \sqrt{2}\alpha_{1i}[E_{\alpha_2},\ H_i] \\
&= -\sqrt{2}(\vec{\alpha_1}\cdot\vec{\alpha_2})E_{\alpha_2} = \frac{1}{\sqrt{2}}E_{\alpha_2}
\end{aligned}$$

i.e.,

$$[E_{-\alpha_1},\ E_{\alpha_3}] = \frac{1}{\sqrt{2}}E_{\alpha_2}$$

Similarly (Please check it yourself),

$$[E_{-\alpha_2},\ E_{\alpha_3}] = -\frac{1}{\sqrt{2}}E_{\alpha_1}$$

By taking the hermitian conjugation of above commutation relations, we further get

$$\begin{aligned}
&[E_{\alpha_1},\ E_{-\alpha_2}] = 0, && [E_{-\alpha_1},\ E_{-\alpha_2}] = -\frac{1}{\sqrt{2}}E_{-\alpha_3}, \\
&[E_{\alpha_1},\ E_{-\alpha_3}] = -\frac{1}{\sqrt{2}}E_{-\alpha_2}, && [E_{\alpha_2},\ E_{-\alpha_3}] = \frac{1}{\sqrt{2}}E_{-\alpha_1}.
\end{aligned}$$

Defintions :

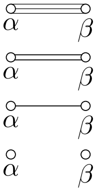
Cartan Matrix A : Let $\{\vec{\alpha_i}\}$ be simple roots of a Lie algebra $\mathfrak{g}$, its Cartan matrix is defined as,

$$A = (A_{ij}), \quad A_{ij} = \frac{2\vec{\alpha_i}\cdot\vec{\alpha_j}}{\alpha_j^2}$$

Dynkin Diagrm : A Dykin diagram is a short-hand notation for writing down the simple roots.

Rules :

1. Each simple root is expressed as an open or solid circle.
2. Pairs of circles are connected by lines, depending on the angle between the pair of roots to which the circles correspond ($\pi/2 \leqslant \theta_{\alpha\beta} < \pi$):



$\theta_{\alpha\beta} = 5\pi/6$

$\theta_{\alpha\beta} = 3\pi/4$

$\theta_{\alpha\beta} = 2\pi/3$

$\theta_{\alpha\beta} = \pi/2$

Meaning of Cartan Matrix A_{ij} :

Let $\{\vec{\alpha_i}\}$ be simple roots of a Lie algebra $\mathfrak{g}$. The accessory $su(2)$ generators related to simple root $\vec{\alpha_j}$ are

$$E_3 = \alpha_j^{-2}\vec{\alpha_j}\cdot\vec{H}, \quad E_\pm = \alpha_j^{-1}E_{\pm\alpha_j}.$$

Therefore, in $\mathfrak{g}$'s adjoint representation, on the state $|E_{\alpha_i}\rangle$ related to some simple root $\vec{\alpha_i}$,

$$E_3\,|E_{\alpha_i}\rangle = \frac{\vec{\alpha_i}\cdot\vec{\alpha_j}}{\alpha_j^2}\,|E_{\alpha_i}\rangle = \frac{A_{ij}}{2}\,|E_{\alpha_i}\rangle,$$

i.e., A_{ij} is twice of the eigenvalue of E_3 on state $|E_{\alpha_i}\rangle$.

Example : $su(3)$'s Dynkin diagram and Cartan matrix:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\theta_{\alpha_1\alpha_2} = 2\pi/3$

G_2 :

Example: G_2

The algebra G_2 has 2 simple roots,

$$\vec{\alpha_1} = (0, 1), \quad \vec{\alpha_2} = (\sqrt{3}/2, -3/2).$$

Obviously,

$$(\alpha_1)^2 = 1, \quad (\alpha_2)^2 = 3, \quad \vec{\alpha_1} \cdot \vec{\alpha_2} = -3/2.$$

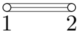
The Cartan matrix is,

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

The angle θ_{12} between two simple roots is calculated through,

$$\cos\theta_{12} = \frac{\vec{\alpha_1} \cdot \vec{\alpha_2}}{\alpha_1 \alpha_2} = -\sqrt{3}/2 \quad \rightsquigarrow \quad \theta_{12} = \frac{5\pi}{6}.$$

G_2's Dynkin diagram is:



$\theta_{12} = 5\pi/6$

Starting point :

We now search for all positive root vectors of G_2 algebra based on the its simple roots $\{\phi_1\}$,

$$\vec{\alpha_1} = (0, 1), \quad \vec{\alpha_2} = (\sqrt{3}/2, -3/2), \qquad (k = 1).$$

Finding $\{\phi_2\}$:

Is $\vec{\alpha_1} + \vec{\alpha_2}$ a positive root vector of $k = 2$?

To answer this question, we examine the properties of states $E_{\pm\alpha_1} |E_{\alpha_2}\rangle$ in G_2's adjoint representation. Construct an accessory $su(2)$ algebra based on simple root $\vec{\alpha_1}$,

$$E_3 = \alpha_1^{-2} \vec{\alpha_1} \cdot \vec{H}, \quad E_\pm = \alpha_1^{-1} E_{\pm\alpha_1}.$$

We claim that the states $E_{\pm\alpha_1} |E_{\alpha_2}\rangle$ are in the spin-j representation of this $su(2)_{\alpha_1}$. Because $(\vec{\alpha_1} - \vec{\alpha_2})$ is not a root, we have

$$E_{-\alpha_1} |E_{\alpha_2}\rangle = 0, \quad \rightsquigarrow \quad |E_{\alpha_2}\rangle = |j, -j\rangle_{\alpha_1}$$

So,

$$-j |E_{\alpha_2}\rangle = E_3 |E_{\alpha_2}\rangle = \frac{1}{2} A_{21} |E_{\alpha_2}\rangle = -\frac{3}{2} |E_{\alpha_2}\rangle$$

i.e., $j = 3/2$ and

$$|E_{\alpha_2}\rangle = |3/2, -3/2\rangle_{\alpha_1}$$

Assuming

$$(E_{\alpha_1})^p\,|E_{\alpha_2}\rangle \neq 0, \quad (E_{\alpha_1})^{p+1}\,|E_{\alpha_2}\rangle = 0,$$

i.e.,

$$(E_+)^p\,|3/2, -3/2\rangle_{\alpha_1} = |3/2, 3/2\rangle_{\alpha_1}$$

This gives that $p = 3\ (> 0)$. Therefore, $\vec{\phi_2} = (\vec{\alpha_1} + \vec{\alpha_2})$ is a root vector of G_2 with $k = 2$.

Corollaries : Relying on the facts,

$$(E_{\alpha_1})^3\,|E_{\alpha_2}\rangle \neq 0, \quad (E_{\alpha_1})^4\,|E_{\alpha_2}\rangle = 0,$$

the algebra G_2 has the following positive root vectors as well,

$$\begin{cases} \vec{\alpha_2} + 2\vec{\alpha_1}, & k = 3; \\ \vec{\alpha_2} + 3\vec{\alpha_1}, & k = 4. \end{cases}$$

Finding $\{\phi_3\}$:

We have found out a positive root vector of $k = 3$: $\vec{\alpha_2} + 2\vec{\alpha_1}$. The remaining candidate is then unique, which is $\vec{\alpha_1} + 2\vec{\alpha_2}$.
We define another accessory $su(2)$ related to the simple root $\vec{\alpha_2}$,

$$E'_3 = \alpha_2^{-2}\vec{\alpha_2}\cdot\vec{H}, \quad E'_\pm = \alpha_2^{-1}E_{\pm\alpha_2}.$$

Notice that $\vec{\alpha_1} + 2\vec{\alpha_2} = (\vec{\alpha_1} + \vec{\alpha_2}) + \vec{\alpha_2}$. In adjoint representation of G_2, assume that

$$(E'_+)^{p'} |\alpha_1 + \alpha_2\rangle \neq 0, \quad (E'_+)^{p'+1} |\alpha_1 + \alpha_2\rangle = 0,$$

and

$$(E'_-)^{q'} |\alpha_1 + \alpha_2\rangle \neq 0, \quad (E'_-)^{q'+1} |\alpha_1 + \alpha_2\rangle = 0.$$

Because the difference between two simple roots is not a root vector,

$$(E_{-\alpha_2})^2 |\alpha_1 + \alpha_2\rangle = 0, \qquad \rightsquigarrow \quad q' = 1.$$

Besides,

$$(q' - p') = \frac{2\vec{\alpha_2} \cdot (\vec{\alpha_1} + \vec{\alpha_2})}{\alpha_2^2} = 2 + A_{12} = 1, \qquad \rightsquigarrow \quad p' = 0.$$

As a result, $\vec{\alpha_1} + 2\vec{\alpha_2}$ is not a root vector of G_2.

Finding $\{\phi_4\}$:

G_2 has a unique positive root vector of $k = 4$, which is the one founded previously,

$$\vec{\phi_4} = \vec{\alpha_2} + 3\vec{\alpha_1}.$$

Finding $\{\phi_5\}$:

There is a unique candidate for the positive root vector of $k=5$,

$$\vec{\phi_5} = 2\vec{\alpha_2} + 3\vec{\alpha_1} = (\vec{\alpha_2} + 3\vec{\alpha_1}) + \vec{\alpha_2}.$$

Is it really a root vector of G_2 ?

As before, in G_2's adjoint representation, assume that

$$(E'_+)^{p''}\left|3\alpha_1+\alpha_2\right\rangle \neq 0, \quad (E'_+)^{p''+1}\left|3\alpha_1+\alpha_2\right\rangle = 0,$$

and

$$(E'_-)^{q''}\left|3\alpha_1+\alpha_2\right\rangle \neq 0, \quad (E'_-)^{q''+1}\left|3\alpha_1+\alpha_2\right\rangle = 0.$$

Because the integer multiple of a simple root is not a root vector,

$$E_{-\alpha_2}\left|3\alpha_1+\alpha_2\right\rangle = 0, \qquad \rightsquigarrow \quad q''=0.$$

Furthermore,

$$(q''-p'') = \frac{2\vec{\alpha_2}\cdot(3\vec{\alpha_1}+\vec{\alpha_2})}{\alpha_2^2} = 2+3A_{12} = -1, \qquad \rightsquigarrow \quad p''=1.$$

Hence, $(2\vec{\alpha_2}+3\vec{\alpha_1})$ is a true positive root vector of G_2 with $k=5$.

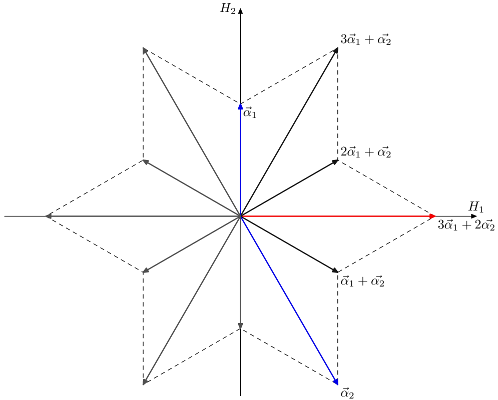
It is easy to know that G_2 has no more positive roots $\vec{\phi_k}$ with $k \geqslant 6$.

In conclusion, G_2 has 12 non-zero root vectors. They are listed as

$$\pm\vec{\alpha_1} = (0, \pm 1), \quad \pm\vec{\alpha_2} = (\pm\sqrt{3}/2, \mp 3/2),$$

and $\pm(\vec{\alpha_1} + \vec{\alpha_2})$, $\pm(2\vec{\alpha_1} + \vec{\alpha_2})$, $\pm(3\vec{\alpha_1} + \vec{\alpha_2})$ and $\pm(3\vec{\alpha_1} + 2\vec{\alpha_2})$.

In weight diagram,



Constructing G_2 :

Generators :

$$\begin{array}{llll} H_1, & H_2, & & \\ E_{\pm\alpha_1}, & E_{\pm\alpha_2}, & & \\ E_{\pm(\alpha_1+\alpha_2)}, & E_{\pm(2\alpha_1+\alpha_2)}, & E_{\pm(3\alpha_1+\alpha_2)}, & E_{\pm(3\alpha_1+2\alpha_2)}. \end{array}$$

Two $su(2)$ subalgebras based on simple roots :

1. $su(2)_{\alpha_1}$: $E_3 = \vec{\alpha_1} \cdot \vec{H}, \quad E_{\pm} = E_{\pm\alpha_1}.$
2. $su(2)_{\alpha_2}$: $E_3' = \frac{1}{3}\vec{\alpha_2} \cdot \vec{H}, \quad E_{\pm}' = \frac{1}{\sqrt{3}} E_{\pm\alpha_2}.$

Construction procedure :

Step 1 :

Obviously,

$$[E_{\alpha_1}, E_{-\alpha_2}] = [E_{-\alpha_1}, E_{\alpha_2}] = 0.$$

Step 2 :

Starting from the state $|E_{\alpha_2}\rangle$ in G_2's adjoint representation. For $su(2)_{\alpha_1}$, this state has:

$$q = 0, \quad p = 3, \quad j = (p+q)/2 = 3/2.$$

In the standard notation of $su(2)_{\alpha_1}$ representation, we rewrite this state as,

$$|E_{\alpha_2}\rangle = |3/2, -3/2\rangle_{\alpha_1}$$

Hence,

$$|[E_{\alpha_1},\ E_{\alpha_2}]\rangle = E_{\alpha_1}\,|E_{\alpha_2}\rangle = E_+\,|3/2, -3/2\rangle_{\alpha_1} = \sqrt{\frac{3}{2}}\,|3/2, -1/2\rangle_{\alpha_1}$$

Ignoring the possible phase factor, we define:

$$|E_{\alpha_1+\alpha_2}\rangle = |3/2, -1/2\rangle_{\alpha_1}$$

Consequently,

$$[E_{\alpha_1},\ E_{\alpha_2}] = \sqrt{\frac{3}{2}} E_{\alpha_1+\alpha_2}$$

- It is better to regard this commutator as the definition of generator $E_{\alpha_1+\alpha_2}$.

Applying E_+ once more gives,

$$\begin{aligned} |[E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]]\rangle &= E_{\alpha_1}\,|[E_{\alpha_1},\ E_{\alpha_2}]\rangle = \sqrt{\tfrac{3}{2}} E_{\alpha_1}\,|E_{\alpha_1+\alpha_2}\rangle \\ &= \sqrt{\tfrac{3}{2}} E_+\,|3/2, -1/2\rangle_{\alpha_1} \\ &= \sqrt{3}\,|3/2, 1/2\rangle_{\alpha_1} \end{aligned}$$

Defining:

$$|E_{\alpha_2+2\alpha_1}\rangle = |3/2, 1/2\rangle_{\alpha_1}$$

Then,

$$E_{\alpha_2+2\alpha_1} = \frac{1}{\sqrt{3}}[E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]]$$

Repeating this procedure, we get,

$$\begin{aligned}
|[E_{\alpha_1}, [E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]]\rangle &= E_{\alpha_1} |[E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]\rangle \\
&= \sqrt{3} E_{\alpha_1} |E_{\alpha_2+2\alpha_1}\rangle \\
&= \sqrt{3} E_+ |3/2, 1/2\rangle_{\alpha_1} \\
&= \tfrac{3}{\sqrt{2}} |3/2, 3/2\rangle_{\alpha_1}
\end{aligned}$$

Defining:

$$|E_{\alpha_2+3\alpha_1}\rangle = |3/2, 3/2\rangle_{\alpha_1}$$

Then,

$$E_{\alpha_2+3\alpha_1} = \frac{\sqrt{2}}{3}[E_{\alpha_1}, [E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]]$$

Step 3 :

In view of $su(2)_{\alpha_2}$, the state $|E_{\alpha_2+3\alpha_1}\rangle$ in G_2's adjoint representation has the properties,

$$0 = E_{-\alpha_2}\,|E_{\alpha_2+3\alpha_1}\rangle \backsimeq E'_-\,|E_{\alpha_2+3\alpha_1}\rangle,$$
$$0 = (E_{\alpha_2})^2\,|E_{\alpha_2+3\alpha_1}\rangle \backsimeq (E'_+)^2\,|E_{\alpha_2+3\alpha_1}\rangle.$$

we see,

$$q' = 0, \quad p' = 1, \quad j' = (p'+q')/2 = 1/2$$

i.e.,

$$|E_{\alpha_2+3\alpha_1}\rangle = |1/2, -1/2\rangle_{\alpha_2}$$

Consequently,

$$\begin{aligned} |[E_{\alpha_2},\ E_{\alpha_2+3\alpha_1}]\rangle &= E_{\alpha_2}\,|E_{\alpha_2+3\alpha_1}\rangle = \sqrt{3}E'_+\,|E_{\alpha_2+3\alpha_1}\rangle \\ &= \sqrt{3}E'_+\,|1/2, -1/2\rangle_{\alpha_2} \\ &= \sqrt{\tfrac{3}{2}}\,|1/2, 1/2\rangle_{\alpha_2} \end{aligned}$$

Defining:

$$|E_{3\alpha_1+2\alpha_2}\rangle = |1/2, 1/2\rangle_{\alpha_2}$$

we get,

$$
\begin{aligned}
E_{3\alpha_1+2\alpha_2} &= \sqrt{\tfrac{2}{3}}[E_{\alpha_2},\ E_{\alpha_2+3\alpha_1}] \\
&= \frac{2}{3\sqrt{3}}[E_{\alpha_2},\ [E_{\alpha_1},\ [E_{\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]]]]
\end{aligned}
$$

The above are enough for determining all the commutation relations of G_2. For example,

$$
\begin{aligned}
[E_{-\alpha_1},\ E_{\alpha_1+\alpha_2}] &= \sqrt{\tfrac{2}{3}}[E_{-\alpha_1},\ [E_{\alpha_1},\ E_{\alpha_2}]] \\
&= -\sqrt{\tfrac{2}{3}}[E_{\alpha_2},\ [E_{-\alpha_1},\ E_{\alpha_1}]] \\
&= \sqrt{\tfrac{2}{3}}\alpha_{1i}[E_{\alpha_2},\ H_i] \\
&= -\sqrt{\tfrac{2}{3}}(\vec{\alpha_1}\cdot\vec{\alpha_2})E_{\alpha_2} \\
&= \sqrt{\tfrac{3}{2}}E_{\alpha_2}
\end{aligned}
$$

Highest weights representation D:

Let $\{\vec{\alpha}_i \,| i = 1, 2, \cdots, m\}$ be the simple roots of a simple Lie algebra $\mathfrak{g}$. Consider an irreducible representation D of $\mathfrak{g}$, in which there is a state $|M\rangle$ satisfying,

$$E_{\alpha_i} |M\rangle = 0, \quad H_i |M\rangle = M_i |M\rangle$$

where $\vec{M} = (M_1, M_2, \cdots, M_m)$ is the weight vector related to $|M\rangle$.

Properties of $\vec{M}$:

- $\vec{M}$ is the highest weight vector in Representation D.
- There must exist some non-negative integers $\{l_i\}$ so that,

$$\frac{2\vec{M} \cdot \vec{\alpha}_i}{\alpha_i^2} = l_i$$

[$\{l_i\}$ are called Dynkin coefficients.]

Definition : The fundamental weights $\{\vec{M}_i\}$ of a simple Lie algebra $\mathfrak{g}$ is defined by,

$$\frac{2\vec{M}_i \cdot \vec{\alpha}_j}{\alpha_j^2} = \delta_{ij}, \qquad (i, j = 1, 2, \cdots, m.)$$

Properties of $\{\vec{M}_i\}$:

- Each $\vec{M}_i$ defines an irreducible representation of $\mathfrak{g}$, in which $\vec{M}_i$ is the highest weight vector.
- $\#\vec{M}_i = m$ (rank of $\mathfrak{g}$).
- The highest weight vectors $\{\vec{M}_i\}$ are called the fundamental weights of $\mathfrak{g}$. The cooresponding irreducible representations are called the fundamental representations.
- The highest weight vector $\vec{M}$ of an arbitrary irreducible representation D can be expressed as

$$\vec{M} = \sum_i l_i \vec{M}_i$$

or equivalently,

$$\vec{M} = (l_1, l_2, \cdots, l_m).$$

- The highest weight state $|M\rangle$ in an irreducible representation D is unique.

Proof : Obviously, if

$$H_i\,|M\rangle = M_i\,|M\rangle\,, \quad H_i\,|M\rangle' = M_i\,|M\rangle'\,,$$

there will be some positive root vectors $\{\vec{\alpha}, \vec{\beta}, \cdots\}$ so that

$$|M\rangle' = E_\alpha \cdots E_\beta E_{-\alpha} \cdots E_{-\beta}\,|M\rangle\,.$$

It is enough to consider $\{\vec{\alpha}, \vec{\beta}, \cdots\}$ as the simple roots here, because

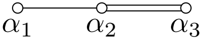
$$E_{\alpha+\beta} = [E_\alpha,\ E_\beta]/\mathcal{N}_{\alpha,\beta}$$

Hence, these two highest weight states are actually the same one:

$$|M\rangle' = (\vec{\alpha}\cdot\vec{M})\cdots(\vec{\beta}\cdot\vec{M})\,|M\rangle\,.$$

Homework :

1. Consider the algebra C_3 corresponding to the following Dynkin diagram. Let $\alpha_1^2 = \alpha_2^2 = 1$ and $\alpha_3^2 = 2$. Find the Cartan matrix A and all of the positive root vectors.



现代数学物理方法

第三章, $SU(3)$

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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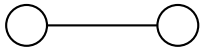
Fundamental weights of $su(3)$:

The algebra $su(3)$ is specified by Dynkin diagram

$su(3)$: 

It has two simple roots $\vec{\alpha_1}$ and $\vec{\alpha_2}$, with properties $\alpha_1^2 = \alpha_2^2 = 1$ and $\vec{\alpha_1} \cdot \vec{\alpha_2} = -1/2$. Therefore, $su(3)$ has 2 fundamental weight vectors :

$$\vec{M}_i = (a_i,\ b_i), \qquad \left\{ i = 1,\ 2. \right\}$$

To find $\vec{M}_i$ $(i = 1,\ 2)$, we first parameterize the simple roots as follows,

$$\vec{\alpha_1} = \left(1/2,\ \sqrt{3}/2\right), \qquad \vec{\alpha_2} = \left(1/2,\ -\sqrt{3}/2\right).$$

Because

$$\delta_{i1} = \frac{2\vec{M}_i \cdot \vec{\alpha_1}}{\alpha_1^2} = a_i + \sqrt{3}b_i, \quad \delta_{i2} = \frac{2\vec{M}_i \cdot \vec{\alpha_2}}{\alpha_2^2} = a_i - \sqrt{3}b_i$$

we see

$$\begin{cases} a_1 + \sqrt{3}b_1 = 1 \\ a_1 - \sqrt{3}b_1 = 0 \end{cases} \qquad \begin{cases} a_2 + \sqrt{3}b_2 = 0 \\ a_2 - \sqrt{3}b_2 = 1 \end{cases}$$

The solution to this system of algebraic equations is unique,

$$\begin{cases} a_1 = 1/2 \\ b_1 = 1/2\sqrt{3} \end{cases}$$

$$\begin{cases} a_2 = 1/2 \\ b_2 = -1/2\sqrt{3} \end{cases}$$

We conclude that :

1. $su(3)$ has 2 fundamental weight vectors. One reads,

$$\vec{M}_1 = \left[\frac{1}{2}, \frac{1}{2\sqrt{3}}\right]$$

   and the another reads,

$$\vec{M}_2 = \left[\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right]$$

2. $su(3)$ has 2 fundamental representations, D_1 and D_2. D_1 is defined by fundamental weight vector $\vec{M}_1$, and can be recast as

$$\text{Rep.}(1, 0)$$

   D_2 is defined by $\vec{M}_2$, and can be recast as

$$\text{Rep.}(0, 1)$$

Fundamental Rep. D_1 of $su(3)$

We now want to find all of the basis states of this representation. Our starting point is the highest weight state $|M_1\rangle$ satisfying

$$E_{\alpha_1}|M_1\rangle = E_{\alpha_2}|M_1\rangle = 0.$$

Procedure :
Build *two* $su(2)$ algebras associated to simple roots $\vec{\alpha_1}$ and $\vec{\alpha_2}$.
$su(2)_1$ consists of

$$E_3 = \vec{\alpha_1}\cdot\vec{H},\ E_\pm = E_{\pm\alpha_1}$$

but $su(2)_2$ consists of

$$E_3' = \vec{\alpha_2}\cdot\vec{H},\ E_\pm' = E_{\pm\alpha_2}$$

The state $|M_1\rangle$ could be embedded into the spin-j representation of $su(2)_1$ with

$$j = \frac{1}{2}\left[p + q\right]$$

or the spin-j' representation of $su(2)_2$ with

$$j' = \frac{1}{2}\left[p' + q'\right]$$

so that

$$\begin{cases} (E_+)^{p+1}|M_1\rangle = (E_-)^{q+1}|M_1\rangle = 0 \\ (E'_+)^{p'+1}|M_1\rangle = (E'_-)^{q'+1}|M_1\rangle = 0 \end{cases}$$

Since $E_{\alpha_1}|M_1\rangle = 0$ and $2\vec{M}_1 \cdot \vec{\alpha}_1 = 1$, we have $p = 0$, $q = 1$ and $j = 1/2.$

Hence,

$$|M_1\rangle = |1/2, 1/2\rangle_1$$

The second basis state in D_1 is found to be:

$$E_{-\alpha_1}|M_1\rangle = E_-|1/2, 1/2\rangle_1 = \frac{1}{\sqrt{2}}|1/2, -1/2\rangle_1$$

Similarly, the state $E_{-\alpha_1}\left|M_1\right\rangle$ can also be embedded into the spin-j'' representation of $su(2)_2$ with

$$j'' = \frac{1}{2}\left[p'' + q''\right]$$

where

$$(E'_+)^{p''+1} E_{-\alpha_1}\left|M_1\right\rangle = (E'_-)^{q''+1} E_{-\alpha_1}\left|M_1\right\rangle = 0.$$

Alternatively, $(q'' - p'')$ is given by

$$q'' - p'' = 2(\vec{M}_1 - \vec{\alpha_1}) \cdot \vec{\alpha_2} = -2\vec{\alpha_1} \cdot \vec{\alpha_2} = 1.$$

The difference of two simple roots is not a root vector,

$$\left[E_{-\alpha_1},\ E_{\alpha_2}\right] = 0$$

Therefore,

$$E_{\alpha_2}\left[E_{-\alpha_1}\left|M_1\right\rangle\right] = E_{-\alpha_1}\left[E_{\alpha_2}\left|M_1\right\rangle\right] = 0, \quad \rightsquigarrow \quad p'' = 0,\ q'' = 1$$

i.e., $j'' = 1/2$.

The state $E_{-\alpha_1}|M_1\rangle$ can be equivalently cast as,

$$E_{-\alpha_1}|M_1\rangle = \frac{1}{\sqrt{2}}|1/2, 1/2\rangle_2$$

The third state in D_1 reads,

$$E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle = E'_{-}\left[\frac{1}{\sqrt{2}}|1/2, 1/2\rangle_2\right] = \frac{1}{2}|1/2, -1/2\rangle_2$$

There are no more basis states in D_1.

Conclusions:

- Rep. D_1 or Rep.(1, 0) is 3-dimensional.
- D_1 is conveniently written as $\mathbf{3}$.

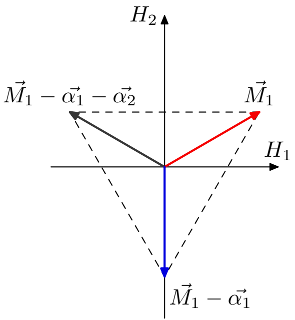
- The weight vectors in D_1 are,

$$\vec{M}_1 = \left[\frac{1}{2},\ \frac{1}{2\sqrt{3}}\right] \qquad \text{(highest weight)}$$

$$\vec{M}_1 - \vec{\alpha}_1 = \left[0,\ -\frac{1}{\sqrt{3}}\right]$$

$$\vec{M}_1 - \vec{\alpha}_1 - \vec{\alpha}_2 = \left[-\frac{1}{2},\ \frac{1}{2\sqrt{3}}\right]$$

In weight diagram,



- In D_1, three orthogonal basis states vectors are

$$|M_1\rangle, \quad E_{-\alpha_1}|M_1\rangle, \quad E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle.$$

Let $\langle M_1|M_1\rangle = 1$. Then,

$$\begin{aligned}
\langle M_1| E_{\alpha_1}E_{-\alpha_1}|M_1\rangle &= \langle M_1|[E_{\alpha_1},\ E_{-\alpha_1}]|M_1\rangle \\
&= \langle M_1|(\vec{\alpha_1}\cdot\vec{H})|M_1\rangle \\
&= (\vec{\alpha_1}\cdot\vec{M}_1) \\
&= 1/2
\end{aligned}$$

and

$$\begin{aligned}
&\langle M_1| E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle \\
&\qquad = \langle M_1| E_{\alpha_1}[E_{\alpha_2},\ E_{-\alpha_2}]E_{-\alpha_1}|M_1\rangle \\
&\qquad = \alpha_{2i}\langle M_1| E_{\alpha_1}H_iE_{-\alpha_1}|M_1\rangle \\
&\qquad = \alpha_{2i}(\vec{M}_1-\vec{\alpha_1})_i\langle M_1| E_{\alpha_1}E_{-\alpha_1}|M_1\rangle \\
&\qquad = \frac{1}{2}\vec{\alpha_2}\cdot(\vec{M}_1-\vec{\alpha_1}) \\
&\qquad = 1/4
\end{aligned}$$

Consequently,

$$|M_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E_{-\alpha_1}|M_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

D_2 or Rep.(0, 1) of $su(3)$ is defined by the fundamental weight vector $\vec{M}_2$:

$$\vec{M}_2 = \left[\frac{1}{2}, \ -\frac{1}{2\sqrt{3}}\right]$$

Highest weight state in D_2 :

The highest weight state $|M_2\rangle$ in D_2 satisfies

$$E_{\alpha_1}|M_2\rangle = E_{\alpha_2}|M_2\rangle = 0.$$

Besides,

$$\frac{2\vec{M}_2 \cdot \vec{\alpha_2}}{\alpha_2^2} = 1$$

Thus, $|M_2\rangle$ is also the highest weight state in the spin-$\frac{1}{2}$ Rep. of the accessory $su(2)_2$,

$$|M_2\rangle = |1/2, 1/2\rangle_2$$

Other basis states in D_2 :

The second basis state in D_2 is

$$E_{-\alpha_2}|M_2\rangle = E'_-|M_2\rangle = \frac{1}{\sqrt{2}}|1/2, -1/2\rangle_2$$

Notice that $E_{\alpha_1}(E_{-\alpha_2}|M_2\rangle) = 0$. Moreover,

$$\frac{2(\vec{M}_2 - \vec{\alpha_2}) \cdot \vec{\alpha_1}}{\alpha_1^2} = -2\vec{\alpha_2} \cdot \vec{\alpha_1} = 1$$

Because of these two equalities, $E_{-\alpha_2}\left|M_2\right\rangle$ is not only the lowest weight state in spin-1/2 representation of $su(2)_2$, it is also the highest weight state in spin-1/2 representation of $su(2)_1$:

$$E_{-\alpha_2}\left|M_2\right\rangle = \frac{1}{\sqrt{2}}\left|1/2, 1/2\right\rangle_1$$

As a result, the third basis state in D_2 is probably to be,

$$E_{-\alpha_1}E_{-\alpha_2}\left|M_2\right\rangle = \frac{1}{2}\left|1/2, -1/2\right\rangle_1$$

There are no more basis states in D_2.

Conclusion :

- D_2 of $su(3)$ is also 3-dimensional.
- D_2 is conveniently recast as $\bar{\mathbf{3}}$.

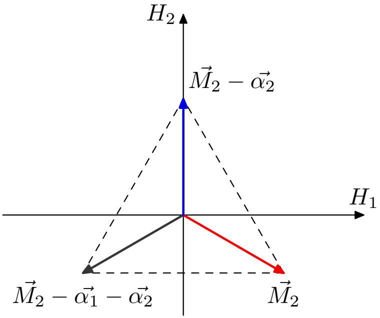
- The weight vectors in D_2 are,

$$\vec{M}_2 = \left[\tfrac{1}{2},\ -\tfrac{1}{2\sqrt{3}}\right] \qquad \text{(highest)}$$

$$\vec{M}_2 - \vec{\alpha}_2 = \left[0,\ \tfrac{1}{\sqrt{3}}\right]$$

$$\vec{M}_2 - \vec{\alpha}_1 - \vec{\alpha}_2 = \left[-\tfrac{1}{2},\ -\tfrac{1}{2\sqrt{3}}\right]$$

In weight diagram,



Complex conjugation :

The weight vectors of $\bar{\mathbf{3}}$ are just the negatives of those of $\mathbf{3}$.

Weights in 3 :

$$\vec{M}_1 = \left[\frac{1}{2},\ \frac{1}{2\sqrt{3}}\right],$$
$$\vec{M}_1 - \vec{\alpha}_1 = \left[0,\ -\frac{1}{\sqrt{3}}\right],$$
$$\vec{M}_1 - \vec{\alpha}_1 - \vec{\alpha}_2 = \left[-\frac{1}{2},\ \frac{1}{2\sqrt{3}}\right].$$

Weights in $\bar{\mathbf{3}}$:

$$\vec{M}_2 = \left[\frac{1}{2},\ -\frac{1}{2\sqrt{3}}\right],$$
$$\vec{M}_2 - \vec{\alpha}_2 = \left[0,\ \frac{1}{\sqrt{3}}\right],$$
$$\vec{M}_2 - \vec{\alpha}_1 - \vec{\alpha}_2 = \left[-\frac{1}{2},\ -\frac{1}{2\sqrt{3}}\right].$$

Question : What does this mean ?

This means that the two representations $\mathbf{3}$ and $\bar{\mathbf{3}}$ are related by complex conjugation.

Insight 1 :

Let X_a be the generators of some representation D of some Lie group $\mathbb{G}$. The group elements can be expressed as

$$e^{i\alpha_a X_a}$$

As a result, we have the following expressions for the group elements of its complex conjugate $\bar{D}$:

$$(e^{i\alpha_a X_a})^* = e^{-i\alpha_a X_a^*} = e^{i\alpha_a(-X_a^*)}$$

Besides, $-X_a^*$ obey the same Lie brackets as X_a,

$$[X_a,\ X_b] = if_{abc}X_c \qquad \rightsquigarrow \ [(-X_a^*),\ (-X_b^*)] = if_{abc}(-X_c^*)$$

Therefore, $-X_a^*$ are the generators of the complex conjugate Rep. $\bar{D}$ of the representation D.

Insight 2 :

The Cartan generators of the complex conjugate representation are $-H_i^*$. Because each H_i are Hermitian matrices, H^* have the same eigenvalues as H_i.

Conclusion:

If $\vec{\mu}$ is a weight vector of Rep.D, $-\vec{\mu}$ is a weight vector of the complex conjugate Rep.$\bar{D}$.

For $su(3)$, we have seen:

$$\text{Rep. } (1,0) = \mathbf{3}, \qquad \text{Rep. } (0,1) = \bar{\mathbf{3}}.$$

In general, for $su(3)$, the complex conjugate of Rep.(n, m) is Rep.(m, n).

Proof :

Because the lowest weight vector of Rep.$(1, 0)$ is the minus of the highest weight vector of Rep.$(0, 1)$, and vice versa. We have for Rep.(n, m),

$$\begin{array}{ll} \text{Highest weight :} & n\vec{M}_1 + m\vec{M}_2 \\ \text{Lowest weight :} & -n\vec{M}_2 - m\vec{M}_1 \end{array}$$

Consequently, the highest weight vector of its complex conjugate representation should be,

$$n\vec{M}_2 + m\vec{M}_1$$

Hence, Rep.(m, n) is the complex conjugate of Rep.(n, m).

Corollary:

- Rep.(n, n) are real representations of $su(3)$.

Rep.$(1, 1)$ of $su(3)$:

We now look for the basis states of the real irreducible representation Rep.$(1, 1)$ of $su(3)$.

Rep.$(1, 1)$ is defined by the highest weight vector,

$$\vec{M} = \vec{M}_1 + \vec{M}_2 = (1, 0)$$

so $2\vec{M} \cdot \vec{\alpha_1}/\alpha_1^2 = 1$, $2\vec{M} \cdot \vec{\alpha_2}/\alpha_2^2 = 1$.

Consider the highest weight state $|M\rangle$ in Rep.$(1, 1)$, which satisfies,

$$E_{\alpha_1} |M\rangle = E_{\alpha_2} |M\rangle = 0.$$

$|M\rangle$ can also be regarded as the highest weight state of the spin-$1/2$ representations of either $su(2)_1$ or $su(2)_2$,

$$|M\rangle = |1/2, 1/2\rangle_1 = |1/2, 1/2\rangle_2 .$$

Consequently, the second and the third basis states in Rep.$(1, 1)$ are found to be:

$$E_{-\alpha_1} |M\rangle = \frac{1}{\sqrt{2}} |1/2, -1/2\rangle_1$$

To find out the 4-th basis state in Rep.$(1,1)$, we examine $E_{-\alpha_1}|M\rangle$ in view of $su(2)_2$.

Notice that

$$E_{\alpha_2}\left\{E_{-\alpha_1}|M\rangle\right\} = 0$$

and

$$\frac{2(\vec{M}-\vec{\alpha_1})\cdot\vec{\alpha_2}}{\alpha_2^2} = 1 - \frac{2\vec{\alpha_1}\cdot\vec{\alpha_2}}{\alpha_2^2} = 1 - 2\left[-\frac{1}{2}\right] = 2$$

we alternatively have

$$E_{-\alpha_1}|M\rangle = \frac{1}{\sqrt{2}}|1,1\rangle_2$$

It leads to the following 4-th and 5-th basis states in Rep.$(1,1)$:

$$E_{-\alpha_2}E_{-\alpha_1}|M\rangle = \frac{1}{2}|1,0\rangle_2\,,\quad (E_{-\alpha_2})^2E_{-\alpha_1}|M\rangle = \frac{1}{2\sqrt{2}}|1,-1\rangle_2\,.$$

Similarly,

$$E_{-\alpha_2}|M\rangle = \frac{1}{\sqrt{2}}|1,1\rangle_1$$

The 6-th and 7-th basis states of Rep.$(1,1)$ should be:

$$E_{-\alpha_1}E_{-\alpha_2}|M\rangle = \frac{1}{2}|1,0\rangle_1\,, \quad (E_{-\alpha_1})^2 E_{-\alpha_2}|M\rangle = \frac{1}{2\sqrt{2}}|1,-1\rangle_1\,.$$

Recall that

$$E_{-\alpha_2}E_{-\alpha_1}|M\rangle = \frac{1}{2}|1,0\rangle_2$$

Remark :

The basis states $E_{-\alpha_1}E_{-\alpha_2}|M\rangle$ and $E_{-\alpha_2}E_{-\alpha_1}|M\rangle$ are linearly independent of each other, although they are not orthogonal.

Question :

Are there any other independent states in Rep.$(1,1)$ *?*

To answer this question, we reexamine the 7-th basis state

$$(E_{-\alpha_1})^2 E_{-\alpha_2} |M\rangle = \frac{1}{2\sqrt{2}} |1, -1\rangle_1$$

in view of $su(2)_2$.
Since $E_{-\alpha_1} |M\rangle \approx |1/2, -1/2\rangle_1$, we have $(E_{-\alpha_1})^2 |M\rangle = 0$.
Consequently,

$$\begin{aligned} E_{\alpha_2}(E_{-\alpha_1})^2 E_{-\alpha_2} |M\rangle &= (E_{-\alpha_1})^2 \big[E_{\alpha_2},\ E_{-\alpha_2}\big] |M\rangle \\ &= (\vec{\alpha_2} \cdot \vec{M})(E_{-\alpha_1})^2 |M\rangle \\ &= 0 \end{aligned}$$

and

$$2\vec{\alpha_2} \cdot (\vec{M} - 2\vec{\alpha_1} - \vec{\alpha_2})/\alpha_2^2 = 1 + 2 - 2 = 1.$$

This implies that

$$(E_{-\alpha_1})^2 E_{-\alpha_2} |M\rangle = \frac{1}{2\sqrt{2}} |1/2, 1/2\rangle_2$$

Followed which is the 8-th basis state in Rep.$(1,1)$,

$$E_{-\alpha_2}(E_{-\alpha_1})^2 E_{-\alpha_2}\,|M\rangle = \frac{1}{4}\,|1/2,-1/2\rangle_2$$

The procedure ends here[1].

Conclusion :

Rep.$(1,1)$ of $su(3)$ is 8-dimensional (i.e., adjoint), $\mathbf{8}$. It is spanned by the following independent basis states:

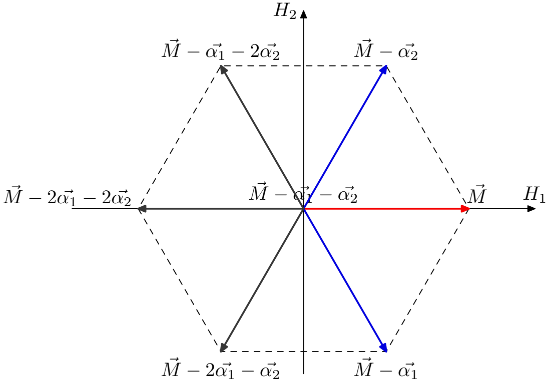
$$\begin{array}{ll}
|M\rangle\,, & E_{-\alpha_1}E_{-\alpha_2}\,|M\rangle\,, \\
(E_{-\alpha_1})^2 E_{-\alpha_2}\,|M\rangle\,, & E_{-\alpha_1}\,|M\rangle\,, \\
E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle\,, & (E_{-\alpha_2})^2 E_{-\alpha_1}\,|M\rangle \\
E_{-\alpha_2}\,|M\rangle & E_{-\alpha_2}(E_{-\alpha_1})^2 E_{-\alpha_2}\,|M\rangle
\end{array}$$

[1] Because the 8-th state and $E_{-\alpha_1}(E_{-\alpha_2})^2 E_{-\alpha_1}\,|M\rangle$ are linearly dependent.

The corresponding weight vectors read,

$$\begin{aligned}
&\vec{M} = (1,0), && \vec{M}-\vec{\alpha}_1 = (1/2,-\sqrt{3}/2),\\
&\vec{M}-\vec{\alpha}_2 = (1/2,\sqrt{3}/2), && \vec{M}-2\vec{\alpha}_1-\vec{\alpha}_2 = (-1/2,-\sqrt{3}/2\\
&\vec{M}-\vec{\alpha}_1-\vec{\alpha}_2 = (0,0), && \text{(Degenerate)}\\
&\vec{M}-\vec{\alpha}_1-2\vec{\alpha}_2 = (-1/2,\sqrt{3}/2), && \vec{M}-2\vec{\alpha}_1-2\vec{\alpha}_2 = (-1,0).
\end{aligned}$$

Rep.$(1,1)$ of $su(3)$ is real. Its weight diagram is:



Appendix :

Now we examine the linear dependence between the basis states of Rep.$(1,1)$ of $su(3)$.

Theorem :

Two states $|A\rangle$ and $|B\rangle$ are linearly dependent iff

$$\langle A|B\rangle\langle B|A\rangle = \langle A|A\rangle\langle B|B\rangle .$$

Proof:
Consider the linear equation,

$$c_1\,|A\rangle + c_2\,|B\rangle = 0$$

The coefficients c_1 and c_2 can be viewed as the unknown quantities of

$$\begin{aligned} \langle A|A\rangle c_1 + \langle A|B\rangle c_2 &= 0, \\ \langle B|A\rangle c_1 + \langle B|B\rangle c_2 &= 0. \end{aligned}$$

Having non-zero c_1 and c_2 requires,

$$\begin{vmatrix} \langle A|A\rangle & \langle A|B\rangle \\ \langle B|A\rangle & \langle B|B\rangle \end{vmatrix} = 0. \qquad \text{(QED)}$$

Firstly, we examine the linear dependence of states $|A\rangle = E_{-\alpha_1} E_{-\alpha_2} |M\rangle$ and $|B\rangle = E_{-\alpha_2} E_{-\alpha_1} |M\rangle$.

Because

$$\begin{aligned}
\langle A|A\rangle &= \langle M| E_{\alpha_2} E_{\alpha_1} E_{-\alpha_1} E_{-\alpha_2} |M\rangle \\
&= \big(\vec{\alpha_1} \cdot (\vec{M} - \vec{\alpha_2})\big)(\vec{\alpha_2} \cdot \vec{M}) = (1/2 + 1/2)1/2 = 1/2 \\
\langle B|B\rangle &= 1/2 \\
\langle A|B\rangle &= \langle M| E_{\alpha_2} E_{\alpha_1} E_{-\alpha_2} E_{-\alpha_1} |M\rangle \\
&= (\vec{\alpha_1} \cdot \vec{M})(\vec{\alpha_2} \cdot \vec{M}) = (1/2) \cdot (1/2) = 1/4 \\
\langle B|A\rangle &= 1/4
\end{aligned}$$

we see,

$$\begin{vmatrix} \langle A|A\rangle & \langle A|B\rangle \\ \langle B|A\rangle & \langle B|B\rangle \end{vmatrix} = (1/2)^2 - (1/4)^2 = \frac{3}{16} \neq 0.$$

Hence, these two states are linearly independent.

Secondly, we examine the linearly dependence of states

$$|\xi\rangle = E_{-\alpha_1} (E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle, \qquad |\eta\rangle = E_{-\alpha_2} (E_{-\alpha_1})^2 E_{-\alpha_2} |M\rangle.$$

The norm of $|\xi\rangle$ is calculated below,

$$\begin{aligned}
\langle \xi|\xi\rangle &= \langle M| E_{\alpha_1} (E_{\alpha_2})^2 E_{\alpha_1} E_{-\alpha_1} (E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle \\
&= \langle M| E_{\alpha_1} (E_{\alpha_2})^2 (\vec{\alpha_1} \cdot \vec{H} + E_{-\alpha_1} E_{\alpha_1})(E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle \\
&= \left[\vec{\alpha_1} \cdot (\vec{M} - \vec{\alpha_1} - 2\vec{\alpha_2})\right] \langle M| E_{\alpha_1} (E_{\alpha_2})^2 (E_{-\alpha_2})^2 E_{-\alpha_1} |M\rangle
\end{aligned}$$

where,

$$\begin{aligned}
\text{Term 2} &= (\vec{\alpha_1}\cdot\vec{M})^2 \langle M|\,(E_{\alpha_2})^2(E_{-\alpha_2})^2\,|M\rangle \\
&= (\vec{\alpha_1}\cdot\vec{M})^2 \langle M|\,E_{\alpha_2}(\vec{\alpha_2}\cdot\vec{H}+E_{-\alpha_2}E_{\alpha_2})E_{-\alpha_2}\,|M\rangle \\
&= (\vec{\alpha_1}\cdot\vec{M})^2(\vec{\alpha_2}\cdot\vec{M})\big[\vec{\alpha_2}\cdot(\vec{M}-\vec{\alpha_2})+\vec{\alpha_2}\cdot\vec{M}\big] \\
&= (1/2)^2(1/2)(1/2-1+1/2) \\
&= 0.
\end{aligned}$$

Rep.$(1,1)=\mathbf{8}$ is the adjoint representation of $su(3)$. Its highest weight vector is nothing but the positive root vector of the highest rank,

$$\vec{M}=\vec{\alpha_1}+\vec{\alpha_2}.$$

Consequently,

$$\begin{aligned}
\langle\xi|\xi\rangle &= \big[\vec{\alpha_1}\cdot(\vec{M}-\vec{\alpha_1}-2\vec{\alpha_2})\big]\langle M|\,E_{\alpha_1}(E_{\alpha_2})^2(E_{-\alpha_2})^2E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\langle M|\,E_{\alpha_1}(E_{\alpha_2})^2(E_{-\alpha_2})^2E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\langle M|\,E_{\alpha_1}E_{\alpha_2}(\vec{\alpha_2}\cdot\vec{H}+E_{-\alpha_2}E_{\alpha_2})E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\big[\vec{\alpha_2}\cdot(\vec{M}-\vec{\alpha_1}-\vec{\alpha_2})\big]\langle M|\,E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&\quad -(\vec{\alpha_1}\cdot\vec{\alpha_2})\langle M|\,E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\langle M|\,E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\big[\vec{\alpha_2}\cdot(\vec{M}-\vec{\alpha_1})\big]\langle M|\,E_{\alpha_1}E_{\alpha_2}E_{-\alpha_2}E_{-\alpha_1}\,|M\rangle \\
&= -(\vec{\alpha_1}\cdot\vec{\alpha_2})\big[\vec{\alpha_2}\cdot(\vec{M}-\vec{\alpha_1})\big]^2(\vec{\alpha_1}\cdot\vec{M}) \\
&= (1/2)(1/2+1/2)^2(1/2)
\end{aligned}$$

i.e. $\langle \xi | \xi \rangle = 1/4$. Similar calculations yield,

$$\langle \xi | \eta \rangle = \langle \eta | \xi \rangle = \langle \eta | \eta \rangle = 1/4$$

Therefore,

$$\begin{vmatrix} \langle \xi | \xi \rangle & \langle \xi | \eta \rangle \\ \langle \eta | \xi \rangle & \langle \eta | \eta \rangle \end{vmatrix} = (1/4)^2 - (1/4)^2 = 0$$

The involved two states $|\xi\rangle$ and $|\eta\rangle$ are linearly dependent.

Rep.$(2,0)$ of $su(3)$:

Rep.$(2,0)$ of $su(3)$ is defined by the highest weight vector

$$\vec{M} = 2\vec{M}_1 = \left[1, \frac{1}{\sqrt{3}}\right]$$

that obeys the master formulae $2\vec{M} \cdot \vec{\alpha_1}/\alpha_1^2 = 2$ and $2\vec{M} \cdot \vec{\alpha_2}/\alpha_2^2 = 0$.

- In Rep.$(2,0)$, the highest weight state $|M\rangle$ satisfies,

  $$E_{\alpha_1}|M\rangle = E_{\alpha_2}|M\rangle = 0.$$

  As a product of the Master formula $2\vec{M} \cdot \vec{\alpha_2}/\alpha_2^2 = 0$, it also satisfies,

  $$E_{-\alpha_2}|M\rangle = 0.$$

In view of the accessory $su(2)_1$ related to the simple root $\vec{\alpha_1}$, $|M\rangle$ can be formulated as,

$$|M\rangle = |1,1\rangle_1$$

Then two other basis states of Rep.$(2,0)$ follow,

$$E_{-\alpha_1}\left|M\right\rangle = \left|1,0\right\rangle_1 , \qquad \left(E_{-\alpha_1}\right)^2\left|M\right\rangle = \left|1,-1\right\rangle_1 .$$

- Relying on the facts

$$E_{\alpha_2}E_{-\alpha_1}\left|M\right\rangle = 0, \qquad \frac{2(\vec{M}-\vec{\alpha_1})\cdot\vec{\alpha_2}}{\alpha_2^2} = 1,$$

  the second basis state $E_{-\alpha_1}\left|M\right\rangle$ can alternatively be regarded as the highest weight state

$$E_{-\alpha_1}\left|M\right\rangle = \left|1/2,1/2\right\rangle_2$$

  in the spin-$1/2$ representation of $su(2)_2$.

This observation leads to the 4-th basis state of Rep.$(2,0)$,

$$E_{-\alpha_2}E_{-\alpha_1}\left|M\right\rangle = \frac{1}{\sqrt{2}}\left|1/2,-1/2\right\rangle_2$$

- Notice that

$$E_{\alpha_2}(E_{-\alpha_1})^2 \left|M\right\rangle = 0, \qquad \frac{2(\vec{M} - 2\vec{\alpha_1}) \cdot \vec{\alpha_2}}{\alpha_2^2} = 2,$$

  the third basis state $(E_{-\alpha_1})^2 \left|M\right\rangle$ can alternatively be viewed as the highest weight state

$$(E_{-\alpha_1})^2 \left|M\right\rangle = \left|1, 1\right\rangle_2$$

  in the spin-1 representation of $su(2)_2$.

As a result of $su(2)_2$, the 5-th and 6-th basis states of Rep.$(2, 0)$ emerge. They are

$$E_{-\alpha_2}(E_{-\alpha_1})^2 \left|M\right\rangle = \left|1, 0\right\rangle_2$$

and

$$(E_{-\alpha_2})^2(E_{-\alpha_1})^2 \left|M\right\rangle = \left|1, -1\right\rangle_2$$

respectively.

Question:

Does Rep.$(2, 0)$ contain any more basis states ?

Let us examine the 4-th basis state $E_{-\alpha_2}E_{-\alpha_1}|M\rangle$.

Obviously,

$$E_{\alpha_1}\left\{E_{-\alpha_2}E_{-\alpha_1}|M\rangle\right\} = (\vec{\alpha_1}\cdot\vec{M})E_{-\alpha_2}|M\rangle = 0,$$

$$\frac{2}{\alpha_1^2}\left[(\vec{M}-\vec{\alpha_1}-\vec{\alpha_2})\cdot\vec{\alpha_1}\right] = 2-2+1 = 1.$$

This suggests that $E_{-\alpha_2}E_{-\alpha_1}|M\rangle$ forms the highest weight state

$$E_{-\alpha_2}E_{-\alpha_1}|M\rangle = \frac{1}{\sqrt{2}}|1/2,1/2\rangle_1$$

of the spin-$1/2$ representation of $su(2)_1$.

Therefore, Rep.$(2,0)$ does probably have the 7-th basis state as follows:

$$E_{-\alpha_1}E_{-\alpha_2}E_{-\alpha_1}|M\rangle = \frac{1}{2}|1/2,-1/2\rangle_1 .$$

However[2], $E_{-\alpha_1} E_{-\alpha_2} E_{-\alpha_1} |M\rangle$ and $E_{-\alpha_2} (E_{-\alpha_1})^2 |M\rangle$, the 5-th basis state in Rep.$(2,0)$ are not only of the same weight, but linearly dependent also.

Conclusion :

Rep.$(2,0)$ of $su(3)$ is a 6-dimensional irreducible representation,

$$\text{Rep.}(2,0) = \mathbf{6}$$

Its 6 independent basis states read,

$$\begin{aligned}
&|M\rangle\,, \\
&E_{-\alpha_2} E_{-\alpha_1} |M\rangle\,, \\
&E_{-\alpha_1} |M\rangle\,, \\
&E_{-\alpha_2} (E_{-\alpha_1})^2 |M\rangle\,, \\
&(E_{-\alpha_1})^2 |M\rangle\,, \\
&(E_{-\alpha_2})^2 (E_{-\alpha_1})^2 |M\rangle\,.
\end{aligned}$$

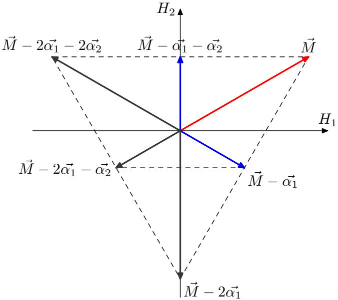
[2]Please check this claim yourself.

The weight vectors of Rep.(2, 0) are as follows:

$$\vec{M} = (1, 1/\sqrt{3}), \qquad \left\{ \text{Highest} \right\}$$

$$\begin{aligned}
\vec{M} - \vec{\alpha_1} - \vec{\alpha_2} &= (0, 1/\sqrt{3}), \\
\vec{M} - \vec{\alpha_1} &= (1/2, -1/2\sqrt{3}), \\
\vec{M} - 2\vec{\alpha_1} - \vec{\alpha_2} &= (-1/2, -1/2\sqrt{3}), \\
\vec{M} - 2\vec{\alpha_1} &= (0, -2/\sqrt{3}), \\
\vec{M} - 2\vec{\alpha_1} - 2\vec{\alpha_2} &= (-1, 1/\sqrt{3}).
\end{aligned}$$

Its weight diagram is



Homework :

1. Consider the following matrices defined in the 6-dimensional tensor product space of the Gell-Mann matrices λ_a and the Pauli matrices σ_i,

$$\frac{1}{2}\lambda_a\sigma_2, \qquad \text{for } a = 1, 3, 4, 6 \text{ and } 8;$$

$$\frac{1}{2}\lambda_a, \qquad \text{for } a = 2, 5, 7 \text{ and } 7.$$

   Show that these matrices generate a reducible representation of $su(3)$ and reduce it.

2. Decompose the tensor product of $\mathbf{3} \times \mathbf{3}$, using the highest weight techniques.

现代数学物理方法

第四章, $SU(N)$

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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Tensor methods :

Lower & upper indices:

- We begin with relabeling the basis states of $su(3)$ fundamental representation $(1,0) = \mathbf{3}$,

$$\left.\begin{array}{rcccl} |M_1\rangle &=& |1/2,\ 1/2\sqrt{3}\rangle &=& |_1\rangle \\ E_{-\alpha_1}|M_1\rangle &=& |0,\ -1/\sqrt{3}\rangle &=& |_3\rangle \\ \sqrt{2}E_{-\alpha_2}E_{-\alpha_1}|M_1\rangle &=& |-1/2,\ 1/2\sqrt{3}\rangle &=& |_2\rangle \end{array}\right\}$$

- The basis states of another $su(3)$ fundamental representation $(0,1) = \bar{\mathbf{3}}$ are re-labelled as:

$$\left.\begin{array}{rcccl} |M_2\rangle &=& |1/2,\ -1/2\sqrt{3}\rangle &=& |^2\rangle \\ E_{-\alpha_2}|M_2\rangle &=& |0,\ 1/\sqrt{3}\rangle &=& |^3\rangle \\ \sqrt{2}E_{-\alpha_1}E_{-\alpha_2}|M_2\rangle &=& |-1/2,\ -1/2\sqrt{3}\rangle &=& |^1\rangle \end{array}\right\}$$

In Rep. $\mathbf{3}$, the matrices of $SU(3)$ generators X_a are expressed as

$$(X_a)^i{}_j$$

so that :

$$X_a \left|_j\right\rangle = \left|_i\right\rangle \, (X_a)^i{}_j$$

Because the Rep. $\bar{\mathbf{3}}$ is the complex conjugate of Rep. $\mathbf{3}$, with generators $-X_a^*$, i.e.,

$$-(X_a^*)_j{}^i = -(X_a^T)_j{}^i = -(X_a)^i{}_j$$

Then,

$$\begin{aligned} X_a \left|^i\right\rangle &= \left|^j\right\rangle \left(-X_a^*\right)_j{}^i \\ &= -\left|^j\right\rangle (X_a)^i{}_j \end{aligned}$$

Now, we can define the tensor product representation of $su(3)$.

A typical tensor product representation of $su(3)$ is:

$$\underbrace{\mathbf{3}\times\mathbf{3}\times\cdots\times\mathbf{3}}_{n}\times\underbrace{\bar{\mathbf{3}}\times\bar{\mathbf{3}}\times\cdots\times\bar{\mathbf{3}}}_{m}$$

The basis states of tensor product representation are:

$$\left|\begin{matrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{matrix}\right\rangle = \left|{}^{i_1}\right\rangle \left|{}^{i_2}\right\rangle \cdots \left|{}^{i_m}\right\rangle \; \left|{}_{j_1}\right\rangle \left|{}_{j_2}\right\rangle \cdots \left|{}_{j_n}\right\rangle$$

Recalling

$$X_a^{D_1\times D_2} = X_a^{D_1}\times 1 + 1\times X_a^{D_2}$$

under the generator action, these basis states transform as follows:

$$\begin{aligned} X_a \left|\begin{matrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{matrix}\right\rangle &= \sum_{l=1}^{n} \left|\begin{matrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_{l-1} k j_{l+1} \cdots j_n \end{matrix}\right\rangle (X_a)^{k}{}_{j_l} \\ &\quad - \sum_{l=1}^{m} \left|\begin{matrix} i_1 i_2 \cdots i_{l-1} k i_{l+1} \cdots i_m \\ j_1 j_2 \cdots j_n \end{matrix}\right\rangle (X_a)^{i_l}{}_{k} \end{aligned}$$

An arbitrary state in this tensor product space is,

$$|v\rangle = \left|{}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n}\right\rangle \; v^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m}$$

Discussions :

- $v = \left(v^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m}\right)$ is called a $SU(3)$ tensor.
- In analogy with the concept of *wave function* in QM, we can express the tensor's components as:

$$v^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = \left\langle {}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} \middle| v \right\rangle$$

- We can think of the action of the generator X_a on state $|v\rangle$ as an effective action of X_a on the tensor components:

$$X_a\,|v\rangle = |X_a v\rangle$$

Consequently,

$$
\begin{aligned}
(X_a v)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} &= \left\langle {}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} \middle| X_a v \right\rangle = \left\langle {}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} \middle| X_a \middle| v \right\rangle \\
&= \left\langle {}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} \middle| X_a \middle| {}^{k_1 k_2 \cdots k_m}_{l_1 l_2 \cdots l_n} \right\rangle v^{l_1 l_2 \cdots l_n}_{k_1 k_2 \cdots k_m} \\
&= \sum_{q=1}^{n} \left\langle {}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} \middle| {}^{k_1 k_2 \cdots k_m}_{l_1 \cdots l_{q-1} p l_{q+1} \cdots l_n} \right\rangle (X_a)^{p}{}_{l_q} v^{l_1 l_2 \cdots l_n}_{k_1 k_2 \cdots k_m} \\
&\quad - \sum_{q=1}^{m} \left\langle {}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} \middle| {}^{k_1 \cdots k_{q-1} p k_{q+1} \cdots k_m}_{l_1 l_2 \cdots l_n} \right\rangle (X_a)^{k_q}{}_{p} v^{l_1 l_2 \cdots l_n}_{k_1 k_2 \cdots k_m} \\
&= \sum_{q=1}^{n} (X_a)^{p}{}_{l_q} \; v^{j_1 \cdots j_{q-1} l_q j_{q+1} \cdots j_n}_{i_1 i_2 \cdots i_m} \delta^{j_q}_{p} \\
&\quad - \sum_{q=1}^{m} (X_a)^{k_q}{}_{p} \; v^{j_1 j_2 \cdots j_n}_{i_1 \cdots i_{q-1} k_q i_{q+1} \cdots i_m} \delta^{p}_{i_q}
\end{aligned}
$$

The action of the $SU(3)$ generators on an arbitrary tensor reads,

$$
(X_a v)^{j_1 \cdots j_n}_{i_1 \cdots i_m} = \sum_{l=1}^{n} (X_a)^{j_l}{}_{k} v^{j_1 \cdots j_{l-1} k j_{l+1} \cdots j_n}_{i_1 i_2 \cdots i_m} - \sum_{l=1}^{m} (X_a)^{k}{}_{i_l} v^{j_1 j_2 \cdots j_n}_{i_1 \cdots i_{l-1} k i_{l+1} \cdots i_m}
$$

Invariant tensors :

An invariant tensor of $SU(3)$ is referred to one that does not change under any $SU(3)$ transformations.

$SU(3)$ invariant tensors :

For $SU(3)$, three invariant tensors exist,

1. δ^i_j
2. ϵ_{ijk}
3. ϵ^{ijk}

Proof :

The invariance of δ^i_j is obvious,

$$\begin{aligned} \left(X_a\delta\right)^i{}_j &= \left(X_a\right)^i{}_k\delta^k_j - \left(X_a\right)^k{}_j\delta^i_k \\ &= \left(X_a\right)^i{}_j - \left(X_a\right)^i{}_j \\ &= 0 \end{aligned}$$

Next we consider the invariance of ϵ^{ijk} and ϵ_{ijk}. e.g.,

$$(X_a\epsilon)^{ijk} = (X_a)^i{}_l\epsilon^{ljk} + (X_a)^j{}_l\epsilon^{ilk} + (X_a)^k{}_l\epsilon^{ijl}$$

By definition,

$$\epsilon^{ijk} = \epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123) \\ 0 & \text{other cases} \end{cases}$$

Hence,

$$\begin{aligned}
(X_a\epsilon)^{123} &= (X_a)^1{}_i\epsilon^{i23} + (X_a)^2{}_j\epsilon^{1j3} + (X_a)^3{}_k\epsilon^{12k} \\
&= (X_a)^1{}_1 + (X_a)^2{}_2 + (X_a)^3{}_3 \\
&= \mathrm{Tr}(X_a) \;= 0 \\
(X_a\epsilon)^{112} &= (X_a)^1{}_3\epsilon^{312} + (X_a)^1{}_3\epsilon^{132} + (X_a)^2{}_k\epsilon^{11k} \\
&= (X_a)^1{}_3 - (X_a)^1{}_3 \;= 0 \\
(X_a\epsilon)^{111} &= (X_a)^1{}_i\epsilon^{i11} + (X_a)^1{}_j\epsilon^{1j1} + (X_a)^1{}_k\epsilon^{11k} \;= 0
\end{aligned}$$

Therefore, for arbitrary $i, j, k = 1, 2, 3$, we have

$$\left(X_a \epsilon\right)^{ijk} = 0$$

and similarly,

$$\left(X_a \epsilon\right)_{ijk} = 0$$

Namely, ϵ_{ijk} and ϵ^{ijk} are two *invariant tensors* of $SU(3)$.

Warning :

Though δ^i_j is a $SU(3)$ invariant, both δ^{ij} and δ_{ij} are not invariant under $SU(3)$ transformations.

Explanation :

Since,

$$\left(X_a \delta\right)^{ij} = \left(X_a\right)^i{}_k \delta^{kj} + \left(X_a\right)^j{}_k \delta^{ik}$$

we have:

$$\left(X_a \delta\right)^{11} = \left(X_a\right)^1{}_k \delta^{k1} + \left(X_a\right)^1{}_k \delta^{1k} = 2\left(X_a\right)^1{}_1 \neq 0$$

Irreducible representations and symmetry :

We now pick out the states in *tensor product representation* according to the irreducible Rep.(n, m).

The highest weight of Rep.(n, m) of $SU(3)$ reads:

$$\vec{M} = n\vec{M}_1 + m\vec{M}_2$$

where $\vec{M}_1 = \left(\frac{1}{2},\ \frac{1}{2\sqrt{3}}\right)$ and $\vec{M}_2 = \left(\frac{1}{2},\ -\frac{1}{2\sqrt{3}}\right)$. Therefore, the highest weight state of Rep. (n, m) is

$$\left|\begin{smallmatrix} 222\cdots \\ 111\cdots \end{smallmatrix}\right\rangle, \qquad \left\{\#2 = m, \quad \#1 = n\right\}$$

which corresponds to the tensor v_H below,

$$\begin{aligned} (v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} &= \left\langle \begin{smallmatrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{smallmatrix} \middle| \begin{smallmatrix} 222\cdots \\ 111\cdots \end{smallmatrix} \right\rangle \\ &= \mathcal{N}\ \delta^{j_1 1}\delta^{j_2 1} \ldots \delta^{j_n 1}\delta_{i_1 2}\delta_{i_2 2} \cdots \delta_{i_m 2} \end{aligned}$$

with $\mathcal{N}$ the normalization constant.

Discussions :

- The tensor v_H is symmetric for the exchange of any two upper indices, and also symmetric for the exchange of any two lower indices.

$$
\begin{aligned}
(v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} &= \mathcal{N}\, \delta^{j_1 1} \delta^{j_2 1} \cdots \delta^{j_n 1} \delta_{i_1 2} \delta_{i_2 2} \cdots \delta_{i_m 2} \\
&= (v_H)^{j_2 j_1 \cdots j_n}_{i_1 i_2 \cdots i_m} = (v_H)^{j_1 j_2 \cdots j_n}_{i_2 i_1 \cdots i_m}
\end{aligned}
$$

- The tensor v_H is *traceless* for one upper and one lower indices,

$$
\delta^{i_1}_{j_1} (v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = 0
$$

Both properties of v_H are preserved by $SU(3)$ transformations, under which $v_H \rightsquigarrow X_a v_H$:

$$
\left.
\begin{aligned}
&(X_a v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = (X_a v_H)^{j_2 j_1 \cdots j_n}_{i_1 i_2 \cdots i_m} = (X_a v_H)^{j_1 j_2 \cdots j_n}_{i_2 i_1 \cdots i_m}, \\
&\delta^{i_1}_{j_1} (X_a v_H)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} = 0.
\end{aligned}
\right\}
$$

Dimension of $SU(3)$ Rep.(n,m) :

In Rep.(n,m) of $SU(3)$, the tensor related to the state $\left|\begin{smallmatrix} i_1 i_2 \cdots i_m \\ j_1 j_2 \cdots j_n \end{smallmatrix}\right\rangle$ is

$$v = v^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m}$$

- v has n upper and m lower indices.
- v is separately symmetric in each type of the indices. *If there were no further constraints*, the number of independent components of v would be:

$$B(n,m) = \frac{(n+2)!}{n!2!}\frac{(m+2)!}{m!2!} = \frac{1}{4}(n+1)(n+2)(m+1)(m+2)$$

- Unfortunately, v *has to be traceless*. As a result, v has to satisfy $B(n-1,m-1)$ additional constraints such as $v^{k j_2 j_3 \cdots j_n}_{i_1 k i_3 \cdots i_m} = 0$.

The correct number of independent components of $SU(3)$ tensor in its irreducible Rep. (n, m) is then,

$$\begin{aligned} D(n,m) &= B(n,m) - B(n-1,m-1) \\ &= \tfrac{1}{4}(n+1)(m+1)\big[(n+2)(m+2) - nm\big] \\ &= \tfrac{1}{2}(n+1)(m+1)(n+m+2) \end{aligned}$$

$D(n, m)$ could also be interpreted as the dimension of the irreducible Rep.(n, m).

Examples :

$$\begin{aligned} &D(1,0) = D(0,1) = 3, \\ &D(1,1) = 8, \\ &D(2,0) = D(0,2) = 6, \\ &D(2,1) = D(1,2) = 15, \\ &D(2,2) = 27, \\ &D(3,0) = D(0,3) = 10. \end{aligned}$$

Clebsch-Gordan decomposition :

Suppose u and v are two $SU(3)$ tensors in Rep. (n, m) and Rep. (p, q), respectively,

$$u = \left(u^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m}\right), \quad v = \left(v^{a_1 a_2 \cdots a_p}_{b_1 b_2 \cdots b_q}\right)$$

The tensor product of these two tensors

$$u \otimes v = (u \otimes v)^{j_1 \cdots j_n a_1 \cdots a_p}_{i_1 \cdots i_m b_1 \cdots b_q} = \left(u^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_m} v^{a_1 a_2 \cdots a_p}_{b_1 b_2 \cdots b_q}\right)$$

yields a $SU(3)$ tensor in a *reducible* representation.

Strategy for picking out *irreducible representations* from the above reducible representation is,

- Making irreducible representations out of the product of tensors u and v;
- Expressing $u \otimes v$ as a sum of such terms that are proportional to some irreducible representations of $SU(3)$.

Consider the CG-decomposition of $\mathbf{3} \times \mathbf{3}$.

Because $\mathbf{3}$ is Rep.$(1,0)$, the tensor of $\mathbf{3}$ has the form of $u = (u^i)$. Consequently, an arbitrary $SU(3)$ tensor of $\mathbf{3} \times \mathbf{3}$ can be written as

$$(u \otimes v)^{ij} = u^i v^j, \qquad i,j = 1,\ 2,\ 3$$

We do the Clebsch-Gordan decomposition as follows:

$$u^i v^j = \frac{1}{2}(u^i v^j + u^j v^i) + \frac{1}{2}(u^i v^j - u^j v^i)$$

- The number of the independent components of symmetric combination $\frac{1}{2}(u^i v^j + u^j v^i)$ is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. This tensor belongs to the irreducible representation $\mathbf{6} = \text{Rep.}(2,0)$.
- The second term (anti-symmetric combination) can be recast as

$$\frac{1}{2}(u^i v^j - u^j v^i) = \frac{1}{2}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) u^k v^l = \frac{1}{2}\epsilon^{ijm}\epsilon_{klm} u^k v^l$$

- In view of product $u^i v^j$, ϵ^{ijm} is an invariant tensor. The remaining factor $\epsilon_{klm} u^k v^l$ forms a tensor in $\bar{\mathbf{3}} = \text{Rep.}(0,1)$ as it has only one *bare* lower index.

We conclude that

$$\mathbf{3} \times \mathbf{3} = \mathbf{6} + \bar{\mathbf{3}}$$

Alternatively but equivalently,

$$(1,0) \otimes (1,0) = (2,0) \oplus (0,1)$$

Consider the tensor product of $\mathbf{3} \times \bar{\mathbf{3}}$.

Because the tensors of $\mathbf{3}$ and $\bar{\mathbf{3}}$ are $u = (u^i)$ and $v = (v_j)$, respectively, the tensor in $\mathbf{3} \times \bar{\mathbf{3}}$ should be

$$(u \otimes v)^i_j = u^i v_j$$

The Clebsch-Gordan decomposition is,

$$u^i v_j = \left[u^i v_j - \frac{1}{3}\delta^i_j u^k v_k \right] + \frac{1}{3}\delta^i_j u^k v_k$$

As a result,

$$(1,0) \otimes (0,1) = (1,1) \oplus (0,0)$$

or

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1}$$

Consider the tensor product of $\mathbf{3} \times \mathbf{8}$.

The tensors of $\mathbf{3}$ and $\mathbf{8}$ are $\boldsymbol{u} = (u^i)$ and $\boldsymbol{v} = ({v^j}_k)$, respectively[1]. Therefore, the tensor of $\mathbf{3} \times \mathbf{8}$ has the form

$$(u \otimes v)^{ij}_k = u^i {v^j}_k$$

[1] The tensor of $\mathbf{8}$ must be traceless, i.e., ${v^j}_j = 0$.

The Clebsch-Gordan decomposition is carried out in the way,

$$
\begin{aligned}
u^i v^j{}_k &= \frac{1}{2}(u^i v^j{}_k + u^j v^i{}_k) + \frac{1}{2}(u^i v^j{}_k - u^j v^i{}_k) \\
&= \frac{1}{2}(u^i v^j{}_k + u^j v^i{}_k) + \frac{1}{2}\epsilon^{ijm}\epsilon_{mnl}u^n v^l{}_k
\end{aligned}
$$

- The first term

$$
\text{term 1} = \frac{1}{2}(u^i v^j{}_k + u^j v^i{}_k)
$$

has been symmetrized about the upper indices i and j. To make it traceless further, we recast it as

$$
\begin{aligned}
\text{term 1} &= \frac{1}{2}\left[(u^i v^j{}_k + u^j v^i{}_k) - a\delta^i_k u^l v^j{}_l - b\delta^j_k u^l v^i{}_l\right] \\
&\quad + \frac{1}{2}\left(a\delta^i_k u^l v^j{}_l + b\delta^j_k u^l v^i{}_l\right)
\end{aligned}
$$

The first row is expected to be in Rep.$(2, 1)$ but the second row in Rep.$(1, 0)$.

The traceless condition in Rep.$(2,1)$ requires,

$$u^l v^j{}_l (1 - 3a - b) = 0, \quad u^l v^i{}_l (1 - a - 3b) = 0.$$

Hence $a = b = 1/4$. We finally recast the *first* term as:

$$\text{term 1} \quad = \frac{1}{2}\left[(u^i v^j{}_k + u^j v^i{}_k) - \frac{1}{4}(\delta^i_k u^l v^j{}_l + \delta^j_k u^l v^i{}_l) \right] + \frac{1}{8}\left(\delta^i_k u^l v^j{}_l + \delta^j_k u^l v^i{}_l \right)$$

In the previous formula for decomposition of tensor product $u^i v^j{}_k$, the second term reads,

$$\text{term 2} = \frac{1}{2} \epsilon^{ijm} \epsilon_{mnl} u^n v^l{}_k$$

After discarding the invariant tensor ϵ^{ijm}, it has only two lower indices m and k, effectively.

- Irreducibility requires the symmetrization about these two indices. Therefore,

$$\begin{aligned}
\text{term 2} &= \frac{1}{2}\epsilon^{ijm}\left[\frac{1}{2}(\epsilon_{mnl}u^n v^l{}_k + \epsilon_{knl}u^n v^l{}_m)\right. \\
&\qquad \left. + \frac{1}{2}(\epsilon_{mnl}u^n v^l{}_k - \epsilon_{knl}u^n v^l{}_m)\right] \\
&= \frac{1}{4}\epsilon^{ijm}\left(\epsilon_{mnl}u^n v^l{}_k + \epsilon_{knl}u^n v^l{}_m\right) \\
&\qquad + \frac{1}{4}\epsilon^{ijm}\epsilon_{pnl}u^n v^l{}_q(\delta^p_m\delta^q_k - \delta^q_m\delta^p_k) \\
&= \frac{1}{4}\epsilon^{ijm}\left(\epsilon_{mnl}u^n v^l{}_k + \epsilon_{knl}u^n v^l{}_m\right) \\
&\qquad + \frac{1}{4}\epsilon^{ijm}\epsilon_{pnl}u^n v^l{}_q\epsilon_{mkr}\epsilon^{pqr}
\end{aligned}$$

On RHS, the first row stands for a symmetric tensor in Rep.$(0,2)$. Let us now focus on the second row.

$$\frac{1}{4}\epsilon^{ijm}\epsilon_{pnl}u^n v^l{}_q \epsilon_{mkr}\epsilon^{pqr} = \frac{1}{4}u^n v^l{}_q \left(\delta^i_k\delta^j_r - \delta^j_k\delta^i_r\right)\left(\delta^q_n\delta^r_l - \delta^r_n\delta^q_l\right)$$

$$= \frac{1}{4}u^n v^l{}_q \left[\delta^i_k(\delta^j_l\delta^q_n - \delta^j_n\delta^q_l) - \delta^j_k(\delta^i_l\delta^q_n - \delta^i_n\delta^q_l)\right]$$

$$= \frac{1}{4}\left[\delta^i_k(u^l v^j{}_l - u^j v^l{}_l) - \delta^j_k(u^l v^i{}_l - u^i v^l{}_l)\right]$$

$$= \frac{1}{4}\left(\delta^i_k u^l v^j{}_l - \delta^j_k u^l v^i{}_l\right)$$

which stands for the tensor of Rep.$(1,0)$.

In summary,

$$u^i v^j{}_k = \frac{1}{2}\left[(u^i v^j{}_k + u^j v^i{}_k) - \frac{1}{4}(\delta^i_k u^l v^j{}_l + \delta^j_k u^l v^i{}_l)\right]$$
$$+\frac{1}{4}\epsilon^{ijm}\left(\epsilon_{mnl}u^n v^l{}_k + \epsilon_{knl}u^n v^l{}_m\right)$$
$$+\frac{1}{8}\left(3\delta^i_k u^l v^j{}_l - \delta^j_k u^l v^i{}_l\right)$$

It implies:

$$(1,0) \otimes (1,1) = (2,1) \oplus (0,2) \oplus (1,0)$$

Equivalently,

$$\mathbf{3} \times \mathbf{8} = \mathbf{15} + \bar{\mathbf{6}} + \mathbf{3}$$

Consider the CG-decomposition of $\mathbf{6} \times \mathbf{3}$.

The tensors of $\mathbf{6}$ and $\mathbf{3}$ are $u = (u^{ij})$ and $v = (v^k)$, respectively. Consequently, the tensor of $\mathbf{6} \times \mathbf{3}$ has the form

$$\left(u \otimes v\right)^{ijk} = u^{ij} v^k$$

where u is a symmetric tensor of $SU(3)$ in Rep.$(2,0)$,

$$u^{ij} = u^{ji}$$

By symmetrizing all of the upper indices,

$$u^{ij}v^k = \frac{1}{3}\left(u^{ij}v^k + u^{jk}v^i + u^{ki}v^j\right) + \frac{1}{3}\left(2u^{ij}v^k - u^{jk}v^i - u^{ki}v^j\right)$$

The first term on RHS

$$\frac{1}{3}\left(u^{ij}v^k + u^{jk}v^i + u^{ki}v^j\right)$$

is symmetric for exchanging any two indices. It describes a tensor in irreducible Rep.$(3,0)$ of $SU(3)$.

The second term is recast as:

$$\frac{1}{3}(2u^{ij}v^k - u^{jk}v^i - u^{ki}v^j) = \frac{1}{3}\left(u^{ij}v^k - u^{jk}v^i\right) + \frac{1}{3}\left(u^{ij}v^k - u^{ki}v^j\right)$$

$$= \frac{1}{3}\left(\delta^i_m\delta^k_n - \delta^i_n\delta^k_m\right)u^{mj}v^n + \frac{1}{3}\left(\delta^j_m\delta^k_n - \delta^j_n\delta^k_m\right)u^{im}v^n$$

$$= \frac{1}{3}\Bigg[\epsilon^{ikl} \underbrace{\epsilon_{lmn}u^{mj}v^n}_{\text{traceless } \epsilon_{lmn}u^{ml}=0} + \epsilon^{jkl} \underbrace{\epsilon_{lmn}u^{im}v^n}_{\text{traceless } \epsilon_{lmn}u^{lm}=0}\Bigg]$$

Apart from the invariant tensors ϵ^{ikl} and ϵ^{jkl}, the term is involved in some traceless tensors

$$\epsilon_{lmn}u^{mj}v^n, \quad \epsilon_{lmn}u^{im}v^n$$

Hence, it describes a tensor in the $SU(3)$ irreducible Rep.$(1, 1)$.

In summary,

$$u^{ij}v^k = \frac{1}{3}\left(u^{ij}v^k + u^{jk}v^i + u^{ki}v^j\right) + \frac{1}{3}\left(\epsilon^{ikl}\epsilon_{lmn}u^{mj}v^n + \epsilon^{jkl}\epsilon_{lmn}u^{im}v^n\right)$$

It implies that,

$$(2,0) \otimes (1,0) = (3,0) \oplus (1,1)$$

Equivalently,

$$\mathbf{6} \times \mathbf{3} = \mathbf{10} + \mathbf{8}$$

Corollary :

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = (\mathbf{6} + \bar{\mathbf{3}}) \times \mathbf{3} = \mathbf{10} + \mathbf{8} + \mathbf{8} + \mathbf{1}$$

Equivalently,

$$(1,0) \otimes (1,0) \otimes (1,0) = (3,0) \oplus (1,1) \oplus (1,1) \oplus (0,0)$$

Problems :

1. Decompose the product of tensor components $u^i v^{jk}$, where $v^{jk} = v^{kj}$ transforms like a tensor in Rep.**6** of $SU(3)$.
2. Find the matrix elements $\langle u|X_a|v\rangle$, where X_a stand for the $SU(3)$ generators and $|u\rangle$ and $|v\rangle$ are states in the adjoint representation of $SU(3)$ with tensor components u^i_j and v^i_j. Write the result in terms of the tensor components and the Gell-Mann Matrices.
3. In Rep. **6** of $SU(3)$, for each weight find the corresponding tensor component v^{ij}.

Young tableaux in $SU(3)$:

Young tableaux is very convenient in dealing with the Clebsch-Gordan decomposition of the Lie group representations. Here we consider its application in $SU(3)$.

A *crucial* observation:

The representation $\bar{\mathbf{3}}$ of $SU(3)$ is the antisymmetric product of two $\mathbf{3}$'s,

$$w_i = \epsilon_{ijk} u^j v^k$$

An irreducible $SU(3)$ tensor $\mathscr{A}$ in Rep.(n, m) has the component structure

$$\mathscr{A}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m}$$

1. $\mathscr{A}$ is symmetric in upper and lower indices, separately.
2. $\mathscr{A}$ is traceless for one upper and one lower indices.

We can raise all the lower tensor indices by using the invariant tensor ϵ^{ijk} of $SU(3)$,

$$\epsilon^{j_1 k_1 l_1} \epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \mathscr{A}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m} = \mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n}$$

- $\mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n}$ is antisymmetric in each pair $\{k_a, l_a\}$ for interchange

$$k_a \leftrightsquigarrow l_a, \quad (a = 1,\ 2, \cdots, m)$$

and symmetric for exchange of pairs

$$\Big\{k_a, l_a\Big\} \leftrightsquigarrow \Big\{k_b, l_b\Big\}, \quad (a,\ b = 1,\ 2, \cdots, m)$$

- Traceless condition of $\mathscr{A}$ becomes:

$$\begin{aligned} &\epsilon_{i_1 k_1 l_1} \mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \\ &\quad = \epsilon_{i_2 k_2 l_2} \mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \\ &\quad = \cdots = 0 \end{aligned}$$

The traceless condition of tensor $\mathscr{B}$ could be shown as follows:

$$
\begin{aligned}
&\epsilon_{i_1 k_1 l_1} \mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \\
&\quad = \epsilon_{i_1 k_1 l_1} \epsilon^{j_1 k_1 l_1} \; \epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \; \mathscr{A}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m} \\
&\quad = 2\delta^{j_1}_{i_1} \; \epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \; \mathscr{A}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m} \\
&\quad = 2\epsilon^{j_2 k_2 l_2} \ldots \epsilon^{j_m k_m l_m} \mathscr{A}^{i_1 i_2 \cdots i_n}_{i_1 j_2 \cdots j_m} \\
&\quad = 0
\end{aligned}
$$

With such a $SU(3)$ tensor $\mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n}$ in Rep.(n, m), we associate a Young tableau

k_1	k_2	$\cdots$	k_m	i_1	i_2	$\cdots$	i_n
l_1	l_2	$\cdots$	l_m				

The Young tableau

k_1	k_2	$\cdots$	k_m	i_1	i_2	$\cdots$	i_n
l_1	l_2	$\cdots$	l_m				

describes a tensor

$$\mathscr{B} = \left(\mathscr{B}^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n}\right)$$

with the following properties:

- It has $(n + 2m)$ upper indices.
- It is antisymmetric for index interchange in every pair $\left\{k_a, l_a\right\}$, where $a = 1, 2, \cdots, m$.
- It is symmetric under arbitrary permutations of the indices i_b and k_a, and separately symmetric under arbitrary permutations of l_a, where $a = 1, 2, \cdots, m$ and $b = 1, 2, \cdots, n$.

Question : Why ?

Because $\mathscr{A} = E_{-} v_H$[2], and the $SU(3)$ transformation preserves the permutational symmetries in tensor indices, we are necessary to analyze the claimed symmetries for tensor $\mathscr{B}_H$,

$$\begin{aligned}
&\mathscr{B}_H^{k_1 l_1 k_2 l_2 \cdots k_m l_m i_1 i_2 \cdots i_n} \\
&\quad = \epsilon^{j_1 k_1 l_1} \epsilon^{j_2 k_2 l_2} \cdots \epsilon^{j_m k_m l_m} \left(v_H\right)^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m} \\
&\quad = \mathcal{N}\ \epsilon^{j_1 k_1 l_1} \epsilon^{j_2 k_2 l_2} \cdots \epsilon^{j_m k_m l_m} \delta^{i_1 1} \delta^{i_2 1} \cdots \delta^{i_n 1} \delta_{j_1 2} \delta_{j_2 2} \cdots \delta_{j_m 2} \\
&\quad = \mathcal{N}\ \epsilon^{2 k_1 l_1} \epsilon^{2 k_2 l_2} \cdots \epsilon^{2 k_m l_m} \delta^{i_1 1} \delta^{i_2 1} \cdots \delta^{i_n 1}
\end{aligned}$$

The **independent** components of $\mathscr{B}_H$ read,

$$\mathscr{B}_H^{1313 \cdots 1311 \cdots 1} = \mathcal{N}\ \epsilon^{213} \epsilon^{213} \cdots \epsilon^{213} = \pm \mathcal{N}$$

corresponding to

$$\begin{aligned}
&k_1 = k_2 = \cdots = k_m = i_1 = i_2 = \cdots = i_n = 1 \\
&l_1 = l_2 = \cdots = l_m = 3
\end{aligned}$$

[2] E_{-} stands for some $SU(3)$ generator.

Therefore,

1. $\mathscr{B}_H$ is symmetric for interchanging the indices in the same rows of the corresponding Young tableau.
2. $\mathscr{B}_H$ is antisymmetric for exchanging the indices in the same columns of the corresponding Young tableau.
3. Young tableaux can be directly used to represent the irreducible representations of $SU(3)$.

Example 1:

Young tableau

$$\boxed{i}$$

can be used to stand for *either* a $SU(3)$ tensor u^i of irreducible representation $\mathbf{3}$ *or* $\mathbf{3}$ itself[3].

[3] For $SU(3)$, $\mathbf{3}$ is Rep.$(1, 0)$. Similarly, $\mathbf{6}$ = Rep.$(2, 0)$.

Example 2:

Young tableau

$$\begin{array}{|c|c|}\hline i & j \\ \hline \end{array}$$

describes *either* a symmetric $SU(3)$ tensor

$$u^{ij} = u^{ji}$$

in Rep.$(2, 0) = \mathbf{6}$ *or* $\mathbf{6}$ itself.

Example 3:

Young tableau

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$$

describes *either* the antisymmetric $SU(3)$ tensor

$$u^{ij} = -u^{ji} = \epsilon^{ijk} v_k$$

in Rep.$(0, 1) = \bar{\mathbf{3}}$ *or* $\bar{\mathbf{3}}$ itself.

Example 4:

Young tableau

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

describes *either* a $SU(3)$ tensor

$$u^{ijk} = u^{jik} = -u^{kji} = \epsilon^{ikl} v_l^j$$

in Rep.$(1, 1) = \mathbf{8}$ *or* $\mathbf{8}$ itself.

Example 5:

Young tableau

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array}$$

is related to the invariant $SU(3)$ tensor ϵ^{ijk}. It represents the trivial Rep.$(0, 0) = \mathbf{1}$.

Example 6:

Young tableau

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline l \\ \hline\end{array}$$

is not allowed in $SU(3)$. The antisymmetric $SU(3)$ tensor

$$u^{ijkl}, \quad \left\{ i, j, k, l = 1,\ 2,\ 3 \right\}$$

does not exist in any of its representations.

Warning :

1. In Young tableaux of $SU(3)$, any columns with 3 boxes contribute a factor proportional to ϵ^{123} and should be ignored. e.g,

a	b	c	d	e	f	g
h	i	j				
k	l					

should be reduced to

c	d	e	f	g
j				

2. The $SU(3)$ tensor which relates to a Young tableau with more than 3 boxes in any column vanishes!

Calculating $D(n,m)$ by using Young tableaux :

The irreducible Rep.(n,m) of $SU(3)$ has dimension

$$D(n,m) = \frac{1}{2}(n+1)(m+1)(n+m+2)$$

Question:

Can $D(n,m)$ be deduced from the corresponding Young tableau ?

The answer is absolutely *yes*. We draw the corresponding Young tableau

k_1	k_2	$\cdots$	k_m	i_1	i_2	$\cdots$	i_n
l_1	l_2	$\cdots$	l_m				

and represent $D(n,m)$ as a fraction:

$$D(n,m) = \frac{a(n,m)}{b(n,m)}$$

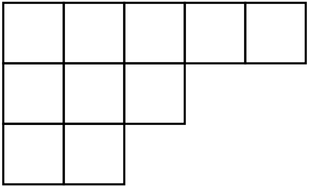
We now introduce the rules for calculating $a(n, m)$ and $b(n, m)$. To this end, we need define two concepts:

1. Content m_{ij}
2. Hook number h_{ij}

for related Young tableau. For later convenience, consider $SU(N)$ for a generic $N \geqslant 3$. The content m_{ij} for a box at the j-th column of the i-th row is,

$$m_{ij} = j - i$$

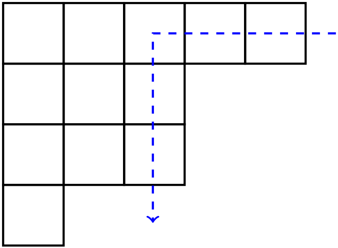
Example : For Young tableau



we have $m_{23} = 1,\ m_{14} = 3$ but $m_{32} = -1$.

To define hook number h_{ij}, we have to introduce the so-called *hook* for each box in Young tableau.

Here is the hook for box at the third column of the first row,



The hook number h_{ij} is the total number of boxes along the hook of the box at the j-th column of the i-th row in the Young tableau.

In given example, we have:

$$h_{13} = 5, \quad h_{22} = 3, \quad h_{21} = 5.$$

Dimensions of $SU(N)$ irreducible representations :

$d_{[\lambda]}(SU(N))$:

The dimension of the irreducible representation of $SU(N)$ described by Young tableau $[\lambda]$ is expressed by a quotient,

$$d_{[\lambda]}(SU(N)) = \prod_{ij} \frac{N + m_{ij}}{h_{ij}}$$

- For $SU(3)$, this formula reduces to:

$$D(n,m) = \frac{a(n,m)}{b(n,m)}$$

where

$$a(n,m) = \prod_{ij}(3 + m_{ij}), \qquad b(n,m) = \prod_{ij} h_{ij}.$$

By define the so-called Numerator Young tableau:

3	4	$\cdots$	$m+2$	$m+3$	$m+4$	$\cdots$	$m+n+2$
2	3	$\cdots$	$m+1$				

we can easily get:

$$a(n,m) = \prod_{i=3}^{n+m+2} \prod_{j=2}^{m+1} ij = \frac{1}{2}(n+m+2)!(m+1)!$$

We introduce the denominator Young tableau as follows:

h_{11}	h_{12}	$\cdots$	h_{1m}	n	$n-1$	$\cdots$	1
h_{21}	h_{22}	$\cdots$	h_{2m}				

where $h_{11} = n+m+1$, $h_{12} = n+m$, $h_{1m} = n+2$, $h_{21} = m$, $h_{22} = m-1$ and $h_{2m} = 1$. Therefore,

$$b(n,m) = \frac{(n+m+1)!m!}{(n+1)}$$

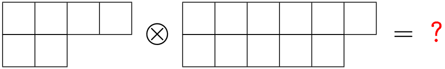
Consequently,

$$\begin{aligned} D(n,m) &= \frac{a(n,m)}{b(n,m)} \\ &= \frac{(n+m+2)!(m+1)!}{2} \cdot \frac{(n+1)}{(n+m+1)!m!} \\ &= \frac{1}{2}(n+1)(m+1)(n+m+2) \end{aligned}$$

This is what we have expected.

Clebsch-Gordan decomposition :

Let us now to discuss the Young tableau rules for decomposing the tensor product of two $SU(3)$ irreducible representations. e.g.,



CG-decomposition rules :

- Mark each box of the second empty tableau with the corresponding number of its row. e.g.,

1	1	1	1	1	1
2	2	2	2	2	

- Continue by adding all the boxes of the second tableau to the first one. These boxes may only be added to the right or the bottom of the first tableau.

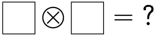
- Each resulting tableau has to be an allowed configuration, i.e., no row is longer than the row above.
- In the case of $SU(N)$, no column must contain more than N boxes.
- Within a row, the numbers in the boxes originating from the second tableau must not decrease from left to right.
- Within a column, the numbers in the boxes originating from the second tableau must increase from top to bottom.
- A box of the i-th row of the second Young tableau must not be attached to the first $(i-1)$ rows of the first Young tableau.
- If two tableaux of the same shape are produced, they are counted as different only if the labels are different.

- Reading along the rows from right to left and from the top row down to the bottom row, the number of 1s must be greater than or equal to the number of 2s.

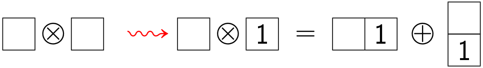
Examples :

Focus on the tensor products of some irreducible representations of $SU(3)$.

The first example is,



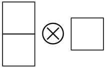
By the studied rules,



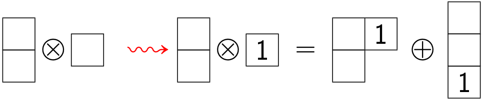
Namely,

$$\mathbf{3} \times \mathbf{3} = \mathbf{6} + \bar{\mathbf{3}}$$

Our second example is about the CG-decomposition of



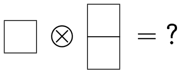
By the studied rules,



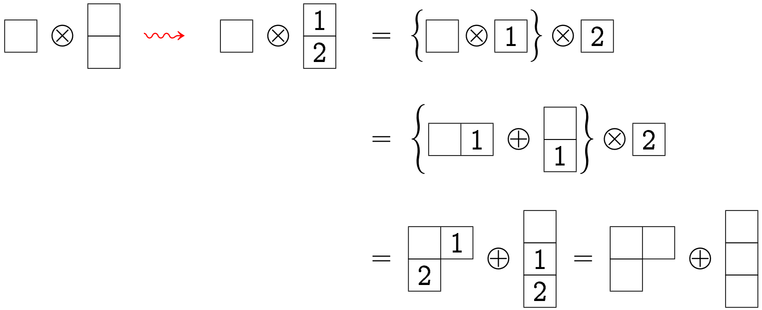
i.e.,

$$\bar{\mathbf{3}} \times \mathbf{3} = \mathbf{8} + \mathbf{1}$$

Another example is to ask



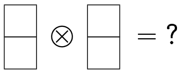
By the studied rules, we have:



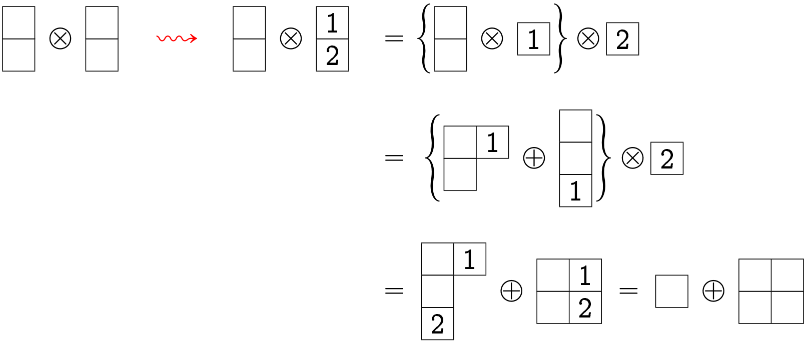
i.e.,

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1}$$

As the 4-th example in $SU(3)$, we consider



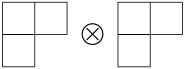
By the studied rules, we see that



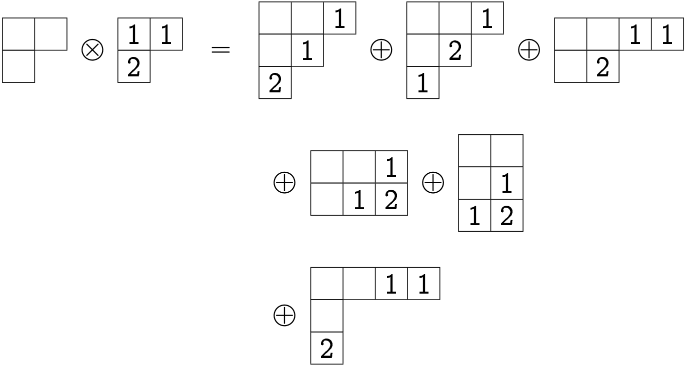
i.e.,

$$\bar{\mathbf{3}} \times \bar{\mathbf{3}} = \mathbf{3} + \bar{\mathbf{6}}$$

Finally, we consider the CG-decomposition of tensor product of



By the studied rules, we have :



i.e.,

$$\mathbf{8} \times \mathbf{8} = \mathbf{8} + \mathbf{8} + \mathbf{27} + \overline{\mathbf{10}} + \mathbf{1} + \mathbf{10}$$

Problems :

1. Find $(2,1) \otimes (2,1)$ for $SU(3)$. Can you determine which representations appear anti-symmetrically in the tensor product, and which appear symmetrically?
2. Find $\mathbf{10} \times \mathbf{8}$.
3. For any Lie group, the tensor product of the adjoint representation with any arbitrary nontrivial representation D must contain D (think about the action of the generators on the states of D and see if you can figure out why this is so.). In particular, you know that for any nontrivial $SU(3)$ representation D. How can you see this using Young tableaux?

现代数学物理方法

第三章, $SU(N)$

杨焕雄

中国科学技术大学近代物理系

hyang@ustc.edu.cn

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$SU(N)$:

Special unitary group $SU(N)$ has (N^2-1) hermitian generators T_a, $a = 1, 2, \cdots, (N^2-1)$.

In defining Rep., T_a are hermitian, traceless, $N \times N$ matrices with normalization

$$\mathrm{Tr}\left\{T_a T_b\right\} = \frac{1}{2}\delta_{ab}$$

They can be defined as a generalization of the Gell-Mann matrices:

$$[T^{(1)}_{ab}]_{ij} = \frac{1}{2}\left\{\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi}\right\}$$

$$[T^{(2)}_{ab}]_{ij} = -\frac{i}{2}\left\{\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\right\}$$

$$[T^{(3)}_{c}]_{ij} = \begin{cases} \delta_{ij}\frac{1}{\sqrt{2c(c-1)}}, & \text{if } i < c\,; \\ -\delta_{ij}\sqrt{\frac{(c-1)}{2c}}, & \text{if } i = c\,; \\ 0, & \text{if } i > c. \end{cases}$$

where $a, b = 1, 2, \cdots, N$ but $a < b$, and $c = 2, 3, \cdots, N$.

The $N-1$ generators $T_c^{(3)}$ form the Cartan subalgebra of $su(N)$. We relabel them as $H_m = T_{m+1}^{(3)}$, so $m = 1, 2, \cdots, N-1$. In defining Rep.,

$$[H_m]_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{k=1}^{m} \delta_{ik} - m\delta_{i,m+1} \right] \delta_{ij}$$

The generators of the raising and lowering operators are defined by,

$$E_{\pm\alpha_{ab}} = \frac{1}{\sqrt{2}} \left[T_{ab}^{(1)} \pm i T_{ab}^{(2)} \right]$$

so that

$$E_{\pm\alpha_{ab}}^{\dagger} = E_{\mp\alpha_{ab}}, \qquad \mathrm{Tr}\left\{ E_{\alpha_{ab}} E_{-\alpha_{cd}} \right\} = \frac{1}{2}\delta_{ac}\delta_{bd}.$$

In defining Rep.,

$$[E_{\alpha_{ab}}]_{ij} = \frac{1}{\sqrt{2}}\delta_{ai}\delta_{bj}, \qquad [E_{-\alpha_{ab}}]_{ij} = \frac{1}{\sqrt{2}}\delta_{aj}\delta_{bi}.$$

Weights of defining Rep. of $SU(N)$:

The defining Rep. of $SU(N)$ has dimension N. It can be characterized by N (dependent) weights

$$\nu^j, \qquad j = 1, 2, \cdots, N$$

Each weight ν^j is a $(N-1)$-dimensional vector in weight space, whose m-th component reads,

$$\left[\nu^j\right]_m = \left[H_m\right]_{jj} = \frac{1}{\sqrt{2m(m+1)}}\left[\sum_{k=1}^{m}\delta_{jk} - m\delta_{j,m+1}\right]$$

They satisfy,

$$\nu^i \cdot \nu^j = -\frac{1}{2N} + \frac{1}{2}\delta_{ij}$$

So the weights all have the same length, $|\nu^i|^2 = (N-1)/2N$, and the angles between any two distinct weights are equal:

$$\nu^i \cdot \nu^j = -\frac{1}{2N} \;\text{ for }\; i \neq j.$$

Proof :

For $j = 1, 2, \cdots, N$, we have

$$
\begin{aligned}
(\nu^j)^2 &= \sum_{m=1}^{N-1} [\nu^j]_m [\nu^j]_m = \sum_{m=1}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \right]^2 \\
&= \sum_{m=1}^{j-1} \frac{1}{2m(m+1)} [-m\delta_{j,m+1}]^2 \\
&\qquad + \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \right]^2 \\
&= \frac{(j-1)^2}{2j(j-1)} + \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} \\
&= \frac{(j-1)}{2j} + \frac{1}{2} \sum_{m=j}^{N-1} \left(\frac{1}{m} - \frac{1}{m+1} \right) = \frac{(j-1)}{2j} + \frac{1}{2} \left(\frac{1}{j} - \frac{1}{N} \right) \\
&= \frac{N-1}{2N}
\end{aligned}
$$

and for $i < j$,

$$\begin{aligned}
\nu^i \cdot \nu^j &= \sum_{m=1}^{N-1} [\nu^i]_m [\nu^j]_m \\
&= \sum_{m=1}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{ik} - m\delta_{i,m+1} \right] \left[\sum_{l=1}^{m} \delta_{jl} - m\delta_{j,m+1} \right] \\
&= -\frac{1}{2j} \sum_{m=1}^{j-1} \left[\sum_{k=1}^{m} \delta_{ik} - m\delta_{i,m+1} \right] \delta_{m,j-1} \\
&\qquad + \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} \left[\sum_{k=1}^{m} \delta_{ik} - m\delta_{i,m+1} \right] \\
&= -\frac{1}{2j} + \sum_{m=j}^{N-1} \frac{1}{2m(m+1)} \\
&= -\frac{1}{2j} + \frac{1}{2} \left(\frac{1}{j} - \frac{1}{N} \right) = -\frac{1}{2N}
\end{aligned}$$

Explicitly, the m-th component[1] of $su(N)$ weights in its defining representation read

$$
\begin{aligned}
[\nu^1]_m &= \frac{1}{\sqrt{2m(m+1)}} \\
[\nu^2]_m &= \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{k2} - \delta_{m,1} \right) \\
[\nu^3]_m &= \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{k3} - 2\delta_{m,2} \right) \\
\ldots \\
[\nu^j]_m &= \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{k=1}^{m} \delta_{kj} - (j-1)\delta_{m,j-1} \right] \\
\ldots \\
[\nu^N]_m &= -\sqrt{\frac{N-1}{2N}}\, \delta_{m,N-1}
\end{aligned}
$$

[1] Evidently, $1 \leqslant m \leqslant N-1$.

We see, for all possible m $(1 \leqslant m \leqslant N-1)$,

$$\begin{aligned} \sum_{j=1}^{N} \left[\nu^j\right]_m &= \frac{1}{\sqrt{2m(m+1)}} \sum_{j=1}^{N} \left[\sum_{k=1}^{m} \delta_{kj} - m\delta_{j,m+1}\right] \\ &= \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{j,k=1}^{m} \delta_{kj} - m \sum_{j=1}^{N} \delta_{j,m+1}\right] \\ &= \frac{1}{\sqrt{2m(m+1)}} [m-m] \\ &= 0 \end{aligned}$$

It turns out to be the traceless condition of the Cartan generator H_m. Namely,

$$\sum_{j=1}^{N} \nu^j = 0$$

This result is an implication of the fact that *in $(N-1)$-dimensional weight space, the maximum number of independent vectors is $N-1$.*

The $su(N)$ weights in its defining representation are listed below:

$$\nu^1 = \left[\frac{1}{2},\ \frac{1}{2\sqrt{3}},\ \cdots,\ \frac{1}{\sqrt{2m(m+1)}},\ \cdots,\ \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\nu^2 = \left[-\frac{1}{2},\ \frac{1}{2\sqrt{3}},\ \cdots,\ \frac{1}{\sqrt{2m(m+1)}},\ \cdots,\ \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\nu^3 = \left[0,\ -\frac{1}{\sqrt{3}},\ \frac{1}{2\sqrt{6}}\ \cdots,\ \frac{1}{\sqrt{2m(m+1)}},\ \cdots,\ \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\cdots$$

$$\nu^m = \left[0,\ 0,\ \cdots,\ \frac{1}{\sqrt{2m(m+1)}},\ \cdots,\ \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\nu^{m+1} = \left[0,\ 0,\ \cdots,\ -\frac{m}{\sqrt{2m(m+1)}},\ \cdots,\ \frac{1}{\sqrt{2N(N-1)}}\right]$$

$$\cdots$$

$$\nu^N = \left[0,\ 0,\ \cdots,\ 0,\ \cdots,\ -\frac{N-1}{\sqrt{2N(N-1)}}\right]$$

Discussions :

- ν^1 is the highest weight of the defining representation of $su(N)$

$$\nu^1 = \left[\frac{1}{2}, \frac{1}{2\sqrt{3}}, \cdots, \frac{1}{\sqrt{2m(m+1)}}, \cdots, \frac{1}{\sqrt{2N(N-1)}}\right]$$

  and

$$\nu^1 > \nu^2 > \nu^3 > \cdots > \nu^{N-1} > \nu^N$$

- The raising and lowering operators take us from one weight to another, so the $su(N)$ roots α_{ij} are differences of its weights, $\alpha_{ij} = \nu^i - \nu^j$ for $i \neq j$.
- The roots all have length 1.

$$\begin{aligned}(\nu^i - \nu^j)^2 &= (\nu^i)^2 + (\nu^j)^2 - 2\nu^i \cdot \nu^j \\ &= 2\left(\frac{N-1}{2N}\right) - 2\left(\frac{1}{2}\delta_{ij} - \frac{1}{2N}\right) \\ &= 1\end{aligned}$$

  The last step has used the fact $i \neq j$.

For $su(N)$, the positive roots are $\alpha_{ij} = \nu^i - \nu^j$ for $i < j$. As expected, their number is $N(N-1)/2$.

The simple roots of $su(N)$ are

$$\alpha^i = \nu^i - \nu^{i+1}, \qquad i = 1,\ 2,\ \cdots,\ N-1.$$

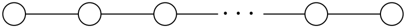
Relying on the fact,

$$\begin{aligned}
\alpha^i \cdot \alpha^j &= (\nu^i - \nu^{i+1}) \cdot (\nu^j - \nu^{j+1}) \\
&= \nu^i \cdot \nu^j + \nu^{i+1} \cdot \nu^{j+1} - \nu^i \cdot \nu^{j+1} - \nu^{i+1} \cdot \nu^j \\
&= \delta_{ij} - \frac{1}{2}\left(\delta_{i,j+1} + \delta_{i,j-1}\right)
\end{aligned}$$

$$\rightsquigarrow \qquad \theta_{i,i\pm 1} = 2\pi/3$$

the Dynkin diagram of $su(N)$ is:

$su(N)$:



Explicit forms of positive roots of $su(N)$:

For completeness, we give the explicit expressions of $su(N)$ positive roots:

$$[\alpha_{ij}]_m = \frac{1}{\sqrt{2m(m+1)}}\left[\sum_{k=1}^{m}(\delta_{ki}-\delta_{kj}) - m(\delta_{m,i-1}-\delta_{m,j-1})\right]$$

where $m, i = 1,\ 2,\ \cdots,\ N-1;\ j = 2,\ 3,\ \cdots,\ N$ and $i < j$.

Equivalently,

$$[\alpha_{ij}]_m = \begin{cases} [-m\delta_{m,i-1}]/\sqrt{2m(m+1)} & \text{if } \ m < i\ ; \\ [1+m\delta_{m,j-1}]/\sqrt{2m(m+1)} & \text{if } \ i \leqslant m < j\ ; \\ 0 & \text{if } \ m \geqslant j\ . \end{cases}$$

Exercise (optional) :

Please check

$$[H_m,\ E_{\pm\alpha_{ij}}] = \pm[\alpha_{ij}]_m E_{\pm\alpha_{ij}}$$

for $SU(N)$.

Fundamental weights of $su(N)$:

Group $SU(N)$ has $(N-1)$ inequivalent irreducible fundamental Reps. Each of them is characterized by a fundamental weight. e.g., D^j by μ^j, satisfying

$$\frac{2\alpha^i \cdot \mu^j}{(\alpha^i)^2} = \delta_{ij}$$

The $su(N)$ fundamental weights read explicitly,

$$\mu^j = \sum_{k=1}^{j} \nu^k, \qquad j = 1,\ 2,\ 3,\ \cdots,\ N-1.$$

$\mu^1 = \nu^1$ is the highest weight of D^1, the defining Rep. of $su(N)$.

1. The highest weight of any irreducible Rep. of $su(N)$ can be written as

$$\mu = \sum_{i=1}^{N-1} q_i \mu^i$$

q_is are non-negative integers, called the Dynkin coefficients.

Checking :

$$
\begin{aligned}
\frac{2\alpha^i \cdot \mu^j}{(\alpha^i)^2} &= 2(\nu^i - \nu^{i+1}) \cdot \sum_{k=1}^{j} \nu^k \\
&= 2\sum_{k=1}^{j} \left[(\nu^i \cdot \nu^k) - (\nu^{i+1} \cdot \nu^k) \right] \\
&= 2\sum_{k=1}^{j} \left[\left(-\frac{1}{2N} + \frac{1}{2}\delta_{ki} \right) + \left(\frac{1}{2N} - \frac{1}{2}\delta_{k,i+1} \right) \right] \\
&= \sum_{k=1}^{j} \left[\delta_{ki} - \delta_{k,i+1} \right] \\
&= \delta_{ij}
\end{aligned}
$$

In the last step, we have analyzed three cases of $i < j$, $i = j$ and $i > j$.

$SU(N)$ tensors :

As in $SU(3)$, we can associate $SU(N)$ states with $SU(N)$ tensors.

The basis vectors of $SU(N)$ defining Rep. are $\left|\nu^i\right\rangle$, $i = 1, 2, \cdots, N$.

$$H_m \left|\nu^i\right\rangle = [\nu^i]_m \left|\nu^i\right\rangle$$

where $m = 1, 2, \cdots, N-1$ and

$$[\nu^i]_m = \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{k=1}^{m} \delta_{ki} - m\delta_{i,m+1} \right]$$

Let us relabel the basis states $\left|\nu^i\right\rangle$ as $\left|_i\right\rangle$. An arbitrary state in $SU(N)$ defining Rep. could be

$$\left|u\right\rangle = u^i \left|_i\right\rangle$$

The *wave function* u^i is called a $SU(N)$ vector.

The arbitrary representations of $SU(N)$ could be built as the *tensor products* of the defining Reps.

Consider the antisymmetric tensor product of m defining Reps.. The basis vectors of such a tensor Rep. are

$$|_{i_1 i_2 \cdots i_m}\rangle = |_{i_1}\rangle \wedge |_{i_2}\rangle \wedge \cdots \wedge |_{i_m}\rangle$$

The general states in this Rep. are:

$$|A\rangle = A^{[i_1 i_2 \cdots i_m]}\, |_{i_1 i_2 \cdots i_m}\rangle$$

where the wave function $A^{[i_1 i_2 \cdots i_m]}$ forms a completely antisymmetric $SU(N)$ tensor.

- Because of the antisymmetry, this set of states forms an irreducible representation of $SU(N)$.
- Because of antisymmetry, no two indices among $i_1, i_2, \cdots, i_m$ can take on the same value.

Consequently, the highest weight state in such Rep. is,

$$|A_H\rangle = A_H^{12\cdots m}\,|_{12\cdots m}\rangle \propto \left[\,|\nu^1\rangle \wedge |\nu^2\rangle \wedge \cdots \wedge |\nu^m\rangle\,\right]$$

The highest weight of this tensor Rep. reads,

$$\mu_{\text{highest}} = \sum_{k=1}^{m} \nu^k$$

It turns out to be the fundamental weight μ^m if $1 \leqslant m \leqslant N-1$.

Insight:

The antisymmetric tensor products of m defining Reps. of $SU(N)$ for $1 \leqslant m \leqslant N-1$ are the fundamental representations D^m.

Question :

What is the lowest weight of Rep. D^m ?

To answer this question, we have to notice the facts that

- Rep. D^m is the antisymmetric tensor product of m Rep. D^1s.
- In defining Rep. D^1, the weight sequence is:

$$\nu^1 > \nu^2 > \cdots > \nu^N$$

Thereby, the lowest weight state $|A_L\rangle$ in Rep. D^m should be:

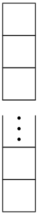
$$|A_L\rangle \propto \left[\left|\nu^{N-m+1}\right\rangle \wedge \left|\nu^{N-m+2}\right\rangle \wedge \cdots \wedge \left|\nu^N\right\rangle \right]$$

The lowest weight of this tensor Rep. reads,

$$\mu_{\text{lowest}} = \sum_{k=N-m+1}^{N} \nu^k$$

The $SU(N)$ tensor $A^{[i_1 i_2 \cdots i_m]}$ associated with the fundamental Rep. D^m could be denoted as a Young tableau with one column of m boxes:

$$A^{[i_1 i_2 \cdots i_m]} \sim$$



- We will sometimes denote the representation corresponding to a Young tableau by giving the number of boxes in each column of the tableau, a series of non-increasing integers, $[l_1, l_2, \cdots]$. In this notation, D^m is $[m]$.

- The dimension of fundamental Rep.$[m]$ of $SU(N)$ is,

$$d_{[m]} = C_N^m = \frac{N!}{m!(N-m)!}$$

where $1 \leqslant m \leqslant N-1$. As expected,

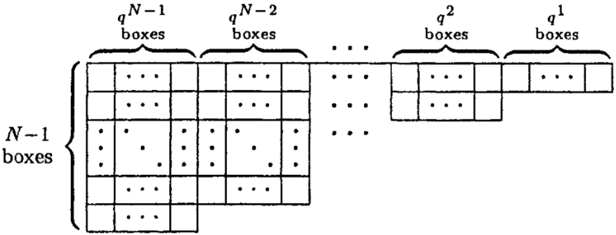
$$d_{[1]} = N$$

Now consider a general $SU(N)$ irreducible Rep. of highest weight

$$\mu = \sum_{k=1}^{N-1} q_k \mu^k$$

The Dynkin coefficients q_k are some non-negative integers.

- The tensor associated with this representation has, for each k from 1 to $N-1$, q_k sets of k indices that are antisymmetric within each set.
- The tensor can be identified to a Young tableau with q_k columns of k boxes:

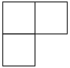


Example :

Consider the $SU(N)$ irreducible Rep. with highest weight[2]

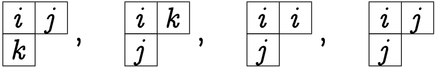
$$\mu = \mu^1 + \mu^2$$

The tensor associated with this Rep. is represented by Young tableau



so the Rep. can be denoted as $[2, 1]$.

Let us study the dimension of Rep.$[2, 1]$ now. $[2, 1]$ tensor does only allow the following independent components:



where $i,\ j,\ k = 1,\ 2,\ \cdots,\ N$ but $i < j < k$.

[2]This highest weight can alternatively be cast as: $\mu = 2\nu^1 + \nu^2$.

The number of tensor components

$$\begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|}\hline i & k \\ \hline j \\ \cline{1-1}\end{array}$$

for $i < j < k$ are clearly,

$$d_1 = 2 \cdot C_N^3 = 2 \cdot \frac{N(N-1)(N-2)}{3!} = \frac{1}{3}N(N-1)(N-2)$$

The number of tensor components

$$\begin{array}{|c|c|}\hline i & i \\ \hline j \\ \cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|}\hline i & j \\ \hline j \\ \cline{1-1}\end{array}$$

for $i < j$ are,

$$\begin{aligned} d_2 &= 2\Big[(N-1) + (N-2) + (N-3) + \cdots + 1\Big] \\ &= 2 \cdot \frac{1}{2}N(N-1) = N(N-1) \end{aligned}$$

Consequently, the dimension of $SU(N)$ Rep.$[2,1]$ is,

$$d_{[2,1]} = d_1 + d_2 = \frac{1}{3}N(N-1)(N-2) + N(N-1) = \frac{1}{3}N(N+1)(N-1)$$

If $N = 3$, $d_{[2,1]} = 8$. As is well known, $[2, 1]$ is the adjoint Rep. of $SU(3)$.

Example :

Consider the $SU(N)$ irreducible Rep. with highest weight[3]

$$\mu = 3\mu^1$$

The tensor associated with this Rep. is represented by Young tableau

so the Rep. can be denoted as $[1, 1, 1]$.

The dimension of Rep.$[1, 1, 1]$ is calculated as follows. It is known that *the independent components of a tensor correspond to the standard Young tableaux.* Consequently,

[3]This highest weight can alternatively be cast as: $\mu = 3\nu^1$.

The tensor of Rep. $[1,1,1]$ has the following independent components:

i	j	k

where $i,\ j,\ k =\ 1,\ 2,\ \cdots,\ N$ and $i \leqslant j \leqslant k$. In other words,

$$i < j+1 < k+2$$

are 3 *different* integers from the set $1, 2, \cdots, (N+2)$.

The number of independent components of $SU(N)$ tensor $[1,1,1]$ is *therefore* equal to the number of ways of selecting 3 different integers from the set $1, 2, \cdots, (N+2)$:

$$d_{[1,1,1]} = C^3_{N+2} = \frac{(N+2)!}{3!(N-1)!} = \frac{1}{6}N(N+1)(N+2)$$

If $N = 3$,

$$d_{[1,1,1]} = 10.$$

Adjoint Rep. of $SU(N)$:

By definition, the adjoint Rep. of $SU(N)$ has dimension $(N^2 - 1)$. Because $SU(N)$ is compact, its adjoint Rep. is real.

In adjoint Rep., the $SU(N)$ tensor should have one upper index and one lower index, u^i_j, satisfying the traceless condition:

$$u^i_i = 0$$

Therefore,

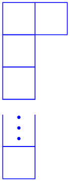
$$u^i_j \propto \epsilon_{j i_2 i_3 \cdots i_N} \left[v^i \otimes v^{i_2} \wedge v^{i_3} \wedge \cdots \wedge v^{i_N} \right]$$

where v^i is the $SU(N)$ vector in its defining Rep.[1], and

$$\epsilon_{i_1 i_2 \cdots i_N} = \begin{cases} 1 & \text{if } (i_1 i_2 \cdots i_N) \text{ is an even permutation of } (12 \cdots N); \\ -1 & \text{if } (i_1 i_2 \cdots i_N) \text{ is an odd permutation of } (12 \cdots N); \\ 0 & \text{other cases} \end{cases}$$

is an invariant tensor of $SU(N)$.

This implies that the $SU(N)$ tensor in its adjoint Rep. can be described by Young tableau[4]



The adjoint Rep. of $SU(N)$ is therefore denoted as Rep.$[N-1,1]$.

Question :

How to calculate the dimension $d_{[N-1,1]}$ of $SU(N)$ adjoint Rep. directly from the given Young tableau ?

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[4] Hence, the $SU(N)$ adjoint Rep. is not among its fundamental irreducible representations.

Factors over hooks Rule :

The dimension of an irreducible Rep. of $SU(N)$ specified by a Young tableau can simply be calculated with the **factors over hooks** rule,

$$d = \frac{F}{H}$$

1. The factors are defined as follows. Put an N in the upper left hand corner of the Young tableau. Then put factors in all the other boxes, by adding 1 each time you move to the right, and subtracting 1 each time you move down. The product of all these factors is F.

2. There is one hook for each box. Call the number of boxes the hook passes through h. The product of all these hs for all hooks is H.

Sample : Please calculate the dimension $d_{[2,1]}$ of $SU(N)$ irreducible Rep.[2, 1] by using factors over hooks rule.

Solution :

The $SU(N)$ tensor in Rep.[2, 1] corresponds to Young tableau,

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Hence[5],

$$F = \begin{array}{|c|c|}\hline x & y \\ \hline z & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = xyz = (N+1)N(N-1)$$

$$H = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = 3$$

Therefore,

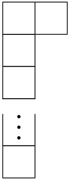
$$d_{[2,1]} = F/H = \frac{1}{3}N(N+1)(N-1)$$

[5] Here we set $x = N$, $y = N+1$ and $z = N-1$.

Sample : Please calculate the dimension $d_{[N-1,1]}$ of $SU(N)$ adjoint Rep.$[N-1,1]$ by using factors over hooks rule.

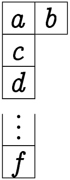
Solution :

The $SU(N)$ tensor in Rep.$[N-1,1]$ corresponds to Young tableau,



Hence, the product of factors is[6],

$F =$



$= bacd\cdots f = (N+1)!$

[6]Here we set $a = N$, $b = N+1$, $c = N-1$, $d = N-2$, $e = N-3$ and $f = 1$.

The product of hooks is[7],

$$H = \begin{array}{|c|c|} \hline a & 1 \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1} e & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} f & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = ade \cdots f = N(N-2)!$$

As expected,

$$d_{[N-1,1]} = \frac{F}{H} = \frac{(N+1)!}{N(N-2)!} = (N+1)(N-1) = N^2 - 1$$

[7] Recall that $a = N$, $b = N+1$, $c = N-1$, $d = N-2$, $e = N-3$ and $f = 1$.

Complex Reps. of $SU(N)$:

Most of the representations of $SU(N)$ are complex.

Example :
The lowest weight of the $SU(N)$ defining Rep. is ν^N. It follows from the traceless conditions of Cartan generators H_m that

$$\sum_{j=1}^{N} \nu^j = 0$$

Thus

$$\nu^N = -\sum_{j=1}^{N-1} \nu^j = -\mu^{N-1}$$

Therefore the Rep.[1] is complex. Its complex conjugate is Rep.$[N-1]$ or D^{N-1},

$$\overline{[1]} = [N-1]$$

Example :

The lowest weight of Rep.$[m]$ is the sum of the m smallest ν^is,

$$\mu_{\text{lowest}} = \sum_{j=N-m+1}^{N} \nu^j = -\sum_{j=1}^{N-m} \nu^j = -\mu^{N-m}$$

This result yields,

$$\overline{[m]} = [N-m]$$

General conclusion :

The complex conjugate of Rep. $[l_1, \cdots, l_n]$ of $SU(N)$ is,

$$\overline{[l_1, \cdots, l_n]} = [N-l_n, \cdots, N-l_1]$$

The Young tableau corresponding to a Rep. and its complex conjugate fit together into a rectangle N boxes high.

The adjoint Rep. $[N-1, 1]$ of $SU(N)$ is real,

$$\overline{[N-1, 1]} = [N-1, 1]$$

Symmetry breaking in $SU(N)$:

Symmetry breaking is a crucial concept in modern physics.

- The typical example in particle physics is the spontaneous breaking of electroweak gauge symmetries

$$SU(2) \times U(1) \; \rightarrow \; U(1)$$

- Another example is

$$SU(5) \; \rightarrow \; SU(3) \times SU(2) \times U(1)$$

in GUT, the so-called *Grand Unification Theory*. It is among the research frontiers beyond SM.

To understand the symmetry breaking mechanism better, we now study the subgroup structure of $SU(N)$.

$su(2) \times u(1) \in su(3)$:

We begin with the defining Rep.[1] of $SU(3)$.

Rep.[1] is generated by $T_a = \lambda_a/2$ $(a = 1, 2, \cdots, 8)$, with λ_a the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Generators T_a for $1 \leqslant a \leqslant 3$ could be recast as

$$T_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \qquad (a = 1,\ 2,\ 3.)$$

Since

$$[\sigma_a,\ \sigma_b] = 2i\epsilon_{abc}\sigma_c$$

these generators generate a subgroup $SU(2)$ in $SU(3)$.

Besides, we can define a so-called hypercharge Y from the generator T_8, $Y = 2T_8/\sqrt{3}$, which could generate a subgroup $U(1) \in SU(3)$. By introducing the 2×2 unit matrix, we can rewrite Y as

$$Y = \frac{1}{3}\begin{pmatrix} I & 0 \\ 0 & -2 \end{pmatrix}$$

Hence,

$$[Y,\ T_a] = 0, \quad 1 \leqslant a \leqslant 3.$$

Totally speaking, $SU(3)$ has a subgroup $SU(2) \times U(1)$.

Now we study the decomposition of a $SU(3)$ irreducible Rep. in terms of the irreducible Reps. of its subgroup $SU(2) \times U(1)$.

First consider the defining Rep.**3** of $SU(3)$. The $SU(3)$ vector in **3** is written as

$$v^{\mu}, \qquad (\mu = 1,\ 2,\ 3)$$

In terms of $SU(2) \times U(1)$,

$$v^{\mu} = \begin{cases} v^{i}, & \text{if} \quad \mu = i, \qquad Y = +1/3 \\ v^{a}, & \text{if} \quad \mu = a, \qquad Y = -2/3 \end{cases}$$

where $\mu = 1,\ 2,\ 3,\ i = 1,\ 2$ and $a = 3$.

With Young tableaux, this decomposition reads:

$$\square = \left(\square \quad \bullet\right) \oplus \left(\bullet \quad \square\right)$$

where $\bullet$ stands for the trivial tableau with no boxes. Equivalently,

$$\mathbf{3} = \mathbf{2}_{1/3} \oplus \mathbf{1}_{-2/3}$$

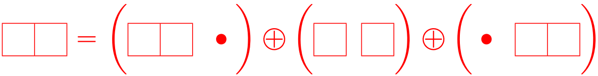
Second look at the **6**. The $SU(3)$ tensor in Rep.**6** is of rank-2

$$S^{\mu\nu}, \qquad (\mu,\ \nu = 1,\ 2,\ 3)$$

with symmetry $S^{\mu\nu} = S^{\nu\mu}$. In terms of subgroup $SU(2) \times U(1)$,

$$S^{\mu\nu} = \begin{cases} S^{ij}, & \text{if } \mu = i,\ \nu = j, & Y = +2/3 \\ S^{ib}, & \text{if } \mu = i,\ \nu = b, & Y = -1/3 \\ S^{ab}, & \text{if } \mu = a,\ \nu = b, & Y = -4/3 \end{cases}$$

where $i,\ j = 1,\ 2$ but $a,\ b = 3$.
With Young tableaux, this decomposition reads:



Equivalently,

$$\mathbf{6} = \mathbf{3}_{2/3} \oplus \mathbf{2}_{-1/3} \oplus \mathbf{1}_{-4/3}$$

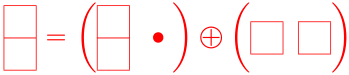
Thirdly we consider the $\bar{\mathbf{3}}$. The $SU(3)$ tensor in Rep.$\bar{\mathbf{3}}$ is of rank-2

$$A^{\mu\nu}, \qquad (\mu,\ \nu = 1,\ 2,\ 3)$$

with symmetry $A^{\mu\nu} = -A^{\nu\mu}$. In terms of subgroup $SU(2) \times U(1)$,

$$A^{\mu\nu} = \begin{cases} A^{ij}, & \text{if } \mu = i,\ \nu = j, \quad Y = +2/3 \\ A^{ib}, & \text{if } \mu = i,\ \nu = b, \quad Y = -1/3 \\ A^{ab}, & \text{if } \mu = a,\ \nu = b, \quad Y = -4/3 \end{cases}$$

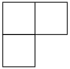
where $i,\ j = 1,\ 2$ but $a,\ b = 3$. Obviously, $A^{ab} = 0$.
With Young tableaux, this decomposition reads:



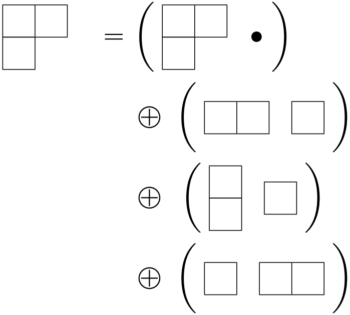
Equivalently,

$$\bar{\mathbf{3}} = \bar{\mathbf{1}}_{2/3} \oplus \mathbf{2}_{-1/3}$$

Next we consider the adjoint Rep.$\mathbf{8}$ of $SU(3)$. The $SU(3)$ tensor in $\mathbf{8}$ is represented by Young tableau



In terms of subgroup $SU(2) \times U(1)$,



$\mathbf{2}_1$

$\mathbf{3}_0$

$\mathbf{1}_0$

$\mathbf{2}_{-1}$

Namely,

$$\mathbf{8} = \mathbf{2}_1 \oplus \mathbf{3}_0 \oplus \mathbf{1}_0 \oplus \mathbf{2}_{-1}$$

Hypercharge :

Question :

How to determine the hypercharge of a tensor component in $SU(3) \rightarrow SU(2) \times U(1)$?

The $SU(3)$ tensor u in some irreducible Rep. forms the common eigenstates of $T_3 \in su(2)$ and hypercharge operator $Y \in u(1)$.

Hence,

$$Yu = yu$$

Consider a tensor u represented by a Young tableau of n boxes. We examine its components with j boxes belong to $su(2)$ and $(n-j)$ boxes belong to $u(1)$. The hypercharge of such components is:

$$y = \frac{j}{3} - \frac{2(n-j)}{3} = j - \frac{2}{3}n$$

Warning :

For $U(1) \in SU(3)$, the antisymmetric tensor such as

$$A^{ab} \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

does not exist. Because $a = b = 3$, we see that $A^{ab} = -A^{ba} = 0$.

Problems :

1. Show that the $su(N)$ algebra has an $su(N-1)$ subalgebra. How do the fundamental Rep.[1] of $SU(N)$ decompose into $SU(N-1)$ representations ?
2. Find $[3] \otimes [1]$ in $SU(5)$. Check that the dimensions work out.
3. Find $[3,1] \otimes [2,1]$ in $SU(6)$.
4. Find $[2] \otimes [1,1]$ in $SU(N)$, using the factors over hooks rule to check that the dimensions work out for arbitrary N.