第二章　随机变量及其分布

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1.

$P(X=k)=\dfrac{C_m^k C_n^{r-k}}{C_{m+n}^r}$,　$0\le k\le \min\{r,m\}$

3.

$P(\xi=k)=\dfrac{a}{a+b}\cdot\dfrac{a-1}{a+b-1}\cdot\ \cdots\ \cdot\dfrac{a-k+1}{a+b-k+1}\cdot\dfrac{b}{a+b-k}\qquad(0\le k\le a)$

6.

$P(\text{至少出现一个6点})=\dfrac{36-25}{36}=\dfrac{11}{36}$

ξ服从几何分布，$P(\xi=k)=(1-\dfrac{11}{36})^{k-1}\dfrac{11}{36}\qquad(k\ge 1)$

10.

令$\dfrac{P(X=k+1)}{P(X=k)}=\dfrac{C_n^{k+1}p^{k+1}(1-p)^{n-k-1}}{C_n^k p^k(1-p)^{n-k}}=\dfrac{n-k}{k+1}\cdot\dfrac{p}{1-p}>1$

得: $k<(n+1)p-1$

$\therefore k<(n+1)p-1$时递增，$k>(n+1)p-1$时递减

当$(n+1)p$是整数时，最大值点$k=(n+1)p-1$或$(n+1)p$

当$(n+1)p$非整数时，最大值点$k=\lfloor(n+1)p\rfloor$，$\therefore (n+1)p-1<k<(n+1)p$

11.

$P(X\ge 1)=1-p(X=0)=1-C_2^0(1-p)^2=\dfrac{5}{9}$

$\therefore p=\dfrac{1}{3}$

$P(Y\ge 1)=1-P(Y=0)=1-C_3^0(1-\dfrac{1}{3})^3=\dfrac{19}{27}$

14.
发生事故次数$X \sim B(n,p)$，其中$n=1000$，$p=0.001$
$\because n$较大，p较小，且$np=1$
$\therefore X$近似服从$Poi(1)$
$\therefore$发生事故的次数不少于2的概率为: $1-P(X=0)-P(X=1)=1-e^{-1}-e^{-1}=1-\dfrac{2}{e}$

15.
设有X个人不来，则$X \sim B(n,p)$，其中$n=52$，$p=0.05$
$\because n$较大，p较小，且$np=2.6$
$\therefore X$近似服从$Poi(2.6)$
$\therefore$每个出现的旅客都有位置的概率为:
$P(X\ge 2)=1-P(X=0)-P(X=1)=1-e^{-2.6}-e^{-2.6}\cdot 2.6$

16.
必要性: 略

充分性:
$\because$对任意非负整数m和n有:

$$P(\xi=m+n|\xi\ge n)=P(\xi=m)$$

$\therefore$令$n=1$可得$P(\xi=m+1|\xi\ge 1)=P(\xi=m)$
$\therefore \dfrac{P(\xi=m+1,\xi\ge 1)}{P(\xi\ge 1)}=P(\xi=m)$
$\therefore P(\xi=m+1)=P(\xi\ge 1)P(\xi=m)=[1-P(\xi=0)]P(\xi=m)$
令$p=P(\xi=0)$可以推出$P(\xi=k)=(1-p)^k p$，$k=0,1,2,\ldots$

注: 这里推得的分布取值从0开始，与常见的几何分布略有不同，
若将题目中的条件$\xi\ge n$改为$\xi>n$则与常见的几何分布相同。

17.
(3)
$$\int_{-1}^{1}\frac{c}{\sqrt{1-x^2}}dx=2c\int_0^1\frac{1}{\sqrt{1-x^2}}dx=2c\arcsin x\Big|_0^1=c\pi=1$$
$\therefore c=\dfrac{1}{\pi}$

(4)
$$c\int_0^{+\infty}x^2e^{-x^2/\alpha}dx\xlongequal{t=x^2/\alpha}\frac{c\alpha^{\frac{3}{2}}}{2}\int_0^{+\infty}t^{\frac{1}{2}}e^{-t}dt=\frac{c\alpha^{\frac{3}{2}}}{2}\Gamma(\frac{3}{2})=\frac{c\alpha^{\frac{3}{2}}}{4}\Gamma(\frac{1}{2})=\frac{c\alpha^{\frac{3}{2}}\sqrt{\pi}}{4}=1$$
$\therefore c=\dfrac{4}{\alpha^{\frac{3}{2}}\sqrt{\pi}}$

18.

(1) $c\int_0^2 (4x-2x^2)dx = 1 \Longrightarrow c = \frac{3}{8}$

(2) $P\Big(\frac{1}{2} < X < \frac{3}{2}\Big) = \int_{1/2}^{3/2} \frac{3}{8}(4x-2x^2)dx = \frac{11}{16}$

19.

$\because X$只在$(0,1)$中取值

$\therefore F(0)=0$，$F(1)=1$

$\because F(b)-F(a)$仅与$b-a$有关

$\therefore F(x+y)-F(x)=F(y)-F(0)$

$\therefore F(x+y)=F(x)+F(y)$ (柯西方程)，又由F的单调性 (解柯西方程的一个充分条件)

$\therefore$解为$F(x)=cx$，由$F(1)=1$可得$c=1$

$\therefore F(x)=x,\ x\in(0,1)$，即$X\sim U(0,1)$

21.

(1)

$P(\xi<2)=\phi(2)=0.97725$

$P(|\xi|\le 2)=\phi(2)-\phi(-2)=2\phi(2)-1=0.9545$

(2)

$P(|\xi-\mu|\le\sigma)=\phi(1)-\phi(-1)=2\phi(1)-1=0.6826$

$P(|\xi-\mu|\le 2\sigma)=\phi(2)-\phi(-2)=2\phi(2)-1=0.9545$

(3)

$P(2<\xi\le 5)=P\Big(\frac{2-3}{2}<\frac{\xi-3}{2}\le\frac{5-3}{2}\Big)=\phi(1)-\phi(-0.5)=\phi(1)+\phi(0.5)-1=0.5328$

$P(\xi>3)=0.5$

$P(|\xi-c|<c)=0.01 \Longleftrightarrow P(0<\xi<2c)=0.01 \Longleftrightarrow P\Big(\frac{0-3}{2}<\frac{\xi-3}{2}<\frac{2c-3}{2}\Big)=0.01$

$\therefore \phi(c-\frac{3}{2})-\phi(-\frac{3}{2})=0.01$

$\therefore \phi(c-\frac{3}{2})=0.07681 \Longrightarrow \phi(\frac{3}{2}-c)=0.92319$

查表知，表中没有与0.92319对应的值，但可以知道$\frac{3}{2}-c$应该在$1.42\sim 1.44$之间

利用线性插值法可得:

$$\frac{3/2-c-1.42}{0.92319-0.92220}=\frac{1.44-1.42}{0.92507-0.92220}$$

解得$c = 0.07310$

22.

由对称性易得$x_2 = 60$且x_1,x_3关于$x = 60$对称

又$P(\xi < x_3) = P\Big(\frac{\xi - 60}{3} < \frac{x_3 - 60}{3}\Big) = \phi(\frac{x_3 - 60}{3}) = 0.7$

解得$x_3 = 61.59$

$\therefore\ x_1 = 58.41$

27.

(1)当$y \le 0$时，$P(Y \le y) = 0$

$y > 0$时，$P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = \begin{cases} 0 & , \quad \ln y \le 0 \\ \ln y & , \quad 0 < \ln y < 1 \\ 1 & , \quad \ln y > 1 \end{cases}$

$$\therefore\ F_Y(y) = \begin{cases} 0 & , \quad y \le 1 \\ \ln y & , \quad 1 < y < e \\ 1 & , \quad y \ge e \end{cases}$$

$$\therefore\ f_Y(y) = \begin{cases} \frac{1}{y} & , \quad 1 < y < e \\ 0 & , \quad otherwise \end{cases}$$

(2) $F_Y(y) = P(Y \le y) = P(-2\ln X \le y) = P(X \ge e^{-y/2}) = \begin{cases} 1 - e^{-y/2} & , \quad 0 < e^{-y/2} < 1 \\ 0 & , \quad e^{-y/2} \ge 1 \end{cases}$

$$\therefore\ f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & , \quad y > 0 \\ 0 & , \quad y \le 0 \end{cases}$$

28.

$y \ge 1$时，$F_Y(y) = 1$

$y \le 0$时，$F_Y(y) = 0$

$0 < y < 1$时，$F_Y(y) = P(Y \le y) = P(\sin X \le y) = \int_0^{\arcsin y} \frac{2x}{\pi^2}\,dx + \int_{\pi - \arcsin y}^{\pi} \frac{2x}{\pi^2}\,dx = \frac{2\arcsin y}{\pi}$

$$\therefore\ f_Y(y) = \begin{cases} \frac{2}{\pi\sqrt{1 - y^2}} & , \quad 0 < y < 1 \\ 0 & , \quad otherwise \end{cases}$$

35.

$$f_X(x) = \int_0^{2\pi}\int_0^{2\pi}\frac{1}{8\pi^3}\Big(1-\sin x\sin y\sin z\Big)dydz = \frac{1}{2\pi}\mathrm{I}_{[0,2\pi]}(x)$$

同理$f_Y(y) = \frac{1}{2\pi}\mathrm{I}_{[0,2\pi]}(y)$，$f_Y(z) = \frac{1}{2\pi}\mathrm{I}_{[0,2\pi]}(z)$

$$f_{XY}(x,y) = \int_0^{2\pi}\frac{1}{8\pi^3}\Big(1-\sin x\sin y\sin z\Big)dz = \frac{1}{4\pi^2}\mathrm{I}_{[0,2\pi]}(x)\mathrm{I}_{[0,2\pi]}(y)$$

可得$f_{XY}(x,y) = f_X(x)f_Y(y)$，其他同理

$\therefore X, Y, Z$两两独立

但$f(x,y,z) \neq f_X(x)f_Y(y)f_Z(z)$

$\therefore X, Y, Z$不相互独立

36.

$$(1)k\int_0^{\infty}\int_0^{\infty} e^{-(3x+4y)}dxdy = 1 \Longrightarrow k = 12$$

(2)$x>0, y>0$时，$F(x,y) = 12\int_0^{y}\int_0^{x} e^{-(3x+4y)}dxdy = (1-e^{-3x})(1-e^{-4y})$

$$\therefore F(x,y) = \begin{cases} (1-e^{-3x})(1-e^{-4y}) & x>0, y>0; \\ 0 & \text{其他.} \end{cases}$$

(3)$P(0<X\leq 1, 0<Y\leq 2) = F(1,2) = (1-e^{-3})(1-e^{-8})$

37.

(1)$f_X(x) = \int_{-\infty}^{+\infty} 4xy\mathrm{I}_{(0,1)}(x)\mathrm{I}_{(0,1)}(y)dy = 2x\mathrm{I}_{(0,1)}(x)$

同理$f_Y(y) = 2y\mathrm{I}_{(0,1)}(y)$

(2)$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = 2y\mathrm{I}_{(0,1)}(y)$，与X无关(X，Y独立)

(3)

$\because (X,Y)$有密度，$\therefore P(X=Y)=0$

又$\because X, Y$独立同分布，$\therefore P(X<Y) = P(X>Y) = \frac{1}{2}$

$$P(0<X<0.5, 0.25<Y<1) = \int_0^{0.5}\int_{0.25}^{1} 4xy\, dxdy = \frac{15}{64}$$

38.

$$f_X(x) = \int_{x^2}^{1} \frac{21}{4} x^2 y \mathrm{I}_{[-1,1]}(x)\, dy = \frac{21}{8} x^2(1-x^4)\mathrm{I}_{[-1,1]}(x)$$

$$f_{Y|X}(y|x) = \frac{p(x,y)}{f_X(x)} = \frac{2y}{1-x^4}\mathrm{I}_{[x^2,1]}(y)$$

$$\therefore P(Y \ge 0.75|X=0.5) = \int_{0.75}^{1} \frac{2y}{1-0.5^4} dy = \frac{7}{15}$$

39.

令$Z = XY$，则$P(Z \le z) = P(XY \le z) = \int_1^2 P(XY \le z|X=x)dx = \int_1^2 P(Y \le \frac{z}{x}|X=x)dx$

$\because Y|X = x \sim Exp(x)$

$\therefore$当$z \le 0$时，$P(Y \le \frac{z}{x}|X=x) = 0,\ \therefore P(Z \le z) = 0$

当$z > 0$时，$P(Y \le \frac{z}{x}|X=x) = 1 - e^{-z},\quad P(Z \le z) = \int_1^2 P(Y \le \frac{z}{x}|X=x)\, dx = 1 - e^{-z}$

$\therefore XY \sim Exp(1)$

40.

(1)$P(\xi = k_1, \eta = k_2) = \dfrac{C_{13}^{k_1} C_{13}^{k_2} C_{26}^{13-k_1-k_2}}{C_{52}^{13}}$,　$k_1, k_2 = 0, 1, \dots, 13$且$0 \le k_1 + k_2 \le 13$

(2)$P(\eta = k|\xi = 1) = \dfrac{C_{13}^{k} C_{26}^{13-k}}{C_{39}^{12}}$,　$k = 0, 1, \dots, 12$

41.

(1)$P(Y = m|X = n) = C_n^m p^m (1-p)^{n-m}$

(2)$P(X = n, Y = m) = P(Y = m|X = n)P(X = n) = e^{-\lambda}\dfrac{\lambda^n}{n!} C_n^m p^m (1-p)^{n-m}$, $m \le n$

42.

$P(X_1 = 0, X_2 = 0) = P(Y \le 1) = 1 - \dfrac{1}{e}$

$P(X_1 = 0, X_2 = 1) = P(Y \le 1, Y > 2) = 0$

$P(X_1 = 1, X_2 = 0) = P(1 < Y \le 2) = \dfrac{1}{e} - \dfrac{1}{e^2}$

$P(X_1 = 1, X_2 = 1) = P(Y > 2) = \dfrac{1}{e^2}$

43.

(1)

η \ ξ	-1	0	1	
0	0.25	0	0.25	0.5
1	0	0.5	0	0.5
	0.25	0.5	0.25	

(2)

$\xi+\eta$ \ $\xi-\eta$	-2	-1	0	1
-1	0	0.25	0	0
0	0	0	0	0
1	0	0.5	0	0.25
2	0	0	0	0

(3)

$P(Z=0)=P(\xi=0,\eta=0)+P(\xi=-1,\eta=0)=0.25$

$P(Z=1)=0.75$

44.

(1)$P(\xi=1)=0.15$, $P(\xi=2)=0.23$, $P(\xi=3)=0.62$

(2)$P(\eta=1)=0.58$, $P(\eta=2)=0.33$, $P(\eta=3)=0.09$

(3)

ξ \ η	1	2	3
1	0.15	0	0
2	0.16	0.07	0
3	0.27	0.26	0.09

45.

$$P(\zeta \le z) = P(|\xi-\eta| \le z) = \begin{cases} 0 & , \quad z \le 0 \\ 1-\dfrac{(2-z)^2}{4} & , \quad 0<z<2 \\ 1 & , \quad z \ge 2 \end{cases}$$

$$\therefore f(z) = \frac{2-z}{2}\mathrm{I}_{(0,2)}(z)$$

46.

$\because U = \dfrac{X+Y}{2}$, $V = Y - X$

$\therefore X = \dfrac{2U-V}{2}$, $Y = \dfrac{2U+V}{2}$

$\therefore g_{UV}(u,v) = e^{-2u}\mathbf{I}_{(\frac{2u-v}{2}>0)}\mathbf{I}_{(\frac{2u+v}{2}>0)}$

$\therefore g_U(u) = \int_{-2u}^{2u} e^{-2u}\mathbf{I}_{(u>0)}dv = 4ue^{-2u}\mathbf{I}_{(u>0)}$

$$g_V(v) = \begin{cases} \int_{v/2}^{+\infty} e^{-2u}du & ,v>0 \\ \int_{-v/2}^{+\infty} e^{-2u}du & ,v\le 0 \end{cases} = \begin{cases} \frac{1}{2}e^{-v} & ,v>0 \\ \frac{1}{2}e^{v} & ,v\le 0 \end{cases}$$

47.

(1)不独立

(2)

令$Z = X + Y,\ W = X$，则$X = W, Y = Z - W$

$\therefore g(z,w) = \frac{1}{2}ze^{-z}|J|\mathbf{I}_{(z>0,0<w<z)}(z,w) = \frac{1}{2}ze^{-z}\mathbf{I}_{(z>0,0<w<z)}(z,w)$

$\therefore g_Z(z) = \int_0^z \frac{1}{2}ze^{-z}\mathbf{I}_{(z>0)}(z)\,dw = \frac{1}{2}z^2e^{-z}\mathbf{I}_{(z>0)}(z)$

49.

(1)

$$P(Z\le z) = P(X+Y\le z) = \begin{cases} 0 & , & z\le 0 \\ \int_0^z\int_0^{z-y} dx\,dy = \frac{1}{2}z^2 & , & 0<z\le 1 \\ \int_0^{z-1}\int_0^1 dx\,dy + \int_{z-1}^1\int_0^{z-y} dx\,dy = -\frac{1}{2}z^2+2z-1 & , & 1<z\le 2 \\ 1 & , & z>2 \end{cases}$$

$$\therefore f_Z(z) = \begin{cases} z & , & 0<z\le 1 \\ 2-z & , & 1<z\le 2 \\ 0 & , & \text{otherwise} \end{cases}$$

(2)

$$P(Z\le z) = P(X+Y\le z) = \begin{cases} 0 & , & z\le 0 \\ \int_0^z\int_0^{z-x} e^{-y}dy\,dx = z+e^{-z}-1 & , & 0<z\le 1 \\ \int_0^1\int_0^{z-x} e^{-y}dy\,dx = 1-e^{1-z}+e^{-z} & , & z>1 \end{cases}$$

$$\therefore f_Z(z) = \begin{cases} 0 & , \quad z \le 0 \\ 1 - e^{-z} & , \quad 0 < z \le 1 \\ e^{-z}(e-1) & , \quad z > 1 \end{cases}$$

50.

$Z = X - Y$, $W = Y$

$g(z, w) = f(z + w, w) = \mathbf{I}_{(\theta-1/2<z+w<\theta+1/2)}\mathbf{I}_{(\theta-1/2<w<\theta+1/2)}$

$$g_Z(z) = \int_{-\infty}^{+\infty} g(z, w)dw = \begin{cases} 1 + z & , \quad -1 < z < 0 \\ 1 - z & , \quad 0 \le z < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

53.

$(1) c\int_2^6 \int_0^5 (2x + y)dydx = 210c \Rightarrow c = \dfrac{1}{210}$

(2)

$$f_X(x) = \int_0^5 \frac{1}{210}(2x + y)\mathbf{I}_{(2,6)}(x)dy = \left(\frac{x}{21} + \frac{5}{84}\right)\mathbf{I}_{(2,6)}(x)$$

$$f_Y(y) = \int_2^6 \frac{1}{210}(2x + y)\mathbf{I}_{(0,5)}(y)dx = \left(\frac{2y}{105} + \frac{16}{105}\right)\mathbf{I}_{(0,5)}(y)$$

(3) $P(3 < X < 4, Y > 2) = \dfrac{1}{210}\displaystyle\int_3^4 \int_2^5 (2x + y)dydx = \dfrac{3}{20}$

(4) $\displaystyle\int_2^4 \int_{4-x}^5 \frac{1}{210}(2x + y)dydx + \int_4^6 \int_0^5 \frac{1}{210}(2x + y)dydx = \frac{33}{35}$

(5)$f(x, y) \ne f_X(x)f_Y(y) \Rightarrow$不独立

54.

$\because X_1 - 2X_2 \sim N(0, 20) \quad , \quad 3X_3 - 4X_4 \sim N(0, 100)$

$\therefore a = 0, b = \dfrac{1}{100}$时，$T \sim \chi_1^2$

$a = \dfrac{1}{20}, b = 0$时，$T \sim \chi_1^2$

$a = \dfrac{1}{20}, b = \dfrac{1}{100}$时，$T \sim \chi_2^2$

55.

设$X_1, X_2, \cdots, X_9 \quad i.i.d \sim N(\mu, \sigma^2)$

则$Y_1 \sim N\left(\mu, \dfrac{\sigma^2}{6}\right)$, $Y_2 \sim N\left(\mu, \dfrac{\sigma^2}{3}\right)$且$Y_1, Y_2$独立

$\therefore Y_1 - Y_2 \sim N\left(0, \dfrac{\sigma^2}{2}\right)$, $\therefore \dfrac{\sqrt{2}(Y_1 - Y_2)}{\sigma} \sim N(0, 1)$

$\because Y_1$只与$X_1, \cdots, X_6$有关，$\therefore Y_1$与S独立

又Y_2与S独立(样本均值与样本方差独立)

$\therefore \dfrac{\sqrt{2}(Y_1 - Y_2)}{\sigma}$与$S$独立

$\because \dfrac{2S^2}{\sigma^2} \sim \chi_2^2$

$$\therefore Z = \frac{\sqrt{2}(Y_1 - Y_2)}{S} = \frac{\dfrac{\sqrt{2}(Y_1 - Y_2)}{\sigma}}{\sqrt{\dfrac{S^2}{\sigma^2}}} \sim t_2$$

56.

$\dfrac{X_i}{2} \sim N(0, 1)$, $i = 1, 2, \cdots, 15$

$\therefore \dfrac{1}{4}(X_1^2 + X_2^2 + \cdots + X_{10}^2) \sim \chi_{10}^2$, $\dfrac{1}{4}(X_{11}^2 + X_{12}^2 + \cdots + X_{15}^2) \sim \chi_5^2$且两部分独立

$$\therefore \frac{\dfrac{1}{10} \cdot \dfrac{1}{4}(X_1^2 + X_2^2 + \cdots + X_{10}^2)}{\dfrac{1}{5} \cdot \dfrac{1}{4}(X_{11}^2 + X_{12}^2 + \cdots + X_{15}^2)} \sim F_{10,5}$$

即$Y \sim F_{10,5}$