Graph Theory Final Exam, 2023Fall

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Problem 1. Show that the number of the spanning trees of $K_n - e$ is $(n-2)n^{n-3}$. (20 points)

Problem 2. (a) T is a tree with m vertices. Show that every graph with minimum degree at least $m-1$ contains T as a subgraph. (10 points)

(b) T is a tree with m vertices. Show that $R(T, K_n) = (m-1)(n-1)+1$. (10 points)

Problem 3. In a bipartite graph $G = (L \cup R, E)$, we say a subset $S \subset L \cup R$ is expanding if $|N(S)| \geq |S|$. Show that if $|L| = |R| = n$ and there are two nonnegative integers p, q such that $p + q = n$ and every subset of L with size at most p as well as every subset of R with size at most q are expanding, then G has a perfect matching. (10 points)

Problem 4. (a) Show that $\chi(G) + \chi(\overline{G}) \leq |G| + 1$. (10 points)

(b) Show that $\chi(G)\chi(\overline{G}) \leq \left\lfloor \dfrac{(|G|+1)^2}{4} \right\rfloor$, and for every $n \in \mathbb{N}^+$, construct a graph G with n vertices such that $\chi(G)\chi(\overline{G}) = \left\lfloor \dfrac{(n+1)^2}{4} \right\rfloor$. (10 points)

Problem 5. (a) For a graph H and two positive integers m, n, we denote the maximum number of edges of the H-free bipartite graph with m vertices in the left side and n vertices in the right side by $\mathrm{ex}(m, n, H)$.

Show that (10 points):

$$\mathrm{ex}(m, n, K_{s,t}) \leq (t-1)^{1/s} m n^{1-1/s} + (s-1)n$$

(b) S is a set with n elements. $S_1, S_2, \ldots, S_m$ are m subsets of S with average size $\dfrac{n}{\omega}$. Show that if $m > 2k\omega$, then there exists k integers $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ such that $|S_{i_1} \cap \cdots \cap S_{i_k}| \geq \left\lfloor \dfrac{n}{(2\omega)^k} \right\rfloor$. (10 points)

Problem 6. Show that $K_{k,m}$ is k-choosable if and only if $m < k^k$. (20 points)