Exercise 3 for 2022∼ 2023 USTC Course

'Introduction to Quantum Information'

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1. Please describe the EPR paradox introduced by Einstein, Podolsky, Rosen at 1935, and explain the contradiction between quantum theory and local realism theory.

   Answer: Assumption by local realism theory:

   (a). Locality: If two measurements are performed in space-like separated locations, their outcomes should not be causal correlated.

   (b). Realism: Every element of the physical reality must have a counter part in the physical theory.

   Contraction: In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality.

   Consider that Alice and Bob share a singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$, once Alice obtains a measurement outcome by measuring particle along arbitrary direction, she could correctly predict the corresponding observable value for Bob's particle, and all observables can be predicted, they should have definite values. Following the realism assumption, every observable corresponding Bob's particle, such as $\sigma_x^B, \sigma_y^B, \sigma_z^B$, is a physical realism element. While following quantum theory, only commutative observables may have eigenvalues simultaneously, i.e. $\sigma_x^B, \sigma_y^B, \sigma_z^B$ can't have definite values simultaneously.

2. (1) Prove the CHSH inequality

$$|E(A_1B_1) + E(A_1B_2) + E(A_2B_1) - E(A_2B_2)| \leq 2,$$

in which $E(A_iB_j)$ is the expectation value of the correlation experiment A_i, B_j.

(2) For the singlet state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

prove that the correlation function $E(A_iB_j)_{quantum} = \langle\psi^-| A_i \otimes B_j |\psi^-\rangle \equiv \langle\psi^-| (\vec{a}_i \cdot \vec{\sigma}) \otimes (\vec{b}_j \cdot \vec{\sigma}) |\psi^-\rangle$ is

$$E(A_iB_j)_{quantum} = -\vec{a}_i \cdot \vec{b}_j.$$

(3) What's the maximal violation of the CHSH inequality allowed by quantum mechanics? Give the corresponding quantum state and specify the measurement operators.

Answer:

(1)

$$\begin{aligned}
&E(A_1B_1) + E(A_1B_2) + E(A_2B_1) - E(A_2B_2)\\
=&E(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2)\\
=&E(A_1(B_1 + B_2) + A_2(B_1 - B_2))\\
=&\sum_{a_1,a_2,b_1,b_2} p(a_1, a_2, b_1, b_2)\left[a_1(b_1 + b_2) + a_2(b_1 - b_2)\right].
\end{aligned}$$

Note that $a_1, a_2, b_1, b_2 \in \{+1, -1\}$. If $b_1 = b_2$, then $b_1 + b_2 = \pm 2,\ b_1 - b_2 = 0$, thus $a_1(b_1 + b_2) + a_2(b_1 - b_2) = \pm 2a_1 \in \{+2, -2\}$. If $b_1 = -b_2$, then $b_1 + b_2 = 0,\ b_1 - b_2 = \pm 2$, thus $a_1(b_1 + b_2) + a_2(b_1 - b_2) = \pm 2a_2 \in \{+2, -2\}$.
In either case, $a_1(b_1+b_2)+a_2(b_1-b_2) = \pm 2$. Since $\sum_{a_1,a_2,b_1,b_2} p(a_1, a_2, b_1, b_2) = 1$,

$$\begin{aligned}
&\sum_{a_1,a_2,b_1,b_2} p(a_1, a_2, b_1, b_2)\left[a_1(b_1 + b_2) + a_2(b_1 - b_2)\right] \in [-2, +2]\,,\\
&\Rightarrow |E(A_1B_1) + E(A_1B_2) + E(A_2B_1) - E(A_2B_2)| \leq 2.
\end{aligned}$$

(2) Due to the asymmetric property of $|\psi^-\rangle$, we have

$$I \otimes (\vec{b}_j \cdot \vec{\sigma})\,|\psi^-\rangle = -(\vec{b}_j \cdot \vec{\sigma}) \otimes I\,|\psi^-\rangle$$
$$\Rightarrow (\vec{a}_i \cdot \vec{\sigma}) \otimes (\vec{b}_j \cdot \vec{\sigma})\,|\psi^-\rangle = -(\vec{a}_i \cdot \vec{\sigma})(\vec{b}_j \cdot \vec{\sigma}) \otimes I\,|\psi^-\rangle\,.$$

Thus,

$$\begin{aligned} E(A_iB_j)_{quantum} &= \langle\psi^-|\,(\vec{a}_i \cdot \vec{\sigma}) \otimes (\vec{b}_j \cdot \vec{\sigma})\,|\psi^-\rangle \\ &= -\langle\psi^-|\,(\vec{a}_i \cdot \vec{\sigma})(\vec{b}_j \cdot \vec{\sigma}) \otimes I\,|\psi^-\rangle \\ &= -a_{ik}b_{jl}\,\langle\psi^-|\,\sigma_k\sigma_l \otimes I\,|\psi^-\rangle \\ &= -a_{ik}b_{jl}\,\langle\psi^-|\,(\mathrm{i}\epsilon_{klm}\sigma_m + \delta_{kl}I) \otimes I\,|\psi^-\rangle \\ &= -a_{ik}b_{jl}\delta_{kl} = -a_{ik}b_{jk} = -\vec{a}_i \cdot \vec{b}_j, \end{aligned}$$

where a_{ik} denotes the kth component of $\vec{a}_i$, similar for b_{jl}.

(3) The maximal violation allowed by quantum mechanics is $2\sqrt{2}$. The corresponding state is $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and the measurement operators are

$$A_1 = X,\;\; A_2 = Z,\;\; B_1 = \frac{X+Z}{\sqrt{2}},\;\; B_2 = \frac{X-Z}{\sqrt{2}}.$$

3. (Tsirelson's inequality) Suppose $Q = \vec{q}\cdot\vec{\sigma}$, $R = \vec{r}\cdot\vec{\sigma}$, $S = \vec{s}\cdot\vec{\sigma}$, $T = \vec{t}\cdot\vec{\sigma}$, where $\vec{q}$, $\vec{r}$, $\vec{s}$ and $\vec{t}$ are real unit vectors in three dimensions and $\vec{\sigma} = (\sigma_x\ \sigma_y\ \sigma_z)$. Show that

$$(Q\otimes S + R\otimes S + R\otimes T - Q\otimes T)^2 = 4I + [Q,R]\otimes[S,T].$$

Use this result to prove that

$$\langle Q\otimes S\rangle + \langle R\otimes S\rangle + \langle R\otimes T\rangle - \langle Q\otimes T\rangle \le 2\sqrt{2}.$$

Answer: For a real unit vector $\vec{n}$ in three dimensions, $(\vec{n}\cdot\vec{\sigma})^2 = I$, then

$$\begin{aligned} &(Q\otimes S + R\otimes S + R\otimes T - Q\otimes T)^2 \\ =&4I + (Q\otimes S)\cdot(R\otimes S) + (R\otimes S)\cdot(Q\otimes S) - (R\otimes T)\cdot(Q\otimes T) - (Q\otimes T)\cdot(R\otimes T) \\ &-(Q\otimes S)\cdot(Q\otimes T) - (Q\otimes T)\cdot(Q\otimes S) + (Q\otimes T)\cdot(Q\otimes S) + (R\otimes S)\cdot(R\otimes T) \\ &+(Q\otimes S)\cdot(R\otimes T) - (R\otimes S)\cdot(Q\otimes T) - (Q\otimes T)\cdot(R\otimes S) + (R\otimes T)\cdot(Q\otimes S) \\ =&4I + QR\otimes I + RQ\otimes I - RQ\otimes I - QR\otimes I - I\otimes ST - I\otimes TS + I\otimes ST + I\otimes TS \\ &+QR\otimes ST - RQ\otimes ST - QR\otimes TS + RQ\otimes TS \\ =&4I + [Q,R]\otimes[S,T]. \end{aligned}$$

As $[Q,R] \leq 2,\ [S,T] \leq 2$, we have $4I + [Q,R] \otimes [S,T] \leq 8$. That is

$$\langle Q \otimes S\rangle + \langle R \otimes S\rangle + \langle R \otimes T\rangle - \langle Q \otimes T\rangle \leq 2\sqrt{2}.$$

4. Derive the Bell's theorem without inequalities from the GHZ state

$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle).$$

Answer:

Read page 40-41 in the lecture ”QIP2022chapt_3_1_Kai Chen.pdf” for reference.

5. Consider the CHSH game in which the referee chooses questions $r, s \in \{0,1\}$ uniformly, and Alice and Bob must each answer a single bit: a for Alice, b for Bob, in which $a, b \in \{0,1\}$. They win if $a \oplus b = r \wedge s$ and lose otherwise.

(1) Give the maximum probability of winning with the classical strategy.

(2) Suppose Alice and Bob share a maximum quantum entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, please derive the maximum probability of winning and give the corresponding quantum strategy.

Answer:

Read page 22-26 in the lecture “QIP2022chapt_3_1_Kai Chen.pdf” for reference.

6. Consider the GHZ game in which the referee chooses questions $rst \in \{000, 011, 101, 110\}$ uniformly, and Alice, Bob and Charles must each answer a single bit: a for Alice, b for Bob, c for Charles, in which $a, b, c \in \{0,1\}$. They win if $a \oplus b \oplus c = r \vee s \vee t$ and lose otherwise. Suppose Alice, Bob and Charles share a GHZ state $|\psi\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$, give a quantum strategy that maximize the probability of winning.

Answer:

Read page 18-21 in the lecture “QIP2022chapt_3_1_Kai Chen.pdf” for reference.

7. Two players, Alice and Bob, are required to independently fill a 3×3 magic square. As shown in Fig. 1, the referee randomly sends two queries $x, y \in \{0, 1, 2\}$ to Alice and Bob, respectively. Here, x labels rows and y labels columns. Alice and Bob are required to reply with three numbers with specific conditions. Denote Alice's answers in a row as $[a_0^x, a_1^x, a_2^x]$ and Bob's answers in a column as $[b_0^y, b_1^y, b_2^y]$, where $a_i^x, b_j^y \in \{-1, +1\}$ for $i, j \in \{0, 1, 2\}$. Alice's answers must satisfy $\prod_i a_i^x = +1$, while Bob's should satisfy $\prod_j b_j^y = -1$ for any x and y. During the game, Alice and Bob are forbidden to communicate with each other. They win the game if the overlapped entry of Alice's row and Bob's column is always the same, i.e.,$a_y^x = b_x^y$ for each x and y.

   (1) Give the maximum probability of winning with the classical strategy.

   (2) Suppose Alice and Bob share a maximum quantum entangled state $|\phi\rangle_{A_1A_2B_1B_2} = |\psi\rangle_{A_1B_1} \otimes |\psi\rangle_{A_2B_2}$ with $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ where Alice has systems A_1A_2 and Bob has B_1B_2. Please derive the maximum probability of winning and give the corresponding quantum strategy.

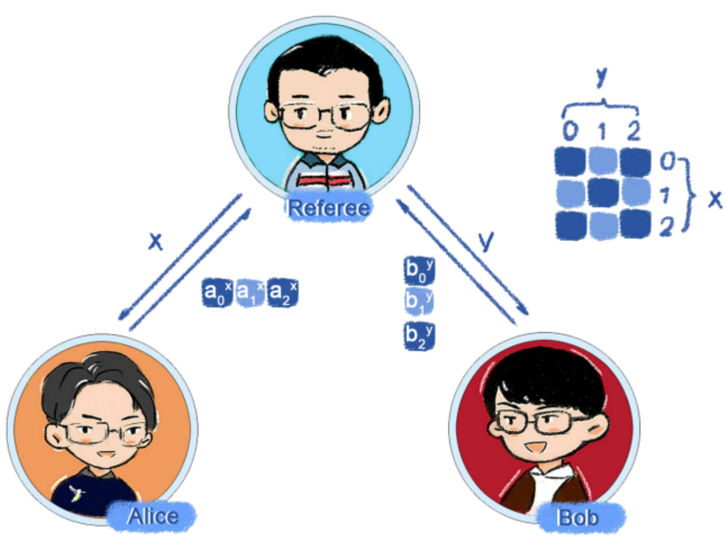

FIG. 1. The Mermin-Peres magic square game.

Answer:

Read page 29-32 in the lecture "QIP2022chapt_3_1_Kai Chen.pdf" for ref-

x \ y	0	1	2
0	$I \otimes Z$	$Z \otimes I$	$Z \otimes Z$
1	$X \otimes I$	$I \otimes X$	$X \otimes X$
2	$-X \otimes Z$	$-Z \otimes X$	$Y \otimes Y$

FIG. 2. The quantum strategy of Mermin-Peres magic square game.

erence.

(1) The maximum probability of winning with the classical strategy is 8/9.

(2) The maximum probability of winning with the quantum strategy is 1, the strategy is shown in Fig. 2.

8. Define the rotation operator $R_{\hat{n}}(\theta) = exp(-i\theta\hat{n} \cdot \vec{\sigma}/2)$, where $\hat{n}$ is a real three-dimensional unit vector. Prove that an arbitrary single qubit unitary operator can be written in the form $U = exp(i\alpha)R_{\hat{n}}(\theta)$. Find values for α, θ, and $\hat{n}$ giving the Hadamard gate H.

Answer:

A single qubit operator $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be expanded in terms of $\{X, Y, Z, I\}$ as $U = a_0 I + a_1 X + a_2 Y + a_3 Z$, where

$$a_0 = \frac{a+d}{2}, a_1 = \frac{b+c}{2}, a_2 = \frac{c-b}{2i}, a_3 = \frac{a-d}{2},$$

U is unitary if $U^\dagger U = I$. This requires that

$$\begin{aligned} |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 &= 1 \\ a_0^* a_1 + a_1^* a_0 + i a_2^* a_3 - i a_3^* a_2 &= 0 \\ a_0^* a_2 - i a_1^* a_3 + a_2^* a_0 + i a_3^* a_1 &= 0 \\ a_0^* a_3 + i a_1^* a_2 - i a_2^* a_1 + a_3^* a_0 &= 0 \end{aligned}$$

Define $\cos(\frac{\theta}{2}) = |a_0|$, then $|a_1|^2 + |a_2|^2 + |a_3|^2 = \sin^2(\frac{\theta}{2})$. Define

$$n_x = |a_1| \,/\, \left|\sin(\frac{\theta}{2})\right|$$
$$n_y = |a_2| \,/\, \left|\sin(\frac{\theta}{2})\right|$$
$$n_z = |a_3| \,/\, \left|\sin(\frac{\theta}{2})\right|$$

Define $\exp(i\alpha) = a_0/\cos(\frac{\theta}{2})$. Denote the phase of a_1, a_2, a_3 as $\alpha_1, \alpha_2, \alpha_3$ respectively. We can get $\cos(\alpha - \alpha_1) = 0$, $\sin(\alpha_2 - \alpha_3) = 0$. Hence, $\alpha_1 = \alpha - \pi/2$, $\alpha_2 = \alpha_3$. Similarly, we find, $\alpha_2 = \alpha_3 = \alpha - \pi/2$, $\alpha_1 = \alpha_2 = \alpha_3$. Therefore,

$$\begin{aligned} \alpha_0 &= \quad \exp(i\alpha)\cos(\frac{\theta}{2}) \\ \alpha_1 &= -\,i\exp(i\alpha)\sin(\frac{\theta}{2})n_x \\ \alpha_2 &= -\,i\exp(i\alpha)\sin(\frac{\theta}{2})n_y \\ \alpha_3 &= -\,i\exp(i\alpha)\sin(\frac{\theta}{2})n_z \end{aligned}$$

That is

$$U = \exp(i\alpha)(\cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})(n_xX + n_yY + n_zZ)) = \exp(i\alpha)R_{\hat{n}}(\theta).$$

9. (1) A state ρ is a pure state if and only if $\mathrm{tr}(\rho^2) = 1$. Prove that this is equivalent to $S(\rho) = 0$, where $S(\rho)$ is the Von Neumann entropy.

   (2) Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1.

   (3) Prove that $|\psi\rangle$ is a product state if and only if ρ^A (and thus ρ^B) are pure states.

Answer:

(1) If $\mathrm{tr}(\rho^2) = 1$,

$$\sum_k \lambda_k^2 = \sum_k \lambda_k = 1.$$

Therefore,

$$\sum_k \lambda_k(\lambda_k - 1) = 0.$$

Since $0 \leq \lambda_k \leq 1, \forall k$, we know that $\lambda_k(\lambda_k - 1) \geq 0, \forall k$, ans thus the only way for the above condition to be satisfied is for $\lambda_k = 0, 1, \forall k$. Therefore $\text{tr}(\rho^2) = 1$ if and only if ρ has a single eigenvalue of 1 with all other eigenvalues 0.

$$S(\rho) = -\sum_k \lambda_k \log_2(\lambda_k) = 0.$$

Since $0 \leq \lambda_k \leq 1, \forall k$, we know that $\lambda_k \log_2(\lambda_k) \geq 0, \forall k$. Therefore, the only way for the above condition to be satisfied is for $\lambda_k = 0, 1, \forall k$, and thus $S(\rho) = 1$ if and only if ρ has a single eigenvalue of 1 with all other eigenvalues 0. Therefore, for density matrices, $\text{tr}(\rho^2) = 1$ and $S(\rho) = 0$ are equivalent statements.

(2) A state is a product state if and only if it can be represented as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. If a state has a Schmidt number 1, it can be represented as a product state $\sum_k \sqrt{\lambda_k}|k_A\rangle|k_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ since only one Schmidt coefficient is nonzero. If it has a Schmidt number greater than 1, it has no such representation as $|\psi_A\rangle \otimes |\psi_B\rangle$, because if it did it would have a Schmidt number of 1 through the above representation.

(3) If an entangled state between Alice and Bob has the Schmidt decomposition $\sum_k \sqrt{\lambda_k}|k_A\rangle|k_B\rangle$. Then Alice's reduced density matrix is $\rho^A = \sum_k \lambda_k |k_A\rangle\langle k_A|$. Therefore, if $|\psi\rangle$ has a Schmidt number of 1, the reduced density matrices ρ^A, ρ^B have only one non-zero eigenvalue and are pure states. If $|\psi\rangle$ has a Schmidt number greater than 1, the reduced density matrices ρ^A, ρ^B have multiple non-zero eigenvalues and are mixed states.

10. Let $\vec{n}$ be a normalized real vector in three dimensions and let θ be real. Prove that the equality

$$f(\theta\vec{n} \cdot \vec{\sigma}) = \frac{f(\theta) + f(-\theta)}{2} I + \frac{f(\theta) - f(-\theta)}{2} \vec{n} \cdot \vec{\sigma}$$

holds for any function $f(\cdot)$.

Answer: The spectral decomposition of $\vec{n}\cdot\vec{\sigma}$ is $\vec{n}\cdot\vec{\sigma} = |n,+\rangle\langle n,+| - |n,-\rangle\langle n,-|$. Thus,

$$\begin{aligned}
f(\theta\vec{n}\cdot\vec{\sigma}) =& f(\theta|n,+\rangle\langle n,+| - \theta|n,-\rangle\langle n,-|) \\
=& f(\theta)|n,+\rangle\langle n,+| + f(-\theta)|n,-\rangle\langle n,-| \\
=& \left(\frac{f(\theta)+f(-\theta)}{2} + \frac{f(\theta)-f(-\theta)}{2}\right)|n,+\rangle\langle n,+| \\
+& \left(\frac{f(\theta)+f(-\theta)}{2} - \frac{f(\theta)-f(-\theta)}{2}\right)|n,-\rangle\langle n,-| \\
=& \frac{f(\theta)+f(-\theta)}{2}(|n,+\rangle\langle n,+| + |n,-\rangle\langle n,-|) \\
+& \frac{f(\theta)-f(-\theta)}{2}(|n,+\rangle\langle n,+| - |n,-\rangle\langle n,-|) \\
=& \frac{f(\theta)+f(-\theta)}{2}I + \frac{f(\theta)-f(-\theta)}{2}\vec{n}\cdot\vec{\sigma}.
\end{aligned}$$

11. Consider a particle with initial state $|0\rangle$. We perform N sequential measurements $\sigma_k \equiv \vec{n}_k\cdot\vec{\sigma}$ with $\vec{n}_k = \left(\sin\left(\frac{k\pi}{2N}\right), 0, \cos\left(\frac{k\pi}{2N}\right)\right)$ and $k = 1, 2, \cdots, N$. What's the probability that all outcomes are $+1$? What if $N \to \infty$?

Answer: The probability that the first measurement gives outcome $+1$ is

$$\begin{aligned}
P_1 &= \mathrm{tr}\left[\frac{I+\vec{n}_1\cdot\vec{\sigma}}{2}|0\rangle\langle 0|\right] \\
&= \frac{1}{2}\mathrm{tr}(|0\rangle\langle 0|) + \frac{1}{2}n_{1i}\,\mathrm{tr}(\sigma_i|0\rangle\langle 0|) \\
&= \frac{1}{2} + \frac{1}{2}n_{1i}\cdot\langle 0|\sigma_i|0\rangle \\
&= \frac{1}{2} + \frac{1}{2}n_{13} = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2N}\right) = \cos^2\left(\frac{\pi}{4N}\right).
\end{aligned}$$

The resulting state is thus $|n_1,+\rangle\langle n_1,+| = (I+\vec{n}_1\cdot\vec{\sigma})/2$. Thus, the probability that the $(k+1)$th measurement gives outcome $+1$ *conditioned on* the kth measurement gives outcome $+1$ is

$$\begin{aligned}
P_{K+1} &= \mathrm{tr}\left[\frac{I+\vec{n}_{k+1}\cdot\vec{\sigma}}{2}\frac{I+\vec{n}_k\cdot\vec{\sigma}}{2}\right] \\
&= \frac{1}{4}\mathrm{tr}\,I + \frac{1}{4}\vec{n}_{k+1}\cdot\vec{n}_k\,\mathrm{tr}\,I \\
&= \frac{1}{2} + \frac{1}{2}\left[\sin\left(\frac{k\pi}{2N}\right)\sin\left(\frac{(k+1)\pi}{2N}\right) + \cos\left(\frac{k\pi}{2N}\right)\cos\left(\frac{(k+1)\pi}{2N}\right)\right] \\
&= \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2N}\right) = \cos^2\left(\frac{\pi}{4N}\right).
\end{aligned}$$

Therefore, the probability that all outcomes are +1 is $P_1 \cdot P_2 \cdots P_N = \cos^{2N}\left(\frac{\pi}{4N}\right)$. The limitation is

$$\begin{aligned}\lim_{N\to\infty}\cos^{2N}\left(\frac{\pi}{4N}\right) &= \exp\left(\lim_{N\to\infty} 2N\ln(\cos\left(\frac{\pi}{4N}\right))\right)\\ &\overset{N=1/t}{=} \exp\left(\lim_{t\to 0} 2\ln(\cos\left(\frac{\pi t}{4}\right))/t\right)\\ &= \exp\left(\lim_{t\to 0} 2\frac{-\frac{\pi}{4}\sin\left(\frac{\pi t}{4}\right)}{\cos\left(\frac{\pi t}{4}\right)}\right) = 1.\end{aligned}$$

12. Consider a 2-qubit quantum state $\rho_{AB} = \frac{1}{8}I + \frac{1}{2}|\psi^-\rangle\langle\psi^-|$, where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

(1) Give the spectral decomposition of ρ_{AB}.

(2) Suppose one measures $\vec{n}\cdot\vec{\sigma}_A$ and measures $\vec{m}\cdot\vec{\sigma}_B$ with $\vec{n}\cdot\vec{m} = \cos\theta$, calculate the probability that both outcomes are +1.

(3) Use the realignment criterion to find out whether ρ_{AB} is entangled or not.

Answer:

(1) The density matrix of ρ_{AB} in computational basis is

$$\rho_{AB} = \begin{pmatrix} 1/8 & 0 & 0 & 0\\ 0 & 3/8 & -1/4 & 0\\ 0 & -1/4 & 3/8 & 0\\ 0 & 0 & 0 & 1/8\end{pmatrix}.$$

The above matrix has eigenvalues $\lambda_{1,2,3} = 1/8$, $\lambda_4 = 5/8$, with corresponding eigenvectors $|\phi^+\rangle$, $|\phi^-\rangle$, $|\psi^+\rangle$, $|\psi^-\rangle$. Thus, $\rho_{AB} = \frac{1}{8}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|) + \frac{5}{8}|\psi^-\rangle\langle\psi^-|$.

(2) The probability that both outcomes are +1 is given by

$$\begin{aligned}P &= \mathrm{tr}\left[\frac{I+\vec{n}\cdot\vec{\sigma}_A}{2}\otimes\frac{I+\vec{m}\cdot\vec{\sigma}_B}{2}\rho_{AB}\right]\\ &= \frac{1}{8}\mathrm{tr}\left[\frac{I+\vec{n}\cdot\vec{\sigma}_A}{2}\otimes\frac{I+\vec{m}\cdot\vec{\sigma}_B}{2}\right] + \frac{1}{2}\mathrm{tr}\left[\frac{I+\vec{n}\cdot\vec{\sigma}_A}{2}\otimes\frac{I+\vec{m}\cdot\vec{\sigma}_B}{2}|\psi^-\rangle\langle\psi^-|\right]\\ &= \frac{1}{8}\mathrm{tr}\left(\frac{I}{4}\right) + \frac{1}{2}\left[\mathrm{tr}\left(\frac{1}{4}|\psi^-\rangle\langle\psi^-|\right) + \frac{1}{4}\langle\psi^-|(\vec{n}\cdot\vec{\sigma}_A)\otimes(\vec{m}\cdot\vec{\sigma}_B)|\psi^-\rangle\right]\\ &= \frac{1}{8} + \frac{1}{8} - \frac{1}{8}\vec{n}\cdot\vec{m} = \frac{1}{4} - \frac{1}{8}\cos\theta,\end{aligned}$$

In the third line, we used $\mathrm{tr}(M\otimes N) = \mathrm{tr}(M)\cdot\mathrm{tr}(N)$, $\mathrm{tr}\left(I \otimes (\vec{m}\cdot\vec{\sigma}_B)\left|\psi^-\right\rangle\left\langle\psi^-\right|\right) = \mathrm{tr}\left((\vec{n}\cdot\vec{\sigma}_A)\otimes I\left|\psi^-\right\rangle\left\langle\psi^-\right|\right) = 0$ since unilateral Pauli operation changes one Bell state into another, and the cyclic property of trace operation. In the fourth line, we used the result of question 2.(2).

(3)

$$\tilde{\rho}_{AB} = \begin{pmatrix} 1/8 & 0 & 0 & 3/8 \\ 0 & 0 & -1/4 & 0 \\ 0 & -1/4 & 0 & 0 \\ 3/8 & 0 & 0 & 1/8 \end{pmatrix}.$$

The eigenvalues are $\left\{\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right\}$. Thus, $\|\tilde{\rho}_{AB}\| = \frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{5}{4} > 1$. Therefore, ρ_{AB} is entangled.

13. Consider a composite system consisting of two qubits. Find the Schmidt decompositions of the states $|\phi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |11\rangle)$

Answer:

We can see that Alice's reduced density matrix is $\rho^A = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, having eigenvalues

$$\lambda_1 = \frac{3+\sqrt{5}}{6}, \lambda_2 = \frac{3-\sqrt{5}}{6},$$

eigenvectors

$$|\phi_{A1}\rangle = \frac{(\frac{1+\sqrt{5}}{2}, 1)^T}{\sqrt{1+(\frac{1+\sqrt{5}}{2})^2}}, |\phi_{A2}\rangle = \frac{(\frac{1-\sqrt{5}}{2}, 1)^T}{\sqrt{1+(\frac{1-\sqrt{5}}{2})^2}}.$$

Likewise, Bob's reduced density matrix is $\rho^B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, having eigenvalues

$$\lambda_1 = \frac{3+\sqrt{5}}{6}, \lambda_2 = \frac{3-\sqrt{5}}{6},$$

eigenvectors

$$|\phi_{B1}\rangle = \frac{(1, \frac{1+\sqrt{5}}{2})^T}{\sqrt{1+(\frac{1+\sqrt{5}}{2})^2}}, |\phi_{B2}\rangle = \frac{(-1, \frac{\sqrt{5}-1}{2})^T}{\sqrt{1+(\frac{\sqrt{5}-1}{2})^2}}.$$

Therefore, the Schmidt decomposition will be $|\phi\rangle = \sqrt{\lambda_1}|\phi_{A1}\rangle|\phi_{B1}\rangle + \sqrt{\lambda_2}|\phi_{A2}\rangle|\phi_{B2}\rangle$.

14. Consider the density matrix $\rho_w = r|\phi^+\rangle\langle\phi^+| + \frac{1-r}{4}I_4$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is Bell state and $0 \le r \le 1$. Calculate the concurrence of ρ_w.

    Answer: We have

$$\rho_w = \begin{pmatrix} (1+r)/4 & 0 & 0 & r/2 \\ 0 & (1-r)/4 & 0 & 0 \\ 0 & 0 & (1-r)/4 & 0 \\ r/2 & 0 & 0 & (1+r)/4. \end{pmatrix}$$

    The eigenvalues of ρ_w are $(1+3r)/4$ and $(1-r)/4$ with multiplicity 3, then $\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 = (3r-1)/2$. The concurrence is $C(\rho_w) = \max\{(3r-1)/2, 0\}$.

15. (1) For the 3-qubit W state $|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, if one particle is lost, what's the reduced density matrix of the remaining two particles?

    (2) For the n-qubit W state $|W_n\rangle = \frac{1}{\sqrt{n}}(|10\cdots0\rangle + |01\cdots0\rangle + \cdots + |00\cdots1\rangle)$, if $n-2$ particles are lost, what's the reduced density matrix of the remaining two particles? Use the PPT criterion to find out whether the remaining two particles are entangled or not.

    Answer: Without loss of generality, we suppose the remaining two particles are particle 1 and 2.

    (1) $\rho_{12} = \text{tr}_3(|W_3\rangle\langle W_3|) = \frac{1}{3}\left(2|\psi^+\rangle\langle\psi^+| + |00\rangle\langle 00|\right)$.

    (2) $\rho_{12} = \text{tr}_{3\cdots n}(|W_n\rangle\langle W_n|) = \frac{1}{n}\left(2|\psi^+\rangle\langle\psi^+| + (n-2)|00\rangle\langle 00|\right)$.

$$\rho_{12} = \begin{pmatrix} \frac{n-2}{n} & 0 & 0 & 0 \\ 0 & \frac{1}{n} & \frac{1}{n} & 0 \\ 0 & \frac{1}{n} & \frac{1}{n} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{12}^{T_1} = \begin{pmatrix} \frac{n-2}{n} & 0 & 0 & \frac{1}{n} \\ 0 & \frac{1}{n} & 0 & 0 \\ 0 & 0 & \frac{1}{n} & 0 \\ \frac{1}{n} & 0 & 0 & 0 \end{pmatrix}.$$

    The eigenvalues of $\rho_{12}^{T_1}$ are $\left\{\frac{1}{n}, \frac{1}{n}, \frac{n-2+\sqrt{n^2-4n+8}}{2n}, \frac{n-2-\sqrt{n^2-4n+8}}{2n}\right\}$, where the last one is negative. Thus, the remaining two particles are still entangled.

16. Suppose P_i is a complete set of orthogonal projectors and ρ is a density operator. Prove that the entropy of the state $\rho' \equiv \sum_i P_i \rho P_i$ of the system after

the measurement is at least as great as the orignal entropy, $S(\rho') \geq S(\rho)$, with equality if and only if $\rho = \rho'$.

Answer:

The proof is to apply Klein's inequality to ρ and ρ',

$$0 \leq S(\rho||\rho') = -S(\rho) - tr(\rho \log \rho').$$

The result will follow if we can prove that $-tr(\rho \log \rho') = S(\rho')$. We apply the completeness relation $\sum_i P_i = I$, the relation $P_i^2 = P_i$, and the cyclic property of the trace, to obtain

$$\begin{aligned} -tr(\rho \log \rho') &= -\, tr(\sum_i P_i \rho \log \rho') \\ &= -\, tr(\sum_i P_i \rho \log \rho' P_i). \end{aligned}$$

Note that $\rho' P_i = P_i \rho P_i = P_i \rho'$. That is, P_i commutes with ρ' and thus with $\log \rho^p rime$, so

$$\begin{aligned} -tr(\rho \log \rho') &= -\, tr(\sum_i P_i \rho P_i \log \rho') \\ &= -\, tr(\rho' \log \rho') = S(\rho'). \end{aligned}$$

This completes the proof.