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# Title:

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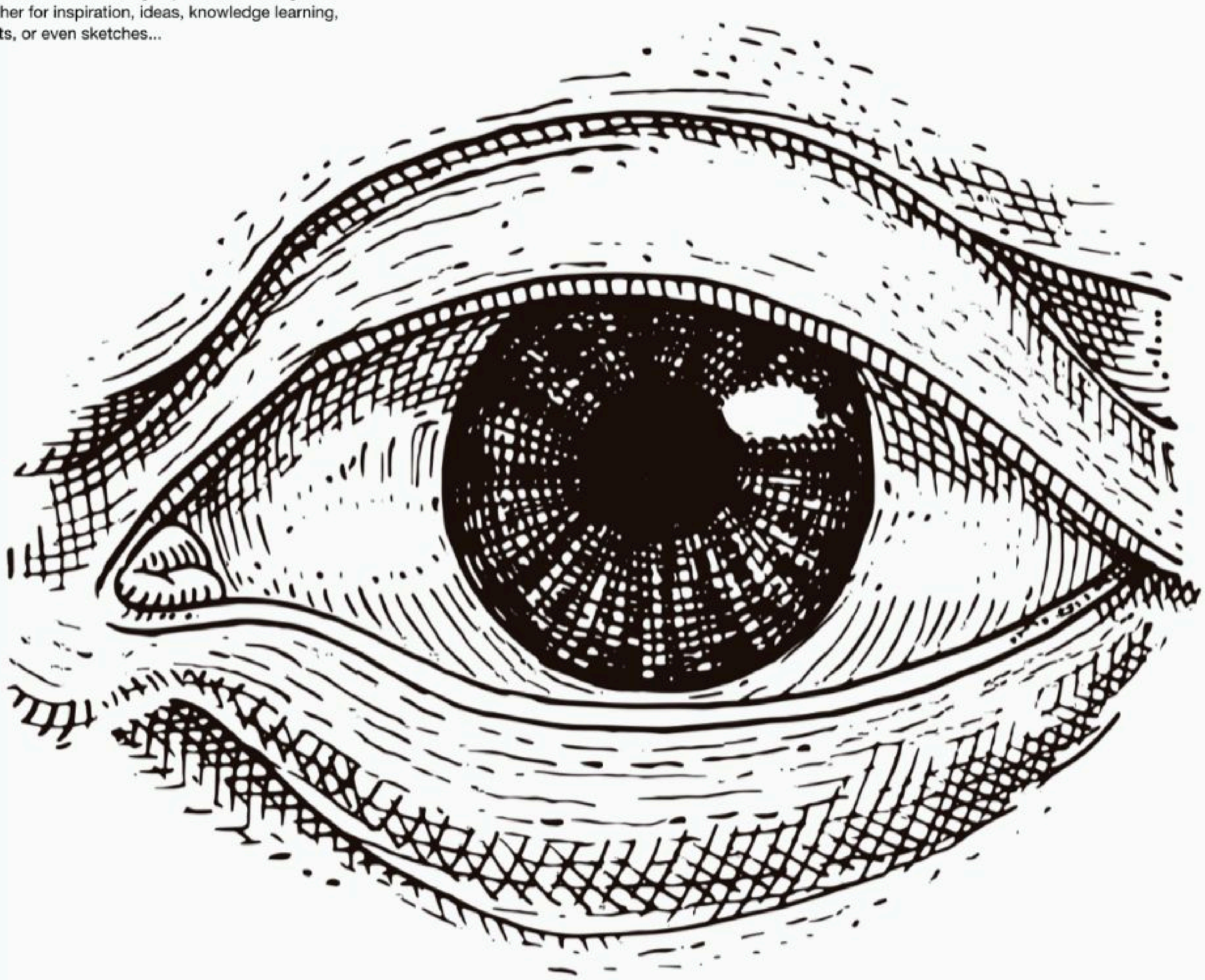


NOTE IN

**Creative notes**

By reading we enrich the mind;
by writing we polish it.

量子力学：

量子测量 t. Hoof

厄米算符 $A$ $\quad A=A^\dagger \xrightarrow{\text{定义}} A|u_n\rangle = A_n|u_n\rangle$ 测量量

$$|\psi\rangle \xrightarrow{A} \begin{cases} A_n \\ |u_n\rangle \end{cases}$$

$|\psi\rangle = \sum_n C_n|u_n\rangle$ $\quad |C_n|^2$ 概率

实际测量结果 $\langle A\rangle = \sum_n |C_n|^2 A_n$

Quantum Trajectory

其中 $\sum_n |u_n\rangle\langle u_n| = 1$ 正交归一

$P_n = |u_n\rangle\langle u_n|$ $\quad P_n P_m = \delta_{nm}$ $\quad \sum_n P_n = 1$

$P_n|\psi\rangle$ 意为 $|\psi\rangle$ 投影到 $|u_n\rangle$ 上

$\langle\psi|P_n|\psi\rangle$ 即为概率

$|\psi\rangle$ 态矢量 如何用数值表征？

引入正交完备集 $\sum_n |u_n\rangle\langle u_n| = 1$ $\quad \langle u_n|u_m\rangle = \delta_{nm}$.

$|\psi\rangle = \sum_n |u_n\rangle\langle u_n|\psi\rangle = \sum_n C_n|u_n\rangle$

存在无穷多组 $|u_n\rangle\langle u_n|$

量子力学中通常以位置集 $\int|x\rangle\langle x|dx = 1$ 展开： $\langle x|x'\rangle = \delta(x-x')$

$\hat{x}|x\rangle = x|x\rangle$

Schrödinger 方程： $i\frac{\partial}{\partial t}|\psi\rangle = \hat{A}|\psi\rangle$

In $x$ : $\langle x|i\frac{\partial}{\partial t}|\psi\rangle = \langle x|\hat{A}|\psi\rangle$

$$\Rightarrow i\frac{\partial}{\partial t}\psi(x) = \int\langle x|\hat{A}|x'\rangle\langle x'|\psi\rangle dx'$$

$$H=\frac{p^2}{2m}+V(x) \qquad \Rightarrow \quad i\frac{\partial}{\partial t}\psi(x)=\int\left[\left(\frac{-i\partial x'}{2m}\right)^2+V(x')\right]\delta(x-x')\psi(x')dx'$$

$$=\left[\left(\frac{-i\partial x}{2m}\right)+V(x)\right]\psi(x)$$

动量表象： $\hat{p}|p\rangle=p|p\rangle \qquad \int|p\rangle\langle p|dp=1$

$$i\frac{\partial}{\partial t}|\psi\rangle=\hat{H}|\psi\rangle$$

$$\Rightarrow \langle p|i\frac{\partial}{\partial t}|\psi\rangle=\langle p|\hat{H}|\psi\rangle$$

$$\Rightarrow \quad i\partial_t\psi(p)=\int \langle p|\hat{H}|p'\rangle\langle p'|\psi\rangle dp'$$

$$=\left[\frac{p^2}{2m}+V\left(i\frac{\partial}{\partial p}\right)\right]\psi(p)$$

$$[\hat{x},\hat{p}]=i\hbar$$

$$\hat{H}=\frac{p^2}{2m}+V(x)$$

$|x\rangle$ 坐标： $i\frac{\partial}{\partial t}\psi(x)=\left(\frac{(-i\partial_x)^2}{2m}+V(x)\right)\psi(x)$

$|p\rangle$ 动量： $i\frac{\partial}{\partial t}\psi(p)=\left(\frac{p^2}{2m}+V(i\partial_p)\right)\psi(p)$

$$|x\rangle=\int dp'\,|p'\rangle\langle p'|x\rangle$$

$$=\int e^{ip'x}|p'\rangle dp$$

两组基

$\sum_n|u_n\rangle\langle u_n|=1 \qquad |u_n\rangle\langle u_m|=\delta_{nm}$

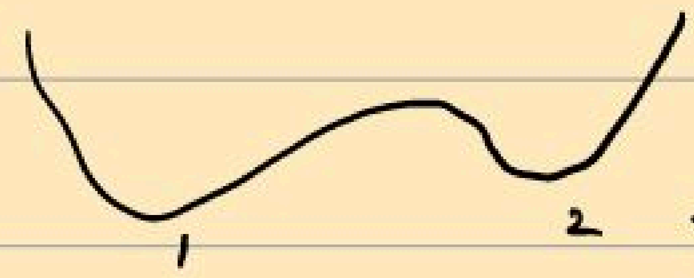
$\sum_n|v_n\rangle\langle v_n|=1 \qquad |v_n\rangle\langle v_m|=\delta_{nm}$

两组基间一定存在幺正变换

$|u_n\rangle=\sum_m U_{nm}|v_m\rangle \qquad \{U_{nm}\}$ 幺正矩阵 $UU^{\dagger}=U^{\dagger}U=1$

谐振子：$\hat{H} = \hat{p}^2 + \hat{x}^2$

为什么研究谐振子？ $\hat{H} = \hat{p}^2 + \hat{V}(x)$




平衡点

$= \underbrace{\hat{p}^2 + \hat{x}^2}_{\text{严格可解}} + ?$

$\hat{H} = p^2 + V(x)$

$= p^2 + V(x_0) + \frac{\partial}{\partial x} V(x_1) \Delta x + \frac{\partial^2}{\partial x^2} V(x_2) \Delta x^2 + O(x_3)$ 微扰求解

束缚条件：$\psi(x) \to 0$ 当 $x \to \infty$ 量子力学中的隐含条件

$\hat{H} \psi(x) = E \psi(x)$

$\Rightarrow [(-i\partial_x)^2 + \hat{x}^2] \psi(x) = E \psi(x)$ $\quad \phi(x) \to 0 \quad x \to \infty$ 束缚态

散射态

$\hat{H} = \hat{p}^2 + \hat{x}^2$

引入 $\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$ $\quad \hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$ $\quad [\hat{x}, \hat{p}] = i\hbar = i$

*所有量子化都是在找正则动量和正则坐标

产生 $a^+$ 湮灭 $a$ 算符 $[a \ a^+] = 1$

$\Rightarrow x = \frac{1}{\sqrt{2}}(a + a^+) \quad p = \frac{i}{\sqrt{2}}(a - a^+)$

$\Rightarrow H = p^2 + x^2 = 2w(a^+a + \frac{1}{2})$

多出一项即真空能 Casimir

去掉真空能 $\begin{cases} H = a^+a \\ [a \ a^+] = 1 \end{cases}$

$[a \ \ a^+a] = [a \ \ \hat{H}] = a$

$aH - Ha = a$

$\Rightarrow aH = (H+1)a$

$\Rightarrow aH|E\rangle = aE|E\rangle = (Ha + a)|E\rangle = Ha|E\rangle + a|E\rangle$

$\Rightarrow Ha|E\rangle = (E-1)a|E\rangle$

由 $H|E\rangle = E|E\rangle$ $[a\ H] = a$

故 $Ha^n|E\rangle = (E-n)a^n|E\rangle$ 当前计算与谐振子无关

$n \to \infty$ ?

考虑谐振子 $H = a^+a$

$H|E\rangle = E|E\rangle$ $\langle E|a^+a|E\rangle = E$

若 $\phi = a|E\rangle$ 有:

$E = \langle E|a^+a|E\rangle = \langle\phi|\phi\rangle \geqslant 0$

说明E是整数, $a|E_0\rangle = 0$ 即 $(\hat{x}+i\hat{p})|E_0\rangle = 0$

$H = p^2 + x^2$ $a = \frac{1}{\sqrt{2}}(x+ip)$ $a^+ = \frac{1}{\sqrt{2}}(x-ip)$

$[a\ a^+] = 1$ $H = a^+a$

$[a, H] = a$ $aH = Ha + a$

$x = \frac{1}{\sqrt{2}}(a + a^+)$

$\hat{H}|E\rangle = E|E\rangle$

$\hat{H}\hat{a}|E\rangle = (E-1)\hat{a}|E\rangle \Rightarrow \hat{H}\hat{a}^n|E\rangle = (E-n)\hat{a}^n|E\rangle$ 说明 $(E \in N)$

$a^+a|E\rangle = E|E\rangle$ $E = \langle E|a^+a|E\rangle \geqslant 0$

$\exists E_0 \in N$ 使 $a|E_0\rangle = 0$

$\Rightarrow (x+ip)|E_0\rangle = 0 \Rightarrow (x + (-i\partial_x)i)|E_0\rangle = 0 \Rightarrow (x+\partial_x)|E_0\rangle = 0$

$\Rightarrow |E_0\rangle = e^{-\frac{x^2}{2}}$

$$H|E_0\rangle = a^\dagger a|E_0\rangle = 0$$

In $\langle x|$ $\Rightarrow \left[(-i\partial_x)^2 + x^2\right]\varphi_0(x) = 0$

But in Schrödinger Eq. $\left[(-i\partial_x)^2 + x^2\right]\varphi(x) = E\phi(x)$ when $x\to\infty$ $\phi(x)\to 0$

$$[H, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger$$

$\Rightarrow Ha^\dagger - a^\dagger H = a^\dagger$

$\Rightarrow a^\dagger H = Ha^\dagger - a^\dagger$

提 $(Ha^\dagger - a^\dagger)|E\rangle = Ea^\dagger|E\rangle$

有 $Ha^\dagger|E\rangle = (E+1)a^\dagger|E\rangle$

由 $H|E\rangle = E|E\rangle$

$\Rightarrow Ha^\dagger|E\rangle = (E+1)a^\dagger|E\rangle$

$\Rightarrow H(a^\dagger)^n|E\rangle = (E+n)(a^\dagger)^n|E\rangle$

由 $H|E_0\rangle = 0 \Rightarrow H(a^\dagger)^n|E_0\rangle = n(a^\dagger)^n|E_0\rangle$

定义 $|E_0\rangle = |0\rangle \Rightarrow |n\rangle = (a^\dagger)^n|0\rangle$

$$\langle x|n\rangle = \langle x|(a^\dagger)^n|0\rangle = \left(x - \frac{\partial}{\partial x}\right)^n \phi_0(x) = \left(x - \frac{\partial}{\partial x}\right)^n e^{-\frac{x^2}{2}}$$

$a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$

Fock态 $|n\rangle = (a^\dagger)^n|0\rangle$ $a^\dagger$产生 $a$湮灭 $|0\rangle$真空

$H|n\rangle = \hbar\omega n|n\rangle \Rightarrow E_n = n\hbar\omega$

Heisenberg 矩阵力学 无法定义问题边界条件

Schrödinger Eq. $\left(-\partial_x^2+V(x)\right)\phi(x)=E\phi(x)$ 易解出边界条件.

$$H=\frac{p^2}{2m}+V(x)$$

时间演化：

$i\partial_t|\psi\rangle=H(t)|\psi\rangle$ 含时系统

引入 $U(t,0)$ $\quad|\psi(t)\rangle=U(t,0)|\psi(0)\rangle$

$\Rightarrow i\frac{dU}{dt}=HU$

$\Rightarrow i\int_0^t\frac{dU}{dt'}dt'=\int_0^t H(t)U(t)dt' \Rightarrow U(t)-1=-i\int_0^t H(t')U(t')dt'$

$\Rightarrow U(t)=1-i\int H(t')U(t')dt' = 1-i\int H(t')\left[1-i\int_0^{t'}H(t'')U(t'')dt''\right]dt'$

$=\cdots\cdots$

$U(t,0)=T\left[e^{-i\int_0^t H(t')dt'}\right]$ $\quad T$为编时算符

若 $H(t')=H$ 不含时，则 $U(t)=e^{-iHt}$

量子光学描述系统大多不含时.

量子力学中的三种图景：

(1) Schrödinger Picture

(2) Heisenberg Picture

(3) Interaction Picture

存在幺正变换

由 $i\partial_t|\psi\rangle=H|\psi\rangle=(H_0+H_1)|\psi\rangle$

$|\psi'\rangle=V(t)|\psi\rangle \qquad V^\dagger(t)V=VV^\dagger=1$

有：

$i\partial_t|\psi'\rangle=i\partial_t(V|\psi\rangle)=(i\partial_t V)|\psi\rangle+Vi\partial_t|\psi\rangle=(i\partial_t V)|\psi\rangle+VH|\psi\rangle$

$=(i\partial_t V)V^\dagger V|\psi\rangle+VHV^\dagger V|\psi\rangle=\left[i\partial_t VV^\dagger+VHV^\dagger\right]|\psi'\rangle=H'|\psi'\rangle$

Where $H' = VHV^\dagger + i\partial_t V V^\dagger = VHV^\dagger - iV\partial_t V^\dagger$

$\partial_t(VV^\dagger) = 0$

So $|\psi'\rangle = V|\psi\rangle$

$\partial_t V V^\dagger = -V\partial_t V^\dagger \qquad VV^\dagger = 1$

In Schrödinger Pic: $i\partial_t|\psi\rangle = H|\psi\rangle$

In Heisenberg Pic: $H' = 0$

态 $|\psi\rangle = V|\psi\rangle \qquad VHV^\dagger - iV\partial_t V^\dagger = 0 \Rightarrow i\partial_t V^\dagger = HV^\dagger$

$i\partial_t V^\dagger = HV^\dagger$ 类比时间演化算符 $i\partial_t U = HU$

$\Rightarrow |\psi'\rangle = U^\dagger|\psi\rangle = |\psi(0)\rangle$

In Interaction Pic

$i\partial_t|\psi'\rangle = (VHV^\dagger - iV\partial_t V^\dagger)|\psi'\rangle = (V(H_0+H_1)V^\dagger - iV\partial_t V^\dagger)|\psi'\rangle$

$\hat{H}' = VHV^\dagger - iV\partial_t V^\dagger$

相互作用表象 $\hat{H} = \hat{H}_0 + \hat{H}_1$

$\Rightarrow \hat{H}' = VHV^\dagger - iV\partial_t V^\dagger = V(\hat{H}_0 + \hat{H}_1)V^\dagger - iV\partial_t V^\dagger$

$= \underbrace{VH_0V^\dagger - iV\partial_t V^\dagger}_{=0} + VH_1V^\dagger$

$VH_0V^\dagger - iV\partial_t V^\dagger = 0$

$\Rightarrow V_0H^\dagger - i\partial_t V^\dagger = 0 \Rightarrow i\partial_t V^\dagger = H_0V^\dagger \Rightarrow V^\dagger = e^{-iH_0t},\ V = e^{iH_0t}$

由 $i\partial_t U = H_0 U \Rightarrow U = e^{-iH_0t}$

$H' = e^{iH_0t}H_1e^{-iH_0t}$

即 $H_{int} = e^{iU_0t}H_1e^{-iU_0t}$

在量子光学中看中态的演化：

$$i\partial_t|\psi\rangle = H|\psi\rangle \qquad 其中\ |\psi(t)\rangle = U(t,0)|\psi(0)\rangle$$

$$i\partial_t|\psi'\rangle = H'|\psi'\rangle \qquad |\psi(t)\rangle = U'(t,0)|\psi'(0)\rangle$$

$$V(t)|\psi(t)\rangle = U(t,0)V(0)|\psi(0)\rangle$$

$$\Rightarrow |\psi(t)\rangle = \underbrace{V^+(t)U'(t,0)V(0)}_{U(t,0)}|\psi(0)\rangle$$

算符平均值在任意表象下不变：

$$\langle\psi(t)|A|\psi(t)\rangle = \langle\psi_I(t)|A_I(t)|\psi_I(t)\rangle$$

$$\Rightarrow \langle\psi(0)|U^+AU|\psi(t)\rangle = \langle\psi_I(0)|U_I(t)A_I(t)U_I(t)|\psi_I(0)\rangle$$

$$\Rightarrow \qquad = \langle\psi(0)|V^+U_I(t)A_I(t)U_I(t)V|\psi(0)\rangle$$

$$\Rightarrow V^+AV = A_I$$

测不准关系：$[x,p] = i\hbar \qquad \Delta x^2\cdot\Delta p^2 \geq \frac{1}{4}$

证明：

$$|\psi'\rangle = (\xi F' + iG')|\psi\rangle$$

$F' = F - \langle F\rangle \qquad G' = G - \langle G\rangle$ 为厄米算符.

由于：$\langle\psi'|\psi'\rangle \geq 0$ 恒成立

$$\Rightarrow \langle\psi|(\xi F' - iG')(\xi F' + iG')|\psi\rangle \geq 0$$

$$\Rightarrow \langle\psi|(\xi F')^2 + (G')^2 + i\xi(F'G' - G'F')|\psi\rangle \geq 0$$

$$\Rightarrow \xi^2\langle\psi|(F')^2|\psi\rangle + \langle\psi|(G')^2|\psi\rangle + i\xi\langle\psi|F'G' - G'F'|\psi\rangle$$

$$\langle\psi|(F')^2|\psi\rangle = \langle\psi|(F-\langle F\rangle)^2|\psi\rangle = \langle\psi|F^2 - F\langle F\rangle - \langle F\rangle F + \langle F\rangle^2|\psi\rangle$$

$$= \langle\psi|F^2|\psi\rangle - \langle F\rangle^2 - \langle F\rangle$$

$$\Rightarrow \xi^2\Delta F^2 + \Delta G^2 + i\xi\langle\psi|[F,G]|\psi\rangle \geq 0 \quad 即\ K(\xi) \geq 0$$

$$\Rightarrow \quad b^2-4ac \le 0 \Rightarrow |i[F,G]|^2 - 4\Delta F^2 \Delta G^2 \le 0 \Rightarrow \Delta F^2 \Delta G^2 \ge \frac{1}{4}[F,G]^2$$

这个不等关系是紧致的吗？

定义：$|e\rangle = (A-\langle A\rangle)|\psi\rangle$

$|g\rangle = (B-\langle B\rangle)|\psi\rangle$

Schwartz 不等式：$\langle e|e\rangle\langle g|g\rangle \ge |\langle e|g\rangle|^2$

$\langle e|e\rangle = \langle\psi|(A-\langle A\rangle)^2|\psi\rangle = \Delta A^2 \qquad \langle g|g\rangle = \Delta B^2$

$\Delta A^2 \Delta B^2 \ge |\langle e|g\rangle|^2$

$$\langle e|g\rangle = \underbrace{\frac{1}{2}(\langle e|g\rangle + \langle g|e\rangle)}_{\text{实数}} + \underbrace{\frac{1}{2}(\langle e|g\rangle - \langle g|e\rangle)}_{\text{纯虚数}}$$

$$\Rightarrow |\langle e|g\rangle|^2 = \left|\frac{1}{2}(\langle e|g\rangle + \langle g|e\rangle)\right|^2 + \left|\frac{1}{2}(\langle e|g\rangle - \langle g|e\rangle)\right|^2$$

$\langle e|g\rangle = \langle BA\rangle - \langle A\rangle\langle B\rangle$

$$= \left|\frac{1}{2}(AB+BA-2\langle A\rangle\langle B\rangle)\right|^2 + \left|\frac{1}{2}(AB-BA)\right|^2$$

$$\Delta A^2 \Delta B^2 \ge \underbrace{\frac{1}{4}|AB+BA-2\langle A\rangle\langle B\rangle|^2}_{\ge 0} + \frac{1}{4}|[A,B]|^2$$

$\Delta A^2 \Delta B^2 \ge \frac{1}{4}|[A,B]|^2 \qquad$ 对 $[x,p]=i\hbar \quad \Delta x^2 \Delta p^2 \ge \frac{1}{4}\hbar^2$

爱因斯坦：　单个系统之对易　　两个系统之不对易？

$[x_1, p_1] = i \quad [x_2, p_2] = i \longrightarrow [x_1+x_2, p_1-p_2] = 0$

对于谐振子：$[x,p]=i\hbar \qquad \Delta x^2 \Delta p^2 \ge \frac{1}{4}$　取等号即压缩态

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时间演化：$i\hbar\partial_t|\psi\rangle = H(t)|\psi\rangle$

不含时系统 $H(t)=H(0)=H$

求解本征态：$H|n\rangle = E_n|n\rangle$

$$\left.\begin{aligned}|\psi(0)\rangle &= \sum_n C_n|n\rangle \\ |\psi(t)\rangle &= \sum_n C_n e^{iE_nt}|n\rangle\end{aligned}\right\} \Rightarrow |\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

含时系统：$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle$

能否将含时态按不含时态展开？量子绝热演化.

当$H(t)$满足绝热条件时

$$|\psi(t)\rangle = \sum_n C_n(t)|n,t\rangle$$

$$H|n,t\rangle = E_n(t)|n,t\rangle \quad \Rightarrow \quad i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$$\Rightarrow i\partial_t\Big(\sum_n C_n(t)|n,t\rangle\Big) = H\sum_n C_n|n,t\rangle = \sum_n C_n E_n|n,t\rangle$$

$$\Rightarrow \sum_n i(\partial_t C_n)|n,t\rangle + \underbrace{\sum_n C_n(i\partial_t|n,t\rangle)}_{\text{不含时为0，两边}\langle m,t|\text{可解}} = \sum_n C_n E_n|n,t\rangle$$

$$\Rightarrow \langle m,t|\sum_n i(\partial_t C_n)|n,t\rangle + \langle m,t|\sum_n C_n\, i\partial_t|n,t\rangle = \langle m,t|\sum_n C_n E_n|n,t\rangle$$

$$\Rightarrow i\partial_t C_m + \langle m,t|i\partial_t|m,t\rangle C_m + \underbrace{\sum_{n\neq m}\langle m,t|i\partial_t|n,t\rangle C_n}_{\text{舍去}} = C_m E_m$$

$$\Rightarrow i\partial_t C_m + \langle m,t|i\partial_t|m,t\rangle C_m = C_m E_m$$

$$\Rightarrow C_m(t) = \mathrm{Exp}\Big[-i\int_0^t E_n(t')dt' \underbrace{-i\int_0^t\langle m,t|i\partial_t|m,t\rangle dt'}_{\text{为一相位.}}\Big]C_m(0)$$

$H(t)|n,t\rangle = E_n(t)|n,t\rangle$ 绝热定理 app.量子退火

(1) 非简并

$i\frac{dU}{dt} = H(t)U$ 封闭体系(马尔柯夫)

$H(t) = (1-\frac{t}{T})\sum_i a_{x_i} + t\sum_{ij} a_{x_i} a_{x_j}$. $t\to 0$

几乎一直在基态, 要求能隙不封闭, 或能隙无穷大

绝热条件: $|\langle n|\partial_t|m\rangle| \ll 1$ 在多体体系下几乎不成立

Quantum Annealing → Classical Annealing. 优化.

D-wave公司 Bose采样

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Baker-Hausdorff公式:

$e^A B e^{-A}$

要么先解 $\psi(t)$

$\langle\psi(0)|\underbrace{e^{iHt} A e^{-iHt}}_{\text{要么解这个}}|\psi(0)\rangle = \langle\psi(t)|A|\psi(t)\rangle$

还有解 $D(\alpha)$

$e^B A e^{-B}$:

$i\frac{d}{dt}U = HU \longrightarrow U = 1 - i\int H(t') + \cdots = T\left[e^{-i\int H(t')dt'}\right]$

$f(x) = \exp(Bx)A\exp(-Bx)$ 当 $f(1) = \exp(B)A\exp(-B)$

$f(0) = A$

有 $\frac{\partial}{\partial x}f = \partial_x\left[e^{Bx}Ae^{-Bx}\right] = Be^{Bx}Ae^{-Bx} - e^{Bx}Ae^{Bx}B$

$$\Rightarrow \quad = e^{xB}(BA-AB)e^{-Bx} = e^{xB}[B,A]e^{-Bx} = [B, f(x)]$$

$$\Rightarrow \quad \frac{\partial}{\partial x} f(x) = [B, f(x)]$$

$$\Rightarrow \quad f(x) - f(0) = \int_0^x [B, f(x')]\, dx'$$

$$\Rightarrow \quad f(x) = A + \int_0^x [B, f(x')]\, dx'$$

$$\Rightarrow \quad f(x) = A + \int_0^x [B,A]\, dx' + \int_0^x \Big[B, \int_0^{x_1} [B, f(x_2)]\Big]\, dx_1\, dx_2$$

$\Rightarrow$ ....

$$\Rightarrow \quad f(x) = A + [B,A]x + [B,[B,A]] \int_0^x dx_1 \int_0^x dx_2 + \cdots + \overbrace{[B,[B,\cdots[B,A]\cdots]]}^{n\text{个对易子}} \int_0^x \cdots \int_0^{x_{n-1}} dx_1 \cdots dx_n$$

$$\Rightarrow \quad f(1) = e^B A e^{-B} = A + [B,A] + \cdots + \frac{1}{n!}[B[B\cdots[B,A]]\cdots]$$

即Haustoff公式

e.g. $D(\alpha) = \exp(\alpha a - \alpha^* a^+)$ 位移算符.

$$\Rightarrow \quad D(\alpha)\, a\, D(\alpha^*) = a + [\alpha a - \alpha^* a^+] = a + \alpha^*$$

$$e^{iQa^+a}\, a\, e^{-iQa^+a} = a + (-iQ)a + \cdots + \frac{1}{n!}(-iQ)^n a + \cdots = e^{-iQ} a$$ 相移

高阶项很小 $(-iQ)^n$ 下降.

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$U = e^{A(t)}$ $\quad i\frac{dU(t)}{dt} = H(t)U(t)$

已知 $U = e^{-iA(t)}$ 计算 $\frac{\partial}{\partial t}U = \frac{\partial}{\partial t} e^{-iA(t)}$

或 $U = e^{A(t)}$ 求 $\partial_t e^{A(t)}$

由 $e^A = \sum_n \frac{A^n}{n!} \Rightarrow \partial_t e^{A(t)} = \sum_n \partial_t \left(\frac{A^n}{n!}\right)$

已知 $\partial_t A^n = (\partial_t A)A^{n-1} + A(\partial_t A)A^{n-2} + \cdots + A^{n-1}(\partial_t A)$

$$\Rightarrow \partial_t e^{A(t)} = \sum_n \frac{1}{(n+1)!} \sum_{k=0}^{n} A^k (\partial_t A) A^n$$

再由 $\sum_{n=0}^{\infty}\sum_{k=0}^{n} f_{n,k} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} f_{n,k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f_{n+k,k}$

$$\Rightarrow \partial_t e^A = \sum_{nk} \frac{1}{(n+k+1)!} A^k (\partial_t A) A^n$$

$$\int_0^1 e^{\mu A} \frac{\partial}{\partial t} A e^{(1-\mu)A} d\mu = \int_0^1 \sum_n \frac{\mu^n A^n}{n!} \frac{\partial A}{\partial t} \sum_m \frac{(1-\mu)^m}{m!} A^m$$

$$= \sum_{nm} \int_0^1 \frac{\mu^n}{n!} \frac{(1-\mu)^m}{m!} d\mu \, A^n \frac{\partial A}{\partial t} A^m$$

已知：$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$　　$\Gamma(n)=n!$

所以：$\partial_t e^A = \int_0^1 e^{\mu A} \frac{\partial A}{\partial t} e^{(1-\mu)A} d\mu = \int_0^1 \underbrace{e^{\mu A} \frac{\partial A}{\partial t} e^{-\mu A}}_{\to\, e^B A e^{-B}} e^A$

于是：$\partial_t e^A = H e^A \Rightarrow \frac{dU}{dt} = HU$　　$H = \int_0^1 e^{\mu A} \frac{\partial A}{\partial t} e^{-\mu A}$

e.g. 绝热条件：$|\langle m|\partial_t|n\rangle| \ll 1$　　瞬时本征态

$H|m\rangle = E_m|m\rangle$

$\Rightarrow \frac{\partial H}{\partial t}|m\rangle + H\frac{\partial}{\partial t}|m\rangle = \underbrace{\frac{\partial E_m}{\partial t}|m\rangle}_{\to\, 0} + E_m \frac{\partial}{\partial t}|n\rangle$

左侧乘 $\langle n|$ 有

$\Rightarrow \langle n|\frac{\partial H}{\partial t}|m\rangle + E_n \langle n|\partial_t|m\rangle = E_m \langle n|\partial_t|m\rangle$

$\Rightarrow \langle n|\frac{\partial}{\partial t}H|m\rangle = (E_m - E_n)\langle n|\partial_t|m\rangle$

其中 $|\langle n|\partial_t|m\rangle| = \left|\frac{\langle n|\frac{\partial}{\partial t}H|m\rangle}{E_m - E_n}\right| \ll 1$　　即绝热条件.

---

$U = e^A$　　$\partial_t U = \underbrace{\int_0^1 e^{\mu A} \frac{\partial}{\partial t} A e^{-\mu A}}_{H} \underbrace{e^A}_{U}$　　$\partial_t U = HU$

---

Lewis 不变量理论（求解含时系统 $H(t)$）非简并系统.

$H(0)|n\rangle = E_n|n\rangle$

对含时 $H(t)$ 引入不变量 $I(t)$，有 $\frac{\partial I}{\partial t} = -i[I,H]$　　与 $\frac{\partial A}{\partial t} = i[A\ H]$ 正好相反.

海森堡算符

不含时

$I(t)$ 不变量，厄米含时，$\Rightarrow$ $I(t)|n\ t\rangle = \lambda_n|n,t\rangle$

验证：

求导：显然有 $\frac{\partial}{\partial t}I|n,t\rangle + I\frac{\partial}{\partial t}|n,t\rangle = \frac{\partial\lambda_n}{\partial t}|n,t\rangle + \lambda_n\frac{\partial}{\partial t}|n,t\rangle$

左乘 $\langle n,t|$ 有：$\langle n,t|\frac{\partial I}{\partial t}|n,t\rangle = -i\langle n,t|HI-IH|n,t\rangle$

$$= -i\lambda_n\langle n,t|H|n,t\rangle + i\lambda_n\langle n,t|H|n\ t\rangle = 0$$

$$\langle n,t|I\frac{\partial}{\partial t}|n,t\rangle = \lambda_n\langle n,t|\frac{\partial}{\partial t}|n,t\rangle$$

$\Rightarrow \frac{\partial\lambda_n}{\partial t} = 0 \Rightarrow \lambda_n$ 不含时.

$I(t)$ 与 $H(t)$ 的差别在于本征值不含时.

绝热条件：$\left|\frac{\langle n\ t|\partial_t|m\ t\rangle}{E_n - E_m}\right| \ll 1$

$\frac{\partial I}{\partial t}|n,t\rangle + I\frac{\partial}{\partial t}|n,t\rangle = \lambda_n\frac{\partial}{\partial t}|n,t\rangle$

左乘 $\langle m,t|$ 有：$\langle m,t|\frac{\partial I}{\partial t}|n,t\rangle + \langle m\ t|I\frac{\partial}{\partial t}|n,t\rangle = \lambda_n\langle m,t|\frac{\partial}{\partial t}|n,t\rangle$

$\Rightarrow -i(\lambda_m-\lambda_n)\langle m|H|n\rangle + \lambda_m\langle m\ t|\frac{\partial}{\partial t}|n,t\rangle = \lambda_n\langle m\ t|\frac{\partial}{\partial t}|n,t\rangle$

$\Rightarrow -i(\lambda_m-\lambda_n)\langle m|H|n\rangle = -(\lambda_m-\lambda_n)\langle m\ t|\frac{\partial}{\partial t}|n,t\rangle$ 要求非简并

$\Rightarrow \langle m\ t|\frac{\partial}{\partial t}|n\ t\rangle = i\langle m|H|n\rangle$

即跃迁只与 $H$ 有关，与时间无关.

对 $i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$ $\qquad |\psi\rangle = \sum_n C_n|n\ t\rangle$

有 $\sum_n i\frac{\partial}{\partial t}C_n|n,t\rangle + \sum_n C_n\, i\frac{\partial}{\partial t}|n,t\rangle = \sum_n C_n H|n\rangle$

左乘 $\langle m,t|$ 有：

$$i\frac{\partial}{\partial t}C_m + C_m\langle m|i\frac{\partial}{\partial t}|m\rangle + \sum_{n\neq m}\langle m|i\frac{\partial}{\partial t}|n\rangle = C_m\langle m|H|m\rangle + \sum_{n\neq m}C_n\langle m|H|n\rangle$$

相等

所以没有跃迁项，即 $i\frac{\partial}{\partial t}C_m + C_m\langle m|i\frac{\partial}{\partial t}|m\rangle = C_m\langle m|H|m\rangle$

求解得：$C_m(t) = \exp\left[-i\int_0^t\langle m|H|m\rangle - i\int_0^t\langle m|i\frac{\partial}{\partial t}|m\rangle\right]C_m(0)$

AA项，非绝热几何相

考虑简并系统（如 $\lambda_0=\lambda_1$）

$$i\frac{\partial}{\partial t}C_0 + C_0\langle 0|i\frac{\partial}{\partial t}|0\rangle + C_1\langle 1,t|i\frac{\partial}{\partial t}|0,t\rangle = \lambda_0 C_0$$

$$i\frac{\partial}{\partial t}C_1 + C_0\langle 0\,t|i\frac{\partial}{\partial t}|1\,t\rangle + C_1\langle 1\,t|i\frac{\partial}{\partial t}|1,t\rangle = \lambda_1 C_1$$

$$\Rightarrow i\frac{\partial}{\partial t}\begin{pmatrix}C_0\\C_1\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}C_0\\C_1\end{pmatrix} + \lambda_0\begin{pmatrix}C_0\\C_1\end{pmatrix}$$

e.g. $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2 \qquad m=m_0e^{-\gamma t}$ 含时系统求解

$$\frac{\partial}{\partial t}I = -i[I,H] \qquad I=\lambda_1p^2+\lambda_2(xp+px)+\lambda_3x^2$$

$$\Rightarrow \frac{\partial}{\partial t}I = \frac{\partial\lambda_1}{\partial t}p^2+\frac{\partial\lambda_2}{\partial t}(xp+px)+\frac{\partial\lambda_3}{\partial t}x^2$$

$$= -i\left[\lambda_1p^2+\lambda_2(xp+px)+\lambda_3x^2,\ \frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2\right] = -i(?)p^2+(?)x^2+(?)(px+xp)$$

由
$$\begin{cases}[p^2\ x^2]=-2i(px+xp)\\ [p^2\ xp+px]=\cdots\\ [x^2\ xp+px]=\cdots\end{cases}$$

$$\frac{\partial}{\partial t}\lambda_1=? \qquad \frac{\partial}{\partial t}\lambda_2=? \qquad \frac{\partial}{\partial t}\lambda_3=?$$

然后利用 $I(t)$ 求解 Schrödinger 方程.（非常难解）

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2 \qquad I(t)=\frac{1}{2}\left[\left(\frac{x}{\rho}\right)^2+\left(\rho p-\mu\frac{\partial\rho}{\partial t}x\right)^2\right]$$

$$\ddot\rho+\gamma\dot\rho+\omega^2=\frac{1}{M^2}\rho^3 \qquad \gamma=\frac{d}{dt}mM(t)$$

$$\Rightarrow I|n,t\rangle=\lambda_n|n,t\rangle$$

引入 $V=e^{-i\mu\frac{\dot\rho}{2\rho}x^2}$ 有 $VIV^\dagger=\left(\frac{x}{\rho}\right)^2+\left[\rho[p+\frac{\mu\dot\rho}{\rho}]\,\mu\dot\rho x\right]^2$

$$=\frac{1}{2}\left[\left(\frac{x}{\rho}\right)^2+(\rho p)^2\right]$$

记 $y=\frac{x}{\rho}$ 有：$I_n=\frac{1}{2}(p_y^2+y^2)\Rightarrow\lambda_n=n+\frac{1}{2}$

引入谐振子的好处 $(p^2\ x^2\ (xp+px))\Rightarrow SO(1,1)$ 代数 $\boxed{\text{封闭}}$

角动量: $L_x = xP_y - yP_x$ $L_y = yP_z - zP_y$ $L_z = zP_x - xP_z$

SU(2)代数 $\begin{cases}[S_x\ S_y] = iS_z \\ [S_y\ S_z] = iS_x \\ [S_z\ S_x] = iS_y\end{cases}$ Paoli 算符. $a_x$ $a_y$ $a_z$.

$H = a_z + \cos wt\ a_x$ 或 $H = S_z + \cos wt\ S_x$ Floquet 定理由该模型导出.

依旧非常难解.

总结: Lewis 不变量求解谐振子

$\frac{\partial I}{\partial t} = i[I, H]$ $I(t)|n\ t\rangle = \lambda_n |n\ t\rangle$ $H = \frac{p^2}{2m} + \frac{m w^2}{2} x^2$ $m(t)$

$\Rightarrow H = wS_z + G(S_+ e^{i\varphi(t)} + S_- e^{-i\varphi(t)})$

$[S_z\ S_\pm] = \pm S_\pm$ $[S_+, S_-] = 2S_z$

找 $I(t) = \lambda_x S_x + \lambda_y S_y + \lambda_z S_z$ 代入 $\frac{\partial}{\partial t} I = i[I, H]$

$\Rightarrow \frac{\partial}{\partial t}\begin{pmatrix}\lambda_x \\ \lambda_y \\ \lambda_z\end{pmatrix} = \begin{pmatrix}\cdots \\ \cdots \\ \cdots\end{pmatrix}\begin{pmatrix}S_x \\ S_y \\ S_z\end{pmatrix}$ $\Rightarrow I(t) = R(t) I_0 R(t)^+$ $R(t) = e^{-\frac{r}{2}(S_+ e^{-i\beta} - S_- e^{i\beta})}$

where $\begin{cases}\frac{\partial r}{\partial t} = 2G\sin(\theta + \beta) \\ \frac{1}{2}\left(\frac{\partial \beta}{\partial t} - w\right)\sin(r) = G\cos(r)\cos(\theta + \beta)\end{cases}$

$\Leftrightarrow \frac{\partial}{\partial t}\begin{pmatrix}C_1 \\ C_2\end{pmatrix} = \begin{pmatrix}w & Ge^{i\theta} \\ Ge^{-i\theta} & -w\end{pmatrix}\begin{pmatrix}C_1 \\ C_2\end{pmatrix}$

$I(t)$ 代回 $\frac{\partial I}{\partial t} = i[I, H]$ 得:

$\frac{\partial R}{\partial t} I_0 R^+ + R I_0 \frac{\partial R^+}{\partial t} = i[RI_0R^+, H] = iRI_0R^+H - iHRI_0R^+$

两边左乘 $R^+$ 右乘 $R$ 得:

$R^+ \frac{\partial R}{\partial t} I_0 R^+ R + R^+ I_0 \frac{\partial R^+}{\partial t} R = iR^+ R I_0 R^+ H R - iR^+ H R I_0 R^+ R$

$\Rightarrow R^+ \frac{\partial R}{\partial t} I_0 - I_0 R^+ \frac{\partial}{\partial t} R = iI_0 R^+ H R - iR^+ H R I_0$

$\Rightarrow [iR^+ \frac{\partial R}{\partial t} + R^+ H R, I_0] = 0$ 即 $[H_{new}, I_0] = 0$

---

$$H=\Delta\sigma_z+g\cos\omega t\,\sigma_x \Rightarrow \frac{\partial}{\partial t}\begin{pmatrix}c_0\\c_1\end{pmatrix}=\begin{pmatrix}\Delta & g\cos\omega t\\ g\cos\omega t & -\Delta\end{pmatrix}\begin{pmatrix}c_0\\c_1\end{pmatrix}\Rightarrow\begin{pmatrix}c_0(t)\\c_1(t)\end{pmatrix}=\sum_n\begin{pmatrix}c_{0n}\\c_{1n}\end{pmatrix}e^{in\omega t}$$

$H(t)=H(t+T)$ Floquet 定理

如果 $H(r)=H(r+R) \Longleftrightarrow H=\frac{p^2}{2m}+V(r) \quad V(r)=V(r+R)$ Bloch定理

固体材料中广泛存在

$H\psi(r)=E\psi(r)$

$\psi_{nk}=e^{ik\cdot r}u_{nk}(r)$ 其中 $\begin{cases}e^{ik\cdot R}=1\\ u_{nk}(r+R)=u_{nk}(r)\end{cases}$

Bloch定理：$H(t)=H(t+T)$

时间演化算符：$U(t,0)=\hat{T}e^{-i\int_0^t H(\tau)d\tau}$ $\qquad U(T,0)$ ?

由 $U(t_1\ t_2)U(t_2\ t_3)=U(t_1\ t_3) \Rightarrow U(t+T,\ t'+T)=U(t\ t')$

定义：$H_{eff}=H-i\frac{\partial}{\partial t} \Rightarrow H_{eff}(t+T)=H_{eff}(t)$

于是：$U(t+T,t)H_{eff}(t)=U(t+T,t)H_{eff}(t)U^+(t+T,t)U(t+T,t)$

$=H_{eff}(t+T)U(t+T,t)=H_{eff}(t)U(t+T,t)$

即 $[U(t+T,t)\ H_{eff}(t)]=0$ 两者拥有共同本征态。

$U^+U=I \qquad U(t+T,t)$ 幺正 $\qquad U(t+T,t)|\varphi(t)\rangle=e^{i\varphi(T)}|\varphi(t)\rangle$

显然：$U(t+nT,t)=[U(t+T,t)]^n=U(t+nT,\ t+(n-1)T)U(t+(n-1)T,t)$

$\Rightarrow e^{iQ(nT)}|\psi\rangle=U(t+nT,t)|\psi\rangle=e^{i\varphi(t)n}|\psi\rangle$

$\Rightarrow e^{iQ(n,T)}=e^{iQ(t)n} \Rightarrow Q$与$T$成正比 $\Rightarrow Q=-\varepsilon T$

$\Rightarrow |\psi(t)\rangle=e^{-i\varepsilon t}|u(t)\rangle$ 准能谱

$i\frac{\partial}{\partial t}|\psi\rangle=H|\psi(t)\rangle \qquad H(t)=H(t+T) \qquad$ 代入 $|\psi(t)\rangle=e^{-i\varepsilon t}|u(t)\rangle$

得 $(H-i\frac{\partial}{\partial t})|u(t)\rangle=\varepsilon u(t) \qquad$ where $|u(t)\rangle=\sum_n e^{in\omega t}u_n$ $\qquad$ Floquet定理

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Magnus 展开

$$\frac{dU}{dt} = A(t)U \Rightarrow U = \hat{T} e^{\int A(t)dt}$$

如何获得 $U(t)$ 具体形式？

或按 $U = e^{\Omega(t)}$ 展开？Magnus 展开

人为引入

$$\frac{dU}{dt} = \lambda A U$$

$$\Rightarrow U = 1 + \sum_{n=1}^{\infty} \lambda^n P_n(t) \Rightarrow \begin{cases} \frac{d}{dt} P_1 = A & \Rightarrow P_1 = \int A(t) dt \\ \frac{d}{dt} P_n = A P_{n-1} \end{cases}$$

$$\Rightarrow U = e^{\Omega} \quad \Omega = \ln U = \ln(1 + U - 1) = \ln\left(1 + \sum_{n=1}^{\infty} P_n\right)$$

$$= \sum_i P_i - \frac{1}{2} \sum_{ij} P_i P_j + \frac{1}{3} \sum_{ijk} P_i P_j P_k.$$

$$\Rightarrow \begin{cases} \Omega_1 = P_1 \\ \Omega_2 = P_2 - \frac{1}{2} P_1^2 \\ \Omega_3 = P_3 - \frac{1}{2}(P_1 P_2 + P_2 P_1) + \frac{1}{3} P_1^3 \end{cases} \Rightarrow \begin{cases} \Omega_1 = \int A(t_1) dt_1 \\ \Omega_2 = \frac{1}{2} \iint dt_1 dt_2 [A(t_1), A(t_2)] dt_1 dt_2 \\ \Omega_3 = \frac{1}{3!} \iiint dt_1 dt_2 dt_3 \left[ [A(t_1) [A(t_2) A(t_3)]] \quad [[A(t_1) A(t_2)] \quad A(t_3)] \right] \end{cases}$$

$$\Rightarrow \Omega = \Omega_1 + \Omega_2 + \cdots$$

$$\Omega_n = P_n - \sum_{k=2}^{n} \frac{(-1)^k}{k!} R_n^k \qquad R_n^k = \sum P_{i_1} P_{i_2} \cdots P_{i_k} \qquad (i_1 + \cdots + i_k) = n$$

能做展开的条件是满足微扰论，即 $|A\Delta t| \ll 1$

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杨振宁：η 配对. Lewis Invariant 求解谐振子 $[H\ A] = \omega A$

总结：

不变量理论：几何相 $\longleftrightarrow$ Floquet 定理. $\begin{cases} H(t+T) = H(t) \\ H(R(0)) = H(R(T)) \end{cases}$ 参数空间

$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \Rightarrow i\frac{\partial}{\partial R}|\psi\rangle \frac{\partial R}{\partial t} = H|\psi\rangle$ R 在参数空间内

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Hausdoff：$e^{B} A e^{B}$

如果 X 是 $N \times N$ 矩阵：$e^{\theta X} = \sum_{m=0}^{N-1} C_m X^m$

$$= \sum_{m=0}^{\infty} \frac{(\theta X)^m}{m!} = \sum_{m=0}^{\infty} \frac{\theta^m X^m}{m!}$$

有本征方程：$X^0 \cdots X^{N-1}$ $\prod_{i=1}^{N}(X-\lambda_i)=0$

$$\Rightarrow X^N = \sum_{m=0}^{N-1} a_m X^m$$

$e^{\theta X} = \sum_{m=0}^{N-1} C_m(\theta) X^m$ 对$\theta$求导

左 $X e^{\theta X} = X \sum_{m=0}^{N-1} C_m(\theta) X^m = \sum_{m=0}^{N-1} C_m(\theta) X^{m+1} = \sum_{m=0}^{N-2} C_{m-1} X^m + C_{N-1} X^N$

右 $= \sum_m \frac{\partial C_m}{\partial \theta} X^m = \sum_m C_{m-1} X^m + C_{N-1} \sum_m a_m X^m$

$\Rightarrow \frac{\partial C_m}{\partial \theta} = a_m C_{N-1} + C_{m-1}$ 初始条件 $e^{0X} = \sum_m C_m(0) X^m$

$[X_i \ X_j] = \sum_{ijk} \varepsilon_{ijk} X_k$

若 $Z = \sum_i \alpha_i X_i$ 试求 $X_i(\theta) = e^{-\theta Z} X_i e^{\theta Z}$

解：$\frac{\partial X_i}{\partial \theta} = -e^{-\theta Z} Z X_i e^{\theta Z} + e^{-\theta Z} X_i Z e^{\theta Z} = e^{-\theta Z}[Z \ X_i] e^{\theta Z}$

代入 $Z = \sum_i \alpha_i X_i$

得 $e^{-\theta Z}[Z \ X_i] e^{\theta Z} = e^{-\theta Z}[\sum_j \alpha_j X_j, X_i] e^{\theta Z} = \sum_j \alpha_j e^{-\theta Z}[X_j \ X_i] e^{\theta Z}$

$= \sum_j \alpha_j e^{-\theta Z}[-\sum_k \varepsilon_{ijk} X_k] e^{\theta Z}$

$= \sum_j \sum_k -\alpha_j \varepsilon_{ijk} e^{-\theta Z} X_k e^{\theta Z}$

$\Rightarrow \frac{\partial X_i(\theta)}{\partial \theta} = -\sum_{jk} \alpha_j \varepsilon_{ijk} X_k(\theta)$

$\Rightarrow \frac{\partial}{\partial \theta}\begin{pmatrix} X_1(\theta) \\ \vdots \\ X_m(\theta) \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}\begin{pmatrix} X_1(\theta) \\ \vdots \\ X_m(\theta) \end{pmatrix}$ 初始：$(X_1(0) \cdots X_m(0))^T = (X_1 \cdots X_m)^T$

$\Rightarrow (X_1(\theta) \cdots X_m(\theta))^T = e^{M\theta}(X_1 \cdots X_m)^T$

例：$[a \ a^+] = 1$ $N = a^+ a$ $Z = \alpha_1 a + \alpha_2 a^+ + \alpha_3 a^+ a$

$H = w a^+ a + g(a + a^+)$ 求 $a(\theta) = e^{-\theta Z} a e^{\theta Z}$

解：$\frac{\partial a}{\partial \theta} = \alpha_3 a + \alpha_2 \Rightarrow a(\theta) = e^{\alpha_3 \theta} a + \frac{\alpha_2}{\alpha_3}(e^{\alpha_3 \varphi} - 1)$

---

SU(2) 代数：

$$S_a = a\cdot S = a_x S_x + a_y S_y + a_z S_z$$

$$S_b = b\cdot S = b_x S_x + b_y S_y + b_z S_z$$

$$[S_a, S_b] = [a_x S_x + a_y S_y + a_z S_z\ ,\ b_x S_x + b_y S_y + b_z S_z]$$

$$= a_x b_y (iS_z) - a_x b_z (iS_y) = i(a\times b)\cdot S$$

$$- a_y b_x (iS_z) + a_y b_z (iS_x)$$

$$+ a_z b_x (iS_y) - a_z b_y (iS_x)$$

$$S_a(\theta) = e^{-i\theta n\cdot S}\, a\cdot S\, e^{i\theta n\cdot S} \qquad H = nS = n_x S_x + n_y S_y + n_z S_z$$

$$\Rightarrow \frac{\partial S_a}{\partial\theta} = -i e^{-i\theta n\cdot S}[n\cdot S, a\cdot S] e^{i\theta n\cdot S}$$

$$= e^{-i\theta n\cdot S}(n\times a)\cdot S\, e^{i\theta n\cdot S}$$

$$= \frac{\partial}{\partial\theta}(n\times S)\cdot S(\theta) = -i e^{-i\theta n\cdot S}[n\cdot S, (n\times a)\cdot S] e^{i\theta n\cdot S}$$

由 $[n\cdot S, (n\times a)\cdot S] = i(n\times(n\times a))\cdot S$ 再由 $a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$

$\Rightarrow \qquad = i[(n\cdot a)n - a]\cdot S$

所以 $$= e^{-i\theta n\cdot S}[(n\cdot a)n\cdot S - a\cdot S] e^{i\theta n\cdot S}$$

$$\Rightarrow \frac{\partial}{\partial\theta}(n\times S)\cdot S(\theta) = -S_a + (a\cdot n)(n\cdot S(\theta))$$

$$e^{-i\theta n\cdot S}\, a\cdot S\, e^{i\theta n\cdot S} = \cos\theta\, a\cdot S + (n\times a)\cdot S\sin\theta + (1-\cos\theta)(n\cdot S)(n\cdot a)$$

任意变换 $e^{VHV^+} = Ve^H V^+$ where $W^+ = V^+V = I$

证：$e^{VHV^+} = \sum_{m=0}^{\infty}\frac{(VHV^+)^m}{m!} = \sum_{m=0}^{\infty}\frac{VH^mV^+}{m!} = V\left(\sum_{m=0}^{\infty}\frac{H^m}{m!}\right)V^+ = Ve^HV^+$ 证毕

对 SU(2)

$$H = V_1 V_2 S_z V_2^+ V_1^+ \qquad V_2 = e^{i\theta S_y} \qquad V_1 = e^{i\phi S_z}$$

$$e^{i\theta H} = e^{i\theta V_1 V_2 S_z V_2^+ V_1^+} = V_1 V_2 e^{i\theta S_z} V_2^+ V_1^+$$

$n\cdot S = S_x \quad a\cdot S = S_z$ 解 $e^{-i\theta S_x} S_z e^{i\theta S_x} = \cos\varphi S_z + S_y \sin\varphi$

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SU(1,1) 代数：

$[S_x\ S_y] = iS_z \quad [S_y\ S_z] = iS_x \quad [S_z\ S_x] = iS_y$ SU(2)群

$[k_x\ k_y] = -ik_z \quad [k_y\ k_z] = ik_x \quad [k_z\ k_x] = ik_y$ SU(1,1)群

可描述谐振子系统 $k_x = a^+a \quad k_y = (a^2 + a^{2+}) \quad k_z = (a^2 - a^{2+})$

重新定义点积： $a\cdot b = a_x b_x + a_y b_y - a_z b_z$

叉积： $(a\times b)_i = -\sum_{jk} \varepsilon_{ijk} a_j b_k \quad (i\neq z)$ $\varepsilon_{ijk} = \begin{cases} 1 & \text{偶置换} \\ -1 & \text{奇置换} \end{cases}$

$(a\times b)_z = \sum_{jk} \varepsilon_{zjk} a_j b_k$

$[a\cdot k, b\cdot k] = i(a\times b)\cdot K$

$e^{-i\theta n\cdot k} a\cdot k\, e^{i\theta n\cdot k} = \cosh\theta\, a\cdot k + (n\times a)\cdot k \sinh\theta - (\cosh\theta - 1)(n\cdot a)(n\cdot k)$

$e^{-i\theta n\cdot S} a\cdot S\, e^{i\theta n\cdot S} = \cos\theta\, a\cdot S + (n\times a)\cdot S \sin\theta + (1-\cos\theta)(n\cdot S)(n\cdot a)$ SU(2)

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绝热定理补充：

Transitionless(无跃迁) driving approach 2009年 M.V. Berry. J. Phys A

Shortcut to transfer

$H(t)|n,t\rangle = E_n(t)|n,t\rangle$

$|n(0)\rangle \xrightarrow{\text{绝热}} |n,t\rangle$ 绝热条件 $|\langle n(t)|\frac{\partial}{\partial t}|m(t)\rangle| \ll 1$

若用演化算符表示： $U(t) = \sum_{n=0}^{\infty} e^{i\alpha_n(t)} |n(t)\rangle\langle n(0)|$ 绝热条件 $|\langle n(t)|\frac{\partial}{\partial t}|m(t)\rangle| \ll 1$

Berry：满足时间演化算符的 Hamitian 量是什么？

$i\frac{\partial}{\partial t} U = \hat{H} U \Rightarrow H = i\frac{\partial}{\partial t} U U^+$

$\Rightarrow \left[\sum_n \left(-\frac{\partial}{\partial t}\right) e^{i\alpha_n(t)} |n(t)\rangle\langle n(0)| + \sum_n e^{i\alpha_n(t)}\, i\frac{\partial}{\partial t}|n(t)\rangle\langle n(0)|\right]\left[\sum_n e^{-i\alpha_n(t)} |n(0)\rangle\langle n(t)|\right]$

$= \sum_n \left(-\frac{\partial \alpha_n}{\partial t}\right)|n(t)\rangle\langle n(t)| + \sum_n \frac{\partial}{\partial t}|n(t)\rangle\langle n(t)|$

其中 $\alpha_n(t) = -\int_0^t E_n(t) + i\underbrace{\int \langle n_0(t)|\frac{\partial}{\partial t}|n_0(t)\rangle}_{\text{几何相}}$

$$\Rightarrow H(t) = H_0(t) + i\sum_n \left[ |\partial_t n(t)\rangle\langle n(t)| - \langle n(t)|\partial_t|n(t)\rangle\, |n(t)\rangle\langle n(t)| \right]$$

其中 $H_0(t) = \sum_n E_n(t)\,|n(t)\rangle\langle n(t)|$ 瞬时本征量: $H_0(t)|n(t)\rangle = E_n|n(t)\rangle$

设 $U(t) = \sum_n e^{i\alpha_n(t)}|n(t)\rangle\langle n(0)|$

若无几何相 $\alpha_n(t) = -\int_0^t E_n(t)$, 则 $H(t) = H_0(t) + i\sum_n \partial_t|n(t)\rangle\langle n(t)|$

正命题 ↓

$\begin{cases} \text{绝热定理}: H_0(t)|n(t)\rangle = E_n|n(t)\rangle \\ \text{绝热条件}: |\langle n(t)|\partial_t|m(t)\rangle| \ll 1 \end{cases}$

$|n(0)\rangle \Rightarrow e^{i\alpha_n(t)}|n(t)\rangle \qquad U(t) = \sum_n e^{i\alpha_n(t)}|n(t)\rangle\langle n(0)|$

↑ 逆命题 Berry 在 2009年提出. $H = H_0 + G(t)$

$I(t)|n(t)\rangle = \lambda_n|n(t)\rangle$
$\langle n|\partial_t|m\rangle = 0$ Lewis 不变量自然满足.

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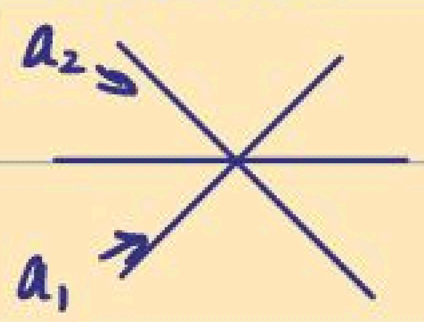
SU(2) 泡利算符. e.g. 二能级系统.

SU(m) 代数: $\{A_{ij}\}$ $i,j = 1\cdots m$ e.g. M能级系统.

$$[A_{ij}, A_{kl}] = \delta_{jk}A_{il} - \delta_{il}A_{kj} \qquad N = \sum_i A_{ii} \quad [N, A_{ij}] = 0$$

e.g. 分束器:




产生 $a_1^+a_1$ $a_2^+a_2$ $a_1^+a_2$ $a_1a_2^+$ $\qquad A_{ij} = a_i^+a_j$ $ij = 1, 2$

守恒量 $N = a_1^+a_1 + a_2^+a_2$

扩广至M个入射模: 守恒量 $N = \sum_{ij} a_i^+a_j$ $\quad A_{ij} = a_i^+a_j$ $\quad ij = 1\cdots M$.

$$A_{kl}(\theta) = e^{-iZ}A_k e^{iZ} \qquad Z = \sum_{ij}\alpha_{ij}A_{ij}$$

$$\frac{\partial A_{kl}}{\partial\theta} = e^{-iZ}[iZ, A_{kl}]e^{iZ} = \sum_{jk}\left(\quad\right)A_{kl}(\theta)$$

即 $\frac{\partial}{\partial\theta}\left(A_{kl}\right)_{n\times n} = \left(\quad\right)\left(\quad\right)_{n\times 1}$

利用 $A_{kl} = a_k^+a_l$ Bose 记法:

$$e^{-iZ}A_{kl}e^{iZ} = e^{-iZ}a_k^+ a_l e^{iZ} = e^{-iZ}a_k^+ e^{iZ} e^{-iZ} a_l e^{iZ}$$

$\Rightarrow e^{iZ}a_k e^{-iZ}$ $\quad a_k(\theta) = e^{i\theta Z} a_k e^{-iZ\theta}$

$\Rightarrow \frac{d}{d\theta} a_k = e^{i\theta Z}[iZ, a_k]e^{-i\theta Z}$ $\quad Z = \sum_{ij} \alpha_{ij} a_i^+ a_j$

由 $[i\sum_{ij}\alpha_{ij} a_i^+ a_j , a_k] = i\sum_{ij}\alpha_{ij}[a_i^+ a_j , a_k]$

$\Rightarrow \frac{d}{d\theta} a_k = i\sum_j \alpha_{kj} a_j(\theta)$ 即解: $\frac{d}{d\theta}\begin{pmatrix} a_1(\theta) \\ \vdots \\ a_m(\theta)\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}_{m\times m}\begin{pmatrix} a_1 \\ \vdots \\ a_m\end{pmatrix}$

---

$SU(m\ n)$ 代数:

$\{a_k\ a_k^+\ b_p\ b_p^+\}$ $\quad k=1\cdots m \quad p=1\cdots n.$ $\quad$ SU(1 1) 代数

其中 $[a_k\ a_k^+]=1 \quad [b_p\ b_p^+]=1$

$X_{jk} = a_j^+ a_k \quad Y_{pq} = b_p^+ b_q \quad Z_{jp} = a_j b_p \quad Z_{jp}^+ = a_j^+ b_p^+$

算符是线性算符，物理过程为非线性过程.

$N = \sum_i a_i^+ a_i - \sum_p b_p^+ b_p$

$Z = \sum_{ij}\alpha_{ij} a_i^+ a_j + \sum_{pq}\beta_{pq} b_p^+ b_q + \sum_{ip}\gamma_{ip} a_i b_p + \sum_{ip}\gamma_{ip} a_i^+ b_p^+$

计算 $e^{iZ} a_i a_j e^{-iZ} = e^{iZ} a_i e^{-iZ} e^{iZ} a_j e^{-iZ}$

$a_i(\theta) = e^{-i\theta Z} a_i e^{i\theta Z}$

$\frac{\partial}{\partial\theta} a_i = e^{-i\theta Z}[Z, a_i]e^{i\theta Z}$ $\quad$ 其中 $[Z, a_i] = a_j + b_p^+$

于是 $\frac{d}{d\theta} b_p^+ = b_p^+ \quad a_i \Rightarrow \frac{\partial}{\partial\theta}\begin{pmatrix} a_1 \\ \vdots \\ a_m \\ b_1 \\ \vdots \\ b_p^+\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}_{(m+n)^2}\begin{pmatrix} \\ \end{pmatrix}$

$S_+ = a_1^+ a_2 \quad S_- = a_1 a_2^+ \quad S_z = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2)$

$a_i(\theta) = e^{i\theta Z} a_i e^{-i\theta Z}$ $\quad Z = w_1 a_1^+ a_1 + w_2 a_2^+ a_2 + \xi_+ a_1^+ a_2 + \xi_- a_1 a_2^+$

$\Rightarrow \begin{cases} i\frac{d}{d\theta} a_1 = w_1 a_1 + \xi_+ a_2 \\ i\frac{d}{d\theta} a_2 = w_2 a_2 + \xi_- a_1 \end{cases} \Rightarrow i\frac{d}{d\theta}\begin{pmatrix} a_1 \\ a_2\end{pmatrix} = \begin{pmatrix} w_1 & \xi_+ \\ \xi_- & w_2\end{pmatrix}\begin{pmatrix} a_1 \\ a_2\end{pmatrix}$

简化：$w_1=w_2=0 \Rightarrow \begin{cases} a_1(\theta)=\cos\sqrt{\xi_+\xi_-}\,\theta a_1 - i\sqrt{\frac{\xi_+}{\xi_-}}\sin\sqrt{\xi_+\xi_-}\,\theta a_2 \\ a_2(\theta)=\cos\sqrt{\xi_+\xi_-}\,\theta a_2 + i\sqrt{\frac{\xi_+}{\xi_-}}\sin\sqrt{\xi_+\xi_-}\,\theta a_1 \end{cases}$

$= e^{-i\theta Z}[Z, a_1^+a_2]\, e^{i\theta Z}$

计算 $e^{i\theta Z}a_1^+a_2e^{-i\theta Z} = e^{i\theta Z}a_1^+e^{-i\theta Z}\,e^{i\theta Z}a_2e^{-i\theta Z} = e^{-i\theta Z}[Z\ a_1^+]e^{i\theta Z}\,e^{-i\theta Z}[Z\ a_2]\,e^{i\theta Z}$

---

$H=w_a a^+a + w_b b^+b + g(ab+a^+b^+)$

SU(1,1) 代数 $k_+=a^+b^+$ $\quad k=ab$ $\quad k_z=\frac{1}{2}(a^+a-b^+b+1)$

$H=w_a a^+a + w_b b^+b + g(a^+b+ab^+)$ 只有粒子数交换,无产生

SU(2) 代数 $S_+=a^+b$ $\quad S_-=ab^+$ $\quad S_z=\frac{1}{2}(a^+a-b^+b)$

在SU(1,1)代数下：$e^{i\theta Z}Ae^{-i\theta Z}$ $\quad Z=w_a a^+a+w_b b^+b+\xi_+a^+b^+ +\xi_- ab$ (多模光子)

$\begin{cases} i\frac{da}{d\theta}=w_a a+\xi_+b^+ \\ i\frac{db^+}{d\theta}=-w_b b^+-\xi_- a \end{cases}$ $\quad w_a=w_b=0$

$\Rightarrow \begin{cases} a(\theta)=\cosh\left(\sqrt{\xi_+\xi_-}\,\theta\right)a - i\sqrt{\frac{\xi_+}{\xi_-}}\sinh\left(\sqrt{\xi_+\xi_-}\,\theta\right)b^+ \\ b^+(\theta)=\cosh\left(\sqrt{\xi_+\xi_-}\,\theta\right)b^+ + i\sqrt{\frac{\xi_+}{\xi_-}}\sinh\left(\sqrt{\xi_+\xi_-}\,\theta\right)a \end{cases}$

$|\psi\rangle = e^{-igt(ab+a^+b^+)}|0\rangle|0\rangle$

$\Rightarrow e^{i\theta Z}Ae^{-i\theta Z} = \langle\psi_0|e^{igt(a^+b^++ab)}A\,e^{-igt(ab+a^+b^+)}|\psi_0\rangle$

$H=w a^+a+g(a^2+a^{+2})$ $\quad k_+=\frac{1}{2}a^{+2}$ $\quad k_-=\frac{1}{2}a^2$ $\quad k_z=\frac{1}{2}(a^+a+\frac{1}{2})$ (单模光子)

$A(\theta)=e^{i\theta Z}Ae^{-i\theta Z}$ $\quad Z=wa^+a+\frac{\xi_+}{2}a^{+2}+\frac{\xi_-}{2}a^2$

$\begin{cases} i\frac{da}{d\theta}=wa+\xi_+a^+ \\ i\frac{da}{d\theta}=-wa^+-\xi_-a \end{cases} \Rightarrow i\frac{d}{d\theta}\begin{pmatrix}a\\a^+\end{pmatrix}=\underbrace{\begin{pmatrix}w & \xi_+\\ \xi_- & -w\end{pmatrix}}_{\text{非厄米}}\begin{pmatrix}a\\a^+\end{pmatrix}$

$\xi_+=\xi_-=$ 常数 $\quad Z=wa^+a+\frac{\xi}{2}(a^2+a^{+2}) \Rightarrow Z=Z^+$ 厄米

$a(\theta)=\cosh\left(\sqrt{\xi_+\xi_-}\,\theta\right)a - i\sqrt{\xi_+\xi_-}\ \sin\left(\sqrt{\xi_+\xi_-}\,\theta\right)a^+$

---

Bose 相干态 $D(\alpha) = e^{\alpha(a-a^+)} = e^{-\alpha a} e^{-\alpha a^+} e^{-\frac{|\alpha|^2}{2}}$

* $e^{\varphi\sum_i \alpha_i x_i} = e^{f_1 x_1} e^{f_2 x_2}\cdots e^{f_n x_n}$ $\quad f_i(0)=0 \quad i=1,\cdots,n$

$$\frac{\partial}{\partial\theta}\left[e^{\theta\sum_i\alpha_i x_i}\right] = \sum_i \alpha_i x_i e^{\theta\sum_i \alpha_i x_i} = \frac{\partial f_1}{\partial\theta}x_1 e^{f_1x_1}e^{f_2x_2}\cdots e^{f_nx_n} + e^{f_1x_1}\frac{\partial f_2}{\partial\theta}x_2 e^{f_2x_2}\cdots e^{f_nx_n}+\cdots$$

$$= \sum_i \alpha_i x_i e^{f_1x_1}e^{f_2x_2}\cdots e^{f_nx_n}$$

$$\Rightarrow \sum_i\alpha_i x_i = \frac{\partial f_1}{\partial\theta}x_1 e^{f_1x_1}e^{f_2x_2}\cdots e^{f_nx_n}\times e^{-f_nx_n}\times\cdots\times e^{-f_1x_1} + e^{f_1x_1}\frac{\partial f_2}{\partial\theta}x_2e^{f_2x_2}\cdots e^{f_nx_n}e^{-f_nx_n}\cdots e^{-f_2x_2}e^{-f_1x_1}+\cdots$$

$$\Rightarrow \sum_i\alpha_i x_i = \frac{\partial f_1}{\partial\theta}x_1 + \frac{\partial f_2}{\partial\theta}\underbrace{e^{f_1x_1}x_2e^{-f_1x_1}}_{\text{求解Haustoff公式.}} + \frac{\partial f_3}{\partial\theta}e^{f_1x_1}e^{f_2x_2}x_3e^{-f_2x_2}e^{-f_1x_1}+\cdots+\frac{\partial f_n}{\partial\theta}e^{f_1x_1}\cdots e^{f_nx_n}x_ne^{-f_nx_n}\cdots e^{-f_1x_1}$$

$[a\ \ a^+]=1 \qquad e^{\theta[\alpha_1 a+\alpha_2 a^+a+\alpha_3 a^+]} = e^{f_1a^+}e^{f_2a^+a}e^{f_3a}e^{f_4}$

$$\begin{cases}\frac{\partial f_1}{\partial\theta} - f_1\frac{\partial f_2}{\partial\theta} = \alpha_3 \\ \frac{\partial f_2}{\partial\theta}=\alpha_2 \qquad \frac{\partial f_3}{\partial\theta} = \alpha_1 e^{\alpha_2\theta}\end{cases} \Rightarrow f_1= \quad f_2= \quad f_3= \quad f_4=$$

$$e^{\alpha_1a+\alpha_3a^+} = e^{\alpha_3a^+}e^{\alpha_1a}e^{\frac12\alpha_1\alpha_3} = e^{\alpha_1a}e^{\alpha_3a^+}e^{-\frac12\alpha_1\alpha_3}$$

SU(2)代数 $e^{i\theta[\alpha_+S_++\alpha_-S_-+\alpha_zS_z]} = e^{f_+S_+}e^{f_zS_z}e^{f_-S_-} = e^{\varphi_-S_-}e^{\varphi_zS_z}e^{\varphi_+S_+}$

$$\Rightarrow\begin{cases}\frac{\partial f_+}{\partial\theta} - f_+\frac{\partial f_z}{\partial\theta} - f_+^2\frac{\partial f_-}{\partial\theta}e^{f_z} = i\alpha_+ \\ \frac{\partial f_z}{\partial\theta}+2f_+\frac{\partial f_-}{\partial\theta}e^{-f_z} = i\alpha_z \\ \frac{\partial f_-}{\partial\theta}e^{-f_z} = i\alpha_-\end{cases} \Rightarrow \begin{cases} f_+(\theta) = \frac{i\alpha_+}{T_1}\frac{\sin T_1\theta}{\cos T_1\theta - i\frac{\alpha_z\sin T_1\theta}{2T_1}} \qquad T_1=\sqrt{\alpha_+\alpha_- + \frac{\alpha_z^2}{4}} \\ f_z(\theta) = -2\ln\left[\cos T_1\theta - \frac{i\alpha_z}{2T_1}\sin T_1\theta\right] \\ f_-(\theta) = \frac{i\alpha_-}{T_1}\frac{\sin T_1\theta}{\cos T_1\theta - \frac{i\alpha_z\sin T_1\theta}{2T_1}}\end{cases}$$

当 $\alpha_z=0 \quad \alpha_+=\alpha_- \Rightarrow \begin{cases} f_z = -2\ln(\cos\alpha\theta) \\ f_- = i\tan(\alpha\theta) \qquad f_+ = i\tan(\alpha\theta)\end{cases}$

---

$$e^{\theta aa^+} = \sum_{m=0}^{\infty} \frac{X^m(\theta)}{m!}(a^+)^m(a)^m \Rightarrow \sum_m \frac{\theta(a^+a)^m}{m!}$$

$$\langle\alpha|(a^+a)^n|\alpha\rangle = \langle\alpha|a^+a\,a^+a\cdots a^+a|\alpha\rangle$$

$$\frac{\partial}{\partial\theta}(e^{\theta a^+a}) = a^+ae^{\theta a^+a} = \sum_{m=0}^{\infty}\frac{X^{m-1}}{(m-1)!}\frac{\partial X}{\partial\theta}(a^+)^m(a)^m$$

$$\Rightarrow \sum_m\frac{X^m}{m!}a^+a(a^+)^m(a)^m = \sum_m\frac{X^{m-1}}{(m-1)!}\frac{\partial X}{\partial\theta}(a^+)^m(a)^m$$

由 $[a\ a^+]=1 \Rightarrow [a\ (a^+)^m] = m(a^+)^{m-1}$

有 $\Rightarrow = \sum_m \frac{x^m}{m!}(a^+)^{m+1}a^{m+1} + \frac{x^m}{(m-1)!}(a^+)^m a^m = \sum_m \frac{x^{m-1}}{(m-1)!}\frac{\partial x}{\partial \theta}(a^+)^m a^m$

$\frac{\partial x}{\partial \theta} = x+1 \quad x(0)=0 \quad x(\theta)=e^\theta - 1$

$|0\rangle\langle 0| = \rho = \frac{e^{\theta a^+ a}}{\sum_n \xi^n} = \sum_m (\xi-1)^m / \sum_m \xi^m \ (a^+)^m a^m$ 其中 $\xi = e^\theta$

$\theta \to -\infty$ 时 $= \sum_m (-1)^m (a^+)^m a^m$

$|0\rangle\langle 0| = \sum_m (-1)^m (a^+)^m a^m$

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散射理论：

$[H_0+V]|\psi\rangle = E|\psi\rangle$ 求解束缚态(通常) $\psi(r\to\infty)=0$ 近散射核

$H_0|\varphi\rangle = E|\varphi\rangle$ 散射态 $\varphi(r\to\infty)\neq 0$ 无穷远处

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$H = \frac{p^2}{2m} - \frac{e^2}{r} = \sum_n E_n |n\rangle\langle n|$

缀饰态 Dressed State

$H|\psi\rangle = \sum_n E_n|n\rangle\langle n| \approx E_1|1\rangle\langle 1| + E_0|0\rangle\langle 0|$

光与物质相互作用：

$\begin{cases} H = \frac{p^2}{2m} + V(r) & \text{原子} \\ E = -\nabla\phi - \frac{\partial A}{\partial t} \quad B = \nabla\times A & \text{光} \end{cases} \Rightarrow H = \frac{1}{2m}(p-eA)^2 + V(r) + e\phi(r)$ $E = -\nabla\phi - \frac{\partial A}{\partial t}$ $B = \nabla\times A$

简化 $e=1$ 有 $H = \frac{1}{2m}(p-A)^2 + V(r) + \phi(r)$

$i\frac{\partial}{\partial t}\psi = H\psi = \left[\frac{1}{2m}(p-A)^2 + V(r) + \phi(r)\right]\psi$

有幺正变换 $\psi' = e^{if}\psi \Rightarrow i\frac{\partial}{\partial t}\psi' = \left[\frac{1}{2m}(p-A')^2 + V(r) + \phi'\right]\psi'$

若 $f = -r\cdot A$ 有 $A' = A + \nabla f = 0$ 且 $\phi' = \phi - \frac{\partial A}{\partial t} = r\cdot E$ 电偶矩近似

$\Rightarrow H' = \frac{p^2}{2m} + V(r) + r\cdot E = H_0 + H_{int}$ 通常原子更适用(原子尺寸 << 光波长)

$H' = \sum_n E_n|n\rangle\langle n| + \sum_{n,m} |n\rangle\langle n| r |m\rangle\langle m| E(r,t)$

$= \underbrace{\sum_n E_n|n\rangle\langle n|}_{\text{无相互作用}} + \sum_{n,m} \underbrace{\langle n|r|m\rangle}_{\text{跃迁矩阵元}} |m\rangle\langle n| E(r,t)$

跃迁矩阵元：$\langle n|r|m\rangle = 0 \Leftrightarrow |n\rangle$ 和 $|m\rangle$ 不能跃迁

$$\langle n|r|m\rangle = \int \varphi_n(r)\,\hat{r}\,\varphi_m(r)\,d^3r$$

电场：$\vec{E} = \vec{e}_x E_x + \vec{e}_y E_y + \vec{e}_z E_z$

记 $\begin{cases} \vec{e}_+ = \vec{e}_x + i\vec{e}_y & \text{左旋} \\ \vec{e}_- = \vec{e}_x - i\vec{e}_y & \text{右旋} \end{cases} \Rightarrow r\cdot E = \rho \cdot Y_{1,\pm 1}(\theta\ \phi)$

$\langle n|r\cdot E|m\rangle = \int \langle Y_{L'm_L'}|Y_{1q}|Y_{L\,m_L}\rangle$

$\begin{cases} q=1 & \Delta m = m_L - m_L' = +1 & a_+ \text{右偏振} \\ q=-1 & \Delta m = m_L - m_L' = -1 & a_- \text{左偏振} \end{cases}$

$q=0 \quad \Delta m = 0 \quad$ Z 光

电偶极近似下，原子与场耦合：$\hat{H} = \frac{p^2}{2m} + V(r) + r\cdot E = \sum_n E_n|n\rangle\langle n| + \sum_n |n\rangle\langle n|\,r\cdot E\,|m\rangle\langle m|$

两能级系统：$\Rightarrow \underline{E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1|} + \underline{|0\rangle\langle 0|r|1\rangle\langle 1|E + |1\rangle\langle 1|r|0\rangle\langle 0|E}$

$\cancel{\frac{E_1+E_0}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)} + \frac{E_1-E_0}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|)$ $\quad \Omega(|0\rangle\langle 1| + |1\rangle\langle 0|)E(t) \quad E(t) = \quad \varepsilon e^{i\omega_0 t} + \varepsilon e^{-i\omega_0 t}$

$\Rightarrow \hat{H} = \omega_0 a_z + g(a_+ + a_-)\cos\omega t \qquad$ 求解？Floquet 定理？

1990年以前大多采用近似方法，90年以后多用数值计算

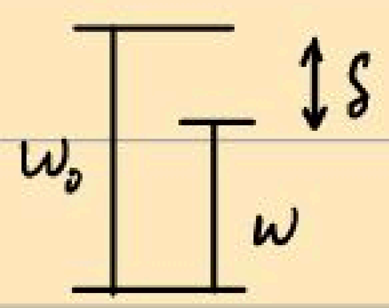
相互作用绘景：

$H = H_0 + H_1 \qquad H_{int} = e^{iH_0 t} H_1 e^{-iH_0 t} \qquad$ （要求 $H_0 \gg H_1$）

$\Rightarrow H_{int} = g a_+ e^{i(\omega_0 - \omega)t} + g a_+ e^{i(\omega_0 + \omega)t} + c.c.$

$\quad = g a_+ e^{i\delta t} + g a_+ e^{i(\omega + \omega_0)t} + c.c.$

其中 $\delta = \omega_0 - \omega$




旋波近似（RWA）：略去高频振荡 $g a_+ e^{i(\omega+\omega_0)t}$

$(\omega + \omega_0 \gg g\ ,\ \omega - \omega_0 \sim g)$

$\Rightarrow H_{int} = g a_+ e^{i\delta t} + c.c. \quad \Rightarrow H = \omega_0 a_z + g a_x \cos\omega t$

(1) 共振情况下：$\delta = 0$ 有：

$H = g a_x \Rightarrow e^{-iHt} = \cos gt - i\sin gt\, a_x \qquad |\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$

$\begin{cases} |\psi(0)\rangle = |e\rangle \\ |\psi(t)\rangle = \cos gt\,|e\rangle - i\sin gt\,|g\rangle \end{cases} \Rightarrow P(t) = |\langle e|\psi\rangle|^2 = \cos^2 gt \quad$ 拉比振荡.

(2) 非共振情况：$\delta \neq 0$.

$\hat{H} = \delta a_z + g a_x$

$\delta \gg g$ 时发生大失谐： $U(t) = e^{-i\hat{H}t} = e^{-i(\delta a_z + g a_x)t}$

由 $\delta a_z + g a_x = \sqrt{\delta^2+g^2}\left[\underbrace{\frac{\delta}{\sqrt{\delta^2+g^2}}}_{\cos\theta} a_z + \underbrace{\frac{g}{\sqrt{\delta^2+g^2}}}_{\sin\theta} a_x\right] = \sqrt{\delta^2+g^2}\, V a_z V^+$ 其中 $V(\theta) = e^{i\frac{\theta}{2}a_y}$

$\Rightarrow U(t) = e^{-i\sqrt{\delta^2+g^2}\, t\, V a_z V^+} = V\left[e^{-i\sqrt{\delta^2+g^2}\, t\, a_z}\right]V^+ = V\left[\cos\sqrt{\delta^2+g^2}\, t - i\sin\sqrt{\delta^2+g^2}\, t\, a_z\right]$

$= \cos\sqrt{\delta^2+g^2}\, t - i\sin\sqrt{g^2+\delta^2}\, t\left[\cos\theta\, a_z + \sin\theta\, a_x\right]$

$= \cos\sqrt{\delta^2+g^2}\, t - i\sin\sqrt{g^2+\delta^2}\, t\left[\frac{\delta a_z}{\sqrt{g^2+\delta^2}} + \frac{g a_x}{\sqrt{\delta^2+g^2}}\right]$

态的演化：$|\psi(0)\rangle = |e\rangle$

$|\psi(t)\rangle = U(t)|e\rangle = \left[\cos\sqrt{g^2+\delta^2}\, t - i\sin\sqrt{g^2+\delta^2}\, t\frac{\delta}{\sqrt{g^2+\delta^2}}\right]|e\rangle - i\sin\sqrt{g^2+\delta^2}\, t\frac{g}{\sqrt{\delta^2+g^2}}|g\rangle$

$P(g) = |\langle g|\psi(t)\rangle|^2 = \sin\sqrt{\delta^2+g^2}\, t\frac{g}{\sqrt{g^2+\delta^2}}$ $(\delta \gg g)$

当 $\delta \gg g$ ： $\sqrt{\delta^2+g^2} = \delta\sqrt{1+(g/\delta)^2} = \delta + \frac{g^2}{2\delta}$ $\frac{\delta}{\sqrt{\delta^2+g^2}} \sim 1$ $\frac{g}{\sqrt{\delta^2+g^2}} \sim 0$

$U = e^{-i\sqrt{\delta^2+g^2}\, a_z t} = e^{-i(\delta + \frac{g^2}{2\delta})a_z}$

正常 $H = \delta a_z + g a_x \Rightarrow U(t) = e^{-i(\delta a_z + g a_x)t} \xrightarrow{\delta \gg g} e^{-i(\delta + \frac{g^2}{2\delta})t a_z}$

有效模型：$H_{eff} = \delta a_z + \frac{g^2}{2\delta}a_x$ Stark-Shift $\frac{g^2}{2\delta}a_z$

再看旋波近似：(其实就是微扰论)

$H = w_0 a_z + g a_x \cos wt \xrightarrow{\text{相互作用}} H_{int} = g a_+ e^{i\delta t} + g a_+ e^{i(w+w_0)t} + c.c.$

$\xrightarrow{RWA} H_{int} = g a_+ e^{i\delta t} + c.c. \xrightarrow[RWA]{\delta \gg g} H_{int} = 0$ 不正确

$\xrightarrow{\delta \gg g} H_{int} = \frac{g^2}{2\delta}a_z$

---

$H = \Delta a_z + g a_x$ $\Delta \gg g$ $H_{eff} = \Delta a_z + \frac{g^2}{\Delta}a_z$ 有效Hamitonian

$$\begin{cases} H_0 = a_z \\ H_1 = g a_x = \lambda a_x \end{cases}$$

微扰：$(\lambda \ll 1)$

$H_0|n\rangle = n|n\rangle$ $\quad |\psi_n\rangle = |n\rangle + \lambda|n'\rangle + \lambda^2|n''\rangle$ $\quad |E_n\rangle = n + \lambda E_n' + \lambda^2 E_n''$

$\Rightarrow H_{eff} = \sum_n E_n' |\psi_n\rangle\langle\psi_n|$ 即有效Hamitonian量 $H_{eff} = \Delta a_z + \underline{\frac{g^2}{\Delta} a_z}$ Stark-Shift

SW变换：

$$H \Rightarrow e^S H e^{-S} = H + [S,H] + \frac{1}{2}[S,[S,H]] + \cdots$$

由 $H = H_0 + H_1$ 有 $e^S H e^{-S} = (H_0 + H_1) + [S, H_0 + H_1] + \frac{1}{2}[S[S, H_0 + H_1]]$

选择S使 $[S,H_0] + H_1 = 0$ $\Rightarrow e^S H e^{-S} = H_0 + [S,H_1] + \frac{1}{2}[S, [S, H_0 + H_1]]$

$e^S H e^{-S} = H_0 + [S,H_1] + \frac{1}{2}[S, -H_1] = H_0 + \frac{1}{2}[S,H_1]$

$$H \xrightarrow{\text{SW变换}} e^S H e^{-S} = H_0 + \frac{1}{2}[S,H_1]$$

确定S：$[S\ H_0] = -H_1$ 即解 $SH_0 - H_0 S = -H_1$

$H_0|n\rangle = E_n|n\rangle$

$\langle n|SH_0 - H_0 S|m\rangle = -\langle n|H_1|m\rangle \Rightarrow \langle n|S|m\rangle E_m - E_n\langle n|S|m\rangle = -\langle n|H_1|m\rangle$

$\Rightarrow \langle n|S|m\rangle = -\frac{\langle n|H_1|m\rangle}{E_m - E_n}$ 非简并时 注：$\langle n|S|n\rangle = ?$ 未知.

SW变换应用在 $H = \Delta a_z + g a_x$ 上时： Schrödinger Picture

$$\begin{cases} [S\ H_0] + H_1 = 0 \\ [S, \Delta a_z] + g a_x = 0 \end{cases} \qquad S = \frac{ig}{\Delta} a_y$$

$\Rightarrow H_{eff} = H_0 + \frac{1}{2}[S\ H_1] = \Delta a_z + \frac{1}{2}[i\frac{g}{\Delta} a_y\ \ g a_x] = \Delta a_z + \frac{g^2}{2\Delta} a_z$

$H = H_0 + H_1$ 微扰论.

---

$H_{int} = g(a_+ e^{i\Delta t} + a_- e^{-i\Delta t})$ $(|gt| \ll 1)$ 演化时间和能态 Interaction Picture

$i\frac{\partial U}{\partial t} = H_{int} U$ $\quad U = 1 - i\int H_{int}(t)dt + (-i)^2\int_0^t H(t_1)\int_0^{t_1} H(t_2)\, dt_1\, dt_2 \cdots = \hat{T}\left[e^{-\int_0^t H(t)dt'}\right]$

若不含时，$U = e^{-iH_{eff}\Delta t}$

含时演化：$U = 1 - i\underbrace{\int g(a_+ e^{i\Delta t} + a_- e^{-i\Delta t})\, dt}_{\text{旋波近似为0}} - g^2\int_0^t dt_1 \int_0^{t_1} dt_2 (a_+ e^{i\Delta t_1} + a_- e^{-i\Delta t_1})(a_+ e^{i\Delta t_2} + a_- e^{-i\Delta t_2})$

$$\Rightarrow U = 1 - g^2\int_0^t dt_1 (a_+ e^{i\Delta t_1} + a_- e^{-i\Delta t_1})\left[\frac{a_+ e^{i\Delta t_1}}{i\Delta} - \frac{a_- e^{-i\Delta t_1}}{i\Delta}\right]$$

$$= 1 - g^2\int_0^t \left(\frac{-a_+ a_-}{i\Delta} + \frac{a_- a_+}{i\Delta}\right) = 1 + g^2\frac{t}{i\Delta}[a_+ a_- - a_- a_+] = 1 + g^2\frac{t}{i\Delta} a_z \approx e^{-iH_{eff} t}$$

$\Rightarrow H_{eff} = \frac{g^2}{2\Delta}\sigma_z \quad \Rightarrow H = g(\sigma_+ e^{i\Delta t} + \sigma_- e^{-i\Delta t}) \simeq \frac{g^2}{2\Delta}\sigma_z$

$\begin{cases} \text{(1) RWA} \\ \text{(2) SW 变换} \end{cases}$ 微扰论 (3) 演化算符.

(1) $H_{int} = \sum_m H_m e^{i\omega_m t} + H_m e^{-i\omega_m t}$

(2) RWA 比较 $\omega_m$ 与 $H_m$ 丢高频,留低频.

若 $H_{eff}=0$ 即都为高频振荡时

(3) 二阶微扰论 $H_{eff} = \sum_m \frac{1}{\omega_m}[H_m, H_m^{+}]$

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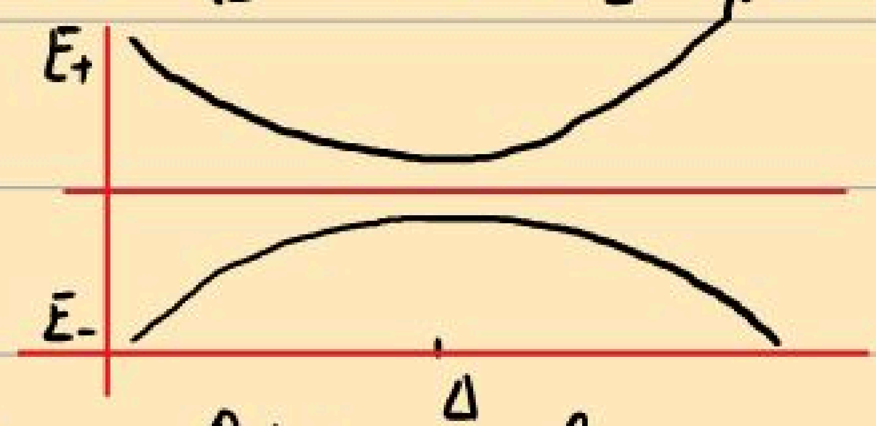
$$H = \Delta\sigma_z + g\sigma_x = \sqrt{\Delta^2+g^2}\,[\cos\theta\,\sigma_z + \sin\theta\,\sigma_x] = \sqrt{\Delta^2+g^2}\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix}$$

本征值: $E_\pm = \pm\sqrt{\Delta^2+g^2}$

失谐量: $\Delta = -\infty \longrightarrow +\infty$




I $\Delta$ $\Delta=0$时共振

本征态: $\begin{cases} E_+ = \sqrt{\Delta^2+g^2} & |\psi_+\rangle = \cos\frac{\theta}{2}|g\rangle + \sin\frac{\theta}{2}|e\rangle \\ E_- = -\sqrt{\Delta^2+g^2} & |\psi_-\rangle = \sin\frac{\theta}{2}|g\rangle - \cos\frac{\theta}{2}|e\rangle \end{cases}$ 其中 $\begin{cases}\cos\theta = \frac{\Delta}{\sqrt{\Delta^2+g^2}} \\ \sin\theta = \frac{g}{\sqrt{\Delta^2+g^2}}\end{cases}$

当 $\Delta\to-\infty$ 时 $\theta\to-0$ $\begin{cases}|\psi_+\rangle = |g\rangle \\ |\psi_-\rangle = -|e\rangle\end{cases}$

当 $\Delta\to0$ 时 $\theta\to\frac{\pi}{2}$ $\begin{cases}|\psi_+\rangle = |g\rangle+|e\rangle \\ |\psi_-\rangle = |g\rangle-|e\rangle\end{cases}$

当 $\Delta\to+\infty$ 时 $\theta\to+0$ $\begin{cases}|\psi_+\rangle = |g\rangle \\ |\psi_-\rangle = |e\rangle\end{cases}$

若驱动场为多频光: $H = \omega\sigma_z + \sigma_x(g_1\cos\omega_1 t + g_2\cos\omega_2 t)$

$$H = \sigma_+[g_1 e^{i\Delta_1 t} + g_2 e^{i\Delta_2 t}] + \sigma_-[g_1 e^{-i\Delta_1 t} + g_2 e^{-i\Delta_2 t}]$$

$$+\sigma_+[g_1 e^{i(\omega_0+\omega_1)t} + g_2 e^{i(\omega_0+\omega_2)t}] + \sigma_-[g_1 e^{i(\omega_0+\omega_1)t} + g_2 e^{i(\omega_0+\omega_2)t}]$$ 高频项,忽略(RWA)

$$\Rightarrow H = \sigma_+[g_1 e^{i\Delta_1 t} + g_2 e^{i\Delta_2 t}] + \sigma_-[g_1 e^{-i\Delta_1 t} + g_2 e^{-i\Delta_2 t}]$$

$$\Rightarrow H = \underbrace{\Delta\sigma_z + g_1\sigma_x}_{\text{视为}H_0} + g_2\sigma_+ e^{i(\Delta_2-\Delta_1)t} + g_2\sigma_- e^{-i(\Delta_2-\Delta_1)t}$$

即缀饰态 (在 $\omega+\omega_1$ 上将 $\omega_2$ 视为微扰)

高频含时系统：PRB, 93. 144307 (2016)

$$H = H_0 + \sum_m H_m e^{imwt} + c.c.$$

RWA近似：$H = H_0$

$$\Rightarrow H = H_0 + \frac{1}{mw}[H_m, H_m^+]$$

SW变换：$S = \lambda S_1 + \lambda^2 S_2 + \cdots \qquad H = H_0 + \lambda H_1 + \lambda^2 H_2$

代入 $e^S H e^{-S} = H + [S, H] + \frac{1}{2!}[S, [S, H]] + \cdots$

凝聚态物理：$H(k) \Rightarrow H(k - eA(t)) = H_0(k) + \sum_m H e^{imwt} + \cdots \Rightarrow H_{eff}.$

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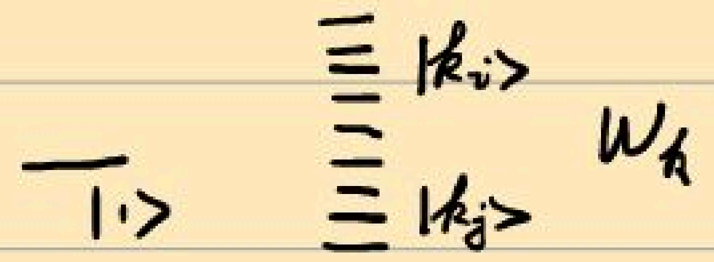


单能级与无穷多能级耦合.

$$H = \sum_k W_k |k\rangle\langle k| + \sum_k (\Omega_k e^{-iV_k t} |k\rangle\langle 1| + c.c.)$$

$$\Rightarrow u = e^{-i \sum_k V_k t |k\rangle\langle k|}$$

$$H' = \sum_k (W_k - V_k) |k\rangle\langle k| + \sum_k (\Omega_k |k\rangle\langle 1| + c.c.)$$

$$|\psi\rangle = C_1 |1\rangle + \sum_k C_k |k\rangle$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t} C_1 = i \sum_k \Omega_k^* C_k \\ \frac{\partial}{\partial t} C_k = i \delta_k C_k + i \Omega_k C_1 \end{cases} \qquad k = 1 \cdots N$$

$$\Rightarrow \begin{cases} C_1(t) = C_1(0) + i \int \Omega_k^* C_k(t') dt' \\ C_k(t) = e^{i\delta_k(t)} C_k(0) + i \int C_1(t) e^{-i\delta_k(t'-t)} dt' \end{cases}$$

取 $C_1(0) = 1 \qquad C_k(0) = 0$

近似

$$\frac{\partial C_1}{\partial t} = -\sum_k |\Omega_k|^2 \int_0^t C_1(t') e^{-i\delta_k(t'-t)} dt' = -\sum_k |\Omega_k|^2 C_1(t) \int_0^t e^{-i\delta_k(t'-t)} dt'$$

$$\lim_{\varepsilon \to 0} \int_0^t e^{-i\delta_k(t'-t)} dt' = \lim_{\varepsilon \to 0} \frac{i}{\delta_k + i\varepsilon} = \pi \delta(\delta_k) + \frac{P}{\delta_k}$$ P为主值.

$$\frac{\partial}{\partial t} C_1 = -C_1 \Big[ \underbrace{\frac{\pi \sum_k |\Omega_k|^2 \delta(V - W_k)}{\gamma/2}}_{\text{耗散项}} + \underbrace{i \sum_k |\Omega_k|^2 \frac{P}{V - W_k}}_{\text{Stark位移项}\ \Delta} \Big]$$

$$\Rightarrow C_1(t) = C_1(0) e^{-\frac{\gamma}{2} t + i \Delta_{stark} t}$$

$$C_k(t) = i\Omega_k\int_0^t C_1(t')\, e^{-i\delta_k(t'-t)}dt' = i\frac{\Omega_k}{\frac{\Gamma}{2}+i\delta_k}\left[1-e^{-(\frac{\Gamma}{2}+i\delta_k)t}\right]e^{i\delta_k t}$$

$$= \frac{i\Omega_k}{\frac{\Gamma}{2}+i\delta_k}\, e^{i\delta_k t}$$

演化初始: $C_1|1\rangle + C_k|k\rangle = \sum_k C_k|k\rangle$　单能级向波包演化

正向 Wigner定理　　反向量子存储

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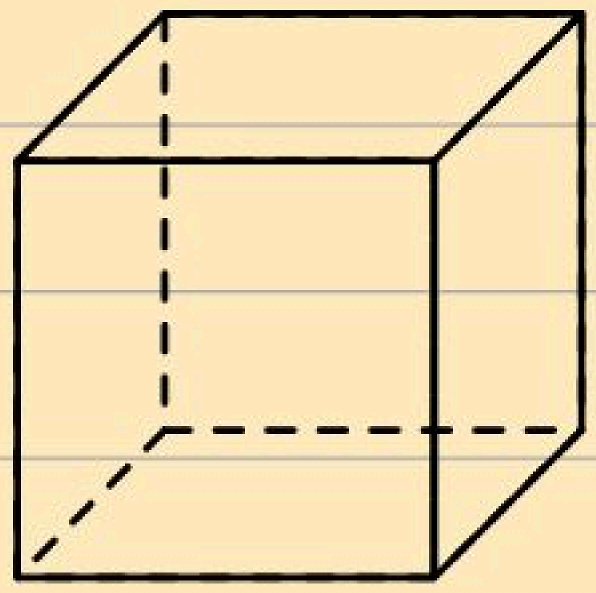
电磁场量子化:(本征模量子化)



周期性条件

$L$

$$\begin{cases}\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}\\ \nabla\times B = \frac{1}{c^2}\frac{\partial E}{\partial t}\end{cases}$$

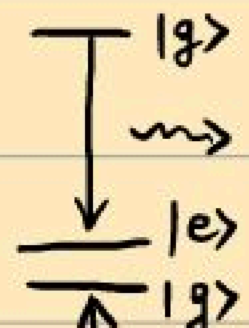
$H = \omega\sigma_z + g\cos\omega t\,\sigma_x$ (Floquet)　　$H = \frac{p^2}{2m} + V(r) + \vec{d}\cdot\vec{E} + H_{电磁场}$

$\sum_n E_n|n\rangle\langle n|$　$\sum_{n,m}|n\rangle\langle n|d|m\rangle\langle m|$　$\sum_k \omega_k a_k^\dagger a_k$

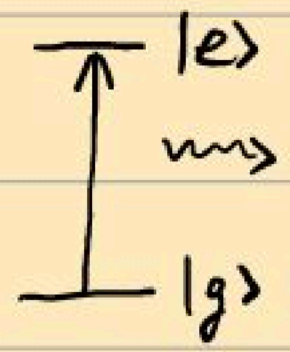
量子化 ⇒ $H = \omega_0\sigma_z + g(a+a^\dagger)\sigma_x + \omega a^\dagger a$ (JC Model)

$(a+a^\dagger)(\sigma_+ + \sigma_-) = a\sigma_+ + a^\dagger\sigma_+ + a\sigma_- + a^\dagger\sigma_-$

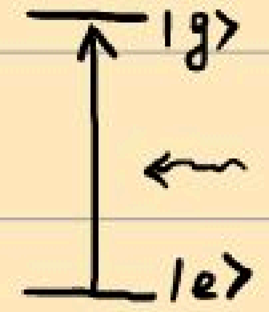
$a^\dagger|g\rangle\langle e|$



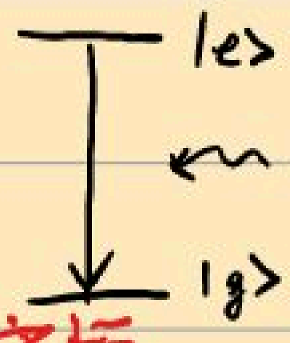

$a^\dagger|e\rangle\langle g|$




$a|e\rangle\langle g|$




$a|g\rangle\langle e|$




满足能量守恒　　能量不守恒　　当 $\omega+\omega_0 \gg g$ 时可忽略.

经典场:　$H = \omega_0\sigma_z + g(\sigma_+ + \sigma_-)\cos\omega t$　　场: $E = \varepsilon e^{i\omega t} + \varepsilon^* e^{-i\omega t}$

量子场:　$H = \omega a^\dagger a + \omega_0\sigma_z + g(a+a^\dagger)(\sigma_+ + \sigma_-)$　　场: $E = a + a^\dagger$

经典场: RWA近似　$H_1 = \sigma_+ e^{i(\omega-\omega_0)t} + \sigma_+ e^{-i(\omega+\omega_0)t} + c.c.$

量子场: RWA 近似　$H = H_0 + H_1$

$$H_1 = g(ae^{i\omega t} + a^\dagger e^{-i\omega t})[a_+ e^{-i\omega_0 t} + a_- e^{i\omega_0 t}]$$
$$= g a a_+ e^{i(\omega-\omega_0)} + g a^\dagger a_+ e^{-i(\omega+\omega_0)t} + c.c.$$

当强场时：$a|\alpha\rangle = \alpha|\alpha\rangle \quad a^\dagger|\alpha\rangle \approx \alpha|\alpha\rangle$

RWA近似：$\omega+\omega_0 \gg g$ 略去 $g a^\dagger a_+ e^{-i(\omega+\omega_0)t}$

$H_1 = g a a_+ e^{i(\omega-\omega_0)t} + c.c$ 记 $\delta = \omega - \omega_0$ 为失谐量

$\Rightarrow H = \delta a_z + g(a a_+ + a^\dagger a_-) \qquad N = a^\dagger a + \frac{1+a_z}{2} \qquad [N, H] = 0$ N为守恒量

注：无旋波近似的守恒量为宇称守恒：$P = e^{i\pi(a^\dagger a + a_z)} \quad [P, H] = 0 \quad P = \begin{cases} +1 \\ -1 \end{cases}$

$N = a^\dagger a + \frac{1+a_z}{2}$ $\quad N=0$ 时 态为 $|0, g\rangle$

$N = m \neq 0$ 时 态为 $|m, g\rangle$ 或 $|m-1, e\rangle$

当 $m=N=0$ 时 $H = -\delta$ $\qquad$ 当 $N=m\neq 0$ 时 $H = \begin{pmatrix} -\delta & g\sqrt{m} \\ g\sqrt{m} & \delta \end{pmatrix}$

$m=N=0$ 时 $|0, g\rangle \longrightarrow e^{-i\delta t}|0, g\rangle$

$m\neq 0$ 时 $H = \delta a_z + g\sqrt{m}\, a_x$ 相互作用项，当m越大时，相互作用越强，旋波近似越正确.

$|m, g\rangle \longrightarrow \cos(\sqrt{m}\, gt)|m, g\rangle - i\sin(\sqrt{m}\, gt)|m-1, e\rangle$

$|m-1, e\rangle \longrightarrow \cos(\sqrt{m}\, gt)|m-1, e\rangle - i\sin(\sqrt{m}\, gt)|m, g\rangle$

当 $m=1$ 时：$|0, g\rangle \to |0, g\rangle \qquad |0, e\rangle \to \cos\sqrt{g}t\,|0, e\rangle - i\sin\sqrt{g}t\,|1, g\rangle$

$[\alpha|g\rangle + \beta|e\rangle]|0\rangle \to \alpha|g\rangle|0\rangle + i\beta|g\rangle|1\rangle = |g\rangle[\alpha|0\rangle + i\beta|1\rangle] \qquad gt = \frac{\pi}{2}$

腔量子电动力学 (Walter. Maser)

大失谐：$\delta \gg g$

经典大失谐：$H = \delta a_z + g a_x \Rightarrow H = \frac{g^2}{\delta}[a_+ a_-] = \frac{g^2}{\delta} a_z$

量子大失谐：$H = \delta a_z + g a a_+ + a^\dagger a_- \longrightarrow H = \frac{g^2}{\delta}[a^\dagger a_-, a a_+]$

$$= \frac{g^2}{\delta}[a^\dagger a a_- a_+ - a a^\dagger a_+ a_-]$$

$$= \frac{g^2}{\delta} a^\dagger a\, a_z - \frac{g^2}{\delta} a_+ a_- \quad \text{stark 位移}$$

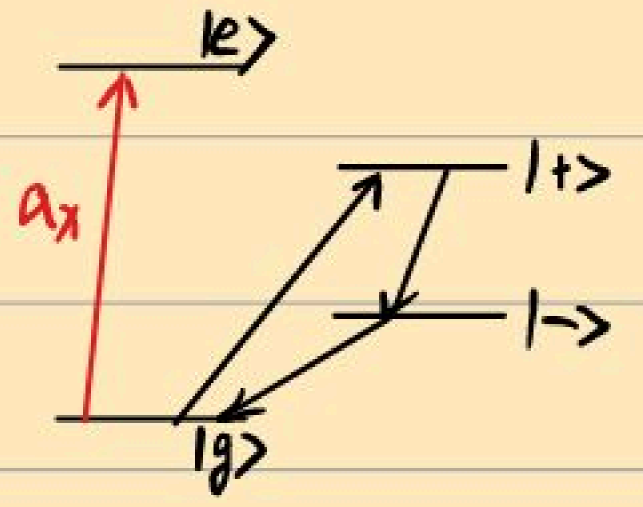
$\begin{cases} H = a^\dagger a\, a_z & \text{dispersive 相互作用} \\ H = g(a a_+ + a^\dagger a_-) & \text{Resonant 相互作用} \end{cases}$

色散相互作用：$H = a^\dagger a\, a_z$

Cat State：$e^{iHt}(|g\rangle + |e\rangle)|\alpha\rangle \Rightarrow |g\rangle|\alpha e^{-igt}\rangle + |e\rangle|\alpha e^{igt}\rangle$  Schrödinger's Cat 实验

where：$|\alpha\rangle = e^{\alpha(a-a^\dagger)}|0\rangle \Rightarrow e^{igta^\dagger a}|0\rangle = |\alpha e^{-igt}\rangle$

记 $|+\rangle = |g\rangle + |e\rangle$  $|-\rangle = |g\rangle - |e\rangle$ 能级测量：




$\Rightarrow |g\rangle\left(|\alpha e^{igt}\rangle + |\alpha e^{-igt}\rangle\right) + |e\rangle\left(|\alpha e^{igt}\rangle - |\alpha e^{-igt}\rangle\right)$

若 $gt = \frac{\pi}{2}$ 即 $\begin{cases} 测|g\rangle: |i\alpha\rangle + |-i\alpha\rangle \\ 测|e\rangle: |i\alpha\rangle - |-i\alpha\rangle \end{cases}$

若 $gt = \pi$ 即 $\begin{cases} 测|g\rangle: |\alpha\rangle + |-\alpha\rangle \\ 测|e\rangle: |\alpha\rangle - |-\alpha\rangle \end{cases}$

$H = \delta a_z + g(a a_+ + a^\dagger a_-)$  $\delta = -\infty \rightarrow +\infty$ 绝热演化

$H = \delta a_z + g\sqrt{m}\, a_x$  $\tan\theta = \frac{g\sqrt{m}}{\delta}$  $|m-1, e\rangle \rightarrow |m, g\rangle$ 为绝热输运过程.

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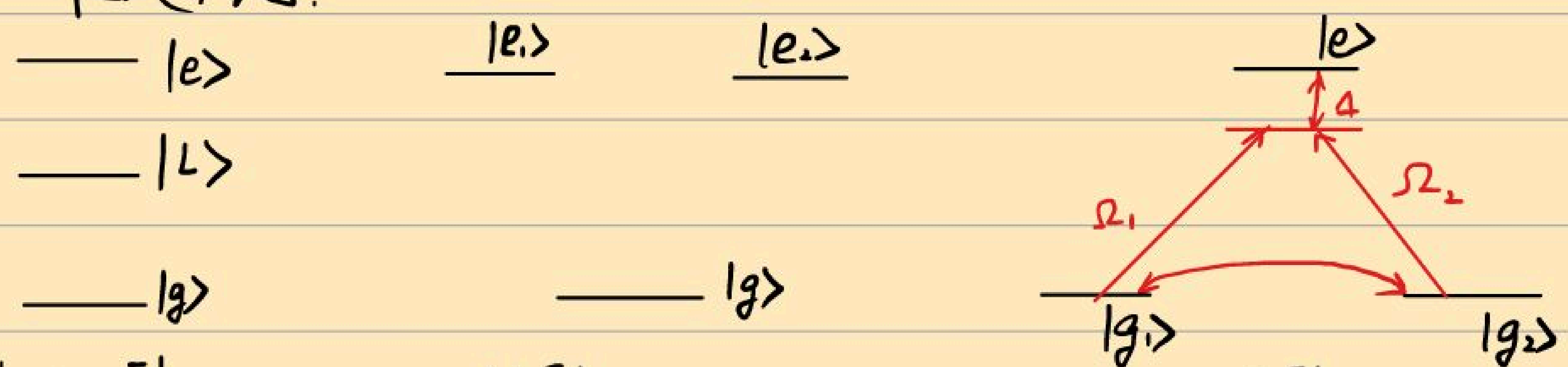
三能级系统：




Latter型  V型  Λ型

Stimulated Raman Transfer : Raman Couple

Λ型：双光子共振

$H = \Delta_1|g_1\rangle\langle g_1| + \Omega_1\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right)$  $\Delta_1 = E_0 - \Omega_1$

$+ \Delta_2|g_2\rangle\langle g_2| + \Omega_2\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)$  $\Delta_2 = E_0 - \Omega_2$

当 $\Delta_1 = \Delta_2 = \Delta$ 时发生双光子共振

发生双光子共振时,有：$H = e^{i\Delta t}\left[\Omega_1|e\rangle\langle g_1| + \Omega_2|e\rangle\langle g_2|\right] + c.c.$

当 $\Delta \gg \Omega_1, \Omega_2$ 时, $H=\frac{1}{\Delta}\left[\Omega_1|e\rangle\langle g_1|+\Omega_2|e\rangle\langle g_2|,\ \Omega_1|g_1\rangle\langle e|+\Omega_2|g_2\rangle\langle e|\right]$

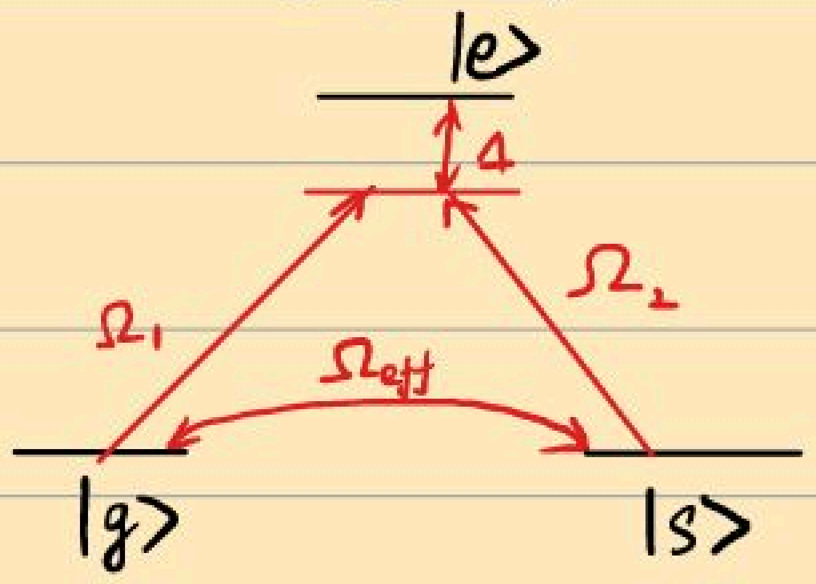
$$H=\frac{1}{\Delta}\left[\Omega_1^2(|e\rangle\langle e|-|g_1\rangle\langle g_1|)-\Omega_1\Omega_2|g_1\rangle\langle g_2|-\Omega_1\Omega_2|g_2\rangle\langle g_1|+\Omega_2^2(|e\rangle\langle e|-|g_2\rangle\langle g_2|)\right]$$

当仅考虑 $|g_1\rangle|g_2\rangle$ 时 $H=\frac{1}{\Delta}\left[\underbrace{-\Omega_1^2|g_1\rangle\langle g_1|-\Omega_2^2|g_2\rangle\langle g_2|}_{\text{Stark 位移}}-\Omega_1\Omega_2(|g_1\rangle\langle g_2|+|g_2\rangle\langle g_1|)\right]$

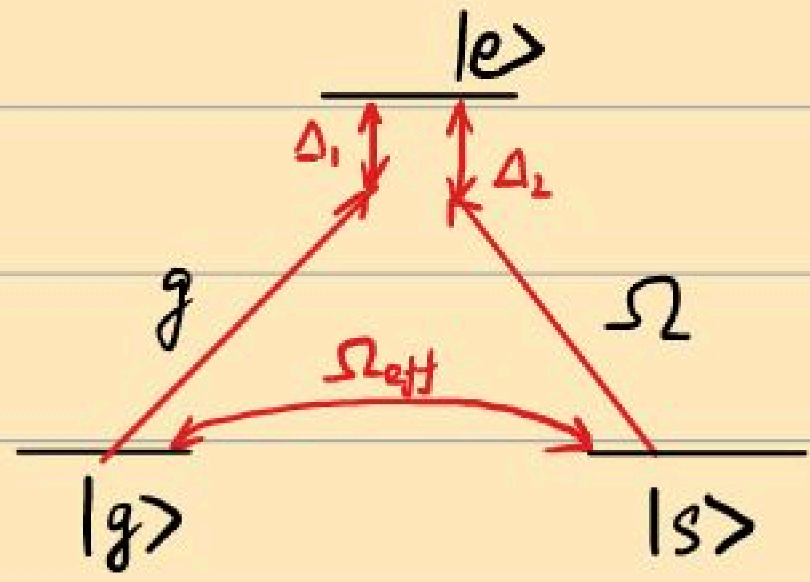
$$\approx g_H(|g_1\rangle\langle g_2|+|g_2\rangle\langle g_1|)$$

若考虑上能级自发辐射寿命：$i\Gamma$

$\Rightarrow H_{int}=\frac{1}{\Delta+i\Gamma}\left[\Omega_1|g_1\rangle\langle e|+\Omega_2|g_2\rangle\langle e|,\ \Omega_1|e\rangle\langle g_1|+\Omega_2|e\rangle\langle g_2|\right]$




$\Omega_{eff}=\frac{\Omega_1\Omega_2}{\Delta}$




量子化： $H=w_a a^\dagger a+g(a^\dagger|g\rangle\langle e|+c.c)+w_e|e\rangle\langle e|+w_g|g\rangle\langle g|+w_s|s\rangle\langle s|+\Omega(|s\rangle\langle e|+|e\rangle\langle s|)$

$H_{int}=g\left[a^\dagger e^{i\Delta_1 t}|g\rangle\langle e|+c.c.\right]+\Omega\left[e^{i\Delta_2 t}|s\rangle\langle e|+h.c.\right]$ $+\delta(|g\rangle\langle g|-|s\rangle\langle s|)$ 高阶小量

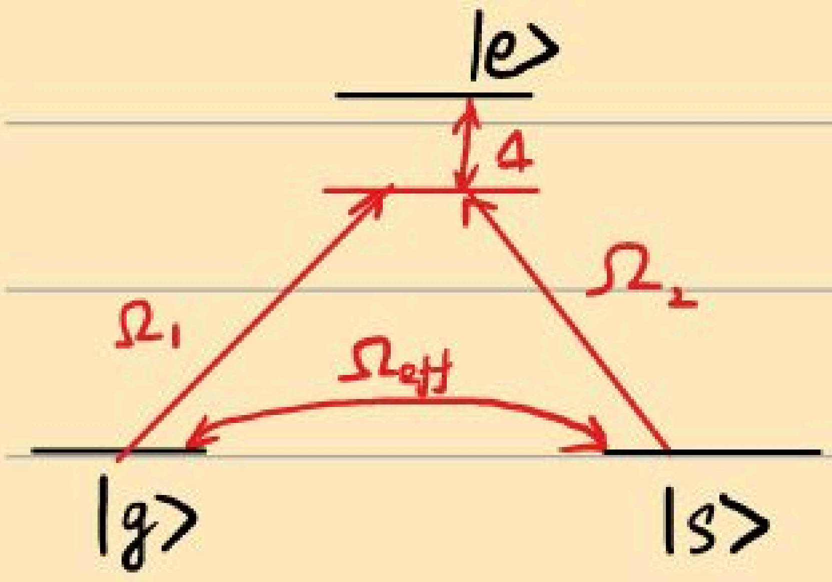
当 $\Delta=\Delta_1=\Delta_2$ 做微扰：

$H=\frac{1}{\Delta}\left[g a^\dagger|g\rangle\langle e|+\Omega|s\rangle\langle e|,\ g a|e\rangle\langle g|+\Omega|e\rangle\langle s|\right]+\delta(|g\rangle\langle g|-|s\rangle\langle s|)$ 一阶

$$\Rightarrow H=\delta(|g\rangle\langle g|-|s\rangle\langle s|)+\frac{1}{\Delta}\left[g^2(a^\dagger a|g\rangle\langle g|-aa^\dagger|e\rangle\langle e|)+\Omega g a^\dagger|g\rangle\langle s|+\Omega g a|s\rangle\langle g|+\Omega^2(|s\rangle\langle s|-|e\rangle\langle e|)\right]$$

$$H=\underbrace{\frac{g^2}{\Delta}a^\dagger a|g\rangle\langle g|+\frac{\Omega^2}{\Delta}|s\rangle\langle s|}_{\text{Stark 位移项}}+\underbrace{\frac{g\Omega}{\Delta}a^\dagger|g\rangle\langle s|+c.c.}_{\text{拉曼耦合项}}$$

S.M.Gate 耗散

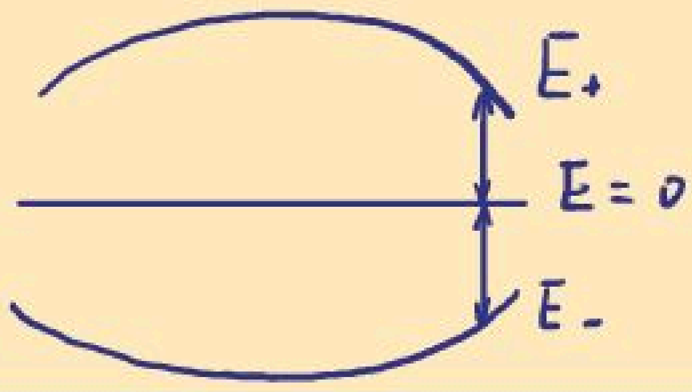



Stimulated Raman Transfer：受激拉曼输运

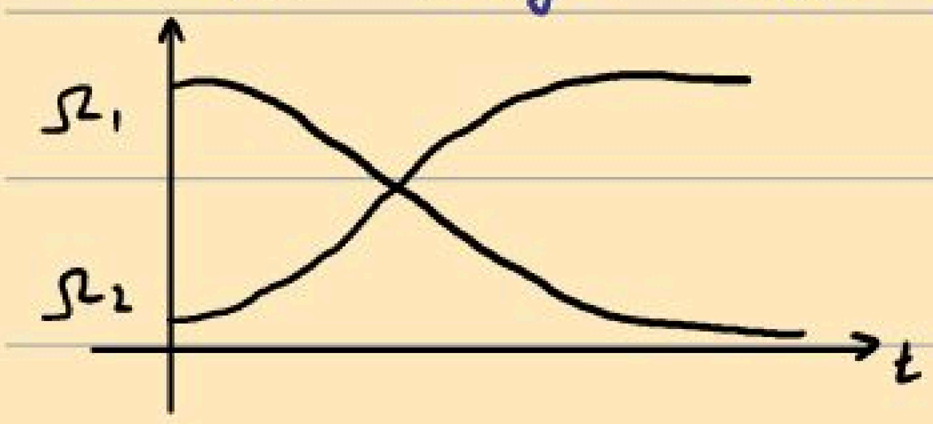
$H=\Delta|e\rangle\langle e|+\Omega_1(|e\rangle\langle g|+c.c.)+\Omega_2(|e\rangle\langle s|+c.c.)$

Dark State：$H|\varphi\rangle=0$ 能量为零　$\langle e|\varphi\rangle=0$ 不含激发态.

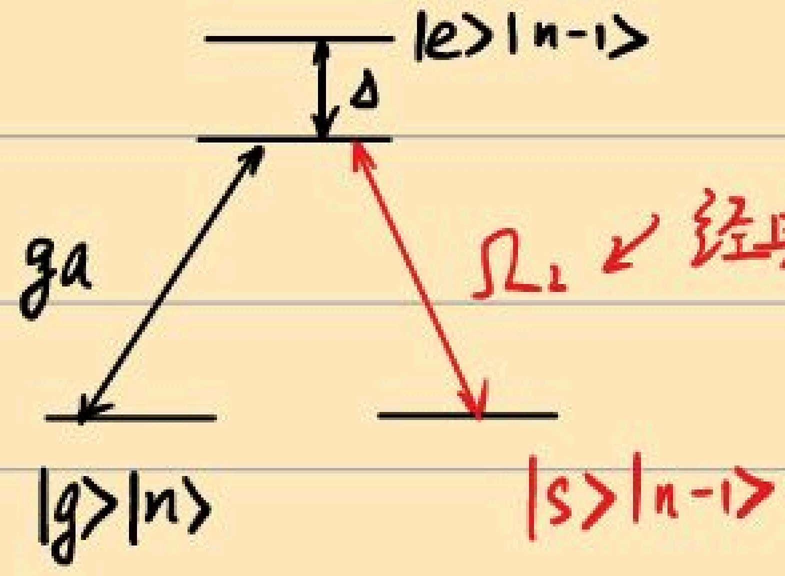
例如 $|\varphi\rangle=\frac{1}{\sqrt{\Omega_1^2+\Omega_2^2}}(\Omega_1|s\rangle-\Omega_2|g\rangle)$




记 $|+\rangle = \Omega_1|g\rangle + \Omega_2|s\rangle$ 有：$H = \Delta|e\rangle\langle e| + |e\rangle\langle +| + c.c.$




量子化：(Raman 耦合的 J.C. Model)




Fock态表征量子场

$\Omega_2$ ↙ 经典光(较大且非相干)不做量子化

Dark State：$|\varphi_n\rangle = g\sqrt{n}|s, n-1\rangle - \Omega_2|g, n\rangle$

当 $\Omega_2 \gg g\sqrt{n}$ 即初始态在 $|g\rangle$ 态上　　则：$|\varphi_n\rangle = |g, n\rangle \xrightarrow{\Omega_2\to 0} |s\ \ n-1\rangle$

$$\sum_n C_n|g, n\rangle \longrightarrow \sum_n C_n|s, n-1\rangle$$

当 $\Omega_2 = 0$ 即初始态在 $|s\rangle$ 态上　　$|\varphi_n\rangle = |s, n-1\rangle \xrightarrow{\Omega_2\to\infty} |g, n\rangle$

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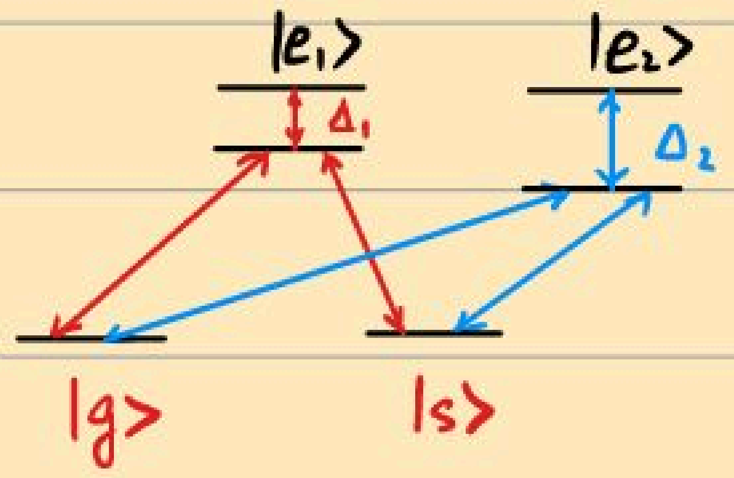



双拉曼耦合

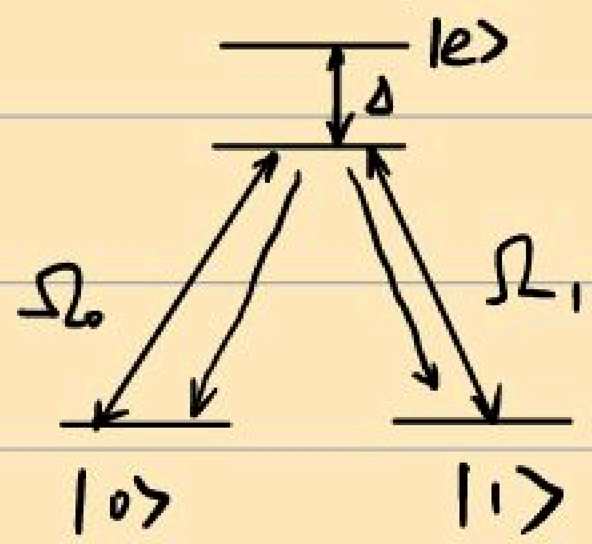
$$H = \frac{g_1\Omega_2}{\Delta_2}\left(a^\dagger|g\rangle\langle s| + c.c.\right) + \frac{g_1\Omega_1}{\Delta_1}\left(a|g\rangle\langle s| + c.c.\right)$$

$$\simeq g_1(aa_+ + a^\dagger a_-) + g_2(a^\dagger a_+ + aa_-) \simeq (a+a^\dagger)(a_+ + a_-)$$

$$\Rightarrow H = wa^\dagger a + w_1 a_z + g(a+a^\dagger)(a_+ + a_-) = g(aa_+ + a^\dagger a_-) - g(a^\dagger a_+ + aa_-)$$

PRA 85, 032111 (2012)　F. Reiter, A. Sorensen.

Effective Operator Formalism for open quantum system.




$$L_{eff} = \frac{\sqrt{\gamma}\,\Omega_0}{\Delta_0 + i\gamma}|0\rangle\langle 0| + \frac{\Omega_1}{\Delta - i\gamma}|0\rangle\langle 1|$$

$$L_{eff} = \frac{\Omega_1}{\Delta_0 + i\gamma}|1\rangle\langle 0| + \frac{\Omega_0}{\Delta - i\gamma}|1\rangle\langle 1|$$

腔QED： C.K. Law

$\sum_{n=0}^{N} C_n|n\rangle$

$H = g(aa_+ + a^+a_-) = \begin{pmatrix} 0 & g\sqrt{n} \\ g\sqrt{n} & 0 \end{pmatrix}$

$[\alpha|g\rangle+\beta|e\rangle]|0\rangle = \alpha|g\rangle|0\rangle+\beta|e\rangle|0\rangle \Rightarrow \alpha|g\rangle|0\rangle+\beta|g\rangle|1\rangle \Rightarrow |g\rangle[\alpha|0\rangle+\beta|1\rangle]$

C.K想法：制备 $C_0|0\rangle+C_1|1\rangle+C_2|2\rangle$

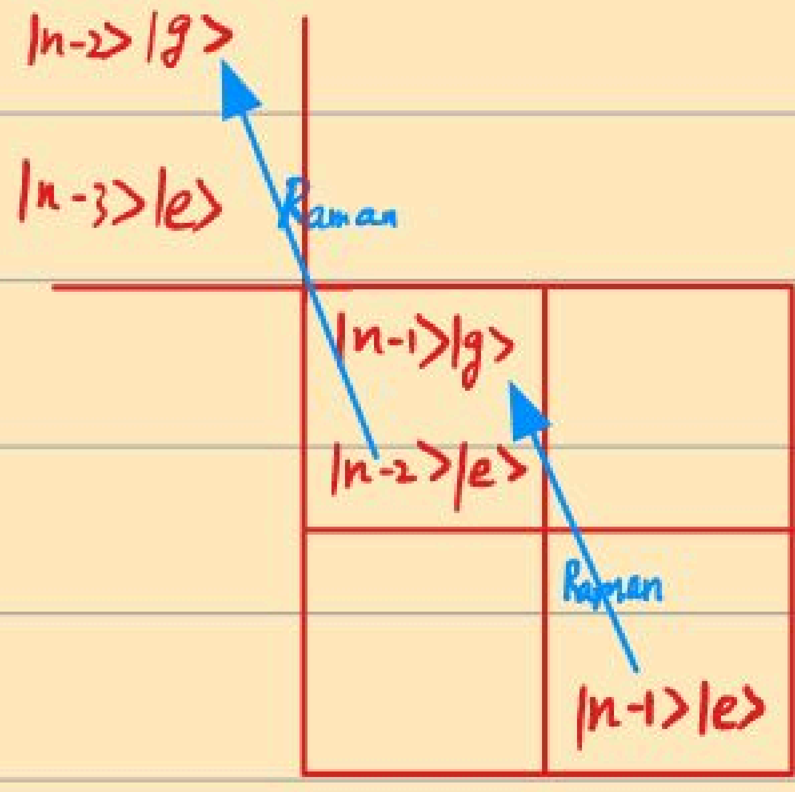
$C_0|0\rangle|g\rangle + C_1[\lambda_{11}|1\rangle|g\rangle+\lambda_{12}|e\rangle|0\rangle] + C_2[\overset{0}{\boxed{\lambda_{21}|2\rangle|g\rangle}} + \overset{1}{\boxed{\lambda_{22}}}|e\rangle|1\rangle]$

$\Rightarrow C_0|0\rangle|g\rangle + C_1[\boxed{\lambda_{11}|1\rangle|g\rangle}+\lambda_{12}|e\rangle|0\rangle] + \boxed{C_2|e\rangle|1\rangle}$

$\Rightarrow C_0|0\rangle|g\rangle + C_1\lambda_{12}|0\rangle|e\rangle + |1\rangle[\boxed{C_1\lambda_{11}|g\rangle+C_2|e\rangle}]$ 做Rabi振荡.

$\Rightarrow C_0|0\rangle[\lambda_{00}|g\rangle+\lambda_{01}|e\rangle] + C_1|0\rangle[\lambda_{10}|g\rangle+\lambda_{11}|e\rangle] + C_2|1\rangle|g\rangle$

$\Rightarrow |0\rangle|g\rangle + \lambda_1|0\rangle|e\rangle+\lambda_2|1\rangle|g\rangle$




$H=\Delta a^+a + g(aa_+ + a^+a_-)$ JC Model 单原子

$H=\Delta a^+a+g(aJ_+ + a^+J_-)$ Diche Model 多原子

$J_+ = \sum_{i=1}^{N} a_{+i}$

(1) $\Delta=0$ $H=g(aJ_+ + a^+J_-)$

N原子：$2^N$基矢 Diche态 $|N,m\rangle = J_+^m|\downarrow\cdots\downarrow\rangle$

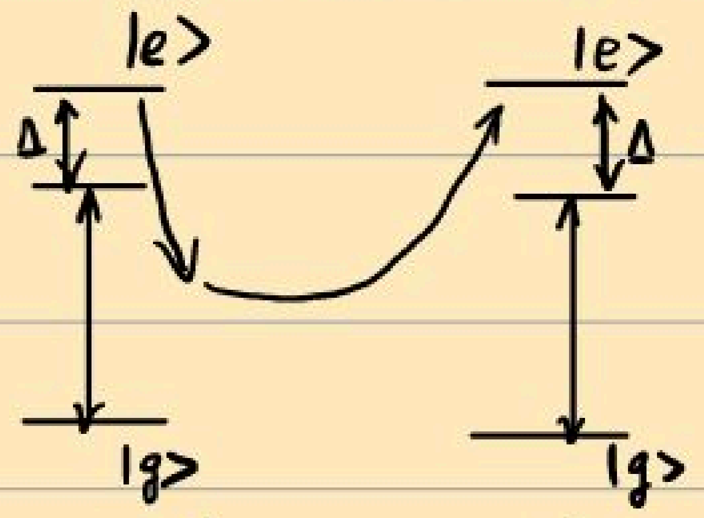
$J_z=\sum_i a_{iz}$ $J_z|N,-N\rangle = -N|0\rangle$

守恒量 $K=a^+a+\frac{J_z+1}{2}$ (N态对易，求解N×N矩阵) JC: $K=a^+a+\frac{a_z+1}{2}$

(2) $\Delta\gg g\sqrt{N}$, $H=\Delta a^+a+g(aJ_+ + a^+J_-)$

$\Rightarrow H=\frac{\Delta^2}{2}[aJ_+, aJ_-]$

$$H=\frac{g^2}{\Delta}\left(aa^\dagger J_+J_- - a^\dagger a J_-J_+\right)=\frac{g^2}{\Delta}\left(J_+J_- + a^\dagger a J_z\right)=\frac{g^2}{\Delta}J_+J_-$$




虚激发：某一原子受激辐射出光被另一原子吸收并激发

PRA. Squeezed Spin State

One axis's Interaction

M.Ueda

$$H=\frac{g^2}{\Delta}J_+J_-=\frac{g^2}{\Delta}(J_x+iJ_y)(J_x-iJ_y)$$

态全同

$$=\frac{g^2}{\Delta}\left(J_x^2+J_y^2+i(J_xJ_y-J_yJ_x)\right)=\frac{g^2}{\Delta}\left(J_x^2+J_y^2+J_z\right)=\frac{g^2}{\Delta}\left(J^2-J_z^2+J_z\right)=\frac{g^2}{\Delta}J_z^2$$

通常态不同：$aJ_+ + a^\dagger J_- + \sum_i \Delta_i \sigma_{zi}$

$$e^{-iJ_z^2 t}\sum_m C_m|m\rangle \Rightarrow \sum_m C_m e^{-im^2 t}|m\rangle$$ Fock态

G.S. Agarwal 1989

$$|\alpha\rangle+|-\alpha\rangle=\sum_m \alpha^m|m\rangle+\sum_m(-1)^m\alpha^m|m\rangle$$

Atomic Schrödinger Cat State

Two Axis Interaction: $H=J_x^2-J_y^2=J_+^2+J_-^2$

---

$$H=\Delta a^\dagger a+g(aS_+ + a^\dagger S_-)\Rightarrow H=\frac{g^2}{2}[aS_+ . aS_-]=\frac{g^2}{2}(S_+S_- + a^\dagger a S_z)$$

PRA. 56, 2249(1997) G.S. Agurwul Atomic Schrödinger cat state

$$H=S_+S_-=(S_x+iS_y)(S_x-iS_y)=S_x^2+S_y^2=S^2-S_z^2$$

$$e^{-iS_z^2 t}\sum_n C_n|n\rangle=\sum_n C_n e^{-in^2 t}|n\rangle \qquad e^{-in^2 t}=e^{-i\frac{n^2\pi}{m}} \qquad t=\frac{\pi}{m}$$

$$\begin{cases}\text{当}m\text{为奇数时：}e^{i\frac{\pi}{m}n(n+1)}=\sum_{g=0}^{m-1}f_g^o\exp\left(\frac{2\pi i g}{m}n\right)\\ \text{当}m\text{为偶数时：}e^{i\frac{\pi}{m}n^2}=\sum_{g=0}^{m-1}f_g^e\exp\left(\frac{2\pi i g}{m}n\right)\end{cases}$$

$$\Rightarrow=\sum_{g=0}^{m-1}f_g^e\sum_n C_n e^{\frac{2\pi i g}{m}n}|n\rangle+\sum_{g=0}^{m-1}\left(e^{\frac{2\pi i g S_z}{m}}|\psi\rangle\right)$$

$$\Rightarrow|\alpha\rangle=e^{\alpha(S_+-S_-)}|-N\rangle=\sum_{g=0}^{m-1}\left|\alpha e^{i\frac{2\pi g}{m}}\right\rangle$$ Cat State.

当 $m=2$ 时 $|\alpha\rangle+|-\alpha\rangle \longleftrightarrow |g+e\rangle\otimes|g+e\rangle\cdots\otimes|g+e\rangle+|g-e\rangle\otimes|g-e\rangle\otimes\cdots\otimes|g-e\rangle$ $\quad gt=\frac{\pi}{m}$

自旋压缩 PRA 47, 5138(1993) M. Kitagawa M.Ueda Squeezed Spin State

单轴 Hamilton：$H=S_z^2$ 双轴 Hamilton：$H=S_+^2-S_-^2=S_x^2-S_y^2$

自旋压缩：

$$H=\frac{g^2}{\Delta}\left[S_+S_-+a^\dagger a S_z\right]=\frac{g^2}{\Delta}\left[\overset{|1\rangle\langle e|}{S_+}S_-+a^\dagger a(|e\rangle\langle e|-|1\rangle\langle 1|)\right]-\frac{g^2}{\Delta}\left[S_+S_-+a^\dagger a(|e\rangle\langle e|-|2\rangle\langle 2|)\right]$$

$$\simeq\frac{g^2}{\Delta}\left(a^\dagger a(|2\rangle\langle 2|-|1\rangle\langle 1|)\right)$$

可解模型：$H=a^\dagger a S_z+g(a+a^\dagger)$ $\quad H=\Delta a^\dagger a+g(a+a^\dagger)S_x$

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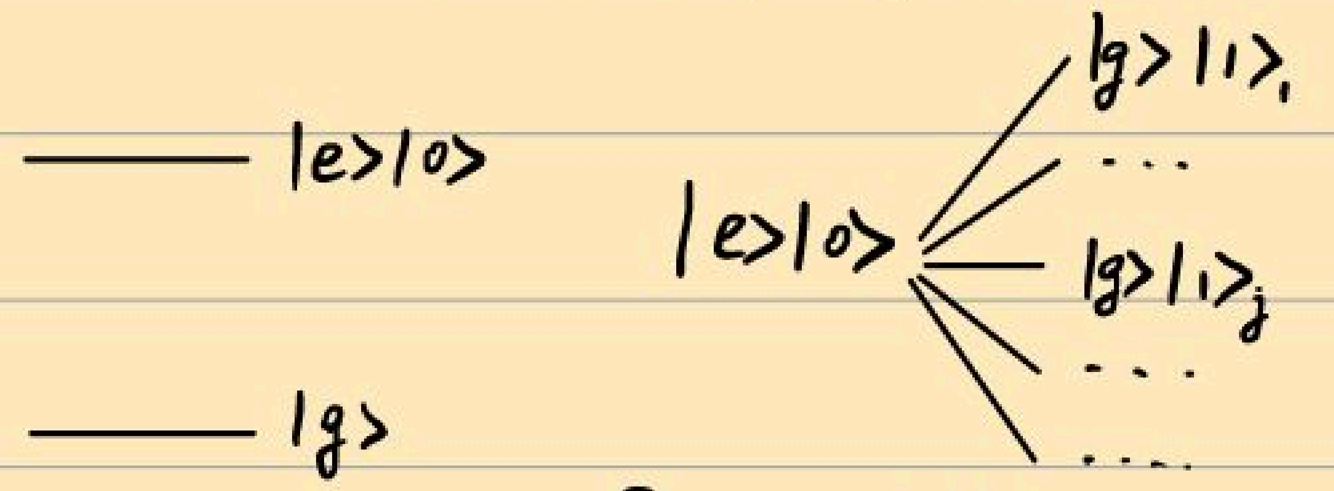
自发辐射：

$$H=\sum_k \omega_k|k\rangle\langle k|+\sum_k\Omega_k e^{i\nu_k t}|k\rangle\langle 1|+c.c.$$

$$H=\frac{\omega}{2}a_z+\sum_j \nu_j a_j^\dagger a_j+\sum_j g_j|e\rangle\langle g|a_j^\dagger+c.c.$$




$$|\psi\rangle=|g\rangle\otimes\sum_k\frac{g_k}{(\nu_k-\omega)+i\gamma/2}e^{-i\nu_k t}|1_k\rangle \quad (t\to\infty)$$ 单光子波包

存储过程：$|\psi\rangle=|g\rangle\otimes\left(\sum_k C_k|k\rangle\right)\Rightarrow|\psi\rangle=|e\rangle\otimes|0_k\rangle$

开放量子系统，引入密度算符

$\rho=\sum_\alpha P_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|$ 混合叠加

$|\psi\rangle=\alpha|\psi_1\rangle+\beta|\psi_2\rangle$ 相干叠加

$$H=\omega a_z+\sum_j\nu_j a_j^\dagger a_j+\sum_j g_j a_j^\dagger a_-+c.c.$$

记 $F(t)=\sum_j g_j a_j^\dagger$ 　做 Markov 近似.（环境不受相互作用影响）

$t=0$ $\rho(t)=\rho_0\otimes\rho_B$ 　　$t=t$ $\rho(t)=\rho_0(t)\otimes\rho_B$

$$\frac{\partial\rho}{\partial t}=-i[H,\rho] \qquad \rho(t)=\rho(0)-i\int[H,\rho(t')]dt'=\rho(0)-i\int[H(t'),\rho(0)]+(-i)^2\int[H(t'),[H(t''),\rho(t'')]]$$

$$H=\sum_j g_j e^{i\nu t}a_j^\dagger a_-+c.c.=F(t)a_-+c.c.$$

$$\Rightarrow\rho(t)=\rho(0)-i\int[F_+a_-+F_-a_+,\rho(0)]+(-i)^2\int[H(t'),[H(t''),\rho(t'')]]$$

$$=\rho(0)-i\int[F_+a_-+F_-a_+,\rho(0)]+(-i)^2\int[F_+a_-+F_-a_+,[F_+a_-+F_-a_+,\rho_s\otimes\rho_B]]$$

$$\Rightarrow\rho_s(t)=Tr_B\,\rho(0)+(-i)^2Tr_B[\qquad]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\frac{\gamma}{2}\bar{n}\left(a_- a_+ \rho - a_+ \rho a_-\right) - \frac{\gamma}{2}(\bar{n}+1)\left[a_+ a_- \rho - a_- \rho a_+\right] + c.c.$$

$$\begin{cases} |\psi(0)\rangle = e \\ \rho_{ee} = e^{-\gamma t}\rho_{ee} \end{cases}$$

自发辐射的主方程

当 $\bar{n}=0$ 真空场时: $\frac{\partial \rho}{\partial t} = \frac{\gamma}{2}\left[a_+ a_- \rho - a_- \rho a_+\right] + \frac{\gamma}{2}\left[\rho a_+ a_- - a_- \rho a_+\right]$

$a_- \sim \sum_j g_j a_j$

---

Landau Zenner Tunnel 精确可解模型

$$H = -\frac{\partial}{\partial t}|2\rangle\langle 2| - \Omega\left(|1\rangle\langle 2| + c.c\right)$$


$P = 1 - e^{-\frac{2\pi|\Omega|^2}{|\alpha|^2}}$ Kibble-Zurek

$$H = \Delta + Vt \qquad \Delta = \begin{pmatrix} \Delta_{11} & \cdots & \Delta_{1n} \\ \vdots & \cdots & \vdots \\ \Delta_{n1} & \cdots & \Delta_{nn} \end{pmatrix} \qquad V = \begin{pmatrix} V_1 & & 0 \\ & \ddots & \\ 0 & & V_n \end{pmatrix}$$

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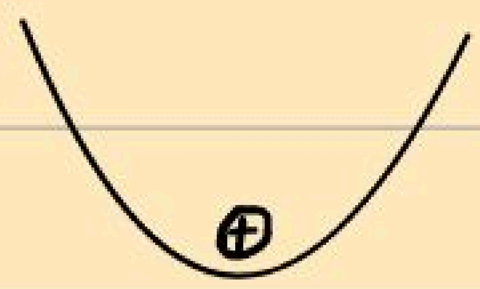
离子阱:

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(H_a \cos\omega t\right)x^2$$



$$H = \overbrace{\boxed{\frac{p^2}{2m} + \frac{m\omega^2}{2}x^2}}^{H_0} + \omega_0 a_z + a_+ e^{ikx + i\omega t} + c.c$$ 行波

$$H = \boxed{\frac{p^2}{2m} + \frac{m\omega^2}{2}x^2} + a_z \cos(\omega t + kz) + c.c.$$ 驻波

对行波做RWA:

$$\Rightarrow H = \underbrace{\nu a^\dagger a + \omega_0 a_z}_{H_0} + \underbrace{g a_+ e^{-\eta(a+a^\dagger) + i\omega t} + c.c.}_{H_1}$$

其中 $kx = \eta$ 称为 Lamb-Diche 参数

$$H_1 = g a_+ e^{i\Delta t + i\eta\left(a e^{i\nu t} + a^\dagger e^{-i\nu t}\right)} + c.c.$$

假定 $\eta \ll 1$: $\Delta = \omega - \omega_0$ 有:

$$H_1 = g a_+ e^{i\Delta t}\left[1 + \underbrace{\eta\left(a e^{i\nu t} + a^\dagger e^{-i\nu t}\right) + c.c.}_{\text{一阶}} \underbrace{- \frac{\eta^2}{2}\left(a e^{i\nu t} + a^\dagger e^{-i\nu t}\right)^2 + c.c.}_{\text{二阶}} \cdots\right.$$

(1) $\Delta=\nu$ $H=ga_+\cdot\eta a^\dagger+c.c.\rightarrow a^\dagger a_+ + c.c.$

(2) $\Delta=-\nu$ $H=ga_+\cdot\eta a+c.c.\rightarrow aa_+ + c.c.$

(3) $\Delta=2\nu$ $H=\eta^2 a_+ a^{2\dagger}+c.c.\longrightarrow a^{\dagger 2}a_+ + c.c.$

(4) $\Delta=-2\nu$ $H=-\eta^2 g a_+ a^2 + c.c.\longrightarrow a^2 a_+ + c.c.$

(5) $\Delta=0$ $H=ga_x - \eta^2 g a^\dagger a a_x$

应用：离子冷却 $a\longrightarrow|0\rangle$

两束光与原子耦合：

光1 光2

$$H=\frac{p^2}{2m}+\frac{m\omega^2}{2}x^2+\omega_0 a_z+g_1 e^{ik_1x}e^{i\omega_1 t}a_+ + c.c. + g_2 e^{ik_2x}e^{i\omega_2 t}a_+ + c.c.$$

$$H=\nu a^\dagger a+\omega a_z+g_1 e^{i\eta_1(a+a^\dagger)}e^{i\omega_1 t}a_+ + c.c. + g_2 e^{i\eta_2(a+a^\dagger)}e^{i\omega_2 t}a_+ + c.c.$$

$$H=g_1 e^{i\eta_1(ae^{i\nu t}+a^\dagger e^{-i\nu t})}e^{i\Delta_1 t}a_+ + c.c. + g_2 e^{i\eta_2(ae^{i\nu t}+a^\dagger e^{-i\nu t})}e^{i\Delta_2 t} + c.c.$$

$$=g_1 e^{i\Delta_1 t}a_+\left[1-\eta_1(ae^{i\nu t}+a^\dagger e^{i\nu t})-\eta_1^2(\cdots)^2+\cdots\right]+g_2 e^{i\Delta_2 t}a_+\left[1+i\eta_2(ae^{-i\nu t}+a^\dagger e^{i\nu t})+(\cdots)^2+\cdots\right]$$

(1) 当 $\Delta_1=\nu$ $\Delta_2=-\nu$ 时：

$$H=g_1 a_+ a+g_2 a_+ a^\dagger = a_+[\eta g_1 a+\eta g_2 a^\dagger]+c.c.$$

(2) 当 $\Delta_1=\nu$ $\Delta_2=2\nu$ 时：

$$H=a_+(g_1 a+g_2\eta^2 a^2)+c.c.$$

Civac P. Zoller R. Blatt. 激光冷却.

$$\partial(\rho)=k[A\rho A^\dagger-\tfrac{1}{2}A^\dagger A\rho-\tfrac{1}{2}\rho A^\dagger A]$$ 考虑耗散.

$$H=g(\mu a+\nu a^\dagger)a_-+c.c.=A^\dagger a_- + Aa_+$$

$$\Rightarrow \partial(\rho)=k(a_-\rho a_+ -\tfrac{1}{2}a_+a_-\rho-\tfrac{1}{2}\rho a_+a_-)$$ 单向自发辐射耗散.

$$\frac{\partial\rho}{\partial t}=i[H,\rho]+k\frac{\partial}{\partial t}\rho$$

若 $k\gg g$ 耗散强度远大于相干强度 $\Rightarrow \overset{A}{\boxed{(\mu a+\nu a^\dagger)}}(\sum_k g_k a_k^\dagger)+c.c.$

有 $\frac{\partial\rho}{\partial t}=i[H,\rho]+(A\rho A^\dagger-\tfrac{1}{2}A^\dagger A\rho-\tfrac{1}{2}\rho A^\dagger A)$

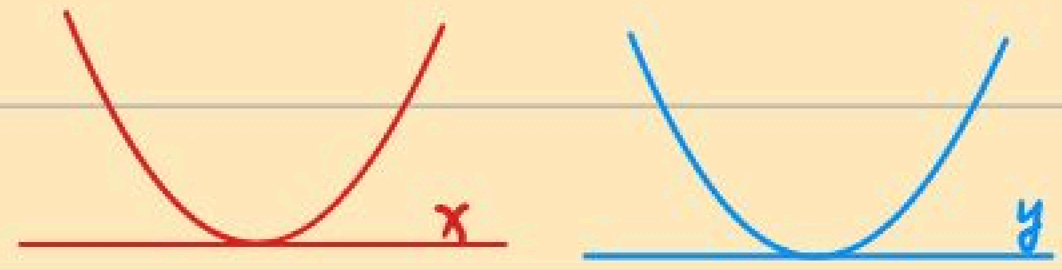
做稳态解：$\frac{\partial\rho}{\partial t}=0$

(1) $A=(\mu a+\nu a^{+})$

$(\mu a+\nu a^{+})|\psi\rangle=0 \Rightarrow$ 解为 $|\psi\rangle=e^{s(a^2-a^{+2})}$ 纯态

(2) $A=(a^2-\alpha)$

解为 $(|+\alpha\rangle+|-\alpha\rangle)$ $(|+\alpha\rangle-|-\alpha\rangle)$ 混态.




二维离子阱束缚

$$H=w_x a_x^{+}a_x+w_y a_y^{+}a_y+w_0 a_z+e^{i\eta_x(a_x+a_x^{+})+i\eta_y(a_y^{+}+a_y)}e^{iwt}+c.c.$$

其中 $\eta_x$ $\eta_y$ 为 Lamb-Diche 参量.

$$H=e^{i\Delta t}a_{+}\,e^{i\eta_x(a_x e^{-iw_x t}+a_x^{+}e^{iw_x t})+i\eta_y(a_y e^{-iw_y t}+a_y^{+}e^{iw_y t})}$$

$$\Rightarrow\quad e^{i\eta(ae^{-iwt}+a^{+}e^{iwt})}=e^{-\frac{\eta^2}{2}}e^{i\eta ae^{-iwt}}e^{i\eta a^{+}e^{iwt}}$$

$$=\sum_{mn}\eta^{n+m}a^{n}(a^{+})^{m}e^{i(n-m)wt}=\sum_{n}\frac{i\eta ae^{-iwt}}{n}\sum_{m}\frac{i\eta a^{+}e^{iwt}}{m}$$

当 $m-n=k$ $\Delta=kw \Longrightarrow H=e^{i\Delta t}\sum_{m_1 n_1}e^{i(m_1-n_1)w_x t}\sum_{m_2 n_2}e^{i(m_2-n_2)w_y t}$ $\quad\Delta=k_1w_x+k_2w_y$.

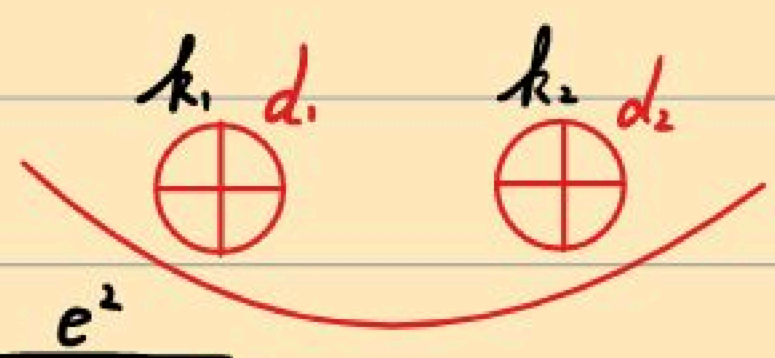
$$\begin{cases}\Delta=k_1w_x+k_2w_y\\ k_1=m_1-n_1\\ k_2=m_2-n_2\end{cases}\Rightarrow (m_1-n_1)w_x+(m_2-n_2)w_y=k_1w_x+k_2w_y$$

$\Rightarrow(m_1-n_1-k_1)w_x=(m_2-n_2-k_2)w_y$ $\neq 0$

当满足 $w_x$ 与 $w_y$ 互质时 $(m_1-n_1-k_1)w_x=(m_2-n_2-k_2)w_y$ $=0$

实验上通常无法做到互质.

两离子离子阱:




$$H=\frac{P_1^2}{2m}+\frac{P_2^2}{2m}+wx_1^2+wx_2^2-\frac{e^2}{|x_1-x_2+d|}$$

$d\gg x_1\,x_2 \Rightarrow H=\frac{P_1^2}{2m}+\frac{P_2^2}{2m}+wx_1^2+wx_2^2+\lambda(x_1-x_2)^2$ 质心系 $x_c=\frac{x_1+x_2}{2}$ $\quad x_r=\frac{x_1-x_2}{2}$

$$=\frac{P_c^2}{4m}+\frac{nx_c^2}{2}+\frac{P_r^2}{2m}+\frac{nx_r^2}{2}+\lambda x_r^2$$

独立操控 $k_1k_2$: $H=\frac{P^2}{2m}+\frac{mw^2}{2}x^2+w_1a_{z1}+w_2a_{z2}+g_1e^{iw_1t}e^{ik_1x}a_{1+}+g_2e^{iw_2t}e^{ik_2x}a_{2+}+c.c.$

$H_o = g_1 e^{i\Delta_1 t} e^{i\eta_1(a+a^+)} a_{1+} + g_2 e^{i\Delta_2 t} e^{i\eta_2(a+a^+)} a_{2+}$

当 $\Delta_1 = w$ $\Delta_2 = w$ 有 $H = ga(a_{1+} + a_{2+}) + c.c.$

当 $\Delta_1 = -w$ $\Delta_2 = -w$ 有 $H = ga^+(a_{1+} + a_{2+}) + c.c.$

当 $\Delta_1 = w+\delta$ $\Delta_2 = w+\delta$ 有 $H = ga\, e^{i\delta t}(a_{1+} + a_{2+}) + c.c.$

当 $\Delta_1 = -w-\delta$ $\Delta_2 = -w-\delta$ 有 $H = ga^+ e^{-i\delta t}(a_{1+} + a_{2+}) + c.c.$

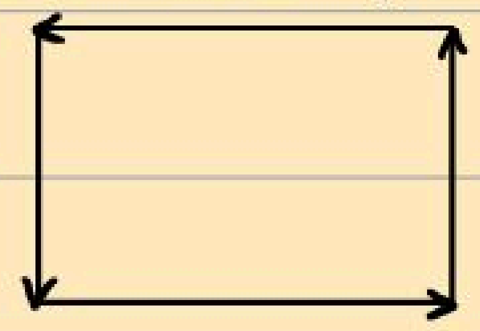
SM Gate $H = g(a+a^+)(a_{x1} + a_{x2}) \Rightarrow H = g(a e^{i\delta t} + a^+ e^{-i\delta t})(a_{x1} + a_{x2})$ Molmer

当 $\delta \gg g$ 时大失谐有：$H = g(ae^{i\delta t} + a^+ e^{-i\delta t}) J_x = \frac{g^2}{\delta} J_x^2$

偏离共振点时：$H_o = \delta a^+ a + g(a+a^+) J_x$ 共振时无第一项

$\xrightarrow{\text{幺正变换}} H = g(ae^{i\delta t} + a^+ e^{-i\delta t}) J_x \longrightarrow H = (a+a^+) J_x\, e^{iPJ_x} e^{ixJ_x} e^{-iPJ_x} e^{-ixJ_x}$

讨论共振：$H = (a+a^+) J_x$



$e^{iP\alpha J_x} e^{ix\alpha J_x} e^{-iP\alpha J_x} e^{-ix\alpha J_x} = e^{-i\alpha J_x^2}$

几何门

$H_o = \delta a^+ a + g(a+a^+) J_x$

① $\delta \gg g$ $H = \frac{g^2}{\delta} J_x^2$

② $H = \delta(a^+ + \frac{gJ_x}{\delta})(a + \frac{gJ_x}{\delta}) - \frac{g^2 J_x^2}{\delta} = D(\frac{gJ_x}{\delta})\, \delta a^+ a D^+ - \frac{g J_x^2}{\delta}$ 其中 $D(\frac{gJ_x}{\delta}) = e^{\frac{gJ_x}{\delta}(a-a^+)}$

腔QED：$H = \Delta a_z + gaS_+ + ga^+S_- \Rightarrow H = \frac{g^2}{\Delta} a^+ a J_z + J_+ J_-$ 与离子阱极为相似.

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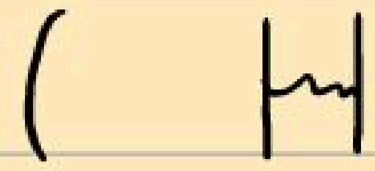
光力

共振相互作用：$H = aa_+ + a^+ a_-$

色散相互作用：$H = a^+ a\, a_z$

C.K. Law H. Kimble Vahala



$L = L_0 + g\cos wt$

腔：$H = w_0(L)\, a^+ a$ e.g. FP腔；回音壁腔.

$H = w_0 a^+ a + wb^+ b + ga^+ a(b+b^+)$ 不依赖于光子数交换，仅依赖于光子数多少

光力系统：宏观量子效应 量子冷却

冷却：$E=\hbar\omega$ $\quad kT \quad$ 量子化条件：$\hbar\omega \gg kT_c$

激光冷却：$H=a^\dagger b+ab^\dagger$ $\quad |n_b\rangle|0\rangle \longleftrightarrow |n_b-1\rangle|1\rangle \xrightarrow{a\text{耗散}} |n_b-1\rangle|0\rangle$

光力系统：

$\frac{\partial\rho}{\partial t}=i[H,\rho]+\mathcal{L}(a)+\mathcal{L}(b)$ 其中 $\mathcal{L}(a)=\gamma_1(a\rho a^\dagger-\frac{1}{2}a^\dagger a\rho-\frac{1}{2}\rho a^\dagger a)$ Schrödinger Pic

$\langle a\rangle=Tr(a\rho) \Rightarrow \frac{\partial}{\partial t}\langle a\rangle=Tr(a\frac{\partial}{\partial t}\rho)=Tr(a\,\mathcal{L}(\rho))=Tr(\mathcal{L}^\dagger(a)\rho)$ $\quad \frac{\partial\rho}{\partial t}=\mathcal{L}\rho$

推导详见 Quantum Noise P. Zoller

$\frac{\partial\langle a\rangle}{\partial t}=(i\omega_a-\gamma_a)\langle a\rangle+g\langle a(b+b^\dagger)\rangle$

$\frac{\partial\langle b\rangle}{\partial t}=(i\omega_b-\gamma_b)\langle b\rangle+g\langle a^\dagger ab\rangle$

稳态解：由 $\langle AB\rangle=\langle A\rangle\langle B\rangle$ $\quad \frac{\partial}{\partial t}\begin{pmatrix}\langle a\rangle\\ \langle b\rangle\end{pmatrix}=0 \Rightarrow \begin{pmatrix}\langle a\rangle\\ \langle b\rangle\end{pmatrix}=0$

加入经典驱动：$H=\Delta a^\dagger a+\omega_b b^\dagger b+ga^\dagger a(b+b^\dagger)+\eta(a+a^\dagger)$

$$\Rightarrow\begin{cases}\frac{\partial\langle a\rangle}{\partial t}=(i\omega_a-\gamma_a)\langle a\rangle+g\langle a(b+b^\dagger)\rangle+\eta\\ \frac{\partial\langle b\rangle}{\partial t}=(i\omega_b-\gamma_b)\langle b\rangle+g\langle a^\dagger ab\rangle\end{cases}$$ 求解 $\frac{\partial}{\partial t}\begin{pmatrix}\langle a\rangle\\ \langle b\rangle\end{pmatrix}=0$

$$\Rightarrow\begin{cases}\langle a\rangle=\frac{\eta}{i\Delta-\gamma_a}\\ \langle b\rangle=0\end{cases}$$

$$\begin{cases}a=\alpha+a'\\ b=\beta+b'\end{cases}$$ 做微扰展开：$H=\Delta a^\dagger a+\omega b^\dagger b+ga^\dagger a(b+b^\dagger)+(a+a^\dagger)$

注：非线性方程线性化 $\begin{cases}\frac{\partial x}{\partial t}=f(x,y)=0\\ \frac{\partial y}{\partial t}=g(x,y)=0\end{cases}$ 置零得到 $(x_0,y_0)$ 解 $\Rightarrow\begin{cases}x=x_0+\delta_x\\ y=y_0+\delta_y\end{cases}$

$\begin{cases}x=x_0+x'\\ y=y_0+y'\end{cases}$ 代入 得 $\begin{cases}\frac{\partial x'}{\partial t}=f(x_0+x',y_0+y')=\overset{0}{\overset{\|}{f(x_0,y_0)}}+\frac{\partial f}{\partial x}\Big|_{x_0}x'+\frac{\partial f}{\partial y}\Big|_{y_0}y'\\ \frac{\partial y'}{\partial t}=g(x_0+x',y_0+y')=\overset{\|}{g(x_0,y_0)}+\frac{\partial g}{\partial x}\Big|_{x_0}x'+\frac{\partial g}{\partial y}\Big|_{y_0}y'\end{cases}$

$\Rightarrow \frac{\partial}{\partial t}\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}f_x(x_0,y_0) & f_y(x_0,y_0)\\ g_x(x_0,y_0) & g_y(x_0,y_0)\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix}$ 求解本征值 $\gamma\begin{cases}>0 & (x_0,y_0)\text{是发散点}\\ <0 & (x_0,y_0)\text{是收敛点}\end{cases}$ 等于0特殊情况

上述即为非线性光学系统线性化的过程

代入 $\begin{cases}a=\alpha+a'\\ b=\beta+b'\end{cases} \Rightarrow H=\Delta a'^\dagger a'+\omega b'^\dagger b'+g(a'^\dagger+\alpha^\dagger)(a'+\alpha)(b+b^\dagger)$

$\Rightarrow H = \Delta a^\dagger a + \omega b^\dagger b + g(a^\dagger \alpha + \alpha^* a)(b + b^\dagger)$ 线性系统

注：其中略去线性项，并在$|\alpha| >> 1$下有：$H = a^\dagger a(b+b^\dagger)$

(1) $\Delta = \omega >> g\alpha$

$H = \alpha(ae^{i\omega t} + a^\dagger e^{-i\omega t})(be^{i\omega t} + b^\dagger e^{-i\omega t})$

$= g\alpha[abe^{2i\omega t} + a^\dagger b^\dagger e^{-2i\omega t} + a^\dagger b + ab^\dagger]$

$H = a^\dagger b + ab^\dagger$ 光声子交换

(2) $\Delta = -\omega >> g\alpha$

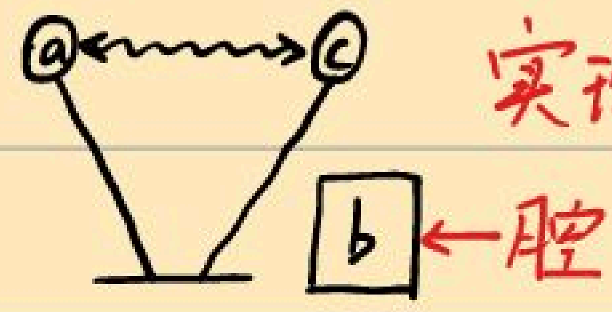
$H = g\alpha(ab + a^\dagger b^\dagger)$ 压缩

双振子：  |b  a)

(c

$H = \Delta_a a^\dagger a + \Delta_c c^\dagger c + \omega b^\dagger b + a^\dagger a(b+b^\dagger) + c^\dagger c(b+b^\dagger) + \eta_1(a+a^\dagger) + \eta_2(c+c^\dagger)$

当 $\Delta_a = \Delta_c = \omega$ 时 $H = g_1(a^\dagger b + ab^\dagger) + g_2(c^\dagger b + cb^\dagger)$  $\omega_a - \omega_a' = \omega_b - \omega_b' = \omega$




实现了两光的耦合（可以实现光波和微波的转换）

相较于Raman共振耦合，频差更大

若无驱动：$H = \omega a^\dagger a + g a^\dagger a(b+b^\dagger) + \omega_b b^\dagger b$ 且无耗散.

$U = e^{-i\omega a^\dagger a t} e^{-i(\omega_b b^\dagger b + g a^\dagger a(b+b^\dagger))t}$

$H = \omega b^\dagger b + g(b+b^\dagger)a^\dagger a = \omega[b^\dagger + \frac{g}{\omega}a^\dagger a][b + \frac{g}{\omega}a^\dagger a] - \frac{g^2}{\omega}(a^\dagger a)^2$

$H = D(\alpha)b^\dagger b D^\dagger(\alpha) - \frac{g^2}{\omega}a^\dagger a$  $D(\alpha) = e^{\frac{g}{\omega}a^\dagger a(b - b^\dagger)}$

$D(\alpha)aD^\dagger(\alpha) = a - \alpha$  $D(\alpha)a^\dagger D^\dagger(\alpha) = a^\dagger - \alpha$  $\alpha = \frac{g}{\omega}a^\dagger a$

$\Rightarrow D(\alpha)e^{-ib^\dagger bt}D^\dagger(\alpha)e^{-i\frac{g^2}{\omega}(a^\dagger a)^2} + e^{-\frac{g}{\omega}a^\dagger a(b-b^\dagger)}e^{-b^\dagger bt}D^\dagger(\alpha)\rho$

类比SM：$H = \delta a^\dagger a + g(a+a^\dagger)J_x$  $H = \omega b^\dagger b + g(b+b^\dagger)a^\dagger a$

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光镊：  c)  $H = b(g_1 a^\dagger + g_2 c^\dagger) + h.c.$

(a  非线性光学系统

$$H = g a b^{2+} + g a^+ b^2 + \eta (a + a^+)$$

$a \longrightarrow \boxed{\phantom{x}} \longrightarrow b$ $\qquad \frac{\partial \rho}{\partial t} = i[H, \rho] + \kappa\, \mathcal{D}\, a$

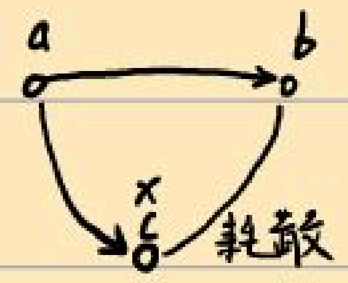
$\langle \frac{\partial \rho}{\partial t} \rangle = 0 \Rightarrow a = \frac{\eta}{\kappa} \Rightarrow H = g \frac{\eta b^{2+}}{\kappa} + g \frac{\eta b^2}{\kappa}$

$$H = b(g_1 a + g_2 c^+) + h.c.$$

非互易 Quantum Noise.

$\frac{\partial \rho}{\partial t} = i[H_0, \rho] + \mathcal{D}(a + e^{i\varphi} b)\rho$

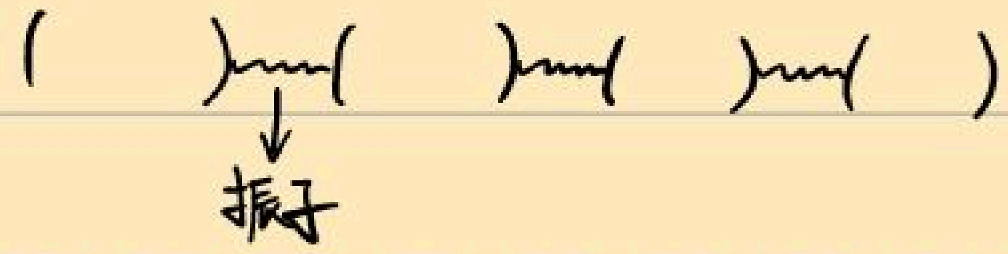
$H_0 = a^+ b + a b^+ \qquad \alpha = a + e^{i\varphi} b$ 非相干耦合




$c(a^+ + e^{i\varphi} b) + h.c. + \mathcal{D}(c) \Rightarrow \mathcal{D}(a + e^{i\varphi} b)$

分离器

实验：




布拉格晶体.

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Rydberg 原子： Saffman Milukin

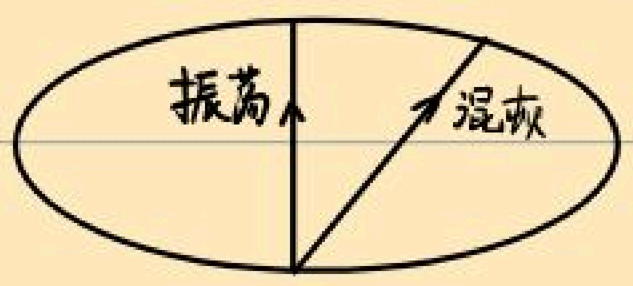
$H = \frac{1}{|r_1 - r_2|}\left(a_i a_j - \frac{a_i n\, a_j n}{3}\right)$

$H = \frac{a_{zi} a_{zj}}{|r_i - r_j|^3} + a_{xi}$

$H = \sum_i (a_{xi} a_{xj} + a_{yi} a_{yj}) + \Delta a_{zi} a_{xj}.$

$H = \sum_i a_i a_{i+1} + a_{zi} + \Delta a_{zi}$

Quantum body Scar. (Papic) Ising gauge theory




混沌系统

Scar Heller. (rare event)

$H = \sum_i |0\rangle_i\langle 0| \otimes |0\rangle_{i+1}\langle 0| + \sum_i g a_{xi}$ (初始)

$\Rightarrow$ 相互作用表象： $\sum_i e^{i(\Omega|0\rangle\langle 0| + \Omega|0\rangle\langle 0|)t} a_{+i} + c.c.$ M. Lukin.

$\Rightarrow H = [|0\rangle\langle 0| e^{i\Omega t} + |1\rangle\langle 1|]\, a_{+i}\, [|0\rangle\langle 0| e^{i\Omega t} + |1\rangle\langle 1|] + c.c.$

$= |1\rangle_{i-1}\langle 1|\, a_{+i}\, |1\rangle_{i+1}\langle 1| + c.c. = \sum_i P_{i-1} x_i P_{i+1}$ PXP 模型

$$\begin{cases} H = \sum_i P_i X_{i+1} P_{i+1} \\ A = \sum_i (-1)^i P_{i-1} a_{+i} P_{i+1} \end{cases} \qquad [H, A] = A$$

Sachdev 提出

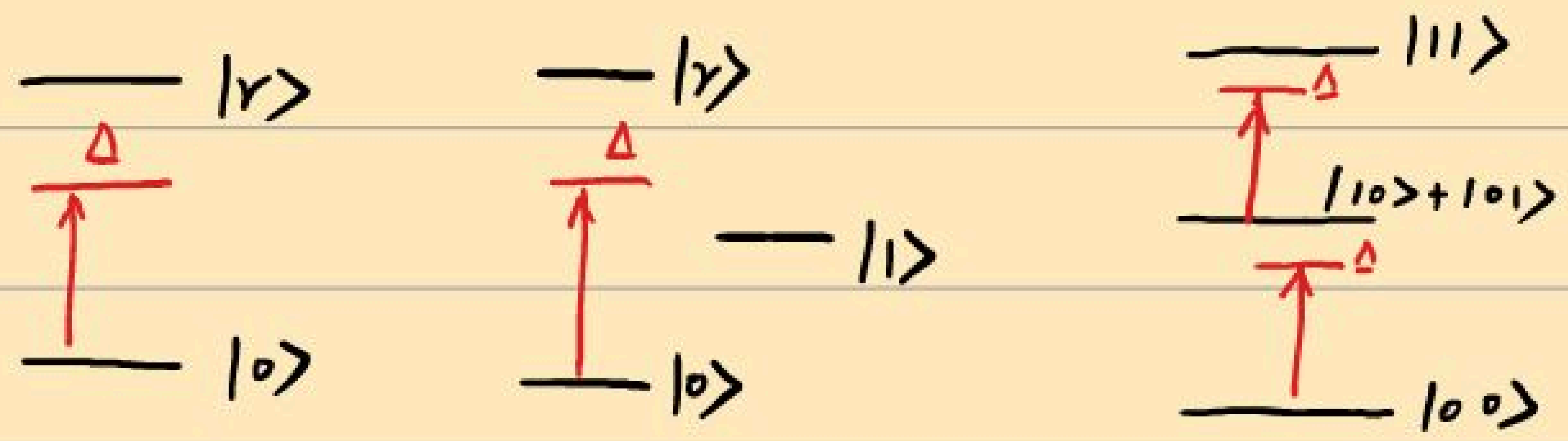
Many body Scar $\Rightarrow$ 本征态热化 $\Longrightarrow$ 实验呈对数 $\Rightarrow$ 熵

Eth假定: $\langle A\rangle = Tr(A\rho)$

Many body scar: 受限量子系统

Rydberg 原子: M. Lukin 8bit 编码保证度90%以上 2023年12月7日

PXP:




产生Stark位移

$H=\sum_i \Omega |1\rangle\langle 1|\otimes|1\rangle\langle 1| + \boxed{\Delta n_i + \Omega_i |1\rangle\langle 1| + c.c}$ 激光驱动.

经典: $H = \sigma_z \sigma_z$ Ising Model

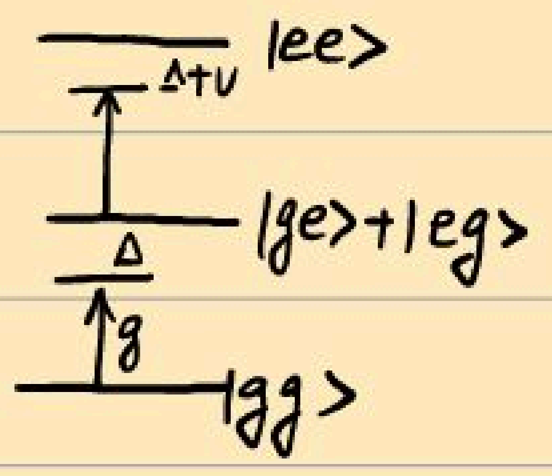
量子: $H=\sum_i \sigma_{zi}\sigma_{zi} + g\sigma_{xi}$

应用: 经典优化: NP 难题.

经典计算 { 模拟计算: 量子
数字计算: 电子计算机

Quantum Blockade 量子阻塞:

$H=\Delta(\sigma_{z1}+\sigma_{z2}) + V\sigma_{z1}\sigma_{z2} + g(\sigma_{x1}+\sigma_{x2})$




若$\Delta=0$ 无失谐, $V>>g$.

$H=(\Delta+V)|ee\rangle\langle ee| + \Delta|+\rangle\langle +| + \sqrt{2}g(|ee\rangle\langle +| + |+\rangle\langle g|) + c.c.$

$V>>g \Rightarrow H=|+\rangle\langle gg| + c.c.$ 形成阻塞.

$H=\sum_n \Delta|n\rangle\langle n| + \sum_n n^2|n\rangle\langle n| + g(|0\rangle\langle 1| + \sqrt{2}|1\rangle\langle 2| + c.c.)$

$\Rightarrow H = g(|0\rangle\langle 1| e^{i(V+\Delta)t} + \sqrt{2}|1\rangle\langle 2| e^{i(3V+\Delta)t} + c.c.)$

在$V+\Delta=0$时 $H=g(|0\rangle\langle 1| + \sqrt{2}|1\rangle\langle 2| e^{2iVt} + c.c$

$V>>g$时 形成阻塞.(很难实现)

$$H=\sum_i V a_z a_z + \sum_i \Delta a_z + \sum_i g a_x$$

$$\Rightarrow H = g e^{i\Delta t} e^{iVt a_{z i-1}} e^{iVt a_{z i+1}} a_{+i} + c.c.$$

$$= e^{i\Delta t}\left(e^{iVt}\frac{1+a_{zi-1}}{2} + e^{-iVt}\frac{1-a_{zi-1}}{2}\right)\left(e^{iVt}\frac{1+a_{zi+1}}{2} + e^{iVt}\frac{1-a_{zi-1}}{2}\right) + c.c.$$

(1) $\Delta=2V$ 时： $H=\frac{1-a_{zi-1}}{2} a_x \frac{1-a_{zi+1}}{2}$ $=|0\rangle\langle 0| a_x |0\rangle\langle 0|$ 基态.

(2) $\Delta=-2V$ 时： ...

Lukin (发现现象) Papic (解释)

— $|r\rangle$ $\quad H = g\sum_i |r\rangle\langle a| + c.c. + \sum_i V|rr\rangle\langle rr|$ 等同 JC 模型

— $|a\rangle$ Deutsch

超导：PRX Quantum 2, 040204 (2021)

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超导：基尔霍夫定理 工程.

D-wave

Quantum Circuit 量子电路 (Google)

传统物理： $H=\sum_i H_i$ 连续模型

Google： $U = U_e S U_o$ $\quad e^{-iHt} = e^{-i\sum_i H_i t}$

$U = e^{-i(a_x a_x + a_y a_y + \Delta a_z a_z)\Delta t}$ 80多个Bit

理论很成熟，工程上很困难.

K.P Zhang Univerial

Measurement Induced Phase transition.

纠缠熵： $\rho_A = Tr_B\, \rho_{AB}$ $\quad E(\rho_A) = -Tr\, \rho_A \ln \rho_A$ $\quad \rho = \sum_n \lambda_n |\psi_n\rangle\langle\psi_n|$

$\Rightarrow \rho_A = \sum_n \lambda_n Tr_B(|\psi_n\rangle\langle\psi_n|) = \sum_n \lambda_n E_n$ 纠缠熵是关于ρ的非线性函数

两条计算路径

S(L) ∝ 常数 面积率 ∝L 即体积率.

冷原子. Mr. Fisher

$H=-\sum_i t(a_i^+ a_{i+1}+c.c)+U\sum_i n(n-1)$ 一维光子晶格

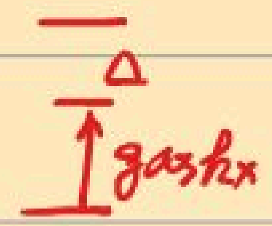
Bose-Hubburd Model

$t\ll u.$ ⟵

$t\gg u$ $(a_1^+ + a_2^+ + \cdots + a_n^+)^n|0\rangle$ 所有原子都摊开

Superfluid-Mott 相变

$H=\Delta a_z+g\cos kx\, a_x \xRightarrow{\Delta\gg g} H=\frac{g^2}{\Delta}\cos^2 kx\, a_z$




大失谐

$\Rightarrow H=\frac{g^2}{\Delta}\cos kx$ 驻波模拟势场 (Bose爱因斯坦凝聚)

$H=\frac{p^2}{2m}+E\cos kx$ D. Jackson Cirac P. Zoller 2000

用冷原子模拟Bose子. 二次量子化. $H=\int\psi^+(x)[H_0\psi(x)]+\int g(\psi^+(x))^2(\psi(x))^2$


$H=\frac{p^2}{2m}+\cos kx$ Bloch理论 ⇒ 能带 (Bloch波函数) ⇒ Wannier基 ⇒ 紧束缚近似

紧束缚近似: $\psi(x)=\sum_n C_n\phi(x-x_n)+\int g(\psi^+(x))^2(\psi(x))^2$ 技巧

$\int C_n^+\phi(x-x_n)H_0 C_m\phi(x-x_m)$ 其中 $\int\phi(x-x_n)\phi(x-x_m)dx=\delta_{mn}$ $H_0\varphi(x-x_n)\neq\varphi(x-x_m)$

近似: $H=\cos kx=1-\frac{1}{2}(kx)^2=p^2+x^2$ 谐振子

$t\propto E$ $u\propto g$ 可通过调整使 $t\ll u\Rightarrow t\gg u$

Greiner Bloch

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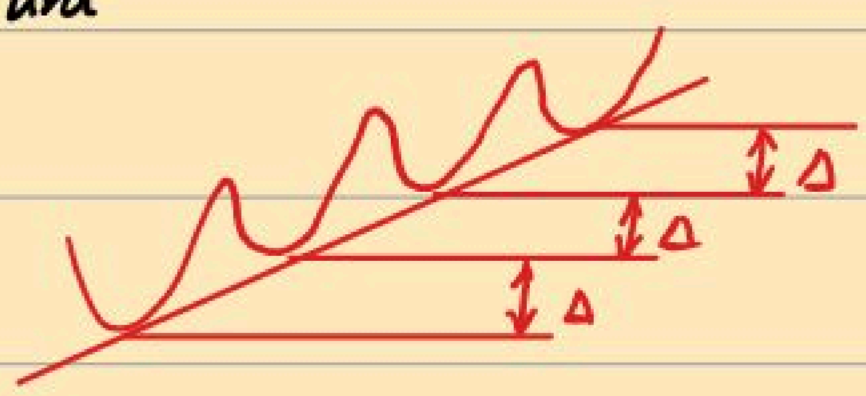
PXP 模型. $H=\sum_i|0\rangle\langle 0|a_x|0\rangle\langle 0|$ Bose-Hubburd

$H=-t\sum_i(a_i^+a_{i+1}+c.c.)+\sum_i U n_i^2+\Delta\sum_i i n_i$

$a=|0\rangle\langle 1|+\sqrt{2}|1\rangle\langle 2|+\cdots$ Fock态展开




做相互作用表象: $a\sum_n|n\rangle\langle n|\overset{e.g.}{=\!=}a_1^+a_2=e^{i\Delta t}(\sqrt{2}|2\rangle\langle 1|+|1\rangle\langle 0|)(|0\rangle\langle 1|+\sqrt{2}|1\rangle\langle 2|)$

$=(e^{2iUt}|2\rangle\langle 1|+|1\rangle\langle 0|)(|0\rangle\langle 1|+\sqrt{2}e^{-2iUt}|1\rangle\langle 2|)$

$= e^{i\Delta t}\left[e^{2iut}|2\rangle\langle 1|\otimes|0\rangle\langle 1| + |2\rangle\langle 1|\otimes|1\rangle\langle 2| + |1\rangle\langle 0|\otimes|0\rangle\langle 1| + e^{-2iut}|1\rangle\langle 0|\otimes|1\rangle\langle 2|\right]$

$H = |1\rangle\langle 0|\otimes|1\rangle\langle 2| \Longleftrightarrow |0\rangle\langle 0|\, a_x\, |0\rangle\langle 0|$

SYK

作业交到 xbz@ustc.edu.cn

Fermi Hubbard Model:

$H = -t\sum_i C_i^+ a\, C_i a + V\sum_i n_{i\uparrow}n_{i\downarrow}$ 一维

=维?

杨振宁: $\eta = \sum_i (-1)^i C_\uparrow^+ C_\downarrow^+$

$[H,\eta] = u\eta \qquad \eta^{+N}|0\rangle$ 本征态

$H = \sum_i i a_{zi} a_{zj} + a_{xi} = \sum_i a_{zi} a_{zi+1} + a_{xi}$ 里德堡原子

$H = \sum_i a_i a_i = \sum_i (a_{xi} a_{xj} + a_{yi} a_{yj} + a_{zi} a_{zj})$

$\Rightarrow H = \sum_i a_i a_i \Rightarrow H = H_0 + H_1$ 二阶微扰.

$H = C_{1\uparrow}^+ C_{2\uparrow} + C_{1\downarrow}^+ C_{2\downarrow} + c.c.$

$C_{1\uparrow}^+ \longrightarrow C_{1\uparrow}^+ e^{iVn_{i\downarrow}} = C_{1\uparrow}^+\left[1 - n_{1\downarrow} + e^{iut} n_{i\downarrow}\right] \qquad C_{i\downarrow} \to C_{i\downarrow}(1 - n_{i\uparrow} + e^{-iut} n_{i\uparrow})$

物理研究: 有效 $H_{eff} \Rightarrow$ 原始 $H_0$

写文章

$\Rightarrow C_{1\uparrow}^+ C_{2\uparrow} = (1 - n_{1\downarrow} + e^{iut} n_{1\downarrow})\, C_{1\uparrow}^+ (1 - n_{2\downarrow} + e^{-iut} n_{2\downarrow}) C_{2\uparrow}$

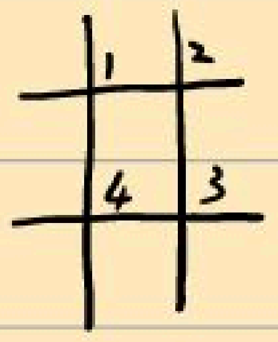
$= \left[(1-n_{1\downarrow})(1-n_{2\downarrow}) + n_{1\downarrow} n_{2\downarrow}\right] C_{1\uparrow}^+ C_{2\uparrow} + e^{-iut}(1-n_{1\downarrow}) n_{2\downarrow} (C_{1\uparrow}^+ C_{2\uparrow})$

$+ e^{iut}(n_{1\downarrow}(1-n_{2\uparrow}))\, C_{1\uparrow}^+ C_{2\uparrow}$

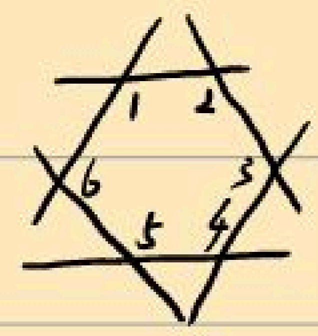
$\Rightarrow H = H_0 + H_1 e^{iut} + H_{-1} e^{-iut}$

$\Rightarrow H = H_0 + \frac{1}{u}[H_1, H_{-1}]$

$H = \sum_i (a_{xi})^2 + \sum_i a_{xi}$



$a_1 a_2 a_3 a_4$



$a_1 a_2 a_3 a_4 a_5 a_6$

Quantum Jump.
Quantum Stochastic Wave function
Quantum trajectory

求解主方程. $\frac{\partial \rho}{\partial t} = i[\rho, H] + \mathcal{L}(\rho)$ 其中 $\mathcal{L}(\rho) = \frac{\gamma}{2}[C^+C\rho + \rho C^+C - 2C\rho C^+]$

形式解. $\mathcal{L}_0(\rho) = i[\rho, H]$ $\frac{\partial \rho}{\partial t} = \mathcal{L}_0\rho + \mathcal{L}\rho$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \mathcal{L}\rho \quad \Rightarrow \rho(t) = e^{\mathcal{L}_0+\mathcal{L}}\rho(0) = e^{(\mathcal{L}_0+\mathcal{L}+J-J)t} = e^{(\mathcal{L}_0+\mathcal{L}-J)t + Jt}\rho(0)$$ $J\rho = c\rho c^+$

$$\Rightarrow \rho(t) = \sum_{m=0}^{\infty}\int_0^t dt_m \int_0^{t_{m-1}} dt_{m-1} \cdots \int_0^{t_2} dt_1 [S(t-t_m) J S(t_m - t_{m-1}) J \cdots J S(t_2-t_1) J S(t_1)\rho(0)]$$

其中 $S(t) = e^{(\mathcal{L}_0+\mathcal{L}-J)t}$ $S(t)\rho = e^{H_{eff}(t)}\rho e^{H_{eff}^+}$ $H_{eff} = -iH_0 - \gamma C^+C$

$J\rho = c\rho c^+ \Rightarrow S(t)\rho = e^{H_{eff}}\rho e^{H_{eff}^+}$ $H_{eff} = iH - \gamma c^+c = -i(H - ic^+c)$

$\rho(t) = e^{-iHt}\rho(0)e^{iHt}$ $H \Rightarrow H - ic^+c$ Schrödinger 演化

$$\Rightarrow \frac{\partial \rho}{\partial t} = i[H, \rho] + \gamma(c^+c\rho + \rho c^+c - 2\gamma c\rho c^+) = -iH\rho + i\rho H = -i(H - \gamma c^+c)\rho + \rho(H + i\gamma c^+c)$$

物理解释

$H = H_0 - i\gamma C^+C$

$\delta t \ll 1$: $|\psi(t+\Delta t)\rangle_1 = e^{-iH\delta t}|\psi(t)\rangle = (1 - iH\delta t)|\psi(t)\rangle$

$\langle\psi(t+\delta t)|\psi(t+\delta t)\rangle = \langle\psi|(1+iH^+\delta t)(1-iH\delta t)|\psi\rangle = 1 - \delta p$ 其中 $\delta p = \delta t\langle\psi|C^+c|\psi\rangle$

$|\psi(t)\rangle_2 \Rightarrow C|\psi(t)\rangle$ $\delta p$

$\sigma(t+\delta t) = (1-\delta p)|\psi(t+\delta t)\rangle_1\langle\psi(t+\delta t)| + \delta p|\psi(t)\rangle_2\langle\psi(t)|$

代入有: $= (1 - iH\delta t)|\psi\rangle\langle\psi|(1 + iH^+\delta t) + C|\psi\rangle\langle\psi|\delta t$ $H = H_0 - i\gamma C^+C$

$= |\psi\rangle\langle\psi| - iH|\psi\rangle\langle\psi| + i|\psi\rangle\langle\psi|H^+ + \delta t\, C|\psi\rangle\langle\psi|C^+$

$\Rightarrow$ $= |\psi\rangle\langle\psi| + \Delta t\gamma c|\psi\rangle\langle\psi| + i(H_0 - \frac{i\gamma}{2}C^+C)|\psi\rangle\langle\psi| - i|\psi\rangle\langle\psi|(H + \frac{i\gamma}{2}C^+C)$

$\Rightarrow$ 记: $\sigma(t) = |\psi(t)\rangle\langle\psi(t)|$ $\Rightarrow \frac{\sigma(t+\Delta t) - \sigma(t)}{\Delta t} = \gamma c\sigma c^+ - \frac{\gamma}{2}[C^+C\sigma + \sigma c^+c]$

$$\Rightarrow \frac{\partial \sigma}{\partial t} = [\sigma, H] + \mathcal{L}(\rho)$$