

复习 3

要点:

1. 分部积分法;
2. 换元法;
3. Newton-Leibniz 公式;
4. 积分中值定理;
5. 变上限积分的应用;
6. 广义积分的概念和计算.

例 1 计算 $I = \int e^{-x^2/2} \frac{\cos x - 2x \sin x}{2\sqrt{\sin x}} dx.$

解

$$\begin{aligned} I &= \int e^{-x^2/2} \frac{\cos x}{2\sqrt{\sin x}} dx + \int (-xe^{-x^2/2}) \sqrt{\sin x} dx \\ &= \int e^{-x^2/2} \frac{\cos x}{2\sqrt{\sin x}} dx \\ &\quad + \left(2e^{-x^2/2} \sqrt{\sin x} - \int 2e^{-x^2/2} \cdot \frac{1}{2} (\sin x)^{-\frac{1}{2}} \cos x dx \right) \\ &= \int e^{-x^2/2} \frac{\cos x}{2\sqrt{\sin x}} dx + \left(2e^{-x^2/2} \sqrt{\sin x} - \int e^{-x^2/2} \frac{\cos x}{2\sqrt{\sin x}} dx \right) \\ &= 2e^{-x^2/2} \sqrt{\sin x} + C \end{aligned}$$

例 2 计算 $I = \int \cos x \cos 2x \cos 3x \, dx$.

解

$$\begin{aligned}\cos x \cos 2x \cos 3x &= \cos 2x \frac{\cos 4x + \cos 2x}{2} \\&= \frac{1}{2} \cos 2x \cos 4x + \frac{1}{2} \cos^2 2x \\&= \frac{1}{4} (\cos 6x + \cos 2x) + \frac{1}{4} (1 + \cos 4x) \\&= \frac{1}{4} \cos 6x + \frac{1}{4} \cos 4x + \frac{1}{4} \cos 2x + \frac{1}{4}.\end{aligned}$$

故,

$$I = \frac{1}{24} \sin 6x + \frac{1}{16} \sin 4x + \frac{1}{8} \sin 2x + \frac{1}{4}x + C.$$

例 3 计算 $I = \int \sqrt{\frac{x}{1-x\sqrt{x}}} dx.$

解 作变换 $t = x^{\frac{3}{2}}$. 则 $dt = \frac{3}{2}\sqrt{x}dx$. 故,

$$\begin{aligned} I &= \frac{2}{3} \int \frac{dt}{\sqrt{1-t}} \\ &= \frac{2}{3} (-2\sqrt{1-t}) + C \\ &= -\frac{4}{3}\sqrt{1-t} + C \\ &= -\frac{4}{3}\sqrt{1-x\sqrt{x}} + C. \end{aligned}$$

例 4 证明 $\int_0^1 x^m(1-x)^n dx \leq \frac{m^m n^n}{(m+n)^{m+n}}$, m, n 为正常数.

证明 设 $f(x) = x^m(1-x)^n$. 则 f 在 $(0, 1)$ 为正, 在 $0, 1$ 为零, 因而 f 的最大值必在 $(0, 1)$ 中驻点取到. 因为

$$f'(x) = mx^{m-1}(1-x)^n - nx^m(1-x)^{n-1}.$$

令 $f(x) = 0$, 得

$$m(1-x) - nx = 0.$$

唯一的驻点为 $x_0 = \frac{m}{m+n}$. 于是

$$f(x) \leq f(x_0) = \left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n = \frac{m^m n^n}{(m+n)^{m+n}}.$$

故,

$$\int_0^1 x^m(1-x)^n dx \leq \frac{m^m n^n}{(m+n)^{m+n}}.$$

例 5 估计积分 $I = \int_0^{100} \frac{e^{-x}}{x+100} dx$ 精确到 0.0001.

解: 作变换, 可得 $I = \int_0^1 \frac{e^{-100x}}{x+1} dx$. 记

$$A = \int_0^1 (1-x)e^{-100x} dx.$$

则

$$\begin{aligned} A &= -\frac{e^{-100x}}{100}(1-x) \Big|_0^1 - \int_0^1 \frac{e^{-100x}}{100} dx \\ &= \frac{1}{100} - \frac{1}{100} \cdot \frac{-1}{100} e^{-100x} \Big|_0^1 \\ &= \frac{1}{100} - \frac{1 - e^{-100}}{100^2} \\ &= \frac{1}{100} - \frac{1}{100^2} + \frac{1}{100e^{100}} \end{aligned}$$

$$\begin{aligned}
0 &< I - \left(\frac{1}{100} - \frac{1}{100^2} \right) = I - \left(A - \frac{1}{100e^{100}} \right) \\
&= \int_0^1 \left(\frac{1}{1+x} - (1-x) \right) e^{-100x} dx + \frac{1}{100e^{100}} \\
&= \int_0^1 \frac{x^2}{1+x} e^{-100x} dx + \frac{1}{100e^{100}} \\
&< \int_0^1 x e^{-100x} dx + \frac{1}{100e^{100}} \\
&= -\frac{1}{100} e^{-100x} x \Big|_0^1 - \int_0^1 \left(-\frac{1}{100} e^{-100x} \right) dx + \frac{1}{100e^{100}} \\
&= -\frac{1}{100} e^{-100} - \frac{1}{100^2} e^{-100x} \Big|_0^1 + \frac{1}{100e^{100}} \\
&= \frac{1}{100^2} - \frac{1}{100} e^{-100} - \frac{1}{100^2} e^{-100} + \frac{1}{100e^{100}} \\
&< 0.0001.
\end{aligned}$$

故, $I \approx \frac{1}{100} - \frac{1}{100^2}$. 误差小于 0.0001.

例 6 计算

$$\lim_{n \rightarrow \infty} \left(\frac{n}{1^2 + \sqrt{n} + n^2} + \frac{n}{2^2 + 2\sqrt{n} + n^2} + \cdots + \frac{n}{n^2 + n\sqrt{n} + n^2} \right).$$

解：对任意 $\varepsilon \in (0, 1)$, 存在自然数 N , 当 $n > N$ 时, $1/\sqrt{n} < \varepsilon$. 因此, 当 $n > N$ 时, 有

$$\begin{aligned} \sum_{k=1}^n \frac{n}{k^2 + k\sqrt{n} + n^2} &> \sum_{k=1}^n \frac{1}{(k/n)^2 + \varepsilon + 1} \cdot \frac{1}{n} \\ &\rightarrow \int_0^1 \frac{dx}{x^2 + \varepsilon + 1}, \quad (n \rightarrow \infty). \end{aligned}$$

上式右端的积分当 ε 趋于零时, 趋于 $\frac{\pi}{4}$. 另一方面, 有

$$\sum_{k=1}^n \frac{n}{k^2 + k\sqrt{n} + n^2} < \sum_{k=1}^n \frac{1}{(k/n)^2 + 1} \cdot \frac{1}{n} \rightarrow \int_0^1 \frac{dx}{x^2 + 1} = \frac{\pi}{4}.$$

于是所求极限为 $\frac{\pi}{4}$.

例 7 设 $f(x)$ 在 $[0, 1]$ 上连续, 证明:

$$\int_0^\pi x f(\sin x) dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx,$$

并计算 $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$.

证明 对下面右端第二个积分作变换 $x = \pi - t$.

$$\begin{aligned}\int_0^\pi x f(\sin x) dx &= \int_0^{\pi/2} x f(\sin x) dx + \int_{\pi/2}^\pi x f(\sin x) dx \\&= \int_0^{\pi/2} x f(\sin x) dx - \int_{\pi/2}^0 (\pi - t) f(\sin t) dt \\&= \int_0^{\pi/2} x f(\sin x) dx + \int_0^{\pi/2} (\pi - x) f(\sin x) dx \\&= \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx.\end{aligned}$$

由此可得

$$\begin{aligned}\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx &= \pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx \\&= -\pi \int_1^0 \frac{1}{1 + t^2} dt \quad (t = \cos x) \\&= \pi \int_0^1 \frac{1}{1 + t^2} dt \\&= \pi \arctan t \Big|_0^1 \\&= \frac{\pi^2}{4}.\end{aligned}$$

例 8 设 $f(x)$ 处处连续, $f(0) = 0$, 且 $f'(0)$ 存在. 记 $F(x) = \int_0^1 f(xy) dy$.

证明 $F(x)$ 处处可导, 并求 $F'(x)$.

证明 显然 $F(0) = 0$. 当 $x \neq 0$ 时,

$$F(x) = \frac{1}{x} \int_0^x f(t) dt.$$

由于 $f(x)$ 处处连续, 故, 当 $x \neq 0$ 时, $F(x)$ 可导, 且

$$F'(x) = \frac{f(x)}{x} - \frac{1}{x^2} \int_0^x f(t) dt.$$

因为

$$\lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{1}{x} \int_0^x f(t) dt = \lim_{x \rightarrow 0} \frac{f(x)}{2x} = \frac{1}{2} f'(0).$$

故, $F'(0) = \frac{1}{2} f'(0)$.

例 9 设 $f(x)$ 和 $g(x)$ 在 $[0, 1]$ 上单调递增, 求证

$$\int_0^1 f(x)g(x) dx \geq \int_0^1 f(x) dx \int_0^1 g(x) dz.$$

证明: 对于 $x, y \in [0, 1]$, 有

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

即,

$$f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x).$$

将上式关于变量 x 和 y 分别在 $[0, 1]$ 积分, 可得

$$2 \int_0^1 f(x)g(x) dx \geq 2 \int_0^1 f(x) dx \int_0^1 g(x) dx.$$

例 10 设 $f(x)$ 在 $[0, \pi]$ 连续. 证明: 不能同时有

$$\int_0^\pi |f(x) - \sin x|^2 dx < \frac{\pi}{4}, \quad \int_0^\pi |f(x) - \cos x|^2 dx < \frac{\pi}{4}. \quad (1)$$

又问何时上面的两个不等式成为等式?

证明 利用 Cauchy-Schwartz 不等式, 有

$$\begin{aligned} & \int_0^\pi (\sin x - f(x))(f(x) - \cos x) dx \\ & \leq \left(\int_0^\pi |\sin x - f(x)|^2 dx \right)^{1/2} \left(\int_0^\pi |f(x) - \cos x|^2 dx \right)^{1/2}. \end{aligned}$$

因此当 (1) 中的两个不等式同时成立时, 有

$$\begin{aligned}\int_0^\pi |\sin x - \cos x|^2 dx &= \int_0^\pi |\sin x - f(x) + f(x) - \cos x|^2 dx \\&= \int_0^\pi |\sin x - f(x)|^2 dx + \int_0^\pi |f(x) - \cos x|^2 dx \\&\quad + 2 \int_0^\pi (\sin x - f(x))(f(x) - \cos x) dx \\&< \frac{\pi}{4} + \frac{\pi}{4} + 2 \cdot \frac{\pi}{4} = \pi.\end{aligned}$$

但是, 另一方面,

$$\int_0^\pi |\sin x - \cos x|^2 dx = \int_0^\pi (1 - \sin 2x) dx = \pi.$$

于是所证结论成立.

当 (1) 中两个不等式都是等式时, 应有

$$\begin{aligned} & \int_0^\pi (\sin x - f(x))(f(x) - \cos x) dx \\ &= \left(\int_0^\pi |\sin x - f(x)|^2 dx \right)^{1/2} \left(\int_0^\pi |f(x) - \cos x|^2 dx \right)^{1/2} = \frac{\pi}{4}. \end{aligned}$$

此时, 有

$$\begin{aligned} & \int_0^\pi \left(f(x) - \frac{\sin x + \cos x}{2} \right)^2 dx = \int_0^\pi \left(\frac{\sin x - f(x)}{2} - \frac{f(x) - \cos x}{2} \right)^2 dx \\ &= \frac{1}{4} \int_0^\pi |\sin x - f(x)|^2 dx + \frac{1}{4} \int_0^\pi |f(x) - \cos x|^2 dx \\ &\quad - \frac{1}{2} \int_0^\pi (\sin x - f(x))(f(x) - \cos x) dx \\ &= \frac{\pi}{16} + \frac{\pi}{16} - \frac{\pi}{8} = 0. \end{aligned}$$

于是当 $f(x)$ 为连续函数时, 有 $f(x) = \frac{\sin x + \cos x}{2}$.