§1 Measures

1. 一些单词

injection 单射　surjection 满射　bijection 双射.

$f(U)$ image　$f^{-1}(U)$ ploimage

2. Measure should:

① countable additivity and nonnegative.

② unchange for rigid motion

③ $\exists B_r(x)$, s.t. the measure of $B_r(x) > 0$

④ There could be some sets that can't be measured.

3. Can't be measured.

Measure : $\mu : \mathcal{A} \to [0, +\infty)$ satisfied.

① $\forall U_1, U_2, \cdots, U_n, \cdots$ disjointed. then $\mu(\bigsqcup_{n=1}^{\infty} U_n) = \sum_{n=1}^{\infty} \mu(U_n)$

② If V is a rigid motion, then $\mu(V(U)) = \mu(U)$.

③ $\exists B_r(x)$, s.t. $\mu(B_r(x)) > 0$.

Def　$x \sim y \Leftrightarrow x - y \in \mathbb{Q}$, It is well-defined.

$[x] = \{y \in [0,1] \mid y \sim x\}$

We pick a y_α from every $[y_\alpha]$, $[0,1] = \bigsqcup_{\alpha \in \Lambda} [y_\alpha]$

Then $A = \bigsqcup_{\alpha \in \Lambda} \{y_\alpha\}$. $\mu(A+\beta) = \mu(A)$. $\beta \in \mathbb{Q}$.

Then $[0,1] \subset \bigsqcup_{\beta \in \mathbb{Q} \cap [0,1]} A + \beta \subset [-1, 2]$.

$\Rightarrow 1 \le \mu(\bigsqcup_{\beta \in \mathbb{Q} \cap [0,1]} A + \beta) \le 3 \Rightarrow 1 \le \sum_{n=1}^{\infty} \mu(A) \le 3$.

So A can't be measured.

4. An algebra of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements. A σ-algebra is an algebra that is closed under countable unions

Then $\phi, X \in \mathcal{A}$

e.g. ① $\{\phi, X\}$　② $P(X)$　③ σ-algebra generated by open sets.

④ X is uncountable, $\mathcal{A} = \{U \subset X, U$ is countable or U^c countable$\}$

5. Def σ-algebra generated by X, $\mathcal{A} = \bigcap_{X \subset B} B$, B is also σ-algebra.
Different X can generate the same σ-algebra.
e.g. $\mathcal{B}(\mathbb{R}) \Leftarrow$ ① (a,b) ② $[a,b]$ ③ $[a,b)$ ④ $(a,+\infty)$
⑤ $[a,+\infty)$ generate.

6. consider $X = \prod_{\alpha\in\Lambda} X_\alpha$, projection $\pi_\alpha : X \to X_\alpha$.
$\forall X_\alpha$, $\exists$ σ-algebra $\mathcal{A}_\alpha$. Product σ-algebra is generated by $\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{A}_\alpha, \alpha\in\Lambda\}$, we call it $\bigotimes_{\alpha\in\Lambda} \mathcal{A}_\alpha$

$\bigotimes_{\alpha\in\Lambda} \mathcal{A}_\alpha$ can be generated by $\{\pi_\alpha^{-1}(E_\alpha), E_\alpha \in \mathcal{A}_\alpha, \alpha\in\Lambda\}$.

If $\mathcal{A}_\alpha$ can be generated by $\mathcal{E}_\alpha$, then $\{\prod_{\alpha\in\alpha} E_\alpha, E_\alpha \in \mathcal{E}_\alpha\}$ generate $\bigotimes_{\alpha\in\Lambda} \mathcal{A}_\alpha$.

7. Measure
Def: A measure μ on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0,+\infty)$ which is satisfied that.
① $\mu(\phi) = 0$
② If $\{U_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ disjoint then $\mu(\bigcup_{n=1}^{\infty} U_n) = \sum_{n=1}^{\infty} \mu(U_n)$
(Countable addtivity)

e.g. consider $\mathcal{A} = \mathcal{P}(X)$, $\mu(U) = \sum_{x\in U} f(x)$.
① $f(x) = 1$, counting measure
② $f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$ Dirac measure

8. ① For $U, V \in \mathcal{A}$, and $U \subset V$
then $\mu(U) \leq \mu(V)$
Proof: $\mu(V) = \mu(U) + \mu(V\setminus U) \geq \mu(U)$

② For $\{U_n\}_{n=1}^{\infty} \subset \mathcal{A}$, $\mu(\bigcup_{n=1}^{\infty} U_n) \leq \sum_{n=1}^{\infty} \mu(U_n)$
Proof: Def: $V_1 = U_1$, $V_2 = U_2 \setminus V_1$, ..., and $V_n \subseteq U_n$.
$\mu(\bigcup_{n=1}^{\infty} U_n) = \mu(\bigcup_{n=1}^{\infty} V_n) = \sum_{n=1}^{\infty} \mu(V_n) \leq \sum_{n=1}^{\infty} \mu(U_n)$

③ Consider $\{U_n\}_{n=1}^{\infty}$ in $\mathcal{A}$

$U_1 \subset U_2 \subset \cdots$, then $\mu(\bigcup_{n=1}^{\infty} U_n) = \lim_{n\to\infty} \mu(U_n)$

Proof: $\mu(\bigcup_{n=1}^{\infty} U_n) = \mu(U_1 \cup (U_2\setminus U_1) \cup (U_3\setminus U_2) \cup \cdots \cup (U_n\setminus U_{n-1}) \cup \cdots)$

$= \mu(U_1) + \sum_{n=1}^{\infty} (\mu(U_{n+1}) - \mu(U_n)) = \lim_{n\to\infty} \mu(U_n)$

④ For $U_1 \supset U_2 \supset U_3 \supset \cdots$

and $\mu(U_1) < \infty$. then

$\mu(\bigcap_{n=1}^{\infty} U_n) = \lim_{n\to\infty} \mu(U_n)$. Proof is the same as ③.

9. For $(X, \mathcal{A}, \mu)$, if $U \in \mathcal{A}$ s.t. $\mu(U) = 0$.
then U is called null set.

If a statement for points $x \in X$ is true except for the points in a null set, we called that it is true almost everywhere. (μ-a.e.)

8. For $(X, \mathcal{A}, \mu)$, if for any null set $U \in \mathcal{A}$, and any $V \subset U$, we have $V \in \mathcal{A}$, then we say $(X, \mathcal{A}, \mu)$ is complete.

Thm: $(X, \mathcal{A}, \mu)$ measure space. $\mathcal{N} = \{N \in \mathcal{A}, \mu(N) = 0\}$

Let $\bar{\mathcal{A}} = \{U \cup V : U \in \mathcal{A}, V \subset N$ for any $N \in \mathcal{N}\}$

Then $\bar{\mathcal{A}}$ is an σ-algebra, and μ can be uniquely extended to $\mathcal{A}$

Pf: ① $\phi, X \in \bar{\mathcal{A}}$

② $U \in \bar{\mathcal{A}} \Rightarrow U^c \in \bar{\mathcal{A}}$

$\exists V \in \mathcal{A}$, $W \subset N \in \mathcal{N}$ s.t. $U = V \cup W$, make $V \cap W = \phi$.

$U^c = (V \cup W)^c = (V \cup (N \setminus (N \setminus W)))^c = ((V \cup N) \setminus (N \setminus W))^c$

$= ((V \cup N) \cap (N \setminus W)^c)^c = (V \cup N)^c \cup (N \setminus W) \in \bar{\mathcal{A}}$

③ $\{U_n\}_{n=1}^{\infty} \subset \bar{\mathcal{A}} \Rightarrow \bigcup_{n=1}^{\infty} U_n \in \mathcal{A}$

$U_n = V_n \cup W_n \in \bar{\mathcal{A}}$, $W_n \subset N_n$, 则 $\bigcup_{n=1}^{\infty} U_n = (\bigcup_{n=1}^{\infty} V_n) \cup (\bigcup_{n=1}^{\infty} W_n)$

$\bigcup_{n=1}^{\infty} W_n \in \mathcal{N}$.

9. Outer measure

Def: $\mu^*: P(X) \to [0,+\infty)$

① $\mu^*(\phi)=0$

② if $U \subset V$, then $\mu^*(U) \le \mu^*(V)$

③ $\mu^*(\bigcup_{n=1}^{\infty} U_n) \le \sum_{n=1}^{\infty} \mu^*(U_n)$

Thm: Let $\mathcal{E} \in P(X)$, $\rho: \mathcal{E} \to [0,+\infty)$

s.t. $\phi, X \in \mathcal{E}$, $\rho(\phi)=0$, premeasure.

For any $A \subset X$, define $\mu^*(A) = \inf\{\sum_{n=1}^{\infty} \rho(U_n), U_n \in \mathcal{E}, A \subset \bigcup_{n=1}^{\infty} U_n\}$

then μ^* is an outer measure.

Pf: ① trivial

② If V is covered, then U is covered.

③ $\forall \varepsilon > 0$, $\exists U_n \subset \bigcup_{n=1}^{\infty} V_{nj}$, s.t. $\mu^*(U_n) \geqslant \sum_{n=1}^{\infty} \rho(V_{nj}) - \frac{\varepsilon}{2^n}$

$$RHS \geqslant \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} \rho(V_{nj}) - \frac{\varepsilon}{2^n}\right) = \sum_{k,j=1}^{\infty} \rho(V_{nj}) - \varepsilon$$

$$\ge \mu^*(\bigcup_{n=1}^{\infty} U_n) - \varepsilon, \quad \varepsilon \to 0.$$

e.g. Lebesgue outer measure

$\mathcal{E} = \{\text{open interval}\}$ $I_n = (a_n, b_n)$, $\rho(I_n) = b_n - a_n$.

$\mu^*(A) = \inf\{\sum_{n=1}^{\infty} |I_n| : A \subset \bigcup_{n=1}^{\infty} I_n\}$.

10. Def: $U \subset X$ is called μ^*-measurable if $\forall V \in P(X)$

$\mu^*(V) = \mu^*(V \cap U) + \mu^*(V \cap U^c)$ (Caratheodory Thm)

① μ^*-measurable sets form σ-algebra $\mathcal{A}$

② μ^* restricted to μ^*-measurable sets becomes a equitable measure

Pf: ① $\phi, X \in \mathcal{A}$.

② if $U \in \mathcal{A}$, $U^c \in \mathcal{A}$.

Consider $\mathcal{U}, V \in \mathcal{A}$ want to show $\mathcal{U} \cup V \in \mathcal{A}$

$\mu^*(W) = \mu^*(W \cap \mathcal{U}) + \mu^*(W \cap \mathcal{U}^c)$

$= \mu^*(W \cap \mathcal{U} \cap V) + \mu^*(W \cap (\mathcal{U} \cap V)^c)$

$+ \mu^*(W \cap \mathcal{U}^c \cap V) + \mu^*(W \cap \mathcal{U}^c \cap V^c)$

$\geq \mu^*(W \cap \mathcal{U} \cup V) + \mu^*(W \cap (\mathcal{U} \cup V)^c)$

So $\mathcal{U} \cup V \in \mathcal{A}$.

For $\{A_n\}_{n=1}^{\infty}$ $B_n = \bigcup_{i=1}^{n} A_i$. $\forall E \subset X$. $B = \bigcup_{n=1}^{\infty} A_n$.

$\mu^*(E \cap B_n) = \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c)$

$= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \Rightarrow \mu^*(E \cap B_n) = \sum_{i=1}^{n} \mu^*(E \cap A_i)$

$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) = \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E \cap B_n^c)$

$n \to \infty$. $\mu^*(E) \geq \sum_{n=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c)$

$\geq \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E)$.

So $B = \bigcup_{n=1}^{\infty} A_n$ is μ^*-measurable.

11. e.g. Lebesgue measure

$\mathcal{E} = \{\text{open interval } (a,b)\}$, $\rho((a,b)) = b-a$.

$\forall \mathcal{U} \in \mathbb{R}$, $\mu^*(\mathcal{U}) = \inf\{\sum_{n=1}^{\infty} \rho(I_n), \mathcal{U} \subset \bigcup_{n=1}^{\infty} I_n\}$.

if V satisfies the Caratheodory Thm in Lebesgue outer measure

We say V is an Lebesgue measurable set, $V \in \mathcal{L}$.

$\mu^*|_{\mathcal{L}} = m$.

12. Thm $\forall \mathcal{U} \in \mathcal{B}(\mathbb{R})$

$m(\mathcal{U}) = \inf\{m(V), V \text{ open and } \mathcal{U} \subset V\}$

$= \sup\{m(K), K \text{ compact and } K \subset \mathcal{U}\}$

Pf: ① $\mathcal{U} \subset V \Rightarrow m(\mathcal{U}) \leq m(V) \Rightarrow m(\mathcal{U}) \leq \inf\{m(V), V \text{ open}, \mathcal{U} \subset V\}$

$m(\mathcal{U}) = \mu^*(\mathcal{U}) \geq \sum_{n=1}^{\infty} m(I_n) - \varepsilon \geq m(\bigcup_{n=1}^{\infty} I_n) - \varepsilon$

$\varepsilon \to 0$, demonstrated.

② suppose $\mathcal{U}$ bold, not closed.

$\bar{U}\setminus U \in B(\mathbb{R})$

$\forall \varepsilon>0, \exists V$ open, s.t. $m(\bar{U}\setminus U) \geq m(V)-\varepsilon$.

$K=\bar{U}\setminus V$, compact

$m(K)= m(U)-m(U\cap V) = m(U) - (m(V)-m(U\setminus V))$

$\geq m(U) - m(V) + m(\bar{V}\setminus V) \geq m(U)-\varepsilon$

13. Thm: Let $U\in B(\mathbb{R})$, define

$U+s=\{x+s, x\in U\}$ $\quad rU=\{rx, x\in U\}$.

Then $m(U+s)=m(U)$, $m(rU)=r\,m(U)$

Pf: We know that $U+s,\ rU \in B(\mathbb{R})$.

$m(U+s) = \inf\{\sum_{n=1}^{\infty} |I_n|,\ U+s\subset \bigcup_{n=1}^{\infty} I_n\} = \inf\{\sum_{n=1}^{\infty} |I_n - s|,\ U\subset \bigcup_{n=1}^{\infty} (I_n-s)\}$

$= m(U)$.

The same theory as rU.

14. Cantor sets $\mathcal{C}\subset[0,1]$.

$\forall x\in[0,1]$, $x=\frac{a_1}{3}+\frac{a_2}{3^2}+\cdots+\frac{a_n}{3^n}+\cdots$

$\mathcal{C}=\{x\in[0,1],\ a_n=0,2\}$.

Prop ① $\mathcal{C}$ is compact and discountable. $m(\mathcal{C})=0$

If $\mathcal{C}$ is countable. $\{x_1,x_2,\cdots,x_n,\cdots\}=\mathcal{C}$.

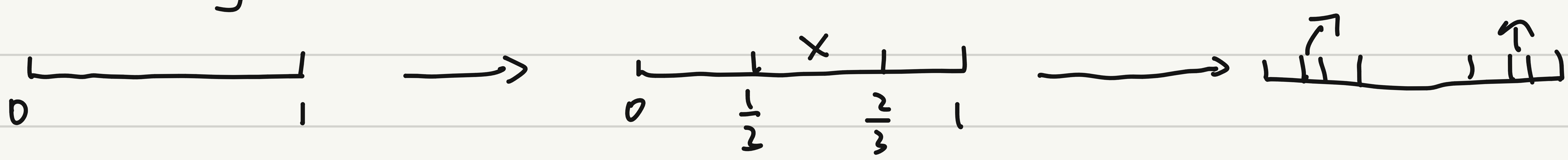
$x_1=\frac{a_{11}}{3}+\frac{a_{12}}{3^2}+\cdots+\frac{a_{1n}}{3^n}+\cdots$

$x_2=\frac{a_{21}}{3}+\frac{a_{22}}{3^2}+\cdots+\frac{a_{2n}}{3^n}+\cdots$

choose $a_n\neq a_{nn}$. So $\frac{a_1}{3}+\frac{a_2}{3^2}+\cdots+\frac{a_n}{3^n}+\cdots\notin\mathcal{C}$.

It's ridiculous.

② A way to create a Cantor set.



Question: How to adjust the measure to make $|\mathcal{C}|>0$.

15. Hausdorff measure

we have a metric space (X, ρ).

$\forall U \subset X$, we define

$$H^d_\delta(U) = \inf\{\sum_{n=1}^{\infty} (\mathrm{diam}(U_n))^d,\ U \subset \bigcup_{n=1}^{\infty} U_n,\ U_n \in P(X), \mathrm{diam}(U_n) < \delta\}.$$

as $\mathrm{diam}(U) = \sup_{x,y \in U} \rho(x,y)$.

Let $H^d(U) = \lim_{\delta \to 0} H^d_\delta(U)$ (Hausdorff outer measure)

Pf: ① $H^d(\phi) = 0$

② If $U \subset V$, then $H^d(U) \le H^d(V)$

③ $\forall \delta$ fixed, $\forall U_n$, $\forall \varepsilon > 0$

$\exists \{U_{nj}\}$ s.t. $U_n \subset \bigcup_{j=1}^{\infty} U_{nj}$

and $H^d_\delta(U_n) \geq \sum_{j=1}^{\infty} (\mathrm{diam}\, U_{nj})^d - \varepsilon \cdot 2^{-n}$

$\bigcup_{n=1}^{\infty} U_n \subset \bigcup_{n,j=1}^{\infty} U_{nj}$. $\sum_{n=1}^{\infty} H^d_\delta(U_n) \geq \sum_{\substack{n=1\\ j=1}}^{\infty} (\mathrm{diam}(U_{nj}))^d - \varepsilon$

$\geq H^d_\delta(\bigcup_{n=1}^{\infty} U_n) - \varepsilon$.

So $H^d_\delta(\bigcup_{n=1}^{\infty} U_n) \le \sum_{n=1}^{\infty} H^d_\delta(U_n)$.

16. Hausdorff dimension $-$ d. $\dim_H(A) = \inf\{0 \le d < \infty \mid H^d(A) = 0\}$.

Thm: ① If for $p > 0$, $H^p(U) < \infty$

then $\forall q > p$, $H^q(U) = 0$

② If for $p > 0$, $H^p(U) = 0$

then $\forall q < p$, $H^q(U) = \infty$.

Pf: ① $H^q_\delta(U) = \inf\{\sum_{n=1}^{\infty} (\mathrm{diam}(U_n))^q,\ U \subset \bigcup_{n=1}^{\infty} U_n,\ \mathrm{diam}\, U_n < \delta\}$

$\le \delta^{q-p} \inf\{\sum_{n=1}^{\infty} (\mathrm{diam}(U_n))^p,\ U \subset \bigcup_{n=1}^{\infty} U_n,\ \mathrm{diam}\, U_n < \delta\}$

$= \delta^{q-p} H^p_\delta(U)$, $\delta \to 0 \Rightarrow H^q(U) = 0$

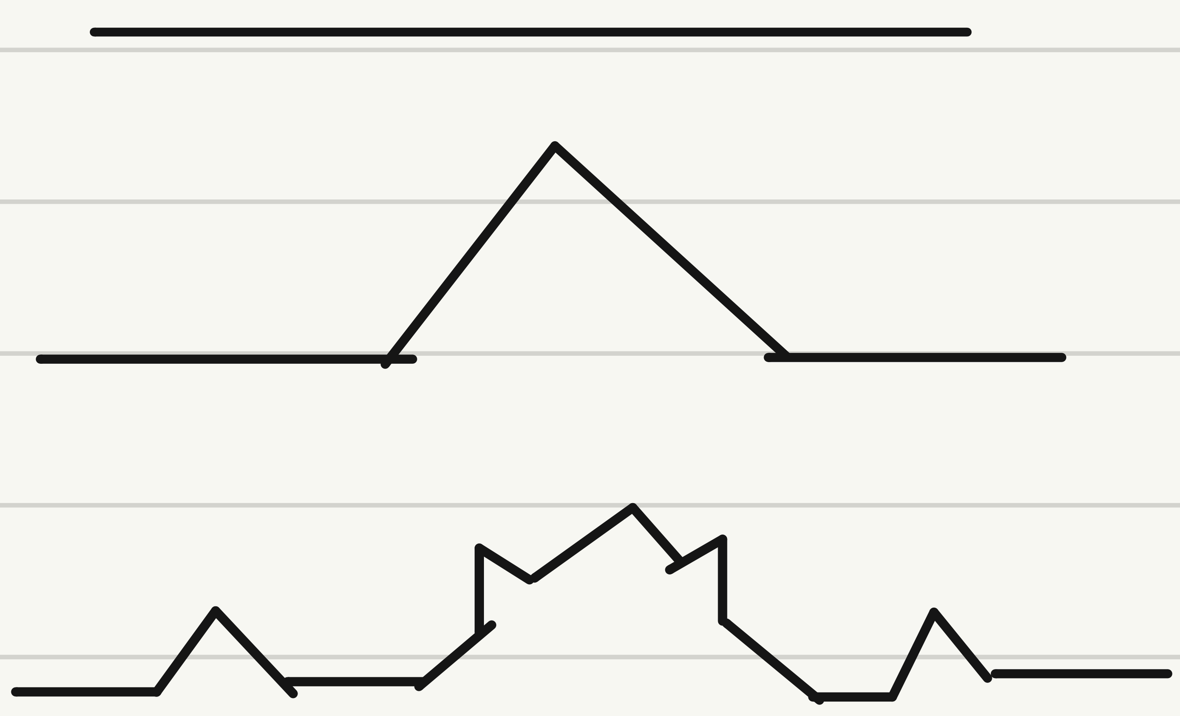
② the same as ①.

17. e.g C, dim $\log_3 2$

$$\left(\frac{1}{3^n}\right)^p \cdot 2^n \Rightarrow p = \log_3 2$$

e.g. Snowflake

$$\left(\frac{1}{3^n}\right)^p \cdot 4^n. \Rightarrow p = \log_3 4$$



18. The saturation of the measure.

$(X, \mathcal{M}, \mu)$ is a measure space. $E \subset X$ is called locally measurable if $E \cap A \in \mathcal{M}$, $\forall A \in \mathcal{M}$, $\mu(A) < \infty$. Let $\tilde{\mathcal{M}}$ be the collection of all locally measurable sets. If $\tilde{\mathcal{M}} = \mathcal{M}$, then μ is called saturated.

(1) If μ is σ-finite, then μ is saturated.

Pf: $\forall E \subset X$ locally measurable.

Because μ is σ-finite, $\exists \{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $X = \bigcup_{n=1}^{\infty} A_n$. $\mu(A_n) < \infty$. So $E \cap A_n \in \mathcal{M}$

Therefore $E = \bigcup_{n=1}^{\infty} (E \cap A_n) \in \mathcal{M} \Rightarrow \tilde{\mathcal{M}} = \mathcal{M}$. μ is saturated.

(2) $\tilde{\mathcal{M}}$ is σ-algebra.

Pf: (i) $\phi \in \tilde{\mathcal{M}}$

(ii) For $E \in \tilde{\mathcal{M}}$, $\forall A \in \mathcal{M}$, $\mu(A) < \infty$.

we have $E^c \cap A = A \setminus (E \cap A) \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$.

(iii) For $\{E_n\}_{n=1}^{\infty} \subset \tilde{\mathcal{M}}$, $\forall A \in \mathcal{M}$, $\mu(A) < \infty$.

$$\bigcup_{n=1}^{\infty} E_n \cap A = \bigcup_{n=1}^{\infty} (E_n \cap A) \in \mathcal{M} \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \tilde{\mathcal{M}}.$$

Define $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ by $\tilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ for otherwise. Then $\tilde{\mu}$ is a saturated measure on $\tilde{\mathcal{M}}$, called the saturation of μ.

(3) $\tilde{\mu}$ is saturated.

Pf: It is obvious that $\tilde{\mu}$ is a measure on $\tilde{\mathcal{M}}$.

For $(X, \tilde{\mathcal{M}}, \tilde{\mu})$, E is locally measurable.

So $\forall A \in \tilde{\mathcal{M}}$, $\tilde{\mu}(A) < \infty$, $\tilde{\mu}(E \cap A) \leq \tilde{\mu}(A) < \infty$.

$\Rightarrow E \cap A \in \tilde{\mathcal{M}}$, $E \cap A \in \mathcal{M}$. so $\tilde{\tilde{\mathcal{M}}} = \tilde{\mathcal{M}}$. $\tilde{\mu}$ is saturated.

(4). If μ is complete, so is $\tilde{\mu}$.

Pf: For $\tilde{\mu}$-null set $N \in \tilde{\mathcal{M}}$, $N \in \mathcal{M}$, $\mu(N) = \tilde{\mu}(N) = 0$.

$\mathcal{M}$ is complete, so $\forall F \subset N$, $\mu(F) = \tilde{\mu}(F) = 0$.

$\Rightarrow \tilde{\mathcal{M}}$ is complete.

(5) Suppose that μ is semifinite, i.e. $\forall E$, $\mu(E) = \infty$.

$\exists F \subset E$, $F \in \mathcal{M}$, $0 < \mu(F) < \infty$.

Define μ_0 on $\mathcal{M}$ by $\mu_0(E) = \sup\{\mu(F) : F \subset E, \mu(F) < \infty\}$.

Then $\mu = \mu_0$.

Pf: If $\mu(E) < \infty$. then $\mu(E) = \mu_0(E)$.

For $E \in \mathcal{M}$, $\mu(E) = \infty$. Then $\mu_0(E) > 0$.

If $\mu_0(E) < \mu(E) = \infty$.

We choose $F_1 \subset E$, s.t. $\frac{\mu_0(E)}{2} < \mu(F_1) \leq \mu_0(E)$.

So $\mu_0(E \setminus F_1) = \mu_0(E) - \mu(F_1) < \frac{1}{2}\mu_0(E)$

We pick $F_2 \subset E \setminus F_1$, s.t. $\mu(F_2) > \frac{1}{2}\mu(E \setminus F_1) > \frac{\mu_0(E)}{2^2}$.

So we get $\{F_n\}_{n=1}^{\infty}$ disjoint, $\mu(F_n) > \frac{1}{2^n}\mu_0(E)$

We choose $F = \bigcup_{n=1}^{\infty} F_n$. $F \subset E$.

$\mu(F) = \sum_{n=1}^{\infty} \mu(F_n) = \mu_0(E) < \infty$.

So $\mu(E \setminus F) = \infty$, but $\mu_0(E \setminus F) = 0$.

It is contradict with μ is semifinite.

(6) Suppose that μ is semifinite. For $E \in \tilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M} \text{ and } A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\tilde{\mathcal{M}}$ that extends μ.

Pf: We claims that $\underline{\mu}(E) = \sup\{\mu(A) . A \in \mathcal{M}, \mu(A) < \infty, A \subset E\} \triangleq \mu_0(E)$

$\forall E \in \mathcal{M}$, it is trivial

We need to prove that if $\exists A \subset E$, $A \in \mathcal{M}$, s.t. $\mu(A) = \infty$, then $\mu_0(A)=\infty$. Because μ is semifinite, according to (5), $\mu_0(A)=\mu(A)=\infty$. In this case, $\mu_0(E)\geq\mu_0(A)=\infty \Rightarrow \mu_0(E)=\infty$.

And $\underline{\mu}|_{\mathcal{M}}=\mu$, we need to show $\underline{\mu}$ is measure on $\tilde{\mathcal{M}}$.

(i) For disjoint set sequence $\{A_n\}\subset\mathcal{M}$.

$\forall E\subset A \triangleq \bigcup_{n=1}^{\infty} A_n$, satisfied $\mu(E)<\infty$, $E\cap A_n\in\mathcal{M}$.

So $\mu(E)\leq\sum_{n=1}^{\infty}\mu(E\cap A_n)\leq\sum_{n=1}^{\infty}\underline{\mu}(A_n)$.

$\Rightarrow \underline{\mu}(E)\leq\sum_{n=1}^{\infty}\underline{\mu}(A_n)$

(ii) W.L.O.G $\underline{\mu}(A_n)\leq\underline{\mu}(A)<\infty$.

$\forall\varepsilon>0$, $n\in\mathbb{N}$. $\exists E_n\subset A_n$, s.t. $\mu(E_n)<\infty$, $\mu(E_n)>\underline{\mu}(A_n)-\frac{\varepsilon}{2^n}$

$\{E_n\}_{n=1}^{\infty}$ are also disjoint, $\underline{\mu}|_{\mathcal{M}}=\mu$.

$$\underline{\mu}(E)\geq\underline{\mu}\Big(\bigcup_{n=1}^{\infty}E_n\Big)=\mu\Big(\bigcup_{n=1}^{\infty}E_n\Big)=\sum_{n=1}^{\infty}\mu(E_n)>\sum_{n=1}^{\infty}\underline{\mu}(E_n)-\varepsilon.$$

$\varepsilon\to0$, $\underline{\mu}(E)=\sum_{n=1}^{\infty}\underline{\mu}(E_n)$.

Finally, we need to show $\underline{\mu}$ is saturated.

For locally measurable set E in $(X,\tilde{\mathcal{M}},\underline{\mu})$, we claim that $E\in\tilde{\mathcal{M}}$. For $A\in\mathcal{M}$, $\mu(A)<\infty$. Due to $\underline{\mu}|_{\mathcal{M}}=\mu$. So $A\in\tilde{\mathcal{M}}$, $\underline{\mu}(A)<\infty$. $E\cap A\in\tilde{\mathcal{M}} \Rightarrow E\cap A\in\mathcal{M} \Rightarrow E\in\tilde{\mathcal{M}}$.

(7) Let X_1, X_2 be disjoint uncountable sets, $X=X_1\cup X_2$ and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be counting measure on $P(X_1)$, and define μ on $\mathcal{M}$ by $\mu(E)=\mu_0(E\cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\tilde{\mathcal{M}}=P(X)$, and $\tilde{\mu}\neq\underline{\mu}$.

Pf: $\mathcal{M}$ is a σ-algebra, and μ is measure on $\mathcal{M}$.

We first show $\tilde{\mathcal{M}}=P(X)$. $\forall E\subset X$, and $A\in\mathcal{M}$, $\mu(A)<\infty$.

If A is co-countable, then

$$\mathrm{card}(X_1)=\mathrm{card}(A)+\mathrm{card}(X_1\setminus A)\leq\mathrm{card}\,A+\mathrm{card}\,A^c=\aleph_0$$

But X_1 is uncountable. A is countable.

So $E\cap A\in\mathcal{M}$, $E\in\tilde{\mathcal{M}}$

$X_2^c=X_1$ is uncountable, X_2 is also uncountable.

$X_2\notin\mathcal{M}$, $\tilde{\mu}(X_2)=\infty$. But $\forall A\in\mathcal{M}$, $A\subset X_2$, $A\cap X_1=\phi$ $\mu(A)=0$.

So $\underline{\mu}(X_2)=0$, $\tilde{\mu}\neq\underline{\mu}$

19. The equal state of measurable.

$\mathcal{A} \subset P(X)$ is an algebra, $\mathcal{A}_\sigma$ is the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ is the countable intersections of sets in $\mathcal{A}_\sigma$. μ_0 is a premeasure on $\mathcal{A}$ and μ^* is induced outer measure.

(1) $\forall E \subset X$, $\varepsilon > 0$. $\exists A \in \mathcal{A}_\sigma$, $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

Pf: $\mu^*(E) = \inf\{ \sum_{n=1}^{\infty} \mu_0(A_n), E \subset \bigcup_{n=1}^{\infty} A_n, A_n \in \mathcal{A} \}$.

So $\forall \varepsilon > 0$, $\exists A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_\sigma$, s.t. $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n) \leq \mu^*(E) + \varepsilon$.

(2) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

Pf: "$\Rightarrow$". we consider $\{A_n\}_{n=1}^{\infty}$, s.t. $\mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$

$A_n \in \mathcal{A}_\sigma$. We pick $A = \bigcap_{n=1}^{\infty} A_n$. so $A \in \mathcal{A}_{\sigma\delta}$, $E \subset A$.

$\mu^*(E) \leq \mu^*(A) \leq \mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$.

$n \to \infty$. So $\mu^*(A) = \mu^*(E)$.

And E is measurable, so $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$

$\Rightarrow \mu^*(A \setminus E) = \mu^*(A) - \mu^*(E) = 0 \Rightarrow \mu^*(A \setminus E) = 0$

"$\Leftarrow$" Because $B \in \mathcal{A}_{\sigma\delta}$, B is measurable.

So $\forall S \subset X$ $\mu^*(S) = \mu^*(B \cap S) + \mu^*(S \setminus B)$

$\mu^*(B \cap S) \geq \mu^*(E \cap S)$

$\mu^*(S \setminus B) = \mu^*((S \setminus E) \setminus (B \setminus E)) \geq \mu^*(S \setminus E)$

So $\mu^*(E \cap S) + \mu^*(S \setminus B) = \mu^*(S) \geq \mu^*(B \cap S) + \mu^*(S \setminus E)$

(3) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (2) is superfluous.

Pf: $X = \bigcup_{n=1}^{\infty} X_n$, $X_n \in \mathcal{A}$, $\mu_0(X_n) < \infty$.

$E_n = E \cap X_n$, then $\mu^*(E \cap X_n) < \infty$.

$\exists A_{nm} \in \mathcal{A}_\sigma$, $E_n \subset A_{nm}$, $\mu^*(A_{nm}) < \mu^*(E_n) + \frac{1}{2^{n+m}}$

$\mu^*(A_{nm} \setminus E_n) < \frac{1}{2^{n+m}}$.

So $\mu^*(\bigcup_{n=1}^{\infty} A_{nm} \setminus E) < \sum_{n=1}^{\infty} \frac{1}{2^{n+m}} = \frac{1}{2^m}$.

Consider $B = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{nm}$, $\mu^*(B \setminus E) < \frac{1}{2^m} \Rightarrow \mu^*(B \setminus E) = 0$

(4) $\mathcal{M}^*$ is the σ-algebra of μ^*-measurable set.

$\bar{\mu} = \mu^*|_{\mathcal{M}^*}$. If μ is σ-finite, then $\bar{\mu}$ is the completion of μ.

Pf: Obviously, if $F \subset E$, $\mu(E) = 0$, then $\mu^*(F) = 0$

$\overline{\mathcal{M}} \subset \mathcal{M}^*$. $\overline{\mathcal{M}}$ is the completion σ-algebra of $\mathcal{M}$.

$\forall E \in \mathcal{M}^*$, due to μ is σ-finite, $\exists B \in \mathcal{M}_{\sigma\delta}$,

s.t. $E \subset B$, $\mu^*(B \setminus E) = 0 \Rightarrow \mu^*(E^c \setminus B^c) = 0$.

So $\forall n$, $\exists A_n \in \mathcal{M}$, s.t. $E^c \setminus B^c \subset A_n$, $\mu^*(A_n) \le \frac{1}{n}$

$A = \bigcap_{n=1}^{\infty} A_n$, $E^c \setminus B^c \subset A \in \mathcal{M}$, $\mu(A) = 0$.

So $E^c = B^c \cup (E^c \setminus B^c) \in \overline{\mathcal{M}}$, so $E \in \overline{\mathcal{M}}$.

(5) In general, $\tilde{\mu}$ is the saturation of the completion of μ.

Pf: We claim that $\tilde{\overline{\mathcal{M}}} = \mathcal{M}^*$

Step 1: $\tilde{\overline{\mathcal{M}}} \subset \mathcal{M}^*$.

$\forall E \in \tilde{\overline{\mathcal{M}}}$, we need to prove $\forall F \subset X$,

$\mu^*(F) \ge \mu^*(F \cap E) + \mu^*(F \setminus E)$

W.L.O.G $\mu^*(F) < \infty$. $\forall n$, $\exists A_n \in \mathcal{M}$, s.t. $\mu(A_n) \le \mu^*(F) + \frac{1}{n}$, $F \subset A_n$

Pick $A = \bigcap_{n=1}^{\infty} A_n$, so $\mu(A) = \mu^*(F) < \infty$.

So $E \cap A \in \overline{\mathcal{M}}$, $\mu^*(F) = \mu(A) = \mu(E \cap A) + \mu(A \setminus E)$

$\ge \mu^*(F \cap E) + \mu^*(F \setminus E)$. $\Rightarrow E \in \mathcal{M}^*$.

Step 2: $\mathcal{M}^* \subset \tilde{\overline{\mathcal{M}}}$

$\bar{\mu}$ is the completion of μ.

Pick $A \cup F \in \overline{\mathcal{M}}$, $\tilde{\mu}(A \cup F) = \mu(A) < \infty$

$E \in \mathcal{M}^*$, $E \cap (A \cup F) = (E \cap A) \cup (E \cap F)$. $\tilde{\mu}(E \cap F) = 0$.

$\mu^*(E \cap A) < \infty$, $E \cap A \in \mathcal{M}^*$.

According to (4), $E \cap A \in \overline{\mathcal{M}}$. So $E \in \tilde{\overline{\mathcal{M}}}$.

Therefore, $\mathcal{M}^* = \tilde{\overline{\mathcal{M}}}$.

Now we should show $\bar{\mu}(E) = \begin{cases} \tilde{\mu}(E) & E \in \overline{\mathcal{M}} \\ \infty & \text{otherwise} \end{cases}$ (*)

$E = A \cup F \in \overline{\mathcal{M}}$, $A \in \mathcal{M}$, $F \subset N$, $N \in \mathcal{M}$, $\mu(E) = 0$

$\tilde{\mu}(E) = \mu(A)$. $\mu(A) \le \tilde{\mu}(E) \le \tilde{\mu}(A \cup N) = \mu(A)$

(*) holds for $E \in \overline{\mathcal{M}}$.

For $E \in \tilde{\overline{\mathcal{M}}} \setminus \overline{\mathcal{M}}$, $\tilde{\mu}(E) = \infty$. if $\tilde{\mu}(E) < \infty$.

Similarly, diractory.

§ 2 Integration

1. measurable function

For measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$

$f: X \to Y$, we say f is $(\mathcal{A}, \mathcal{B})$ measurable, if $\forall\, U \in \mathcal{B}$, $f^{-1}(U) \in \mathcal{A}$.

2. Some properties

① composition of measurable functions are measurable.

i.e. For $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$, $(Z, \mathcal{C}, \tau)$

$f: X \to Y$, $g: Y \to Z$, are all measurable.

Then $g \circ f$ is measurable.

② If $\mathcal{B}$ is generated by $\mathcal{E}$

we only need to check $\forall\, U \in \mathcal{E}$, $f^{-1}(U) \in \mathcal{A}$.

$\mathcal{B}$ is a σ- algebra, then $\{f^{-1}(U), U \in \mathcal{B}\}$ is σ-algebra.

③ $f: (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ continuous.

then f is $(\mathcal{B}_X, \mathcal{B}_Y)$ measurable

④ $f: (X, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

f is $(\mathcal{A}, \mathcal{B}_{\mathbb{R}})$ measurable $\Leftrightarrow f^{-1}((a, +\infty)) \in \mathcal{A}$, $\forall a \in \mathbb{R}$.

⑤ $f: (\mathbb{R}, \mathcal{L}, m) \to (\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ is $(\mathcal{L}, \mathcal{B}_{\mathbb{C}})$ measurable.

Then we say f is Lebesgue measurable.

$f: (X, \mathcal{A}, \mu) \to (Y, \mathcal{B}, \nu)$.

if for U, $\forall\, V \in \mathcal{B}$, $f^{-1}(V) \cap U$ is measurable.

Then we say f is measurable on U.

3. $f: (X, \mathcal{A}, \mu) \to (Y, \mathcal{B}, \nu)$ $\quad Y = \prod_{\alpha \in \Lambda} Y_\alpha$

Denote $\pi_\alpha: Y \to Y_\alpha$ as the projection

Then f is measurable iff. $f_\alpha = \pi_\alpha f$ is measurable.

Pf: "$\Rightarrow$" $\forall\, U_\alpha \in \mathcal{B}_\alpha$, $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{B}$

$f_\alpha^{-1} = f^{-1} \circ \pi_\alpha^{-1}: Y_\alpha \to Y \to X$ is measurable.

"$\Leftarrow$" $\forall\, U \in \mathcal{B}$, s.t. $\exists\, \alpha$, $U_\alpha \in \mathcal{B}_\alpha$, $U = \pi_\alpha^{-1}(U_\alpha)$

$U_\alpha \to \pi_\alpha^{-1}(U_\alpha) \to f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ measurable.

4. Thm: $f, g : (X, \mathcal{A}, \mu) \to (\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ are measurable.

then so are $f \pm g$, $f \cdot g$.

Pf: $f+g : X \to \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $x \xrightarrow{T_1} (f(x), g(x)) \xrightarrow{T_2} f(x)+g(x)$.

T_1 is measurable, T_2 is continuous.

So $T_2 \circ T_1 = f+g$ is measurable.

The same as $f-g$, $f \cdot g$.

5. Given $\{f_n\} : (X, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ measurable.

Then $\sup\limits_n f_n$, $\inf\limits_n f_n$, $\limsup\limits_n f_n$, $\liminf\limits_n f_n$ are all measurable

Pf: $\{x : \sup\limits_n f_n > \lambda\} = \bigcup\limits_{n=1}^{\infty} \{x : f_n > \lambda\}$

$\{x : \inf\limits_n f_n > \lambda\} = \bigcap\limits_{n=1}^{\infty} \{x : f_n > \lambda\}$.

$\limsup\limits_n f_n = \inf\limits_n \sup\limits_{j \geq n} f_n$.

$\{x : \limsup\limits_n f_n > \lambda\} = \bigcap\limits_{n=1}^{\infty} \{x : \sup\limits_{j \geq n} f_n > \lambda\}$

$= \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{j=n}^{\infty} \{x : f_n > \lambda\}$.

6. Definition: $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$.

$f = f^+ - f^-$.

7. Simple function (Step function)

$f(x) = \sum\limits_{k=1}^{n} a_j \chi_{U_j}$, $\{U_j\}_{j=1}^n \subset \mathcal{A}$, $\{a_j\}$ are mutually distinct

① If $f : X \to [0, \infty)$ measurable. Then $\exists \{\phi_n\}$ simple function, s.t. $0 < \phi_1 \leq \phi_2 \leq \cdots \leq f$ $\phi_n \xrightarrow{u.n.} f$.

Pf: Let $\phi_n = 2^n \chi_{V_n} + \sum\limits_{j=0}^{2^{2n}-1} \frac{j}{2^n} \chi_{U_j}$.

$V_n = \{x \in X, f(x) \geq 2^n\}$ $U_j = \{x \in X, \frac{j}{2^n} \leq f < \frac{j+1}{2^n}\}$.

$\phi_n \leq \phi_{n+1}$, $\phi_n \to f$.

② $f: (X, \mathcal{A}, \mu) \to (\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ measurable

$\exists \phi_n$ simple, $|\phi_1| \le |\phi_2| \le \cdots \le |f|$. $\phi_n \xrightarrow{u.n.} f$.

Pf: $f = u + iv = (u^+ - u^-) + i(v^+ - v^-)$

$\phi_n^+ \to u^+$, $\phi_n^- \to u^-$, $\psi_n^+ \to v^+$, $\psi_n^- \to v^-$.

8. For simple function $\phi = \sum_{j=1}^{n} a_j \chi_{U_j}$

define $\int \phi \, d\mu = \sum_{j=1}^{n} a_j \mu(U_j)$. $\int_\Omega \phi \, d\mu = \sum_{j=1}^{n} a_j \mu(U_j \cap \Omega)$

Prop: ① $\int c\phi = c \int \phi$.

② $\int \phi + \psi = \int \phi + \int \psi$

Pf:

$\phi = \sum_{j=1}^{n} a_j \chi_{U_j}$, $\psi = \sum_{k=1}^{m} b_k \chi_{E_k}$.

So $\int \phi + \psi = \int \sum_{j=1}^{n} a_j \sum_{k=1}^{m} \chi_{U_j \cap E_k} + \sum_{k=1}^{m} b_k \sum_{j=1}^{n} \chi_{U_j \cap E_k}$

$= \int \sum_{\substack{j=1 \\ k=1}}^{nm} (a_j + b_k) \chi_{(U_j \cap E_k)} = \sum_{\substack{j=1 \\ k=1}}^{nm} (a_j + b_k) \mu(U_j \cap E_k)$

$= \int \phi + \int \psi$.

③ If $\phi \le \psi$, then $\int \phi \le \int \psi$

LHS $= \sum_{j=1}^{n} a_j \mu(U_j) = \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{m} \mu(U_j \cap V_k) \right)$

$\le \sum_{\substack{j=1 \\ k=1}}^{nm} b_j \mu(U_j \cap V_k) = \int \psi$.

9. For nonnegative function f.

$\int f \, d\mu \overset{def}{=} \sup\{ \int \phi \, d\mu, \ \phi$ simple, $0 \le \phi \le f \}$.

Prop: ① $\int cf = c \int f$

② $\int f + g = \int f + \int g$.

③ If $f \leq g$, then $\int f \leq \int g$

10. For real valued function f.

$f = f^+ - f^-$.

$\int f = \int f^+ - \int f^-$

For complex valued function f.

$f = u + iv = (u^+ - u^-) + i(v^+ - v^-)$

$\int f = \int u^+ - \int u^- + i(\int v^+ - \int v^-)$

if $\int |f| < \infty$, we say that f is integrable.
and $f \in L^1(X, \mu)$.

11. $|\int f| \leq \int |f|$

Pf: $LHS = \dfrac{(\overline{\int f}) \int f}{|\int f|} = \int \dfrac{(\overline{\int f})}{|\int f|} f = \int Re\left(\dfrac{\overline{\int f}}{|\int f|}\right) f$

$\leq \int \left|\dfrac{\overline{\int f}}{|\int f|} f\right| = \int |f|$

12. The following statements are equivalent.

① $\int_\Omega f = \int_\Omega g$ for $\forall \Omega \in \mathcal{A}$.

② $\int |f-g| = 0$

③ $f = g$ μ a.e.

Pf: ① ⇒ ② $\int |f-g| = \int_{\{f>g\}} f-g + \int_{\{f<g\}} f-g = 0$.

② ⇒ ③ Suppose $\exists\, \mu(U) > 0$, s.t. $f \neq g$.
then $\int |f-g| = \int_U |f-g| + \int_{U^c} |f-g| > 0$.

③ ⇒ ① $\int_\Omega f = \int_{\Omega \cap U} f + \int_{\Omega \cap U^c} f = \int_{\Omega \cap U^c} f = \int_\Omega g$.
$U = \{f \neq g\}$.

13. MCT

nonnegative valued function s.t. $f_n \le f_{n+1}$ $\forall n$.

$f = \lim_{n\to\infty} f_n$ a.s. , then $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$.

Pf: Step 1: $f_n \le f$. $\Rightarrow \int f_n\,d\mu \le \int f\,d\mu$

$\Rightarrow \lim_{n\to\infty} \int f_n\,d\mu \le \int f\,d\mu$.

Step 2: For any simple function ϕ, $0 \le \phi \le f$.

Consider $U_n = \{x \in X,\ f_n \ge \alpha\phi,\ \alpha \in (0,1]\}$.

$n\to\infty$, $U_n \to X$

$\int f_n \ge \int_{U_n} f_n \ge \alpha \int_{U_n} \phi$

$\lim_{n\to\infty} \int f_n \ge \alpha \int \phi$, $\Rightarrow \lim_{n\to\infty} \int f_n \ge \int \phi$.

So $\lim_{n\to\infty} \int f_n = \int f$.

14. $\{f_n\}$, $f_n \ge 0$. $f = \sum_{n=1}^{\infty} f_n$, then $\int f = \sum_{n=1}^{\infty} \int f_n$.

Pf: $\int f = \int \lim_{N\to\infty} \sum_{k=1}^{N} f_k = \lim_{N\to\infty} \int \sum_{k=1}^{N} f_k = \lim_{N\to\infty} \sum_{k=1}^{N} \int f_k$

$= \sum_{k=1}^{\infty} \int f_n$.

15. Fatou's Lemma.

$\{f_n\}$, $f_n \ge 0$ so $\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$.

Pf: LHS $= \int \lim_{n\to\infty} \inf_{k\ge n} f_k \overset{MCT}{=} \lim_{n\to\infty} \int \inf_{k\ge n} f_k$

$\le \lim_{n\to\infty} \inf_{k\ge n} \int f_k = \liminf_{n\to\infty} \int f_n$.

16. Lebesgue Dominated converagence (DCT).

$\{f_n\}$ in $L^1(X,\mu)$ satisfied that.

a) $f_n \xrightarrow{a.e.} f$

b) $\exists g$ in $L^1(X,\mu)$, s.t. $|f_n| \leq g$.

Then $f \in L^1(X,\mu)$, and $\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$.

Pf: We know that $\begin{cases} f_n + g \geq 0 \\ g - f_n \geq 0 \end{cases}$

According to Fatou's Lemma,

$$\int \liminf_n (g - f_n)\, d\mu \leq \liminf_n \int g - f_n\, d\mu.$$

$$\Rightarrow \int g - f\, d\mu \leq \int g\, d\mu - \limsup_n \int f_n\, d\mu.$$

$$\Rightarrow \int f\, d\mu \geq \limsup_n \int f_n\, d\mu.$$

$$\int \liminf_n (g + f_n)\, d\mu \leq \liminf_n \int g + f_n\, d\mu$$

$$\Rightarrow \int f\, d\mu \leq \liminf_n \int f_n\, d\mu.$$

$$\Rightarrow \int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu.$$

17. Thm: ① Simple functions are dense in $L^1(X,\mu)$

② When $X = \mathbb{R}$ (or $\mathbb{R}^n$), $\mu = m$. then continuous functions are dense in $L^1(\mathbb{R}, m)$.

Pf: ① Need to show that $\forall f \in L^1$, $\exists \{\phi_n\}$ simple function s.t. $\|f - \phi_n\|_1 \to 0$. $(n \to \infty)$

$f = f^+ - f^-$, $\varphi_n \nearrow f^+$, $\psi_n \nearrow f^-$. $\phi_n = \varphi_n - \psi_n$

so $\int |\phi_n - f|\, d\mu = \int |(\varphi_n - f^+) - (\psi_n - f^-)|\, d\mu$

$= \int (f^+ - \varphi_n)\, d\mu + \int f^- - \psi_n\, d\mu \to 0$ (DCT)

② $\exists \{\phi_n\}$ simple function, $\phi_n \xrightarrow{L^1} f$.

$\phi_n = \sum_{j=1}^{N} a_j \chi_{U_j}$, $U_j \in A_j$, $\exists \{V_{jm}\}$ open set. s.t $m(V_{jm} \setminus U_j) \to 0$.

According to Urython's Therome.

$\forall U \subset \mathbb{R}^n$ close set, $F \subset \mathbb{R}^n$ open set. $F \subset G$.

There is a continuous function satisfied that.

i) $0 \le f \le 1$ on $\mathbb{R}^n$. ii) $f = 1$ on F iii) $f = 0$ on $\mathbb{R}^n \setminus G$.

$\psi_{nm} = \sum_{j=1}^{N} a_j \chi_{V_{jm}}$

Then there is a continuous function g_{nmk}.

$g_{nmk} \xrightarrow{L^1} \psi_{nm}$. $(k \to \infty)$

So $g_{nnn} \xrightarrow{L^1} f$.

18. $f : X \times [a,b] \to \mathbb{C}$

$f(\cdot, t) : X \to \mathbb{C}$ is integrable for each t.

$F(t) = \int_X f(x,t)\, d\mu_x$.

① If $\exists g \in L^1$, s.t. $|f(x,t)| \le |g(x)|$ $\forall x, t$.

if $\lim_{t \to t_0} f(x,t) = f(x, t_0)$ $\forall x$.

then $\lim_{t \to t_0} F(t) = F(t_0)$.

② Suppose $\frac{\partial f}{\partial t}$ exists. and $\exists g \in L^1$, s.t. $\left|\frac{\partial f}{\partial t}(x,t)\right| \le |g(x)|$.

$\forall x, t$. Then F is differential, $F'(t) = \int_X \frac{\partial f}{\partial t}\, d\mu_x$

Pf: ① $\lim_{t \to t_0} F(t) = \lim_{t \to t_0} \int_X f(x,t)\, d\mu_x$

$\overset{DCT}{=\!=} \int_X \lim_{t \to t_0} f(x,t)\, d\mu_x = \int_X f(x, t_0)\, d\mu_x = F(t_0)$

② $\lim_{\varepsilon \to 0} \dfrac{F(t+\varepsilon) - F(t)}{\varepsilon} = \lim_{\varepsilon \to 0} \int_X \dfrac{f(x, t+\varepsilon) - f(x,t)}{\varepsilon}\, d\mu_x$

$= \lim_{\varepsilon \to 0} \int_X f'(x, t + \varepsilon\theta)\, d\mu_x \overset{DCT}{=\!=} \int_X \frac{\partial f}{\partial t}\, d\mu_x$

19. e.g. $(\mathbb{R}^n, m)$

$I(f) = \int_{\mathbb{R}^n} |\nabla f|^2 dm$ (Dirichlet energy)

Variational method.

Suppose I attain its min at f.

$I(f+\varepsilon\varphi) \geq I(f) \quad \forall \varepsilon \geq 0$

$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I(f+\varepsilon\varphi) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\mathbb{R}^n} |\nabla f + \varepsilon \nabla\varphi|^2 dm$

For $|\nabla(f+\varepsilon\varphi)| \in L^2$ u.n. bdl. (H^1).

$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I(f+\varepsilon\varphi) = \int_{\mathbb{R}^n} \frac{d}{d\varepsilon} |\nabla f + \varepsilon\nabla\varphi|^2 dm$

$= \int_{\mathbb{R}^n} 2(\nabla f + \varepsilon\nabla\varphi)(\nabla\varphi) dm$

So $\int_{\mathbb{R}^n} \nabla f \nabla\varphi = 0 \quad \forall \varphi \in H^1$.

If f has compact surpport, $\int_{\mathbb{R}^m} \Delta f \cdot \varphi = 0$, $\Delta f = 0$.

20. Thm: f is bdd real valued function on $[a,b]$.

① If f is Riemann-integrable, then f is $\mathcal{L}$-integrable.

② f is R-integrable $\Leftrightarrow \{x \in [a,b], f \text{ is discontinuous at } x\}$ is of m-measure zero.

Pf: ① f R-integrable

Consider $L_n(x) = \sum_{j=0}^{n-1} l_j \chi_{[a_j, a_{j+1}]}$, $l_j = \min_{[a_j, a_{j+1}]} f$, $a_j = a + \frac{b-a}{n}$.

$R_n(x) = \sum_{j=0}^{n-1} S_j \chi_{[a_j, a_{j+1}]}$, $S_j = \max_{[a_j, a_{j+1}]} f$, $a_j = a + \frac{b-a}{n}$.

L_n, R_n are simple function. $L_n \uparrow$, $R_n \downarrow$.

so $\lim_{n\to\infty} \int_a^b L_n dx = \int_{[a,b]} f dm = \lim_{n\to\infty} \int_a^b R_n dx$

$\Rightarrow \int_{[a,b]} f d\mu = \int_a^b f dx$.

② $\lim_{n\to\infty}\int_{[a,b]} L_n \, dm = \lim_{n\to\infty}\int_{[a,b]} R_n \, dm$

From MCT, $\int_{[a,b]} \lim_{n\to\infty} R_n - \lim_{n\to\infty} L_n \, dm = 0$.
$\Rightarrow \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$, a.e. $\Rightarrow f$ is continuous a.e..

21. Convergence in measure
$f_n \xrightarrow{meas} f$, if $\forall \varepsilon > 0$, $\mu(\{x \mid |f_n(x) - f(x)| > \varepsilon\}) \to 0$.

$\{f_n\}$ is Cauchy in the sense of convergence in measure.
i.e. $\forall \varepsilon > 0$, $\mu\{|f_n - f_m| > \varepsilon\} \to 0$ as $m, n \to \infty$.
Then ① $\exists f$, s.t. $f_n \xrightarrow{meas} f$.
② $\exists$ subsequence $\{f_{n_j}\}$, s.t. $f_{n_j} \xrightarrow{a.e.} f$.
③ If $f_n \xrightarrow{meas} g$, then $f = g$ a.e.

Pf: Pick $\{f_{n_j}\}$ s.t. $\mu\{|f_{n_j} - f_{n_{j+1}}| \geq 2^{-j}\} = \mu(E_j) \leq 2^{-j}$.
Then for k large, any $j, \ell \geq k$.
$|f_{n_j}(x) - f_{n_k}(x)| \leq \sum_{\alpha=j+1}^{\ell} |f_\alpha(x) - f_{\alpha-1}(x)| \leq 2^{-(j-1)}$
For $x \notin \bigcup_{j>k} E_j$, we have $|f_{n_j}(x) - f_{n_\ell}(x)| \leq 2^{-j+1}$.
so for $x \notin \limsup E_j$. $\{f_{n_j}\}$ is Cauchy sequence.
definite $f(x) = \begin{cases} \lim f_{n_j}(x) & x \notin \limsup E_j \\ 0 & \text{else} \end{cases}$
Then ①② are demonstrated.
③ $\mu\{|f-g| \geq \varepsilon\} \leq \mu\{|f_n - f| \geq \varepsilon\} + \mu\{|f_n - g| \leq \varepsilon\} \to 0$
$\Rightarrow f = g$ a.e.

22. For $(X, \mathcal{A}, \mu)$ $(Y, \mathcal{B}, \nu)$
If $A \in \mathcal{A}$, $B \in \mathcal{B}$, the $A \times B$ is the rectangle in $X \times Y$.
Define $\pi(A \times B) = \mu(A)\nu(B)$
we claims that $\pi|$rectangle satisfies countable additivity.
Let $A \times B = \bigcup_{n=1}^{\infty} (A_n \times B_n)$, each $A_n \times B_n$ is disjoint.

$$\chi_{A\times B}(x,y) = \sum_{n=1}^{\infty} \chi_{A_n\times B_n}(x,y) = \sum_{n=1}^{\infty} \chi_{A_n}(x)\,\chi_{B_n}(y) = \chi_A(x)\chi_B(y).$$

$$\int_X \left(\int_Y \chi_A(x)\chi_B(y)\, d\nu_y \right) d\mu_x = \int_X \left(\int_Y \sum_{n=1}^{\infty} \chi_{A_n}(x)\, \chi_{B_n}(y)\, d\nu_y \right) d\mu_x$$

$$\overset{MCT}{=} \sum_{n=1}^{\infty} \int_X \chi_{A_n}(x) \left(\int_Y \chi_{B_n}(y)\, d\nu_y \right) d\mu_x$$

$$= \sum_{n=1}^{\infty} \int_X \chi_{A_n}(x)\, \nu(B_n)\, d\mu_x = \sum_{n=1}^{\infty} \mu(A_n)\,\nu(B_n) = \sum_{n=1}^{\infty} \pi(A_n\times B_n)$$

23. Fubini - Tonelli:

$(X, \mathcal{A}, \mu)$ $(Y, \mathcal{B}, \nu)$ are σ-finite.

① (Tonelli) If $f \in L^+(X\times Y)$

then $g(x) = \int_Y f(x,y)\, d\nu_y$, $h(y) = \int_X f(x,y)\, d\mu_x$.

are perspectively in $L^+(X)$ and $L^+(Y)$.

And we have

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\, d\nu_y \right) d\mu_x = \int_Y \left(\int_X f(x,y)\, d\mu_x \right) d\nu_y$$

② (Fubini) Tonelli holds for $L^1(X\times Y)$

The proof is later.

24. Thm $(X, \mathcal{A}, \mu)$ $(Y, \mathcal{B}, \nu)$. are both σ-finite.

For $U \in \mathcal{A}\times\mathcal{B}$.

$U^x = \{y \in Y : (x,y) \in U\}$, $U^y = \{x \in X : (x,y) \in U\}$.

Then U^x is ν-measurable for every $x \in U$.

U^y is μ-measurable for every $y \in U$.

Pf: ① we claim $U^x \in \mathcal{B}$, $U^y \in \mathcal{A}$.

consider $\mathcal{R} = \{U \subset X\times Y, U^x \in \mathcal{B}, U^y \in \mathcal{A}\}$.

we only need to show $\mathcal{R}$ is σ-algebra.

$\phi, X\times Y \in \mathcal{R}$; $U \in \mathcal{R}$, $U^c \in \mathcal{R}$ $((U^c)^x = (U^x)^c)$.

$\bigcup_{n=1}^{\infty} U_n \in \mathcal{R}$: $\left(\bigcup_{n=1}^{\infty} U_n\right)^x = \bigcup_{n=1}^{\infty} U_n^x \in \mathcal{B}$

② As a result, if f is $\mu\times\nu$ measurable.

Then $f(x,y)$ is ν-meas for fixed x
$f(x,y)$ is μ-meas for fixed y

Tips: $\mu\times\nu$ is not complete now.

25. The proof of Fubini Thm.
we need to show.

$$\int_{X\times Y} \chi_U \, d(\mu\times\nu) = \int_Y \left(\int_X \chi_U \, d\mu_x \right) d\nu_y \quad (*).$$

First, show this holds for rectangles
$U = A\times B$, $(\mu\times\nu)(U) = \mu(A)\nu(B) = \int_Y \left(\int_X \chi_U \, d\mu_x \right) d\nu_y$

$R = \{U \in \mathcal{A}\times\mathcal{B},\ (*) \text{ holds for } U\}$.
we need to show that R is a σ-algebra.

Lemma (Monotonicity Lemma) $\mathcal{A}$ is algebra, $\mathcal{C}$ is the monotone class generated by $\mathcal{A}$; $\mathcal{M}$ is the σ-algebra generated by $\mathcal{A}$. Then $\mathcal{C} = \mathcal{M}$.

The proof of the Lemma:
$\mathcal{M}$ is a monotone class. then $\mathcal{C} \subset \mathcal{M}$
For $E \in \mathcal{C}$, define $C(E) = \{F \in C.\ E\setminus F,\ F\setminus E,\ E\cap F \in \mathcal{C}\}$
① $\phi, E \in C(E)$
② $E \in C(F) \Leftrightarrow F \in C(E)$
③ $\mathcal{C}(E)$ is a monotone class
$F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ in $C(E)$

Consider $\bigcup_n F_n$
$E\setminus \bigcup_n F_n = E \cap (\bigcup_n F_n)^c = E \cap (\bigcap_n F_n^c) = \bigcap_n (E\cap F_n^c) = \bigcap_n (E\setminus F_n) \in \mathcal{C}$.
If $E \in \mathcal{A}$, then $\forall F \in \mathcal{A}$, $F \in \mathcal{C}(E)$
so $\mathcal{A} \subset C(E)$, $E \in \mathcal{A}$. so $\mathcal{C} \subset \mathcal{C}(E)$
$\forall F \in \mathcal{C}$, $F \in \mathcal{C}(E) \rightarrow \mathcal{A} \subset \mathcal{C}(F)$, $\forall F \in \mathcal{C}$.

And $\mathcal{C}(F) \subset \mathcal{C} \Rightarrow \mathcal{C} = \mathcal{C}(F)$

$\mathcal{C}$ is closed under $E \setminus F$, $E \cap F$

$\bigcup_n U_n = \bigcup_n V_n \in \mathcal{C}$, so $\mathcal{C}$ is a σ-algebra.

we need to show that R is monotone class.

For $U_1 \subset U_2 \subset \cdots \subset U_n$, $U_i \in R$

Then $\int_{X\times Y} \chi_{\bigcup_{n=1}^{\infty} U_n} \, d(\mu\times\nu) \overset{MCT}{=} \lim_{n\to\infty} \int_{X\times Y} \chi_{U_n} \, d(\mu\times\nu)$

$= \lim_{n\to\infty} \int_Y \left(\int_X \chi_{U_n} d\mu_x\right) d\nu_y = \int_Y \left(\int_X \chi_U d\mu_x\right) d\nu_y$

And we know that for $f \in \mathcal{L}^1(X\times Y)$.

$f = U + iV = (U^+ - U^-) + i(V^+ - V^-)$ $\qquad i = 1,2.$

$\exists\, \phi_k^{(1)} \uparrow U^+, \phi_k^{(2)} \uparrow U^-, \psi_k^{(1)} \uparrow V^+, \psi_k^{(2)} \uparrow V^-$. $\phi_k^{(i)}, \psi_k^{(i)}$ simple

So $\int_{X\times Y} f \, d(\mu\times\nu) = \int_{X\times Y} U^+ d(\mu\times\nu) - \int_{X\times Y} U^- d(\mu\times\nu)$

$+ i \int_{X\times Y} V^+ d(\mu\times\nu) - i \int_{X\times Y} V^- d(\mu\times\nu)$

$\overset{MCT}{=} \lim_{n\to\infty} \int_{X\times Y} \phi_k^{(1)} d(\mu\times\nu) - \lim_{n\to\infty} \int_{X\times Y} \phi_k^{(2)} d(\mu\times\nu)$

$+ i \lim_{n\to\infty} \int_{X\times Y} \psi_k^{(1)} d(\mu\times\nu) - i \lim_{n\to\infty} \int_{X\times Y} \psi_k^{(2)} d(\mu\times\nu)$

$= \lim_{n\to\infty} \int_Y \left[\int_X \phi_k^{(1)} - \phi_k^{(2)} + i(\psi_k^{(1)} - \psi_k^{(2)}) \, d\mu_x \right] d\nu_y$

$= \int_Y \left[\int_X f \, d\mu_x \right] d\nu_y$.

26. Applications of Fubini Thm.

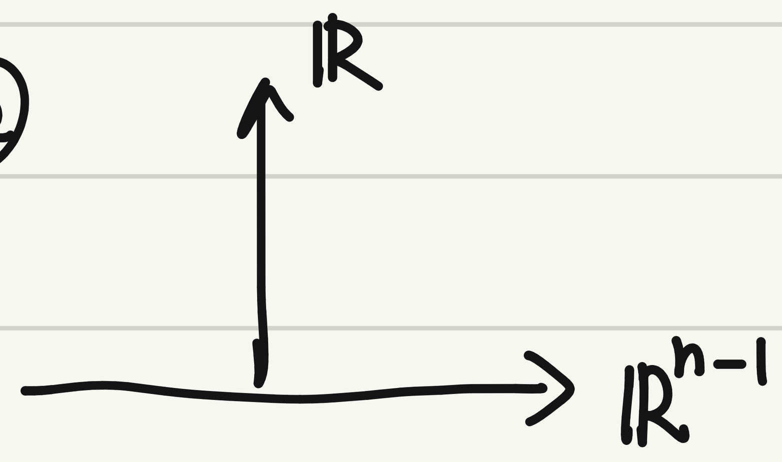
① $\int_X |f|^p d\mu = \int_0^\infty p\lambda^{p-1} \mu(|f| > \lambda) \, d\lambda$.

Pf: RHS $= \int_0^\infty \int_X p\lambda^{p-1} \chi_{\{|f|>\lambda\}} \, d\mu \, d\lambda$

$= \int_X \int_0^\infty p\lambda^{p-1} \chi_{\{|f|>\lambda\}} \, d\lambda \, d\mu = \int_X \int_0^{|f|} p\lambda^{p-1} d\lambda \, d\mu$

$= \int_X |f|^p d\mu$.

②



$$\begin{cases} -\Delta u = 0 \\ u = f \quad x \in \mathbb{R}^{n-1}. \end{cases}$$

Cone : $\Gamma_h(Q) = \{ x \in \mathbb{R}^n_1 \ , |x_n| = h\,|x - Q| \ , x = (x_1, \cdots, x_n) \}$.

Define square function

$$S(u)(Q) = \left(\int_{\Gamma_h(Q)} |\nabla u|^2 \frac{1}{x_n^{n-2}} \, dx \right)^{\frac{1}{2}}$$

$$\| S(u)(Q) \|^2 = \int_{\mathbb{R}^{n-1}} \left(\int_{\Gamma_h(Q)} |\nabla u|^2 \frac{1}{x_n^{n-2}} \, dx \right) dQ.$$

$$= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n_1} |\nabla u|^2 \frac{1}{x_n^{n-2}} \chi_{\{ x_n > h|x - Q| \}} \, dx \, dQ.$$

$$= \int_{\mathbb{R}^n_1} \int_{\mathbb{R}^{n-1}} \frac{|\nabla u|^2}{x_n^{n-2}} \chi_{\{ x_n > h|x - Q| \}} \, dQ \, dx.$$

$$= \int_{\mathbb{R}^n_1} \frac{|\nabla u|^2}{x_n^{n-2}} C \left| \frac{x_n}{h} \right|^{n-1} dx = C \int_{\mathbb{R}^n_1} |\nabla u|^2 x_n \, dx. \ C \text{ is const.}$$

§ 3 signed measure & differential

1. Def: $(X, \mathcal{A})$ A signed measure.
is a function. $\nu: \mathcal{A} \to [-\infty, +\infty]$.
① $\nu(\phi) = 0$
② ν cannot achieve $\pm\infty$ simultaneously.
③ $\{U_n\}$ in $\mathcal{A}$ disjoint. Then $\nu(\bigcup_n U_n) = \sum_n \nu(U_n)$
If $\nu(\bigcup_n U_n)$ is finite, then $\sum_n \nu(U_n)$ convergences absolutely

Prop: ν signed measure
$U_1 \subset U_2 \subset \cdots \subset U_n \subset \cdots$ in $\mathcal{A}$.
then $\nu(\bigcup_{n=1}^{\infty} U_n) = \lim_{n\to\infty} \nu(U_n)$

Pf: $\nu(\bigcup_{n=1}^{\infty} U_n) = \nu(U_1 \cup (U_2 \setminus U_1) \cup \cdots \cup (U_n \setminus U_{n-1}) \cup \cdots)$
$= \nu(U_1) + \sum_{n=1}^{\infty} \nu(U_{n+1} \setminus U_n) = \nu(U_1) + \sum_{n=1}^{\infty} (\nu(U_{n+1}) - \nu(U_n))$
$= \lim_{n\to\infty} \nu(U_n)$

2. ν signed measure
$U \in \mathcal{A}$ is called positive, if $\forall V \in \mathcal{A}, V \subseteq U$, we have $\nu(V) \geq 0$

Lemma: The union of countable many positive sets is positive

Pf: $\{U_n\}$ positive set
WLOG U_n is disjoint.
$\forall V \subset \bigcup_n U_n$, $V = \bigcup_n (U_n \cap V) \Rightarrow \nu(V) = \sum_n \nu(U_n \cap V) \geq 0$.

Thm (Hahn decomposition) $(X, \mathcal{A}, \nu)$ signed measure.
Then $\exists P$ positive, N negative s.t. $X = P \cup N$ and
$P \cap N = \phi$; If $\exists P', N'$ another such pair, then
$P \triangle P'$ is null set.

Pf: Define $m = \sup_{E \text{ positive}} \{\nu(E)\}$, $\exists \{P_n\}$ positive, s.t. $\nu(P_n) \uparrow m$.

Then let $P = \bigcup_n P_n$, then P is positive. $\nu(P) = m$

$N = X / P$.

Suppose N is not negative.

First, N cannot contain any nonnull positive subset otherwise, then extract that set out of N and combine it with P. Contradiction.

Second, if $\exists A \subset N$, s.t. $\nu(A) \geq 0$.

then $\exists B \subset A$, s.t. $\nu(B) \geq \nu(A)$

Construct $\{A_j\} \subset N$, $\{n_j\}$, n_1 is the smallest integer s.t. $\exists B \subset N$, s.t. $\nu(B) > \frac{1}{n_1}$, name $A_1 = B$

... n_j is the smallest integer s.t. $\exists B \subset A_{j-1}$, s.t. $\nu(B) > \nu(A_{j-1}) > \frac{1}{n_j}$. $A = \bigcap_j A_j$

Then $\infty > \nu(A) = \lim_{j \to \infty} \nu(A_j) > \sum_{j=1}^{\infty} \frac{1}{n_j}$, then $n_j \uparrow \infty$

Then there exists no subset of A, s.t. $\nu(B) > \nu(A)$

otherwise, take $\nu(B) > \nu(A) + \frac{1}{n}$, then for $n \leq n_j$, $B \subset A_{j-1}$

then contradict the smallest of n_j.

If $X = P \cup N = P' \cup N'$.

then $P \setminus P' \subset P$ positive, $P \setminus P' \subset N'$ negative

so $P \setminus P'$ is null set

3. μ, ν signed measures, are mutually singular if $\exists U, V \in \mathcal{A}$, s.t. $U \cap V = \phi$ and $U \cup V = X$, U is null for μ and V is null for ν. we say that $\mu \perp \nu$.

4. Thm (Jordan decomposition)

ν signed measure, $\exists!$ positive measure ν^+, ν^-.

s.t. $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Pf: $X = P \cup N$, let $\nu^+(U) = \nu(U \cap P)$, $\nu^-(U) = -\nu(U \cap N)$

then $\nu = \nu^+ - \nu^-$, $\nu^+ \perp \nu^-$.

If $\nu = \mu^+ - \mu^-$ is another decomposition

$\exists X = E \cup F$, s.t. $E \cap F = \phi$, $\mu^+(F) = \mu^-(E) = 0$.

then $E \cup F$ is another Hahn decomposition.
$P \triangle E$ is ν-null.
$\forall A \in \mathcal{A}$, $\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$
So $\mu^+ = \nu^+$, $\mu^- = \nu^-$

5. Def: ν is a signed measure, μ is a positive measure. We say ν an absolutely continuous w.r.t. μ ($\nu \ll \mu$), if $\mu(U) = 0$ implies $\nu(U) = 0$ holds for any U.

Thm: $\nu \ll \mu$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. when $\mu(E) < \delta$, we have $|\nu(E)| < \varepsilon$.

Pf: $\Leftarrow$: ✓
$\Rightarrow$: Suppose $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists$ set E
s.t. $\mu(E) < \delta$, $|\nu(E)| > \varepsilon$
Hence we can find $\{E_n\}$, s.t. $\mu(E_n) < 2^{-n}$
but $|\nu(E_n)| > \varepsilon$ $\forall n$.
$V_k = \bigcup_{n=k} E_n$, $V = \bigcap_{k=1}^{\infty} V_k$
$\mu(V_n) \le \sum_{n=k}^{\infty} \mu(E_n) \le \sum_{n=k}^{\infty} 2^{-n} = 2^{1-k} \to 0$ $(k \to \infty)$.
So $\mu(V) = 0$. but $|\nu(V)| \geq \varepsilon$, contradiction

6. Thm Lebesgue - Random - Nikodym.
ν σ-finite signed measure, μ σ-finite positive measure, Then $\exists!$ σ-finite signed measure λ, ρ.
s.t. $\nu = \lambda + \rho$
① $\lambda \perp \rho$
② $\rho \ll \mu$. $\exists f \in L^1(d\mu)$ s.t. $d\rho = f\,d\mu$. $f \triangleq \frac{d\rho}{d\mu}$.

Pf: Case I: ν, μ are all finite, positive
$\mathcal{F} = \{f : X \to [0, +\infty),\ \int_E f\,d\mu = \nu(E),\ \text{for all } E \in \mathcal{A}\}$

① $\mathcal{F}$ nonempty
② If $f, g \in \mathcal{F}$, then $\max(f,g) \in \mathcal{F}$.

Define $a = \sup\{\int_X f d\mu : f \in \mathcal{F}\}$
then $a \leq \nu(X)$. Pick a sequence.
$\{f_n\} \subset \mathcal{F}$, s.t. $\int_X f_n d\mu \to a$.

Let $g_n = \max\{f_1, f_2, \cdots, f_n\} \in \mathcal{F}$. $f = \sup_n f_n$. $g_n \uparrow f$.
$\lim_n \int g_n d\mu = \int f d\mu = a$.

Consider $d\lambda = d\nu - f d\mu$. Claim: $\lambda \perp \mu$.

Lemma: ν, μ finite positive
Then either $\nu \perp \mu$ or $\exists\, \varepsilon > 0, E \in \mathcal{A}$, s.t. $\mu(E) > 0$, E is positive set of $\nu - \varepsilon\mu$.

Pf: Let $X = P_n \cup N_n$ be a Hahn decomposition of $\nu - \frac{1}{n}\mu$.
Let $P = \bigcup_n P_n$ $N = \bigcap_n N_n$
Then N is negative for $\nu - \frac{1}{n}\mu$. $\forall n$.
So $\nu(N) \leq \frac{1}{n}\mu(N)$, $\nu(N) = 0$

① If $\mu(P) = 0$ then $\nu \perp \mu$.
② If $\mu(P) > 0$.
then $\mu(P_n) > 0$ for large n then P_n is positive for $\nu - \frac{1}{n}\mu$. $\square$

Consider λ and μ.
① $\lambda \perp \mu$ ✓
② $\exists E$, s.t. $\lambda - \varepsilon\mu \geq 0$ for E $d\nu = f d\mu + d\lambda$.
Then $\varepsilon \chi_E d\mu \leq d\nu - f d\mu$. $(f + \varepsilon\chi_E) d\mu \leq d\nu$
So $f + \varepsilon\chi_E \in \mathcal{F}$ $\int f + \varepsilon\chi_E d\mu = a + \varepsilon\mu(E) > a$, Contradiction

Case 2. for λ, μ σ-finite.
if $d\nu = f d\mu + d\lambda = f' d\mu + d\lambda'$ $d\lambda - d\lambda' = (f' - f) d\mu$.

$\lambda \perp \mu$, $\lambda' \perp \mu$ $\Rightarrow f' - f = 0$ μ-a.e.

e.g. $\mu = m$ $\quad \nu =$ point mass at x_0

$\nu \perp \mu$. $d\nu = 0\,d\mu + d\nu = f\,d\mu$. ($f$ is dirac function)

f is called distribution.

harmonic measure

$G(x,y)$ is Green function of $\begin{cases} \Delta u = 0 \\ u|_{\partial \mathbb{R}^n_+} = f. \end{cases}$

$$u(x) = \int_{\partial \mathbb{R}^n_+} \frac{\partial G(x,y)}{\partial \vec{n}} f(y)\, dH^{n-1}(y)$$

Def $d\nu^* = \dfrac{\partial G(x,y)}{\partial \vec{n}}\, dH^{n-1}(y)$

For $\Omega = \{(x', x_n) : x_n = \varphi(x')\}$. $\quad \begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = f. \end{cases}$

$$u(x) = \int_{\partial\Omega} f(y)\, d\nu^*$$

7. Prop: ν signed measure σ-finite

μ, λ σ-finite , $\nu \ll \mu \ll \lambda$.

① If $g \in L^1(d\nu)$

then $\int_X g\, d\nu = \int_X g \dfrac{d\nu}{d\mu}\, d\mu$.

② $\dfrac{d\nu}{d\lambda} = \dfrac{d\nu}{d\mu} \cdot \dfrac{d\mu}{d\lambda}$ $\quad \lambda$-a.e.

Pf: ① $\forall \chi_E$, $\nu(E) = \int_E \dfrac{d\nu}{d\mu}\, d\mu$

so it is also holded for $g \in L^1(d\nu)$.

② $\nu(E) = \int_E \dfrac{d\nu}{d\mu}\, d\mu$. Replace ① g by $\dfrac{d\nu}{d\mu}$, $d\nu$ by $d\mu$

So $\dfrac{d\nu}{d\mu}\, d\mu = \dfrac{d\nu}{d\mu} \dfrac{d\mu}{d\lambda}\, d\lambda = d\nu$.

So $\dfrac{d\nu}{d\lambda} = \dfrac{d\nu}{d\mu} \cdot \dfrac{d\mu}{d\lambda}$.

8. Back to $(\mathbb{R}^n, \mathcal{B}, m)$

ν is a signed measure $\ll m$.

From R-N. $d\nu = f\,dm$ for some $f \in L^1$.

Consider $\frac{1}{m(B_r(x))} \int_{B_r(x)} d\nu = \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dm_y$

$\triangleq F_r(x)$.

Vitali Lemma: $\mathcal{C}$ = some open balls in $\mathbb{R}^n$

$U = \bigcup_{B \in \mathcal{C}} B$ If $\exists c$, s.t. $0 < c \leq m(U)$

then $\exists$ disjoint $B_1, \cdots, B_k \in \mathcal{C}$, s.t. $\sum_{j=1}^{k} m(B_j) > \frac{c}{3^n}$

$A_r f(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dm(y) = ⨍_{B_r(x)} f(y)\,dm(y)$.

Thm If $f \in L^1_{Loc}(\mathbb{R}^n)$, then $A_r f(x)$ is continue with x, r.

Pf: Fix r_0, x_0. let $(r, x) \to (r_0, x_0)$. So $m(B_r(x)) \to m(B_{r_0}(x_0))$

$$\left| \int_{B_r(x)} f\,dm - \int_{B_{r_0}(x_0)} f\,dm \right|$$

$$= \left| \int_{\mathbb{R}^n} f(\chi_{B_r(x)} - \chi_{B_{r_0}(x_0)})\,dm \right|$$

$$\leq \int_{\mathbb{R}^n} |f|\,|\chi_{B_r(x)} - \chi_{B_{r_0}(x_0)}|\,dm$$

$|f|\,|\chi_{B_r(x)} - \chi_{B_{r_0}(x_0)}| \leq |f|\,\chi_{B_{2r_0}(x_0)} \in L^1$.

From DCT, $\int_{\mathbb{R}^n} |f|\,|\chi_{B_r(x)} - \chi_{B_{r_0}(x_0)}|\,dm \to 0$.

9. Maximal function (Hardy - Littlewood)

$Hf(x) = \sup_{r>0} A_r |f|(x)$.

Thm weak L^1 boundary.

$m(Hf > \alpha) \leq \frac{c}{\alpha} \|f\|_1$. (i.e. $\|Hf\|_{L^{1,\infty}} \leq C\|f\|_1$)

Pf: $E_\alpha = \{Hf > \alpha\}$. $\exists r_x$, s.t. $A_{r_x}|f|(x) > \alpha$.

$B_{r_x}(x)$ Covers E_α.

$m(E_\alpha) > c > 0$.

$\exists \{B_{r_{x_1}}(x_1), \cdots, B_{r_{x_k}}(x_k)\}$ disjoint, $\sum_{i=1}^{k} B_{r_{x_i}}(x_i) > \frac{c}{3^n}$.

So $c < 3^n \sum_{i=1}^{k} B_{r_{x_i}}(x_k) \leq 3^n \sum_{j=1}^{k} \frac{1}{\alpha} \int_{B_{r_{x_j}}(x_j)} |f| dm \leq \frac{3^n}{\alpha} \|f\|_1$.

10. If $f \in L^1_{Loc}(\mathbb{R}^n)$, then $\lim_{r \to 0} A_r f(x) = f(x)$.

Pf: WLOG, only need to prove for $|x| < N$ and $f \in L^1$.

Since $C(\mathbb{R}^n) \overset{\text{dense}}{\subset} L^1$.

$\forall \varepsilon > 0, \exists g$ continuous, s.t. $\int_{\mathbb{R}^n} |f-g| dm < \varepsilon$.

$|A_r f(x) - f(x)| \leq |A_r f(x) - A_r g(x)| + |A_r g(x) - g(x)| + |g(x) - f(x)|$

$\leq H(f-g)(x) + |g(x) - f(x)|$

$\{x : \limsup_{r \to 0} |A_r f(x) - f(x)| > \alpha\} \subseteq \{x : H(f-g) > \frac{\alpha}{2}\} \cup \{x : |g-f| > \frac{\alpha}{2}\}$

$m(x, H(f-g)(x) > \frac{\alpha}{2}) < \frac{C}{\alpha/2} \int_{\mathbb{R}^n} |f-g| dx \leq \frac{2C\varepsilon}{\alpha}$

$m(x, |g-f| > \frac{\alpha}{2}) < \frac{1}{\alpha/2} \int_{\mathbb{R}^n} |f-g| dx \leq \frac{2\varepsilon}{\alpha}$.

$\Rightarrow m(\{x : \limsup_{r \to 0} |A_r f(x) - f(x)| > \alpha\}) < \frac{2(C+1)\varepsilon}{\alpha} \to 0$.

So $\lim_{r \to 0} A_r f = f$ a.e. $L_f = \{x, \lim_{r \to 0} A_r f(x) = f(x)\}$

11. Lebesgue differentiation

Assume that $f \in L^1_{Loc}(\mathbb{R}^n)$, then for $x \in L_f$. $\lim_{r \to 0} ⨍_{E_r} |f(y) - f(x)| dm_y = 0$

where E_r shrink nicely to x.

Pf: $\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm_y \leq \frac{\alpha}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dm_y$

$\to 0$ a.e.

We say that $\{E_r\}$ Borel sets shrink to x nicely if $E_r \subset B_r(x)$ $\forall r$, $\exists \alpha > 0$, indepedent for r, $m(E_r) > \alpha m(B_r(x))$

12. BV functions

$T_f(x) = \sup\{\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|, n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\}$

$T_f(x)$ is the oscillication of function.

if $T_f(x) < \infty$. we say f is a B.V. function on $[x_0, x_n]$.

13. Def A Borel measure ν on $\mathbb{R}^n$ is regular if.

① $\forall K$ compact, $\nu(K) < \infty$.

② $\forall E \in \mathcal{B}_{\mathbb{R}^n}$, $\nu(E) = \inf\{\nu(U),\ E \subset U \text{ open}\}$.

e.g. $d\nu = f\,dm$.

14. Thm ν regular signed measure

$d\nu = d\lambda + f\,dm$, Then for m-a.e. x, we have

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x),\quad E_r \text{ shrink nicely to } x.$$

Pf: ν is regular, then λ and $f\,dm$ are both regular.

$d\nu^+ = d\lambda^+ + f^+ dm$

need to show $\dfrac{\lambda(E_r)}{m(E_r)} + \fint_{E_r} f\,dm \xrightarrow{a.e.} f(x)$

we need to show $\dfrac{\lambda^+(E_r)}{m(E_r)} \to 0$.

Since $\lambda^+ \perp m^-$, $\exists$ Borel set A, s.t. $\lambda(A) = m(A^c) = 0$

Consider $F_k = \{x \in A,\ \limsup_{r\to 0} \dfrac{\lambda(B_r(x))}{m(B_r(x))} > \frac{1}{k}\}$.

Want to show $m(F_k) = 0$

$\forall \varepsilon > 0$, $\exists A \subset U_\varepsilon$, s.t. $\lambda(U_\varepsilon) < \varepsilon$.

$\forall x \in F_k$, $\exists B_{r_x}(x) \subset U_\varepsilon$, s.t. $\dfrac{\lambda(B_{r_x}(x))}{m(B_{r_x}(x))} > \frac{1}{k}$

Consider $m(\bigcup_{x\in F_k} B_{r_x}(x)) > 0$.

$\exists B_{r_{x_1}}(x_1), B_{r_{x_2}}(x_2), \cdots, B_{r_{x_n}}(x_n)$ disjoint.

$$m\Big(\bigcup_{x\in F_k} B_{r_x}(x)\Big) \leq 3^n \sum_{j=1}^{n} m(B_{r_{x_j}}(x_j)) \leq 3^n k \sum_{j=1}^{n} \lambda(B_{r_{x_j}}(x_j))$$

$$< 3^n k\, \lambda\Big(\bigcup_{x\in F_k} B_{r_x}(x)\Big)$$

15. Thm: $f: \mathbb{R} \to \mathbb{R}$ $\uparrow$

$g(x) = f(x^+) = \lim_{y\to x^+} f(y)$

① The set of discontinuous point of f is countable

② f, g are differential a.e., $f' = g'$ a.e.

Pf: ① Since $f \uparrow$

$(f(x^-), f(x^+))$ are disjoint

So $m\left(\bigcup_{|x|<N} (f(x^-), f(x^+))\right) = m((f(-N), f(N)))$

$\sum_{|x|<N} (f(x^+) - f(x^-)) = f(N) - f(-N) < \infty$.

Then $\{x : f(x^-) \neq f(x^+)\}$ is at most countable.

② $g \uparrow$ continuous.

Define $g(x+h) - g(x) = \begin{cases} \mu_g((x, x+h]) & h>0 \\ -\mu_g((x+h, x]) & h<0. \end{cases}$

induces a Borel measure.

μ_g is regular. $\forall K$ compact, $\mu_g(K) < \infty$

$\forall E$ Borel, $\exists$ open set U s.t. $\mu_g(U \setminus E) < \varepsilon$.

$\varphi = g - f$, want to show

$\varphi' = 0$ a.e. and $\varphi' = 0$ a.e.

$f \neq g$ at $\{x_j\}_{j=1}^{\infty}$, Then $\varphi(x_j) > 0$

set $\mu = \sum_{j=1}^{\infty} \varphi(x_j) \delta_j$ then μ is regular. $\mu \perp m$.

L-R-N, $d\mu = d\lambda + \phi\, dm$, $\phi = 0$.

$$\frac{\varphi(x+h) - \varphi(x)}{h} \leqslant \frac{\varphi(x+h) + \varphi(x)}{h} \leqslant \frac{2\mu(x-2h, x+2h)}{h}$$

$$= \frac{2\mu(B_{2h}(x))}{m(B_{2h}(x))} \to 0 \text{ as } h \to 0.$$

16. For $F: \mathbb{R} \to \mathbb{C}$, the total variation of F is

$T_F(x) = \sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|, n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\}$

If $T_F(\infty) = \lim_{x \to \infty} T_F(x)$ is finite.

then we say F is of bdd variation.

17. Thm: ① $F: \mathbb{R} \to \mathbb{C}$ BV iff $\operatorname{Re} F, \operatorname{Im} F \in BV$.

② $F: \mathbb{R} \to \mathbb{R}$, BV iff F is the difference of 2 bdd $\uparrow$ function

$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ (Jordan decomposition)

③ If $F \in BV$, then $F(x^+), F(x^-)$ exists.

④ If $F \in BV$, then the discontinuous points are at most countable.

⑤ If $F \in BV$, $G(x) = F(x^+)$, then F', G' exists and are equal a.e.

18. Def: $NBV = \{ F \in BV, F \text{ is right continuous and } F(-\infty) = 0 \}$.

Lemma: If $F \in BV$, then $T_F(-\infty) = 0$. If F is right continuous, the T_F is also.

Thm: μ complex Borel measure on $\mathbb{R}$ $F(x) = \mu((-\infty, x])$ then $F \in NBV$. If $F \in NBV$, $\exists$: complex Borel measure μ_F, s.t. $F(x) = \mu_F((-\infty, x])$.

Pf: ① From Hahn decomposition.

$\mu = \mu^+ - \mu^-$

Def $F^+(x) = \mu^+((-\infty, x])$, $F^-(x) = \mu^-((-\infty, x])$.

$F(x) = F^+(x) - F^-(x)$.

② $F = G - H$, $\mu^+((-\infty, x]) = G(x)$, $\mu^-((-\infty, x]) = H(x)$.

§ 4.5 Point Set Topology

1. X is a nonempty set. A topology on X is a family $\mathcal{T}$ of subsets of X that contains ϕ and X. And it is closed under arbitrary unions and finite intersections. $(X, \mathcal{T})$ is called topological space.

e.g. ① $\mathcal{P}(X)$ (called discrete topology)

② $\{\phi, X\}$ (called trivial topology).

③ X is an infinite set. $\{U \subset X : U = \phi$ or U^c is finite$\}$ is a topology on X, called cofinite topology.

④ X metric space. the collection of all open sets with respect to metric space is a topology on X.

⑤ $(X, \mathcal{T})$ is a topological space and $Y \subset X$, then $\mathcal{T}_Y = \{U \cap Y, U \in \mathcal{T}\}$ is a topology on Y.

2. The members of $\mathcal{T}$ are called open sets, and their complements are called closed sets. If $Y \subset X$, the open (closed) subsets of Y in the relative topology are called relatively open.

3. If $A \subset X$, $\Lambda = \{\alpha : O_\alpha$ open set, $O_\alpha \subset A\}$.

$\bigcup_{\alpha \in \Lambda} O_\alpha$ is called the interior of A, denoted A°

$\Pi = \{\beta : C_\beta$ closed set, $A \subset C_\beta\}$.

$\bigcap_{\beta \in \Pi} C_\beta$ is called the closure of A, denoted $\bar{A}$.

$\bar{A} \setminus A^\circ$ is called the boundary of A, denoted ∂A.

If $\bar{A} = X$, A is called dense in X.

If $(\bar{A})^\circ = \phi$, A is called nowhere dense.

4. $x \in X$ (or $E \subset X$), a neighborhood of x is a set $A \subset X$, s.t. $x \in A^\circ$. (or $E \subset A^\circ$).

A point x is called an accumulation point of A if $A \cap (U \setminus \{x\}) \neq \phi$, for every neighborhood U of x.

5. $acc(A) = \{$the accumulation points of $A\}$.
Then $\bar{A} = A \cup acc(A)$, A is closed iff $acc(A) \subset A$.
Pf: If $x \notin \bar{A}$, then A^c is a neighborhood of x
Then $A \cap (A^c \setminus \{x\}) = \phi \Rightarrow x \notin acc(A)$.
So $A \cup acc(A) \subset \bar{A}$
If $x \in A \cup acc(A)$. So $\exists\, U$ open set containing x.
$U \cap A = \phi$, so $\bar{A} \subset U^c$ and $x \notin \bar{A} \Rightarrow \bar{A} \subset A \cup acc(A)$

6. $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, $\mathcal{T}_1 \subset \mathcal{T}_2$, we say that $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$, $\mathcal{T}_2$ is stronger that $\mathcal{T}_1$.
If $\mathcal{E} \subset P(X)$, there is a unique weakest topology $\mathcal{T}(\mathcal{E})$ on X that contains $\mathcal{E}$. It is called the topology generated by $\mathcal{E}$, $\mathcal{E}$ is called a subbase of $\mathcal{T}(\mathcal{E})$.

7. $(X, \mathcal{T})$ is a topological space.
a neighborhood base for $\mathcal{T}$ at $x \in X$ is a family $\mathcal{N} \subset \mathcal{T}$
s.t. ① $x \in V$, $\forall V \in \mathcal{N}$
② if $U \in \mathcal{T}$ and $x \in U$, there exists $V \in \mathcal{N}$, s.t. $V \subset U$
A base for $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$.
$\mathcal{B} = \bigcup_{x \in X} \{$The neighborhood base of $x\}$.

e.g. $B_r(x)$ in metric space.

8. $\mathcal{E}$ is a base for $\mathcal{T}$ iff every nonempty $U \in \mathcal{T}$ is a union of members of $\mathcal{E}$.
Pf: If $\mathcal{E}$ is a base. $x \in U \in \mathcal{T}$. $\exists V_x \in \mathcal{E}$, $x \in V_x \subset U$.
so $U = \bigcup_{x \in U} V_x$.
The converse statement is trivial

9. $\mathcal{E}$ is the base for a topology on $X \Leftrightarrow$ the following two conditions.
① each $x \in X$ is contained in some $V \in \mathcal{E}$.
② if $U, V \in \mathcal{E}$ and $x \in U \cap V$, $\exists\, W \in \mathcal{E}$, $x \in W \subset U \cap V$.

Pf: "$\Rightarrow$" trivial

"$\Leftarrow$" let $\mathcal{T} = \{ U \subset X : \forall x \in U, \exists V \in \mathcal{E}$ with $x \in V \subset \mathcal{E} \}$

Then $\phi, X \in \mathcal{T}$. $\mathcal{T}$ is closed under union.

If $U_1, U_2 \in \mathcal{T}$, $x \in U_1 \cap U_2$. $\exists V_1, V_2 \in \mathcal{E}$. $x \in V_1 \subset U_1$, $x \in V_2 \subset U_2$. so $\exists W \subset V_1 \cap V_2$, $x \in W$. $W \in \mathcal{E}$.

Thus $U_1 \cap U_2 \in \mathcal{T}$. so $\mathcal{T}$ is a topology.

10. $(X, \mathcal{T})$ is a topological space, it is first countable if there is a countable neighborhood base for $\mathcal{T}$ at every point of X. It is second countable, if $\mathcal{T}$ has a countable base. $(X, \mathcal{T})$ is seperable if X has a countable dense subset.

11. Every second countable space is seperable.

Pf: X is second countable.

$\mathcal{E}$ is the countable base of $\mathcal{T}$.

$\forall U \in \mathcal{E}$, pick a point $x_U \in U$.

Then $\overline{\{x_U : U \in \mathcal{E}\}}^c$ is an open set that does not include any $U \in \mathcal{E}$. So $\overline{\{x_U : U \in \mathcal{E}\}} = X$.

12. A sequence $\{x_j\}$ in a topological space X converges to $x \in X$ ($x_j \to x$) if $\forall$ neighborhood U of X, $\exists N \in \mathbb{N}$, s.t. $x_j \in U$, $\forall j > N$.

If X is first countable and $A \subset X$, then $x \in \bar{A}$ iff there is a sequence $\{x_j\}$ in A that converges to x.

13. If X has property T_j, we say that X is a T_j space.

T_0: If $x \neq y$, there is an open set containing x but not y or an open set containing y but not x.

T_1: If $x \neq y$, $\exists$ open set containing y but not x.

T_2: If $x \neq y$, $\exists$ disjoint open sets U, V, s.t. $x \in U$, $y \in V$

T_3: X is a T_1 space, $\forall$ closed set $A \subset X$, $\forall x \in A^c$, $\exists$ disjoint open sets U, V, $x \in U$, $A \subset V$.

T_4: X is a T_1 space, $\forall$ disjoint closed sets A, B in X, $\exists$ disjoint open sets U, V, $A \subset U$, $B \subset V$.

T_2 space is also called Hausdorff space.
T_3 space is called regular space.
T_4 space is called normal space.
X is a T_1 space iff $\{x\}$ is closed for every $x \in X$

14. X, Y are topological spaces.
$f: X \mapsto Y$, then f is called continuous if $\forall V$ openset of Y, $f^{-1}(V)$ is an open set in X.
$\Leftrightarrow$ $\forall A$ closed set of Y, $f^{-1}(A)$ is a closed set in X.
Denote $f \in C(X, Y)$.
If $x \in X$, f is called continuous at x if $\forall$ neighborhood V of $f(x)$, $\exists$ neighborhood U of x, s.t. $f(U) \subset V$.
$f \in C(X, Y) \Leftrightarrow f$ is continuous at every $x \in X$.
If the topology on Y is generated by $\mathcal{E}$. $f \in C(X, Y) \Leftrightarrow$ $\forall V \in \mathcal{E}$, $f^{-1}(V)$ is open.

15. If $f: X \to Y$ is bijective and f, f^{-1} are both continuous, f is called a homeomorphism. and X & Y are called homeomorphic.
If $f: X \to Y$ is injective but not surjective, $f: X \to f(X)$ is a homeomorphism when $f(X) \subset Y$ is given the relative topology, f is called an embedding.

16. $\{f_\alpha : X \to Y_\alpha\}$ is a family of maps from X into some topological spaces Y_α, there is a weakest topology $\mathcal{J}$ on X that makes all the f_α continuous, it is called the weak topology generated by $\{f_\alpha\}_{\alpha \in A}$

e.g. product topology $\Rightarrow$ Cartesian product of topological spaces.

17. If $\forall \alpha \in A$, X_α is Hausdorff, then $X = \prod_{\alpha \in \Lambda} X_\alpha$ is Hausdorff.
Pf: $\forall x \neq y$, $x, y \in X$. $\exists \alpha \in A$, $\pi_\alpha(x) \neq \pi_\alpha(y)$.
Let U, V be disjoint neighborhoods of $\pi_\alpha(x)$, $\pi_\alpha(y)$ in X_α.
So $\pi_\alpha^{-1}(U)$, $\pi_\alpha^{-1}(V)$ are disjoint neighborhoods of x, y.

18. Urysohn's Lemma (in topological space).
Let X be a normal space. If A and B are disjoint closed sets in X, $\exists f \in C(X)$, $f \in [0,1]$, s.t. $f = 0$ on A and $f = 1$ on B

$\Rightarrow$ Tietze Extension Thm
X normal space, closed subset A, $f \in C(A)$, $f \in [a,b]$
$\exists F \in C(X)$, $F \in [a,b]$, s.t. $F|_A = f$.

19. Def A topological space X is compact if whenever $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X, $\exists B$ finite subset of A, s.t. $\{U_\alpha\}_{\alpha \in B}$ is also the open cover of X.
A subset Y of X is compact if it is compact in the relative topology. If its closure is compact, Y is called precompact.

20. X topological space is compact iff every family $\{F_\alpha\}_{\alpha \in A}$ of closed sets with finite intersection prop, $\bigcap_{\alpha \in A} F_\alpha \neq \phi$.
Pf: $U_\alpha = (F_\alpha)^c$ open. $\bigcap_{\alpha \in A} F_\alpha \neq \phi \Leftrightarrow \bigcup_{\alpha \in A} U_\alpha \neq X$
$\{F_\alpha\}_{\alpha \in A}$ has finite intersection prop iff no finite subcover of X. $\square$

21. If F is a compact subset of Hausdorff space X and $x \notin F$, $\exists$ disjoint open sets U, V, s.t. $x \in U$, $F \subset V$.
Pf: $\forall y \in F$, choose disjoint open set U_y, V_y, $x \in U_y$, $y \in V_y$.
So $\{V_y\}$ is the open cover of F, finite subcover $\{V_{y_i}\}_{i=1}^n$.

Pick $U=\bigcap_{i=1}^{n} U_{y_i}$, $V=\bigcup_{i=1}^{n} V_{y_i}$.

22. Compact subset of Hausdorff is closed.
Pf: If F is compact, so F^c is a neighborhood of every point of it. so F is closed.

23. Compact Hausdorff space is normal.
Pf: X compact Hausdorff space.
E, F are disjoint closed subsets of X.
$\forall x\in E$, $\exists U_x, V_x$ disjoint and open, s.t. $x\in U_x$, $E\subset U_x$.
$\{U_x\}_{x\in E}$ is the open cover of E. We pick finite subcover $\{U_{x_i}\}_{i=1}^{n}$, $U=\bigcup_{i=1}^{n} U_{x_i}$, $V=\bigcap_{i=1}^{n} V_{x_i}$, Then U, V open, $E\subset U$, $F\subset V$

24. Prop. ① If X is compact, $f: X\to Y$ is continuous, then $f(X)$ is compact
② X is compact, $f\in C(X)$, then f is bdd.

25. A directed set is a set A equipped with binary relation $\precsim$ s.t.
- $\alpha\precsim\alpha$ $\forall\alpha\in A$ · if $\alpha\precsim\beta$ and $\beta\precsim\gamma$ then $\alpha\precsim\gamma$
- $\forall\alpha,\beta\in A$, $\exists\gamma\in A$, s.t. $\alpha\precsim\gamma$, $\beta\precsim\gamma$.

A net in a set X is a mapping $\alpha\mapsto x_\alpha$ from a dircected A into X, denoted $\langle x_\alpha\rangle_{\alpha\in A}$ or $\langle x_\alpha\rangle$, $\langle x_\alpha\rangle$ is indexed by A.

26. E is a subset of topological space X. A net $\langle x_\alpha\rangle_{\alpha\in A}$ is eventually in E if $\exists\alpha_0\in A$ s.t. $x_\alpha\in E$, $\alpha\succsim\alpha_0$. $\langle x_\alpha\rangle$ is frequently in E if $\forall\alpha\in A$ $\exists\beta\succsim\alpha$, s.t. $x_\beta\in E$.
A point $x\in X$ is a limit of $\langle x_\alpha\rangle$ (i.e. $x_\alpha\to x$) if $\forall U$ neighborhood of x, $\langle x_\alpha\rangle$ is eventually in U.
A point $x\in X$ is a cluster point of $\langle x_\alpha\rangle$ if $\forall U$ neighborhood of x, $\langle x_\alpha\rangle$ is frequently in U.

27. X topological space, $E \subset X$, $x \in X$, then x is an accumulation point of E iff $\exists$ net in $E \setminus \{x\}$ that converges to x, and $x \in \overline{E}$ iff $\exists$ net in E converges to x.

Pf: If x is an accumulation point of E, let $\mathcal{N}$ be the set of neighborhoods of x, directed by reverse inclusion. $\forall U \in \mathcal{N}$, pick $x_U \in (U \setminus \{x\}) \cap E$, $x_U \to x$. Conversely, if $x_\alpha \in E \setminus \{x\}$ and $x_\alpha \to x$, then $\forall$ punctured neighborhood of x contains some x_α, so x is the accumulation point of E.

28. X, Y topological spaces. $f \in C(X, Y)$ iff $\forall$ net $\langle x_\alpha \rangle$ converging to x, $\langle f(x_\alpha) \rangle$ converges to $f(x)$.

29. A subnet of net $\langle x_\alpha \rangle_{\alpha \in A}$ is a net $\langle y_\beta \rangle_{\beta \in B}$ together with a map $\beta \mapsto \alpha_\beta$ from B to A s.t.

- $\forall \alpha_0 \in A$, $\exists \beta_0 \in B$ s.t. $\alpha_\beta \gtrsim \alpha_0$ whenever $\beta \gtrsim \beta_0$.
- $y_\beta = x_{\alpha_\beta}$.

If $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in a topological space X, then $x \in X$ is a cluster point of $\langle x_\alpha \rangle$ iff has a subnet that converges to x.

30. Thm X is a topological space, the following are equivalent

a. X is compact

b. $\forall$ net in X has a cluster point.

c. $\forall$ net in X has a convergent subnet.

Pf: $b \Leftrightarrow c$ we have got it.

If X is compact and $\langle x_\alpha \rangle$ is a net in X.

Let $E_\alpha = \{x_\beta : \beta \gtrsim \alpha\}$ Since $\forall \alpha, \beta \in A$, $\exists \gamma \in A$ s.t. $\gamma \gtrsim \alpha, \gamma \gtrsim \beta$.

So $\{E_\alpha\}_{\alpha \in A}$ has the finite intersection property.

So $\bigcap_{\alpha \in A} \overline{E_\alpha} \neq \emptyset$ If $x \in \bigcap_{\alpha \in A} \overline{E_\alpha}$, and U is a neighborhood of x, then U intersects each E_α, so $\langle x_\alpha \rangle$ is frequently in U. x is a cluster point of $\langle x_\alpha \rangle$.

On the other hand, if X is not compact.

Let $\{U_\beta\}_{\beta\in B}$ be an open cover of X with no finite subcover.
Let $\mathcal{A}$ be the collection of finite subset of B, directed by inclusion. $\forall A\in\mathcal{A}$, x_A is a point in $(\bigcup_{\beta\in A} U_\beta)^c$.
Then $\langle x_A\rangle_{A\in\mathcal{A}}$ is a net with no cluster point.
Indeed, if $x\in X$, choose $\beta\in B$ with $x\in U_\beta$. If $A\in\mathcal{A}$ and $A \gtrsim \{\beta\}$ then $x_A \notin U_\beta$, so x is not a cluster point of $\langle x_A\rangle$.

31. X is called countablly compact if $\forall$ countable open cover of X has a finite subcover.
X is called sequentially compact if $\forall$ sequence in X has a subsequence.

32. A topological space is called locally compact if every point has a compact neighborhood.
We call locally compact Hausdorff space LCH for short.

33. Prop. ① X LCH, $U\subset X$ is open, $x\in U$, $\exists$ compact neighborhood N of x, s.t. $N\subset U$.
② X LCH, $K\subset U\subset X$, K is compact and U is open. $\exists$ precompact open set V s.t $K\subset V\subset \bar{V}\subset U$
③ X LCH, then X is a normal space.

34. X is a topological space, $f\in C(X)$.
$\mathrm{supp}(f)$ is called the surpport of f, it is the closure of $\{x : f(x)\neq 0\}$.
If $\mathrm{supp}(f)$ is compact, we say f is compactly surpported
$C_c(X)=\{f\in C(X) : \mathrm{supp}(f) \text{ is compact}\}$.
We say f vanishes at infinity if $\forall\varepsilon>0$, $\{x : |f(x)|\geq\varepsilon\}$ is compact, $C_0(X)=\{f\in C(X) . f \text{ vanishes at infinity}\}$.

35. X LCH, $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric.

Pf: If $\{f_n\}$ is a sequence in $C_c(X)$ that converges uniformly to $f \in C(X)$.

$\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $\|f_n - f\|_\infty < \varepsilon$. Then $|f(x)| < \varepsilon$ if $x \notin \operatorname{supp} f_n$

So $f \in C_0(X)$.

Conversely, if $f \in C_0(X)$. let $K_n = \{x : |f(x)| \geq \frac{1}{n}\}$. compact.

So $\exists g_n \in C_c(X)$, $g_n \in [0,1]$, and $g_n|_{K_n} = 1$.

So $f_n = g_n f \xrightarrow{\infty} f$. (Urysohn's Lemma in LCH)

36. If X is a noncompact LCH space, let ∞ denote a point that is not an element of X, $X^* = X \cup \{\infty\}$, and let $\mathcal{T}$ be the collection of all subsets of X^* s.t. either (i) U is an open set of X, or (ii) $\infty \in U$ and U^c is compact then $(X^*, \mathcal{T})$ is a compact Hausdorff space, and the inclusion map $i: X \to X^*$ is an embedding.

X^* is called the one-point compactification or Alexandroff compactification of X.

37. X LCH space, $E \subset X$ closed iff $\forall K$ compact, $E \cap K$ is closed.

Pf: "$\Rightarrow$" E is closed, K is compact so K is closed

So $E \cap K$ is closed.

"$\Leftarrow$" If E is not closed.

Pick $x \in \bar{E} \setminus E$, K is a compact neighborhood of x.

Then x is an accumulation point of $E \cap K$, but $x \notin E \cap K$.

So $E \cap K$ is not closed.

38. X LCH space. $C(X)$ is a closed subspace of $\mathbb{C}^X$ in the topology of uniform convergence on compact sets.

Pf: If $f \in \overline{C(X)}$, then f is a uniform limit of continuous functions on each compact $K \subset X$. So $f|_K$ is continuous.

If $E \subset \mathbb{C}$ closed, $f^{-1}(E) \cap K = (f|_K)^{-1}(E)$ closed. Thus

$f^{-1}(E)$ is closed, f is continuous

39. A topological space X is called σ-compact if it is a countable union of compact sets.

If X is a σ-compact LCH space, $\exists \{U_n\}_{n=1}^{\infty}$, U_n open, s.t. $\overline{U_n} \subset U_{n+1}$ and $X = \bigcup_{n=1}^{\infty} U_n$.

Pf: $X = \bigcup_{n=1}^{\infty} K_n$, K_n is compact.

$\exists U_1$ precompact open neighborhood of K_1.

$\exists U_n$ precompact open neighborhood of $\overline{U_{n-1}} \cup K_n$.

So $X = \bigcup_{n=1}^{\infty} U_n$, $\overline{U_n} \subset U_{n+1}$

40. X topological space, $E \subset X$.

A partition of unity on E is a collection $\{h_\alpha\}_{\alpha \in A}$ of functions in $C(X, [0,1])$ s.t.

① $\forall x \in X$, $\exists$ neighborhood of x, only finitely many $h_\alpha \neq 0$.

② $\sum_{\alpha \in A} h_\alpha = 1$, $\forall x \in E$

A partition of unity $\{h_\alpha\}$ is subordinate to an open cover $\mathcal{U}$ of E if $\forall \alpha$, $\exists U \in \mathcal{U}$ s.t. $\mathrm{supp}(h_\alpha) \subset U$.

41. Tychonoff's Thm

If $\{X_\alpha\}_{\alpha \in A}$ is any family of compact topological spaces, then $X = \prod_{\alpha \in A} X_\alpha$ is compact in product topology.

42. Arzelà-Ascoli Thm.

$\mathcal{F} \subset C(X)$, $\mathcal{F}$ is called equicontinuous at $x \in X$, if $\forall \varepsilon > 0$ $\exists$ neighborhood U of x s.t. $\forall y \in U$, $f \in \mathcal{F}$, $|f(y) - f(x)| < \varepsilon$.

I. X compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C(X)$, then $\mathcal{F}$ is totally bounded in the uniform metric, and $\mathcal{F}$ is precompact.

II. X σ-compact LCH space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $C(X)$, $\exists f \in C(X)$ and a subsequence of $\{f_n\}$ converge to f uniformly in compact sets.

43. The Stone-Weierstrass Thm

A subset $\mathcal{A}$ of $C(X,\mathbb{R})$ or $C(X)$ is said to separate points if $\forall x, y \in X$, $x \neq y$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.

$\mathcal{A}$ is called an algebra if it is a real vector subspace of $C(X,\mathbb{R})$, s.t. if $f, g \in \mathcal{A}$, $fg \in \mathcal{A}$.

If $\mathcal{A} \subset C(X,\mathbb{R})$, $\mathcal{A}$ is called a lattice if $\max(f,g)$, $\min(f,g)$ are in $\mathcal{A}$ if $f, g \in \mathcal{A}$.

X compact Hausdorff space. If $\mathcal{A}$ is a closed subalgebra of $C(X,\mathbb{R})$ the separates points, then either $\mathcal{A} = C(X,\mathbb{R})$ or $\mathcal{A} = \{f \in C(X,\mathbb{R}), f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative holds iff $\mathcal{A}$ contains the constant functions.

§ 5 Functional Analysis

1. Def · X vector space
$\|\cdot\|: X \to [0,+\infty)$
s.t. $\|u+v\| \leq \|u\|+\|v\|$, $\|\lambda u\| = |\lambda| \|u\|$.
$\|\cdot\|$ is a semi-norm.
If $\|u\|=0$, iff $u=0$, Then $\|\cdot\|$ is a norm.
$(X, \|\cdot\|)$ is a normed vector space.

2. A normed vector space that is complete, is called Banach space.
A linear map $T: X \to Y$ is called bdd iff $\|Tu\| \leq C\|u\|$.
the following statements are equivalent.
① T continuous
② T continuous at 0
③ T is bdd.

3. Def: ① $\|T\| = \inf\{c : \|Tu\| \leq c\|u\|\} = \sup\{\frac{\|Tu\|}{\|u\|}, u \in X\}$
② $\mathcal{L}(X,Y)$: {bdd linear map from X to Y}.
$(\mathcal{L}(X,Y), \|\cdot\|)$ is normed vector space.

4. Thm Hahn-Banach
X is a real vecotor space. p is a sublinear functional. $M \subset X$ subspace. f is a linear functional in M, s.t. $f(x) \leq p(x)$. Then f can be extended to X (call F), s.t.
$F|_M = f$, $F \leq p$.

Pf: f extend to $M \oplus \text{span}\{x\}$.
then $F(y+\lambda x) = F(y) + \lambda F(x) = f(y) + \lambda F(x) \leq p(y+\lambda x)$
so $F(x) \leq \inf\{p(y+x) - f(x), y \in M\}$.
$F(x) \geq \sup\{f(y) - p(y-x), y \in M\}$.
consider $\sup\{f(y) - p(y-x), y \in M\} \leq \beta \leq \inf\{p(y+x) - f(x), y \in M\}$.
$\tilde{f}: M \oplus \text{span}\{x\} \to \mathbb{R}$, $y + \lambda x \to f(y) + \lambda\beta$. satisfied. (check)

According to Zorn's Lemma, we can extend it to X.

5. e.g. $\begin{cases} -\Delta u = u^{q-1} & \Omega \\ u=0 & \partial\Omega \end{cases}$ $\quad 1 < q < \frac{2n}{n-2}$ (*) where Ω is bdd.

Variational method.

$$I[u] = \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |u|^q dx)^{\frac{2}{q}}}$$

Consider $u \in W_0^{1,2}(\Omega) = \{u; u \in L^2(\Omega), |\nabla u| \in L^2(\Omega), u \text{ vanishes on } \partial\Omega\}$.

Consider $m = \inf\limits_{u \in W_0^{1,2}} \frac{\int |\nabla u|^2 dx}{(\int |u|^q)^{\frac{2}{q}}}$

Assume $I[u] = m$, then $I[u+\varepsilon\varphi] \geq I[u]$, $\forall \varphi \in C^\infty(\Omega)$

Then $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I[u+\varepsilon\varphi] = 0$.

$$0 = \frac{dI}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\int |\nabla u + \varepsilon\nabla\varphi|^2}{(\int |u+\varepsilon\varphi|^q)^{\frac{2}{q}}} = \frac{(\int 2\nabla(u+\varepsilon\varphi)\nabla\varphi)(\int |u+\varepsilon\varphi|^q)^{\frac{2}{q}} - (\ldots)}{(\int |u+\varepsilon\varphi|^q)^{\frac{4}{q}}}$$

Notice $\int \nabla u \nabla\varphi = \int |u|^{q-2} u \cdot \varphi = \int -\Delta u\, \varphi$. (weak solution).

So $-\Delta u = |u|^{q-2} u$ a.e.

$\exists u_k$ in $W_0^{1,2}$ s.t. $I[u] \downarrow m$.

We need to show $\{u_k\}$ is bdd in $W_0^{1,2}$.

$\|u\|_{W^{1,2}} = \left(\int |u|^2 + |\nabla u|^2 dx\right)^{\frac{1}{2}}$

Let $\|u_k\|_{L^q} = 1$, then $\int_\Omega |\nabla u_k|^2 dx \downarrow m$.

$\begin{cases} \int_\Omega |\nabla u_k|^2 \downarrow m \\ \int |u_k|^2 < C. \end{cases}$ Then $|\nabla u_k|$ is bdd in L^2

$$\int_\Omega |u|^2 \cdot 1 \leq \left(\int_\Omega |u|^{2\cdot\frac{q}{2}}\right)^{\frac{2}{q}} C(\Omega) = C(\Omega)\|u\|_q^2$$

Face: $W_0^{1,2}$ is reflexive.

So $\exists$ subsequence $\{u_{k_j}\}$ weak convergent

$u_{k_j} \xrightarrow{L^q} u$, u is the solution of (*), and $I[u] = m$

Def $\forall \varphi \in C_c^\infty(\Omega)$

$\int \nabla u_{k_j} \nabla \varphi + u_{k_j} \varphi \to \int \nabla u \nabla \varphi + u\varphi$

u is a weak solution of $-\Delta u = u^{q-1}$.

Need: $\|u\|_{L^q} = 1$.

i.e. $u_{k_j} \xrightarrow{L^q} u$.

Sobolev inequality : $\|u\|_q \leq C\|u\|_{W^{1,2}}$.

So $u_{k_j} \xrightarrow{L^q} u$.

6. Thm Baire Category

Complete metrix space is the 2nd category.

Pf: $\{U_n\}$ open dense in X.

then $\overline{\bigcap_{n=1}^\infty U_n} = X$.

$\forall W$ open , $W \cap (\bigcap_{n=1}^\infty U_n) \neq \phi$ (need to prove)

$W \cap U_1$ open $\to B_{r_1}(x_1)$.

$B_{r_1}(x_1) \cap U_2$ open $\to B_{r_2}(x_2)$

...

$B_{r_n}(x_n) \cap U_{n+1}$ open $\to B_{r_{n+1}}(x_{n+1})$; ... $x_j \to x$

$\Rightarrow W \cap (\bigcap_{n=1}^\infty U_n) \neq \phi$.

7. H is complex vector space

$H \times H \to \mathbb{C}$ $(x,y) \to \langle x,y \rangle$.

① $\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$

② $\langle y,x\rangle = \overline{\langle x,y\rangle}$

③ $\langle x,x\rangle = 0 \Leftrightarrow x=0$

consider $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$

If H is complete with $\|\cdot\|$, H is called Hilbert Space

$x \perp y \Leftrightarrow \langle x,y\rangle = 0$.

For $E \subset H$, $E^\perp = \{x \in H, x \perp y, \forall y \in E\}$.

8. Thm: If $f \in H^*$, $\exists!\, y \in H$, s.t. $f(x) = \langle x, y\rangle$, $\forall x \in H$

Thm (Bessel)

$\{u_\alpha\}_{\alpha\in\Lambda}$ orthonormal in H, $\sum_{\alpha\in\Lambda} |\langle x, u_\alpha\rangle|^2 \le \|x\|^2$.

Thm: The followings are equivalent

① $\|u\|^2 = \sum_{\alpha\in\Lambda} |\langle u, u_\alpha\rangle|^2$

② If $\langle u, u_\alpha\rangle = 0$ $\forall \alpha$, then $u = 0$.

9. e.g. For the solution of $-\Delta_{S^2} u = \lambda u$

They are dense in $L^2(S^2)$.

$$\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$u(\theta, \phi) = Y(\theta) Z(\phi)$

$$\text{so } -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) Z - \frac{1}{\sin\theta} Y \frac{\partial Z}{\partial\phi} = \lambda Y Z$$

Consider P_ℓ^m Legendre.

$$u = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}, \quad \ell \in \mathbb{Z}, m \le \ell.$$

It is the spherical harmonic in S^2.

They are dense in $L^2(S^2)$.

10. e.g. $\begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$ (*) Ω is precompact open set.

Then there exists $u \in W_0^{1,2}(\Omega)$, s.t. u is the weak solution of (*), and the first eigenvalue $\lambda_1 = \inf_{u\in W_0^{1,2}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}$.

Pf: The weak solution of (*) satisfies

$$\int_\Omega \nabla u \nabla\varphi\, dx = \lambda \int u\varphi\, dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

Consider $I[u] = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}$. Note $S^2 = \{u \in L^2(\Omega) \mid \|u\|_{L^2(\Omega)} = 1\}$

$Q[u] \triangleq \int_\Omega |\nabla u|^2 dx$, $\inf_{u\in W_0^{1,2}} I[u] = \inf_{u\in S^2} Q[u]$.

$$m \triangleq \inf_{u\in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} = \inf_{u\in W_0^{1,2}(\Omega)} J[u].$$

Then $\exists$ sequence $\{u_k\}_{k=1}^\infty \subset S^2$, s.t. $Q[u_k] \to m$.

So $\{u_k\}_{k=1}^\infty$ is bounded in $W_0^{1,2}(\Omega)$

And bounded sequence in $W_0^{1,2}$ is compact in L^2.

W.L.O.G $u_k \to u$ in $L^2(\Omega)$.

Then $\|u\|_2 = \lim_{k\to\infty} \|u_k\|_2 = 1$. $\Rightarrow u \in S^2$.

We need to prove $u \in W_0^{1,2}$.

We know that $Q[\frac{u_k+u_l}{2}] + Q[\frac{u_k-u_l}{2}] = \frac{1}{2}(Q[u_k]+Q[u_l])$.

Due to $J[u] \geq m \ \forall u \in W_0^{1,2}$.

Then $\int_\Omega |\nabla u|^2 dx \geq m\int_\Omega u^2 dx \ \forall u \in W_0^{1,2}$.

We know that $\|\frac{u_k+u_l}{2} - u\|_2 \to 0$ as $k,l\to\infty$.

So $\|\frac{u_k+u_l}{2}\|_2 \to 1$ as $k,l \to \infty$.

Thus $Q[\frac{u_k-u_l}{2}] = \frac{1}{2}(Q[u_k]+Q[u_l]) - Q[\frac{u_k+u_l}{2}]$

$\leq \frac{1}{2}(Q[u_k]+Q[u_l]) - m\|\frac{u_k+u_l}{2}\|_2^2 \to \frac{1}{2}(m+m) - m = 0$

$\Rightarrow \|\frac{u_k-u_l}{2}\|_{W^{1,2}}^2 \to 0$.

$\{u_k\}_{k=1}^\infty$ is the Cauchy sequence in $W^{1,2}(\Omega)$.

According to the uniqueness of limitation.

$u_k \to u$ in $W^{1,2}(\Omega)$. $u \in W^{1,2}(\Omega)$, $J[u] = m$.

Then considen $f(t) = J[u+t\varphi]$ $t\in\mathbb{R}$, $\varphi \in C_c^\infty(\Omega)$.

so $\frac{d}{dt} f(t)\Big|_{t=0} = 0 \Rightarrow \int_\Omega \nabla u \nabla\varphi\, dx = m\int_\Omega u\varphi\, dx$. $\forall \varphi \in C_c^\infty(\Omega)$

Thus u is the weak solution of $(*)$. $\lambda_1 = m$.

If $\lambda < \lambda_1$ satisfies $(*)$.

Then $\int_\Omega \nabla u \nabla \varphi\, dx = \lambda \int_\Omega u\varphi dx \ \ \forall \varphi \in C_c^\infty(\Omega)$

Because $u \in W_0^{1,2}(\Omega)$, $\exists \{\varphi_n\}_{n=1}^\infty$, $\varphi_n \to u$ in $W^{1,2}(\Omega)$.

Then $\lim_{n\to\infty} \int_\Omega \nabla u \nabla \varphi_n dx = \int_\Omega |\nabla u|^2 dx$

$= \lambda \lim_{n\to\infty} \int_\Omega u\varphi_n\, dx = \lambda \int_\Omega u^2 dx$. $\Rightarrow J[u] < \lambda_1$.

Contradiction.

§ 6 L^p Space

1. Def: $L^p(X,\mu) = \{ f: X \to \mathbb{C} \mid \int |f|^p d\mu < \infty \}$ is L^p space.
   $\|f\|_{L^p} = (\int |f|^p d\mu)^{\frac{1}{p}}$ is a norm. $(1 \le p < \infty)$
   Hölder inequality $\int |fg| \le (\int |f|^p)^{\frac{1}{p}} (\int |g|^q)^{\frac{1}{q}}$, $\frac{1}{p} + \frac{1}{q} = 1$
   L^p is a Banach space

2. Some facts.
   ① Simple functions are dense in L^p.
   ② $L^p \subset L^q + L^r$ $\quad 0 < q < p < r < \infty$
   Pf: $\forall f \in L^p$, Let $E = \{ |f(x)| > 1 \}$, $f = f\chi_E + f\chi_{E^c}$
   $\int |f\chi_E|^q < \int |f\chi_E|^p \le \|f\|_p^{p}$
   ③ $\|f\|_{L^q} \le \|f\|_{L^p}^{\lambda} + \|f\|_{L^q}^{1-\lambda}$ for some $\lambda < 1$
   Pf: Hölder inequality

   ④ $(L^p)^{**} = L^p$
   Pf: $\forall \phi \in (L^p)^{**}$, $\forall f \in L^q$
   $\phi(f) \triangleq \int fg$ for some $g \in L^p$
   $\forall g \in L^p$ can be viewed as a linear functional acting on L^p $(\frac{1}{p} + \frac{1}{q} = 1)$
   $Tg(f) = \int fg \le \|f\|_q \|g\|_p \Rightarrow \|Tg\| \le \|g\|_p$
   Pick $f = (\frac{g}{\|g\|_p})^{q-1}$, $\Rightarrow \|Tg\| = \|g\|_p$. (... in functional analysis)

3. Sobolev inequality.
   $u \in C_0^1(\mathbb{R}^n)$, $\quad \|u\|_{L^{p^*}(\mathbb{R}^n)} \le C_{n,p} \|\nabla u\|_{L^p(\mathbb{R}^n)}$
   $1 \le p < \infty$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Proof 1: We consider $\int_\Omega \nabla u \nabla u = \int_\Omega -\Delta u \cdot u = \int_\Omega (-\Delta)^{\frac{1}{2}} u (-\Delta)^{\frac{1}{2}} u$

$(-\Delta)^{\frac{1}{2}} u = C_n \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-1}} dy$

$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{u(y)\, v(y)}{|x-y|^{n-1}} dy \right) dx \le C \|u\|_{L^p} \|v\|_{L^{p^{*\prime}}}$, $p^{*\prime} = \frac{p^*}{p^*-1}$

It is Hardy – Littlewood – Sobolev inequality.

In fact, it is so hard.

Proof 2: First prove $p=1$

i.e. $\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq C(n) \int_{\mathbb{R}^n} |\nabla u|$

If we replace $|u|$ by $|u|^r$.

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{nr}{n-1}} dx\right)^{\frac{n-1}{n}} \leq C(n) \int |\nabla |u|^r| dx .$$

$$\leq C(n) r \int |u|^{r-1} |\nabla u| \leq C(n) r \left(\int |\nabla u|^p\right)^{\frac{1}{p}} \left(\int \left(|u|^{r-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p'}{p}}$$

Pick r, s.t. $\frac{rn}{n-1} = \frac{(r-1)p}{p-1} \Rightarrow \frac{rn}{n-1} = p^*$.

So we only to show $p=1$

$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \cdots, y_i, \cdots, x_n) dy_i$

$|u(x)| \leq \int_{-\infty}^{+\infty} |\nabla u(x_1, \cdots, y_i, \cdots, x_n)| dy_i$

$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, y_i, \cdots, x_n)| dy_i\right)^{\frac{1}{n-1}}$

$\Rightarrow \int_{-\infty}^{+\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \left(\int_{-\infty}^{\infty} |\nabla u(y_1, \cdots, x_n)| dy_1\right)^{\frac{1}{n-1}}$.

$\int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, y_i, \cdots, x_n)| dy_i\right)^{\frac{1}{n-1}}$

$\overset{\text{Hölder}^*}{\leq} \left(\int_{-\infty}^{\infty} |\nabla u(y_1, \cdots, x_n)| dy_1\right)^{\frac{1}{n-1}} \cdot \prod_{i=2}^{n} \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\nabla u(x_1, \cdots, y_i, \cdots, x_n)| dy_i dx_1\right)^{\frac{1}{n-1}}$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u(x_1, y_2, \cdots, x_n)| dy_2 dx_1\right)^{\frac{1}{n-1}}$

$\cdot \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\nabla u(y_1, \cdots, x_n)| dy_1\right)^{\frac{1}{n-1}}$

$\cdot \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, y_i, \cdots, x_n)| dy_i dx_1\right)^{\frac{1}{n-1}} dx_2$

$\cdots$ (replicate this motion for n times) $\Rightarrow \int |\nabla u|^{\frac{n}{n-1}} dx$

$\leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u(x_1, \cdots, y_i, \cdots, x_n)| dx_i \cdots dy_i \cdots dx_n\right)^{\frac{1}{n-1}}$

$= \left(\int_{\mathbb{R}^n} |\nabla u|\right)^{\frac{n}{n-1}}$, However, we can't get the relationship between c and n, p.

Proof 3: $\|u\|_{L^{p^*}} \le C_{n,p} \|\nabla u\|_{L^p}$

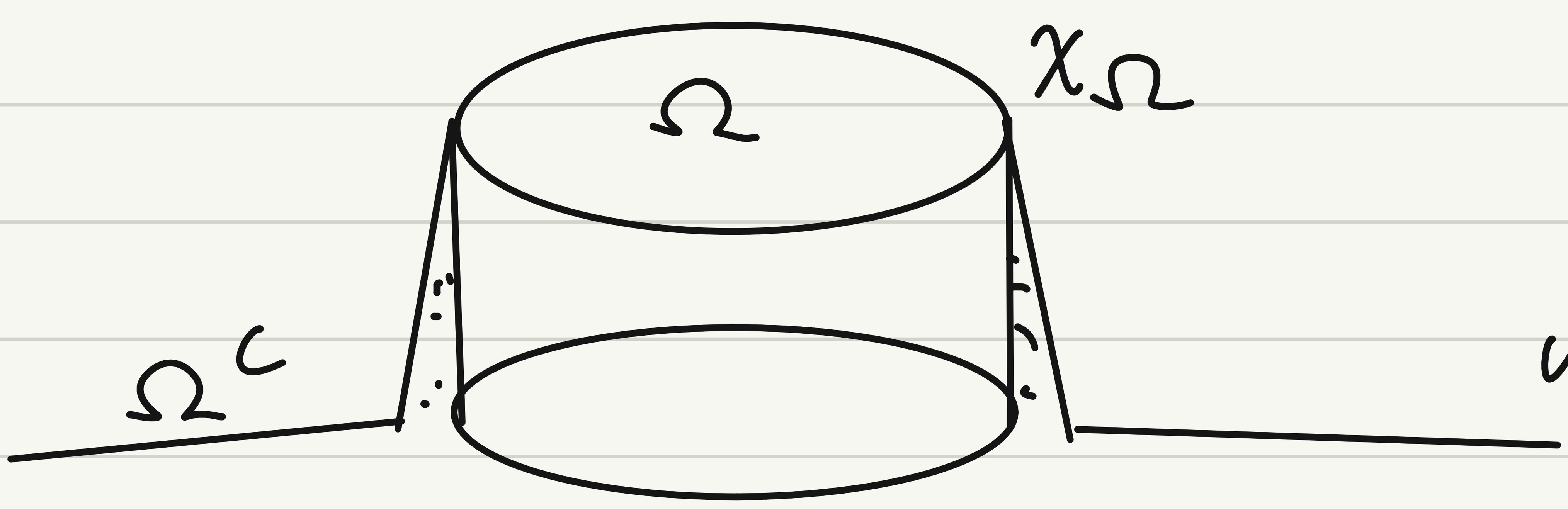
"=" holds iff $u(x) = \dfrac{1}{\left(a+b|x|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}}$

$$C_{n,p} = \frac{1}{\sqrt{\pi}\, n^{\frac{1}{p}}} \left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma(1+\frac{n}{2})\Gamma(n)}{\Gamma(\frac{n}{p})\Gamma(1+n-\frac{n}{p})}\right)^{\frac{1}{n}}$$

when $p=1$, $C_{n,1} = \lim_{p\to 1} C_{n,p}$.

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le C_{n,1} \int_{\mathbb{R}^n} |\nabla u|$$

Let $u = \chi_\Omega \Rightarrow m(\Omega)^{\frac{n-1}{n}} \le C_{n-1} H^{n-1}(\partial\Omega)$



$|\nabla u| \approx \frac{1}{\varepsilon}$

$\int_{\mathbb{R}^n} |\nabla u| = H^{n-1}(\partial\Omega)$

$u^*(x) = \inf\{t \mid m(u>t) \ge B(0,x)\}$

Claim ① $\|u\|_{L^{p^*}} = \|u^*\|_{L^{p^*}}$

② $\|\nabla u\|_{L^p} \ge \|\nabla u^*\|_{L^p}$

So if we prove $\|u^*\|_{L^{p^*}} \le C_{n,p} \|\nabla u^*\|_{L^p}$, QED.

① $\int_X |f|^p d\mu = \int_0^\infty p\lambda^{p-1} \mu(\{|f|>\lambda\})\, d\lambda$

$$\|u\|_{L^{p^*}}^{p^*} = \int_{\mathbb{R}^n} |u|^{p^*} d\mu = \int_0^\infty p^* \lambda^{p^*-1} m(\{|u|>\lambda\})\, d\lambda$$

$$= \int_0^\infty p^* \lambda^{p^*-1} m(\{|u^*|>\lambda\})\, d\lambda = \|u^*\|_{L^{p^*}}^{p^*}.$$

② Polyā – Szegō

$\mu(t) \triangleq m\{u(x)>t\}$

then $\mu(t) = \int_{\{u(x)=t\}} \frac{1}{|\nabla u(x)|} dH^{n-1}(x)$

Prove it. $\int_{\mathbb{R}^n} |\nabla u|^p dx = \int_{\mathbb{R}^n} |\nabla u|^{p-1} |\nabla u| dx = \int_0^\infty \int_{\{u(x)=t\}} |\nabla u|^{p-1} dH^{n-1}(x)\, dt$

$\Rightarrow H^{n-1}\{u(x)=t\} = \int_{\{u(x)=t\}} |\nabla u|^{1-\frac{1}{p}} \frac{1}{|\nabla u|^{p-1}} dH^{n-1} \le$

$$\left(\int_{\{u(x)=t\}} |\nabla u|^{p-1}\right)^{\frac{1}{p}} \left(\int_{\{u(x)=t\}} \frac{1}{|\nabla u|} dH^{n-1}\right)^{\frac{p-1}{p}}$$

$$\int_{\{u(x)=t\}} |\nabla u|^{p-1} \geq \left(H^{n-1}(\{u(x)=t\})\right)^{p} \left(\int_{\{u(x)=t\}} \frac{1}{|\nabla u|}\right)^{p-1}$$

$$= \left(H^{n-1}(u(x)=t)\right)^{p} \mu(t)^{p-1}$$

So $\|\nabla u\|_{L^p} \geq \|\nabla u^*\|_{L^{p^*}}$

Back to original question.

$$I[u] = \frac{\int_0^\infty |u^*(r)|^p r^{n-1} dr}{\left(\int_0^\infty |u^*(r)|^{p^*} r^{n-1} dr\right)^{\frac{p}{p^*}}}$$

$$\left(\int_{\mathbb{R}^n} |u^*(x)|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C_{n,p} \left(\int_{\mathbb{R}} |\nabla u^*|^p dx\right)^{\frac{1}{p}}$$

$$\left(w_n \int_0^\infty |u^*(r)|^{p^*} r^{n-1} dr\right)^{\frac{1}{p^*}} \leq C_{n,p} \left(w_n \int_0^\infty |u^{*\prime}(r)|^p r^{n-1} dr\right)^{\frac{1}{p}}.$$

Assume I attains minimal at u

Then $I[u] < I[u+\varepsilon\varphi_0]$ $\varphi \in C_0^1$

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I[u+\varepsilon\varphi]$$

$$= \frac{\left(\int_0^\infty p|u'+\varepsilon\varphi'|^{p-2}(u'+\varepsilon\varphi')\varphi' r^{n-1} dr\right)\left(\int_0^\infty |u+\varepsilon\varphi|^{p^*} r^{n-1} dr\right)^{\frac{p}{p^*}} -}{M}$$

$$\frac{\left(\int_0^\infty |u'+\varepsilon\varphi'|^p r^{n-1} dr\right)\frac{p}{p^*}\left(\int_0^\infty |u+\varepsilon\varphi|^{p^*} r^{n-1} dr\right)^{\frac{p}{p^*}-1} \int_0^\infty p^*|u+\varepsilon\varphi|^{p^*-2}}{M}$$

$$\frac{(u+\varepsilon\varphi)\cdot\varphi r^n dr}{}$$

$$= \frac{\left(\int_0^\infty |u'|^{p-2} u'\varphi' r^{n-1} dr\right)\left(\int_0^\infty |u|^{p^*} r^{n-1} dr\right)^{p/p^*} - \left(\int_0^\infty |u'|^p r^{n-1} dr\right)\frac{p}{p^*}}{M}$$

$$\frac{\left(\int_0^\infty |u|^{p^*} r^{n-1} dr\right)^{p/p^*-1}\left(\int_0^\infty |u|^{p^*-2} u\varphi r^{n-1} dr\right)}{M}$$

$$\Rightarrow -\left(|u'|^{p-2} u' r^{n-1}\right)' = C|u|^{p^*-2} u r^{n-1}$$

$$\Rightarrow \quad u(r) = \frac{1}{\left(a + b r^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}}$$

Proof 4: Optimal transport

Def: $T: \mathbb{R}^n \to \mathbb{R}^n$ transport

μ to ν if $\forall \Omega$ Borel in $\mathbb{R}^n$, $\nu(\Omega) = \mu(T^{-1}(\Omega))$

$y = Tx$. $\int_\Omega G(y)\,dy = \int_{T^{-1}(\Omega)} F(x)\,dx \Rightarrow G(Tx)\,|J_T(x)| = F(x)$.

Thm Brewer

μ propability measure, $\nu \ll m$, $\exists T = \nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$

when $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex

set T transport μ to ν such convex function is unique.

$$F(x)\, G^{-\frac{1}{n}}(\nabla\varphi(x)) = F^{1-\frac{1}{n}}(x) \det(\nabla^2\varphi)^{\frac{1}{n}} \le \frac{1}{n} F^{1-\frac{1}{n}}(x)\,\Delta\varphi$$

$$\int_{\mathbb{R}^n} G^{-\frac{1}{n}}(\nabla\varphi(x))\, F(x)\,dx \le \frac{1}{n}\int_{\mathbb{R}^n} F^{1-\frac{1}{n}}(x)\,\Delta\varphi(x)\,dx$$

$$\int_{\mathbb{R}^n} G^{-\frac{1}{n}}(y)\, G(y)\,dy \le \frac{1}{n}\int_{\mathbb{R}^n} (\nabla f^{(1-\frac{1}{n})p^*})\,\nabla\varphi\,dx.$$

$$\le -\frac{p(n-1)}{n(n-p)}\int_{\mathbb{R}^n} f^{\frac{n(p-1)}{n-p}}\,\nabla f\,\nabla\varphi\,dx$$

$$\le \frac{p(n-1)}{n(n-p)}\left(\int_{\mathbb{R}^n} |\nabla f|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n} \left(|f|^{\frac{n(p-1)}{n-p}}|\nabla\varphi|\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}$$

$$= \frac{p(n-1)}{n(n-p)}\|\nabla f\|_{L^p}\left(\int_{\mathbb{R}^n} |f|^{p^*}\,|\nabla f(x)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}$$

$$= \frac{p(n-1)}{n(n-p)}\|\nabla f\|_{L^p}\int_{\mathbb{R}^n} y^{\frac{p}{p-1}}\,G(y)\,dy$$

$$\Rightarrow \frac{\int_{\mathbb{R}^n} g^{p^*(1-\frac{1}{n})}(y)\,dy}{\int_{\mathbb{R}^n} |y|^{\frac{p}{p-1}}\, g^{p^*}(y)\,dy} \le \frac{p(n-1)}{n(n-p)}\|\nabla f\|_{L^p}.$$

Pick any g s.t. $g^{p^*}(y)\,dy$ is a propability measure.

$g^{p^*} = e^{-c\frac{|y|^2}{4}} \Rightarrow$

$$\begin{cases} f = g = \dfrac{1}{(a+b|x|^{\frac{p}{p-1}})^{\frac{n-p}{p}}} \\ \varphi(x) = x \end{cases}$$

4. Hardy's inequality $\quad p>1$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p$$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f|^p dx.$$

Pf: Pick $p=2$. Hardy's inequality i.e.

$$\left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} dx \leq \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

Because $\int_{\mathbb{R}^n} |\nabla u|^2 = \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^n} \left|\nabla u + \frac{n-1}{2}\frac{x}{|x|^2}\right|^2 dx$

$$\geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} dx$$

5. Weak L^p norm.

$$\|f\|_{L^{p,\infty}} = \inf\left\{ c>0 : \mu\{|f|>\alpha\} \leq \frac{c^p}{\alpha^p}, \forall \alpha \right\}.$$

Thus, $\forall \alpha$, $\mu\{|f|>\alpha\} \leq \dfrac{\|f\|_{L^{p,\infty}}^p}{\alpha^p}$

6. Young's convolution inequality:

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)g(x-y)h(y)\,dx\,dy\right| \leq \|f\|_p \|g\|_q \|h\|_r.$$

where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$.

Pf: Pick p', q', r'. $\frac{1}{p'}+\frac{1}{p} = \frac{1}{q'}+\frac{1}{q} = \frac{1}{r'}+\frac{1}{r} = 1$.

Then $\frac{1}{p'}+\frac{1}{q'}+\frac{1}{r'} = 3-(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}) = 1 \quad \frac{1}{p'}+\frac{1}{q'} = \frac{1}{r}, \frac{1}{p'}+\frac{1}{r'} = \frac{1}{q}, \frac{1}{q'}+\frac{1}{r'} = \frac{1}{p}$.

Then $\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)g(x-y)h(y)\,dx\,dy\right|$

$$\leq \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x)^p h(y)^r|^{\frac{1}{q'}} |f(x)^p g(x-y)^q|^{\frac{1}{r'}} |g(x-y)^q h(y)^r|^{\frac{1}{p'}} dx\,dy$$

$$\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x)|^p |h(y)|^r dx\right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^n} f(x)^p g(x-y)^q dx\right)^{\frac{1}{r'}} \left(\int_{\mathbb{R}^n} |g(x-y)|^q |h(y)|^r dx\right)^{\frac{1}{p'}} dy$$

$$= \|f\|_p^{\frac{p}{q'}} \|g\|_q^{\frac{q}{p'}} \int_{\mathbb{R}^n} |h(y)|^{\frac{r}{q'}} |h(y)|^{\frac{r}{p'}} \left(\int_{\mathbb{R}^n} |f(x)|^p |g(x-y)|^q dx\right)^{\frac{1}{r'}} dy$$

$$\leq \|f\|_p^{\frac{p}{q'}} \|g\|_q^{\frac{q}{p'}} \left(\int_{\mathbb{R}^n} |h(y)|^r dy\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)|^p |g(x-y)|^q dx\, dy\right)^{\frac{1}{r'}}$$

$$= \|f\|_p \|g\|_q \|h\|_r .$$

7. Young's inequality.

$$\|f * g\|_r \leq \|f\|_p \|g\|_q .$$

where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. $1 \leq p, q, r \leq \infty$.

Pf: We know that $\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{q} - \frac{1}{r}\right) = 1$.

Thus $\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y) g(x-y) dy \right|^r dx$

$$\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)| |g(x-y)| dy\right)^r dx$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)|^{\frac{p}{r}} |g(x-y)|^{\frac{q}{r}} |f(y)|^{1-\frac{p}{r}} |g(x-y)|^{1-\frac{q}{r}} dy\right)^r dx$$

$$\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)|^p |g(x-y)|^q dy\right) \cdot \|f\|_p^{r-p} \|g\|_q^{r-q} dx$$

$$= \|f\|_p^r \|g\|_q^r \Rightarrow \|f * g\|_r \leq \|f\|_p \|g\|_q .$$

§ 6 Fourier Analysis & Distribution

1. Def: $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \xi} dx$, $x, \xi \in \mathbb{R}^n$
is the Fourier transform of f.

2. Schwartz class
$X = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $\partial_i f = \frac{\partial f}{\partial x_i}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ multiindex
Then $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f$, $\alpha! = \prod_{i=1}^n \alpha_i!$, $|\alpha| = \sum_{i=1}^n |\alpha_i|$
$\partial^\alpha (fg) = \sum_{\beta \le |\alpha|} \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_n}{\beta_n} \partial^{\alpha-\beta} f \, \partial^\beta g$.

Def: $f \in C^\infty(\mathbb{R}^n)$ is Schwartz function ($S(\mathbb{R}^n)$) if
$\forall \alpha, \beta$, $\exists C_{\alpha\beta}$, s.t. $\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f| < C_{\alpha\beta} < \infty$.

e.g. ① $e^{-|x|^2}$ ② $e^{-|x|}$ not Schwartz, since nonsmooth at $x=0$.
③ $\frac{1}{(1+|x|)^\alpha}$ not Schwartz .

Remark: $f \in S(\mathbb{R}^n)$ iff $|\partial^\alpha f| \le \frac{C_{\alpha,N}}{(1+|x|)^N}$, $\forall N \in \mathbb{N}^+$.

3. $P_{\alpha\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f|$
$P_{\alpha\beta}(f-g)$ gives a seminorm on $S(\mathbb{R}^n)$
We define $d(f,g) = \sum_{\alpha,\beta} 2^{-(|\alpha|+|\beta|)} \frac{P_{\alpha\beta}(f-g)}{1+P_{\alpha\beta}(f-g)}$

Prop: $f_n \xrightarrow{S} f \Rightarrow f_n \xrightarrow{L^p} f$.
$d(f,g)$ is a metrix on $S(\mathbb{R}^n)$.
Pf: It suffices to prove $\|\partial^\beta f\|_{L^p}^p \le \Big(C_{p,n} \sum_{|\alpha| \le [\frac{n+1}{p}]+1} P_{\alpha\beta}(f)\Big)^p$

$$\|\partial^\beta f\|_{L^p}^p = \int_{|x|<1} |\partial^\beta f|^p + \int_{|x|>1} |\partial^\beta f|^p |x|^{n+100} \frac{1}{|x|^{n+100}} dx$$

$$\le C_{p,n} \|\partial^\beta f\|_{L^\infty}^p + \sup_{x \in \mathbb{R}^n} |x|^{n+100} |\partial^\beta f|^p \int_{|x|>1} \frac{1}{|x|^{n+100}} dx$$

$$\le C\big(P_{\alpha\beta}(f) + P_{\frac{n+100}{p}\beta}(f)\big) < \infty$$

4. $f, g \in S(\mathbb{R}^n)$, Then $f \cdot g$, $f * g \in S(\mathbb{R}^n)$
$\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g)$

Pf: $\partial^\alpha(f*g) = \partial^\alpha\left(\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right) \overset{DCT}{=} \int_{\mathbb{R}^n} \partial^\alpha f(x-y)\; g(y)\,dy$

$\int_{\mathbb{R}^n} f(x-y)g(y)\,dy \leq C\int_{\mathbb{R}^n} \frac{1}{(1+|x-y|)^N}\;\frac{1}{(1+|y|)^{N+n+1}}\,dy$

$\leq C\,\frac{1}{(1+|x|)^N}\int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{n+100}}\,dy < \infty. \quad \forall N.$

5. Def: $f\in S(\mathbb{R}^n)$, $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\xi}\,dx$

$\Rightarrow |\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f|\,dx < \infty.$

e.g. $f(x) = e^{-\pi|x|^2}$, $x\in\mathbb{R}$.

$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx = e^{-\pi\xi^2}\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\,dx = e^{-\pi\xi^2}$

Prop. ① $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

② $\widehat{f*g} = \hat{f}*\hat{g}$

③ $\widehat{bf} = b\hat{f}$

④ $\widehat{f(-x)} = \hat{f}(-\xi)$

⑤ $\hat{\bar{f}} = \overline{\hat{f}(-\xi)}$

⑥ $\widehat{f(x-y)}(\xi) = e^{-2\pi i y\xi}\hat{f}(\xi)$

⑦ $\widehat{e^{2\pi i xy}f(x)}(\xi) = \hat{f}(\xi-y)$

⑧ $\widehat{f(tx)}(\xi) = \frac{1}{t^n}\hat{f}\left(\frac{\xi}{t}\right)$

$\widehat{f(tx)}(\xi) = \int_{\mathbb{R}^n} f(tx)e^{-2\pi i x\xi}\,dx = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \frac{x}{t}\xi}\,d\frac{x}{t}$

$= \frac{1}{t^n}\hat{f}\left(\frac{\xi}{t}\right)$

⑨ $\widehat{\partial^\alpha f(x)}(\xi) = (2\pi i\xi)^\alpha\,\hat{f}(\xi)$

⑩ $\partial^\alpha(\hat{f}(\xi)) = \widehat{(-2\pi i x)^\alpha f(x)}(\xi)$

⑪ $\hat{f}\in S(\mathbb{R}^n)$

⑫ $\widehat{f*g} = \hat{f}*\hat{g}$

⑬ $\widehat{f\circ A}(\xi) = \hat{f}(A\xi)$ $\quad A\in O(n)$

$\int_{\mathbb{R}^n} f(Ax)e^{-2\pi i x\xi}\,dx = \int_{\mathbb{R}^n} f(x)e^{-2\pi i Ax\xi}\,dx = \hat{f}(A\xi)$

6. Inverse Fourier transform

$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \xi} d\xi. = \hat{f}(-x).$

Prop: ① $\int_{\mathbb{R}^n} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^n} \hat{f}(x) g(x)\,dx$ (Fubini)

② $(\hat{f})^{\vee} = f = (\check{f})^{\wedge}$

Pf: Pick $g_\varepsilon(\xi) = e^{2\pi i \xi t} e^{-\pi|\varepsilon\xi|^2}$

$\hat{g}_\varepsilon(\xi) = \frac{1}{\varepsilon^n} e^{-\pi\left(\frac{x-t}{\varepsilon}\right)^2}$ (good kernal).

We need to show $\int_{\mathbb{R}^n} f(x)\hat{g}_\varepsilon(x) \xrightarrow{\varepsilon\to 0} f(t)$

$\|\hat{g}_\varepsilon(\xi)\|_{L^1} = 1$

$\left| \int_{\mathbb{R}^n} f(x+t) \frac{1}{\varepsilon^n} e^{-\pi\left(\frac{x-t}{\varepsilon}\right)^2} dx - \int_{\mathbb{R}^n} f(t) \frac{1}{\varepsilon^n} e^{-\pi\left(\frac{x-t}{\varepsilon}\right)^2} dx \right|$

$= \int_{\mathbb{R}^n} |f(x+t) - f(t)| \frac{1}{\varepsilon^n} e^{-\pi\left(\frac{x-t}{\varepsilon}\right)^2} dx$

$= \int_{|x|\le\delta} |f(x+t) - f(t)| \frac{1}{\varepsilon^n} e^{-\pi\left(\frac{x-t}{\varepsilon}\right)^2} dx$

$+ \int_{|x|>\delta} |f(x+t) - f(t)| \frac{1}{\varepsilon^n} e^{-\pi\left(\frac{x-t}{\varepsilon}\right)^2} dx. \to 0.$

③ $\int f\bar{g}\,d\mu = \int \hat{f}\,\bar{\hat{g}}\,d\mu$. (i.e. $\langle f,g\rangle = \langle \hat{f},\hat{g}\rangle$)

7. Extension of Fourier transform

$\hat{f} = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\xi} dx$ is well defined for $f\in L^1$.

So we can extend from $S(\mathbb{R}^n)$ to L^1.

$S(\mathbb{R}^n) \overset{\text{dense}}{\subset} L^2_1$.

$\forall f\in L^2_1$, $\exists f_n \in S(\mathbb{R}^n)$, s.t. $f_n \xrightarrow{L^2} f$.

$\hat{f}(\xi) = \lim \hat{f}_n(\xi)$. $\|f\|_{L^2} = \lim_{n\to\infty} \|f_n(\xi)\| = \lim_{n\to\infty} \|\hat{f}_n\|_{L^2} = \|\hat{f}\|_{L^2}$

Thus we can extend from $S(\mathbb{R}^n)$ to L^2.

For $p\in(1,2)$, $f\in L^p$, $\exists f_1\in L^1$, $f_2\in L^2$, $f = f_1 + f_2$.

We define $\hat{f} = \hat{f}_1 + \hat{f}_2$.

We can pick $f_1(x) = \begin{cases} 0 & |f(x)|\le 1 \\ f(x) & |f(x)|>1 \end{cases}$ $f_2(x) = \begin{cases} f(x) & |f(x)|\le 1 \\ 0 & |f(x)|>1 \end{cases}$

8. Interpolation Thm

Thm Marcinkiewicz

Def (X,μ), (Y,ν), T is a linear operator from a linear space of $\mathbb{C}$-value meas functions on X to meas function on Y.

① $T(f+g) = Tf + Tg$

② $T(\lambda f) = \lambda Tf$.

If ① $|T(f+g)| \leq |Tf| + |Tg|$

② $|T(\lambda f)| = |\lambda| \, |Tf|$

We say T is sublinear.

$0 < p_0 < p_1 \leq \infty$, T is sublinear on $L^{p_0} + L^{p_1} = \{f_0 + f_1, f_0 \in L^{p_0}, f_1 \in L^{p_1}\}$

Assume $\exists A_0, A_1$, s.t.

$$\left.\begin{aligned} \|T(f)\|_{L^{p_0,\infty}} &\leq A_0 \|f\|_{L^{p_0}} \\ \|T(f)\|_{L^{p_1,\infty}} &\leq A_1 \|f\|_{L^{p_1}} \end{aligned}\right\} \Rightarrow \|Tf\|_{L^p} \leq A \|f\|_{L^p}, \quad p_0 < p < p_1$$

Pf: $f = f_0^\alpha + f_1^\alpha$

$$f_0^\alpha = \begin{cases} f(x) & |f| > \alpha\delta \\ 0 & |f| \leq \alpha\delta \end{cases}, \quad f_1^\alpha = f - f_0^\alpha.$$

$$\int |f_0^\alpha(x)|^{p_0} d\mu = \int_{\{|f|>\alpha\delta\}} |f|^{p_0} d\mu = \int_{\{|f|>\alpha\delta\}} |f|^p |f|^{p_0-p} d\mu$$

$$\leq (\alpha\delta)^{p_0-p} \|f\|_{L^p}^p$$

T is sublinear, $|Tf| \leq |Tf_0^\alpha| + |Tf_1^\alpha|$

$$\|Tf\|_{L^p}^p = \int |Tf|^p d\mu = \int_0^\infty p\alpha^{p-1} \mu(\{|Tf| > \alpha\})\, d\alpha$$

$$\leq \int_0^\infty p\alpha^{p-1} \left(\mu(\{|Tf_0^\alpha| > \tfrac{\alpha}{2}\}) + \mu(\{|Tf_1^\alpha| > \tfrac{\alpha}{2}\}) \right) d\alpha.$$

$$\mu(|Tf_0^\alpha| > \tfrac{\alpha}{2}) \leq \frac{A_0^{p_0}}{(\alpha/2)^{p_0}} \|f_0^\alpha\|_{L^{p_0}}^{p_0}$$

$$\mu(|Tf_1^\alpha| > \tfrac{\alpha}{2}) \leq \frac{A_1^{p_1}}{(\alpha/2)^{p_0}} \|f_1^\alpha\|_{L^{p_1}}^{p_1}$$

则 $\|Tf\|_{L^p}^p \leq p(2A_0)^{p_0} \int_0^\infty \alpha^{p-1-p_0} \|f_0^\alpha\|_{L^{p_0}}^{p_0} d\alpha$

$+ p(2A_1)^{p_1} \int_0^\infty \alpha^{p-1-p_1} \|f_1^\alpha\|_{L^{p_1}}^{p_1} d\alpha.$

$$\le p(2A_0)^{p_0}\int_0^\infty \alpha^{p-1-p_0}\left(\int_{\{|f|>\alpha\delta\}}|f|^p d\mu\right)d\alpha$$
$$+p(2A_1)^{p_1}\int_0^\infty \alpha^{p-1-p_1}\left(\int_{\{|f|<\alpha\delta\}}|f|^p d\mu\right)d\alpha.$$
$$\overset{\text{Fubini}}{=} p(2A_0)^{p_0}\int_X |f|^{p_0}\left(\int_0^{\frac{|f|}{\delta}}\alpha^{p-1-p_0}d\alpha\right)d\mu$$
$$+p(2A_1)^{p_1}\int_X |f|^{p_1}\left(\int_{\frac{|f|}{\delta}}^\infty \alpha^{p-1-p_1}d\alpha\right)d\mu$$
$$=\frac{p(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}}\int_X |f|^{p_0}|f|^{p-p_0}d\mu$$
$$+\frac{p(2A_1)^{p_1}}{p_1-p}\frac{1}{\delta^{p-p_1}}\int_X |f|^{p_1}|f|^{p-p_1}d\mu$$
$$=A\|f\|_p^p$$

9. Riesz – Thorin Thm

(X,μ) (Y,ν) σ-finite.

T linear: Simple functions $\to$ meas functions

$1\le p_0, p_1, q_0, q_1\le\infty$.

$$\left.\begin{array}{l}\|Tf\|_{L^{q_0}}\le M_0\|f\|_{L^{p_0}}\\ \|Tf\|_{L^{q_1}}\le M_1\|f\|_{L^{p_1}}\end{array}\right\}\Rightarrow \|Tf\|_{L^q}\le M_0^{1-\theta}M_1^{\theta}\|f\|_{L^p}.$$

where $0\le\theta\le1$, $\frac{1}{p}=(1-\theta)\frac{1}{p_0}+\theta\frac{1}{p_1}$, $\frac{1}{q}=(1-\theta)\frac{1}{q_0}+\theta\frac{1}{q_1}$.

$T: L^p\to L^q$ is bdd.

Pf: Let $f=\sum_{k=1}^m a_k e^{i\alpha_k}\chi_{A_k}$, $a_k>0$. A_k disjoint.

$$\|Tf\|_{L^q}=\sup_{g\in L^{q'},\|g\|_{L^{q'}}=1,\, g\text{ simple}}\int (Tf)g\,d\nu$$
$$=\sum_{k=1}^m\sum_{j=1}^n a_k b_j e^{i\alpha_k}e^{i\beta_j}\int_Y (T\chi_{A_k})\chi_{B_j}\quad \left(g=\sum_{j=1}^n b_j e^{i\beta_j}\chi_{B_j}\right)$$

Let $P(z)=p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)$, $z=\theta+it$.

$Q(z)=q'\left(\frac{1-z}{q_0'}+\frac{z}{q_1'}\right)$,

$f_z = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k}, \quad g_z = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \chi_{B_j}.$

$F(z) = \int (Tf_z) g_z = \sum a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_j} \int (T\chi_{A_k}) \chi_{B_j}$

$|F(it)| \le \|Tf_{it}\|_{L^{q_0}} \|g_{it}\|_{L^{q_0'}}$

$\le M_0 \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q_0'}} = M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'}$

Similarly, $|F(1+it)| \le M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'}$

Lemma: Hadamard 3 lines Lemma.

let F be analytic on $\{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$. bdd.

$|F(z)| \le B_0$, when $\operatorname{Re} z = 0$, $|F(z)| \le B_1$, when $\operatorname{Re} z = 1$.

Then $|F(z)| \le B_0^{1-\theta} B_1^{\theta}$ when $\operatorname{Re} z = \theta$.

$G(z) \triangleq \frac{F(z)}{B_0^{1-z} B_1^{z}}$, $G_n(z) \triangleq G(z) e^{\frac{z^2-1}{n}}$

$|G_n(x+iy)| = |G(x+iy) e^{\frac{(x+iy)^2-1}{n}}| \le C e^{-\frac{y^2}{n}}$

So $y \to \pm\infty$, $G_n \to 0$, when y large enough, $|G_n| \le 1$.

For $\{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1, -|C_1| \le \operatorname{Im} z \le |C_1|\}$.

From maximine principle, $|G_n(x+iy)| \le 1. \Rightarrow |G_n(z)| \le 1$,

$\forall z \in$ Strip. let $n \to \infty$, $|G(z)| \le 1$.

10 Hausdorff - Young inequality.

$\frac{1}{p} + \frac{1}{p'} = 1$, $p \in [1,2]$

Then $\|\hat{f}(\xi)\|_{p'} \le \|f(x)\|_p$. (Get it by Riesz – Thorin Thm)

11. $C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$.

$f_k \xrightarrow{C^\infty} f \quad \lim_{k\to\infty} \sup_{|x| \le N} |\partial^\alpha (f_k - f)| = 0 \quad \forall \alpha, N.$

$f_k \xrightarrow{S} f \quad \lim_{k\to\infty} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta (f_k - f)| = 0 \quad \forall \alpha, \beta.$

$f_k \xrightarrow{C_0^\infty} f \quad \lim_{k\to\infty} \|\partial^\alpha (f_k - f)\|_\infty = 0 \quad \forall \alpha.$

$(C_0^\infty(\mathbb{R}^n))' = \{$continuous linear functions$\} = D'(\mathbb{R}^n)$ distribution

$(S(\mathbb{R}^n))' = S'(\mathbb{R}^n)$ tempered distribution.

$(C^\infty(\mathbb{R}^n))' = \mathcal{E}'(\mathbb{R}^n)$ distribution with cpt surpport. (untrivial)

$\mathcal{E}'(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$,

$T_k \xrightarrow{D'} T \qquad T_k(f) \to T(f), \forall f \in C_0^\infty(\mathbb{R}^n).$

$T_k \xrightarrow{S'} T \qquad T_k(f) \to T(f), \forall f \in S(\mathbb{R}^n)$

$T_k \xrightarrow{\mathcal{E}'} T \qquad T_k(f) \to T(f) \quad \forall f \in C^\infty(\mathbb{R}^n)$

Prop: ① $u \in D'(\mathbb{R}^n)$ iff

$\forall K$ compact, $\forall m, \exists C(m)$, s.t. $|\langle u, f\rangle| \le C \sum_{|\alpha| \le m} \|\partial^\alpha f\|_{L^\infty(K)}$.

$\forall f \in C^\infty$ surpported in K.

Pf: $\forall f_k \xrightarrow{C_0^\infty} 0$, we have $\langle u, f_k\rangle \to 0$.

$\forall \varepsilon > 0$, $\exists N$ s.t. $k > N$, $|\langle u, f_k\rangle| < \varepsilon$.

$\forall m$. s.t. $\sum_{|\alpha| \le m} \|\partial^\alpha f_k\|_{L^\infty} < \delta$

$|\langle u, f_k\rangle| \le \frac{\varepsilon}{\delta} \sum_{|\alpha| \le m} \|\partial^\alpha f_k\|_{L^\infty}$

② $u \in S'(\mathbb{R}^n)$ iff $\exists m, k$, s.t. $\forall f \in S(\mathbb{R}^n)$

$|\langle u, f\rangle| \le C \sum_{\substack{|\alpha| \le m \\ |\beta| \le k}} \sup |x^\alpha \partial^\beta f|$.

③ $u \in \mathcal{E}'(\mathbb{R}^n)$, iff $\exists m, N$

$|\langle u, f\rangle| \le C \sum_{|\alpha| \le m} \sup_{|x| \le N} |\partial^\alpha f(x)|$.

e.g. ① Dirac meas $\delta_0 \in \mathcal{E}' \subset S' \subset D'$

$\langle \delta_0, f\rangle = f(0)$.

② $1 \in S'$

$|\langle 1, f\rangle| = |\int_{\mathbb{R}^n} f\,dx| \le \|f\|_{L^\infty} m(B_1) + \int_{|x|>1} |f|\,|x|^{n+100} \frac{1}{|x|^{n+100}}\,dx$

$\le \|f\|_{L^\infty} m(B_1) + \sup_{\mathbb{R}^n} |x^{n+100} f| \int_{|x|>1} \frac{1}{|x|^{n+100}}\,dx < \infty$.

③ L^p $1 \le p < \infty \subset S'(\mathbb{R}^n)$.

$\langle u, f\rangle = \int_{\mathbb{R}^n} u f\,dx \le \|u\|_{L^p} \|f\|_{L^{p'}} < \infty$.

④ on $\mathbb{R}$, $S'(\mathbb{R})$

$$\langle u, f\rangle = \lim_{\varepsilon\to 0}\int_{\varepsilon\le|x|\le 1} f(x)\frac{dx}{x} = \lim_{\varepsilon\to 0}\int_{\varepsilon\le|x|\le 1}(f(x)-f(0))\frac{dx}{x} + \lim_{\varepsilon\to 0}\int_{\varepsilon\le|x|\le 1} f(0)\frac{dx}{x}$$

$\int_{\varepsilon\le|x|\le 1} f(0)\frac{dx}{x} = 0$. $\langle u, f\rangle \le \lim_{\varepsilon\to 0}\int_{\varepsilon\le|x|\le 1}\|f'\|_{L^\infty_{[-1,1]}}dx \le 2\|f'\|_{L^\infty_{[-1,1]}}$

Denote $u(x) = \text{p.v.}\frac{1}{x}$ Hilbert transform.

12. Def: $\langle\hat{u}, f\rangle \triangleq \langle u, \hat{f}\rangle$, $u\in S'(\mathbb{R}^n)$, $f\in S(\mathbb{R}^n)$.

Let $u\in S(\mathbb{R}^n)$,

distribution derivative: $\langle\partial^\alpha u, f\rangle \triangleq \langle u, \partial^\alpha f\rangle(-1)^{|\alpha|}$.

We also call it weak derivative

e.g. $\hat{\delta_0}$, $\widehat{\partial^\alpha\delta_0}$

$\langle\hat{\delta_0}, f\rangle = \langle\delta_0, \hat{f}\rangle = \hat{f}(0). = \int_{\mathbb{R}^n} f(x)dx$

$\langle\widehat{\partial^\alpha\delta_0}, f\rangle = \langle\partial^\alpha\delta_0, \hat{f}\rangle = (-1)^{|\alpha|}\langle\delta_0, \partial^\alpha\hat{f}\rangle$

$= (-1)^{|\alpha|}\langle\delta_0, \partial^\alpha(\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\xi}dx)\rangle$

$= (-1)^{|\alpha|}\langle\delta_0, \int_{\mathbb{R}^n}(-2\pi i x)^\alpha f(x)e^{-2\pi i x\xi}dx\rangle$

$= (-1)^{|\alpha|}\int_{\mathbb{R}^n}(-2\pi i x)^\alpha f(x)dx$

$\widehat{\partial^\alpha\delta_0} = (2\pi i x)^\alpha$.

13. u is a tempered distribution

Def: ① $(\tau^t f)(x) = f(x+t)$.

then $\langle\tau^t u, f\rangle \triangleq \langle u, \tau^{-t}f\rangle$

② $(\delta^a f)(x) = f(ax)$.

then $\langle\delta^a u, f\rangle \triangleq \langle u, \frac{1}{a^n}\delta^{\frac{1}{a}}f\rangle$

③ $\tilde{f}(x) = f(-x)$

then $\langle\tilde{u}, f\rangle \triangleq \langle u, \tilde{f}\rangle$

④ $f^A(x) = f(Ax)$, A linear transform

then $\langle u^A, f\rangle \triangleq \frac{1}{|\det A|}\langle u, f^A\rangle$

⑤ $u \in S'(\mathbb{R}^n)$, $h \in S(\mathbb{R}^n)$

$\langle h*u, f\rangle = \langle u, \tilde{h}*f\rangle$.

e.g. $u = \delta x_0$

$\langle h*\delta x_0, f\rangle = \langle \delta x_0, \tilde{h}*f\rangle = \langle \delta x_0, \int_{\mathbb{R}^n} h(x-y) f(x)dx\rangle$

$= \int_{\mathbb{R}^n} h(x-x_0) f(x) dx$

14. $u \in S'(\mathbb{R}^n)$, $h \in C^\infty$ with at most polynormied growth.

i.e. $|(\partial^\alpha h)(x)| \le C_\alpha (1+|x|)^{k_\alpha}$. (slowly increasing).

Def: $\langle hu, f\rangle \triangleq \langle u, hf\rangle$

The surpport of $u \in S'(\mathbb{R}^n)$ is the intersection of all closed set K s.t. $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp}\,\varphi \subseteq K^c$.

$\langle u, \varphi\rangle = 0$. e.g. $\text{supp}\,\delta x_0 = \{x_0\}$.

15. $u \in S'(\mathbb{R}^n)$, $\varphi \in S(\mathbb{R}^n)$, then $(\varphi * u) \in C^\infty(\mathbb{R}^n)$.

Moreover, $|\partial^\alpha(\varphi*u)| \le C_\alpha(1+|x|)^{k_\alpha}$. if $u \in \mathcal{E}'(\mathbb{R}^n)$, $(\varphi*u) \in S(\mathbb{R}^n)$

In fact, if $u \in \mathcal{E}'(\mathbb{R}^n)$, $\hat{u} \in C^\infty(\mathbb{R}^n)$ increasing slowly

Pf: $\langle \varphi*u, f\rangle = \langle u, \tilde{\varphi}*f\rangle = \langle u, \int_{\mathbb{R}^n} \varphi(x-y) f(x) dx\rangle$

$= \int_{\mathbb{R}^n} \langle u, \varphi(x-y)\rangle f(x) dx$

So $(\varphi*u)(x) = \langle u, \varphi(x-y)\rangle$.

$$\lim_{h\to 0} \frac{(\varphi*u)(x+h\vec{e}) - (\varphi*u)(x)}{h} = \langle u, \lim_{h\to\infty} \frac{\varphi(x+h\bar{e}-y) - \varphi(x-y)}{h}\rangle$$

$\to \langle u, \partial_{x_j} \varphi(x-y)\rangle = \partial_{x_j}\varphi * u$.

$|\partial^\alpha(\varphi*u)(x)| = |(\partial^\alpha\varphi*u)(x)| = |\langle u, \partial^\alpha \varphi(x-y)\rangle|$

$\le C \sum_{|\beta|\le k} \sup_{y\in\mathbb{R}^n} (1+|y|)^M |\partial^{\alpha+\beta}\varphi(x-y)| \le C(1+|x|)^M \sum_{|\beta|\le k} (1+|x-y|)^M |\partial^{\alpha+\beta}\varphi(x-y)|$

16. Prop: $u, v \in S'(\mathbb{R}^n)$.

① $\widehat{u+v} = \hat{u} + \hat{v}$

② $\widehat{bu} = b\hat{u}$.

③ If $u_k \xrightarrow{S'} u$, then $\hat{u}_k \xrightarrow{S'} \hat{u}$

④ $(\hat{u})^\vee = (\check{u})^\wedge$

⑤ $(\delta^a u)^\wedge = \frac{1}{a^n} \delta^a \hat{u}$

⑥ $(\tau^y u)^\wedge(\xi) = e^{-2\pi i y \xi} \hat{u}(\xi)$.

⑦ $(e^{2\pi i x y} u)^\wedge(\xi) = \tau^y \hat{u}(\xi)$.

⑧ $(\partial^\alpha u)^\wedge = (2\pi i \xi)^\alpha \hat{u}$

⑨ $\partial^\alpha \hat{u}(\xi) = ((-2\pi i x)^\alpha u)^\wedge$

⑩ $(\hat{u})^\vee = u$

⑪ $\widehat{f * u} = \hat{f} \hat{u}$

⑫ $\widehat{fu} = \hat{f} * \hat{u}$.

⑬ $\partial_j^{\alpha+\beta}(fu) = \sum_{k=0}^{m} \binom{m}{k} \partial_j^k f \partial_j^{m-k} u$.

⑭ $u_k \xrightarrow{L^p} u$, then $u_k \xrightarrow{S'} u$.

17. Thm : Suppose $u \in S'(\mathbb{R}^n)$ is supported in $\{0\}$, then $\exists k$, s.t. $u = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \delta_0$.

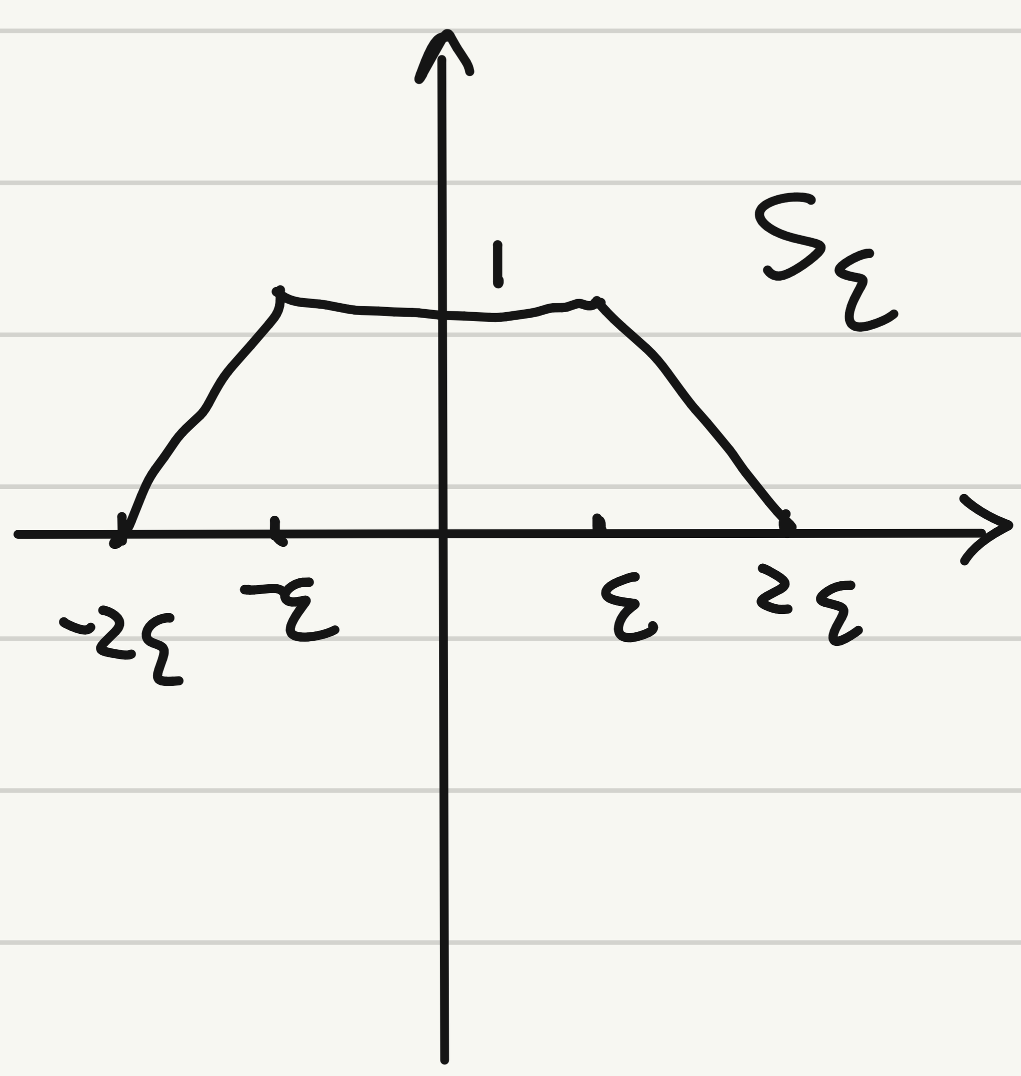
Pf: $\eta = \begin{cases} 1 & |x| \ge 2 \\ 0 & |x| \le 1 \\ \eta & 1 < |x| < 2 \end{cases}$ $\quad \eta \in C^\infty(\mathbb{R}^n)$.

$$\langle u, f \rangle = \langle u, \eta f \rangle + \langle u, (1-\eta) f \rangle$$

$$= \langle u, (1-\eta) \sum_{|\alpha| \le k} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha + O(x^{k+1}) \rangle + \langle u, (1-\eta) O(x^{k+1}) \rangle$$

$$= \sum_{|\alpha| \le k} \frac{\partial^\alpha f(0)}{\alpha!} \langle u, (1-\eta) x^\alpha \rangle + \langle u, h(x) \rangle.$$

$$= \sum_{|\alpha| \le k} (-1)^{|\alpha|} \frac{\langle u, (1-\eta) x^\alpha \rangle}{\alpha!} \langle \partial^\alpha \delta_0, f \rangle + \langle u, h(x) \rangle.$$



$$|\langle u, h \rangle| = |\langle u, S_\varepsilon h \rangle| + |\langle u, (1-S_\varepsilon) h \rangle|$$

$$\le C_N \sum_{|\alpha| \le k} |\partial^\alpha S_\varepsilon h| \le \varepsilon^k. \to 0.$$

$\Rightarrow \langle u, h \rangle = 0$.

So $a_\alpha = (-1)^\alpha \frac{\langle u, (1-\eta) x^\alpha \rangle}{\alpha!}$

$$u = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \delta_0$$

18. e.g. $u \in S'(\mathbb{R}^n)$ s.t. $\Delta u = 0$.

then u is polynominal.

$<\widehat{\Delta u}, f> = <\Delta u, \hat{f}> = <u, \Delta \hat{f}> = <u, \widehat{(2\pi i x)^2 f}>$

$<\Delta u, \hat{f}> = 0$, $<\widehat{\Delta u}, f> = 0$, $<4\pi^2|\xi|^2 \hat{u}, f> = 0$.

$\Rightarrow <\hat{u}, 4\pi^2|\xi|^2 f(\xi)> = 0$

So $\hat{u}$ can only support in $\{0\}$.

$\hat{u} = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \delta_0$, $u = \sum_{|\alpha| \le k} a_\alpha (\partial^\alpha \delta_0)^\vee = \sum_{|\alpha| \le k} a_\alpha (2\pi i x)^\alpha$.

19. Thm: $\Delta \frac{C_n}{|x|^{n-2}} = \delta_0$.

So $u(x) = <\Delta \frac{C_n}{|x-y|^{n-2}}, u(y)> = (\Delta^{-1})(\Delta u)$.

$\Delta^{-1} u \triangleq < \frac{C_n}{|x-y|^{n-2}}, u(y)>$

$(-\Delta)^{-\frac{\alpha}{2}} u \triangleq < \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}}, u(y)>$

Pf:

$[(-\Delta)^1 u(x)]^\wedge(\xi) = -\sum_{i=1}^n \widehat{(\frac{\partial^2}{\partial x_i^2}) u}(\xi) = -\sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i^2} u(x) e^{-2\pi i x\xi} dx$

$= \sum_{i=1}^n -(2\pi i \xi_i)^2 \int_{\mathbb{R}^n} u(x) e^{-2\pi i x \xi} dx = 4\pi^2 |\xi|^2 \hat{u}(\xi)$.

$[(-\Delta)^{\frac{\alpha}{2}} u]^\wedge(\xi) \triangleq (4\pi^2|\xi|^2)^{\frac{\alpha}{2}} \hat{u}(\xi)$.

$(-\Delta)^{\frac{\alpha}{2}} u = [(4\pi^2|\xi|^2)^{\frac{\alpha}{2}} \hat{u}(\xi)]^\vee(x) = [(4\pi^2|\xi|^2)^{\frac{\alpha}{2}}]^\vee * u(x)$.

Now we need to compute $(|\xi|^\alpha)^\vee = (|\xi|^\alpha)^\wedge$.

$\int_{\mathbb{R}^n} \widehat{|\xi|^\alpha}(x) \varphi(x) dx = \int_{\mathbb{R}^n} |\xi|^\alpha \hat{\varphi}(\xi) d\xi$

$= \int_{\mathbb{R}^n} |\xi|^\alpha \left(\int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i x \xi} dx \right) d\xi$

$= \int_{\mathbb{R}^n} \varphi(x) \left(\int_{\mathbb{R}^n} |\xi|^\alpha e^{-2\pi i x\xi} d\xi \right) dx$

$= \int_{\mathbb{R}^n} \varphi(x) \left(\int_{\mathbb{R}^n} |\xi|^\alpha e^{-2\pi i (|x|\xi)\cdot \frac{x}{|x|}} d\xi \right) dx$

Note: $C'_{n,\alpha}$ in $(*)$ is different from $C_{n,\alpha} = \dfrac{\Gamma(\frac{n+\alpha}{2})\,\pi^{-\frac{\alpha}{2}}}{\Gamma(-\frac{\alpha}{2})\,\pi^{\frac{n+\alpha}{2}}}$

$$= \int_{\mathbb{R}^n} \varphi(x) \left(\int_{\mathbb{R}^n} |\xi|^\alpha e^{-2\pi i\, T(x/|\xi|)\, \vec{e}}\, d\xi \right) dx$$

$$= \int_{\mathbb{R}^n} \varphi(x) \left(\int_{\mathbb{R}^n} \frac{|\xi|^\alpha}{|x|^\alpha} e^{-2\pi i\, \xi\, \vec{e}} \frac{1}{|x|^n} d\xi \right) dx$$

$$= \int_{\mathbb{R}^n} \varphi(x) \left(\left(\int_{\mathbb{R}^n} |\xi|^\alpha e^{-2\pi i \xi \vec{e}} d\xi \right) \frac{1}{|x|^{n+\alpha}} \right) dx$$

Consider $\displaystyle\int_{\mathbb{R}^n} \frac{C_{n,\alpha}}{|x|^{n+\alpha}} e^{-\pi|x|^2} dx = \int_{\mathbb{R}^n} |x|^\alpha e^{-\pi|x|^2} dx$

$C_{n,\alpha} = \dfrac{\int_{\mathbb{R}^n} |x|^\alpha e^{-\pi|x|^2} dx}{\int_{\mathbb{R}^n} \frac{1}{|x|^{n+\alpha}} e^{-\pi|x|^2} dx}$, $\mathrm{I} \triangleq \int_{\mathbb{R}^n} |x|^\alpha e^{-\pi|x|^2} dx$, $\mathrm{II} = \int_{\mathbb{R}^n} \frac{e^{-\pi|x|^2}}{|x|^{n+\alpha}} dx$.

Then $\displaystyle\int_{\mathbb{R}^n} \widehat{|\xi|^\alpha}(x)\varphi(x)\, dx = \int_{\mathbb{R}^n} \varphi(x) \left(\int_{\mathbb{R}^n} |\xi|^\alpha e^{-2\pi i \xi \vec{e}} d\xi \right) \frac{1}{|x|^{n+\alpha}} dx$

$$= \int_{\mathbb{R}^n} \varphi(x) \frac{C_{n,\alpha}}{|x|^{n+\alpha}} dx.$$

$$\mathrm{I} = \omega_{n-1} \int_0^\infty r^\alpha e^{-\pi r^2} r^{n-1} dr = \frac{\omega_{n-1}}{2} \frac{\Gamma(\frac{n+\alpha}{2})}{\pi^{\frac{n+\alpha}{2}}}.$$

$$\mathrm{II} = \omega_{n-1} \int_0^\infty \frac{1}{r^{n+\alpha}} e^{-\pi r^2} r^{n-1} dr = \frac{\omega_{n-1}}{2} \frac{\Gamma(-\frac{\alpha}{2})}{\pi^{-\frac{\alpha}{2}}}$$

$$C_{n,\alpha} = \frac{\mathrm{I}}{\mathrm{II}} = \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(-\frac{\alpha}{2})} \frac{\pi^{-\frac{\alpha}{2}}}{\pi^{\frac{n+\alpha}{2}}}, \quad (|\xi|^\alpha)^\wedge = \frac{C_{n,\alpha}}{|x|^{n+\alpha}}$$

So we can define

$$(-\Delta)^{-\frac{\alpha}{2}} u(x) = C'_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}} dy. \qquad (*).$$

20. Fractional Sobolev inequality.

$\|(-\Delta)^{-\frac{\alpha}{2}} u\|_{L^q} \leq C_{n,p,\alpha} \|u\|_{L^p}$, where $1 < p \leq n$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

Pf: $I_\alpha \triangleq (-\Delta)^{\frac{\alpha}{2}}$.

Lemma: If K is radially decreasing and positive.
then $\left| \int K(x-y) u(y)\, dy \right| \leq \|K\|_{L^1} M(u)(x)$.

Pf of Lemma: Step I: when $K = \chi_{B_r(0)}$.

$\left| \int_{\mathbb{R}^n} K(x-y) u(y)\, dy \right| \leq m(B_r(0)) ⨍_{B_r(x)} |u(y)|\, dy \leq \|K\|_{L^1} M(u)(x).$

Step 2: $k=\sum_{j=1}^{n} a_j \chi_{B_{r_j}(0)\setminus B_{r_{j-1}}(0)}$.

$a_j: a_1>a_2>\cdots>a_n$, $r_j: r_1<r_2<\cdots<r_n$, $c_j=a_j-a_{j-1}$

$k=\sum_{j=1}^{n} c_j \chi_{B_{r_j}(0)}$.

$$\left|\int k(x-y)u(y)dy\right| \le \sum_{j=1}^{n} c_j \int_{\mathbb{R}^n} \chi_{B_{r_j}(0)}(x-y)|u(y)|dy$$

$$\lesssim \sum_{j=1}^{n} c_j\, m(B_{r_j}(x)) \fint_{B_{r_j}(0)} |u(y)|dy \le \|k\|_{L^1} M(u)(x).$$

Step 3: Use simple functions to approximate L^1 functions. $\square$

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}}dy\right)^q dx\right)^{\frac{1}{q}}$$

$$=\left(\int_{\mathbb{R}^n}\left(\int_{B_r(x)} \frac{u(y)}{|x-y|^{n-\alpha}}dy+\int_{\mathbb{R}^n\setminus B_r(x)} \frac{u(y)}{|x-y|^{n-\alpha}}dy\right)^q dx\right)^{\frac{1}{q}}$$

$$\le\left(\int_{\mathbb{R}^n}\left(\left\|\frac{1}{|x|^{n-\alpha}}\right\|_{L^1} M(u)(x)+\left(\int_{\mathbb{R}^n\setminus B_r(x)} \frac{1}{|x-y|^{n-\alpha}}dy\right)^{\frac{1}{p'}}\|u\|_{L^p}\right)^q dx\right)^{\frac{1}{q}}$$

$$=\left(\int_{\mathbb{R}^n}\left(\frac{w_{n-1}}{\alpha}M(u)(x)+\left(\frac{w_n(p-1)}{\alpha p-n}\right)^{\frac{1}{p'}} r^{\alpha-\frac{n}{p}}\|u\|_{L^p}\right)^q dx\right)^{\frac{1}{q}}$$

Optimize $\quad r=\left(\dfrac{\|u\|_{L^p}}{M(u)(x)}\right)^{\frac{p}{n}}$

$$\text{LHS}\le\left(\frac{w_{n-1}}{\alpha}+C_{\alpha,n}\right)\|u\|_{L^p}^{\frac{\alpha p}{n}}\left(\int M(u)(x)^{(1-\frac{\alpha p}{n})q}dx\right)^{\frac{1}{q}}$$

$$\le\left(\frac{w_{n-1}}{\alpha}+C_{\alpha,n}\right)C\|u\|_{L^p}^{\frac{\alpha p}{n}}\|u\|_{L^p}^{\frac{p}{q}}=C_{n,\alpha,p}\|u\|_{L^p}.$$

21. e.g. Heat equation

$$\frac{\partial}{\partial t}u(x,t)=\Delta_x u(x,t)$$

So $\frac{\partial}{\partial t}\hat{u}(\xi,t)+4\pi^2|\xi|^2\hat{u}(\xi,t)=0$

$\Rightarrow \hat{u}(\xi,t)=e^{-4\pi^2|\xi|^2 t}\hat{u}(\xi,0)$.

$u(x,t)=\left(\left(e^{-4\pi^2|\xi|^2 t}\right)^{\vee} * u|_{t=0}\right)(x)$.

$$\int_{\mathbb{R}^n} e^{-4\pi^2|\xi|^2 t}e^{2\pi i\xi x}d\xi=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

$$u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}u(y,0)dy.$$

22. Def: $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

For $x, y \in \mathbb{R}^n$, if $x - y \in \mathbb{Z}^n$, $x \sim y$.

So $\mathbb{T}^n = [-\frac{1}{2}, \frac{1}{2}]^n$.

Def: $\hat{f}(m) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i m x} dx$, $m \in \mathbb{Z}^n$.

f is periodic function on $\mathbb{T}^n$, i.e. $f(x+z) = f(x)$, if $z \in \mathbb{Z}^n$.

23. Def: $D_N(x) = \sum_{|m| \leq N} e^{2\pi i m x}$, $|m| = \sum_{i=1}^{n} m_i$

Thus $(D_N * f)(x) = \int_{\mathbb{R}^n} \sum_{|m| \leq N} e^{2\pi i m (x-y)} f(y) dy$

$= \sum_{|m| \leq N} \int_{\mathbb{R}^n} f(y) e^{-2\pi i m y} dy \, e^{2\pi i m x} = \sum_{|m| \leq N} \hat{f}(m) e^{2\pi i m x}$.

D_N is called Dirichlet kernal.

e.g. For $\mathbb{T}^1$.

$D_N(x) = \sum_{|m| \leq N} e^{2\pi i m x} = \frac{e^{2\pi i x (N+1)} - e^{2\pi i N x}}{e^{2\pi i x} - 1} = \frac{\sin(2N+1)\pi x}{\sin \pi x}$.

Def: $F_N(x) = \frac{1}{N+1} \sum_{j=0}^{N} D_j(x)$, $x \in \mathbb{T}^1$. Fejér kernal.

$F_N(x) = \frac{1}{N+1} \sum_{j=0}^{N} \sum_{|m| \leq j} e^{2\pi i m x} = \frac{1}{N+1} \sum_{|m| \leq N} (N - |m| + 1) e^{2\pi i m x}$

$= \sum_{|m| \leq N} (1 - \frac{|m|}{N+1}) e^{2\pi i m x} = \cdots = \frac{1}{N+1} \left(\frac{\sin(N+1)\pi x}{\sin \pi x} \right)^2$.

Def: $F_N^n(x) = \prod_{j=1}^{n} F_N(x_j) = \sum_{\substack{m \in \mathbb{Z}^n \\ |m_j| \leq N}} (1 - \frac{|m_1|}{N+1}) \cdots (1 - \frac{|m_n|}{N+1}) e^{2\pi i m x}$

$= \frac{1}{(N+1)^n} \prod_{j=1}^{n} \left(\frac{\sin \pi (N+1) x_j}{\sin \pi x_j} \right)^2$

$(F_N^n * f)(x) = \int_{|\xi| \leq N} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$.

24. Thm: ① $\| F_N^n \|_{L^1(\mathbb{T}^n)} \leq C$ holds for any N.

② $\int_{\mathbb{T}^n} F_N^n \, dx = 1$

③ $\forall \delta > 0$, $\int_{|x| \geq \delta} F_N^n \, dx \to 0$ as $N \to \infty$

Pf: We know that $F_N^n \geq 0$.

$$\int_{\mathbb{T}^n} \sum_{\substack{m\in\mathbb{Z}^n \\ |m_j|\le N}} \left(1-\frac{|m_1|}{N+1}\right)\cdots\left(1-\frac{|m_n|}{N+1}\right) e^{2\pi i m x}\, dx$$

$$= \sum C_m \int_{\mathbb{T}^n} e^{2\pi i m x}\, dx = \sum C_m \int_{-\frac12}^{\frac12}\int_{-\frac12}^{\frac12}\cdots\int_{-\frac12}^{\frac12} e^{2\pi i m_1 x_1}\cdots e^{2\pi i m_n x_n}\, dx$$

$$\int_{-\frac12}^{\frac12} e^{2\pi i m x} dx = \begin{cases} 0 & m\neq 0 \\ 1 & m=0. \end{cases}$$

Thus $\int F_N^n\, dx = 1$.

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin(N+1)\pi x}{\sin \pi x}\right)^2 \le \frac{1}{N+1}\min\left(\frac{(N+1)\pi x}{|\sin\pi x|}, \frac{1}{|\sin \pi x|}\right)^2$$

$$\le \frac{1}{N+1}\cdot\frac{\pi^2}{4}\min\left(N+1, \frac{1}{|\pi x|}\right)^2$$

$$\int_{|x|\ge\delta} F_N(x) \le \frac{1}{N+1}\frac{\pi^2}{4}\int_{|x|\ge\delta}\frac{1}{\pi^2|x|^2}dx = \frac{1}{N+1}\cdot\frac{\pi^2}{4}\cdot\frac{1}{\pi\delta}\to 0$$

$$\int_{|x|\ge\delta} F_N^n(x) = \int_{|x|\ge\delta}\prod_{i=1}^n F_N(x_i)\,dx$$

$$= \sum_{i=1}^n \int_{|x_j|\ge\delta} F_N(x_j)\prod_{k\neq j} F_N(x_k)\, dx_j\; dx_1\cdots dx_n \to 0.$$

25. Thm: ① If $f\in L^p(\mathbb{T}^n)$ $1\le p<\infty$.
then $f * F_N^n \xrightarrow{L^p} f$ as $N\to\infty$.
② If $f\in L^\infty(\mathbb{T}^n)$ is uniformly continuous.
then $f * F_N^n \xrightarrow{L^\infty} f$.

Pf: ① $f*F_N^n(x) - f(x) = \int_{\mathbb{T}^n}(f(x-y)-f(x))F_N^n(y)\,dy$.

$$\|f*F_N^n(x)-f(x)\|_{L^p} \le \left(\int_{\mathbb{T}^n}\left(\int_{\mathbb{T}^n}|f(x-y)-f(x)|F_N^n(y)\,dy\right)^p dx\right)^{\frac1p}$$

$$\le \int_{\mathbb{T}^n}\left(\int_{\mathbb{T}^n}|f(x-y)-f(x)|^p F_N^n(y)^p\,dy\right)^{\frac1p}dx$$

$$\le \int_{|y|<\delta}\left(\int_{\mathbb{T}^n}|f(x-y)-f(x)|^p dx\right)^{\frac1p} F_N^n(y)\,dy$$

$$+\int_{|y|\ge\delta}\left(\int_{\mathbb{T}^n}|f(x-y)-f(x)|^p dx\right)^{\frac1p} F_N^n(y)\,dy$$

$\forall \varepsilon>0, \exists \delta$, s.t. where $|y|<\delta$, $\left(\int |f(x-y)-f(x)|^p dy\right)^{\frac{1}{p}} < \varepsilon$.
Thus $\|f*F_N^n - f\|_{L^p} \leq \|f\|_{L^p} \int_{|y|\geq\delta} F_N^n(y)\,dy \to 0$ as $N\to\infty$.

② Similarly.

Corollary: ① Trig polynomial is dense in $L^p(\mathbb{T}^n)$.
② If $f,g\in L^1(\mathbb{T}^n)$, $\hat{f}(m)=\hat{g}(m)$, then $f=g$ a.e.
Pf: Consider $f-g\in L^1(\mathbb{T}^n)$
$\widehat{(f-g)}(m)=0$. $(f-g)*F_N^n(x)=0 \xrightarrow{L^1} f-g \Rightarrow f-g=0$ a.e.

26. $\{e^{2\pi imx}\}$ forms a ONB in $L^2(\mathbb{T}^n)$.

Pf: $\int_{\mathbb{T}^n} e^{2\pi imx} e^{-2\pi ikx}\,dx = \delta_{mk}$.
If $\langle f, e^{2\pi imx}\rangle = 0$ holds for any $m\in\mathbb{Z}^n$.
Then $\hat{f}(m)=0$ for every m.
Thus $f(x)=0$ a.e. $\{e^{2\pi imx}\}$ is complete.

Prop: ① Planchend. $\|f\|^2_{L^2(\mathbb{T}^n)} = \sum_{m\in\mathbb{Z}^n} |\hat{f}(m)|^2$.
Pf: f compare with $\sum_{m\in\mathbb{Z}^n} \hat{f}(m) e^{-2\pi imx}$
$\hat{f}(m) = \langle f, e^{2\pi imx}\rangle$
Then $\|f - \sum_{m\in\mathbb{Z}^n} \hat{f}(m) e^{-2\pi imx}\|_{L^2} = 0$.
So $\|f\|^2_{L^2(\mathbb{T}^n)} = \sum_{m\in\mathbb{Z}^n} |\hat{f}(m)|^2$.
② $\langle f,g\rangle = \sum_{m\in\mathbb{Z}^n} \hat{f}(m)\overline{\hat{g}(m)}$.
③ $\widehat{fg}(m) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\hat{g}(m-k)$.

27. Poisson formula.
$f\in C(\mathbb{R}^n)$, $|f(x)| \leq \frac{C}{(1+|x|)^{n+\delta}}$, $\sum_{m\in\mathbb{Z}^n} |\hat{f}(m)| < \infty$, $\delta>0$
Then $\sum_{m\in\mathbb{Z}^n} \hat{f}(m) e^{2\pi imk} = \sum_{k\in\mathbb{Z}^n} f(x+k)$.

Pf: Let $F(x) \triangleq \sum_{k\in\mathbb{Z}^n} f(x+k)$, then $\|F\|_{L^1(\mathbb{T}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$.

$$\hat{F}(m) = \int_{\mathbb{T}^n} \sum_{k\in\mathbb{Z}^n} f(x+k)\, e^{-2\pi i m x}\, dx$$

$$\overset{DCT}{=} \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{T}^n} f(x)\, e^{-2\pi i m (x-k)}\, dx$$

$$= \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{T}^n - k} f(x) e^{-2\pi i m x}\, dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i m x} dx = \hat{f}(m)$$

28. $f: \mathbb{T}^n \to \mathbb{C}$, $f \in L^1(\mathbb{T}^n)$, $\hat{f}(m) \to 0$ as $m \to \infty$.

Pf: $f \in L^1(\mathbb{T}^n)$, then f can be approximated by Trig polynomials.

$f * F_N^n \xrightarrow{L^1} f$.

$f * F_N^n = \sum \hat{f}(m) (1 - \frac{|m_1|}{N+1}) \cdots (1 - \frac{|m_N|}{N+1})\, e^{2\pi i m x}$

$\langle f * F_N^n, e^{2\pi i m x} \rangle \to 0$

Then $\langle f, e^{2\pi i m x} \rangle \to 0$.

29. Thm: If $\partial^\alpha f$ exists, integrable on $\mathbb{T}^n$, $\forall |\alpha| \le s$.

Then. $|\hat{f}(m)| \le \left(\frac{\sqrt{n}}{2\pi}\right)^s \frac{\max_{|\alpha|=s} |\partial^\alpha f(x)|}{|m|^s}$

Pf: $|\hat{f}(m)| = \left| \int_{\mathbb{T}^n} f(x)\, e^{-2\pi i m x} dx \right|$

$$\overset{IBP}{=} \left| \int_{\mathbb{T}^n} \partial^\alpha f(x) \frac{e^{-2\pi i m x}}{(-2\pi i m)^\alpha} dx \right| \le C_n \frac{\max_{|\alpha|=s} |\widehat{\partial^\alpha f}|}{|m|^s} m(\mathbb{T}^n)$$

$|m| = \sqrt{m_1^2 + \cdots + m_n^2}$, $\exists\, m_j$, s.t. $|m_j| \ge \frac{|m|}{\sqrt{n}}$.

We can give that $C_n m(\mathbb{T}^n) = \left(\frac{\sqrt{n}}{2\pi}\right)^s$.

30. Thm: If $\partial^\alpha f$ exists, $\forall |\alpha| \le s$.

when $|\alpha| = s$, $\partial^\alpha f$ is uniformly γ-Lipschiz,

i.e. $\sup_{\substack{x,h\in\mathbb{T}^n \\ h\ne 0}} \frac{|\partial^\alpha f(x+h) - \partial^\alpha f(x)|}{|h|^\gamma} \triangleq \|\partial^\alpha f\|_{Lip\,\gamma} < \infty$

Then $|\hat{f}(m)| \le \frac{(\sqrt{n})^{s+\gamma}}{(2\pi)^s\, 2^{\gamma+1}} \frac{\max_{|\alpha|=s} \|\partial^\alpha f\|_{Lip\,\gamma}}{|m|^{s+\gamma}}$

31. Thm: Let $s>0$, $f \in L^1(\mathbb{T}^n)$.

If $|\hat{f}(m)| \leq C \frac{1}{(1+|m|)^{n+s}}$ $\forall m \in \mathbb{Z}^n$.

then f has partial derivatives up to $|\alpha| \leq [s]-1$.

Pf: $|\hat{f}(m)| \leq C \frac{1}{(1+|m|)^{n+s}}$.

Then $\sum_{m\in\mathbb{Z}^n} |\hat{f}(m)|$ converges.

$$F(x) = \sum_{m\in\mathbb{Z}^n} \hat{f}(m) e^{2\pi i m x} \overset{a.e.}{=} f(x)$$

$$\lim_{h\to 0} \frac{f(x+h\vec{e_j})-f(x)}{h} = \lim_{h\to 0} \sum_{m\in\mathbb{Z}^n} \hat{f}(m) \frac{e^{2\pi i m(x+he_j)} - e^{2\pi i m x}}{h}$$

$$\overset{DCT}{=} \sum_{m\in\mathbb{Z}^n} \hat{f}(m) \cdot 2\pi i m_j \cdot e^{2\pi i m x}$$

Induction shows f is differentable all the way to order $[s]-1$.

32. Thm: If $f \in L^1(\mathbb{T}^n)$, $f(x_0^+)$, $f(x_0^-)$ exist.

then $(F_N * f)(x_0) \to \frac{1}{2}(f(x_0^+)+f(x_0^-))$

Pf: $(F_N*f)(x_0) - \frac{1}{2}(f(x_0^+)+f(x_0^-))$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t) f(x_0-t)\,dt - \int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)\left(\frac{f(x_0^+)+f(x_0^-)}{2}\right)dt$$

$$= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t) f(x_0-t)\,dt + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t) f(x_0+t)\,dt$$

$$- \int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t) \frac{f(x_0^+)+f(x_0^-)}{2}\,dt$$

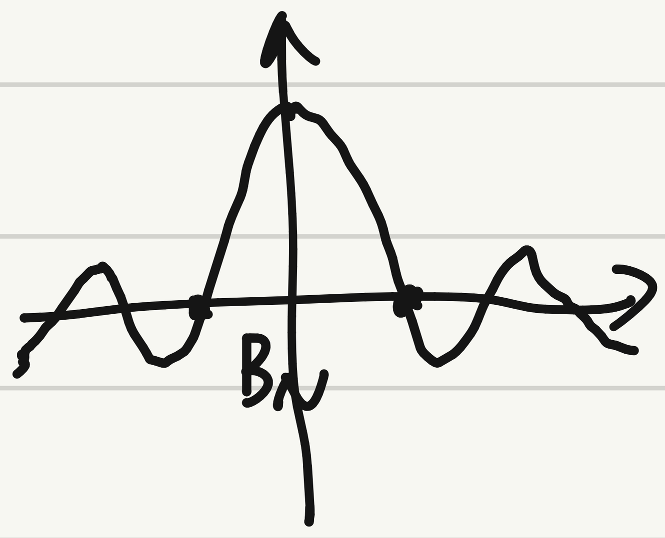
$$= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)\Big(\big(f(x_0+t)-f(x_0^+)\big) + \big(f(x_0-t)-f(x_0^-)\big)\Big)\,dt$$

$$\leq \frac{1}{2}\int_{|t|\leq\delta} F_N(t)\,\varepsilon\,dt + \frac{1}{2}\int_{|t|>\delta} 4|f| \sup_{|t|>\delta}|F_N(t)|\,dt$$

$$\to 0.$$

33. $\exists f \in C(\mathbb{T}^n)$, s.t. $\limsup_{N\to\infty} \sum_{\substack{m\in\mathbb{Z}^n \\ |m_j|\le N}} \hat{f}(m) e^{2\pi i m x} = \infty$.

Pf: Consider $n=1$



$T_N : C(\mathbb{T}^1) \to \mathbb{C}$, $f \mapsto (D_N * f)(0)$.

$D_N(x) = \dfrac{\sin(2N+1)\pi x}{\sin \pi x}$ $\quad B_N$ as picture.

$A_N = [-\frac{1}{2}, \frac{1}{2}] \setminus B_N$. $\quad f_N = \begin{cases} 1 & x \in A_N \ \ D_N(x) > 0 \\ -1 & x \in A_N \ \ D_N(x) < 0. \\ -1 < \quad < 1 & x \in B_N. \end{cases}$

$$T_N f = \left| \int_{\mathbb{T}^1} D_N(0-x) f(x)\, dx \right|$$

$$\geq \left| \int_{A_N} D_N(-x) f(x)\, dx \right| - \left| \int_{B_N} D_N(-x) f(x)\, dx \right|$$

$$= \int_{\mathbb{T}^1} |D_N(x)|\, dx - \left(\left| \int_{B_N} D_N(-x) f(x)\, dx \right| + \int_{B_N} |D_N(x)|\, dx \right)$$

$$\left| \int_{B_N} D_N(-x) f(x)\, dx \right| + \int_{B_N} |D_N(x)|\, dx$$

$$\leq 2 \int_{B_N} |D_N|\, dx \leq 2\,(2N+1)(2N+1) \cdot 2\, \frac{1}{(2N+1)^2} \leq 4.$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{|m|\le N} e^{2\pi i m x} \right| dx$$

$$= \sum_{k=-2N+1}^{2N} \int_{-\frac{1}{2}\frac{k+1}{2N+1}}^{\frac{1}{2}\frac{k+1}{2N+1}} \left| \sum_{|m|\le N} e^{2\pi i m x} \right| dx$$

$$= \sum \int_{-\frac{1}{2}\frac{k+1}{2N+1}}^{\frac{1}{2}\frac{k+1}{2N+1}} \left| \pm 2 \sum_{m=1}^{N} \cos 2\pi m x\, dx \right| = C \sum \frac{1}{k}.$$

Then $\|T_n\| \to \infty$. as $N \to \infty$.

34. Thm: ① $f \in L^1(\mathbb{T}^1)$, $t_0 \in \mathbb{T}^1$

if $f(t_0)$ is defined and $\displaystyle\int_{|t|\le\frac{1}{2}} \frac{|f(t+t_0) - f(t_0)|}{|t|}\, dt < \infty$ (Dini)

Then $(D_N^n * f)(t_0) \longrightarrow f(t_0)$ as $N \to \infty$.

② $f \in L^1(\mathbb{T}^n)$, $a = (a_1, a_2, \cdots, a_n) \in \mathbb{T}^n$

if $\displaystyle\int_{|x_1|\le\frac{1}{2}} \cdots \int_{|x_n|\le\frac{1}{2}} \frac{|f(x+a) - f(a)|}{|x_1||x_2|\cdots|x_n|}\, dx_1 dx_2 \cdots dx_n < \infty$.

then $(D_N^n * f)(a) \to f(a)$ as $N \to \infty$.

Pf: We only need to prove when $a=0$, $f(-y)-f(0) \overset{\circ}{=} g(-y)$

$$D_N^n * f(0) - f(0) = D_N^n * (f - f(0))$$

$$= \int_{\mathbb{T}^n} (f(-y) - f(0)) \prod_{j=1}^{n} \frac{\sin(2N+1)\pi y_j}{\sin \pi y_j} dy_1 dy_2 \cdots dy_n$$

$$= \int_{\mathbb{T}^n} g(-y) \prod_{j=1}^{n} \frac{\sin 2N\pi y_j \cos \pi y_j + \cos 2N\pi y_j \sin \pi y_j}{\sin \pi y_j} dy_1 \cdots dy_n$$

$$= \int_{\mathbb{T}^n} g(-y) \prod_{j=1}^{n} \left(\frac{\cos \pi y_j}{\sin \pi y_j} \sin 2N\pi y_j + \cos 2N\pi y_j \right) dy_1 \cdots dy_n$$

$$= \int_{\mathbb{T}^n} g(-y) \sum_{j \in J} \left(\frac{\cos \pi y_j}{\sin \pi y_j} \sin 2N\pi y_j \right) \prod_{k \in J^c} \cos 2N\pi y_k$$

$$= \sum_{J \subseteq \{1,2,\cdots,n\}} \int_{\mathbb{T}^n} \prod_{\substack{j \in J \\ k \in J^c}} \left[g(-y) \frac{\cos \pi y_j}{\sin \pi y_j} \right] \frac{e^{2N\pi y_j} - e^{-2N\pi y_j}}{2i} \cdot \frac{e^{2N\pi y_i} + e^{-2N\pi y_j}}{2} dy_1 \cdots dy_n$$

For $|y| \geqslant \delta$, $\left| g(-y) \frac{\cos \pi y_j}{\sin \pi y_j} \right| \leqslant |g| \frac{1}{\pi \delta} \in L^1$.

For y closed to zero.

$$g(-y) \frac{\cos \pi y_j}{\sin \pi y_j} \sim g(-y) \frac{1}{\pi y_j} \in L^1 \text{ (For Dini)}.$$

For $f \in L^1$, $|\hat{f}(m)| \to 0$ as $|m| \to \infty$.

Then $\int_{\mathbb{T}^n} \prod_{\substack{j \in J \\ k \in J^c}} \left[g(-y) \frac{\cos \pi y_j}{\sin \pi y_j} \right] \frac{e^{2N\pi y_j} - e^{-2N\pi y_j}}{2i} \cdot \frac{e^{2N\pi y_j} + e^{-2N\pi y_j}}{2} dy_1 \cdots dy_n$

$\to 0$. Thus $D_N^n * f(0) \to f(0)$ as $N \to \infty$.

35. Corollary: $f \in L^1(\mathbb{T}^1)$ vanishes on $I \overset{\text{open}}{\subset} \mathbb{T}^1$.
Then $\forall x \in I$, $(D_N * f)(x) \to 0$.

Pf: Need to check f satisfies Dini at x.
Pick δ, s.t. $(\delta - x, x + \delta) \subset I$.

Then $$\int_{|t| \leq \frac{1}{2}} \frac{|f(t+x) - f(x)|}{|t|} dt = \int_{|t| \geqslant \delta} \frac{|f(t+x) - f(x)|}{|t|} dt$$

$$+ \int_{|t| < \delta} \frac{|f(t+x) - f(x)|}{|t|} dt \leqslant \frac{2}{\delta} \|f\|_1 + \varepsilon < \infty.$$

Thus $(D_N * f)(x) \to f(x) = 0$.

36. Corallery: $a=(a_1,a_2,\cdots,a_n)\in\mathbb{T}^n$

$f\in L^1(\mathbb{T}^n)$ is constant

$\{x\in\mathbb{T}^n : |x_j-a_j|<\delta_j \text{ for some } j\}$.

where $0<\delta_j<\frac{1}{2}$, Then $(D_N^n * f)(a)\to f(a)$.

37. Thm: $R>0$, $m\in\mathbb{Z}^n$, let

$a(m,R): \mathbb{Z}^n\times\mathbb{R}\to\mathbb{R}$, s.t.

① $\forall R>0$, $\exists q_R$ s.t. when $|m|\geq q_R$, $a(m,R)=0$.

② $\forall m\in\mathbb{Z}^n$, $\lim\limits_{R\to 0} a(m,R)=a_m$

③ $\exists M_0$, s.t. $|a(m,R)|\leq M_0$, $\forall m,R$.

For $1\leq p<\infty$, $f\in L^p(\mathbb{T}^n)$

$S_R(f)(x)=\sum\limits_{m\in\mathbb{Z}^n} a(m,R)\hat{f}(m)e^{2\pi imx}$

For $h\in C^\infty(\mathbb{T}^n)$, define $A(h)(x)=\sum_{m\in\mathbb{Z}^n} a_m \hat{f}(m)e^{2\pi imx}$

Then $S_R f$ converges in L^p iff $\exists K$, s.t.

$\sup\limits_{R>0}\|S_R\|_{L^p\to L^p}\leq K$,

Moreover, $\sup\limits_{h\in C^\infty}\dfrac{\|A(h)\|_{L^p}}{\|h\|_{L^p}}\leq K$.

e.g. ① $D_N^n * f$ ② $F_N^n * f(x)=\sum\limits_{\substack{m\in\mathbb{Z}^n\\ |m_j|\leq N}}\left(1-\frac{|m_1|}{N+1}\right)\cdots\left(1-\frac{|m_n|}{N+1}\right)\hat{f}(m)e^{2\pi imx}$

③ Bochnur-Riesz operator

$B_R^\alpha(f)(x)=\sum\limits_{m\in\mathbb{Z}^n}\chi_{\{|x|\leq R\}}\left(1-\frac{|m|^2}{R^2}\right)^a\hat{f}(m)e^{2\pi imx}$.

pf: "$\Rightarrow$" If $S_R f$ converges in L^p.

then $\|S_R(f)\|_{L^p}\leq C_p$.

$|S_R(f)|=\left|\sum_{m\in\mathbb{Z}^n} a(m,R)\hat{f}(m)e^{2\pi imx}\right|$

$\leq M\sum\limits_{|m|\leq q_R}|\hat{f}(m)|\leq M_0\,\#\{|m|\leq q_R\}\|f\|_p$.

So $\|S_R\|_{L^p\to L^p}\leq M_0\,\#\{|m|\leq q_R\}$

According to uniformly bounded priciple.

$\sup\limits_{R>0}\|S_R\|_{L^p\to L^p}<\infty$ Pick $K=\sup\limits_{R>0}\|S_R\|_{L^p\to L^p}$.

"$\Leftarrow$" If $\sup\limits_{R>0}\|S_R\|_{L^p\to L^p}\leq K$

$|a(m,R)\hat{h}(m)| \leq M_0 |\hat{h}(m)|$ and $\sum_{m\in\mathbb{Z}^n} |\hat{h}(m)| < \infty$. $(h \in C^\infty)$.

Thus $\lim_{R\to\infty} S_R(h) = \lim_{R\to\infty} \sum_{|m|\leq q_R} a(m,R)\hat{h}(m) e^{2\pi i m x}$

$\underset{=\!=}{\overset{DCT}{}} \sum_{m\in\mathbb{Z}^n} a_m \hat{h}(m) e^{2\pi i m x} = A(h)(x)$.

So $\|A(h)\|_{L^p} = \|\lim_{R\to\infty} S_R(h)\|_{L^p} \overset{\text{Fatou}}{\leq} \liminf_{R\to\infty} \|S_R(h)\|_{L^p}$

$\leq K\|h\|_{L^p}$.

$\forall f \in L^p(\mathbb{T}^n)$, $\exists h \in C^\infty(\mathbb{T}^n)$, s.t. $\|f-h\|_{L^p} < \varepsilon$, for $R_2 > R_1 > 0$.

$\|S_{R_2}(f) - S_{R_1}(f)\|_{L^p} \leq \|S_{R_2}(f-h)\|_{L^p} + \|S_{R_1}(f-h)\|_{L^p}$

$+ \|(S_{R_2} - S_{R_1})(h)\|_{L^p} \leq 2K\varepsilon + 2M_0 \sum_{|m|>\alpha} |\hat{h}(m)|$

$+ M \sum_{|m|\leq\alpha} |a(m,R_2) - a(m,R_1)| \to 0$.

Thus $\{S_R(f)\}_{R>0}$ Cauchy sequence, converges in $L^p(\mathbb{T}^n)$.

38. Def: $f \in C^\infty(\mathbb{T}^1)$

its covyngate function $\tilde{f} \triangleq -i \sum_{m\in\mathbb{Z}^n} \mathrm{sgn}(m) \hat{f}(m) e^{2\pi i m x}$

$= -i\left(\sum_{m=1}^{\infty} \hat{f}(m) e^{2\pi i m x} + 0 - \sum_{m=-\infty}^{-1} \hat{f}(m) e^{2\pi i m x}\right)$

$$\mathrm{sgn}(m) = \begin{cases} 1 & m>0 \\ 0 & m=0 \\ -1 & m<0 \end{cases}$$

$P_+(f)(x) \triangleq \sum_{m=1}^{\infty} \hat{f}(m) e^{2\pi i m x}$, $P_-(f)(x) \triangleq \sum_{m=-\infty}^{-1} \hat{f}(m) e^{2\pi i m x}$

$f = P_-(f) + \hat{f}(0) + P_+(f)$.

$S_N f \triangleq D_N^n * f$.

Prop: $S_N f \xrightarrow{L^p} f \Leftrightarrow \exists C_p$, s.t. $\forall f \in C^\infty(\mathbb{T}^1)$, $\|\tilde{f}\|_{L^p} \leq C_p \|f\|_p$.

Pf: $S_N f = \sum_{m=-N}^{N} \hat{f}(m) e^{2\pi i m x} = \sum_{m=-N}^{N} \left(\int_{\mathbb{T}^1} f(y) e^{-2\pi i m y} dy\right) e^{2\pi i m x}$

$= \sum_{m=0}^{2N} \left(\int_{\mathbb{T}^1} f(y) e^{-2\pi i (m-N) y} dy\right) e^{2\pi i (m-N) x}$

$= e^{-2\pi i N x} \sum_{m=0}^{2N} \left(\int_{\mathbb{T}^1} [f(y) e^{2\pi i N y}] e^{-2\pi i m y} dy\right) e^{2\pi i m x}$

$\triangleq e^{-2\pi i N x} S_N'(f(y)e^{2\pi i N y})$

According to Thm 37. $D_N * f \xrightarrow{L^p} f$

$\Leftrightarrow \|S_N f\|_{L^p} \le C_p \|f\|_{L^p} \Leftrightarrow \|S_N'(f e^{2\pi i N y})\|_{L^p} \le C_p \|f\|_{L^p}$

$$S_N'(f)(x) = \sum_{m=0}^{2N} \hat{f}(m) e^{2\pi i m x} = \sum_{m=0}^{\infty} \hat{f}(m) e^{2\pi i m x} - \sum_{m=2N+1}^{\infty} \hat{f}(m) e^{2\pi i m x}$$

$$= \sum_{m=1}^{\infty} \hat{f}(m) e^{2\pi i m x} + \hat{f}(0) - e^{2\pi i \cdot 2N x} \sum_{m=1}^{\infty} \hat{f}(m+2N) e^{2\pi i m x}$$

$$= P_+(f)(x) + \hat{f}(0) - e^{2\pi i \cdot 2N x} P_+(e^{-2\pi i \cdot 2N x} f(x))$$

Thus S_N bdd in $L^p \Leftrightarrow S_N'$ bdd in L^p

$\Leftrightarrow P_+$ bdd in L^p, P_- bdd in L^p.

$\Leftrightarrow T: f \to \tilde{f}$ is bdd in L^p, i.e. $\|\tilde{f}\|_{L^p} \le C_p \|f\|_{L^p}$.

39. Thm: For $1<p<\infty$, $f \in C^\infty$, $D_N^n * f \xrightarrow{L^p} f$.

Pf: We need to show $\|\tilde{f}\|_{L^p} \le C \|f\|_{L^p}$.

First we prove a special case.

$$\begin{cases} ① \ f \text{ trig polynomial} \\ ② \ \hat{f}(0) = 0 \\ ③ \ f \text{ is real valued.} \end{cases}$$

$$\tilde{f}(t) = -i \sum_{m \in \mathbb{Z}^n} \operatorname{sgn}(m) \hat{f}(m) e^{2\pi i m t}$$

$$= i \sum_{m=-\infty}^{-1} \hat{f}(m) e^{2\pi i m t} - i \sum_{m=1}^{\infty} \hat{f}(m) e^{2\pi i m t}$$

$$= i \sum_{m=1}^{\infty} \hat{f}(-m) e^{-2\pi i m t} - i \sum_{m=1}^{\infty} \hat{f}(m) e^{2\pi i m t}$$

$$= 2\operatorname{Re}\left(-i \sum_{m=1}^{\infty} \hat{f}(m) e^{2\pi i m t}\right)$$

Consider $f + i\tilde{f} = \sum_{m=-N}^{N} \hat{f}(m) e^{2\pi i m x} + i\left(-i \sum_{m=-N}^{N} \operatorname{sgn}(m) \hat{f}(m) e^{2\pi i m x}\right)$

$$= 2 \sum_{m=-N}^{N} \hat{f}(m) e^{2\pi i m x}$$

$\int_{\mathbb{T}} (f(t) + i\tilde{f}(t))^{2k} dt = 0$. Binary expand, take real point.

$$0 = \sum_{j=0}^{k} C_{2k}^{2j} f(t)^{2j} \tilde{f}(t)^{2k-2j} (-1)^{k-j}$$

$$\tilde f(t)^{2k} = -\sum_{j=1}^{k} (-1)^{k-j} C_{2k}^{2j} f(t)^{2j} \tilde f(t)^{2k-2j}$$

$$\int_{\mathbb{T}^1} |\tilde f(t)|^{2k} dt \le \sum_{j=1}^{k} C_{2k}^{2j} \int_{\mathbb{T}^1} |f^{2j} \tilde f^{2k-2j}| dt$$

$$\overset{\text{Hölder}}{\le} \sum_{j=1}^{k} C_{2k}^{2j} \left(\int_{\mathbb{T}^1} |f|^{2k} dt\right)^{\frac{2j}{2k}} \left(\int_{\mathbb{T}^1} |\tilde f|^{2k} dt\right)^{\frac{2k-2j}{2k}}$$

Thus $\dfrac{\|\tilde f\|_{L^{2k}}^{2k}}{\|f\|_{L^{2k}}^{2k}} \le \displaystyle\sum_{j=1}^{k} C_{2k}^{2j} \frac{\|\tilde f\|_{L^{2k}}^{2k-2j}}{\|f\|_{L^{2k}}^{2k-2j}} < \infty$. (induction).

If f is not real valued.

$$f(t) = \sum_{m=-N}^{N} \hat f(m) e^{2\pi i m t} = \sum_{m=-N}^{N} \frac{\hat f(m) + \overline{\hat f(-m)}}{2} e^{2\pi i m t}$$

$$+ i \left(\sum_{m=-N}^{N} \frac{\hat f(m) - \overline{\hat f(-m)}}{2} e^{2\pi i m t} \right).$$

$$\sum_{m=-N}^{N} \frac{\hat f(m) + \overline{\hat f(-m)}}{2} e^{2\pi i m t}, \quad \sum_{m=-N}^{N} \frac{\hat f(m) - \overline{\hat f(-m)}}{2} e^{2\pi i m t} \in \mathbb{R}.$$

If $\hat f(0) \ne 0$.

$f = f - \hat f(0) + \hat f(0)$. For constant c, $\tilde c = 0$.

$$\|\tilde f\|_{L^{2k}} = \|\widetilde{f - \hat f(0)} + \widetilde{\hat f(0)}\|_{L^{2k}} \le C \|f - \hat f(0)\|_{L^{2k}} \le 2C\|f\|_{L^{2k}}$$

If f is not trig polynomial

$\exists \{P_n\}_{n=1}^{\infty} \subset L^p$, $P_n \xrightarrow{L^p} f$. Then $\tilde P_n \xrightarrow{L^p} \tilde f$.

For $2k < p < 2k+2$.

According to Macinkowisz interpolation.

then $\|\tilde f\|_{L^p} \le \|f\|_{L^p}$.

For $1 < p < 2$. $\displaystyle\int \tilde f \bar g = \sum_{m\in\mathbb{Z}} -i\,\mathrm{sgn}(m) \hat f(m) \hat{\bar g}(m)$

$$= -\sum_{m\in\mathbb{Z}} \hat f(m) \overline{-i\,\mathrm{sgn}(m) \hat g(m)} = -\int f \bar{\tilde g}.$$

Consider duality. $\|\tilde g\|_{L^{p'}} \le C\|g\|_{L^p} \Rightarrow \|\tilde f\|_{L^p} \le C\|f\|_{L^p}$

40. $\|f\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{4\pi}{n}\left(\int_{\mathbb{R}^n} |f(x)|^2 |x-y|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\xi - z|^2 dz\right)^{\frac{1}{2}}$.
For $\forall x, z \in \mathbb{R}^n$.

Pf: $n\|f\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} f\bar{f} \sum_{j=1}^n \frac{\partial}{\partial x_j}(x_j - y_j)\, dx$

$$= -\int_{\mathbb{R}^n} \sum_{j=1}^n \frac{\partial}{\partial x_j}(f\bar{f}) \ (x_j - y_j)\, dx$$

$$= -\sum_{j=1}^n \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} f \cdot \bar{f} + f \cdot \frac{\partial}{\partial x_j} \bar{f}\right)(x_j - y_j)\, dx$$

$$\leq \sum_{j=1}^n \int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_j} f \cdot f\right| |x_j - y_j|\, dx$$

$$\lesssim \int_{\mathbb{R}^n} |f|\,|\nabla f|\,|x-y|\, dx$$

$$\leq \left(\int_{\mathbb{R}^n} |f(x)|^2 |x-y|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2 dx\right)^{\frac{1}{2}}$$

Replace $f(x)$ to $f(x)e^{-2\pi i x z}$, we can notice that $|f(x)e^{-2\pi i x z}| = |f(x)|$.

Thus $n\|f\|_{L^2(\mathbb{R}^n)}^2 \leq 2\left(\int_{\mathbb{R}^n} |f(x)|^2 |x-y|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla(e^{-2\pi i x z} f(x))|^2 dx\right)^{\frac{1}{2}}$

$$= 2\sum_{j=1}^n \left(\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_j}(e^{-2\pi i x z} f(x)\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |f(x)|^2 |x-y|^2 dx\right)^{\frac{1}{2}}$$

$$\underset{\text{Plankerel}}{=} 2\left(\sum_{j=1}^n \int_{\mathbb{R}^n} \left|\widehat{\frac{\partial}{\partial x_j} e^{-2\pi i x z} f(x)}\right|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |f(x)|^2 |x-y|^2 dx\right)^{\frac{1}{2}}$$

$$= 2\left(\int_{\mathbb{R}^n} 4\pi^2 |\xi|^2 |\hat{f}(z+\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |f(x)|^2 |x-y|^2 dx\right)^{\frac{1}{2}}$$

$$= 4\pi\left(\int_{\mathbb{R}^n} |\xi - z|^2 |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |f(x)|^2 |x-y|^2 dx\right)^{\frac{1}{2}}.$$

41. If f is bounded on $\mathbb{R}^n$, $\hat{f}$ surpported on $B_R(0)$. Then there exists $C_{\alpha,n}$, s.t. $\|\partial^\alpha f\|_\infty \leq C_{\alpha,n} R^{|\alpha|} \|f\|_\infty$.

Pf: Pick $\zeta \in C_c^\infty(\mathbb{R}^n)$, $\zeta \equiv 1$ when $x \in B_1(0)$, and ζ surpported on $B_2(0)$. $\varphi(x) = \check{\zeta}(x)$, $\varphi^{\varepsilon}(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$. $\psi(x) \triangleq \varphi^{\frac{1}{R}}(x)$.

Then $\hat{\psi}(\xi) = \int_{\mathbb{R}^n} R^n \varphi(Rx)\, e^{-2\pi i x\xi}\, dx = \int_{\mathbb{R}^n} \varphi(y)\, e^{-2\pi i y \frac{\xi}{R}}\, dy$

$= \hat{\varphi}(\frac{\xi}{R}) = \zeta(\frac{\xi}{R})$.

Thus $\hat{\psi} \equiv 1$ when $|\xi| < R$, and $\operatorname{supp}\hat{\psi} \subset B_{2R}(0)$.

$\widehat{(f*\psi)}(\xi) = \hat{f}(\xi)\hat{\psi}(\xi) = \hat{f}(\xi)$ when $\xi \in B_R(0)$.

And $\operatorname{supp}\hat{f} \subset B_R(0)$, $(f*\psi)^\wedge(\xi) = \hat{f}(\xi) \Rightarrow f*\psi = f$.

$\partial^\alpha f = f * \partial^\alpha \psi$.

$\|\partial^\alpha \psi\|_1 = \int_{\mathbb{R}^n} R^n \cdot R^{|\alpha|} |\partial^\alpha \varphi(Rx)|\, dx = R^{n+|\alpha|} \|\partial^\alpha \varphi\|_1 \triangleq R^{n+|\alpha|} C_{n,\alpha}$.

Thus $\|\partial^\alpha f\|_\infty \le \|f\|_\infty \|\partial^\alpha \psi\|_1 = C_{n,\alpha} R^{n+|\alpha|} \|f\|_\infty$.

42. Hardy-Littlewood inequality.

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}}\, dx\, dy \le C \|f\|_{L^p} \|g\|_{L^q}.$$

where $\frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n} = 1$.

Pf: $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}}\, dx\, dy = \int_{\mathbb{R}^n} (-\Delta)^{-\frac{\alpha}{2}} f(y) \cdot g(y)\, dy$

$\overset{\text{Hölder}}{\le} \|(-\Delta)^{-\frac{\alpha}{2}} f(y)\|_{L^{p^*}} \|g\|_{L^q}$

$\overset{\text{Sobolev}}{\lesssim} C \|f\|_{L^p} \|g\|_{L^q}$.

where $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$.