Exercise 2 for 2022∼ 2023 USTC Course

'Introduction to Quantum Information'

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1. For the singlet state $|\psi^{-}\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$, prove that Alice and Bob's outcomes are always anti-correlated when they measure two particles respectively along the same direction.

   Answer: Refer to the Box 2.7 on the page of 113 of "Quantum computation and quantum information" by Nielsen.

2. PPT(Positive Partial Transposition) criterion is a strong separability criterion for quantum state, which is very convenient and practical for entanglement detection.

(1) Describe the PPT (Positive Partial Transposition) criterion and the realignment criterion.

(2) For the 2-qubit state $\rho = p\left|\phi^-\right\rangle\left\langle\phi^-\right| + (1-p)\frac{\mathbb{I}}{4}$, where, $0 \leq p \leq 1$, $\left|\phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$, calculate the p's lower bound when ρ is entangled state using PPT criterion and realignment criterion respectively.

Answer:

(1) PPT criterion reads: If ρ is separable, then the partial transpose ρ^{T_A} has no negative eigenvalues.

Realignment criterion reads: For any bipartite separable state ρ, $||\tilde{\rho}|| \leq 1$, where $||\tilde{\rho}||$ is the sum of all the singular values of $\tilde{\rho}$, $\tilde{\rho}$ is the realignment of ρ.

(2)

$$\rho = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & -\frac{p}{2} \\ 0 & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & 0 \\ -\frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix}$$

then

$$\rho^{T_A} = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & 0 \\ 0 & \frac{1-p}{4} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{1-p}{4} & 0 \\ 0 & 0 & 0 & \frac{1+p}{4} \end{pmatrix}$$

The eigenvalues of ρ^{T_A} are $\left\{\frac{1}{4}(1-3p), \frac{p+1}{4}, \frac{p+1}{4}, \frac{p+1}{4}\right\}$.

If ρ is entangled, ρ^{T_A} has negative eigenvalues, then we get $1 \geq p > \frac{1}{3}$

$$\tilde{\rho} = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{1-p}{4} \\ 0 & -\frac{p}{2} & 0 & 0 \\ 0 & 0 & -\frac{p}{2} & 0 \\ \frac{1-p}{4} & 0 & 0 & \frac{1+p}{4} \end{pmatrix}$$

The singular values of $\tilde{\rho}$ are $\left\{\frac{1}{2}, \frac{p}{2}, \frac{p}{2}, \frac{p}{2}\right\}$, then $||\tilde{\rho}|| = \frac{3p+1}{2}$. If ρ is entangled, $||\tilde{\rho}|| > 1$, then we get $1 \geq p > \frac{1}{3}$.

3. (1) Calculate the amount of entanglement of the state $\rho = \lambda |\phi^+\rangle \langle\phi^+| + (1 - \lambda) |\psi^+\rangle \langle\psi^+|, (0 \leq \lambda \leq 1)$ with negativity measure, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle, |\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

(2) Derive the value scope for λ when the state ρ is entangled using negativity measure.

Answer:

(1)

$$\rho = \frac{1}{2}\begin{pmatrix} \lambda & 0 & 0 & \lambda \\ 0 & 1-\lambda & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & 0 \\ \lambda & 0 & 0 & \lambda \end{pmatrix}$$

then

$$\rho^{T_A} = \frac{1}{2}\begin{pmatrix} \lambda & 0 & 0 & 1-\lambda \\ 0 & 1-\lambda & \lambda & 0 \\ 0 & \lambda & 1-\lambda & 0 \\ 1-\lambda & 0 & 0 & \lambda \end{pmatrix}$$

The eigenvalues of ρ^{T_A} are $\left\{\frac{1}{2}, \frac{1}{2}, \frac{2\lambda-1}{2}, \frac{1-2\lambda}{2}\right\}$,and the singular values are $\left\{\frac{1}{2}, \frac{1}{2}, |\frac{2\lambda-1}{2}|, |\frac{1-2\lambda}{2}|\right\}$.

So, the amount of entanglement of ρ is:

$$N(\rho) = \frac{||\rho^{T_A}|| - 1}{2} = |\lambda - \frac{1}{2}|$$

(2) If ρ is an entanglement state,

$$N(\rho) = \frac{||\rho^{T_A}|| - 1}{2} = |\lambda - \frac{1}{2}| > 0$$

when $\lambda \neq 1/2$, ρ is entangled.

4. (1) Describe the definition of the Entanglement Witness (EW).

(2) For the three-qubit **GHZ** state,

$$|\mathbf{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

prove that the entanglement witness $\mathcal{W} = \frac{1}{2}\mathbf{I} - |GHZ\rangle\langle GHZ|$ detects three-qubit entanglement around it.

(3) A mixed state $\rho = (1-p)\frac{\mathbf{I}}{8} + p|GHZ\rangle\langle GHZ|$ $(0 \leq p \leq 1)$, calculate the p's lower bound when ρ is entangled state using the EW given above.

Answer:

(1) An entanglement witness is a functional which distinguishes a specific entangled state from separable ones. W can be called an entanglement witness, if it satisfies that

(a). W has at least one negative eigenvalue;

(b). For any separable state ρ_{AB}, $Tr(W\rho_{AB}) \geq 0$

(2) To prove that $\mathcal{W}$ is an EW, one needs to show that $Tr(\rho_{sep}\mathcal{W}) \geq 0$ for all separable states. That is, for all separable states, $Tr(\rho_{sep}|GHZ\rangle\langle GHZ|) \leq \frac{1}{2}$. The maximum value of $Tr(\rho_{sep}|GHZ\rangle\langle GHZ|)$ is given by the square of the Schmidt coefficient which is maximal over all possible bipartite partitions$(1|23, 2|13, 3|12)$ of $|GHZ\rangle$. Then it is easy to calculate

$$max_{\rho_{sep}} Tr(\rho_{sep}|GHZ\rangle\langle GHZ|) = 1/2.$$

So

$$Tr(\rho_{sep}\mathcal{W}) \geq 0.$$

The entanglement witness $\mathcal{W} = \frac{1}{2}\mathbf{I} - |GHZ\rangle\langle GHZ|$ detects three-qubit entanglement around it.

Experimental Detection of Multipartite Entanglement using Witness Operators (PhysRevLett.92.087902).

(3) ρ is an entangled state, them

$$Tr(\rho\mathcal{W}) = \frac{1-p}{2} - \frac{1-p}{8} - \frac{p}{2} < 0,$$

$$p > \frac{3}{7}.$$

5. (1) What conditions should a good entanglement measures meet?

(2) Describe the definition of distillable entanglement and entanglement cost and their relationship.

(3) Write down the monogamy of entanglement and describe its physical meanings.

Answer:

(1) A good entanglement measure $E(\cdot)$ should satisfy that,

(a) For any separable state ρ, $E(\rho) = 0$;

(b) No increase under LOCC, i.e. $E(\Lambda_{LOCC}(\rho)) \leq E(\rho)$;

(c) Continuity, i.e. $E(\rho) - E(\sigma) \to 0$, when $||\rho - \sigma|| \to 0$;

(d) Convexity, i.e. $E(\lambda\rho + (1-\lambda)\sigma) \leq \lambda E(\rho) + (1-\lambda)E(\sigma)$;

(e) Normalization, i.e. $E(P_+^d) = \log d$.

(2) Read the page 62, 63 in the lecture "QIP2022chapt_2_Kai Chen.pdf" for reference.

(3) Monogamy of entanglement says that:

For any tripartite state of systems A, B_1, B_2 we have

$$E(A|B_1) + E(A|B_2) \leq E(A|B_1B_2).$$

If the above inequality holds in general, i.e. not only for qubits, then it can be immediately generalized by induction to the multipartite case:

$$E(A|B_1) + E(A|B_2) + \cdots + E(A|B_N) \leq E(A|B_1B_2\cdots B_N).$$

It means that if two qubits A and B are maximally quantumly correlated they cannot be correlated at all with a third qubit C. In general, there is a trade-off between the amount of entanglement between qubits A and B and the same qubit A and qubit C. Note that, in some cases, entanglement is not monogamay.

6. The four Bell states have the following mathematical expressions on the basis $\{0, 1\}$ (the eigenstates of σ_z),

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

(1) Prove that the four Bell states can be transformed to each other using single qubit rotations $\{I, \sigma_x, \sigma_y, \sigma_z\}$.

(2) Give the representation of the four Bell states on the basis $\{+,-\}$ (the eigenstates of σ_x).

Answer:

(1)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

$$\Phi^+ \begin{cases} \xrightarrow{\sigma_x \otimes I} |\Psi^+\rangle, \\ \xrightarrow{\sigma_y \otimes I} -i|\Psi^-\rangle, \\ \xrightarrow{\sigma_z \otimes I} |\Phi^-\rangle, \end{cases} \tag{2}$$

$$\Phi^- \begin{cases} \xrightarrow{\sigma_x \otimes I} -|\Psi^-\rangle, \\ \xrightarrow{\sigma_y \otimes I} i|\Psi^+\rangle, \\ \xrightarrow{\sigma_z \otimes I} |\Phi^+\rangle, \end{cases} \tag{3}$$

$$\Psi^+ \begin{cases} \xrightarrow{\sigma_x \otimes I} |\Phi^+\rangle, \\ \xrightarrow{\sigma_y \otimes I} -i|\Phi^-\rangle, \\ \xrightarrow{\sigma_z \otimes I} |\Psi^-\rangle, \end{cases} \tag{4}$$

$$\Psi^- \begin{cases} \xrightarrow{\sigma_x\otimes I} -|\Phi^-\rangle, \\ \xrightarrow{\sigma_y\otimes I} i|\Phi^+\rangle, \\ \xrightarrow{\sigma_z\otimes I} |\Psi^+\rangle, \end{cases} \tag{5}$$

(2) The single qubit transformation between the σ_z basis and the σ_x basis is

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \\ |1\rangle &= \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \end{aligned} \tag{6}$$

So,

$$\begin{cases} |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle), \\ |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \\ |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|++\rangle - |--\rangle), \\ |\Psi^-\rangle = -\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle), \end{cases} \tag{7}$$

7. (1) Describe the physical meanings of von Neumann entropy.

(2) Prove that $S(\rho) \leq \log D$, where D is the number of the non-zero eigenvalues of ρ.

(3) Prove the subadditivity of the von Neumann entropy

$$|S(A) - S(B)| \leq S(A, B) \leq S(A) + S(B)$$

(4) Prove the concavity of the von Neumann entropy

$$S(\sum_i p_i\rho_i) \geq \sum_i p_i S(\rho_i)$$

(5) Prove that the two body pure state $|\psi_{AB}\rangle$ is a entangled state if and only if $S(B|A) < 0$, in which $S(B|A) = S(B,A) - S(A)$, $S(\cdot)$ is the von Neumann entropy.

Answer:

(1) The von Neumann entropy quantizes the quantum information of each character of the quantum ensemble. When the signal ρ is pure state, von Neumann entropy $S(\rho)$ is the information quantization of the quantum information source.

(2)

$$S(\rho) = -tr(\rho\log\rho) = -\sum_i \lambda_i \log\lambda_i = \sum_{i=1}^{D} \lambda_i \log\frac{1}{\lambda_i} \leq \log(\sum_{i=1}^{D} \lambda_i \frac{1}{\lambda_i}),$$

in which the concavity of logarithmic function

$$\log(p_1x_1 + p_2x_2) \geq p_1\log x_1 + p_2\log x_2$$

is used.

(3) Consider the relative entropy of ρ_{AB} and $\rho_A \otimes \rho_B$

$$S(\rho_{AB}||\rho_A \otimes \rho_B) = tr(\rho_{AB} \log \rho_{AB}) - tr(\rho_{AB} \log(\rho_A \otimes \rho_B))$$

$$= -S(\rho_{AB}) - tr(\rho_{AB} \log \rho_A) - tr(\rho_{AB} \log \rho_B)$$

$$= -S(\rho_{AB}) + S(\rho_A) + S(\rho_B)$$

$$\geq 0$$

So,

$$S(A, B) \leq S(A) + S(B)$$

Consider a purification of $\rho_{AB} = tr_C \left|\phi\right\rangle_{ABC} \left\langle\phi\right|$, apply subadditivity to ρ_{BC}, we can get that

$$S(B, C) \leq S(B) + S(C).$$

Since $S(B, C) = S(A), S(C) = S(A, B)$, so we get that

$$S(A, B) \geq S(A) - S(B).$$

Similarly, $S(A, B) \geq S(B) - S(A)$.

So,

$$|S(A) - S(B)| \leq S(A, B)$$

(4) Apply subadditivity to

$$\rho_{AB} = \sum_i p_i \rho_i \otimes |i\rangle \langle i|_B$$

we can get that

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) = S(\sum_i p_i \rho_i) + H(p_i)$$

From the joint entropy theorem we can get that

$$S(\rho_{AB}) = S(\sum_i \rho_i \otimes p_i |i\rangle \langle i|_B) = \sum_i p_i S(\rho_i) + H(p_i)$$

so

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i)$$

(5) Since $|\psi_{AB}\rangle$ is a pure state, so $S(A, B) = 0$.

If $|\psi_{AB}\rangle$ is an entangled state, then its Schmidt decomposition can be write as

$$|\psi_{AB}\rangle = \sum_i \sqrt{p_i} |i_A\rangle |i_B\rangle, i \geq 2$$

so

$$\rho_A = \sum_i p_i |i_A\rangle \langle i_A|,$$

$$S(A) = -\sum_i p_i \log p_i > 0,$$

so

$$S(B|A) = S(A,B) - S(A) = -S(A) < 0$$

8. Prove that $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is invariant under transformation $U(\theta,\vec{n}) \otimes U(\theta,\vec{n})$, where $U(\theta,\vec{n}) = e^{-\frac{i}{2}\theta\cdot\vec{n}\cdot\vec{\sigma}}$.

Answer:

$$U(\theta,\vec{n}) = e^{-\frac{i}{2}\theta\cdot\vec{n}\cdot\vec{\sigma}} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}$$

$$U(\theta,\vec{n}) \otimes U(\theta,\vec{n}) = \cos^2\frac{\theta}{2}I \otimes I - i\sin\frac{\theta}{2}\cos\frac{\theta}{2}(n\cdot\vec{\sigma}_B + n\cdot\vec{\sigma}_A) - \sin^2\frac{\theta}{2}(\vec{n}\cdot\vec{\sigma})_A \otimes (\vec{n}\cdot\vec{\sigma})_B.$$

then, we have

$$\cos^2\frac{\theta}{2}I \otimes I|\psi^-\rangle = \cos^2\frac{\theta}{2}|\psi^-\rangle$$

$$\sigma_x \otimes \sigma_x|\psi^-\rangle = \sigma_y \otimes \sigma_y|\psi^-\rangle = \sigma_z \otimes \sigma_z|\psi^-\rangle = -|\psi^-\rangle$$

$$(\vec{n}\cdot\vec{\sigma}_A + \vec{n}\cdot\vec{\sigma}_B)|\psi^-\rangle = 0$$

Hence, $U(\theta,\vec{n}) \otimes U(\theta,\vec{n})|\psi^-\rangle = |\psi^-\rangle$.

9. For the 2-qubit state $\rho = p|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{\mathbb{I}}{4}$, where $0 \leq p \leq 1$, $|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, calculate the EOF(Entanglement of Formation) of ρ.

Answer:

The square roots are $\left\{\frac{1-p}{4}, \frac{1-p}{4}, \frac{1-p}{4}, \frac{1+3p}{4}\right\}$, so the concurrence of ρ is $C(\rho) = max\{0, \frac{3p-1}{2}\}$.

If $p \leq \frac{1}{3}$, the EOF of state ρ is $E(C(\rho)) = H(1) = 0$.

If $p > \frac{1}{3}$, the EOF of state ρ is $E(C(\rho)) = H(\frac{1+\sqrt{1-(\frac{3p-1}{2})^2}}{2})$.

10. Consider the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$, $\rho_A = tr_B(|\psi\rangle\langle\psi|)$. Calculate the Von Neumann entropy of ρ_A.

Answer:

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$S(\rho_A) = -(\frac{1}{2}log(\frac{1}{2}) + \frac{1}{2}log(\frac{1}{2})) = 1$$

11. Give a noisy entanglement state with purity F for the singlet state $|\Psi^-\rangle$,

$$W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|.$$

Supposing $F = \frac{3}{5}$, please design a two-way LOCC purification protocol that can obtain the singlet state $|\Psi^-\rangle$ with as high fidelity as possible from the above mixed state in five steps.

Answer:

Read the page 99 in the lecture "QIP2022chapt_2_Kai Chen.pdf" for reference. An arbitrary mixed two-partite state ρ with fidelity $F = \langle\Psi^-|\rho|\Psi^-\rangle$ can be transformed to the symmetric Werner state with random bilateral rotations,

$$W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|.$$

where $|\Psi^\pm\rangle = \frac{1}{\sqrt{(2)}}(\uparrow\downarrow \pm \downarrow\uparrow), |\Phi^\pm\rangle = \frac{1}{\sqrt{(2)}}(\uparrow\uparrow \pm \downarrow\downarrow)$ and $F = \langle\Psi^-|W_F|\Psi^-\rangle$.

Alice and Bob share two pairs of W_F state, i.e. W_{F12} and W_{F34}, with 1 and 3 in Alice's side, 2 and 4 in Bob's side. The purification protocol is:

(a) Alice and Bob make unilateral transformation σ_y (i.e. $\sigma_y \otimes I$) on their two pairs of W_F state. We get the new state,

$$W_F \xrightarrow{\sigma_y \otimes I} W'_F = F|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-| + \frac{1-F}{3}|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+|.$$

(b) Alice and Bob perform the C-NOT operations on their two pair of W'_F state with 1 and 2 as 'source' particles and 3 and 4 as 'target' particles. The transformation is shown as follow, then, measure two target particles along the Z

Before		After(n.c. = no change)	
Source	Target	Source	Target
$\Phi^\pm$	Φ^+	n.c.	n.c.
$\Psi^\pm$	Φ^+	n.c	Ψ^+
$\Psi^\pm$	Ψ^+	n.c	Φ^+
$\Phi^\pm$	Ψ^+	n.c	n.c
$\Phi^\pm$	Φ^-	$\Phi^\mp$	n.c
$\Psi^\pm$	Φ^-	$\Psi^\mp$	Ψ^-
$\Psi^\pm$	Ψ^-	$\Psi^\mp$	Φ^-
$\Phi^\pm$	Ψ^-	$\Phi^\mp$	n.c

axis. If the target pair's Z spins are parallel, keep the correspond source state; otherwise, discard the source state. As the measurements along the Z axis can only distinguish Φ from Ψ (but can't distinguish $-$ from $+$), we keep the 1, 3, 5, 7 rows' source states.

(c) For $F = \frac{3}{5}$, we get a state $\rho = 0.62|\Phi^+\rangle\langle\Phi^+| + 0.26|\Phi^-\rangle\langle\Phi^-| + 0.06|\Psi^+\rangle\langle\Psi^+| + 0.06|\Psi^-\rangle\langle\Psi^-|$, note that the main noise state is $|\Phi^-\rangle$ now. Change the bases into $\{|+\rangle, |-\rangle\}$, denote $|+\rangle$ as $|0'\rangle$ and $|-\rangle$ as $|1'\rangle$. We can rewrite $\rho =$

$0.62|\Phi'^+\rangle\langle\Phi'^+| + 0.26|\Psi'^+\rangle\langle\Psi'^+| + 0.06|\Phi'^-\rangle\langle\Phi'^-| + 0.06|\Psi'^-\rangle\langle\Psi'^-|$, repeat the step (b), ρ changes into $\rho_1 = 0.68|\Phi'^+\rangle\langle\Phi'^+| + 0.13|\Psi'^+\rangle\langle\Psi'^+| + 0.13|\Phi'^-\rangle\langle\Phi'^-| + 0.06|\Psi'^-\rangle\langle\Psi'^-|$. Go back to $\{|0\rangle, |1\rangle\}$ bases, $\rho_1 = 0.68|\Phi^+\rangle\langle\Phi^+| + 0.13|\Phi^-\rangle\langle\Phi^-| + 0.13|\Psi^+\rangle\langle\Psi^+| + 0.06|\Psi^-\rangle\langle\Psi^-|$, for which $F_1 = 0.68$.

Repeat step (b) and (c), we can get $F_2 = 0.80$, $F_3 = 0.93$, etc. At last, the final state can be converted back to a mostly Ψ^- state by a unilateral σ_y rotation.