Group Theory Homework I

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January 7, 2024

Problem I

Proof:

(1)Closure:

For $(a, b), (c, d) \in G$, where $a, b, c, d \in R \quad a \neq 0,\ c \neq 0$, we have:

$$(a, b)(c, d) = (ac, ad + b)$$

where $ac \in R$ $ac \neq 0 \quad ad + b \in R$ So $(ac, ad + b) \in G$

(2)Associativity:

For $(a, b), (c, d), (e, f) \in G$, we have

$$\left.\begin{aligned} [(a,b)(b,c)](e,f) &= (ac, ad+b)(e,f) \\ &= (ace, acf + ad + b) \\ (a,b)[(b,c)(e,f)] &= (a,b)(ce, cf+d) \\ &= (ace, acf + ad + b) \end{aligned}\right\} \implies [(a,b)(b,c)](e,f) = (a,b)[(b,c)(e,f)]$$

(3)Identity:

Obviously, $(1, 0)$ is the identity of G.

$\forall (a, b) \in G$, we have

$$(1, 0)(a, b) = (a, b)(1, 0) = (a, b)(1, 0)(1, 0) = (1, 0)$$

(4)Invers:

Obviously, $\left(\frac{1}{a}, -\frac{b}{a}\right)$ is the inverse of (a, b)

$$\left(\frac{1}{a}, -\frac{b}{a}\right)(a, b) = (a, b) = (1, 0)$$

In conclusion, G is a Group.

Group Theory Homework II

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January 7, 2024

Problem 2

Proof:

(1)Closure: Assume $\theta_1, \theta_2 \in [0, 2\pi)$, we have

$$R(\theta_1) = \begin{pmatrix} cos\theta_1 & sin\theta_1 \\ -sin\theta_1 & cos\theta_1 \end{pmatrix} \quad R(\theta_2) = \begin{pmatrix} cos\theta_2 & sin\theta_2 \\ -sin\theta_2 & cos\theta_2 \end{pmatrix}$$

$$\Longrightarrow R(\theta_1)R(\theta_2) = \begin{pmatrix} cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 & cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2 \\ -sin\theta_1 cos\theta_2 - cos\theta_1 sin\theta_2 & -sin\theta_1 sin\theta_2 + cos\theta_1 cos\theta_2 \end{pmatrix}$$

$$= \begin{pmatrix} cos(\theta_1+\theta_2) & sin(\theta_1+\theta_2) \\ sin(\theta_1+\theta_2) & cos(\theta_1+\theta_2) \end{pmatrix} = R(\theta_1+\theta_2) \in G$$

(2)Associatity: Since $R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2)$, we have

$$R(\theta_1)R(\theta_2)R(\theta_3) = [R(\theta_1)R(\theta_2)]\, R(\theta_3) = R(\theta_1)\, [R(\theta_2)R(\theta_3)] = R(\theta_1+\theta_2+\theta_3)$$

(3)Identity: Obviously, $R(\theta = 0)$ is the identity, we have:

$$R(0)R(0) = R(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\forall \theta \in G, \quad R(0)R(\theta) = R(\theta)R(0) = R(\theta)$$

(4)Inverse: There is the inverse element of $R(\theta)$ $(\forall \theta \in [0, 2\pi))$, which is given by $R^{-1}(\theta) = R(-\theta)$,

$$R^(-1)(\theta)R(\theta) = R(-\theta)R(\theta) = R(0)$$

$$R(\theta)R^(-1)(\theta) = R(\theta)R(-\theta) = R(0)$$

Which means the left inverse and right inverse are same for all the element in G

In Conclusion, G is Group.

Problem 3

Note that

$$G = \left\{ T(\alpha) \middle| (\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \alpha \in R \right\}$$

(1)Closure: Asssume $\alpha_1, \alpha_2 \in R$, we have $T(\alpha_1), T(\alpha_2) \in G$

$$T(\alpha_1)T(\alpha_2) = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 + \alpha_2 \\ 0 & 1 \end{pmatrix} \in G$$

(2)Associatity: Assume $\alpha_1, \alpha_2, \alpha_3 \in R \implies T(\alpha_1), T(\alpha_2), T(\alpha_3) \in G$

$$T(\alpha_1)T(\alpha_2)T(\alpha_3) = T(\alpha_1)\left[T(\alpha_2)T(\alpha_3)\right] = \left[T(\alpha_1)T(\alpha_2)\right]T(\alpha_3) = \begin{pmatrix} 1 & \alpha_1 + \alpha_2 + \alpha_3 \\ 0 & 1 \end{pmatrix}$$

(3)Identity: Obviously, when $\alpha = 0$, $T(0)$ is the identity

$$T(0)T(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T(0)$$

$$\forall \alpha \in R, \quad T(0)T(\alpha) = T(\alpha)T(0) = T(\alpha)$$

(4)Inverse: There is the inverse element of $T(\alpha)$ $(\forall \alpha \in R)$, which is given by $T^{-1}(\alpha) = T(-\alpha)$,

$$T^(-1)(\alpha)T(\alpha) = T(-\alpha)T(\alpha) = T(0)$$

$$T(\alpha)T^(-1)(\alpha) = T(\alpha)T(-\alpha) = T(0)$$

Which means the left inverse and right inverse are same for all the element in G

In Conclusion, G is Group.

Problem 4

The two multiplication rules for four order group are given by

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Problem 5

Proof: If $x^2 = e \implies x = x^{-1}$, Note that $H_1 = \{y|y \in G, y^2 \neq e\} \quad H_2 = \{x|x \in G, x^2 = e\}$

Obviously, $H_1 \cap H_2 = G$.

We know that $y^2 \neq e \implies y \neq y^{-1}$, but according to the closure of group, there is an element $z \neq y (z \in G)$ satisfied $zy = e$, where $z \in H_1$ due to $z \neq y$

Therefore, y and z always appear together, which means the number of H_1 is even.

As result,

$$\left.\begin{aligned} Num(H_1) + Num(H_2) = Ord(G) \\ Num(H_1)\ mod\ 2 = 0 \\ Ord(G)\ mod\ 2 = 0 \end{aligned}\right\} \implies Num(H_2)\ mod\ 2 = 0$$

Alternatively, $x^2 = e$ has an even number of solutions.

Group Theory Homework III

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January 7, 2024

Problem 6

Proof:

$$\forall a \in G,\ a^{-1}a = e \implies e^{-1}e = e$$

$$\forall g \in G\ (g \neq e),\ g^{-1}g = e \implies e^{-1}g^{-1}g = e^{-1}e = e$$

$$\implies (ge)^{-1}g = e$$

Due to $g^{-1}g = e$, and the element in Set G can not repeat

$$\implies ge = g$$

Therefore, e is also a right identity of G.

Furthermore,

$$\forall g \in G\ (g \neq e),\ g^{-1}g = e \implies gg^{-1}g = g \implies (gg^{-1})g = g$$

$$\implies gg^{-1} = e$$

Which means g^{-1} is also the right inverse of $g \in G$. In conclusion, G is Group.

Problem 7

Proof:

According to the statement, assume $b = a$ such that

$$\left.\begin{matrix} xa = a \\ ay = a \end{matrix}\right\} \implies a \text{ has left and right identity } e \in G$$

Assume $b = e$, we have

$$xa = e \implies x = a_1$$

$$ay = e \implies y = a_2$$

$$\implies a_2 = ea_2 = (a_1 a)a_2 = a_1(aa_2) = a_1 e = a_1$$

Therefore, we have the inverse $a^{-1} = a_1 = a_2$ such that $a^{-1}a = aa^{-1} = e$

Problem 8

(1) Proof:

a) First, $(R_{C_2}\ +)$ is a Abel Group:

$$\forall x, y \in R_{C_2} \quad x + y = y + x = (x_0 + y_0)e + (x_1 + y_1)a \in R_{C_2}$$

$$\text{if } x = 0\ (\forall y \in R_{C_2}) \left.\begin{aligned} x + x = x \\ x + y = y \end{aligned}\right\} \implies x = 0 \text{ is the identity of the Group.}$$

b) Second, $(R_{C_2}\ \cdot)$ is a SemiGroup:

$$\forall x, y \in R_{C_2} \quad x \cdot y = y \cdot x = (x_0 y_0 + x_1 y_1)e + (x_0 y_1 + x_1 y_0)a \in R_{C_2}$$

$$\begin{aligned} \forall x, y, z \in R_{C_2} \quad (x \cdot y) \cdot z = x \cdot (y \cdot z) =& (x_0 y_0 z_0 + x_1 y_1 z_0 + x_0 y_1 z_1 + x_1 y_0 z_1)e \\ &+ (x_0 y_0 z_1 + x_1 y_1 z_1 + x_0 y_1 z_0 + x_1 y_0 z_0)a \end{aligned}$$

c) Third, Distributive law:

$$\begin{aligned} \forall x, y, z \in R_{C_2} \quad (x + y) \cdot z = x \cdot z + y \cdot z = & (x_0 z_0 + y_0 z_0 + x_1 z_1 + y_1 z_1)e \\ & + (x_0 z_1 + y_0 z_1 + x_1 z_0 + y_1 z_0)a \end{aligned}$$

e) Furthermore, $(R_{C_2}\ +\ \cdot)$ is a Commutative Ring:

$$\forall x, y \in R_{C_2}, x \cdot y = y \cdot x = (x_0 y_0 + x_1 y_1)e + (x_0 y_1 + x_1 y_0)a$$

f) Furthermore, $(R_{C_2}\ +\ \cdot)$ has identity e:

$$\forall x \in R_{C_2} \left.\begin{aligned} x \cdot e = e \cdot x = x \\ e \cdot e = e \end{aligned}\right\} \implies e \text{ is identity}$$

In conclusion, $(R_{C_2}\ +\ \cdot)$ is a Commutative Ring with identity.

(2) Solution:

$(R_{C_2}\ +\ \cdot)$ is not a Field:

If $\forall y \in R_{C_2}\ \exists x \in R_{C_2}$ such that $xy = yx = e$, we have

$$(x_0y_0 + x_1y_1)e + (x_0y_1 + x_1y_0)a = e$$

$$\Longrightarrow \begin{bmatrix} y_0 & y_1 \\ y_1 & y_0 \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The system of linear equations has a solution if and only if $det(Y) \neq 0$

Therefore, if $det(Y) = 0$, the system has no solution.

$$\begin{vmatrix} y_0 & y_1 \\ y_1 & y_0 \end{vmatrix} = 0 \Longrightarrow y_0^2 = y_1^2$$

For example, the Nonzero Element $y = e + a \in R_{C_2}$ has no inverse.

(3) Solution: Assume $x = x_0e + x_1a \in R_{C_2}$,

$$\begin{aligned}
exp(x) &= exp(x_0e + x_1a) = exp(x_0e)exp\left[i\frac{x_1}{i}a\right] \\
&= \left[e + x_0e + \frac{1}{2!}(x_0e)^2 + \dots\right] exp\left[i\frac{x_1}{i}a\right] \\
&= e\left[1 + x_0 + \frac{1}{2!} + \dots\right] exp\left[i\frac{x_1}{i}a\right] \\
&= e \cdot exp(x_0)exp\left[i\frac{x_1}{i}a\right] \\
&= e \cdot exp(x_0)\left[e + i\frac{x_1}{i}a - \frac{1}{2!}\left(\frac{x_1}{i}a\right)^2 - i\frac{1}{3!}\left(\frac{x_1}{i}a\right)^3 + \dots\right] \\
&= e \cdot exp(x_0)\left[e\left[1 - \frac{1}{2!}\left(\frac{x_1}{i}\right)^2 + \frac{1}{4!}\left(\frac{x_1}{i}\right)^4 + \dots\right] + a\left[i\left(\frac{x_1}{i}\right) - i\left(\frac{x_1}{i}\right)^3 + i\left(\frac{x_1}{i}\right)^5 + \dots\right]\right] \\
&= e \cdot exp(x_0)\left[e \cdot cos\left(\frac{x_1}{i}\right) + ia \cdot sin\left(\frac{x_1}{i}\right)\right] \\
&= exp(x_0)cos\left(\frac{x}{i}\right) \cdot e + iexp(x_0)sin\left(\frac{x_1}{i}\right) \cdot a
\end{aligned}$$

Group Theory Homework IV

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2024 年 1 月 7 日

Problem 9

Solution:

D_4 群的乘法表为：（其中 t 为对角线二次轴，σ 为对边二次轴）

	e	σ_1	σ_2	t_1	t_2	τ	τ^2	τ^3
e	e	σ_1	σ_2	t_1	t_2	τ	τ^2	τ^3
σ_1	σ_1	e	τ^2	τ	τ^3	t_1	σ_2	t_2
σ_2	σ_2	τ^2	e	τ^3	τ	t_2	σ_1	t_1
t_1	t_1	τ^3	τ	e	τ^2	σ_2	t_2	σ_1
t_2	t_2	τ	τ^3	τ^2	e	σ_1	t_1	σ_2
τ	τ	t_2	t_1	σ_1	σ_2	τ^2	τ^3	e
τ^2	τ^2	σ_2	σ_1	t_2	t_1	τ^3	e	τ
τ^3	τ^3	t_1	t_2	σ_2	σ_1	e	τ	τ^2

根据乘法表可以看出 D_4 群的共轭类为：$\{e\}\{\tau^2\}\{\sigma_1,\sigma_2\}\{t_1,t_2\}\{\tau,\tau^3\}$

Problem 10

Proof:

充分性：

记 $C = AB = \{c = ab|a \in A, b \in B\}$　$D = BA = \{d = ba|a \in A, b \in B\}$, 其中 $C \leq G$，下证 $C = D$：

因为 A, B, C 均是 G 的子群，所以 A, B, C 都满足封闭性、结合律、含幺元和含逆元。

由于 C 中含有幺元 e，而 $C = AB$，所以 $eA = Ae \subseteq C$　$eB = Be \subseteq C$，即：

$$\forall a \in A, \forall b \in B \implies a \in C, b \in C$$

同时，由于群乘法的封闭性，C 中任意两元素相乘结果任在 C 中，所以：

$$\forall a \in A \subseteq C, \forall b \in B \subseteq C \implies ab \in C, ba \in C$$

由于 $D = BA = \{d = ba|a \in A, b \in B\}$, 所以 $D \subseteq C$, 同理 $C \subseteq D$，因此 $C = D$，即 $AB = BA$.

必要性:

记 $C = AB = \{c = ab|a \in A, b \in B\} \quad D = BA = \{d = ba|a \in A, b \in B\}$, 其中 $C = D$，下证 $C < G$：

1. 封闭性：

$$\begin{aligned}
&\forall c_1, c_2 \in C,\ c_1c_2 = a_1b_1a_2b_2 \\
&\because b_1 \in B, a_2 \in A \\
&\therefore b_1a_2 \in D = BA \\
&\because D = BA = AB = C \\
&\therefore \exists c_3 \in C, c_3 = a_3b_3 = b_1a_2 \\
&\implies c_1c_2 = a_1b_1a_2b_2 = a_1a_3b_3b_2 \\
&\because a_1a_3 = a_s \in A, b_3b_2 = bs \in B \\
&\therefore c_s = a_sb_s \in C \\
&\implies \forall c_1, c_2 \in C,\ c_1c_2 \in C
\end{aligned}$$

2. 结合律：显然，由于 A, B 均为 G 的子群，所以由 AB 组成的 C 中的所有元素继承了 G 中的结合律。

3. 含幺：由于 A, B 中都含有幺元 e, 所以 C 也存在幺元 e。

4. 含逆：

$$\begin{aligned}
\forall c \in C,\ c = ab,\ \exists c^{-1} \text{ such that } cc^{-1} = e \\
\implies c^{-1} = b^{-1}a^{-1}
\end{aligned}$$

因此 C 是 G 的一个子群

Problem 11

Proof:

记 $R = \{r\}$，$T = \{t\} = R^{-1} = \{r^{-1}\}$

已知 R 为子群 A 右陪集代表元系，即：

$$\forall r \in R \implies r \notin A$$

显然，$T = R^{-1} = r^{-1}$ 与 A 交集为空，并且由于陪集中存在公共元素则陪集相同，所以

$$\forall r_i, r_j \in R\ \forall a_i, a_j \in A \implies a_ir_i \neq a_jr_j$$

为了证明 T 是 A 的左陪集代表元系，即要求：

$$\forall t_i, t_j \in T\ \forall a_i, a_j \in A \implies t_ia_i \neq t_ja_j$$

不妨假设 $\exists t_i, t_j \in T\ \exists a_i, a_j \in A$ 使 $t_ia_i = t_ja_j$，即

$$r_i^{-1}a_i = r_j^{-1}a_j$$

由于 $r_i, r_j, r_i^{-1}, r_j^{-1}$ 以及 A 中所有元素都在群 G 中，所以

$$\begin{aligned} &\Longrightarrow r_j r_i^{-1} a_i = a_j \\ \forall a_k \in A, \quad & a_k r_j r_i^{-1} a_i = a_k a_j \\ &\Longrightarrow a_k r_j r_i^{-1} = a_k a_j a_i^{-1} \\ &\Longrightarrow a_k r_j = a_k a_j a_i^{-1} r_i \end{aligned}$$

由于群的封闭性，因此上式即：

$$\exists a_x, a_y in A \ \exists r_i, r_j \in R \implies a_x r_j = a_y r_i$$

显然与已知矛盾，所以不存在 $t_i, t_j \in T$ 使 $t_i A = t_j A$，即 T 是子群 A 的一种左陪集代表元系

Group Theory Homework V

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January 7, 2024

Problem 12

Proof:

$$\forall (a, b) \in G \ (a, b)(a, b) = (a^2, ab + b) = (a, b) \implies e = (1, 0)$$

$$\forall (a, b) \in G \ (a, b)(c, d) = (1, 0) \implies (a, b)^{-1} = (c, d) = \left(\frac{1}{a}, -\frac{b}{a}\right)$$

First, $\forall (1, b_1), (1, b_2) \in K \implies (1, b_1)(1, b_2) = (1, b_1)(1, b_2) = (1, b_1 + b_2) \in K$

K is a Group

Furthermore,

$$\forall (x, y) \in G \ \forall (1, b) \in K, \ (x, y)(1, b)(x, y)^{-1} = (x, y)(1, b)\left(\frac{1}{x}, -\frac{y}{x}\right) = (1, xb) \in K$$

Therefore, K is a Invariant Subgroup of G(also Normal Subgroup).

The Quotient Group is given by $G/K = \{gK | \ \forall g \in G\}$

Construct a mapping as $f : G/K \to \mathbb{R}^*$,

$\forall A, B \in \mathbb{R}^*$, $\forall (x, y), (x', y') \in G$, we have mapping:

$$\begin{cases} A \to (x, y)K \\ B \to (x', y')K \end{cases}$$

$$f(AB) = f(A)f(B) = (x, y)K(x', y')K = (x, y)(x', y')KK = (xx', xy' + y)K = f(C)$$

$$(xx', xy' + y)K = (x'', y'')K \in G/K \text{ and } C \in R^*$$

Similarly, the reverse is also correct: $\varphi : \mathbb{R}^* \to G/K$.

Therefore, $G/K \cong R^*$

Problem 13

Proof:

Note that $M = (R^+, \times),\ N = (R, +)$

Construct a mapping as $f : M \to N$

$$\forall a, b \in R^+ \quad x, y \in R \implies \begin{cases} a \to x \\ \\ b \to y \end{cases}$$

$$f(a \times b) = f(a) + f(b) = x + y \in R$$

Therefore, $M \xrightarrow{f} N$

Similarly, construct a mapping as $\varphi : N \to M$

$$\forall a, b \in R^+ \quad x, y \in R \implies \begin{cases} x \to a \\ \\ y \to b \end{cases}$$

$$\varphi(x + y) = \varphi(x)\varphi(y) = ab \in R^*$$

Therefore, $N \xrightarrow{\varphi} M$

In conclusion, $M \cong N$

Problem 14

Proof:

Suppose the order of $g \in G$ is n, which means $g^n = e_G$

Assume $f(g) = k \in H$,

$$f(e_G) = f(g^n) = f(g)^n = k^n = e_H$$

Suppose the order of $k \in H$ is m ($k^m = e_H$), we have to prove that $n\ mod\ m = 0$

We can suppose $n\ mod\ m \neq 0$, which means:

$$\begin{cases} n\ /\ m = i \\ \\ n\ mod\ m = j \end{cases} \qquad i, j \in \mathbb{N}$$

Obviously,

$$k^n = e_H \implies k^{n-im}(k^m)^i = e_H \implies k^j(e_H)^i = e_H \implies k^j = e_H$$

Which means j is the order of $k \in H$ instead of m

This contradicts the assumption, so $n \ / \ m = q \in \mathbb{N}$.

Group Theory Homework VI

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Problem 14

Proof:

$$\forall g_1, g_2 \in G, \pi(g_1 g_2) = \pi(g_1)\pi(g_2) = g_1 N g_2 N$$

$$\because N \lhd G \implies gN = Ng$$

$$\therefore g_1 N g_2 N = g_1 g_2 N N = g_3 N \in G/N$$

Therefore, this mapping is Homomorphic Mapping.

Since $ker\pi = g|\pi(g) = 1_\pi \ \forall g \in G$, if $g \in ker\pi$, it will satisfy

$$\forall g \in G, \pi(gg) = \pi(g)\pi(g) = \pi(g)$$

$$\implies gNgN = gN$$

$$\implies ggNN = ggN = gN$$

Case 1: $g \in N$

$$ggN = N = gN$$

Case 2: $g \notin N$

$$ggN = gN \implies gg = g \implies g = e \in N$$

Therefore, $ker\pi = N$

Problem 15

(1) **Proof**:

$$\forall A_1, A_2 \in SL(2,\mathbb{C}),\ A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\ A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

We have

$$A_3 = A_1A_2 = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

According to Mobius Trasform,

$$\hat{A}_3 z = \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}$$

On the other hand

$$\begin{aligned}\hat{A}_1\hat{A}_2 z = \hat{A}_1\frac{a_2z + b_2}{c_2z + d_2} &= \frac{a_1\frac{a_2z+b_2}{c_2z+d_2} + b_1}{c_1\frac{a_2z+b_2}{c_2z+d_2} + d_1} \\ &= \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}\end{aligned}$$

Therefore,

$$\text{Mobius}(A_1A_2) = \hat{A}_3 = \hat{A}_1\hat{A}_2$$

This mapping is Homomorphic Mapping.

(2) **Solution**: Since $ker\,\hat{} = \{A|\hat{A} = 1\hat{}\ \forall A \in SL(2,\mathbb{C})\}$, if $A \in ker\,\hat{}$, it will satisfy

$$\forall A \in SL(2,\mathbb{C}) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \hat{A}\hat{A}z = \hat{A}z = z$$

Therefore,

$$ker\,\hat{} = \left\{ A \middle| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Problem 16

(1) **Proof**:

Note that $a = a_0 + a_1i_1 + a_2i_2 + a_3i_3 \quad b = b_0 + b_1i_1 + b_2i_2 + b_3i_3$

$$\forall a, b \in \mathbb{H},\ a + b = (a_0 + b_0) + (a_1 + b_1)i_1 + (a_2 + b_2)i_2 + (a_3 + b_3)i_3 = c$$

$$\begin{aligned}\varphi(a+b) = \varphi(a) + \varphi(b) &= \begin{pmatrix} a_0 - a_3i & -a_2 - a_1i \\ a_2 - a_1i & a_0 + a_3i \end{pmatrix} + \begin{pmatrix} b_0 - b_3i & -b_2 - b_1i \\ b_2 - b_1i & b_0 + b_3i \end{pmatrix} \\ &= \begin{pmatrix} (a_0 + b_0) - (a_3 + b_3)i & -(a_2 + b_2) - (a_1 + b_1)i \\ (a_2 + b_2) - (a_1 + b_1)i & (a_0 + b_0) + (a_3 + b_3)i \end{pmatrix} = \varphi(c)\end{aligned}$$

Furthermore,

$$\forall a,b\in\mathbb{H},\ d=ab=(a_0+a_1i_1+a_2i_2+a_3i_3)(b_0+b_1i_1+b_2i_2+b_3i_3)$$

$$=(a_0b_0-a_1b_1-a_2b_2-a_3b_3)+i_1(a_1b_0+a_0b_1+a_2b_3-a_3b_2)+i_2(a_2b_0+a_0b_2+a_3b_1-a_1b_3)+i_3(a_3b_0+a_0b_3+a_1b_2-a_2b_1)$$

$$\varphi(ab)=\varphi(a)\varphi(b)=\begin{pmatrix}a_0-a_3i & -a_2-a_1i\\ a_2-a_1i & a_0+a_3i\end{pmatrix}\begin{pmatrix}b_0-b_3i & -b_2-b_1i\\ b_2-b_1i & b_0+b_3i\end{pmatrix}$$

$$=\begin{pmatrix}(a_0b_0-a_1b_1-a_2b_2-a_3b_3)-(a_3b_0+a_0b_3+a_1b_2-a_2b_1)i & -(a_2b_0+a_0b_2+a_3b_1-a_1b_3)-(a_1b_0+a_0b_1+a_2b_3-a_3b_2)i\\ (a_2b_0+a_0b_2+a_3b_1-a_1b_3)-(a_1b_0+a_0b_1+a_2b_3-a_3b_2)i & (a_0b_0-a_1b_1-a_2b_2-a_3b_3)+(a_3b_0+a_0b_3+a_1b_2-a_2b_1)i\end{pmatrix}$$

$$=\phi(d)$$

This mapping is Homomorphic Mapping.

(2) **Proof**:

$$\varphi(a)=\begin{pmatrix}a_0-a_3i & -a_2-a_1i\\ a_2-a_1i & a_0+a_3i\end{pmatrix}\implies det(\varphi(a))=\begin{vmatrix}a_0-a_3i & -a_2-a_1i\\ a_2-a_1i & a_0+a_3i\end{vmatrix}=a_0^2+a_1^2+a_2^2+a_3^2$$

Obviously, $|a|=det(\varphi(a))$

$$\forall a,b\in Q,\ ab=c$$

$$|c|=|a||b|=det(\varphi(a))det(\varphi(b))$$

$$\because det(\varphi(a))=det(\varphi(b))=1$$

$$\therefore |c|=det(\varphi(a))det(\varphi(b))=1$$

$$\implies c\in Q$$

Furthermore,

$$\begin{cases}\forall a,b,c\in Q,(ab)c=a(bc)=abc\in Q\\ e=\frac{1}{2}(1-i_1-i_2-i_3)\end{cases}$$

Therefore, Q is a Group.

(3) **Proof**:

Note that $a\to A\in\varphi(Q)$

Obviously, $|a|=det(\varphi(a))=det\ A=1$ (Proved in (2))

$$A=\begin{pmatrix}a_0-a_3i & -a_2-a_1i\\ a_2-a_1i & a_0+a_3i\end{pmatrix}\quad A^\dagger=\begin{pmatrix}a_0+a_3i & a_2+a_1i\\ -a_2+a_1i & a_0-a_3i\end{pmatrix}\implies A^\dagger A=\begin{pmatrix}a_0^2+a_1^2+a_2^2+a_3^2 & (a_0a_2-a_0a_2)-(a_1a_3-a_1a_3)i\\ (a_0a_2-a_0a_2)-(a_1a_3-a_1a_3)i & a_0^2+a_1^2+a_2^2+a_3^2\end{pmatrix}=I^{2\times2}$$

Group Theory Homework VII

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Problem 17

D_8 群的展示为：$D_8 = \langle \sigma, \tau | \tau^8 = e, \sigma^2 = e, \sigma\tau\sigma^{-1} = \tau^{-1} \rangle$，作用空间大小为 3^8 种情况

$$\begin{cases} \text{Case 1:} & e & x^e = x & |x^e| = 3^8 \\ \text{Case 2:} & \tau, \tau^7 & x^\tau = x & |x^\tau| = 3 \\ \text{Case 3:} & \tau^2, \tau^6 & x^{\tau^2} = x & |x^{\tau^2}| = 3^2 \\ \text{Case 4:} & \tau^3, \tau^5 & x^{\tau^3} = x & |x^{\tau^3}| = 3^3 \\ \text{Case 5:} & \tau^4 & x^{\tau^4} = x & |x^{\tau^4}| = 3^4 \\ \text{Case 6:} & \sigma, \tau^2\sigma, \tau^4\sigma, \tau^6\sigma & x^\sigma = x & |x^\sigma| = 3^4 \\ \text{Case 7:} & \tau\sigma, \tau^3\sigma, \tau^5\sigma, \tau^7\sigma & x^{\tau\sigma} = x & |x^{\tau\sigma}| = 3^5 \end{cases}$$

因此共计不同种手链数目为 $|X/G| = \dfrac{1}{16}(3^8 + 3 + 3^2 + 3^3 + 3^4 + 3^4 + 3^5) = 501$ 种

Problem 18

(1) Proof:

记仿射变换 $\mathcal{F} : (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2)$

$$\forall \begin{cases} \mathcal{F} : (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) \in \mathbb{R} \\ \mathcal{F}' : (\alpha_1', \alpha_2', \alpha_3', \alpha_4', \beta_1', \beta_2') \in \mathbb{R} \end{cases} \quad \mathcal{F}\mathcal{F}' \begin{pmatrix} x \\ y \end{pmatrix} = \mathcal{F} \left[\begin{pmatrix} \alpha_1' & \alpha_2' \\ \alpha_3' & \alpha_4' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \beta_1' \\ \beta_2' \end{pmatrix} \right]$$

$$= \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} \alpha_1' & \alpha_2' \\ \alpha_3' & \alpha_4' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} \beta_1' \\ \beta_2' \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

显然我们有

$$\exists \ \mathcal{F}'' : (\alpha_1'', \alpha_2'', \alpha_3'', \alpha_4'', \beta_1'', \beta_2'') \in \mathbb{R} \implies \begin{cases} \alpha_1'' = \alpha_1\alpha_1' + \alpha_2\alpha_3' \\ \alpha_2'' = \alpha_1\alpha_2' + \alpha_2\alpha_4' \\ \alpha_3'' = \alpha_3\alpha_1' + \alpha_4\alpha_3' \\ \alpha_4'' = \alpha_3\alpha_2' + \alpha_4\alpha_4' \\ \beta_1'' = \alpha_1\beta_1' + \alpha_2\beta_2' + \beta_1 \\ \beta_2'' = \alpha_3\beta_1' + \alpha_4\beta_2' + \beta_2 \end{cases}$$

满足群封闭性，同时显然满足结合律，单位元为 $\mathcal{F}_I = (1\ 0\ 0\ 1\ 0\ 0)$，有逆，因而 $\forall\mathcal{F}$ 构成群, 记为 $\mathrm{Aff}(2,\mathbb{R})$

(2) Proof:

对于任意出现在 κ 中的分子分母元素

$$\langle a,b,c\rangle = \begin{pmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ 1 & 1 & 1 \end{pmatrix}$$

我们有：

$$\begin{vmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x_a & x_b \\ y_a & y_b \end{vmatrix} + \begin{vmatrix} x_b & x_c \\ y_b & y_c \end{vmatrix} + \begin{vmatrix} x_c & x_a \\ y_c & y_a \end{vmatrix}$$

对于任意的仿射变换 $\mathcal{F} \in \mathrm{Aff}(2,\mathbb{R})$，我们有

$$\begin{pmatrix} x'_a & x'_b \\ y'_a & y'_b \end{pmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \begin{pmatrix} x_a & x_b \\ y_a & y_b \end{pmatrix} + \begin{bmatrix} \beta_1 & \beta_1 \\ \beta_2 & \beta_2 \end{bmatrix}$$

取模

$$\Longrightarrow \begin{vmatrix} x'_a & x'_b \\ y'_a & y'_b \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{vmatrix} \begin{vmatrix} x_a & x_b \\ y_a & y_b \end{vmatrix}$$

如果将转动变换阵 α_i 记为 A，那么上式可写为 $\langle a,b,c\rangle' = |A|\langle a,b,c\rangle$

于是

$$\kappa' = \frac{A\langle 125\rangle A\langle 134\rangle}{A\langle 124\rangle A\langle 135\rangle} = \frac{\langle 125\rangle\langle 134\rangle}{\langle 124\rangle\langle 135\rangle} = \kappa$$

即 κ 为投影不变量

(3) Solution:

在 S_5 群作用下，显然只存在四种置换使投影不变量不发生变化，分别为

$$\begin{cases} \text{单位元 e：} & (1)(2)(3)(4)(5) \\ \text{25 对换，34 对换} & (1)(25)(34) \\ \text{23 对换，45 对换} & (1)(23)(45) \\ \text{24 对换，35 对换} & (1)(24)(35) \end{cases}$$

因此 κ 在 S_5 作用下的小群即包含以上四个元素

(4) Solution:

S_5 群阶为 $5! = 120$，κ 在 S_5 群作用下的小群群阶为 4，根据轨道公式，κ 在 S_5 群作用下不重复的元素个数 $|S_5^\kappa| = \dfrac{|S_5|}{|[\kappa]|} = 30$ 个

Problem 19

(1) Solution:

$$s = \begin{pmatrix} 2 & 3 & 5 & 4 & 6 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

(2) Solution:

$$s = (1\ 6\ 5\ 3\ 2)(4)$$

这是一个长度为 5 的轮换乘一个 1 轮换，是偶置换

(3) Solution:

长度为 5 的轮换，阶为 5.

(4) Solution:

$$s = (1\ 6)(2\ 1)(5\ 3)(3\ 2)(2\ 1)(4)$$

Problem 20

Solution:

$$\begin{aligned} s =& (1\ 2\ 3)(2\ 5\ 3\ 4) \\ =& (1\ 2)(2\ 3)(2\ 5\ 3)(3\ 4) \\ =& (1\ 2)(2\ 3)(3\ 2\ 5)(3\ 4) \\ =& (1\ 2)(2\ 3)(3\ 2)(2\ 5)(3\ 4) \\ =& (1\ 2)(2\ 5)(3\ 4) \\ =& (1\ 2\ 5)(3\ 4) \end{aligned}$$

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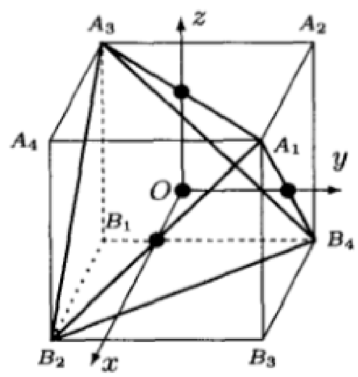
Problem 21

(1) Solution:



如图所示，正四面体共有 7 个对称轴：其中坐标轴 x, y, z 分别为 3 个二次轴，记为 t_x, t_y, t_z，正四面体的四根体对角线为分别为 4 个三次轴，记为 r_1, r_2, r_3, r_4

(2) Solution:

T 群共 12 个元素，其中包括恒元 1 个元素，3 个二次轴给出 3 个元素，4 个三次轴给出 8 个元素。

$$T = \{e, t_x, t_y, t_z, r_1, r_2, r_3, r_4, r_1^2, r_2^2, r_3^2, r_4^2\}$$

其表示同构于 S_4 群的一个子群，表示为：

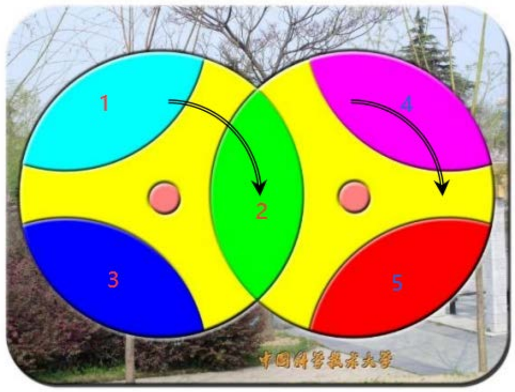
t_x	(1 4)(2 3)	t_y	(1 2)(3 4)	t_z	(1 3)(2 4)		
r_1	(1)(2 3 4)	r_2	(2)(1 3 4)	r_3	(3)(1 2 4)	r_4	(4)(1 2 3)
r_1^2	(1)(2 4 3)	r_2^2	(2)(1 4 3)	r_3^2	(3)(1 4 2)	r_4^2	(4)(1 3 2)

(3) Solution:

根据置换群的共轭类性质可知，相同轮换结构的置换属于同一个共轭类，因此 T 群共 3 个类，其中恒元自成一类 $\{e\}$，三个二次轴一类 $\{t_x, t_y, t_z\}$，四个三次轴给出的置换为一类 $\{r_1, r_2, r_3, r_4, r_1^2, r_2^2, r_3^2, r_4^2\}$

Problem 22

如图所示，给小盘标上序号：
绕左轴顺时针旋转 $\frac{2}{3}\pi$ 操作记为 $l(l^3=e)$，右轴同理记为 $r(r^3=e)$



根据轨道定理，保持 1 位置不动的小群为 e, r, r^2，在 S_5 子群作用下轨道数为 5，因此玩具小盘满足的对成群 W 含有 15 个群元

e	(1)(2)(3)(4)(5)	l	(1 3 2)	l^2	(1 2 3)	r	(2 5 4)	r^2	(2 4 5)
lr	(1 3 2 5 4)	lr^2	(1 3 2 4 5)	l^2r	(3 1 2 5 4)	l^2r^2	(3 1 2 4 5)		
rl	(5 4 2 1 3)	rl^2	(5 4 2 3 1)	r^2l	(4 5 2 1 3)	r^2l^2	(4 5 2 3 1)		
rlr	(1 3 5 2 4)	lrl	(5 4 1 2 3)						

左侧图样可以通过右转 $4/3\pi$，再左转 $2/3\pi$ 得到
右侧图样存在 1、3 色块的镜像操作，所以无法通过旋转得到

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Problem 23

D_4 群的群元包括:$\{e, r, r^2, r^3, a, ar, ar^2, ar^3\}$
$R \otimes M$ 群元为:$\{(e,e)(r,e)(r^2,e)(r^3,e)(e,a)(r,a)(r^2,a)(r^3,a)\}$
显然 $R \otimes M$ 与 D_4 拥有相同的群阶，但乘法无法做到同构映射，因此 D_4 群并不同构于 $R \otimes M$
我们对 R 群建立以 M 群元素为基础的自同构映射:

$$\begin{cases} \hat{e}e \implies e & \hat{a}e \implies e \\ \hat{e}r \implies r & \hat{a}r \implies r^3 \\ \hat{e}r^2 \implies r^2 & \hat{a}r^2 \implies r^2 \\ \hat{e}r^3 \implies r & \hat{a}r^3 \implies r \end{cases}$$

考虑乘法：

$$\forall (k_1, h_1), (k_2, h_2) \in R \otimes M,\ (k_1, h_1)(k_2, h_2) = (k_1 \hat{h_1} k_2, h_1 h_2)$$

满足该乘法关系的半直积 $R \otimes_S M$，与 D_4 群同构，且满足一一映射关系：

$$\begin{matrix} e \leftrightarrow (e,e) & r \leftrightarrow (r,e) & r^2 \leftrightarrow (r^2,e) & r^3 \leftrightarrow (r^3,e) \\ a \leftrightarrow (e,a) & ar \leftrightarrow (r,a) & ar^2 \leftrightarrow (r^2,a) & ar^3 \leftrightarrow (r^3,a) \end{matrix}$$

例如：$(r,a)(e,a) = (e\hat{a}r, a^2) = (r^3, e) \leftrightarrow ara = r^3$

Problem 24

$$675 = 3 \times 3 \times 3 \times 5 \times 5$$

所有的 675 阶 Abel 群包括：

$$\begin{cases} \mathbb{Z}_5^2 \otimes \mathbb{Z}_3^3 & \mathbb{Z}_{25} \otimes \mathbb{Z}_3^3 \\ \mathbb{Z}_5^2 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_9 & \mathbb{Z}_{25} \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_9 \\ \mathbb{Z}_5^2 \otimes \mathbb{Z}_{27} & \mathbb{Z}_{25} \otimes \mathbb{Z}_{27} \end{cases}$$

Problem 1

(1) $detT(G)$

$$detT(gh) = det(T(g)T(h)) = det(T(g))det(T(h))$$

是线性表示

(2) $trT(G)$

$$trT(gh) = tr(T(g)T(h)) \neq tr(T(g))tr(T(h))$$

不是线性表示

(3) $T^\dagger(G)$

$$T^\dagger(gh) = (T(g)T(h))^\dagger = T^\dagger(h)T^\dagger(g) \neq T^\dagger(g)T^\dagger(h)$$

不是线性表示

(4) $T^T(G)$ **转置**

$$T^T(gh) = (T(g)T(h))^T = T^T(h)T^T(g) \neq T^T(g)T^T(h)$$

不是线性表示

(5) $T^*(G)$

$$T^*(gh) = T^*(g)T(h) = T^*(g)T^*(h)$$

是线性表示

(6) $T^{-1}(G)$

$$T^{-1}(gh) = (T(g)T(h))^{-1} = T^{-1}(h)T^{-1}(g) \neq T^{-1}(g)T^{-1}(h)$$

不是线性表示

(7) $T^{-T}(G)$ **转置逆**

$$T^{-T}(gh) = (T(g)T(h))^{-T} = T^{-T}(g)T^{-T}(h)$$

是线性表示

(8) $T^{*-1}(G)$

$$T^{*-1}(gh) = (T^*(g)T^*(h))^{-1} = T^{*-1}(h)T^{*-1}(g) \neq T^{*-1}(g)T^{*-1}(h)$$

不是线性表示

Problem 2

(1) Solution:

对于 $g(\theta)f(\boldsymbol{x}) = f(g^{-1}(\theta)\boldsymbol{x}) = f(g(-\theta)\boldsymbol{x})$

$$\begin{cases} x' = xcos\theta + ysin\theta \\ y' = -xsin\theta + ysin\theta \end{cases} \implies \begin{cases} x'^2 = x^2cos^2\theta + y^2sin^2\theta + xysin2\theta \\ y'^2 = x^2sin^2\theta + y^2cos^2\theta - xysin2\theta \\ x'y' = -x^2sin\theta cos\theta + y^2sin\theta cos\theta + xycos2\theta \end{cases}$$

因此 $g(\theta)$ 在三维空间中的线性表示为：

$$T(g(\theta)) = \begin{pmatrix} cos^2\theta & sin^2\theta & -sin\theta cos\theta \\ sin^2\theta & cos^2\theta & sin\theta cos\theta \\ sin2\theta & -sin2\theta & cos2\theta \end{pmatrix}$$

同理对于 P 有三维空间线性表示：

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(2) Solution:

$$\begin{cases} T^\dagger(g(\theta))T(g(\theta)) \neq I^{3\times3} \\ T_{ij} \in \mathbb{R} \\ \text{由于}sin\text{函数和}cos\text{函数具有周期性，因此为同态} \end{cases}$$

$Tg(\theta)$ 不是幺正表示，是实表示，不是忠实表示。

类似地

$$\begin{cases} P\dagger P = I^{3\times3} \\ P_{ij} \in \mathbb{R} \\ \text{镜像对称与}P\text{线性表示唯一对应，为同构} \end{cases}$$

P 是幺正表示，是实表示，是忠实表示。

Group Theory Homework X

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Problem 3

Proof:
乘法封闭：

$$\begin{aligned}\forall X, Y \in SU(2), \quad x * y = [X, Y] = XY - YX &= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j s_i s_j - \sum_{i=1}^{3}\sum_{j=1}^{3} y_i x_j s_i s_j \\ &= 2i(x_2y_3 - x_3y_2)s_1 + 2i(x_3y_1 - x_1y_3)s_2 + 2i(x_1y_2 - x_2y_1)s_3 \\ &= z_1x_1 + z_2x_2 + z_3x_3 = Z \in SU(2)\end{aligned}$$

分配律：

$$\forall X, Y, Z \in SU(2), \quad \begin{cases}[X, Y+Z] = [X, Y] + [X, Z] \\ [X+Y, Z] = [X, Z] + [Y, Z]\end{cases}$$

数乘：

$$\forall X, Y \in SU(2),\ \alpha \in \mathbb{C}, \quad \alpha[X, Y] = [\alpha X, Y] = [X, \alpha Y]$$

所以 $SU(2)$ 代数是线性代数

Problem 4

(1) Solution:

$$\because \forall \{i, j\} \in \{1, 2\}, \{\theta_i, \theta_j\} = \theta_i\theta_j + \theta_j\theta_i = 0$$

$$\therefore \begin{cases}\theta_1^2 = \theta_2^2 = 0 \\ \theta_1\theta_2 = -\theta_2\theta_1\end{cases}$$

$$\begin{aligned}\implies xy &= (x_0 + x_1\theta_1 + x_2\theta_2 + x_3\theta_1\theta_2)(y_0 + y_1\theta_1 + y_2\theta_2 + y_3\theta_1\theta_2) \\ &= x_0y_0 + (x_1y_0 + x_0y_1)\theta_1 + (x_2y_0 + x_0y_2)\theta_2 + (x_3y_0 + x_0y_3 + x_1y_2 - x_2y_1)\theta_1\theta_2 \\ &= z \in Grassmann\end{aligned}$$

(2) Solution:

在 (1) 中已证 Grassmann 代数对乘法封闭，同样有：

$$\begin{cases} \forall x, y, z \in Grassmann \quad x(y+z) = xy + xz \\ \forall x, y \in Grassmann, \forall \alpha \in \mathbb{C} \quad \alpha xy = (\alpha x)y = x(\alpha y) \end{cases}$$

说明 Grassmann 代数是线性代数，此外

$$\begin{aligned} \forall x, y, z \in Grassmann, (xy)z = x(yz) &= x_0y_0z_0 + (x_0y_0z_1 + x_1y_0z_0 + x_0y_1z_0)\theta_1 \\ &+ (x_0y_0z_2 + x_2y_0z_0 + x_0y_2z_0)\theta_2 \\ &+ (x_3y_0z_0 + x_0y_3z_0 + x_0y_0z_3 + x_1y_0z_2 + x_0y_1z_2 + x_1y_2z_0 - x_2y_1z_0 - x_2y_0z_1 - x_0y_2z_1)\theta_1\theta_2 \end{aligned}$$

满足结合律，因此 Grassmann 代数为结合代数

Problem 5

Suppose

$$(e, P, T, PT) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We have

$$\begin{cases} e(e, P, T, PT) = e(e, P, T, PT) \\ P(e, P, T, PT) = (P, e, PT, T) \\ T(e, P, T, PT) = (T, PT, e, P) \\ PT(e, P, T, PT) = (PT, T, P, e) \end{cases} \implies \begin{cases} \hat{L}(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \hat{L}(P) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \hat{L}(T) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \hat{L}(PT) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

Group Theory Homework XI

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Problem 6

$SU(3)$ 群的中心定义为 $C_{SU(3)} = \{g \in SU(3) | \forall a \in SU(3),\ gag^{-1} = a\}$

$$gag^{-1} = a \implies ga = ag$$

根据舒尔引理 1

$$g = \lambda I_{3\times3}$$

$$\begin{cases} g^\dagger g = I_{3\times3} \\ |g| = 1 \end{cases} \implies \lambda^3 = 1 \implies C_{SU(3)} = I_{3\times3}\{1, e^{-i\frac{2}{3}\pi}, e^{i\frac{2}{3}\pi}\}$$

Problem 7

(1) Solution:

$$\varphi(\theta) = \begin{cases} k\theta & k\theta \in [0, 2\pi) \\ k\theta \ mod\ 2\pi & k\theta \in [2\pi, +\infty) \end{cases} (k \in \mathbb{Z})$$

(2) Solution:

$$f(\theta) = \int_0^{2\pi} C(\alpha) e^{i\varphi(\theta)} d\alpha$$

(3) Solution:

$$C(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) e^{-i\varphi(\theta)} d\theta$$

Problem 8

8 的因子有 $1, 2, 4, 8$

$$\begin{cases} 8 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 & Z_8\text{群} \\ 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 & D_4\text{群} \\ 8 = 2^2 + 2^2 & \text{无对应群 (必有恒等表示)} \end{cases}$$

共有 2 种可能的群，其中 Z_8 群有 8 个一维不等价不可约表示，D_4 群有 4 个一维和 1 个二维不等价不可约表示

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Problem 9:

Solution:

(1)

$$S(e)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} S(d)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} S(f)=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

$$S(a)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} S(b)=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} S(c)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

(2)

$$\chi(e)=3 \quad \chi(d)=0 \quad \chi(f)=0 \quad \chi(a)=1 \quad \chi(b)=1 \quad \chi(c)=1$$

(3)

$$\chi(e)=3 \quad \chi(d)=0 \quad \chi(f)=0 \quad \chi(a)=-1 \quad \chi(b)=-1 \quad \chi(c)=-1$$

(4)

根据等价表示理论，$S(D_3)$和$T(D_3)$无法通过相似变换相互得到，所以线性表示并不等价

(5)

根据特征标定理，$S(D_3)$和$T(D_3)$并不等价

(6)

一个表示是不可约表示的充要条件是其特征标满足方程：

$$\sum_{R\in G}\chi^*(R)\chi(R)=|G|$$

显然$S(D_3)$和$T(D_3)$并不满足，因而是可约表示

(7)

$S(D_3)$包含了 1 个一维表示和 1 个二维表示。

Problem 10

Solution:

D_4群群元记为$\{e,\tau,\tau^2,\tau^3,\sigma_1,\sigma_2,t_1,t_2\}$

共轭类为$\{e\}\{\tau,\tau^3\}\{\tau^2\}\{\sigma_1,\sigma_2\}\{t_1,t_2\}$

类算符：

$$\hat{k}_1=e$$

$$\hat{k}_2=\tau+\tau^3$$

$$\hat{k}_3 = \tau^2$$

$$\hat{k}_4 = \sigma_1 + \sigma_2$$

$$\hat{k}_5 = t_1 + t_2$$

类代数的结构常数：

$$\hat{k}_1\hat{k}_1 = \hat{k}_1 \quad \hat{k}_1\hat{k}_2 = \hat{k}_2 \quad \hat{k}_1\hat{k}_3 = \hat{k}_3 \quad \hat{k}_1\hat{k}_4 = \hat{k}_4 \quad \hat{k}_1\hat{k}_5 = \hat{k}_5$$

$$\hat{k}_2\hat{k}_1 = \hat{k}_2 \quad \hat{k}_2\hat{k}_2 = 2\hat{k}_1 + 2\hat{k}_3 \quad \hat{k}_2\hat{k}_3 = \hat{k}_2 \quad \hat{k}_2\hat{k}_4 = 2\hat{k}_5 \quad \hat{k}_2\hat{k}_5 = 2\hat{k}_4$$

$$\hat{k}_3\hat{k}_1 = \hat{k}_3 \quad \hat{k}_3\hat{k}_2 = \hat{k}_2 \quad \hat{k}_3\hat{k}_3 = \hat{k}_1 \quad \hat{k}_3\hat{k}_4 = \hat{k}_4 \quad \hat{k}_3\hat{k}_5 = \hat{k}_5$$

$$\hat{k}_4\hat{k}_1 = \hat{k}_4 \quad \hat{k}_4\hat{k}_2 = 2\hat{k}_5 \quad \hat{k}_4\hat{k}_3 = \hat{k}_4 \quad \hat{k}_4\hat{k}_4 = 2\hat{k}_1 + 2\hat{k}_3 \quad \hat{k}_4\hat{k}_5 = 2\hat{k}_2$$

$$\hat{k}_5\hat{k}_1 = \hat{k}_5 \quad \hat{k}_5\hat{k}_2 = 2\hat{k}_4 \quad \hat{k}_5\hat{k}_3 = \hat{k}_5 \quad \hat{k}_5\hat{k}_4 = 2\hat{k}_2 \quad \hat{k}_5\hat{k}_5 = 2\hat{k}_1 + 2\hat{k}_3$$

即

$$C_{111} = C_{122} = C_{133} = C_{144} = C_{155} = C_{212} = C_{232} = C_{313} = C_{322} = C_{333} = C_{344} =$$

$$= C_{355} = C_{414} = C_{434} = C_{515} = C_{515} = 1$$

$$C_{221} = C_{223} = C_{245} = C_{254} = C_{425} = C_{441} = C_{443} = C_{452} = C_{524} = C_{542} =$$

$$C_{551} = C_{553} = 2$$

其余结构常数均为 0

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Problem 11:

D_5群是正五边形群，拥有一根五次轴和 5 根一次轴，因而群元记为

$$D_5 = \{e, \tau, \tau^2, \tau^3, \tau^4, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$$

且群元间满足：

$$\sigma_i \tau^j \sigma_i = \tau^{5-j}$$

$$\sigma_{(i+1)\, mod\, 5} = \sigma_i \tau$$

(1) 群元标准格式为$\sigma^m \tau^n$，其中$m \in \{0,1\}$，$n \in \{0,1,2,3,4\}$

(2) 共轭类：$\{e\}\{\tau, \tau^4\}\{\tau^2, \tau^3\}\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$

(3) 类算符：

$$\hat{k}_1 = e$$

$$\hat{k}_2 = \tau + \tau^4$$

$$\hat{k}_3 = \tau^2 + \tau^3$$

$$\hat{k}_4 = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5$$

(4) 结构常数：

$$\hat{k}_1\hat{k}_2 = \hat{k}_2 \quad \hat{k}_1\hat{k}_3 = \hat{k}_3 \quad \hat{k}_1\hat{k}_4 = \hat{k}_4$$

$$\hat{k}_2\hat{k}_1 = \hat{k}_2 \quad \hat{k}_2\hat{k}_3 = \hat{k}_2 + \hat{k}_3 \quad \hat{k}_2\hat{k}_4 = 2\hat{k}_4$$

$$\hat{k}_3\hat{k}_1 = \hat{k}_3 \quad \hat{k}_3\hat{k}_2 = \hat{k}_2 + \hat{k}_3 \quad \hat{k}_3\hat{k}_4 = 2\hat{k}_4$$

$$\hat{k}_4\hat{k}_1 = \hat{k}_4 \quad \hat{k}_4\hat{k}_2 = 2\hat{k}_4 \quad \hat{k}_4\hat{k}_3 = 2\hat{k}_4$$

(5) D_5群有 4 个共轭类，因此有 4 个不等价不可约表示。由完备性定理，表示维数平方和为群阶，可推测$10 = 1^2 + 1^2 + 2^2 + 2^2$，$D_5$群有 2 个一维表示和 2 个二维表示。

第一行为恒等表示，特征标为 1

	$1C_1$	$2C_2$	$2C_3$	$5C_4$
$\chi_A^1(D_5)$	1	1	1	1
$\chi_B^1(D_5)$	1	1	1	-1
$\chi_C^2(D_5)$	2	x	y	z
$\chi_D^2(D_5)$	2			

第一列为恒元特征标，为表示维数。

(6) 对于第二行χ_B^1，首先由于$5C_4$中群元$\sigma_i^2 = e$，可知$\chi_B^1(C_4) = \pm 1$

如果$\chi_B^1(C_4)$为$+1$，将得到$\chi_A^1(D_5)$恒等表示，重复；

如果$\chi_B^1(C_4)$为-1，由特征标定理：特征标模方和为群阶，可以得到$\chi_B^1(C_2)=\chi_B^1(C_3)=1$

为求得二维表示特征标，不妨假设第三行未知元素分别为x,y,z。由特征标表性质得：

$$\begin{cases}2^2+2x^2+2y^2+5z^2=10\\2+2x+2y-5z=0\\2+2x+2y+5z=0\end{cases}$$

其中第一条方程为特征标模方和为群阶，后两条为特征标正交定理。

由后两条方程可知

$$\begin{cases}z=0\\x=-1-y\end{cases}$$

带入第一条方程可以得到

$$y=\frac{-1\pm\sqrt{5}}{2}$$

于是有

$$\begin{cases}y=\dfrac{-1+\sqrt{5}}{2}\\x=\dfrac{-1-\sqrt{5}}{2}\end{cases}\qquad\qquad\begin{cases}y=\dfrac{-1-\sqrt{5}}{2}\\x=\dfrac{-1+\sqrt{5}}{2}\end{cases}$$

分别为$\chi_C^2(D_5)$和$\chi_D^2(D_5)$，于是D_5群的特征标表为：

	$1C_1$	$2C_2$	$2C_3$	$5C_4$
$\chi_A^1(D_5)$	1	1	1	1
$\chi_B^1(D_5)$	1	1	1	-1
$\chi_C^2(D_5)$	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0
$\chi_D^2(D_5)$	2	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0

由特征标表可见，$ker\chi_A^1=D_5$，$ker\chi_B^1=C_5$，$ker\chi_C^2=ker\chi_D^2=\{e\}$

由此可见D_5群的非平庸正规子群为五阶循环群C_5和$\{e,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5\}$

Group Theory Homework XIV

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Problem 12

D_4 群群元记为 $\{e, \tau, \tau^2, \tau^3, a, a\tau, a\tau^2, a\tau^3\}$，共轭类为 $\{e\}\{\tau, \tau^3\}\{\tau^2\}\{a, a\tau^2\}\{a\tau, a\tau^3\}$
根据完备性定理，$2^2 + 1^2 + 1^2 + 1^2 + 1^2 = 8$，所以 D_4 群共有 4 个一维表示及 1 个二维表示
C_4 群为 Abel 群，所有 4 个一维不可约表示构成特征标表，C_2 群也为 Abel 群，所有 2 个一维不可约表示构成特征标表：

$\mathbf{C_4}$	e	τ	τ^2	τ^3
A	1	1	1	1
B	1	i	-1	$-i$
C	1	-1	1	-1
D	1	$-i$	-1	i

$\mathbf{C_2}$	e	a
A	1	1
B	1	-1

由 C_4 和 C_2 群直乘得到的 D_4 群由此拥有四个一维不可约表示写为：

$\mathbf{D_4}$	e	τ	τ^2	τ^3	a	$a\tau$	$a\tau^2$	$a\tau^3$
A	1	1	1	1	1	1	1	1
B	1	i	-1	$-i$	1	i	-1	$-i$
C	1	-1	1	-1	1	-1	1	-1
D	1	$-i$	-1	i	1	$-i$	-1	i
E	1	1	1	1	-1	-1	-1	-1
F	1	i	-1	$-i$	-1	$-i$	1	i
G	1	-1	1	-1	-1	1	-1	1
H	1	$-i$	-1	i	-1	i	1	$-i$

根据特征标定理可知，其中 B、D、F、H 不是 D_4 群的一维表示，因此 D_4 群的所有四个不等价不可约一维表示写为：

$\mathbf{D_4}$	e	τ	τ^2	τ^3	a	$a\tau$	$a\tau^2$	$a\tau^3$
A	1	1	1	1	1	1	1	1
B	1	-1	1	-1	1	-1	1	-1
C	1	1	1	1	-1	-1	-1	-1
D	1	-1	1	-1	-1	1	-1	1

而对于 D_4 群的二维表示，由诱导表示可知，D_4 群生成元 τ 和 a 的表示为：

$$\begin{aligned}
D_{11}^{(2)}(\tau) &= D^{(1)n}(e^{-1}\tau e) = D^{(1)n}(\tau) = exp\left(\frac{2n\pi i}{4}\right) \\
D_{12}^{(2)}(\tau) &= D^{(1)n}(e^{-1}\tau a) = D^{(1)n}(\tau a) = 0 \\
D_{21}^{(2)}(\tau) &= D^{(1)n}(a^{-1}\tau e) = D^{(1)n}(a\tau) = 0 \\
D_{22}^{(2)}(\tau) &= D^{(1)n}(a^{-1}\tau a) = D^{(1)n}(\tau^{-1}) = exp\left(-\frac{2n\pi i}{4}\right)
\end{aligned}$$

$$D^{(2)}(\tau) = \begin{pmatrix} exp\left(\frac{2n\pi i}{4}\right) & 0 \\ 0 & exp\left(-\frac{2n\pi i}{4}\right) \end{pmatrix} \qquad D^{(2)}(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

其中 n 为生成元的第 n 类一维表示，$n = 1, 2, 3, 4$，很显然这个二维不可约表示相互等价，取其中一个即可。

$$D^{(2)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D^{(2)}(\tau) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} D^{(2)}(\tau^2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} D^{(2)}(\tau^3) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$D^{(2)}(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D^{(2)}(a\tau) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} D^{(2)}(a\tau^2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D^{(2)}(a\tau^3) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Group Theory Homework XV

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Problem 13

若 $A \sim B^*$, 也就是说 $\exists X \in GL(2, \mathbb{R})$ 使得 $XAX^{-1} = B^*$

$$\implies \forall R \in G,\ \chi_A(R) = \chi_{B^*}(R) = \chi_B^*(R)$$

为求出 $A \bigotimes B$ 中恒等表示重数，根据特征标定理有：

$$a_E = \frac{1}{|G|} \sum_{R \in G} \chi_E(R) \chi_{A \bigotimes B}(R)$$

由于 $\chi_E(R) = 1$ 以及 $\chi_{A \bigotimes B}(R) = \chi_A(R)\chi_B(R) = \chi_B^*(R)\chi_B(R)$ 因此

$$a_E = \frac{1}{|G|} \sum_{R \in G} \chi_B^*(R) \chi_B(R)$$

根据特征标类函数空间的正交完备定理可知

$$a_E = \frac{1}{|G|} \sum_{R \in G} \chi_B^*(R) \chi_B(R) = 1$$

综上，$A \bigotimes B$ 中恒等表示重数为 1 的条件是 $A \sim B$

Group Theory Homework XVI

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Problem 15

(1)Solution:

$$T=\begin{pmatrix} cos\theta & -sin\theta & a \\ sin\theta & cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$$

T 的代数余子式 $C=\{(-1)^{i+j}C_{ij}\}$ 为

$$C=\begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ -bsin\theta-acos\theta & -bcos\theta+asin\theta & 1 \end{pmatrix} \Longrightarrow \text{伴随矩阵}A=C^T=\begin{pmatrix} cos\theta & sin\theta & -bsin\theta-acos\theta \\ -sin\theta & cos\theta & -bcos\theta+asin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow T^{-1}=\frac{A}{|T|}=\begin{pmatrix} cos\theta & sin\theta & -bsin\theta-acos\theta \\ -sin\theta & cos\theta & -bcos\theta+asin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\because dT=\begin{pmatrix} -sin\theta d\theta & -cos\theta d\theta & da \\ cos\theta d\theta & -sin\theta d\theta & db \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore T^{-1}dT=\begin{pmatrix} 0 & -d\theta & cos\theta da+sin\theta db \\ d\theta & 0 & -sin\theta da+cos\theta db \\ 0 & 0 & 0 \end{pmatrix} \qquad dT\ T^{-1}=\begin{pmatrix} 0 & -d\theta & bd\theta+da \\ d\theta & 0 & -ad\theta+db \\ 0 & 0 & 0 \end{pmatrix}$$

(2)Solution: 由于任意 $T^{-1}dT$3 个矩阵元的外积是 T 矩阵的左不变体积，由此可得左不变积分测度为:

$$\rho_L(a,b,\theta)d\theta dadb=d\theta\wedge(cos\theta da+sin\theta db)\wedge(-sin\theta da+cos\theta db)=d\theta\wedge da\wedge db$$

同理任意 $dT\ T^{-1}$3 个矩阵元的外积是 T 矩阵的右不变体积

$$\rho_R(a,b,\theta)d\theta dadb=d\theta\wedge(bd\theta+da)\wedge(-ad\theta+db)=d\theta\wedge da\wedge db$$

$$\therefore \rho_L=\rho_R=1$$

Group Theory Homework XVII

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Problem 15

Solution:

$$e^{i\varphi(\theta)} = \begin{cases} e^{ik\theta} & k\theta \in [0, 2\pi) \\ e^{k\theta \ mod \ 2\pi} & k\theta \in [2\pi, +\infty) \end{cases} (k \in \mathbb{Z})$$

第三章 Problem 1

Proof:

$$SO(3) = \{R \in \mathbb{R}^{3\times3} | R^T R = I_{3\times3},\ detR = 1\}$$

封闭性：

$$\forall a, b \in SO(3), c = ab \implies \begin{cases} c^T c = (ab)^T ab = b^T a^T ab = I_{3\times3} \\ det|c| = det|ab| = det|a|\ det|b| = 1 \end{cases} \implies c \in SO(3)$$

结合律：

$$\forall a, b, c \in SO(3),\ (ab)c = a(bc)$$

幺元：

$$I_{3\times3} \in SO(3),\ \forall R \in SO(3),\ IR = RI = R$$

逆元：

$$\forall R \in SO(3),\ \exists R^{-1} \in SO(3),\ RR^{-1} = R^{-1}R = I_{3\times3}$$

综上，$SO(3)$ 在矩阵乘法下构成群

Problem 2

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

$$|A-\lambda I| = 0 \implies \begin{cases} \lambda_0 = 0 \\ \lambda_1 = \sqrt{5}i \\ \lambda_2 = -\sqrt{5}i \end{cases}$$

由 Cayley-Hamilton 定理推论可知，$e^{tA} = c_0 I + c_1 tA + c_2 t^2 A^2$ 由特征值得到待定系数方程：

$$\begin{cases} e^{0t} = c_0 + c_1 \times t \times 0 + c_2 \times t^2 \times 0^2 \\ e^{5it} = c_0 + c_1 \times t \times \sqrt{5}i + c_2 \times t^2 \times (\sqrt{5}i)^2 \\ e^{-5it} = c_0 + c_1 \times t \times (-\sqrt{5}i) + c_2 \times t^2 \times (-\sqrt{5}i)^2 \end{cases}$$

$$\implies \begin{cases} c_0 = 1 \\ c_1 = \dfrac{sin\sqrt{5}t}{\sqrt{5}t} \\ c_2 = \dfrac{1-cos\sqrt{5}t}{5t^2} \end{cases}$$

因此，

$$e^{tA} = c_0 I + c_1 tA + c_2 t^2 A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{sin\sqrt{5}t}{\sqrt{5}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} + \frac{1-cos\sqrt{5}t}{5} \begin{pmatrix} -1 & 0 & 2 \\ 0 & -5 & 0 \\ 2 & 0 & -4 \end{pmatrix}$$

Problem 4

Solution:

(1)

$$A_{ad} = (X_j)_{ab}(X_k)_{bd} = \epsilon_{jab}\epsilon_{kbd} = \epsilon_{bja}\epsilon_{bdk} = \delta_{jd}\delta_{ak} - \delta_{jk}\delta_{ad}$$

$$\implies Tr(X_j X_k) = \sum_a A_{aa} = \sum_a \delta_{ja}\delta_{ak} - 3\delta_{jk} = \delta_{jk} - 3\delta_{jk} = -2\delta_{jk}$$

(2)

$$Tr(X_j X_{\boldsymbol{n}}) = \sum_{i=j,k,l} Tr(n_i X_j X_i) = n_j Tr(X_j^2) = -2n_j$$

(3)

$$X_{\boldsymbol{n}}^2 = n_1^2 X_1^2 + n_2^2 X_2^2 + n_3^2 X_3^2 + n_1 n_2 (X_1 X_2 + X_2 X_1) + n_1 n_3 (X_1 X_3 + X_3 X_1) + n_2 n_3 (X_2 X_3 + X_3 X_2)$$

$$= -n_1^2 X_1 - n_2^2 X_2 - n_3^2 X_3 + n_1 n_2 (X_1 X_2 + X_2 X_1) + n_1 n_3 (X_1 X_3 + X_3 X_1) + n_2 n_3 (X_2 X_3 + X_3 X_2)$$

$$Tr(X_j X_{\boldsymbol{n}}^2) = n_k n_l (X_j X_k X_l + X_j X_l X_k) = n_k n_l (X_j X_k X_l - X_j X_k X_l) = 0$$

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Problem 5

Solution:

对转动矩阵参数化，即 $R(\boldsymbol{\psi}) = R(\boldsymbol{n}(\theta,\phi),\psi)$，那么求 $\boldsymbol{\psi}$ 即求其参数表示 $\boldsymbol{\psi}(\theta,\phi,\psi)$

由随体坐标系转动可知，三维空间中的任意转动 $R(\boldsymbol{\psi}(\phi,\theta,\psi))$ 可分解为 $R_z(\phi)R_x(\theta)R_z(\psi)$，其矩阵表示分别为：

$$R_z(\phi) = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix} \qquad R_z(\psi) = \begin{pmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \ R(\boldsymbol{\psi}(\phi,\theta,\psi)) = \begin{pmatrix} cos\phi cos\psi - sin\phi cos\theta sin\psi & -cos\phi sin\psi - sin\phi cos\theta cos\psi & sin\phi sin\theta \\ sin\phi cos\psi + cos\phi cos\theta sin\psi & -sin\phi sin\psi + cos\phi cos\theta cos\phi & -cos\phi sin\theta \\ sin\theta sin\psi & sin\theta sin\psi & cos\theta \end{pmatrix}$$

若已知 R 的全部矩阵元 R_{ij} 那么

$$cos\theta = R_{33} \quad \Longrightarrow \quad \theta = arccos R_{33}$$
$$sin\theta sin\psi = R_{31} \quad \Longrightarrow \quad \psi = arcsin\frac{R_{31}}{sin\theta}$$
$$sin\theta sin\phi = R_{13} \quad \Longrightarrow \quad \phi = arcsin\frac{R_{13}}{sin\theta}$$

即为 $\boldsymbol{\psi}$ 的参数化表示

Problem 6

Proof:

(1) 张量关于某两个指标 l, k 对称，即

$$T_{j_1\ldots j_l\ldots j_k\ldots j_N} = T_{j_1\ldots j_k\ldots j_l\ldots j_N}$$

经历三维空间中的转动 $R(\boldsymbol{\psi})$ 后，有

$$\begin{cases} T'_{j_1\ldots j_l\ldots j_k\ldots j_N} = R_{j_1 j'_1}\ldots R_{j_l j'_l}\ldots R_{j_k j'_k}\ldots R_{j_N j'_N} T_{j'_1\ldots j'_l\ldots j'_k\ldots j'_N} \\ T'_{j_1\ldots j_k\ldots j_l\ldots j_N} = R_{j_1 j'_1}\ldots R_{j_k j'_k}\ldots R_{j_l j'_l}\ldots R_{j_N j'_N} T_{j'_1\ldots j'_k\ldots j'_l\ldots j'_N} \\ T'_{j_1\ldots j_l\ldots j_k\ldots j_N} = T'_{j_1\ldots j_k\ldots j_l\ldots j_N} \end{cases}$$

为了证明转动后 T' 矩阵依旧关于 k,l 指标对称，即证明

$$T_{j_1'\dots j_k'\dots j_l'\dots j_N'} = T_{j_1'\dots j_l'\dots j_k'\dots j_N'}$$

就需要证明三维空间中的 N 阶张量转动群 R 关于指标 k,l 具有交换不变性，即

$$R_{j_1 j_1'} \dots R_{j_l j_l'} \dots R_{j_k j_k'} \dots R_{j_N j_N'} = R_{j_1 j_1'} \dots R_{j_k j_k'} \dots R_{j_l j_l'} \dots R_{j_N j_N'}$$

这显然是正确的，因为张量转动群是单个指标 R_j 的直积，转动操作只会对相应的张量分量起作用，不会影响其他指标，所以 R 具有指标交换不变性，因此转动后的 T' 矩阵仍旧关于指标 k',l' 对称
(2) 张量关于某两个指标 l,k 反对称，即

$$T_{j_1\dots j_l\dots j_k\dots j_N} = -T_{j_1\dots j_k\dots j_l\dots j_N}$$

与第一问类似，经历三维空间中的转动 $R(\boldsymbol{\psi})$ 后，我们有

$$\begin{cases} T'_{j_1\dots j_l\dots j_k\dots j_N} = R_{j_1 j_1'} \dots R_{j_l j_l'} \dots R_{j_k j_k'} \dots R_{j_N j_N'} T_{j_1'\dots j_l'\dots j_k'\dots j_N'} \\ T'_{j_1\dots j_k\dots j_l\dots j_N} = R_{j_1 j_1'} \dots R_{j_k j_k'} \dots R_{j_l j_l'} \dots R_{j_N j_N'} T_{j_1'\dots j_k'\dots j_l'\dots j_N'} \\ T'_{j_1\dots j_l\dots j_k\dots j_N} = -T'_{j_1\dots j_k\dots j_l\dots j_N} \end{cases}$$

同样由于三维空间中的 N 阶张量转动群 R 关于指标 k,l 具有交换不变性，所以

$$R_{j_1 j_1'} \dots R_{j_l j_l'} \dots R_{j_k j_k'} \dots R_{j_N j_N'} = R_{j_1 j_1'} \dots R_{j_k j_k'} \dots R_{j_l j_l'} \dots R_{j_N j_N'}$$

$$\Longrightarrow \quad T_{j_1'\dots j_l'\dots j_k'\dots j_N'} = -T_{j_1'\dots j_k'\dots j_l'\dots j_N'}$$

即转动后的 T' 矩阵依旧关于指标 k',l' 反对称
(3) T 矩阵关于 l,k 无迹，即

$$\sum_s T_{j_1\dots j_l\dots j_k\dots j_N} \delta_{j_k s} \delta_{j_l s} = 0$$

经过旋转操作 R 后，

$$\sum_s R_{j_1 j_1'} \dots R_{j_l j_l'} \dots R_{j_k j_k'} \dots R_{j_N j_N'} T_{j_1'\dots j_l'\dots j_k'\dots j_N'} \delta_{j_k s} \delta_{j_l s} = 0$$

$$\Longrightarrow \sum_s R_{j_1 j_1'} \dots R_{ss'} \dots R_{ss'} \dots R_{j_N j_N'} T_{j_1'\dots s'\dots s'\dots j_N'} = 0$$

$$\Longrightarrow T_{j_1'\dots s'\dots s'\dots j_N'} \sum_s R_{j_1 j_1'} \dots R_{ss'} \dots R_{ss'} \dots R_{j_N j_N'} = 0$$

由于 $R_{ss'}R_{ss'}$ 不一定为 0 (当且仅当转角为 π 时为 0)，因而

$$T_{j_1'\dots s'\dots s'\dots j_N'} = T_{j_1'\dots j_l'\dots j_k'\dots j_N'} \delta_{j_k s} \delta_{j_l s} = 0$$

即转动后张量 T' 仍旧关于指标 k',l' 无迹

Problem 7

Solution:

(1)

$$(\boldsymbol{a}\cdot\boldsymbol{\tau})(\boldsymbol{b}\cdot\boldsymbol{\tau})(\boldsymbol{c}\cdot\boldsymbol{\tau}) = \sum_{j,k,l}(a_j\tau_j)(b_k\tau_k)(c_l\tau_l) = \sum_{j,k,l} a_j b_k c_l \tau_j\tau_k\tau_l$$

$$= \sum_{j,k,l} a_j b_k c_l (\delta_{jk} I + i\epsilon_{jkl}\tau_l)\tau_l = \sum_{j,k,l} a_j b_k c_l \delta_{jk}\tau_l + \sum_{j,k,l} i a_j b_k c_l \epsilon_{jkl}\tau_l^2$$

$$= \sum_{j,k,l} a_j b_k c_l \delta_{jk}\tau_l + \sum_{j,k,l} i a_j b_k c_l \epsilon_{jkl} I$$

$$\Longrightarrow\ Tr(\boldsymbol{a}\cdot\boldsymbol{\tau})(\boldsymbol{b}\cdot\boldsymbol{\tau})(\boldsymbol{c}\cdot\boldsymbol{\tau}) = \sum_{j,k,l} a_j b_k c_l \delta_{jk} Tr(\tau_l) + \sum_{j,k,l} i a_j b_k c_l \epsilon_{jkl} Tr(I)$$

$$= 2i \sum_{j,k,l} a_j b_k c_l \epsilon_{jkl}$$

$$\Longrightarrow\ -\frac{i}{2} Tr(\boldsymbol{a}\cdot\boldsymbol{\tau})(\boldsymbol{b}\cdot\boldsymbol{\tau})(\boldsymbol{c}\cdot\boldsymbol{\tau}) = \sum_{j,k,l} a_j b_k c_l \epsilon_{jkl}$$

$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2$$

(2) 不妨假设

$$-\frac{i}{2} Tr[(\boldsymbol{a}\cdot\boldsymbol{\tau})(\boldsymbol{b}\cdot\boldsymbol{\tau})(\boldsymbol{c}\cdot\boldsymbol{\tau})] = \sum_{j,k,l} \epsilon_{jkl} a_j b_k c_l > 0$$

即拥有右手手征，那么下面证明在 $R(u)$ 变换下，上式的右手手征保持不变:

$$-\frac{i}{2} Tr[(R(u)\boldsymbol{a}\cdot\boldsymbol{\tau})(R(u)\boldsymbol{b}\cdot\boldsymbol{\tau})(R(u)\boldsymbol{c}\cdot\boldsymbol{\tau})]$$

$$= -\frac{i}{2} Tr[u(\boldsymbol{a}\cdot\boldsymbol{\tau})u^{-1}u(\boldsymbol{b}\cdot\boldsymbol{\tau})u^{-1}u(\boldsymbol{c}\cdot\boldsymbol{\tau})u^{-1}]$$

$$= -\frac{i}{2} Tr[u(\boldsymbol{a}\cdot\boldsymbol{\tau})(\boldsymbol{b}\cdot\boldsymbol{\tau})(\boldsymbol{c}\cdot\boldsymbol{\tau})u^{-1}]$$

$$= -\frac{i}{2} Tr[u(\boldsymbol{a}\cdot\boldsymbol{\tau})(\boldsymbol{b}\cdot\boldsymbol{\tau})(\boldsymbol{c}\cdot\boldsymbol{\tau})u^{-1}]$$

$$= u[\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})]u^{-1} = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) > 0$$

即 $R(u)$ 变换下，手征保持不变

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Problem 8

Solution:

$$u(a,b,c) = u_z(a)Hu_z(b)Hu_z(c)$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\frac{a}{2}} & 0 \\ 0 & e^{i\frac{a}{2}} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\frac{b}{2}} & 0 \\ 0 & e^{i\frac{b}{2}} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\frac{c}{2}} & 0 \\ 0 & e^{i\frac{c}{2}} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-\frac{i}{2}(a+b+c)} + e^{-\frac{i}{2}(a-b+c)} & e^{-\frac{i}{2}(a+b-c)} - e^{-\frac{i}{2}(a-b-c)} \\ e^{\frac{i}{2}(a-b-c)} - e^{\frac{i}{2}(a+b-c)} & e^{\frac{i}{2}(a-b+c)} + e^{\frac{i}{2}(a+b+c)} \end{pmatrix}$$

Problem 9

Solution:

伴随矩阵法求得 u 的逆为

$$u^{-1} = \begin{pmatrix} cos\frac{\beta}{2}e^{\frac{i}{2}(\alpha+\gamma)} & sin\frac{\beta}{2}e^{-\frac{i}{2}(\alpha-\gamma)} \\ -sin\frac{\beta}{2}e^{\frac{i}{2}(\alpha-\gamma)} & cos\frac{\beta}{2}e^{-\frac{i}{2}(\alpha+\gamma)} \end{pmatrix}$$

$$du = \frac{\partial u}{\partial \alpha}d\alpha + \frac{\partial u}{\partial \beta}d\beta + \frac{\partial u}{\partial \gamma}d\gamma$$

$$\Longrightarrow u^{-1}du = \begin{pmatrix} -\frac{i}{2}cos\beta d\alpha - \frac{i}{2}d\gamma & isin\frac{\beta}{2}cos\frac{\beta}{2}e^{i\gamma}d\alpha - \frac{1}{2}e^{i\gamma}d\beta + isin\frac{\beta}{2}cos\frac{\beta}{2}e^{i\gamma}d\gamma \\ isin\frac{\beta}{2}cos\frac{\beta}{2}e^{-i\gamma}d\alpha + \frac{1}{2}e^{-i\gamma}d\beta & \frac{i}{2}cos\beta d\alpha + \frac{i}{2}cos\beta d\gamma \end{pmatrix}$$

计算外积得到

$$dV = \frac{sin\beta}{16\pi^2}d\alpha \wedge d\beta \wedge d\gamma$$

积分不变测度即为 $\frac{sin\beta}{16\pi^2}$

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Problem 10

Proof: 由

$$u(\boldsymbol{\psi}) = e^{-\frac{i}{2}\boldsymbol{\psi}\cdot\boldsymbol{\tau}}$$

$$\Longrightarrow\ u(\boldsymbol{\alpha})u(\boldsymbol{\psi})u(\boldsymbol{\alpha}) = e^{-\frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}e^{-\frac{i}{2}\boldsymbol{\psi}\cdot\boldsymbol{\tau}}e^{\frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}$$

根据 BCH 公式推论：$e^A e^B e^{-A} = exp(e^A B e^{-A})$，可得

$$u(\boldsymbol{\alpha})u(\boldsymbol{\psi})u(\boldsymbol{\alpha}) = exp\left[e^{-\frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}\left(-\frac{i}{2}\boldsymbol{\psi}\cdot\boldsymbol{\tau}\right)e^{\frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}\right] = exp\left[-\frac{i}{2}u(\boldsymbol{\alpha})(\boldsymbol{\psi}\cdot\boldsymbol{\tau})u(\boldsymbol{\alpha})^{-1}\right]$$

$$= exp\left[-\frac{i}{2}(R(\boldsymbol{\alpha})\boldsymbol{\psi})\cdot\boldsymbol{\tau}\right] = u\left(R(\boldsymbol{\alpha})\boldsymbol{\psi}\right)$$

证毕

Problem 11

(1)Proof:

$$\forall u \in SE(2),\quad u\begin{pmatrix}x_1\\x_2\\1\end{pmatrix} = \begin{pmatrix}x_1'\\x_2'\\1\end{pmatrix} = \begin{pmatrix}cos\theta & -sin\theta & a\\ sin\theta & cos\theta & b\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\1\end{pmatrix} \Longrightarrow u^{-1} = \begin{pmatrix}cos\theta & sin\theta & -acos\theta - bsin\theta\\ -sin\theta & cos\theta & asin\theta - bcos\theta\\ 0 & 0 & 1\end{pmatrix}$$

所以对于小于 n 次的多项式张成的空间 $\mathbb{F}$

$$f(x_1, x_2) = \sum_{i,j=0}^{n} C_{ij}x_1^i x_2^j$$

在 $SE(2)$ 群作用下保持空间不变，即

$$\forall u \in SE(2), \forall f(\boldsymbol{x}) \in \mathbb{F},\ \hat{u}f(\boldsymbol{x}) = f(u^{-1}\boldsymbol{x})$$

$$= f(x_1cos\theta + x_2sin\theta - acos\theta - bsin\theta, -x_1sin\theta + x_2cos\theta + asin\theta - bcos\theta) = f(\boldsymbol{x'}) \in \mathbb{F}$$

因此小于 n 次的多项式构成 $SE(2)$ 群表示下的不变子空间

(2)Solution:

为方便起见 $SE(2)$ 群取 (1) 中的参数化，因此对于多项式

$$f(\boldsymbol{x}) = \alpha x + \beta y + \gamma$$

有

$$\forall u \in SE(2),\ \hat{u}f(\boldsymbol{x}) = f(\hat{u}^{-1}\boldsymbol{x}) = \alpha x' + \beta y' + \gamma$$

再由参数化可知

$$\begin{cases} x' = cos\theta x + sin\theta y + a \\ y' = -sin\theta x + cos\theta y + b \end{cases}$$

$$\Longrightarrow \ \alpha x' + \beta y' + \gamma = x(\alpha cos\theta - \beta sin\theta) + y(\alpha sin\theta + \beta cos\theta) + \gamma - \alpha(acos\theta + bsin\theta) + \beta(asin\theta - bcos\theta)$$

$$= \alpha' x + \beta' y + \gamma'$$

$$\Longrightarrow \begin{cases} \alpha' = \alpha cos\theta - \beta sin\theta \\ \beta' = \alpha sin\theta + \beta cos\theta \\ \gamma' = -\alpha(acos\theta + bsin\theta) + \beta(asin\theta - bcos\theta) + \gamma \end{cases}$$

$$\Longrightarrow \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ -a\cos\theta - bsin\theta & asin\theta - bcos\theta & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

由此可见 $SE(2)$ 群的三维表示即为

$$\begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ -a\cos\theta - bsin\theta & asin\theta - bcos\theta & 1 \end{pmatrix}$$

(3) Solution:

由第 16 次作业可知 $SE(2)$ 群的左右积分不变测度均为 1，

于是对三维表示特征标 $\chi^{(3)} = 2cos\theta + 1$ 在 $SE(2)$ 群上积分有

$$\langle \chi^{(3)} | \chi^{(3)} \rangle = \frac{1}{2\pi}\int_0^{2\pi} (2cos\theta + 1)^2 d\theta = 3$$

说明该三维表示可约，但由于表示矩阵无法化为 Jordan 标准型，因此无法完全可约

(4) Solution:

由于 $SE(2)$ 群是非紧致李群，因此 $SE(2)$ 群不存在等价的幺正表示

Problem 12

(1) Solution:

$$D^1 = \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 \\ -\sqrt{2}ab^* & aa^* - bb^* & \sqrt{2}a^*b \\ b^{*2} & -\sqrt{2}a^*b^* & a^{*2} \end{pmatrix}$$

其中

$$\begin{cases} a = cos\frac{\psi}{2} - isin\frac{\psi}{2}cos\theta \\ b = -isin\frac{\psi}{2}sin\theta e^{-i\phi} \end{cases}$$

(2) Solution:

$$TrD^1 = \sum_{m=-1}^{1} e^{-im\psi} = \frac{sin\left(\frac{3}{2}\psi\right)}{sin\frac{\psi}{2}} = 1 + 2cos\psi$$

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Problem 13

Solution:

由欧拉角参数的定义有：

$$R(\alpha,\beta,\gamma)=R_z(\alpha)R_y(\beta)R_z(\gamma)$$

$$\Longrightarrow \begin{cases} \dfrac{\partial R(\alpha,\beta,\gamma)}{\partial\alpha}=\dfrac{\partial R_z(\alpha)R_y(\beta)R_z(\gamma)}{\partial\alpha}=\dfrac{\partial R_z(\alpha)}{\partial\alpha}R_y(\beta)R_z(\gamma) \\ \dfrac{\partial R(\alpha,\beta,\gamma)}{\partial\beta}=\dfrac{\partial R_z(\alpha)R_y(\beta)R_z(\gamma)}{\partial\beta}=R_z(\alpha)\dfrac{\partial R_y(\beta)}{\partial\beta}R_z(\gamma) \\ \dfrac{\partial R(\alpha,\beta,\gamma)}{\partial\alpha}=\dfrac{\partial R_z(\alpha)R_y(\beta)R_z(\gamma)}{\partial\alpha}=R_z(\alpha)R_y(\beta)\dfrac{\partial R_z(\gamma)}{\partial\gamma} \end{cases}$$

$$\Longrightarrow \left.\frac{\partial R(\alpha,\beta,\gamma)}{\partial(\alpha,\beta,\gamma)}\right|_{(\alpha,\beta,\gamma=0)}=\begin{cases} \left.\dfrac{\partial R_z(\alpha)}{\partial\alpha}\right|_{\alpha=0}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix} \\ \left.\dfrac{\partial R_y(\beta)}{\partial\beta}\right|_{\beta=0}=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix} \\ \left.\dfrac{\partial R_z(\gamma)}{\partial\gamma}\right|_{\gamma=0}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix} \end{cases}$$

Problem 14

(1) Solution:

$$D^1=\begin{pmatrix} a^2 & \sqrt{2}ab & b^2 \\ -\sqrt{2}ab^* & aa^*-bb^* & \sqrt{2}a^*b \\ b^{*2} & -\sqrt{2}a^*b^* & a^{*2} \end{pmatrix}$$

其中

$$\begin{cases} a=cos\dfrac{\psi}{2}-isin\dfrac{\psi}{2}cos\theta \\ b=-isin\dfrac{\psi}{2}sin\theta e^{-i\phi} \end{cases}$$

$$\Longrightarrow \begin{cases} \left.\frac{\partial D^1}{\partial \psi}\right|_{\boldsymbol{\psi}=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ \left.\frac{\partial D^1}{\partial \theta}\right|_{\boldsymbol{\psi}=0} = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \\ \left.\frac{\partial D^1}{\partial \phi}\right|_{\boldsymbol{\psi}=0} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \end{cases}$$

(2)Solution:

$$S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

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Problem 15

(1)Solution:

$$T = \begin{pmatrix} cos\theta & -sin\theta & a \\ sin\theta & cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \hat{J}_\theta = \frac{\partial T}{\partial \theta}\Big|_{\boldsymbol{\psi}=0} = \begin{pmatrix} -sin\theta & -cos\theta & 0 \\ cos\theta & -sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\Bigg|_{\boldsymbol{\psi}=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{p}_x = \frac{\partial T}{\partial a}\Big|_{\boldsymbol{\psi}=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{p}_y = \frac{\partial T}{\partial b}\Big|_{\boldsymbol{\psi}=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

对易关系：

$$[\hat{p}_x, \hat{p}_y] = \hat{p}_x\hat{p}_y - \hat{p}_y\hat{p}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\hat{J}_\theta, \hat{p}_x] = \hat{J}_\theta\hat{p}_x - \hat{p}_x\hat{J}_\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \hat{p}_y$$

$$[\hat{J}_\theta, \hat{p}_y] = \hat{J}_\theta\hat{p}_y - \hat{p}_y\hat{J}_\theta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{p}_x$$

(2)Proof:

函数空间中，$SE(2)$ 群的表示为

$$\hat{p}_x f(\boldsymbol{x}) = \frac{\partial T}{\partial a}\Big|_0 f(\boldsymbol{x})$$

$$\hat{p}_y f(\boldsymbol{x}) = \left.\frac{\partial T}{\partial b}\right|_0 f(\boldsymbol{x})$$

$$\hat{J}_\theta f(\boldsymbol{x}) = \left.\frac{\partial T}{\partial \theta}\right|_0 f(\boldsymbol{x})$$

$$\Longrightarrow \begin{cases} \hat{p}_x = -i\dfrac{\partial}{\partial x} \\ \hat{p}_y = -i\dfrac{\partial}{\partial y} \\ \hat{J}_\theta = iy\dfrac{\partial}{\partial x} - ix\dfrac{\partial}{\partial y} = i\begin{vmatrix} y & x \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{vmatrix} = -i\theta\dfrac{\partial}{\partial \theta} \end{cases}$$

将 $g(\theta, a, b)$ 作用在 $f(\boldsymbol{x})$ 上

$$\Longrightarrow \hat{g}(\theta, a, b) f(\boldsymbol{x}) = (a\hat{p}_x + b\hat{p}_y + \theta J_\theta) f(\boldsymbol{x}) = \left(-ia\frac{\partial}{\partial x} - ib\frac{\partial}{\partial y} - i\theta\frac{\partial}{\partial \theta}\right) f(\boldsymbol{x})$$

$$\Longrightarrow \hat{g}(\theta, a, b) = e^{-(ia\hat{p}_x + ib\hat{p}_y + i\theta\hat{J}_\theta)}$$

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Problem 16

Solution:

记

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

两自旋为 $\frac{1}{2}$ 的粒子耦合，得到的直积态包括

$$\phi_1 \otimes \phi_1 \quad \phi_1 \otimes \phi_2 \quad \phi_2 \otimes \phi_1 \quad \phi_2 \otimes \phi_2$$

约化后，总自旋为 1 的波函数为

$$\phi_1 \otimes \phi_1 \quad \frac{1}{\sqrt{2}}(\phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_1) \quad \phi_2 \otimes \phi_2$$

自旋为 0 的波函数为

$$\frac{1}{\sqrt{2}}(\phi_1 \otimes \phi_2 - \phi_2 \otimes \phi_1)$$

Problem 17

Solution:

$$\chi(\psi) = Tr[D^{j_1}(\psi) \otimes D^{j_2}(\psi)] = \chi^{(j_1)}(\psi)\chi^{(j_2)}(\psi) = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} e^{-i(m_1+m_2)\psi}$$

对于 $j_1 = j_2 = 2$ 有

$$\chi(\psi) = \chi^{(4)}(\psi) + \chi^{(3)}(\psi) + \chi^{(2)}(\psi) + \chi^{(1)}(\psi) + \chi^{(0)}(\psi)$$

$$\Longrightarrow (\chi, \chi) = |j_1 + j_2| - |j_1 - j_2| + 1 = \begin{cases} 2j_2 + 1 & (j_1 > j_2) \\ 2j_1 + 1 & (j_2 > j_1) \end{cases}$$

$$(\chi, \chi^{(0)}) = (\chi, \chi^{(1)}) = (\chi, \chi^{(2)}) = 1$$

Problem 18

Solution:

$$\chi(\psi) = Tr[D^{j_1}(\psi) \otimes D^{j_2}(\psi)] = \chi^{(j_1)}(\psi)\chi^{(j_2)}(\psi) = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} e^{-i(m_1+m_2)\psi}$$

对于 $j_1 = j_2 = 2$ 有

$$\chi(\psi) = \chi^{(4)}(\psi) + \chi^{(3)}(\psi) + \chi^{(2)}(\psi) + \chi^{(1)}(\psi) + \chi^{(0)}(\psi)$$

$$\Longrightarrow (\chi, \chi) = |j_1 + j_2| - |j_1 - j_2| + 1 = \begin{cases} 2j_2 + 1 & (j_1 > j_2) \\ 2j_1 + 1 & (j_2 > j_1) \end{cases}$$

$$(\chi, \chi^{(0)}) = (\chi, \chi^{(1)}) = 1$$

Problem 19

Proof:

$$\langle j, m'|J_x|j, m\rangle = \frac{1}{2}\sqrt{j(j+1) - m(m+1)}\delta_{m',m+1} + \frac{1}{2}\sqrt{j(j+1) - m(m-1)}\delta_{m',m-1}$$

$$\langle j, m'|J_y|j, m\rangle = -\frac{i}{2}\sqrt{j(j+1) - m(m+1)}\delta_{m',m+1} + \frac{i}{2}\sqrt{j(j+1) - m(m-1)}\delta_{m',m-1}$$

$$\langle j, m'|J_z|j, m\rangle = m\delta_{m',m}$$

很显然

$$trJ_j = \sum_{m=-J}^{J} \langle JM|J_j|JM\rangle = 0$$

然后对于第二个等式

$$tr(J_jJ_k) = \frac{J(J+1)(2J+1)}{3}\delta_{jk}$$

我们有

$$\begin{cases} tr(J_xJ_y) = \frac{1}{4i}\left\langle JM|J_+^2 - J_+J_- + J_-J_+ - J_-^2|JM\right\rangle = \sum_{m=-J}^{J} -\frac{m}{2i} = 0 \\ tr(J_xJ_z) = \frac{1}{2}\langle JM|J_+J_0 + J_-J_0|JM\rangle = \frac{M}{2}\langle JM|J_+ + J_-|JM\rangle = 0 \\ tr(J_yJ_z) = \frac{1}{2i}\langle JM|J_+J_0 - J_-J_0|JM\rangle = \frac{M}{2i}\langle JM|J_+ - J_-|JM\rangle = 0 \end{cases}$$

而

$$\begin{cases} tr(J_x^2) = \frac{1}{4}\left\langle JM|J_+^2 + J_-^2 + J_+J_- + J_-J_+|JM\right\rangle = \frac{1}{4}(|\xi_m|^2 + |\xi_{m-1}|^2) = \frac{1}{2}J(J+1) - \frac{1}{2}\sum_{m=-J}^{J} m^2 = \frac{J(J+1)(2J+1)}{3} \\ tr(J_y^2) = -\frac{1}{4}\left\langle JM|J_+^2 + J_-^2 - J_+J_- - J_-J_+|JM\right\rangle = \frac{1}{4}(|\xi_m|^2 + |\xi_{m-1}|^2) = \frac{1}{2}J(J+1) - \frac{1}{2}\sum_{m=-J}^{J} m^2 = \frac{J(J+1)(2J+1)}{3} \\ tr(J_z^2) = \sum_{m=-J}^{J} m^2 = \frac{J(J+1)(2J+1)}{3} \end{cases}$$

因此

$$tr(J_jJ_k) = \frac{J(J+1)(2J+1)}{3}\delta_{jk}$$

对于第三个等式，同理代入

$$\begin{cases} J_x = \frac{1}{2}(J_+ + J_-),\ J_y = \frac{1}{2i}(J_+ - J_-),\ J_z = J_0 \\ \xi_m^* = \langle JM|J_-|J, M+1\rangle,\ \xi_m = \langle J, M+1|J_+|J, M\rangle \\ \xi_m = \sqrt{J(J+1) - M(M+1)} \end{cases}$$

对对角元 M 指标求和同样得证

$$tr(J_j J_k J_l) = i\frac{J(J+1)(2J+1)}{6}\epsilon_{jkl}$$

第四个等式也同理

$$tr(J_j J_k J_l J_m) = \frac{1}{30}J(J+1)(2J+1)[(2J^2+2J+1)(\delta_{jk}\delta_{lm} + \delta_{jm}\delta_{kl}) + 2(J-1)(J+2)\delta_{jl}\delta_{km}]$$

以此类推

(不会张量分析只能一项一项算了 (哭))

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第 21 题

Solution：

(1) 五阶笛卡尔张量等价于 5 个三维矢量做直积，维度为$3\times3\times3\times3\times3=243$维。

可视为 5 个自旋为 1 的粒子做角动量耦合

$$j_1=1 \quad j_2=1 \quad j_3=1 \quad j_4=1 \quad j_5=1$$

$$2j_n+1=3 \quad \text{(笛卡尔三维坐标)}$$

对直积态做约化，根据角动量耦合理论，耦合态总角动量可以取$0,1,2,3,4,5$，由其表示特征标可知

$$\chi(\psi)=Tr\bigcup_{n=1}^{5}\otimes D^{j_n}=\prod_{n=1}^{5}TrD^{j_n}=\sum_{m_1,m_2,m_3,m_4,m_5=-1}^{1}e^{-i(m_1+m_2+m_3+m_4+m_5)\psi}$$

共计$3^5=243$项，由所有可能的排列组合可得

$$\chi(\psi)=e^{-5i\psi}+5e^{-4i\psi}+15e^{-3i\psi}+30e^{-2i\psi}+45e^{-i\psi}+51e^{0i\psi}$$

$$+e^{5i\psi}+5e^{4i\psi}+15e^{3i\psi}+30e^{2i\psi}+45e^{i\psi}$$

$$=1\times\chi^{(5)}+4\times\chi^{(4)}+10\times\chi^{(3)}+15\times\chi^{(2)}+15\times\chi^{(1)}+6\times\chi^{(0)}$$

每一项对应一个不可约不变子空间，且通过特征标完备定理可以计算出对应维度空间的重数：

特征标$\chi^{(n)}$	空间维数	重数
$\chi^{(0)}$	一维	6
$\chi^{(1)}$	三维	15
$\chi^{(2)}$	五维	15
$\chi^{(3)}$	七维	10
$\chi^{(4)}$	九维	4
$\chi^{(5)}$	十一维	1

共计$1+4+10+15+15+6=51$个不可约不变子空间，其中一维不可约不变子空间有 6 个。

(2) **独立迷向张量数目**等于**一维不变子空间**（对应 0 自旋）个数，因此五阶张量的独立迷向张量个数为 6 个。

$$\delta_{ij}\epsilon_{klm}=\delta_{kj}\epsilon_{ilm}+\delta_{lj}\epsilon_{kim}+\delta_{mj}\epsilon_{kli}$$

第 22 题

Solution:

(1)

SO(3)的三阶张量场描述最高自旋粒子，粒子的最高自旋为 3

(2)

三阶张量一共$3 \times 3 \times 3 = 27$个自由度，记为T_{ijk}

计算方法有很多，我选择直接数

1. $T_{112} = T_{211} = T_{121}$

2. $T_{113} = T_{311} = T_{131}$

3. $T_{122} = T_{212} = T_{221}$

4. $T_{123} = T_{312} = T_{231} = T_{132} = T_{213} = T_{321}$

5. $T_{133} = T_{313} = T_{331}$

6. $T_{223} = T_{322} = T_{232}$

7. $T_{332} = T_{233} = T_{323}$

8. T_{111}

9. T_{222}

10. T_{333}

共有 10 个自由度，但是考虑无迹：

$$T_{111} + T_{112} + T_{113} = 0$$

$$T_{22}\ \ + T_{222} + T_{223} = 0$$

$$T_{331} + T_{33}\ \ + T_{33}\ \ = 0$$

要减去 3 个自由度，因此，独立的三阶无迹对称张量共有 7 个，自由度为 7.

Group Theory Homework XXV

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Problem 1

Proof:

二维晶格：$L = \left\{ \boldsymbol{r} = \sum_{i=1}^{2} n_i a_i | n_1, n_2 \in \mathbb{Z} \right\}, G \in O(2)$

对于二维平面内的一个转动 $R \in G$,

$$R(a_1, a_2) = (a_1, a_2)C^T(R)$$

$$\implies RA = AC^T(R)$$

$$\implies C^T(R) = A^{-1}RA$$

由此得到二维晶体限制定理：

$$Tr(C^T(R)) = Tr(R) = \left| 2cos\left(\frac{\psi}{2}\right) \right| \leq 2$$

$$\implies \psi = 0 \text{ or } \pi$$

因此二维晶体中的转动元素只有 $\{E, C_2\}$, 转动反演元素只有 $\{I, IC_2\}$

Problem 1

Solution:

(1) S_4 群的所有轮换结构包括

$$S_4 \begin{cases} 1^4 = 4 \\ 1^2 + 2^1 = 4 \\ 2^2 = 4 \\ 1 + 3^1 \\ 4^1 = 4 \end{cases} \implies \begin{cases} [\lambda]_1 = [1111] \\ [\lambda]_2 = [211] \\ [\lambda]_3 = [22] \\ [\lambda]_4 = [31] \\ [\lambda]_5 = [4] \end{cases}$$

分别对应一个杨图:

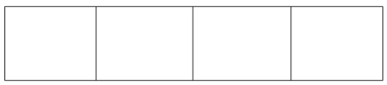

图 1: S4 [4]

图 2: S4 [31]

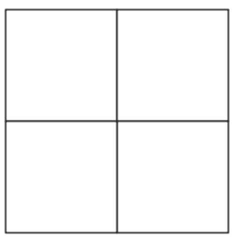

图 3: S4 [22]

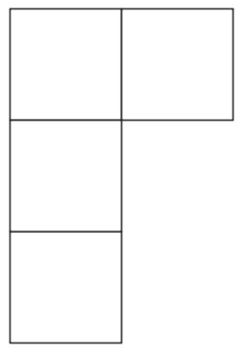

图 4: S4 [211]

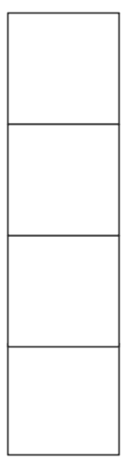

图 5: S4 [1111]

(2) 根据杨图性质可知，不等价不可约表示共有 5 个，维数分别为

$$\begin{cases} [4] & \text{一维} \\ [31] & \text{三维} \\ [22] & \text{二维} \\ [211] & \text{三维} \\ [1111] & \text{一维} \end{cases}$$

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Problem 2

Solution: (1) $S4$ 群的 [22] 标准杨表有两种表示方法：对第一种标准杨表：

1	2
3	4

1	3
2	4

$$\begin{cases} \text{行算符：} & P(T_1^{[22]}) = E + (12) + (34) \\ \text{列算符：} & Q(T_1^{[22]}) = E - (13) - (24) \end{cases}$$

因此杨算符为

$$Y(T_1^{[22]}) = P(T_1^{[22]})Q(T_1^{[22]}) = E + (12) + (34) - (13) - (24) - (132) - (143) - (124) - (234)$$

(2) $S4$ 群标准表示：
对 $S4$ 群所有相邻对换：(12)(23)(34)

$$(12):\ U^{[22]}(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(23):\ U^{[22]}(23) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$(34):\ U^{[22]}(34) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Problem 3

Solution: 四个一轮换有 2 种正则填法，宇称都是 1；
两个一轮换一个二轮换有 2 种正则填法，宇称分别为 1 和-1；

	$1(1^4)$	$6(211)$	$3(2^2)$	$8(13)$	$6(4)$
$T^{[22]}$	2	0	2	-1	0

两个二轮换有 2 种正则填法，字称均为 1
一个一轮换一个三轮换只有一种填法，字称为-1
一个四轮换没有填法，字称为 0

Problem 1

Proof:

(1) $\in \mathcal{T} \quad X \in \mathcal{T}$

(2) $\forall A_1, A_2, \ldots, A_k \in \mathcal{T} \implies A_1 \cap A_2 \cap \cdots \cap A_k \in \mathcal{T}$

(3) $\forall A_1, A_2, \cdots \in \mathcal{T} \implies A_1 \cup A_2 \cup \cdots \in \mathcal{T}$

因此 $(X, \mathcal{T})$ 构成了拓扑空间

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Problem 2

Solution:

$E(2)$ 群包含了二维平面上所有的旋转、平移和镜像操作，是二维欧几里得空间上的一组有界线性变换，并且其中只有一个变换定义了一条带有有限个阶段转换点的曲线。由于存在镜像操作，因此由镜像操作所分割开的两个拓扑空间是不互相连通的，因此 $E(2)$ 群是**非连通群**。

$E(2)$ 群包含的平移没有上确界，因此是**非紧致群**

$E(2)$ 具有 2 个连通分支，每个连通分支都是无穷连通

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Problem 3

(1) **Proof:**

封闭性

$$\forall M_1, M_2 \in GL(2,\mathbb{C}),\ M_1M_2 \in GL(2,\mathbb{C})$$

结合律

$$\forall M_1, M_2, M_3 \in GL(2,\mathbb{C}),\ (M_1M_2)M_3 = M_1(M_2M_3)$$

幺元

$$\forall M \in GL(2,\mathbb{C}),\ MI = IM = M$$

逆元

$$\forall M \in GL(2,\mathbb{C}),\ \exists M^{-1} \in GL(2,\mathbb{C}) \implies MM^{-1} = I$$

因此 $\{M\}$ 在矩阵乘法下构成群

(2) **Solution:**

$M \in GL(2,\mathbb{C})$ 不妨记为

$$M = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x_1 + x_2 i & y_1 + y_2 i \\ z_1 + z_2 i & w_1 + w_2 i \end{pmatrix}$$

拥有 8 个自由度，但 M 需要满足 Bogliubov 变换不变性，因此维度应小于 8

$$\begin{pmatrix} a' \\ a'^\dagger \end{pmatrix} = M \begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \begin{pmatrix} xa + ya^\dagger \\ za + wa^\dagger \end{pmatrix}$$

由对易子不变性

$$[a', a'^\dagger] = 1$$

$$\implies a'a'^\dagger - a'^\dagger a' = (xa + ya^\dagger)(za + wa^\dagger) - (za + wa^\dagger)(xa + ya^\dagger)$$

$$= yz[a^\dagger, a] + xw[a, a^\dagger] = xw - yz = 1$$

$$\implies \begin{cases} x_1w_1 - x_2w_2 - y_1z_1 + y_2z_2 = 1 \\ x_2w_1 + x_1w_2 - y_2z_1 - y_1z_2 = 0 \end{cases}$$

约束方程有两条，因此自由度为 $8 - 2 = 6$

(3) **Solution:**

只要满足上述两条约束方程，$GL(n,\mathbb{C})$ 李群的参数可以取无穷大，因此是非紧致群
同样该群是连通群，连通度为单连通李群
(4) **Solution:**
进一步要求 $(a)^\dagger = a^\dagger$，且满足 M 变换不变性

$$
\begin{aligned}
& (a')^\dagger = a'^\dagger \\
\Longrightarrow \; & (xa + ya^\dagger)^\dagger = za + wa^\dagger \\
\Longrightarrow \; & x^\dagger a^\dagger + y^\dagger a = za + wa^\dagger
\end{aligned}
$$

$$
\Longrightarrow \begin{cases} y^\dagger = z \\ x^\dagger = w \end{cases} \quad \Longrightarrow \begin{cases} z_1 = y_1 \quad z_2 = -y_2 \\ w_1 = x_1 \quad w_2 = -x_2 \end{cases}
$$

而对于之前的对易不变性，要求

$$
xw - yz = 1 \implies xx^\dagger - yy^\dagger = 1
$$

仅给出一个约束条件，因此一共五条约束，M 的李群维度为 8-5=3，是三维李群

Problem 4

$SU(5)$ 群的维数为 $5^2 - 1 = 24$ 维，是紧致李群，连通李群，连通度为单连通