How Linear Algebra B1 Should Have Been

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1 Properties of Matrices

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{1}$$

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} \tag{2}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \tag{3}$$

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \tag{4}$$

Hence,

$$(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1} \tag{5}$$

When $\mathbf{A} = (\mathbf{A}_{ij})_{r\times s}$ and $\mathbf{B} = (\mathbf{B}_{ij})_{s\times t}$,

$$(\mathbf{AB})_{ij} = \mathbf{A}_{ik}\mathbf{B}_{kj} \tag{6}$$

Hence,

$$(\mathrm{diag}(\mathbf{A}_1, ..., \mathbf{A}_r))^{-1} = \mathrm{diag}(\mathbf{A}_1^{-1}, ..., \mathbf{A}_r^{-1}) \tag{7}$$

2 Trace

$$\mathrm{trace}(\mathbf{AB}) = \mathrm{trace}(\mathbf{BA}) \tag{8}$$

$$\mathrm{trace}(\mathbf{A}) = \sum_i \lambda_i \tag{9}$$

3 Determinant

$$\det(\mathbf{A}) = \sum_{i \in S_n} (-1)^{\tau(i_1,...,i_n)} a_{1,i_1} ... a_{n,i_n} \tag{10}$$

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) \tag{11}$$

$$\det(\mathbf{A}^\mathrm{T}) = \det(\mathbf{A}) \tag{12}$$

$$\mathrm{trace}(\mathbf{A}) = \prod_i \lambda_i \tag{13}$$

The determinant of a matrix stays unvaried when a row (or column) times some factor is added to another row (or column). Its sign reserves when two rows (or columns) swap. It is magnified by the same factor by which a row (or column) is magnified.

4 Properties of an Adjugate Matrix

$$(\mathbf{A}^*)_{ij} = A_{ji} \tag{14}$$

where A_{ij} is $\mathbf{A}$'s Algebraic Minor at index i and j, or $(-1)^{i+j}$ times the minor (the determinant of the matrix excluding row i and column j).

$$\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^* = \det(\mathbf{A})\mathbf{I} \tag{15}$$

$$\det(\mathbf{A}) \neq 0 \iff \det(\mathbf{A}^*) \neq 0 \tag{16}$$

5 Rank

$\mathbf{A}$ is defined to have rank r (such an index exists uniquely for all matrices) when reversible matrices $\mathbf{P}$ and $\mathbf{Q}$ exists so that

$$\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \tag{17}$$

The rank of a matrix equals to the highest of the orders of its non-zero minors.

$$\mathrm{rank}(\mathbf{A}^\mathrm{T}) = \mathrm{rank}(\mathbf{A}) \tag{18}$$

The rank of a matrix stays unvaried either when a row (or column) times some factor is added to another row (or column), when two rows (or columns) swap, and when a row (or column) is magnified by a factor.

$$\mathrm{rank}(\mathbf{AB}) \leq \min\{\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{B})\} \leq \mathrm{rank}\begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \leq \mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\mathbf{B}) \tag{19}$$

Frobenius Inequality:

$$\mathrm{rank}(\mathbf{ABC}) + \mathrm{rank}(\mathbf{B}) \geq \mathrm{rank}(\mathbf{AB}) + \mathrm{rank}(\mathbf{BC}) \tag{20}$$

6 Cramer's Rule

The solution for $\mathbf{Ax} = \mathbf{b}$ is

$$x_i = \frac{\Delta_i}{\det(\mathbf{A})} \tag{21}$$

where Δ_i is the determinant of $\mathbf{A}$, but with $\mathbf{b}$ as column i.

7 Positive Definite Matrices

$$\mathbf{A} > 0 \iff \exists \mathbf{P} \text{s.t.} \mathbf{A} = \mathbf{P}^{\mathrm{T}}\mathbf{P} \tag{22}$$

where $\det(\mathbf{P}) \neq 0$.

A real, symmetric matrix $\mathbf{A}$ is positive definite only if

$$Q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0 \tag{23}$$

is satisfied for all $\mathbf{x} \neq \mathbf{0}$.

Real matrices $\mathbf{A}$ and $\mathbf{B}$ are defined to be congruent when a real, reversible matrix $\mathbf{P}$ exists so that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{B} \tag{24}$$

and a positive inertia index r and a negative inertia index s exist uniquely for all real, symmetric matrices, where a real, reversible matrix $\mathbf{Q}$ satisfies

$$\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathrm{diag}(\mathbf{I}_r, -\mathbf{I}_s, \mathbf{O}) \tag{25}$$

Note that $\mathbf{A}$ is positive definite only if r equals to the order of $\mathbf{A}$ and s equals to 0.

A real, symmetric matrix $\mathbf{A}$ is positive definite only if

$$Q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0 \tag{26}$$

is satisfied for all $\mathbf{x} \neq \mathbf{0}$.

A real, symmetric matrix $\mathbf{A}$ is positive definite only if all its eigenvalues are positive.

For a real, symmetric matrix $\mathbf{A}$, the following three propositions are equivalent to one another.

$\mathbf{A}$ is semi positive definite, or r is less than or equal to the order of $\mathbf{A}$ and s equals to 0.

A real matrix $\mathbf{P}$ of the same order as $\mathbf{A}$ exists so that $\mathbf{A} = \mathbf{P}^{\mathrm{T}}\mathbf{P}$.

All the principal minors of $\mathbf{A}$ are greater than or equal to zero. (Note: A minor is principal when the deleted rows and the deleted columns have identical indices.)

8 Orthogonal Matrices

An orthogonal matrix is a real square matrix whose columns and rows are orthonormal vectors.

A real, square matrix $\mathbf{A}$ is defined to be orthogonal when

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I} \tag{27}$$

The eigenvalues of an orthogonal matrix are of magnitude one, and the eigenvectors belonging to different eigenvalues are perpendicular.

9 Real Symmetric Matrices

A real symmetric matrix has only real eigenvalues.

For a real symmetric matrix $\mathbf{A}$, an orthogonal matrix $\mathbf{P}$ exists so that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathrm{diag}(\lambda_1, ..., \lambda_n) \tag{28}$$

10 Properties of Eigenvalues

If $\lambda \neq 0$ is an eigenvalue of $\mathbf{A}$, λ is an eigenvalue of $\mathbf{A}^{\mathrm{T}}$, $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$, and $\frac{\det(\mathbf{A})}{\lambda}$ is an eigenvalue of $\mathbf{A}^{*}$.

A root of $\det(\lambda\mathbf{I} - \mathbf{A})$ must correspond to at least one eigenvector. This is because a square matrix of zero determinant has non-zero solutions.

An eigenvalue's algebraic multiplicity is always greater than or equal to its geometric multiplicity, the latter being at least one.

11 Similarity

Matrices $\mathbf{A}$ and $\mathbf{B}$ are defined to be similar when a reversible matrix $\mathbf{P}$ exists so that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B} \tag{29}$$

Similar matrices have the same eigenvalues, the same trace, the same determinant and the same rank.

For a square matrix $\mathbf{A}$, the following three propositions are equivalent to one another.
$\mathbf{A}$ is similar to a diagonal matrix.

A has orthogonal eigenvectors with a number no less than the order of **A**.

The algebraic multiplicities of all eigenvalues of **A** equal to their geometric multiplicities respectively.

Any square matrix is similar to a upper triangular matrix, with its eigenvalues as the principal diagonal.

12 Matrix as Transformations

If linear transformation $\mathcal{A}$ transforms bases $(\vec{\alpha}_1, ..., \vec{\alpha}_n)$ into $(\vec{\beta}_1, ..., \vec{\beta}_n)$, the transitional matrix is T, where

$$(\vec{\beta}_1, ..., \vec{\beta}_n) = (\vec{\alpha}_1, ..., \vec{\alpha}_n)\mathbf{T} \tag{30}$$

The coordinates of the same vector (expressed as column vectors) is transformed from **x** to **y**, where

$$\mathbf{y} = \mathbf{T}^{-1}\mathbf{x} \tag{31}$$

Suppose a linear transformation corresponds to matrices **A** and **B** in sets of bases $(\vec{\alpha}_1, ..., \vec{\alpha}_n)$ and $(\vec{\beta}_1, ..., \vec{\beta}_n)$ respectively, then

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \tag{32}$$