

数理方程 历年真题汇总

说明

1. 这里收录了若干套中国科学技术大学数理方程(A/B)考试试题, 对扫描质量较差的黑心书店版本试卷内容进行 $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$ 科技排版, 方便读者阅读使用.
2. 按照考试时间先后排序, 其次为A、B卷. 修读数理方程B的同学可以完成大部分数理方程A的试题.
3. 本试题集的主要作用是供同学们考试之前模拟使用, 越靠近现在的考卷越能接近现在的出题风格.
4. 没有参考答案, 希望读者自行思考, 同时熟悉题目类型. 建议助教在考前习题课讲解对应的考试题.
5. 不同试卷的参考公式不一, 教学组没有明确考试会给哪些公式, 读者备考时尽量多记诵一些以防万一.
6. 不同读者的复习备考方法不尽相同, 敬请读者根据自己的需求使用本试题集.
7. 感谢鄢雯哲助教核对试卷! 感谢吴天助教的指导! 预祝读者在期末考试取得满意的成绩!

2019-2020春季学期 数理方程B助教
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2001-2002学年第一学期数理方程期末试题

注：考试时间两小时，前七题中选做六题，第八题必做。试卷中 $a > 0$ 是常数。

一. (15分)解定解问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + 2x, & (t > 0, -\infty < x < \infty), \\ u(t, x)|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 3x^2. \end{cases}$$

二. (15分)线性偏微分算子 $L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x \partial y} - 2 \frac{\partial^2}{\partial y^2}$,

1. 求方程 $L[u] = 0$ 的通解;

2. 解定解问题

$$\begin{cases} L[u] = 0, & (y > 0, -\infty < x < +\infty), \\ u(x, y)|_{y=0} = \sin x, \quad \frac{\partial}{\partial y}|_{y=0} = 0. \end{cases}$$

三. 解定解问题(15分)

1.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, & (t > 0, 0 < x < l), \\ u(t, x)|_{x=0} = \frac{\partial u}{\partial x}|_{x=l} = 0, \\ u(t, x)|_{t=0} = \phi(x), & (\phi(0) = 0). \end{cases}$$

2.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, & (t > 0, 0 < x < l), \\ u(t, x)|_{x=0} = u_0, \quad \frac{\partial u}{\partial x}|_{x=l} = \frac{q_0}{k}, \\ u(t, x)|_{t=0} = u_0. \end{cases}$$

其中 u_0, q_0, k 为常数.

四. (15分)

1. 求解Laplace方程的边值问题

$$\begin{cases} \Delta_2 u = 0, & (r = \sqrt{x^2 + y^2} < 1), \\ \frac{\partial u}{\partial r}|_{r=1} = \cos^2 \theta - \sin^2 \theta. \end{cases}$$

2. 如果把边界条件改为 $\frac{\partial u}{\partial r}|_{r=1} = f(\theta)$, $f(\theta) = f(\theta + 2\pi)$ 且有一阶连续导数及分段二阶连续导数, 上述边值问题是否一定有解? 为什么?

五. (15分)解定解问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, & (t > 0, x > 0), \\ (u - \frac{\partial u}{\partial x})|_{x=0} = 0, \\ u(t, x)|_{t=0} = 1, \quad \frac{\partial u}{\partial t}|_{t=0} = 0. \end{cases}$$

六. (15分)

1. 解定解问题

$$\begin{cases} \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x - \xi, y - \eta), & (x > 0, \xi < +\infty; y > 0, \eta < +\infty), \\ G(x, y; \xi, \eta)|_{x=0} = G(x, y; \xi, \eta)|_{y=0} = 0. \end{cases}$$

2. 利用1)中的 $G(x, y; \xi, \eta)$ 写出定解问题

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & (x > 0; y > 0), \\ u(x, y)|_{x=0} = \phi(y), \quad u(x, y)|_{y=0} = \psi(x). \quad (\phi(0) = \psi(0)) \end{cases}$$

解的积分公式.

七. (15分)求初值问题

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \Delta_2 u + b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + cu + f(t, x, y), & (t > 0, -\infty < x, y < +\infty), \\ u(t, x, y)|_{t=0} = \phi(x, y). \end{cases}$$

的基本解, 并利用基本解写出此定解问题解的积分公式 (b_1, b_2, c 是常数).

八. (10分)用分离变量法求解边值问题

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + x \frac{\partial}{\partial x} (x \frac{\partial}{\partial x}) = 0, & (1 < x < e, 0 < y < 1, 0 < z < +\infty), \\ u(x, y, z)|_{x=1} = u(x, y, z)|_{x=e} = 0, \\ \frac{\partial u}{\partial y}|_{y=0} = \frac{\partial u}{\partial y}|_{y=1} = 0, \\ (u - \frac{\partial u}{\partial z})|_{z=0} = \psi(x, y), \quad \text{且 } z \rightarrow \infty \text{ 时, } u(x, y, z) \text{ 有界.} \end{cases}$$

参考公式

$$\int_0^{+\infty} e^{-a^2 x^2} \cos bxdx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{4a^2}}; \quad L\left[\frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}}\right] = \frac{e^{-a\sqrt{p}}}{\sqrt{p}}; \quad L[l^n] = \frac{n!}{p^{n+1}}, \quad n = 0, 1, 2, 3, \dots;$$

$$L[e^{\lambda t} f(t)] = \bar{f}(p - \lambda); \quad L[f(t - \tau)] = e^{-p\tau} \bar{f}(p), \quad \text{其中 } \bar{f}(p) = L[f(t)].$$

2001-2002学年第二学期数理方程期末试题

一. (20分)

1. 利用镜像法写出上半圆($x^2 + y^2 < a^2, y > 0$)内场位方程第一边值问题的Green函数.
2. 利用达朗贝尔公式求出一维波动方程初值问题的基本解.

二. (45分)解下列定解问题

1.

$$\begin{cases} \Delta_2 u = 0, & (r < 1, 0 < \phi < \pi/4), \\ u|_{\phi=0} = \frac{\partial u}{\partial \phi}|_{\phi=\pi/4} = 0, \\ u|_{r=1} = \sin 2\phi + \sin 6\phi. \end{cases}$$

2.

$$\begin{cases} \Delta_3 u = 0, & (r \neq 1), \\ u|_{r=1} = f(\theta), \\ \lim_{r \rightarrow \infty} u = 0. \end{cases}$$

3.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, & (t > 0, -\infty < x < \infty), \\ \frac{\partial u}{\partial x}|_{x=0} = q(t), & u|_{t=0} = 0, \\ u_x(t, \infty) = u(t, \infty) = 0. \end{cases}$$

三. (20分)

1. 解定解问题($G = G(t, x; \xi)$)

$$\begin{cases} G_{tt} = a^2 G_{xx} + \delta(x - \xi), & (0 < t, 0 < x < l, 0 < \xi < l), \\ G|_{x=0} = G|_{x=l} = 0, \\ G|_{t=0} = 0, & G_t|_{t=0} = 0. \end{cases}$$

2. 利用1)得到的 $G(t, x; \xi)$, 写出定解问题

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x), & (t > 0, 0 < x < l), \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = 0, & u_t|_{t=0} = 0 \end{cases}$$

的解.

四. (15分)(任选一题)

1. 设
- $G(x, y, z; \xi, \eta, \zeta)$
- 为场位方程第三边值问题的Green函数, 即定解问题

$$\begin{cases} \Delta_3 G = -\delta(x - \xi, y - \eta, z - \zeta), ((x, y, z) \in V, (\xi, \eta, \zeta) \in V), \\ (\alpha G + \beta \frac{\partial G}{\partial n})|_S = 0, \alpha, \beta \text{ 是任意常数, } S \text{ 是 } V \text{ 的边界} \end{cases}$$

的解, 试利用第二Green公式, 推出定解问题

$$\begin{cases} \Delta_3 u = 0, ((x, y, z) \in V), \\ (\alpha u + \beta \frac{\partial u}{\partial n})|_S = \phi(x, y, z), \alpha, \beta \text{ 是任意常数, } S \text{ 是 } V \text{ 的边界} \end{cases}$$

的解的积分表达式.

2. 利用积分变换求出三维波动方程初值问题的基本解.

参考公式

1. 设
- $u(x, y, z)$
- 和
- $v(x, y, z)$
- 在区域
- V
- 及边界曲面
- S
- 上有一阶连续偏导数, 在
- V
- 内有二阶连续偏导数, 则有

$$\iiint_V (u\Delta v - v\Delta u) dV = \iint_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

- 2.

$$L[f(t - \tau)] = e^{-p\tau} L[f(t)], \quad L\left[\frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}}\right] = \frac{e^{-a\sqrt{p}}}{\sqrt{p}}$$

- 3.

$$\int_{-\infty}^{+\infty} e^{a\lambda - \beta^2 \lambda^2} d\lambda = \frac{\sqrt{\pi}}{\beta} e^{\frac{a^2}{4\beta^2}}, \quad \beta \neq 0$$

- 4.

$$\int_0^{+\infty} e^{-a^2 x^2} \cos bx dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{4a^2}}$$

2002-2003学年第二学期数理方程期末试题

一. (20分)解定解问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, & (0 < x < l, t > 0), \\ u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = \sin \frac{\pi}{l} x + \sin \frac{2\pi}{l} x, \\ u|_{x=0} = 0, \quad u|_{x=l} = 0. \end{cases}$$

二. (20分)解定解问题

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - u, & (r = \sqrt{x^2 + y^2} < 1, t > 0), \\ u|_{t=0} = x^2 + y^2, \\ u|_{r=1} = e^{-t}. \end{cases}$$

三. (15分)用Laplace变换求解

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} + c^2 u = 0, & (x > 0, y > 0), \quad c > 0 \text{ 为常数}, \\ u|_{x=0} = y, \\ u|_{y=0} = 0. \end{cases}$$

四. (10分)求边值问题

$$\begin{cases} \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x - \xi, y - \eta), & (0 < x, \xi < +\infty, 0 < y, \eta < +\infty), \\ G|_{x=0} = 0, \quad G|_{y=0} = 0 \end{cases}$$

的解 $G(x, y; \xi, \eta)$.

五. (20分)现有初值问题

$$\begin{cases} \frac{\partial u}{\partial t} = 9 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u + f(t, x, y), & ((x, y) \in R^2, t > 0), \\ u|_{t=0} = \phi(x, y), \end{cases}$$

1. 求此初值问题的基本解 $U(t, x, y)$;
2. 利用此基本解写出上述初始问题解的积分表达式.

六. (15分) 设 $L[u] = x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2}$, $xy \neq 0$, 试

1. 求出方程 $L[u] = 0$ 的特征曲线族 $\phi(x, y) = c_1$, $\psi(x, y) = c_2$;
2. 在区域 $x > 0, y > 0$ 内求方程 $L[u] = 0$ 的通解;
3. 求定解问题

$$\begin{cases} L[u] = 0, & (x > 0, xy > 1, y > x), \\ u|_{xy=1} = \frac{1}{x^2}, \\ u|_{y=x^2} = x^2. \end{cases}$$

参考公式

1. 在柱坐标 (r, θ, z) 下,

$$\Delta_3 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

2. 在球坐标 (r, θ, ϕ) 下,

$$\Delta_3 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right]$$

3. ν 阶 Bessel 方程 $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$, 在 $0 < x < +\infty$ 上得基础解组为 $J_\nu(x), N_\nu(x)$, 其中

$$J_\nu(x) = \sum_{k=0}^{+\infty} (-1)^k \frac{1}{k! \Gamma(k + \nu + 1)} \left(\frac{x}{2}\right)^{2k + \nu}$$

2003-2004学年数理方程A期末试题

一. (20分)解定解问题:

$$\begin{cases} \Delta_2 u = 0, \\ u(1, \theta) = 1 + \cos \theta + \cos 2\theta. \end{cases}$$

二. (20分)解定解问题

$$\begin{cases} u_{tt} = u_{xx} + 2xt, \\ u|_{x=0} = 0, \quad u|_{x=1} = -\frac{1}{3}t^3, \\ u|_{t=0} = u_t|_{t=0} = 0. \end{cases}$$

三. (20分)将 $y(x) = x^2 - 1, (|x| \leq 1)$ 按零阶贝塞尔函数展开.

四. (20分)解初值问题

$$\begin{cases} u_t = u_{xx} - 2u_x + u + f(t, x), \\ u|_{t=0} = \varphi(x). \end{cases}$$

五. (10分)用 V 表示区域: $x^2 + y^2 + z^2, z > 0$, S 表示 V 的边界, 求 $\begin{cases} \Delta_3 u = 0, \\ u|_S = 0 \end{cases}$ 的基本解.

六. (10分) 验证:

$$u(t, x) = \int_0^l \phi(\xi)G(t, x; 0, \xi)d\xi + \int_0^t d\tau \int_0^l f(\tau, \xi)G(t, x; \tau, \xi)d\xi$$

是定解问题

$$\begin{cases} u_t = Lu + f(t, x), \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = \phi(x) \end{cases}$$

的解. 其中 $G(t, x; \tau, \xi) = G(t - \tau, x; \xi)$, $G(0, x; \xi) = \delta(x; \xi)$ 是该定解问题的基本解.)

2003-2004学年第一学期数理方程B期末试题

一. (20分)解定解问题

$$\begin{cases} u_{tt} - u_{xx} = \sin 2x, & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = 0, & u_t|_{t=0} = 6x^2. \end{cases}$$

二. (20分)解定解问题

$$\begin{cases} \Delta_3 u = 0, & (1 < r < 2, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi), \\ u|_{r=1} = 1 + \cos^2 \theta, \\ u_r|_{r=2} = 0. \end{cases}$$

三. (20分)解定解问题

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) + u, & (t > 0, 0 < x < 1), \\ u|_{x=0} \text{有界}, & u|_{x=1} = 0, \\ u|_{t=0} = \varphi(x). \end{cases}$$

四. (20分)解定解问题

$$\begin{cases} u_t = a^2 u_{xx} + bu_x + cu + f(t, x), & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \varphi(x). \end{cases}$$

其中 a, b, c 为常数.五. (20分)求平面区域 $D: x > 0, y > 0$ 的格林函数 $G(x, y; \xi, \eta)$, 并求下列定解问题的解:

$$\begin{cases} \Delta_2 u = -f(M), & M(x, y) \in D: x > 0, y > 0, \\ u|_l = \varphi(M), & M(x, y) \in l: l \text{为} D \text{的边界}. \end{cases}$$

$$\text{注: } \Delta_3 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}.$$

2004-2005学年第二学期数理方程A期末试题

一. (30分)填空题

1. 设
- $0 < x_0 < l$
- ,
- $\delta(x - x_0)$
- 在
- $[0, l]$
- 上按照正弦函数系
- $\{\sin \frac{n\pi x}{l}\}$
- 的展开式为

$$\delta(x - x_0) = \underline{\hspace{10cm}},$$

- $\delta'(x - x_0)$
- 在
- $[0, l]$
- 上按照余弦函数系
- $\{\cos \frac{n\pi x}{l}\}$
- 的展开式为

$$\delta'(x - x_0) = \underline{\hspace{10cm}}.$$

- 2.
- $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 \delta(x, y)$
- 的Fourier变换是
- $\underline{\hspace{10cm}}.$

3. 已知
- $f(x)$
- 的Fourier变换为
- $F[f(x)] = \frac{A}{2}(\delta(\lambda + \lambda_0) + \delta(\lambda - \lambda_0))$
- , 则

$$f(x) = \underline{\hspace{10cm}}.$$

- 4.
- $\Delta_2 u = f(x, y)$
- 在平面区域
- $D: 0 < \arg z < 1/3\pi$
- 内第一边值问题的Green函数是
- $\underline{\hspace{10cm}}.$

5. 固有值问题

$$\begin{cases} y'' + \lambda y = 0, & (0 < x < 1), \\ y(0) = 0, & y(1) = 0 \end{cases}$$

的固有值为 $\underline{\hspace{10cm}}$, 固有函数为 $\underline{\hspace{10cm}}$, 固有函数的模平方为 $\underline{\hspace{10cm}}.$

二. 解下列初值问题:

1. (10分)

$$\begin{cases} \frac{\partial u}{\partial t} - e^{-x} \frac{\partial u}{\partial x} = 0, & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x. \end{cases}$$

2. (10分)

$$\begin{cases} u_{xx} - u_{yy} + \cos x = 0, & (-\infty < x, y < +\infty), \\ u(x, 0) = 0, & u_y(x, 0) = 4x. \end{cases}$$

3. (10分)

$$\begin{cases} 3\frac{\partial^2 u}{\partial x^2} + 10\frac{\partial^2 u}{\partial x\partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0, & (-\infty < x < +\infty, y > 0), \\ u|_{y=0} = 0, \quad \frac{\partial u}{\partial y}|_{y=0} = \varphi(x). \end{cases}$$

三. 解下列定解问题:

1. (10分)

$$\begin{cases} u_t - u_{xx} + hu = f(t, x), & (t > 0, -\infty < x < +\infty), \\ u(0, x) = 0. \end{cases}$$

2. (15分)

$$\begin{cases} \Delta_2 u = x^2 - y^2, & (r^2 = x^2 + y^2 < a^2), \\ \left(\frac{\partial u}{\partial r} + u\right)|_{x^2+y^2=a^2} = 0. \end{cases}$$

3. (15分)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), & (t > 0, 0 < r = \sqrt{x^2 + y^2} < b), \\ u|_{r=0} \text{有界}, \quad \frac{\partial u}{\partial r}|_{r=b} = 0, \\ u|_{t=0} = \varphi(r), \quad \frac{\partial u}{\partial t}|_{t=0} = 0. \end{cases}$$

注: $\int_0^{+\infty} e^{-a^2 x^2} \cos bxdx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{4a^2}}, (a > 0).$

2005-2006学年第一学期数理方程B期末试题

一. (30分) 填空

1. 方程 $u_{xy} + u_y = 1$ 的通解是_____.
2. 固有值问题: $y'' + \lambda y = 0, y(0), y'(\pi) = 0$ 的固有值 $\lambda_n =$ _____, 对应的固有函数 $y_n(x) =$ _____.
3. 设 $P_{2006}(x)$ 是 2006 阶勒让德多项式, 计算 $\int_{-1}^1 2^{2005} P_{2006}(x) dx =$ _____.
4. 计算 $\delta(x-a)$ 的傅里叶变换 $F(\delta(x-a)) =$ _____.
5. 试将函数 $f(x) = x^3 (-1 < x < 1)$ 按勒让德多项式展开: $f(x) =$ _____.

二. (15分) 求解定解问题

$$\begin{cases} u_{tt} = u_{xx} + 2x, & (t > 0, -\infty < x < +\infty), \\ u(0, x) = 0, & u_t(0, x) = 0. \end{cases}$$

三. (15分) 求解定解问题

$$\begin{cases} u_t = u_{xx}, & (t > 0, 0 < x < \pi), \\ u(t, 0) = 0, & u(t, \pi) = 100, \\ u(0, x) = \frac{100}{\pi}x + \delta(x - \frac{\pi}{2}). \end{cases}$$

四. (15分) 求解定解问题

$$\begin{cases} \frac{1}{x} \frac{\partial}{\partial x} (x \frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial t^2}, & (0 < x < l, t > 0), \\ u(t, 0) \text{ 有限}, & u(t, l) = 0, \\ u|_{t=0} = f(x), & u_t|_{t=0} = 0. \end{cases}$$

五. (10分)

1. 求出区域 $D = \{(x, y) : x^2 + y^2 < 1, y > 0\}$ 上的格林函数 $G(x, y; \xi, \eta), (\xi, \eta) \in D$, 即求解定解问题

$$\begin{cases} \Delta_2 G = -\delta(x - \xi, y - \eta), & (x, y) \in D, \\ G(x, y)|_{x^2+y^2=1} = 0, & G(x, 0) = 0. \end{cases}$$

2. 写出定解问题

$$\begin{cases} \Delta_2 u = -f(x, y), & (x, y) \in D, \\ u(x, y)|_{x^2+y^2=1} = 0, & u(x, 0) = \phi(x) \end{cases}$$

的解的积分表达式.

六. (15分)

1. 求出方程 $u_t = a^2 u_{xx} + bu$ 的柯西问题的基本解 $U(t, x)$, 其中 a 和 b 是常数, 即求定解问题

$$\begin{cases} u_t = a^2 u_{xx} + bu, & (t > 0, -\infty < x < +\infty), \\ u(0, x) = \delta(x). \end{cases}$$

2. 求解柯西问题

$$\begin{cases} u_t = a^2 u_{xx} + bu, & (t > 0, -\infty < x < +\infty), \\ u(0, x) = 1 + x^2. \end{cases}$$

参考公式

1. 勒让德方程 $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, ($n = 0, 1, 2, \dots, -1 < x < 1$); 勒让德多项式: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$, 特别地, $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^2 - 3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$.

2. 贝塞尔方程是 $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$, ($\nu \geq 0, 0 < x < a$), 贝塞尔函数具有微分关系式:

$$\frac{d}{dx} [x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x)$$

和

$$\frac{d}{dx} \left[\frac{J_\nu(x)}{x^\nu} \right] = -\frac{J_{\nu+1}(x)}{x^\nu}.$$

贝塞尔函数在第一、二类边界条件下的模平方 $N_\nu^2 = \int_0^n x J_\nu^2(\omega x) dx$ 分别是

$$N_{\nu 1}^2 = \frac{a^2}{2} J_{\nu+1}^2(\omega a), \quad N_{\nu 2}^2 = \frac{1}{2} \left[a^2 - \left(\frac{\nu}{\omega} \right)^2 \right] J_\nu^2(\omega a).$$

3. 积分 $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$. $f(x)$ 的傅里叶变换定义为 $F(\lambda) = \int_{-\infty}^{+\infty} f(x) e^{i\lambda x} dx$. $F(\lambda) = e^{-a|\lambda|}$ 的傅里叶反变换是 $f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\lambda) e^{-i\lambda x} d\lambda = \frac{a}{\pi(x^2 + a^2)}$, $F(\lambda) = e^{-\lambda^2 t}$ 的傅里叶反变换是 $f(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$.

2005-2006学年第二学期数理方程A期末试题

一. (10分)求解定解问题

$$\begin{cases} x \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \\ u|_{y=0} = x^2. \end{cases}$$

二. (12分)求解定解问题

$$\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 1, \\ u(x, 0) = 3x^2, \quad u_y(x, 0) = \frac{x}{2}. \end{cases}$$

三. (12分)求解以下固有值问题(计算结果中要明确指出固有值和固有函数)

$$1. \begin{cases} \frac{1}{x}(xY')' + \lambda Y = 0, \quad (0 < x < 1), \\ |Y(0)| < +\infty, \quad Y(1) = 0. \end{cases}$$

$$2. \begin{cases} Y'' + \lambda Y = 0, \quad (0 < x < 2), \\ Y(0) = 0, \quad Y'(2) = 0. \end{cases}$$

四. (14分, 超纲)写出泛函

$$J[u(x, y)] = \iint_{x^2+y^2 \leq 1} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 - 2xyu \right] dx dy$$

的Euler方程并求出满足边界条件 $u|_{x^2+y^2=1} = 1$ 的极小元.

五. (8分) 将函数 $f(x) = \delta(x)$ 在 $[-1, 1]$ 上按Legendre多项式 $P_n(x)$ 展开.

六. (14分)求定解问题

$$\begin{cases} u_{tt} = u_{xx} + \cos 3\pi x, \quad (x \in [0, 1], t > 0), \\ u_x(t, 0) = u_x(t, 1) = 0, \\ u_t(0, x) = 0, \quad u(0, x) = 2 \cos \pi x + 4 \cos 2\pi x. \end{cases}$$

七. (14分)求函数 $f_1(x) = \delta(x-1)$, $f_2(x) = e^{ix}$, $f_3(x) = \cos x$ 的Fourier变换 $F[f_1(x)]$, $F[f_2(x)]$, $F[f_3(x)]$ 并利用Fourier变换求初值问题

$$\begin{cases} u_t = 2u_{xx} + u + f(t, x), \quad (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \phi(x). \end{cases}$$

的基本解, 再利用相应公式解出此初值问题.

八. (10分) 已知半空间的场位方程的第一边值问题为:

$$\begin{cases} \Delta_3 u = -f(x, y, z), & (x > 0), \\ u|_{x=0} = \phi(y, z). \end{cases} \quad (1)$$

1. 写出此边值问题的Green函数 G 满足的定解问题, 并求出Green函数 G .
2. 当在半空间的场位方程的第一边值问题(1)中取 $f(x, y, z) = 0$ 时, 到处解 $u(x, y, z)$ 的积分公式.

九.(6分) 用球函数将以下函数展开:

$$f(\theta, \varphi) = \sin^2 \theta (\cos^2 \varphi + 15 \cos \theta \cos 2\varphi)$$

参考公式

1.

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} P_n(x)t^n, \quad (|t| < 1, |x| \leq 1)$$

2.

$$P_n^m(x) = (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \quad (m \leq n); \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad (n = 0, 1, 2, \dots)$$

2006-2007学年第一学期数理方程B期末试题

一. (20分)求解定解问题

$$\begin{cases} u_{tt} - u_{xx} = x + t, & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \sin x, & u_t|_{t=0} = 4x. \end{cases}$$

二. (20分)求解定解问题

$$\begin{cases} \Delta_3 u = 0, & (1 < r < 2), \\ u|_{r=1} = 0, & u|_{r=2} = 1 + \cos \theta, \end{cases}$$

其中 (r, θ, φ) 为球坐标.

三. (24分)求解以下固有值问题(计算结果中要明确指出固有值和固有函数)

1.

$$\begin{cases} Y''(x) + \lambda Y(x) = 0, & (0 < x < 1), \\ Y'(0) = Y'(1) = 0. \end{cases}$$

2.

$$\begin{cases} x^2 Y'' + x Y' + (\lambda x^2 - 1) Y = 0, & (0 < x < b), \\ |Y(0)| < +\infty, & Y(b) = 0. \end{cases}$$

3.

$$\begin{cases} \Delta_2 u + \lambda u = 0, & (0 < x < 2, 0 < y < 3), \\ u|_{x=0} = u|_{x=2} = u|_{y=0} = u|_{y=3} = 0. \end{cases}$$

四. 设初值问题

$$(*) \begin{cases} u_t = 2u_x + f(t, x), & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \varphi(x). \end{cases}$$

1. (10分)求上述初值问题的基本解 $U(t, x)$.

2. (10分)求初值问题(*)的解.

五. 设平面区域 $D = \{(x, y) | y > x\}$,

1. (10分) 求 D 内格林函数 G :

$$\begin{cases} \Delta_2 G = -\delta(x - \xi, y - \eta), & ((x, y) \in D, (\xi, \eta) \in D), \\ G|_{y=x} = 0. \end{cases}$$

2. (6分) 求边值问题

$$\begin{cases} \Delta_2 u = -f(x, y), & ((x, y) \in D), \\ u|_{y=x} = \varphi(x) \end{cases}$$

的解.

参考公式

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

2007-2008学年第二学期数理方程A期末试题

一. (14分) 设 $u = u(t, x)$, 求解以下定解问题:

1.

$$\begin{cases} u_{tx} = x, & (t > 0, x > 0), \\ u(0, x) = 1 + \sin x, & u(t, 0) = 1. \end{cases}$$

2.

$$\begin{cases} u_{tt} = 9u_{xx}, & (t > 0, -\infty < x < +\infty), \\ u(0, x) = \cos x, & u_t(0, x) = x^2. \end{cases}$$

二. (10分) 求解定解问题

$$\begin{cases} 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0, \\ u|_{x=0} = y^2 - z. \end{cases}$$

三. (14分) 求解定解问题

$$\begin{cases} u_t = 4u_{xx}, & (t > 0, 0 < x < 2), \\ u(t, 0) = u(t, 2) = 0, \\ u(0, x) = \delta(x - 1). \end{cases}$$

四. (10分) 求解以下固有值问题(计算结果中要明确指出固有值和固有函数)

1.

$$\begin{cases} [(1-x^2)y']' + \lambda y = 0, & (0 < x < 1), \\ y(0) = 0, & |y(1)| < +\infty. \end{cases}$$

2.

$$\begin{cases} \Delta_2 u + \lambda u = 0, & (0 < x < 1, 0 < y < 2), \\ \frac{\partial u}{\partial x}|_{x=0} = u|_{x=1} = u|_{y=0} = \frac{\partial u}{\partial y}|_{y=2} = 0. \end{cases}$$

五. (8分, 超纲) 写出泛函

$$J[y(x)] = \int_1^2 (y'^2 - 2xy) dx$$

的Euler方程并求出满足边界条件 $y(1) = 0, y(2) = -1$ 的极值元.

六. (12分)求解定解问题

$$\begin{cases} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, & (0 < r < 1, 0 < z < 1), \\ |u(0, z)| < +\infty, & u(1, z) = 0, \\ u(r, 0) = 0, & u(r, 1) = 1 - r. \end{cases}$$

七. (14分)求初值问题

$$\begin{cases} u_t = u_{xx} + 2u_y + u + f(t, x, y), & (t > 0, -\infty < x, y < +\infty), \\ u|_{t=0} = \phi(x, y) \end{cases}$$

的解的积分表达式.

八. (8分)设空间区域 $V = \{(x, y, z) | x > 0, y > 0\}$, 试求定解问题

$$\begin{cases} \Delta_3 G = -\delta(x - \xi, y - \eta, z - \zeta), & ((x, y, z) \in V, (\xi, \eta, \zeta) \in V), \\ G|_S = 0, & (\text{其中 } S \text{ 是 } V \text{ 的边界}) \end{cases}$$

的解 $G(x, y, z; \xi, \eta, \zeta)$.

九. (10分)求解定解问题

$$\begin{cases} u_{tt} = u_{xx} + \sin \frac{3}{2}x, & (t > 0, 0 < x < \pi), \\ u(t, 0) = 0, & u_x(t, \pi) = 1, \\ u(0, x) = x + \sin \frac{x}{2} + 5 \sin \frac{5x}{2}, & u_t(0, x) = \sin \frac{3x}{2}. \end{cases}$$

参考公式

1.

$$(x^\gamma J_\gamma)' = x^\gamma J_{\gamma-1}, N_{\gamma 1n}^2 = \frac{a^2}{2} J_{\gamma+1}^2(\omega_{1n} a)$$

2.

$$\frac{1}{\pi} \int_0^{+\infty} e^{-a^2 \lambda^2 t} \cos \lambda x d\lambda = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right)$$

2008-2009学年第二学期数理方程A期末试题

一. (12分)求下面方程的通解:

$$u_{xx} - u_{yy} = x^2 - y^2.$$

二. (13分)求解定解问题:

$$\begin{cases} (x^2 + 1) \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0, \\ u|_{x=0} = y^2. \end{cases}$$

三. (15分)求解定解问题:

$$\begin{cases} u_{tt} = u_{xx}, & (0 < x, \xi < 1), \\ u_x|_{x=0} = u_x|_{x=1} = 0, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = \delta(x - \xi). \end{cases}$$

四. (10分)求矩形域 $[0, a] \times [0, b]$ 上问题

$$\begin{cases} u_{xx} + u_{yy} + u_x + \lambda u = 0, \\ u|_{x=0} = u|_{x=a} = u|_{y=0} = u|_{y=b} = 0 \end{cases}$$

的固有值和固有函数.

五. (15分)求解以下定解问题, 其中 (r, θ, φ) 为球坐标:

$$\begin{cases} \Delta_3 u = 1, & (r < 1), \\ u|_{r=1} = \cos 2\theta. \end{cases}$$

六. (15分)先求下面Cauchy问题的基本解, 再求该定解问题解的积分公式:

$$\begin{cases} u_t = u_{xx} + 2u_x + f(t, x), & (t > 0, -\infty < x < +\infty), \\ u(0, x) = \phi(x). \end{cases}$$

七. (20分) 设 D 为圆心在原点, 半径 r_0 的圆盘, 考虑定解问题

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \Delta_2 u, & (t > 0, M(x, y) \in D), \\ \frac{\partial u}{\partial n} |_{M(x, y) \in \partial D} = 0, \\ u(0, M) = \phi(x, y). \end{cases}$$

1. 求 $u(t, M)$.
2. 证明 $\int_D u(t, M) dM = \int_D \phi(M) dM$.
3. 对任意 $M \in D$, 求极限 $\lim_{t \rightarrow \infty} u(t, M)$.
4. 试从物理上说明2,3小题的意义.

参考公式

1. $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, n = 0, 1, 2, \dots$.
2. $\frac{1}{\pi} \int_0^{+\infty} e^{-a^2 \lambda^2 t} \cos \lambda x d\lambda = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right)$.

3. 柱坐标下:

$$\Delta_3 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

4. 球坐标下:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}$$

2013-2014学年第二学期数理方程B期末试题

一. (16分)求下列偏微分方程的通解 $u = u(x, y)$:

$$1. \frac{\partial^2 u}{\partial x \partial y} = x^2 y.$$

$$2. y \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = xy.$$

二. (10分)求下列固有值问题的解, 要求明确指出固有值及其所对应的固有函数:

$$\begin{cases} x^2 y'' + xy' + \lambda x^2 y = 0, & (0 < x < 2), \\ |y(0)| < +\infty, & y'(2) = 0. \end{cases}$$

三. (12分)求第一象限 $D = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$ 的第一边值问题的Green函数.

四. (12分)用积分变换法求解下列方程:

$$\begin{cases} u_t = a^2 u_{xx} + u, & (-\infty < x < +\infty, t > 0), \\ u(0, x) = \varphi(x). \end{cases}$$

五. (15分)用分离变量法求解下列方程:

$$\begin{cases} \Delta_2 u = 0, & (r < 2), \\ u|_{r=2} = \sin \theta + 2 \sin 5\theta - 7 \cos 4\theta. \end{cases}$$

六. (15分)用分离变量法求解下列方程:

$$\begin{cases} u_{tt} = 4u_{xx}, & (0 < x < 1, t > 0), \\ u(t, 0) = 0, & u(t, 1) = 1, \\ u(0, x) = \varphi(x) + x, & u_t(0, x) = \delta(x - \frac{1}{2}). \end{cases}$$

七. (15分)用分离变量法求解下列方程:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = z, & (x^2 + y^2 + z^2 < 1), \\ u|_{x^2 + y^2 + z^2 = 1} = 0 \end{cases}$$

八. (5分)求解下列定解问题:

$$\begin{cases} 4u_{xx} = u_{tt} + 2u_t + u, & (-\infty < x < +\infty, t > 0), \\ u(0, x) = 2 \cos x, & u_t(0, x) = 2x. \end{cases}$$

提示: 先对泛定方程进行变换成为一个较为简单的泛定方程, 再根据初始条件进行求解.

参考公式: 包括极坐标和球坐标下的Laplace算子表达式, Fourier级数及其系数的公式, Laplace和Fourier所有性质和变换公式及求解过程中用到的反变换公式, 勒让德方程的固有值和固有函数以及勒让德函数 $n=1-5$ 时的表达式.

注: 本卷为考后回忆版本, 未给具体公式内容, 请同学自行参考其它卷子的相关公式.

2014-2015学年第二学期数理方程A期末试题

一. (15分) 设 $a \neq b$ 为实常数, 考察二阶线性齐次方程:

$$u_{xx} - (a+b)u_{xy} + abu_{yy} = 0, \quad (-\infty < x, y < +\infty).$$

1. 是判断方程的类型(椭圆/双曲线/抛物线).
2. 试将该方程化成标准型.
3. 求出该方程的解.
4. 求出该方程满足的条件: $u(x, -ax) = \varphi(x)$, $u(x, -bx) = \psi(x)$ 的特解, 其中 $\varphi(0) = \psi(0)$.

二. (10分) 考察一阶线性非齐次方程:

$$\frac{\partial u}{\partial x} + 2x \frac{\partial u}{\partial y} = y, \quad (-\infty < x, y < +\infty).$$

1. 求出此方程的特征线.
2. 求出此方程满足条件 $u(0, y) = 1 + y^2$ 的解.

三. (20分) 考察定解问题:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2} + f(t, x), & (0 < x < \pi, t > 0), \\ u|_{x=0} = 0, \quad u|_{x=\pi} = 0, \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x). \end{cases}$$

1. 当 $f(t, x) = 0$ 时, 求此定解问题的解 u_1 .
2. 当 $f(t, x) = \sin 2x \sin \omega t$ (其中 $\omega \neq 4$), $\varphi(x) = 0$, $\psi(x) = 0$ 时, 求此定解问题的解 u_2 以及 $\lim_{\omega \rightarrow 4} u_2(x, t, \omega)$ 的值.

四. (20分) 考察定解问题:

$$\begin{cases} \Delta_3 u = 0, & (r < a, 0 < \theta < 2\pi, 0 < z < h), \\ u|_{r=a} = 0, \\ u|_{z=0} = g_1(r, \theta), \quad u|_{z=h} = g_2(r, \theta). \end{cases}$$

1. 当 $g_1(r, \theta) = 0$, $g_2(r, \theta) = f(r)$ 时, 求此定解问题的解.
2. 当 $g_1(r, \theta) = \varphi(r, \theta)$, $g_2(r, \theta) = \psi(r, \theta)$ 时, 可作分离变量: $u = R(r)\Theta(\theta)Z(z)$, 分别求出 R, Θ, Z 满足的常微分方程, 并写出此时与定解问题相应的固有值问题.

五. (15分)考察初值问题:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_3 u + 3u + f(t, x, y, z), & (t > 0, -\infty < x, y, z < +\infty), \\ u|_{t=0} = \varphi(x, y, z). \end{cases}$$

1. 求出此问题的基本解.
2. 当 $f(t, x, y, z) = 0$, $\varphi(x, y, z) = e^{-(x^2+y^2+z^2)}$ 时, 求此问题的解.

六. (15分)已知右半平面区域 $S = \{(x, y) | x > 0, -\infty < y < +\infty\}$

1. 求出 S 内 Poisson 方程第一边值问题的 Green 函数.
2. 求解定解问题:

$$\begin{cases} u_{xx} + 25u_{yy} = 0, & (x > 0, -\infty < y < +\infty), \\ u|_{x=0} = \varphi(y). \end{cases}$$

七. (5分)求方程: $Z'(\theta) + \cot \theta Z(\theta) + 20Z(\theta) = 0$, $(0 < \theta < \frac{\pi}{2})$ 满足条件 $Z(0) = 1$ 的解 $Z(\theta)$, 并求 $Z(\frac{\pi}{2})$.

2015-2016学年第二学期数理方程B期末试题

一. (12分)求以下固有值问题的固有值和固有函数:

$$\begin{cases} Y''(x) + \lambda Y(x) = 0, & (0 < x < 16), \\ Y'(0) = 0, & Y'(16) = 0. \end{cases}$$

二. (16分)利用分离变量法求解定解问题:

$$\begin{cases} u_t = 4u_{xx}, & (t > 0, 0 < x < 5), \\ u(t, 0) = u(t, 5) = 0, \\ u(0, x) = \phi(x). \end{cases}$$

并求 $\phi(x) = \delta(x - 2)$ 时此定解问题的解.

三. (14分)考虑初值问题:

$$\begin{cases} u_{tt} = 4u_{xx} + f(t, x), & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x^2, & u_t|_{t=0} = \sin 2x. \end{cases}$$

1. 如取 $f(t, x) = 0$, 求此初值问题的解.
2. 如取 $f(t, x) = t^2 x^2$, 求此初值问题相应的解.

四. (14分)求解以下初值问题

$$\begin{cases} u_{tt} = 4u_{xx} + 5u, & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \phi(x). \end{cases}$$

并指出当 $\phi(x) = e^{-x^2}$ 时此定解问题的解.

五. (16分)求解以下定解问题:

$$\begin{cases} u_t = u_{rr} + \frac{1}{r}u_r, & (0 < r < 1), \\ |u(t, 0)| < +\infty, & u(t, 1) = 0, \\ u|_{t=0} = \phi(r). \end{cases}$$

并算出 $\phi(r) = J_0(ar) + 3J_0(br)$ 时的解, 其中 $0 < a < b$, 且 $J_0(a) = J_0(b) = 0$.

六. (14分)已知下半空间 $V = \{(x, y, z) | x < 0, -\infty < x, y < +\infty\}$.

1. 求出 V 内泊松方程第一边值问题的Green函数.

2. 求解定解问题:

$$\begin{cases} 4u_{xx} + u_{yy} + u_{zz} = 0, & (z < 0, -\infty < x, y < +\infty), \\ u|_{z=0} = \varphi(x, y). \end{cases}$$

七. (6分)对于三维波动方程

$$u_{tt} = a^2 \Delta_3 u, \quad (a > 0, t > 0, -\infty < x, y, z < +\infty)$$

它的形如 $u = u(t, r) = T(t)R(r)$ 的解称为方程的可分离变量的径向对称解, 求方程满足 $\lim_{t \rightarrow +\infty} u = 0$ 的可分离变量的径向对称解, 这里 $r = \sqrt{x^2 + y^2 + z^2}$.

八. (8分)考虑固有值问题

$$\begin{cases} \frac{d}{dx}[(1-x^2)y'] + \lambda y = 0, & (0 < x < 1), \\ y'(0) = 0, |y(1)| < +\infty. \end{cases}$$

1. 求此固有值问题的固有值和固有函数.

2. 把 $f(x) = 2x + 1$ 按此固有值问题所得到的固有函数系展开.

参考公式

1. 直角坐标系: $\Delta_3 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$, 柱坐标系: $\Delta_3 u = \frac{1}{r} r \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$,
球坐标系: $r^2 \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$.

2. 若 ω 是 $J_\nu(\omega a) = 0$ 的一个正根, 则有模平方 $N_{\nu 1}^2 = \|J_\nu(\omega x)\|_1^2 = \frac{a^2}{2} J_{\nu+1}^2(\omega a)$.

若 ω 是 $J'_\nu(\omega a) = 0$ 的一个正根, 则有模平方 $N_{\nu 2}^2 = \|J_\nu(\omega x)\|_2^2 = \frac{1}{2} \left[a^2 - \frac{\nu^2}{\omega^2} \right] J_\nu^2(\omega a)$.

3. 勒让德多项式: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$, $\int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1}$, $n = 0, 1, 2, \dots$,

母函数: $(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} P_n(x) t^n$, 递推公式: $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$.

4. $\frac{1}{\pi} \int_0^{+\infty} e^{-a^2 \lambda^2 t} \cos \lambda x d\lambda = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right)$

5. 设 $G(M; M_0)$ 是三维Poisson方程第一边值问题

$$\begin{cases} \Delta_3 u = -f(M), & (M = (x, y, z) \in V), \\ u|_S = \phi(M) \end{cases}$$

对应的Green函数, 则

$$u(M_0) = - \iint_S \phi(M) \frac{\partial G}{\partial n}(M; M_0) dS + \iiint_V f(M) G(M; M_0) dM,$$

其中 $M_0 = (\xi, \eta, 0)$.

2016-2017学年第二学期数理方程B期末试题

一. (10分)求方程 $u_x + yu_{xy} = 0$ 的一般解.

二. (10分)求解一维半无界弦的自由振动问题:

$$\begin{cases} u_{tt} = 9u_{xx}, & (t > 0, 0 < x < +\infty), \\ u|_{x=0} = 0, \\ u|_{t=0} = x, \quad u_t|_{t=0} = 2 \sin x. \end{cases}$$

三. (20分)考察一维有界限振动问题:

$$\begin{cases} u_{tt} = u_{xx} + f(t, x), & (t > 0, 0 < x < \pi), \\ u|_{x=0} = 0, \quad u_x|_{x=\pi} = 0, \\ u|_{t=0} = \sin \frac{3}{2}x, \quad u_t|_{t=0} = \sin \frac{x}{2}. \end{cases}$$

1. 当 $f(t, x) = 0$ 时, 求出上述定解问题的解 $u_1(x)$.
2. 当 $f(t, x) = \sin \frac{x}{2} \sin \omega t$, ($\omega \neq k + \frac{1}{2}, k \in \mathbb{N}$)时, 求出上述定解问题的解 $u_2(t, x)$.
3. 指出定解问题中方程非齐次项 $f(t, x)$, 边界条件和初始条件的物理意义.

四. (15分)求解定解问题:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x} \right) + u, & (t > 0, 0 < x < 1), \\ u|_{x=0} \text{有界}, \quad u_x|_{x=1} = 0, \\ u|_{t=0} = \varphi(x). \end{cases}$$

五. (15分)求解如下泊松方程的边值问题:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = z, & (x^2 + y^2 + z^2 < 1), \\ u|_{x^2+y^2+z^2=1} = 0. \end{cases}$$

六. (15分)设区域 $\Omega = \{(x, y) | y \geq x\}$.

1. 求区域 Ω 上的Poisson方程Dirichlet边值问题的Green函数.
2. 求解如下Poisson方程的Dirichlet边值问题:

$$\begin{cases} \Delta_2 u = 0, & ((x, y) \in \Omega), \\ u(x, x) = \phi(x). \end{cases}$$

七. (15分)考察定解问题:

$$\begin{cases} u_t = 4u_{xx} + 3u, & (-\infty < x < +\infty, t > 0), \\ u(0, x) = \varphi(x). \end{cases}$$

1. 求出上述定解问题相应的基本解.
2. 当 $\varphi(x) = x$ 时, 求解上述定解问题.

参考公式

1. 拉普拉斯算子 Δ_3 在各个坐标系下的表达式

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.$$

2. 二阶欧拉方程: $x^2 y'' + pxy' + qy = f(x)$, 在作变量代换 $x = e^t$ 下, 可以约化为常系数线性微分方程:

$$\frac{d^2 y}{dt^2} + (p-1) \frac{dy}{dt} + qy = f(e^t).$$

3. Legendre方程: $[(1-x^2)y']' + \lambda y = 0$; n 阶Legendre多项式:

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k} = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$; \text{Legendre多项式的母函数: } (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} P_n(x)t^n, (|t| < 1);$$

$$\text{Legendre多项式的模平方: } \|P_n(x)\|^2 = \frac{2}{2n+1}.$$

4. ν 阶Bessel方程: $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$; ν 阶Bessel函数: $J_\nu(x) = \sum_{l=0}^{+\infty} (-1)^l \frac{1}{l! \Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$

$$\text{Bessel函数的母函数: } e^{\frac{x}{2}(\zeta - \zeta^{-1})} = \sum_{n=-\infty}^{+\infty} J_n(x)\zeta^n; \text{Bessel函数在三类边界条件下的模平方: } N_{\nu 1n}^2 =$$

$$\frac{a^2}{2} J_{\nu+1}^2(\omega_{1n}a), N_{\nu 2n}^2 = \frac{1}{2} \left[a^2 - \frac{\nu^2}{\omega_{2n}^2} \right] J_\nu^2(\omega_{2n}a), N_{\nu 3n}^2 = \frac{1}{2} \left[a^2 - \frac{\nu^2}{\omega_{2n}^2} + \frac{a^2 \alpha^2}{\beta^2 \omega_{3n}^2} \right] J_\nu^2(\omega_{3n}a).$$

5. 傅里叶变换和逆变换: $\mathcal{F}[f](\lambda) = \int_{-\infty}^{+\infty} f(x)e^{i\lambda x} dx$; $\mathcal{F}^{-1}[F](x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\lambda)e^{-i\lambda x} d\lambda$; $\mathcal{F}^{-1}[e^{-\lambda^2}] =$

$$\frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}}.$$

6. 拉普拉斯变换: $L[f(t)] = \int_0^{+\infty} f(t)e^{-pt} dt, p = \sigma + is$; $L[e^{\alpha t}] = \frac{1}{p-\alpha}$, $L[t^\alpha] = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$, $L[\sin t] =$

$$\frac{1}{p^2+1}, L[\cos t] = \frac{p}{p^2+1}, L\left[\frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}}\right] = \frac{e^{-a\sqrt{p}}}{\sqrt{p}}.$$

7. 拉普拉斯方程 $\Delta_3 u = \delta(M)$ 的基本解:

$$\text{二维, } U(x, y) = -\frac{1}{2\pi} \ln \frac{1}{r}, r = \sqrt{x^2 + y^2};$$

$$\text{三维, } U(x, y, z) = -\frac{1}{4\pi r}, r = \sqrt{x^2 + y^2 + z^2}.$$

2018-2019学年第二学期数理方程B期末试题

一. 设有一个均匀圆柱物体, 半径为 a , 高为 h , 侧面在温度为零的空气中自由冷却. 上底绝热, 下底温度为 $g(t, x, y)$, 初始温度为 $\varphi(x, y, z)$, 试写出圆柱体内温度所满足的定解问题. (不用求解, 仅列方程)

二. 求解一维无界弦的振动问题

$$\begin{cases} u_{tt} = u_{xx} - 4t + 2x, & (-\infty < x < +\infty, t > 0), \\ u|_{t=0} = x^2, \quad u_t|_{t=0} = \sin 3x. \end{cases}$$

三. 求解固有值问题

$$\begin{cases} y'' + 2y' + \lambda y = 0, & (0 < x < 9), \\ y(0) = y(9) = 0. \end{cases}$$

四. 求解一维有界弦的振动问题

$$\begin{cases} u_{tt} = u_{xx}, & (0 < x < 1, t > 0), \\ u|_{x=0} = u|_{x=1} = 1, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \end{cases}$$

五. 求解如下泊松方程的边值问题

$$\begin{cases} \Delta_3 u = 0, & (x^2 + y^2 < 1, 0 < z < 1), \\ u|_{x^2+y^2=1} = 0, \\ u|_{x=0} = 0, \quad u|_{x=1} = 1 - (x^2 + y^2). \end{cases}$$

六. 求解热传导问题

$$\begin{cases} u_t = u_{xx} + u, & (-\infty < x < +\infty, t > 0), \\ u(0, x) = e^{-x^2}. \end{cases}$$

七. 设平面区域 $\Omega = \{(x, y) | x + y > 0\}$

1. 求出区域 Ω 的Green函数.

2. 求出区域 Ω 的定解问题:

$$\begin{cases} \Delta_2 u = 0, & (x, y) \in \Omega, \\ u(x, -x) = \varphi(x). \end{cases}$$

八. 计算积分

$$\int_{-1}^1 P_4(x)(1+x+2x^2+3x^3+4x^4)dx$$

参考公式

1. 拉普拉斯算子
- Δ_3
- 在各个坐标系下的表达式

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.$$

2. Legendre方程:
- $[(1-x^2)y']' + \lambda y = 0$
- ;
- n
- 阶Legendre多项式:

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k} = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

; Legendre多项式的母函数: $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} P_n(x)t^n, (|t| < 1)$;

Legendre多项式的模平方: $\|P_n(x)\|^2 = \frac{2}{2n+1}$.

Legendre多项式满足的递推公式($n \geq 1$): $(n+1)P_{n+1}(x) - x(2n+1)P_n(x) + nP_{n-1}(x) = 0$, $nP_n(x) - xP'_n(x) + P'_{n-1}(x) = 0$, $nP_{n-1}(x) - P'_n(x) + xP'_{n-1}(x) = 0$, $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

- 3.
- ν
- 阶Bessel方程:
- $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$
- ;
- ν
- 阶Bessel函数:
- $J_\nu(x) = \sum_{l=0}^{+\infty} (-1)^l \frac{1}{l! \Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$

Bessel函数的母函数: $e^{\frac{x}{2}}(\zeta - \zeta^{-1}) = \sum_{n=-\infty}^{+\infty} J_n(x)\zeta^n$; Bessel函数在三类边界条件下的模平方: $N_{\nu 1n}^2 =$

$\frac{a^2}{2} J_{\nu+1}^2(\omega_{1n}a), N_{\nu 2n}^2 = \frac{1}{2} \left[a^2 - \frac{\nu^2}{\omega_{2n}^2} \right] J_\nu^2(\omega_{2n}a), N_{\nu 3n}^2 = \frac{1}{2} \left[a^2 - \frac{\nu^2}{\omega_{2n}^2} + \frac{a^2 \alpha^2}{\beta^2 \omega_{3n}^2} \right] J_\nu^2(\omega_{3n}a)$. Bessel函数满

足微分关系和递推公式: $\frac{d}{dx} (x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$, $\frac{d}{dx} \left(\frac{J_\nu(x)}{x^\nu} \right) = -\frac{J_{\nu+1}(x)}{x^\nu}$

4. 傅里叶变换和逆变换:
- $\mathcal{F}[f](\lambda) = \int_{-\infty}^{+\infty} f(x)e^{i\lambda x} dx$
- ;
- $\mathcal{F}^{-1}[F](x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\lambda)e^{-i\lambda x} d\lambda$
- ;
- $\mathcal{F}^{-1}[e^{-\lambda^2}] = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}}$
- .

5. 拉普拉斯变换:
- $L[f(t)] = \int_0^{+\infty} f(t)e^{-pt} dt, p = \sigma + is$
- ;
- $L[e^{\alpha t}] = \frac{1}{p-\alpha}$
- ,
- $L[t^\alpha] = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$

6. 拉普拉斯方程
- $\Delta_3 u = \delta(M)$
- 的基本解:

二维, $U(x, y) = -\frac{1}{2\pi} \ln \frac{1}{r}, r = \sqrt{x^2 + y^2}$;

三维, $U(x, y, z) = -\frac{1}{4\pi r}, r = \sqrt{x^2 + y^2 + z^2}$.

7. Green第一公式:
- $= \iiint_V u \Delta v dV + \iint_V \nabla u \nabla v dV$

Green第二公式: $\iint_{\partial V} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS = \iiint_V (u \Delta v - v \Delta u) dV$

2019-2020学年第二学期数理方程B期末试题(毕业年级重修)

一. (18分)求解下列Cauchy问题:

1.

$$\begin{cases} u_{tt} = 4u_{xx}, & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x^2, & u_t|_{t=0} = \cos 2x. \end{cases}$$

2.

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} = 20, \\ u(0, y) = y^2, & u(x, 0) = \sin x. \end{cases}$$

二. (18分)求以下固有值问题的固有值和固有函数:

1.

$$\begin{cases} Y''(x) + \lambda Y(x) = 0, & (0 < x < \pi), \\ Y'(0) = 0, & Y'(\pi) = 0. \end{cases}$$

2.

$$\begin{cases} x^2 Y''(x) + x Y'(x) + \lambda Y(x) = 0, & (1 < x < b), \\ Y(1) = 0, & Y'(b) = 0. \end{cases}$$

三. (18分)

1. 求周期边界条件下

$$\begin{cases} u_{tt} = u_{xx}, & (t > 0, 0 < x < 1), \\ u(t, 0) = u(t, 1), & u_x(t, 0) = u_x(t, 1) \end{cases}$$

的分离变量解 $u = T(t)X(x)$.

2. 求解

$$\begin{cases} u_{tt} = u_{xx}, & (t > 0, 0 < x < 1), \\ u(t, 0) = u(t, 1), & u_x(t, 0) = u_x(t, 1), \\ u(0, x) = \sin 2\pi x, & u_t(0, x) = 2\pi \cos 2\pi x. \end{cases}$$

四. (14分)求解

$$\begin{cases} u_t = u_{xx} + u, & (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \delta(x+1). \end{cases}$$

五. (18分)

1. P_n 为n阶勒让德函数, 写出 $P_0(x), P_1(x), P_2(x)$, 并计算积分 $\int_{-1}^1 (20+x)P_2(x)dx$.

2. 求解以下定解问题, 其中 (r, θ, ϕ) 为球坐标:

$$\begin{cases} \Delta_3 u = 0, & (r < 2), \\ u|_{r=2} = 3 \cos 2\theta. \end{cases}$$

六. (14分) 已知平面区域 $D = \{(x, y) | -\infty < x < +\infty, y < 1\}$.

1. 写出 D 内泊松方程第一边值问题的Green函数所满足的定解问题, 并求出Green函数.

2. 求解定解问题:

$$\begin{cases} u_{xx} + a^2 u_{yy} = 0, & (-\infty < x < +\infty, y < 1), \\ u|_{y=1} = \varphi(x). \end{cases}$$

参考公式

1. 直角坐标系: $\Delta_3 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$,

柱坐标系: $\Delta_3 u = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$,

球坐标系: $\Delta_3 u = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}$.

2. 若 ω 是 $J_\nu(\omega a) = 0$ 的一个正根, 则有模平方 $N_{\nu 1n}^2 = \frac{a^2}{2} J_{\nu+1}^2(\omega a)$.

若 ω 是 $J'_\nu(\omega a) = 0$ 的一个正根, 则有模平方 $N_{\nu 2n}^2 = \frac{1}{2} \left[a^2 - \frac{\nu^2}{\omega^2} \right] J_\nu^2(\omega a)$.

3. 勒让德多项式: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$, $n = 0, 1, 2, 3, \dots$, 母函数: $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} P_n(x)t^n$, ($|t| < 1$), 递推公式: $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$.

4. $\frac{1}{\pi} \int_0^{+\infty} e^{-a^2 \lambda^2 t} e^{i \lambda x} d\lambda = \frac{1}{\pi} \int_0^{+\infty} e^{-a^2 \lambda^2 t} \cos \lambda x d\lambda = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right)$

5. 二维泊松方程基本解为 $u = \frac{1}{2\pi} \ln r$, 这里 (r, θ) 为极坐标.

6. 由平面区域 D 内Poisson方程第一边值问题的Green函数 $G(M; M_0)$, 求得Poisson方程第一边值问题解 $u(M)$ 的公式是:

$$u(M) = \int_S \varphi(M_0) \frac{\partial G}{\partial n}(M; M_0) dS + \iint_D f(M_0) G(M; M_0) dM_0,$$

其中 S 是 D 的边界.

考察定解问题:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2} + f(t, x), & (0 < x < \pi, t > 0), \\ u|_{x=0} = 0, u|_{x=\pi} = 0, & (t > 0), \\ u|_{t=0} = \phi(x), u_t|_{t=0} = \psi(x), & (0 < x < \pi). \end{cases}$$

- 1) 当 $f(t, x) = 0$ 时, 求此定解问题的解 u_1 ;
- 2) 当 $f(t, x) = \sin 2x \sin \omega t$ (其中 $\omega \neq 4$), $\phi(x) = 0, \psi(x) = 0$ 时, 求此定解问题的解 u_2 .