

2023 春季《数学分析 B2》期中试卷解析

一 (本题 15 分) 讨论函数 $z = f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$ 在 $(0,0)$ 处的 (1)

连续; (2) 可偏导; (3) 可微性

解析: (1) 连续性; 即: $0 \leq \left| \frac{x^3 - y^3}{x^2 + y^2} \right| = |x - y| \left| \frac{x^2 + y^2 + xy}{x^2 + y^2} \right| \leq \frac{3}{2} |x - y| \rightarrow 0$

(2) 偏导数; 即: $f'_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x} = 1$

同理: $f'_y(0,0) = \lim_{y \rightarrow 0} \frac{f(0,y) - f(0,0)}{y} = -1$

(3) 全微分; 即: $\lim_{x \rightarrow 0, y \rightarrow 0} \frac{f(x,y) - f(0,0) - x + y}{\sqrt{x^2 + y^2}}$

$$\text{即: } = \lim_{x \rightarrow 0, y \rightarrow 0} \frac{\frac{x^3 - y^3}{x^2 + y^2} - x + y}{\sqrt{x^2 + y^2}}$$

$$\text{即: } = \lim_{x \rightarrow 0, y \rightarrow 0} \frac{x^3 - y^3 - (x - y)(x^2 + y^2)}{(x^2 + y^2)^{\frac{3}{2}}}$$

$$\text{即: } = \lim_{x \rightarrow 0, y \rightarrow 0} \frac{x^3 - y^3 - x^3 - xy^2 + x^2y + y^3}{(x^2 + y^2)^{\frac{3}{2}}}$$

$$\text{即: } = \lim_{x \rightarrow 0, y \rightarrow 0} \frac{x^2y - xy^2}{(x^2 + y^2)^{\frac{3}{2}}} \neq 0$$

二 (本题 12 分, 每小题 6 分)

1. 设二元函数 $z = f(s, t)$ 有二阶连续偏导数, 求 $z = f(xy, x)$ 的偏导数 $\frac{\partial^2 z}{\partial y \partial x}$

解析: $\frac{\partial z}{\partial x} = yf'_1 + f'_2$

进一步: $\frac{\partial^2 z}{\partial y \partial x} = f'_1 + yxf''_{11} + xf''_{21}$

2. 求由方程 $z^3 - 3xyz = a^3$ 在点 $(0,0,1)$ 附近所确定函数 $z(x, y)$ 的偏导数 $\frac{\partial^2 z}{\partial y \partial x}$. 其中 a

为给定的正常数.

解 1: 设 $F(x, y, z) = z^3 - 3xyz - a^3$

则: $\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = \frac{yz}{z^2 - xy}, \quad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z} = \frac{xz}{z^2 - xy}$

$$\text{得: } \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{yz}{z^2 - xy} \right)$$

$$\text{即: } = \frac{(z + yz'_y)(z^2 - xy) - yz(2zz'_y - x)}{(z^2 - xy)^2}$$

$$\text{到: } = \frac{\left(z + \frac{xyz}{z^2 - xy} \right) (z^2 - xy) - yz \left(2z \frac{xyz}{z^2 - xy} - x \right)}{(z^2 - xy)^2}$$

$$\text{即: } = \frac{z(z^4 - 2xyz^2 - x^2y^2)}{(z^2 - xy)^3}$$

$$\text{即: } \frac{\partial^2 z}{\partial y \partial x} \Big|_{(0,0,1)} = 1.$$

三 (本题 12 分, 每小题 6 分)

1. 计算二重积分 $\iint_D \frac{1-x^2-y^2}{1+x^2+y^2} dx dy$, 其中 $D = \{(x,y) | x^2 + y^2 \leq 1, y \geq 0\}$.

$$\text{解析: } \iint_D \frac{1-x^2-y^2}{1+x^2+y^2} dx dy = 2 \iint_{D_1} \frac{1-x^2-y^2}{1+x^2+y^2} dx dy$$

$$\text{即: } = 2 \int_0^{\pi} d\theta \int_0^1 \frac{1-r^2}{1+r^2} r dr$$

$$\text{即: } = \pi \int_0^1 \left(\frac{2}{1+r^2} - 1 \right) r dr$$

$$\text{即: } = \pi \left(\ln(1+r^2) - \frac{1}{2}r^2 \right) \Big|_0^1$$

$$\text{即: } = \pi \left(\ln 2 - \frac{1}{2} \right).$$

2. 计算三重积分 $\iiint_V (x+y+z) dv$, 其中积分区域 V 由曲面 $z = \sqrt{x^2 + y^2}$ 和曲面 $z = \sqrt{1 - x^2 - y^2}$ 所围成.

分析: 用三重积分的对称性, 第一二项的积分为零

$$\text{解析: } I = \iiint_V z dv$$

$$\text{由对称性: } \iiint_V x dv = 0$$

$$\text{和: } \iiint_V y dv = 0$$

$$\text{故有: } \iiint_V z dv = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^1 \rho^3 \sin \varphi \cos \varphi d\rho$$

$$\text{即: } = \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \sin \varphi \cos \varphi d\varphi$$

$$\text{即: } = \frac{\pi}{2} \frac{1}{2} \sin^2 \varphi \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

四 (本题 15 分)

已知 $f(x,y)$ 满足: $f''_{xy}(x,y) = 2(y+1)e^x$, $f'_x(x,0) = (1+x)e^x$, $f(0,y) = y^2 + 2y$,

(1) 求 $f(x,y)$ 的极值; (2) 用拉格朗日乘数法求 $f(x,y)$ 在条件下 $ye^x = 1$ 的极值.

解析: $f''_{xy}(x,y) = 2(y+1)e^x \Rightarrow f'_x(x,y) = (1+y)^2 e^x + g(x)$

由 $f'_x(x,0) = (1+x)e^x \Rightarrow g(x) = xe^x \Rightarrow f'_x(x,y) = (1+y)^2 e^x + xe^x$.

再对 x 积分: $f(x,y) = (1+y)^2 e^x + (x-1)e^x + h(y)$.

由: $f(0,y) = y^2 + 2y \Rightarrow h(y) = 0$

所以: $f(x,y) = (1+y)^2 e^x + (x-1)e^x$.

下面求 $f(x,y)$ 的极值:

由: $\begin{cases} f'_x(x,y) = (1+y)^2 e^x + xe^x = 0 \\ f'_y(x,y) = 2(y+1)e^x = 0 \end{cases} \Rightarrow (x,y) = (0,-1)$

再: $\begin{cases} f''_{xx}(x,y) = (1+y)^2 e^x + (x+1)e^x \\ f''_{xy}(x,y) = 2(y+1)e^x \\ f''_{yy}(x,y) = 2e^x \end{cases}$

有: $A = f''_{xx}(0,-1) = 1$, $B = f''_{xy}(0,-1) = 0$, $C = f''_{yy}(0,-1) = 2$.

由: $B^2 - AC = -2 < 0$, $A = 1 > 0$, 极小值 $f(0,-1) = -1$.

解析 (2): 从条件中解出 $y = e^{-x}$, 代入 $f(x,y)$ 中, 可看出有极小值.

由拉格朗日乘数法

构造: $L(x,y,\lambda) = (1+y)^2 e^x + (x-1)e^x + \lambda(ye^x - 1)$

有: $\begin{cases} (1+y)^2 e^x + xe^x + \lambda ye^x = 0 \\ 2(y+1)e^x + \lambda e^x = 0 \\ ye^x - 1 = 0 \end{cases}$

$$\text{即: } \begin{cases} (1+y)^2 + x + \lambda y = 0 \\ 2(y+1) + \lambda = 0 \\ ye^x = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda = -4 \end{cases}$$

故条件极值点为(0,1), 条件极小值为 3.

五 (本题 16 分, 每小题 8 分)

1. 设 a, b 是正数, Γ 是椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, 计算第一型曲线积分 $\int_{\Gamma} |xy| ds$.

解析: 设 Γ 的参数方程为: $x = a \cos \varphi, y = b \sin \varphi, 0 \leq \varphi \leq 2\pi$.

$$\text{所以: } \int_{\Gamma} |xy| ds = ab \int_0^{2\pi} |\cos \varphi \sin \varphi| \sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi} d\varphi$$

$$\text{即: } = 4ab \int_0^{\pi} \sin \varphi \cos \varphi \sqrt{(a^2 - b^2) \sin^2 \varphi + b^2} d\varphi$$

$$\text{即: } = 2ab \int_0^1 \sqrt{(a^2 - b^2)t + b^2} dt$$

$$\text{即: } = 2ab \frac{2}{3(a^2 - b^2)} ((a^2 - b^2)t + b^2)^{\frac{3}{2}} \Big|_0^1$$

$$\text{即: } = \frac{4ab}{3(a^2 - b^2)} (a^3 - b^3)$$

$$\text{即: } = \frac{4ab}{3(a+b)} (a^2 + b^2 + ab)$$

注: 以上在 $a > b$ 的情况下推出结论; 当 $a < b$ 和 $a = b$ 情况时,

所得结论一样.

2. 设 a, b, c 是常数, S 是圆球面 $x^2 + y^2 + z^2 = R^2, (R > 0)$, 计算第一型曲面积分

$$\iint_S (ax + by + cz)^2 dS.$$

解析: 由对称性可知

$$\text{有: } \iint_S xy dS = \iint_S yz dS = \iint_S zx dS = 0$$

$$\text{且: } \iint_S x^2 dS = \iint_S y^2 dS = \iint_S z^2 dS$$

$$\text{因此: } \iint_S (ax + by + cz)^2 dS = \iint_S (a^2 + b^2 + c^2) x^2 dS$$

$$\text{即: } = \frac{a^2 + b^2 + c^2}{3} \iint_S (a^2 + b^2 + c^2) dS$$

$$\text{即: } = \frac{a^2+b^2+c^2}{3} \iint_S R^2 dS$$

$$\text{即: } = \frac{a^2+b^2+c^2}{3} 4\pi R^4.$$

六 (本题 10 分) 证明曲面 $z + \sqrt{x^2 + y^2 + z^2} = x^3 f(\frac{y}{x})$ 上任意点处的切平面在 Oz 轴上的截距与切点到原点的距离之比为常数, 并求此常数.

证明: 令: $r = \sqrt{x^2 + y^2 + z^2}$, 则 r 表示点 (x, y, z) 到原点的距离

$$\text{设: } u = \frac{y}{x}; \text{ 且 } F(x, y, z) = z + r - x^3 f(u)$$

$$\text{则: } F'_x(x, y, z) = \frac{x}{r} - 3x^2 f(u) + xyf'(u)$$

$$\text{和: } F'_y(x, y, z) = \frac{y}{r} - x^2 f'(u)$$

$$\text{及: } F'_z(x, y, z) = \frac{z}{r} + 1.$$

则曲面在任意点 (x, y, z) 的切平面方程

$$\text{为: } F'_x(x, y, z)(X - x) + F'_y(x, y, z)(Y - y) + F'_z(x, y, z)(Z - z) = 0$$

$$\text{即: } F'_x X + F'_y Y + F'_z Z = F'_x x + F'_y y + F'_z z = -2(r + z)$$

$$\text{转化为截距式方程: } \frac{X}{\frac{2(r+z)}{F'_x}} + \frac{Y}{\frac{2(r+z)}{F'_y}} + \frac{Z}{\frac{2(r+z)}{F'_z}} = 1$$

$$\text{切平面在 } Oz \text{ 轴上的截距为: } c = -\frac{2(r+z)}{F'_z} = -\frac{2(r+z)}{\frac{z}{r} + 1} = -2r$$

故截距与切点到原点的距离为: -2

七 (本题 10 分): 设 $f(x, y)$ 为开区域 $D \subseteq R^2$ 上的连续可偏导函数, u, v 为 R^2 上的夹角为 a 的单位向量. 证明: $((\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2) \sin^2 a \leq 2((\frac{\partial f}{\partial u})^2 + (\frac{\partial f}{\partial v})^2)$.

证明 1: 设方向 u 的方向角为 θ , 则 u 的方向余弦为 $\{\cos \theta, \sin \theta\}$

则: v 的方向余弦为 $\{\cos(\theta + a), \sin(\theta + a)\}$

$$\text{于是: } \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta, \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cos(\theta + a) + \frac{\partial f}{\partial y} \sin(\theta + a)$$

$$\text{有: } \frac{\partial f}{\partial u} \sin(\theta + a) = \frac{\partial f}{\partial x} \cos \theta \sin(\theta + a) + \frac{\partial f}{\partial y} \sin \theta \sin(\theta + a)$$

$$\text{和: } \frac{\partial f}{\partial v} \sin \theta = \frac{\partial f}{\partial x} \cos(\theta + a) \sin \theta + \frac{\partial f}{\partial y} \sin(\theta + a) \sin \theta.$$

$$\text{得: } \frac{\partial f}{\partial u} \sin(\theta + a) - \frac{\partial f}{\partial v} \sin \theta = \frac{\partial f}{\partial x} \cos \theta \sin(\theta + a) - \frac{\partial f}{\partial x} \cos(\theta + a) \sin \theta$$

$$\text{即: } = \frac{\partial f}{\partial x} (\cos \theta \sin(\theta + a) - \cos(\theta + a) \sin \theta)$$

$$\text{即: } = \frac{\partial f}{\partial x} \sin a.$$

设 $\sin a \neq 0$ 时, (当 $\sin a = 0$ 时不定式成立)

$$\text{于是: } \frac{\partial f}{\partial x} = \frac{1}{\sin a} \left(\frac{\partial f}{\partial u} \sin(\theta + a) - \frac{\partial f}{\partial v} \sin \theta \right)$$

$$\text{同理: } \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta, \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cos(\theta + a) + \frac{\partial f}{\partial y} \sin(\theta + a)$$

$$\text{有: } \frac{\partial f}{\partial u} \cos(\theta + a) = \frac{\partial f}{\partial x} \cos \theta \cos(\theta + a) + \frac{\partial f}{\partial y} \sin \theta \cos(\theta + a)$$

$$\text{和: } \frac{\partial f}{\partial v} \cos \theta = \frac{\partial f}{\partial x} \cos(\theta + a) \cos \theta + \frac{\partial f}{\partial y} \sin(\theta + a) \cos \theta.$$

$$\text{得: } \frac{\partial f}{\partial u} \cos(\theta + a) - \frac{\partial f}{\partial v} \cos \theta = \frac{\partial f}{\partial y} \sin \theta \cos(\theta + a) - \frac{\partial f}{\partial y} \sin(\theta + a) \cos \theta$$

$$\text{即: } = \frac{\partial f}{\partial y} (\sin \theta \cos(\theta + a) - \sin(\theta + a) \cos \theta)$$

$$\text{即: } = -\frac{\partial f}{\partial y} \sin a.$$

$$\text{是: } \frac{\partial f}{\partial x} = \frac{1}{\sin a} \left(\frac{\partial f}{\partial u} \sin(\theta + a) - \frac{\partial f}{\partial v} \sin \theta \right)$$

$$\text{和: } \frac{\partial f}{\partial y} = -\frac{1}{\sin a} \left(\frac{\partial f}{\partial u} \cos(\theta + a) - \frac{\partial f}{\partial v} \cos \theta \right)$$

$$\text{故: } \left(\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right) = \frac{1}{\sin^2 a} \left(\left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 - 2 \frac{\partial f}{\partial u} \frac{\partial f}{\partial v} \cos a \right)$$

$$\text{即: } \leq \frac{1}{\sin^2 a} \left(\left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 + 2 \left| \frac{\partial f}{\partial u} \right| \left| \frac{\partial f}{\partial v} \right| \right)$$

$$\text{即: } \leq \frac{2}{\sin^2 a} \left(\left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 \right)$$

$$\text{于是: } \left(\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right) \sin^2 a \leq 2 \left(\left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 \right).$$

八 (本题 10 分):

二元函数 $f(x, y)$ 被称为凸函数是指: 对任意 $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $t \in [0, 1]$, 都有

$$\text{如下不等式: } f(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \leq tf(x_1, y_1) + (1-t)f(x_2, y_2).$$

假设函数 f 是可微的, 求证: f 是凸函数等当且仅当对任意 $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$\text{有 } (x_1 - x_2)f'_x(x_1, y_1) + (y_1 - y_2)f'_y(x_1, y_1) \geq f(x_1, y_1) - f(x_2, y_2).$$

证明:

$$\text{必要性: } f(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)) \leq f(x_1, y_1) + t(f(x_2, y_2) - f(x_1, y_1))$$

$$\text{有: } f(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)) - f(x_1, y_1) \leq t(f(x_2, y_2) - f(x_1, y_1)), t \in (0, 1)$$

上式乘 $\frac{1}{t}$, 令 $t \rightarrow 0^+$, 即得必要性。

充分性: $t \in (0, 1)$, 记 $P_t = ((1-t)x_1 + tx_2, (1-t)y_1 + ty_2)$

$$\text{有: } [(1-t)x_1 + tx_2 - x_2] \frac{\partial f}{\partial x}(P_t) + [(1-t)y_1 + ty_2 - y_2] \frac{\partial f}{\partial y}(P_t)$$

$$: = (1-t)(x_1, x_2) \frac{\partial f}{\partial x}(P_t) + (1-t)(y_1, y_2) \frac{\partial f}{\partial y}(P_t) \geq f(P_t) - f(x_2, y_2) \dots (1)$$

$$: [(1-t)x_1 + tx_2 - x_1] + [(1-t)y_1 + ty_2 - y_1] \frac{\partial f}{\partial y}(P_t) =$$

$$: = t(x_2 - x_1) \frac{\partial f}{\partial x}(P_t) + t(y_2 - y_1) \frac{\partial f}{\partial y}(P_t) \geq f(P_t) - f(x_1, y_1) \dots (2)$$

$$\text{又: } (1) \times t + (2) \times (1-t) \Rightarrow 0 \geq f(P_t) - tf(x_2, y_2) - (1-t)f(x_1, y_1).$$