PDE 学习笔记

樊audit

2024.1.7

三类偏微分方程:

一.波动方程: $\begin{cases} \partial_t^2 u - \Delta u = f \\ u(0)=\varphi,\ u_t(0)=\psi \\ \text{边界条件} \end{cases}$ $x \in \Omega \subset \mathbb{R}^n,\ t \in \mathbb{R}$

① $n=1$ 特征线法 D'Alembert 公式

② $n=3$ 球面平均法 Kirchhoff 公式

③ $n=2$ 升维法 Poisson 公式

$\Omega \in [0,1]$ 分离变量法

能量估计 → 解的唯一性
→ 波的传播性质

二. Poisson 方程

1. Laplace 方程 $\Delta u = 0$ 调和函数的性质

(平均值性质, Harnack 不等式, 梯度估计)

2. 基本解与 Green 函数

① 求 Green 函数 ② 用 Green 函数表示 Poisson 方程的解

3. 唯一性: 极值原理与最大模估计

三、热传导方程 $\begin{cases} \partial_t u - \Delta u = f & x \in \Omega \quad t > 0 \\ u(x,0) = \varphi \\ \text{边值条件} \end{cases}$

· $\Omega \in [0,1]$ 分离变量法.

· $\Omega \in \mathbb{R}^n$ Fourier变换

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\pi}} \varphi(y)\,dy + \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4(t-s)}} f(y,s)\,dy\,ds$$

· 解的唯一性:

①能量估计 (Gronwall 不等式)

②极值原理与最大模估计

§1 波动方程

▽ 特征线法

eg $\begin{cases} 2u_t - u_x + xu = 0 \\ u(x,0) = 2xe^{\frac{x^2}{2}} \end{cases}$

step 1. $\begin{cases} \frac{dx(t)}{dt} = -\frac{1}{2} \\ x(0) = c \end{cases} \Rightarrow x(t) = -\frac{1}{2}t + c \Rightarrow c = x(t) + \frac{1}{2}t$

step 2.

$$\frac{dU}{dt} = -\frac{1}{2}u_x + u_t = \frac{2u_t - u_x}{2} = \frac{(\frac{t}{2}-c)U}{2}$$

$$\Rightarrow \frac{dU}{U} = (\frac{c}{2} - \frac{t}{4})\,dt \Rightarrow \ln|U| = \frac{t^2}{8} - \frac{c}{2}t + \ln U(0) = \frac{t^2}{8} - \frac{(x+\frac{t}{2})t}{2} + \ln U(0)$$

$$= \frac{t^2}{8} - \frac{xt}{2} - \frac{t^2}{4} + \ln(2ce^{\frac{c^2}{2}}) = \frac{x^2 + \frac{t^2}{4} + xt}{2} + \ln(2x+t) - \frac{t^2}{8} - \frac{xt}{2}$$

$$\Rightarrow \ln|U| = \ln(2x+t) + \frac{x^2}{2} \Rightarrow u(x,t) = \pm(2x+t)e^{\frac{x^2}{2}}$$

step 3. 由初值 $u(x,t) = (2x+t)e^{\frac{x^2}{2}}$

△ 波动方程

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \varphi(x) \\ \partial_t u(x,0) = \psi(x) \end{cases}$$

Thm 1: $u = u_1 + u_2 + u_3 = \partial_t M_\psi + M_\varphi + \int_0^t M_{f_\tau}(x, t-\tau)\,d\tau$

设 $u_1 = \partial_t \hat{u}$, $\hat{u}$ 是 ② 中 $u_t(x,0) = \varphi(x)$ 的解

则 $\partial_t^2 u_1 - \Delta u_1 = \partial_t(\Delta \hat{u}) - \Delta(\partial_t \hat{u}) = 0$

$u_1(x,0) = \partial_t \hat{u}(x,0) = \varphi(x)$

$\partial_t u_1(x,0) = \partial_t^2 \hat{u}(x,0) = \Delta \hat{u}(x,0) = 0$

设 $u_3 = \int_0^t M_{f_\tau}(x, t-\tau)\,d\tau$, M_{f_τ} 是 ② 中 $u_t(x,0) = f(t,\tau)$ 的解

$$\boxed{\begin{aligned} &\partial_t u_3 = M_{f_\tau}(x,0) + \int_0^t \partial_t M_{f_\tau}(x,t-\tau)\,d\tau = \int_0^t \partial_t M_{f_\tau}(x,t-\tau)\,d\tau \\ &\partial_t^2 u_3 = f(t,\tau) + \int_0^t \Delta M_{f_\tau}(x,t-\tau)\,d\tau \end{aligned}}$$

$\partial_t^2 u_3 - \Delta u_3 = f(t,\tau)$

$u_3(x,0) = 0$, $\partial_t u_3(x,0) = M_{f_\tau}(x,0) = 0$

Thm 2. $n=1$，特征线法. D'Alembert 公式

Thm 2.1 $\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 \\ u(x,0) = 0 \\ \partial_t u(x,0) = \psi(x) \end{cases}$

$(\partial_t + \partial_x)(\partial_t - \partial_x) u = 0$

step 1 $(\partial_t - \partial_x) u = V \Rightarrow \partial_t V + \partial_x V = 0$

$\begin{cases} \dfrac{dx(t)}{dt} = 1 \\ x(0) = c \end{cases} \Rightarrow x = t + c$

$$\frac{dV(t+c,t)}{dt} = 0 \Rightarrow V = V(0) = \partial_t u(c,0) - \overset{0}{\overset{\Uparrow}{\partial_x u(c,0)}} = \psi(c)$$

step 2 $\partial_t u - \partial_x u = \psi(c) = \psi(x-t)$

$\begin{cases} \dfrac{dx(t)}{dt} = -1 \\ x(0) = a \end{cases} \Rightarrow x(t) = a - t$

$$\frac{du(a-t,t)}{dt} = u_t - u_x = \psi(x-t) \Rightarrow \frac{du(a-t,t)}{dt} = \psi(a-2t)$$

$$\Rightarrow u(a-t,t) - \overset{0}{\overset{\parallel}{u(a,0)}} = \int_0^t \psi(a-2s)\,ds.$$

$$\Rightarrow u(x,t) = \int_0^t \psi(a-2s)\,ds = \int_0^t \psi(x+t-2s)\,ds \Rightarrow$$

$$u(x,t) = \frac{1}{2}\int_0^t [\psi(x+s) + \psi(x-s)]\,ds$$

Thm 2.2: Thm 1 + Thm 2.1 $\Rightarrow$ D'Alembert 公式

$$u_1 = \frac{1}{2}[\varphi(x+t) + \varphi(x-t)]$$

$$u_3 = \frac{1}{2}\int_0^t d\tau \int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\,dy$$

D'Alembert 公式: $\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$ 的形式解为

$$u(x,t) = \frac{1}{2}(\varphi(x+t) + \varphi(x-t)) + \frac{1}{2}\int_0^t [\psi(x+s) + \psi(x-s)]\,ds + \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f$$

$f \equiv 0$ 时: $u(x,t) = \frac{1}{2}(\varphi(x+t) + \varphi(x-t)) + \frac{1}{2}\int_0^t [\psi(t+s) + \psi(t-s)]\,ds$

若令 $F(s)=\frac{1}{2}\varphi(s)+\frac{1}{2}\int_0^s \psi(\xi)\,d\xi$, $G(s)=\frac{1}{2}\varphi(s)-\frac{1}{2}\int_0^s \psi(\xi)\,d\xi$

则 $u(x,t)=F(x+t)+G(x-t)$

Thm 2.3 一维半无界问题

$$\begin{cases} \partial_t^2 u-\partial_x^2 u=f(x,t) \\ u(x,0)=\varphi(x) \\ \partial_t u(x,0)=\psi(x) \\ u(0,t)=g(t) \end{cases}$$

Thm 2.3.1. $g(t)\equiv 0$, 作奇延拓: $f,\varphi,\psi \to \bar{f},\bar{\varphi},\bar{\psi}$

则 $\bar{u}(x,t)=\frac{1}{2}(\bar{\varphi}(x+t)+\bar{\varphi}(x-t))+\frac{1}{2}\int_0^t[\bar{\psi}(x+s)+\bar{\psi}(x-s)]\,ds$

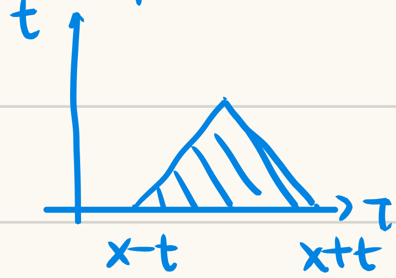
$$+\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\,dy\,d\tau$$

case1 $x\geq t$.

$\bar{u}(x,t)=\frac{1}{2}(\varphi(x+t)+\varphi(x-t))+\frac{1}{2}\int_0^t[\bar{\psi}(x+s)+\bar{\psi}(x-s)]\,ds$

$$+\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\,dy\,d\tau$$

$\to$ 积分区域:

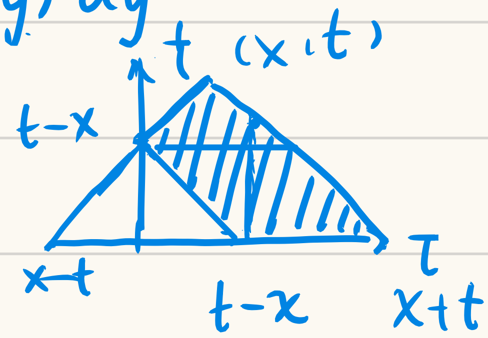


case 2. $x<t$

$\bar{u}(x,t)=\frac{1}{2}(\varphi(x+t)-\varphi(t-x))+\frac{1}{2}\int_{t-x}^{t+x}\psi(y)\,dy$

$$+\frac{1}{2}\int_{t-x}^{t}\int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\,dy\,d\tau$$

$$+\frac{1}{2}\int_0^{t-x}\int_{(t-\tau)-x}^{(t-\tau)+x} f(y,\tau)\,dy\,d\tau$$



▽相容性条件

① $\lim_{x\to 0^+} u(x,0)=\varphi(0)=u(0,0)=\lim_{t\to 0^+} u(0,t)=g(0)=0 \Rightarrow \varphi(0)=0$

② $\lim_{x\to 0^+} \partial_t u(x,0)=\partial_t u(0,0)=\psi(0)=\lim_{t\to 0^+}\partial_t u(0,t)=g'(0)=0 \Rightarrow \psi(0)=0$

③ $\lim_{x\to 0^+} \partial_t^2 u(x,0)=\partial_x^2 u(0,0)+f(0,0)=0 \Rightarrow \varphi''(0)+f(0,0)=0$

Thm 2.3.2 $g(t)\not\equiv 0$. 令 $v(x,t)=u(x,t)-g(t) \Rightarrow$ Thm 2.3.1

$\partial_t^2 v-\partial_x^2 v=f(x,t)-g''(t)$; $v(x,0)=\varphi(x)-g(0)$

$\partial_t V(x,0) = \psi(x) - g'(0)$； $V(x,0)=0$. 更一般的相容性条件由此给出

Thm3. $n=3$ 球面平均法 Kirchoff公式

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = \varphi(x) \\ \partial_t u(x,0) = \psi(x) \end{cases}$$

∇ 在极坐标上 $\Delta u = \partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u$

(Δ_{S^2} 表示 S^2 上 Laplace, $\int_{S^2} \Delta_{S^2} u \, d\omega = \int_{\partial S^2} \frac{\partial u}{\partial n} ds = 0$)

Thm3.1 $f(x,t) \equiv 0$

$\partial_t^2 u - \partial_r^2 u - \frac{2}{r}\partial_r u - \frac{1}{r^2}\Delta_{S^2} u = 0$ 在 S 上积分

$\partial_t^2 \int_{S^2} u \, d\omega - \left(\partial_r^2 \int_{S^2} u \, d\omega + \frac{2}{r}\partial_r \int_{S^2} u \, d\omega\right) = 0$

令 $\bar{u} = \frac{1}{4\pi}\int_{S^2} u \, d\omega \Rightarrow \partial_t^2 \bar{u} - \left(\partial_r^2 \bar{u} + \frac{2}{r}\partial_r \bar{u}\right) = 0$

令 $\bar{u} = r^{-k} V(t,r)$, $\partial_r \bar{u} = -k r^{-k-1} V(t,r) + r^{-k}\partial_t V$

$\partial_r^2 \bar{u} = -k[(-k-1) r^{-k-2} V + r^{-k-1}\partial_t V] + (-k) r^{-k-1}\partial_t V + r^{-k}\partial_t^2 V$

$= k(k+1) r^{-k-2} V - 2k r^{-k-1}\partial_t V + r^{-k}\partial_t^2 V$

$\partial_r^2 \bar{u} + \frac{2}{r}\partial_r u = k(k-1) r^{-k-2} V - 2(k-1) r^{-k-1}\partial_t V + r^{-k}\partial_t^2 V$

令 $r=1$. $V(t,r) = r\bar{u}(t,r)$

$$\begin{cases} \partial_t^2 V - \partial_r^2 V = r\partial_t^2 \bar{u} - \partial_r^2 \bar{u} - 2\partial_r \bar{u} = 0 \\ V(r,0) = r\bar{u}(r,0) = r\bar{\varphi}(r) \\ \partial_t V(r,0) = r\partial_t \bar{u}(r,0) = r\bar{\psi}(r) \end{cases}$$

将 V 关于 r 作偶延拓, 得到 $\bar{V}$

$\bar{V}(r,t) = \frac{1}{2}[(r+t)\bar{\varphi}(r+t) + (r-t)\bar{\varphi}(r-t)] + \frac{1}{2}\int_{r-t}^{r+t} y\bar{\psi}(y)\, dy$

$\bar{V} \to V \to \bar{u} \to u \quad (\bar{u} = \frac{1}{4\pi}\int_{S^2} u \, d\omega)$

Step 1. $\bar{u}(0,t) = u(0,t) = \partial_r(r\bar{u}(r,t))|_{r=0} = \partial_r V|_{r=0}$

$= \frac{1}{2}(\bar{\varphi}(t) + t\bar{\varphi}'(t) + \bar{\varphi}(-t) - t\bar{\varphi}'(-t)) + \frac{1}{2}[t\bar{\psi}(t) + t\bar{\psi}(-t)]$

$= \bar{\varphi}(t) + t\bar{\varphi}'(t) + t\bar{\psi}(t) = \frac{d}{dt}[t\bar{\varphi}(t)] + t\bar{\psi}(t)$

$= \frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2} \varphi(t\omega)\, d\omega\right) + \frac{t}{4\pi}\int_{S^2} \psi(t\omega)\, d\omega$

Step2. $u(x+x_0,t)$ 应用于上一步

$$u(x_0,t)=\frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(x_0+t\omega)\,d\omega\right)+\frac{t}{4\pi}\int_{S^2}\psi(x_0+t\omega)\,d\omega$$

$x_0\to x$, $y=x+t\omega$

Kirchhoff 公式:

$$u(x,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$

Thm 3.2 $f(x,t)\not\equiv 0$

$$u(x,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$
$$+\int_0^t\frac{1}{4\pi(t-\tau)}\int_{|x-y|=t-\tau}f(y,\tau)\,dy\,d\tau$$

Thm 4. $n=2$. 升维法 Poisson 公式

$$\begin{cases}\partial_t^2u-\Delta u=f(x,t)\\ u(x,0)=\varphi(x)\\ \partial_t u(x,0)=\psi(x)\end{cases}$$

Thm 4.1 $f(x,t)\equiv 0$

令 $\hat{u}(\hat{x},t)=u(x_1,x_2,t)$, $\hat{x}=(x_1,x_2,x_3)$

$\hat{\varphi}(\hat{x})=\varphi(x_1,x_2)$, $\hat{\psi}(\hat{x})=\psi(x_1,x_2)$

则 $\hat{u}(\hat{x},t)$ 为三维波动方程 $\begin{cases}\partial_t^2\hat{u}-\Delta u=0\\ \hat{u}(x,0)=\hat{\varphi}(x)\\ \partial_t\hat{u}(x,0)=\hat{\psi}(x)\end{cases}$ 的解

由 Kirchhoff 公式.

$$\hat{u}(\hat{x},t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|\hat{x}-\hat{y}|=t}\hat{\varphi}(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|\hat{x}-y|=t}\hat{\psi}(y)\,dS(y)$$

当 $\hat{x}=\vec{0}$ 时, $\hat{u}(0,t)=u(0,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y|=t}\varphi(y_1,y_2)\,dS(y)\right)$
$$+\frac{1}{4\pi t}\int_{|y|=t}\psi(y_1,y_2)\,dS(y)$$

注意到 $\int_{|y|=t}\psi(y_1,y_2)\,dS(y)=2\int_{y_3=\sqrt{t^2-y_1^2-y_2^2}}\psi(y_1,y_2)\,dS(y)$

$$=2\int_{y_1^2+y_2^2\le t}\psi(y_1,y_2)\sqrt{1+(\partial_{y_1}y_3)^2+(\partial_{y_2}y_3)^2}\,dy_1dy_2$$

$$=2\int_{y_1^2+y_2^2\le t}\psi(y_1,y_2)\frac{t}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2$$

$$u(0,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{B_t(0)}\frac{\varphi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2\right)+\frac{1}{2\pi}\int_{B_t(0)}\frac{\psi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2$$

$$u(0,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{B_t(0)}\frac{\varphi(y)}{\sqrt{t-|y|^2}}dy_1dy_2\right)+\frac{1}{2\pi}\int_{B_t(0)}\frac{\psi(y)}{\sqrt{t^2-|y|^2}}dy_1dy_2$$

$u(x+x_0,t)$ 代入得

$$u(x_0,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{B_t(0)}\frac{\varphi(y+x_0)}{\sqrt{t-|y|^2}}dy_1dy_2\right)+\frac{1}{2\pi}\int_{B_t(0)}\frac{\psi(y+x_0)}{\sqrt{t^2-|y|^2}}dy_1dy_2$$

$\Rightarrow$ Poisson 公式:

$$u(x,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{|y-x|=t}\frac{\varphi(y)}{\sqrt{t-|y-x|^2}}dy\right)+\frac{1}{2\pi}\int_{|y-x|=t}\frac{\psi(y)}{\sqrt{t-|y-x|^2}}dy$$

Thm4.2 性质.

▽ (x_0,t_0) 仅与初值在球面上积分有关

▽ 依赖区域: (x_0,t_0) 依赖于 $\{x \mid |x-x_0|\leq t_0\}$ 的值

决定区域 $\{(x,t) \mid D_{x,t}\subset D\}$ 为 D 的决定区域

影响区域

Thm3. 混合问题 分离变易法.

eg1 考虑 $\begin{cases}\partial_t^2u-\partial_x^2u=f(x,t)\\ u(x,0)=\varphi(x),\ \partial_tu(x,0)=\psi(x)\\ u(0,t)=g_1(t),\ u(l,t)=g_2(t)\end{cases}$

其中 $x\in[0,l]$, $t\geq 0$

step1. 齐次方程 $f,g_1,g_2\equiv 0$. 令 $u(x,t)=X(x)T(t)$

代入: $\frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda \Rightarrow \begin{cases}X''+\lambda X=0\\ T''+\lambda T=0\end{cases}$

由边值 $T(t)X(0)=T(t)X(l)=0 \Rightarrow X(0)=X(l)=0$

case1. $\lambda<0 \Rightarrow X=C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x} \Rightarrow \begin{cases}C_1+C_2=0\\ C_1e^{\sqrt{-\lambda}l}+C_2e^{-\sqrt{-\lambda}l}=0\end{cases}$

$\Rightarrow C_1=C_2=0$, 舍去 or

▽ 能量估计: $0=\int_0^l X(X''+\lambda X)dx=X'X\Big|_0^l-\int_0^l[(X')^2-\lambda X^2]dx$
$=\int_0^l \lambda X^2-(X')^2dx \Rightarrow \lambda\geq 0$

case2. $\lambda=0 \Rightarrow X(x)=C_1x+C_2 \Rightarrow \begin{cases}C_2=0\\ C_1l=0\end{cases} \Rightarrow C_1=C_2=0$

case3. $\lambda>0 \Rightarrow X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x \Rightarrow \begin{cases}C_1=0\\ C_2\sin\sqrt{\lambda}l=0\end{cases} \Rightarrow \sqrt{\lambda}l=n\pi$

$\Rightarrow \sqrt{\lambda}=\frac{n\pi}{l} \Rightarrow X_n(x)=\sin\frac{n\pi}{l}x$, $n=1,2\cdots$

设与 X_n 对应的 $T_n(t)=C_n\cos\frac{n\pi}{l}t+D_n\sin\frac{n\pi}{l}t$

$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}(C_n\cos\frac{n\pi}{l}t+D_n\sin\frac{n\pi}{l}t)\sin\frac{n\pi}{l}x$

$u(x,0)=\varphi(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi}{l}t$, $\partial_t u(x,0)=\psi(x)=\sum_{n=1}^{\infty}\frac{n\pi}{l}D_n\sin\frac{n\pi}{l}x$

将 $\varphi(x)$ 展开：$\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin\frac{n\pi}{l}x \Rightarrow C_n=\varphi_n=\frac{\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\,dx}{\int_0^l\sin^2\frac{n\pi}{l}x\,dx}=\frac{2}{l}\int_0^l\varphi(x)\sin$

同理 $D_n=\frac{l}{n\pi}\psi_n=\frac{2}{n\pi}\int_0^l\psi(x)\sin\frac{n\pi}{l}x\,dx$

Step2. $f\not\equiv 0$, $g_1, g_2\equiv 0$.

设 $u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$, $f(x,t)=\sum_{n=1}^{\infty}f_n(t)\sin\frac{n\pi}{l}x$

$T_n''(t)+\lambda T_n(t)=f_n(t)$, $T_n(0)=\varphi_n$, $T_n'(0)=\psi_n$

$\Rightarrow T_n(t)=\varphi_n\cos\frac{n\pi}{l}t+\frac{l}{n\pi}\psi_n\sin\frac{n\pi}{l}t+\frac{l}{n\pi}\int_0^t f_n(\tau)\sin\frac{n\pi}{l}(t-\tau)\,d\tau$

Step3. $f, g_1, g_2\not\equiv 0$. 令 $v(x,t)=u(x,t)-\frac{(l-x)g_1(t)+xg_2(t)}{l}$

eg2. $\begin{cases}\partial_t u=\partial_x^2 u=0\\ u(x,0)=\varphi(x)\\ u(0,t)=0,\ \partial_x u(l,t)+hu(l,t)=0\end{cases}$

设 $u(x,t)=T(t)X(x)$

$\Rightarrow T'(t)X(x)=T(t)X''(x)\Rightarrow\frac{T'(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda$

$\Rightarrow T'(t)+\lambda T(t)=0$. $X''(x)+\lambda X(x)=0$

$T(t)X(0)=0\Rightarrow X(0)=0$. $T(t)X'(l)+hT(t)X(l)=0$

$\lambda<0$, $X(x)=C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x}\Rightarrow C_1, C_2=0$

$\lambda=0$. 亦舍.

$\lambda>0$: $X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$

$\Rightarrow C_1=0$, $\sqrt{\lambda}\cos\sqrt{\lambda}l+h\sin\sqrt{\lambda}l=0\Rightarrow\tan\sqrt{\lambda}l=-\frac{\sqrt{\lambda}}{h}$

令 $X_n(x)=\sin\sqrt{\lambda_n}x$, $T_n=A_ne^{-\lambda_nt}$

$u(x,t)=\sum_{n=1}^{\infty}A_ne^{-\lambda_nt}\sin\sqrt{\lambda_n}x$.

$u(x,0)=\varphi(x)\Rightarrow A_n=\varphi_n$

eg3. 在圆盘 $\Omega=\{(x,y):x^2+y^2\le 1\}$ 中解位势方程

$$\begin{cases} \Delta u = 0 \\ u(x,t) = \varphi(x) \quad x \in \partial\Omega \end{cases}$$

极坐标换元: $\partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u = 0$, $u|_{r=1} = \varphi(\cos\theta, \sin\theta) = \hat{\varphi}(\theta)$

令 $u(r,\theta) = R(r)\Theta(\theta)$, 则

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\Rightarrow r^2\frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ r^2R''(r) + rR'(r) = \lambda R \end{cases}$$

$\lambda < 0$. $\Theta(\theta) = C_1 e^{-\sqrt{-\lambda}\theta} + C_2 e^{\sqrt{-\lambda}\theta}$ $\Rightarrow$ 不满足周期性条件

$\lambda = 0$ 舍去

$\lambda > 0$ $\Theta(\theta) = C_1\cos\sqrt{\lambda}\theta + C_2\sin\sqrt{\lambda}\theta$ 由 $\Theta(\theta) = \Theta(\theta + 2\pi) \Rightarrow \sqrt{\lambda} \in \mathbb{Z}^+ \Rightarrow \lambda_n = n^2$

$\Theta_n(\theta) = C_1\cos n\theta + C_2\sin n\theta$

Euler方程. 令 $r = e^t$. 则 $\partial_t R = r\partial_r R$. $\partial_t^2 R = r\partial_r R + r^2\partial_r^2 R$

那么: $\partial_t^2 R - r\partial_r R + r\partial_r R - n^2 R = 0 \Rightarrow \partial_t^2 R - n^2 R = 0$

$n \neq 0 \Rightarrow R_n = C_1 e^{nt} + C_2 e^{-nt} \Rightarrow R_n = C_1 r^n + C_2 r^{-n}$

$n = 0 \Rightarrow R_0 = C_1 t + C_2 = C_1 \ln r + C_2$

$$\Rightarrow R_n = \begin{cases} r^n & n \geq 1 \\ 1 & n = 0 \end{cases}$$

令 $u(r,\theta) = \sum_{n=1}^{\infty} r^n(C_n\cos n\theta + D_n\sin n\theta) + C_0$

而 $u(1,\theta) = \hat{\varphi}(\theta) = \sum_{n=1}^{\infty}(C_n\cos n\theta + D_n\sin n\theta) + C_0$

$C_n\int_0^{2\pi}\cos^2 n\theta\, d\theta = \int_0^{2\pi}\hat{\varphi}(\theta)\cos n\theta\, d\theta$ $C_0 = \int_0^{2\pi}\hat{\varphi}(\theta)\, d\theta$

$D_n\int_0^{2\pi}\sin^2 n\theta\, d\theta = \int_0^{2\pi}\hat{\varphi}(\theta)\sin n\theta\, d\theta$

eg4. $$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < l \quad t > 0 \\ u(x,0) = x(x-2l),\ u_t(x,0) = 0, & 0 \leq x \leq l \\ u(0,t) = 0,\ u_x(l,t) = 0, & t \geq 0. \end{cases}$$

$u(x,t) = T(t)X(x)$

由①: $\begin{cases} T''(t) + \lambda T(t) = 0 \end{cases}$

$X''(x)+\lambda X(x)=0$

边值 $T(0)X(x)=x(x-2l)$, $T'(0)X(x)=0$

$X(0)=0$, $X'(l)=0$

能易估计有 $\lambda>0$. 故 $X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$

代入边值 $C_1=0$, $\sqrt{\lambda}l=\frac{(2n+1)\pi}{2} \Rightarrow \lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2$

$X_n(x)=\sin\left(\frac{(2n+1)\pi}{2l}x\right)$

而 $T_n(t)=a_n\cos\left(\frac{(2n+1)\pi}{2l}t\right)+b_n\sin\left(\frac{(2n+1)\pi}{2l}t\right)$

$u(x,t)=\sum_{n=0}^{\infty}X_n(x)T_n(t)=\sum_{n=0}^{\infty}\sin\left(\frac{(2n+1)\pi}{2l}x\right)\left[a_n\cos\left(\frac{(2n+1)\pi}{2l}t\right)+b_n\sin\left(\frac{(2n+1)\pi}{l}t\right)\right]$

$u(x,0)=\sum_{n=0}^{\infty}\sin\left(\frac{(2n+1)\pi}{2l}x\right)a_n=x(x-2l)$

$$a_n=\frac{\int_0^l\left[x(x-2l)\cdot\sin\left(\frac{(2n+1)\pi}{2l}x\right)\right]dx}{\int_0^l\left[\sin\left(\frac{(2n+1)\pi}{2l}x\right)\right]^2dx}=-\frac{32l^2}{(2n+1)^3\pi^3}$$

$u_t(x,0)=\sum_{n=0}^{\infty}b_n\frac{(2n+1)\pi}{2l}\sin\left(\frac{(2n+1)\pi}{2l}x\right)=0 \Rightarrow b_n=0$

故 $u(x,t)=\sum_{n=0}^{\infty}-\frac{32l^2}{(2n+1)^3\pi^3}\sin\left(\frac{(2n+1)\pi}{2l}x\right)\cos\left(\frac{(2n+1)\pi}{2l}t\right)$

egs. $\begin{cases} u_{tt}-u_{xx}=0 & 0<x<l,\ t>0 \\ u(x,0)=\frac{Ax^2}{l^2}\quad u_t(x,0)=0, & 0\le x\le l \\ u_x(0,t)=0,\ u(l,t)=A & t>0 \end{cases}$

! 先将边值零化: $u(x,t)=v(x,t)-A$

则 $\begin{cases} v_{tt}-v_{xx}=0 \\ v(x,0)=\frac{Ax^2}{l^2}-A,\ v_t(x,0)=0 \\ v_x(0,t)=0,\ v(l,t)=0 \end{cases}$

$V=X(x)T(t)$

(略) $X_n(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$

代入边值 $C_2=0$. $\sqrt{\lambda}l=\frac{(2n+1)\pi}{2} \Rightarrow \sqrt{\lambda}=\frac{(2n+1)\pi}{2l}$

故 $X_n(x)=\cos\frac{(2n+1)\pi}{2l}x$ $\quad T_n(t)$ 有形式 $\quad C_n\cos\frac{(2n+1)\pi}{2l}t+D_n\sin\frac{(2n+1)\pi}{2l}t]$

$V(x,t)=\sum_{n=0}^{\infty}\cos\frac{(2n+1)\pi}{2l}x\left[C_n\cos\frac{(2n+1)\pi}{2l}t+D_n\sin\frac{(2n+1)\pi}{2l}t\right]$

$$V(x,0)=\sum_{n=0}^{\infty}\cos\left[\frac{(2n+1)\pi}{2l}x\right]C_n=\frac{Ax^2}{l^2}-A$$

$$\Rightarrow C_n=\frac{\int_0^l\left(\frac{Ax^2}{l^2}-A\right)\left(\cos\frac{(2n+1)\pi}{2l}x\right)dx}{\int_0^l\left[\cos\frac{(2n+1)\pi}{2l}x\right]^2dx} \qquad D_n=0.$$

$$C_n=-\frac{(-1)^n 32A}{(2n+1)^3\pi^3}$$

故 $u(x,t)=\sum_{n=0}^{\infty}-\frac{(-1)^n32A}{(2n+1)^3\pi^3}\cos\left(\frac{(2n+1)\pi}{2l}x\right)\cos\left(\frac{(2n+1)\pi}{2l}t\right)+A$

eg. $\begin{cases}u_t-u_{xx}=0 & 0<x<\pi,\ t>0\\ u(x,0)=0 & 0\le x\le\pi\\ u_x(0,t)=A_1t,\ u_x(\pi,t)=A_2t & t\geq 0\end{cases}$

令 $V=u-\left(\frac{1}{2}\frac{A_2t-A_1t}{\pi}x^2+A_1tx\right)$

$$\begin{cases}V_t-V_{xx}=-\frac{A_2-A_1}{2\pi}x^2-A_1x+\frac{A_2t-A_1t}{\pi}=f(x,t)\\ V(x,0)=0 \quad 0\le x\le\pi\\ V_x(0,t)=0,\ V_x(\pi,t)=0\end{cases}$$

$X''(x)+\lambda X(x)=0$

$\Rightarrow X(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$，代入边值 $C_2=0$

$\sin\sqrt{\lambda}\pi=0\Rightarrow\sqrt{\lambda}\in\mathbb{Z}^+\Rightarrow\lambda=n^2$. $X_n(x)=\cos nx$

$V(x,t)=\sum_{n=0}^{\infty}X_n(x)T_n(t)$.

$V_t-V_{xx}=\sum_{n=0}^{\infty}\cos nx\,T_n'(t)+\sum_{n=0}^{\infty}n^2\cos nx\,T_n(t)=f(x,t)$

故 $T_n'(t)+n^2T_n(t)=f_n(t)$，$T_n(0)=0$

其中 $f_n(t)=\frac{\int_0^{\pi}\cos(nx)f(x,t)dx}{\int_0^{\pi}\cos^2(nx)dx}$

#

Thm 4. 能量估计 解的唯一性 波的传播性质

Thm 4.1. $\mathbb{R}^n$

$u_{tt}-\Delta u=0 \Rightarrow u_t u_{tt}-u_t\Delta u=0$

$\frac{1}{2}\partial_t(\partial_t u)^2-\sum_{i=1}^{n}\partial_t u\,\partial_{x_i}^2 u=\frac{1}{2}\partial_t(\partial_t u)^2-\sum_{i=1}^{n}[\partial_{x_i}(\partial_t u\,\partial_{x_i}u)-\partial_t\partial_{x_i}u\,\partial_{x_i}$

$=\frac{1}{2}\partial_t(\partial_t u)^2-\sum_{i=1}^{n}[\partial_{x_i}(\partial_t u\,\partial_{x_i}u)-\frac{1}{2}\partial_t(\partial_{x_i}u)^2]$

$=\frac{1}{2}\partial_t(\partial_t u)^2-\mathrm{div}(\partial_t u\nabla u)-\frac{1}{2}\partial_t|\nabla u|^2]$

$\Rightarrow\partial_t[\frac{1}{2}(u_t^2)+|\nabla u|^2]-\mathrm{div}(u_t\nabla u)=0$

$\partial_t\int_{\mathbb{R}^n}(\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2)\,dx-\int_{\mathbb{R}^n}\mathrm{div}(u_t\nabla u)=0$

$\Rightarrow\partial_t\int_{\mathbb{R}^n}(\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2)\,dx=0$，令 $E(t)=\int_{\mathbb{R}^n}\underbrace{\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2}_{e}\,dx$，$E(t)\equiv E(0$

Thm 4.2 Ω

Thm 4.2.1 $\begin{cases}\partial_t^2u-\Delta u=0 & x\in\Omega\subset\mathbb{R}^n, t>0\\ u|_{\partial\Omega}=0\\ u(x,0)=\varphi(x) & u_t(x,0)=\psi(x)\end{cases}$

$\frac{d}{dt}\int_\Omega[\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2]=\int_{\partial\Omega}u_t\nabla u\cdot\vec{n}\cdot ds=\int_{\partial\Omega}u_t\frac{\partial u}{\partial n}ds$

$u|_{\partial\Omega}=0\Rightarrow u_t|_{\partial\Omega}=0\Rightarrow E(t)=\int_\Omega\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2$，$E(t)=E(0)$

Thm 4.2.2 $\begin{cases}\partial_t^2u-\Delta u=f(x,t) & x\in\Omega, t>0\\ u(x,0)=\varphi(x) \quad u_t(x,0)=\psi(x) & x\in\Omega \qquad (*)\\ u|_{\partial\Omega}=h(x,t) & t\geq0 \quad (\text{第一类边值})\end{cases}$

△解的唯一性

设*有两个解 u_1, u_2，令 $u(x,t)=u_1(x,t)-u_2(x,t)$

则 $\begin{cases}\partial_t^2u-\Delta u=0\\ u(x,0)=0,\ u_t(x,0)=0\\ u|_{\partial\Omega}=0\end{cases}$

$E(t)=\frac{1}{2}\int_\Omega u_t^2+|\nabla u|^2dx$，$E(t)=E(0)=0$

$\Rightarrow u_t=0,\ \nabla u=0 \quad \forall x \text{ in } \Omega \quad t\geq0$

$\Rightarrow u=\text{const} \quad \text{in } \Omega$

由于 u 边值为 0，为使得 u 连续至边界，$u\equiv0$

Thm 4.2.3.

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = \varphi(x), \partial_t u(x,0) = \psi(x) \\ u(x,t)|_{\partial\Omega} = 0 \end{cases}$$

$\partial_t u(\partial_t^2 u - \Delta u) = \partial_t u f(x,t) \Rightarrow$

$\frac{1}{2}[\partial_t(\partial_t u)^2 + |\nabla u|^2] - \mathrm{div}(u_t \nabla u) = u_t f$

在Ω上对x积分 $\frac{1}{2}\frac{d}{dt}\int_\Omega [(u_t)^2 + |\nabla u|^2]\,dx = \int_\Omega \partial_t u f\,dx$

$\leq \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{2}\int_\Omega f^2\,dx \leq E(t) + \int_\Omega |f|^2 dx$

$\Rightarrow \frac{d}{dt}E(t) \leq E(t) + \frac{1}{2}\int_\Omega |f|^2 dx$

$\frac{d}{dt}\cdot[E(t)e^{-t}] \leq \frac{e^{-t}}{2}\int_\Omega |f|^2 dx \quad \Rightarrow$

$E(t) \leq C_T(E(0) + \frac{1}{2}\int_0^T\int_\Omega |f|^2 dx)$. $E(0) = \frac{1}{2}\int_\Omega(|\nabla\varphi|^2 + \psi^2)\,dx$

令 $E_0(t) = \int_\Omega |u(x,t)^2|\,dx$

$\frac{d}{dt}E_0(t) = 2\int_\Omega u\partial_t u\,dx \leq \int_\Omega u^2 dx + \int_\Omega(\partial_t u)^2 dx \Rightarrow \frac{d}{dt}E_0(t) \leq E_0(t) + \int_\Omega$

$E_0(t) \leq C_T(E_0(0) + \frac{1}{2}\int_0^T\int_\Omega |u_t|^2 dx\,dt)$

$\leq C_T'(E_0(0) + E(0) + \int_0^T\int_\Omega |f(x,t)|^2 dx\,dt)$

Thm 4.2.4 稳定性

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = \varphi(x), \partial_t u(x,0) = \psi(x) \\ u(x,t)|_{\partial\Omega} = 0 \end{cases}$$

存在$\eta(\varepsilon,T)$, $\|\varphi_1 - \varphi_2\|_{L^2(\Omega)} \leq \eta$, $\|\nabla\varphi_1 - \nabla\varphi_2\|_{L^2(\Omega)} \leq \eta$

$\|\psi_1 - \psi_2\|_{L^2(\Omega)} \leq \eta$, $\|f_1 - f_2\|_{L^2((0,T),\Omega)} \leq \eta$

则以φ_1, ψ_1为初值, f_1为右端项的解u_1与以φ_2, ψ_2为初值, f_2为右端项的解u_2的差在$0 \leq t \leq T$满足

$\|u_1 - u_2\|_{L^2(\Omega)} + \|\nabla u_1 - \nabla u_2\|_{L^2(\Omega)} + \|\psi_1 - \psi_2\|_{L^2(\Omega)} < \varepsilon$

$\triangleright \|f\|_{L^2(\Omega)} = (\int_\Omega |f(x)|^2 dx)^{1/2}$

$(\int_0^T\int_\Omega |f(x,t)|^2 dx\,dt)^{1/2}$

令 $u=u_1-u_2$, $f=f_1-f_2$, φ, ψ 同理

$$\begin{cases} \partial_t^2 u - Au = f(x,t) \\ u(x,0)=\varphi(x), \partial_t u(x,0)=\psi(x) \\ u(x,t)|_{\partial\Omega}=0 \end{cases}$$

$$\partial_t \frac{1}{2}\left[\int_\Omega (u_t)^2+|\nabla u|^2\right]=\int_\Omega f(x,t)\partial_t u\,dx$$

$$\left|\int_\Omega f(x,t)\partial_t u\,dx\right| \le \frac{1}{2}\int_\Omega |f|^2dx+\frac{1}{2}\int_\Omega(\partial_t u)^2dx$$

$$\le \frac{1}{2}\int_\Omega |f|^2dx+\frac{1}{2}\int_\Omega((\partial_t u)^2+|\nabla u|^2)dx$$

$$\Rightarrow \frac{d}{dt}E(t)\le \frac{1}{2}\int_\Omega |f|^2dx+E(t) \Rightarrow$$

$$E(t)\le C_T\left(E(0)+\int_0^T\int_\Omega |f|^2dx\,ds\right)$$

再令 $y(t)=\int_\Omega |u|^2dx$, $y'(t)=2\int_\Omega u\partial_t u\,dx \le \int_\Omega u^2dx+\int_\Omega(\partial_t u)^2dx$

$$\Rightarrow \frac{d}{dt}y(t)\le y(t)+2E(t).$$

$$\Rightarrow \frac{d}{dt}y(t)\le y(t)+2e^T\left(E(0)+\int_0^T\int_\Omega |f|^2dx\,ds\right)$$

$$\Rightarrow y(t)\le C^T\left(y(0)+E(0)+\int_0^T\int_\Omega |f|^2dx\,ds\right)$$

这表明 E, y 有同样的界

$$\|u\|^2_{L^2(\Omega)}+\|\nabla u\|^2_{L^2(\Omega)}+\|\partial_t u\|^2_{L^2(\Omega)}\le C^T\cdot\left(\|\varphi\|^2_{L^2(\Omega)}+\|\psi\|^2_{L^2(\Omega)}+\|\nabla\varphi\|^2_{L^2(\Omega)}+\int_0^T\int_\Omega f^2(x,s\right.$$

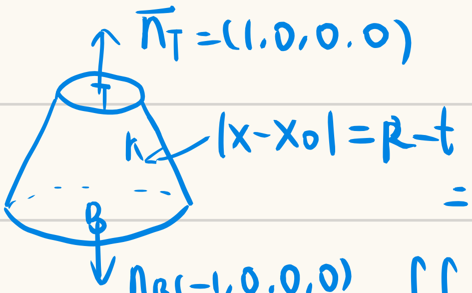
$$\le 4\eta^2 C_T<\varepsilon$$

Thm 4.25 波的传播性质之有限传播速度

$$\begin{cases} \partial_t^2 u-\Delta u=0 \\ u(x,0)=\varphi(x) \\ u_t(x,0)=\psi(x) \end{cases} \quad x\in\mathbb{R}^3$$

$$\partial_t u(\partial_t^2 u)-\partial_t u\Delta u=0 \Rightarrow \frac{1}{2}\partial_t(u_t^2+|\nabla u|^2)-\mathrm{div}(\partial_t u\nabla u)=0$$

$$\Rightarrow \partial_t e(t)-\mathrm{div}(\partial_t u\nabla u)=0$$



$$\iint_\Delta \left(\partial_t\frac{1}{2}[u_t^2+|\nabla u|^2]-\mathrm{div}(\partial_t u\nabla u)\,dx\,dt\right)=0$$

$$=\iint_\Delta(\partial_t,\nabla)\cdot(e(t),-\partial_t u\nabla u)\,dx\,dt$$

$$=\iint_{\partial\Delta}(e(t),-\partial_t u\nabla u)\cdot\vec{n}\,ds$$

$$\Rightarrow \int_B e(0)dx=\int_T e(t_0)dx+\int_h(e(t),-\partial_t u\nabla u)\frac{1}{\sqrt{2}}\left(\frac{R-t}{|R-t|},\frac{x-x_0}{|x-x_0|}\right)ds$$

$$\Rightarrow \int_B e(0)\,dx = \int_T e(t_0)\,dx + \frac{1}{2\sqrt{2}} \int_K \left(u_t^2 + |\nabla u|^2 \right) \frac{R-t}{|R-t|} - 2u_t \nabla u \, \frac{x-x_0}{|x-x_0|} \, ds.$$

$$= \int_T e(t_0)\,dx + \frac{1}{2\sqrt{2}} \int_K \left[\left| u_t - \frac{x-x_0}{|x-x_0|} \nabla u \right|^2 + |\nabla u|^2 - \left| \frac{x-x_0}{|x-x_0|} \cdot \nabla u \right|^2 \right]$$

定义 $F(u \times [0,t_0]) = \int_K \left(\left| u_t - \frac{x-x_0}{|x-x_0|} \nabla u \right|^2 + |\nabla u|^2 - \left| \frac{x-x_0}{|x-x_0|} \cdot \nabla u \right|^2 \right) ds \geq 0$

梯度方向最大.

$$\Rightarrow \int_B e(0)\,dx = \int_T e(t_0)\,dx + F(u \times [0, t_0])$$

($t=0$处能量)　　($t=t_0$处能量)　　(能量溢出)

∇ 若 $(u, u_t)|_{t=0} = 0$，在 B 上能量为 0 $\Rightarrow (u, u_t) \equiv 0$

§2. 位势方程

Thm 2.1 Laplace方程 $\Delta u=0$ 调和函数

Thm 2.1.1 $u:\Omega\to\mathbb{R}^n$ 二阶连续可微, $\Delta u=0$ in Ω

$u(x+x_0)$, $u(\lambda x)$, $u(Ox)$ 均调和.

Thm 2.1.2 第一平均值性质 $\Leftrightarrow$ 第二平均值性质

$\forall B_r(x)\in\Omega$, $u(x)=\dfrac{\int_{B_r(x)}u(y)dy}{|B_r(x)|}\Leftrightarrow u(x)=\dfrac{\int_{\partial B_r(x)}u(y)ds(y)}{|\partial B_r(x)|}$

$\Rightarrow u(x)\frac{4}{3}\pi r^3=\int_{B_r(x)}u(y)dy$, 两边对r求导

$4\pi r^2u(x)=\int_{\partial B_r(x)}u(y)ds(y)\Rightarrow u(x)=\dfrac{\int_{\partial B_r(x)}u(y)ds(y)}{4\pi r^2}$

$\Leftarrow\int_{\partial B_r(x)}u(y)ds(y)=4\pi r^2u(x)$

$\int_{B_r(x)}u(y)dy=\int_0^r\int_{\partial B_\rho(x)}u(y)ds(y)d\rho=\int_0^r 4\pi\rho^2u(x)d\rho=\frac{4}{3}\pi r^3u(x)$

$\Rightarrow u(x)=\dfrac{|\int_{B_r(x)}u(y)dy|}{\frac{4}{3}\pi r^3}$

Thm 2.1.3 调和函数 $\Rightarrow$ 平均值性质

$0=\int_{B_r(x)}(\Delta u)(y)dy=\int_{\partial B_r(x)}\nabla u(y)\frac{y-x}{|y-x|}ds(y)$

令 $|y-x|=r$, $y=x+rw$. $ds(y)=r^2ds(w)$

$r^2\int_{|w|=1}\nabla u(x+wr)ds(w)=r^2\int_{|w|=1}\frac{d}{dr}u(x+wr)ds(w)$

$=r^2\frac{d}{dr}\boxed{\int_{|w|}u(x+wr)ds(w)}$ 与r无关, 取 $r=0$.

$\int_{|w|=1}u(x+wr)ds(w)=\int_{|w|=1}u(x)ds(w)=4\pi u(x)$

$\Rightarrow u(x)=\frac{1}{4\pi}\int_{|w|=1}u(x+wr)ds(w)=\frac{1}{4\pi r^2}\int_{\partial B_r(x)}u(y)ds(y)$

Thm 2.1.4 平均值性质 $\Rightarrow$ 调和函数

Step1 $\forall B_r(x)\in\Omega$. $\int_{B_r(x)}(\Delta u)(y)dy=r^2\int_{|w|=1}\frac{d}{dr}u(x+wr)ds(w)$

而. $u(x)=\frac{1}{4\pi r^2}\int_{|y-x|=r}u(y)ds(y)=\frac{1}{4\pi}\int_{|w|=1}u(x+wr)ds(w)$

故 $\int_{B_r(x)}(\Delta u)y\,dy=r^2\frac{d}{dr}(4\pi u)=0$

Step2 要证 $\Delta u=0$ $\forall x\in\Omega$. 否则 $\exists x_0\in\Omega$. s.t. $(\Delta u)(x_0)>c>0$ 不妨设

则 $\exists r_0>0$, $(\Delta u)(x)>\frac{c}{2}$ $\forall x\in B_{r_0}(x_0)$

则 $\int_{B_{r_0}(x_0)} \Delta u(y)\,dy \geq \frac{c}{2}|B_{r_0}(x_0)|>0$

Thm2.1.5 平均值性质 $\Rightarrow$ 光滑.

令 $\varphi\in C_0^\infty(B_1(0))$, ($\varphi\in C^\infty$, $\operatorname{supp}\varphi\in B_1(0)$)

$\varphi\equiv 1$ on $B_{\frac12}(0)$, $\int_{\mathbb{R}^n}\varphi(x)\,dx=\int_{B_1(0)}\varphi(x)dx=1$

$\varphi(x)=\varphi(|x|) \rightsquigarrow$ radial

则 $1=\int_{\mathbb{R}^n}\varphi(x)\,dx=\int_0^1\int_{|w|=1}\varphi(rw)r^{n-1}dS(w)dr$

$=\int_0^1\int_{|w|=1}\varphi(r)r^{n-1}dS(w)dr=w_n\int_0^1\varphi(r)r^{n-1}dr$

其中 w_n 为 n 维单位球面积

def. $\varphi_\varepsilon(x)=\frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$, $\varphi_\varepsilon\in C_0^\infty(B_\varepsilon(0))$

$\int_{\mathbb{R}^n}\varphi_\varepsilon(x)\,dx=\int_{\mathbb{R}^n}\frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})dx=\int_{\mathbb{R}^n}\varphi(y)dy=1$

claim. $\forall x\in\Omega$, 取 $\varepsilon<\frac14\operatorname{dist}(x,\partial\Omega)$

则 $u(x)=(u*\varphi_\varepsilon)(x)$

$(u*\varphi_\varepsilon)(x)=\int_\Omega u(y)\varphi_\varepsilon(x-y)dy=\int_{\Omega\cap B_\varepsilon(x)}u(y)\frac{1}{\varepsilon^n}\varphi(\frac{x-y}{\varepsilon})dy$

$=\int_{B_\varepsilon(x)}u(y)\frac{1}{\varepsilon^n}\varphi(\frac{x-y}{\varepsilon})dy \overset{y=x+r\varepsilon}{=\!=} \int_0^\varepsilon\int_{|w|=1}u(x+rw)\frac{1}{\varepsilon^n}\varphi(\frac{r}{\varepsilon})\ldots$

$\overset{\frac{r}{\varepsilon}=\rho}{=\!=}\int_0^1\int_{|w|=1}u(x+\varepsilon\rho w)\cancel{\frac{1}{\varepsilon^n}}\varphi(\rho)\cancel{\varepsilon^{n-1}}\rho^{n-1}\cancel{\varepsilon}\,dS(w)d\rho$

$=\int_0^1\int_{|w|=1}u(x+\varepsilon\rho w)dS(w)\varphi(\rho)\rho^{n-1}d\rho$.

由于 u 调和: $\int_{|w|=1}u(x+\varepsilon\rho w)dS(w)=\int_{|w|=1}u(x)dS(w)=w_n\ldots$

故 $(u*\varphi_\varepsilon)(x)=w_n\int_0^1\rho^{n-1}\varphi(\rho)d\rho\,u(x)=u(x)$

φ_ε 光滑. 由卷积性质: u 光滑

Thm2.1.6 Harnack不等式.

对于 Ω 上任何连通紧子集 V, 存在一个仅与距离函数 $d(V,\partial\Omega)=\min_{x\in V,y\in\partial\Omega}|x-\ldots$

和维数 n 有关的正常数 C. s.t. $\sup_V u\leq C\inf_V u$, u 非负且调和

只要证 $u(x)\leq Cu(y)$

step 1. $|x-y|<r$, $B_r(x)\subset B_{2r}(y)$. 令 $r<\frac14 d(V,\partial\Omega)$ 就能保证 $B_r(x), B_{2r}(\ldots$

$$u(y)=\frac{\int_{B_{2r}(y)}u(z)dz}{|B_{2r}(y)|}\geq\frac{1}{2^n}\frac{\int_{B_r(x)}u(z)dz}{|B_r(x)|}=\frac{1}{2^n}u(x)$$

Step 2. $\forall x,y\in Z$，由于 V 紧致，连通，存在有限个球 $B_r(x_i)$ $i=1\cdots N$

$x,y\in\bigcup_{i=1}^{N}B_r(x_i)$，$B_r(x_i)\cap B_r(x_{i+1})\neq\phi\Rightarrow u(y)\geq\frac{1}{2^{nN}}u(x)$

$\Rightarrow\sup_V u(x)\leq C\inf_V u(x)$

Thm 2.1.7 梯度估计

$u\in C(\overline{B_R(x_0)})$ 调和，则 $|Du(x_0)|\leq\frac{n}{R}\max_{B_R}u(x_0)$

在 $\Delta u=0$ 两边同时作用 $\partial x_i\Rightarrow\Delta(\partial_{x_i}u)=0$

而 $\partial_{x_i}u(x)=\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}\partial_{x_i}u(y)dy=\frac{1}{|B_r(x_0)|}\int_{B_R(x_0)}\mathrm{div}(0\cdots u\cdots)dy$

$=\frac{1}{|B_R(x_0)|}\int_{\partial B_r(x_0)}u\cdot r^i\,dS(y)\Rightarrow|\partial_{x_i}u(x)|\leq\frac{1}{|B_r(x_0)|}\int_{\partial B_r(x_0)}|u|\,dS(y)$

$=\frac{n}{R}\max_{B_R(x_0)}|u|$

$|Du(x)|\leq\frac{n}{R}\max_{B_R(x_0)}|u|$ （度另为最大分另）

Thm 2.1.8 (Liouville 定理) 假设 u 是 $\mathbb{R}^n$ 上的有界调和函数，则 $u\equiv$ const.

设 $u(x)\leq M$，$\forall x\in\mathbb{R}^n$. 由于 u 在 $\mathbb{R}^n$ 上调合，u 在 $\forall B_R$ 上调和

$|Du(x)|\leq\frac{n}{R}\max_{\mathbb{R}^n}|u|\leq\frac{n}{R}M\quad\forall x$

令 $R\to+\infty\Rightarrow|\nabla u(x)|=0\ \forall x\Rightarrow u$ 为常数

Thm 2.2 位势方程 $\Delta u=f$.

∇ δ: Dirac 函数 $\langle\delta,f\rangle=f(0)$，$f*\delta=f$

$\Delta u=f*\delta=f*(\Delta\Gamma)=\Delta(f*\Gamma)$ 令 $u=f*\Gamma$，$\Delta u=f\quad\forall x\in\mathbb{R}^n$

Thm 2.2.1 f radial $\xrightarrow{\Delta u=f\text{解唯一}}$ u radial

∇ δ 径向，Γ 为径向函数. 设 $\Gamma=f(r)$

$\Delta=\partial_r^2+\frac{n-1}{r}\partial_r+\frac{1}{r^2}\Delta_{S^n}$. $\Delta_{\mathbb{R}^n}\Gamma=\Delta_r^2\Gamma+\frac{n-1}{r}\Delta_r\Gamma=0$ $r>0$ (δ 在非0处为(

令 $V=\partial_r\Gamma\Rightarrow\partial_r V+\frac{n-1}{r}V=0\Rightarrow\partial_r\Gamma=V=C_1r^{-(n-1)}$

$\Gamma=\begin{cases}C_1\ln r+C_2 & n=2\end{cases}$ 令 $\Gamma(r)=\begin{cases}\frac{\ln r}{2\pi} & n=2\end{cases}$

$C_1 r^{-(n-2)} + C_2 \quad n\geq 3$ $\quad -\frac{1}{4\pi}\frac{1}{r} \quad n=3$

则 $\Delta\Gamma=\delta$, $u=f*\Gamma$

Thm 2.2.2 $\Delta u=f \quad x\in\Omega$; $u|_{\partial\Omega}=\varphi$

Δ Green公式. 若 $u, v\in C^2(\Omega)\cap C^1(\bar\Omega)$, 则:

$$\int_\Omega u\Delta v\,dx=\int_\Omega[\nabla\cdot(u\cdot\nabla v)-\nabla u\nabla v]\,dx$$

$$\int_\Omega v\Delta u\,dx=\int_\Omega[\nabla\cdot(v\nabla u)-\nabla u\nabla v]\,dx$$

$$\Rightarrow \int_\Omega(u\Delta v-v\Delta u)\,dx=\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds$$

Thm 2.2.3. 若 $u\in C^2(\Omega)\cap C^1(\bar\Omega)$ 满足 $\Delta u=0$ in Ω, 则 $\forall x_0\in\Omega$, $0\in\Omega-\{x_0\}$

$$u(x_0)=\int_{\partial\Omega}\left[-\frac{u}{4\pi}\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}\right]ds$$

$$\Leftrightarrow u(0)=\int_{\partial\Omega}\left[-\frac{u}{4\pi}\frac{\partial}{\partial n}\frac{1}{|x|}+\frac{1}{4\pi|x|}\frac{\partial u}{\partial n}\right]ds$$

考虑 $v(x)=u(x+x_0)\in C^2(\Omega-x_0)\cap C^1(\overline{\Omega-x_0})$

$\Delta v=\Delta u(x+x_0)=0 \quad (\because x+x_0\in\Omega)$

$$v(0)=u(x_0)=\int_{\partial\Omega'}\left[-\frac{v}{4\pi}\frac{\partial}{\partial n}\frac{1}{|x|}+\frac{1}{4\pi|x|}\frac{\partial v}{\partial n}\right]ds$$

$$=\int_{\partial\Omega'}\left[-\frac{u(x+x_0)}{4\pi}\frac{\partial}{\partial n}\frac{1}{|x|}+\frac{1}{4\pi|x|}\frac{\partial u(x+x_0)}{\partial n}\right]ds \quad \underline{\underline{y=x+x_0}}$$

$$=\int_{\partial\Omega}\left[-\frac{u(y)}{4\pi}\frac{\partial}{\partial n}\frac{1}{|y-x_0|}+\frac{1}{4\pi|y-x_0|}\frac{\partial u(y)}{\partial n}\right]ds.$$

尝试对 u, $\frac{1}{|x|}$ 用Green公式. 但 $\frac{1}{|x|}$ 有奇性, 故我们取 $\Omega_\varepsilon=\Omega\setminus\overline{B_\varepsilon(0)}$

则 $\int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds=\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)+\int_{|x|=\varepsilon}\left[\left(u\frac{\partial}{\partial r}\frac{1}{4\pi|x|}\right)^{①}-\frac{1}{4\pi}\dots\right.$

$$\int_{|x|=\varepsilon}u\frac{\partial}{\partial r}\frac{1}{4\pi r}=\int_{|x|=\varepsilon}u\left(-\frac{1}{4\pi|x|^2}\right)ds=-\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}u(x)\,ds$$

$$=-\frac{1}{4\pi\varepsilon^2}\left(\int_{|x|=\varepsilon}u(0)\,ds\right)-\frac{1}{4\pi\varepsilon^2}\left(\int_{|x|=\varepsilon}[u(x)-u(0)]\,ds\right)$$

令 $\varepsilon\to 0$, ① $=-u(0)$

$$-\frac{1}{4\pi\varepsilon}\int_{|x|=\varepsilon}\frac{\partial u}{\partial r}ds=0$$

$\hookrightarrow \left|\frac{1}{4\pi\varepsilon}\int_{|x|=\varepsilon}\frac{\partial u}{\partial r}ds\right|\leq\frac{4\pi\varepsilon^2}{4\pi\varepsilon}\max_{\Omega}\left|\frac{\partial u}{\partial r}\right|=0 \quad *$

$\hookrightarrow -\frac{1}{4\pi\varepsilon}\int_{|x|=\varepsilon}\frac{\partial u}{\partial n}ds=-\frac{1}{4\pi\varepsilon}\int_{|x|\leq\varepsilon}\nabla u\,dx=0$

故而: $\varepsilon\to 0$ 时, $u(0)=\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)$ □

▽若 $\Delta u = f$. * ✓

则 $\int_{\partial\Omega_\varepsilon}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) = \int_\Omega(-vf)\,dx = -u(0) + \int_{\partial\Omega}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})$

$\Rightarrow u(x_0) = \int_\Omega -\frac{1}{4\pi|x-x_0|}f(x)\,dx + \int_{\partial\Omega}[\ u\frac{\partial}{\partial n}(-\frac{1}{4\pi|x-x_0|}) - (-\frac{1}{4\pi|x-x_0|}$

若存在 $g(x)$ 在 $\hat{\Omega}$ 上调和, $g|_{\partial\Omega} = \frac{1}{4\pi|x-x_0|}|_{\partial\Omega}$, 对 u, g 在 $\hat{\Omega}$ 上应用

第二Green公式.

$\int_{\tilde{\Omega}} -gf\,dx = \int_{\partial\tilde{\Omega}}(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}) = \int_{\partial\tilde{\Omega}}(u\frac{\partial}{\partial n}\frac{1}{4\pi|x-x_0|} - \frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}]\,ds$

$(A)+(B) \Rightarrow u(x_0) = \int_\Omega f(g - \frac{1}{4\pi|x-x_0|})\,dx + \int_{\partial\Omega} u\frac{\partial}{\partial n}(\boxed{g - \frac{1}{4\pi|x-x_0|}})\,ds$ Green

$u(x) = \int_\Omega f(y)\,G(y,x)\,dy + \int_{\partial\Omega} u\frac{\partial}{\partial n}G(y,x)\,ds$

Thm 2.2.4 Green函数性质

$G(x,x_0)$ 在 $x \neq x_0$ 时二阶连续可微且调和 $G|_{\partial\Omega} = 0$

$G(x,x_0) + \frac{1}{4\pi|x-x_0|}$ 在 Ω 上二阶连续可微且调合

△ $G(x,x_0) = G(x_0,x)$ $\forall x_0, x \in \Omega$

令 $u(x) = g(x,a)$, $v(x) = g(x,b)$, $a, b \in \Omega$

$u(b) \overset{?}{=} v(a)$.

在 $\Omega\setminus\overline{B_\varepsilon(a)} \sqcup \overline{B_\varepsilon(b)} \overset{\Delta}{=} \Omega_\varepsilon$ 上应用第二Green公式:

则 $0 = \int_{\partial\Omega_\varepsilon}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})$

$= \int_{\partial\Omega}\underbrace{(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})}_{=0} + \underbrace{\int_{|x-a|=\varepsilon}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})}_{A_\varepsilon} + \underbrace{\int_{|x-b|=\varepsilon}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})\,ds}_{B_\varepsilon}$

$A_\varepsilon = \int_{|x-a|=\varepsilon}[(u + \frac{1}{4\pi|x-a|})\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}(u + \frac{1}{4\pi|x-a|})$ ① $- \int_{|x-a|=\varepsilon}\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n}$ ②

① $u + \frac{1}{4\pi|x+a|}$, $v + \frac{1}{4\pi|x-a|}$ 在 $|x-a| = \varepsilon$ 上调和 $+ \int_{|x-a|=\varepsilon} v\frac{\partial}{\partial n}\frac{1}{4\pi|x-a|}$ ③

由第二Green公式 ① $= 0$

②. $-\int_{|x-a|=\varepsilon}\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n}\,ds = -\frac{1}{4\pi\varepsilon}\int_{B_\varepsilon(a)}\Delta v\,dx = 0$

③: $\frac{\partial}{\partial n}\frac{1}{4\pi|x-a|} = -\frac{x-a}{|x-a|}\nabla(\frac{1}{4\pi|x-a|}) = \frac{1}{4\pi|x-a|^2}\cdot -\frac{x-a}{|x-a|}\cdot -\frac{x-a}{|x-a|} =$

$\frac{1}{4\pi\varepsilon^2}\int_{|x-a|=\varepsilon} v(x) = v(a) + \frac{1}{4\pi\varepsilon^2}\int_{|x-a|=\varepsilon}[v(x) - v(a)]\,ds = v(a)$ $\varepsilon \to 0$

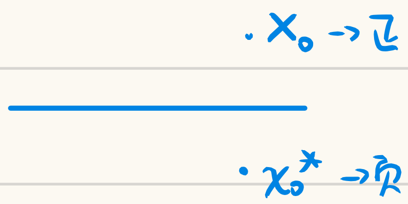
故 $v(b) - u(a) = 0 \Rightarrow G(a,b) = G(b,a)$

Thm 2.2.4 Green 函数的求法

▽ 考虑 G 调和 → 故 f 取 $\frac{1}{4\pi|x-x_0^*|}$ 型，$x_0^* \notin \Omega$

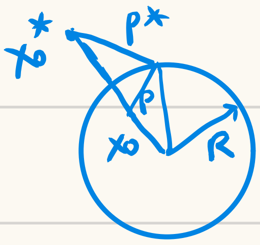
$G|_{\partial\Omega} = 0$ → 正负电荷势能为 0

一. 半空间 $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \mid x_3 > 0\}$ $x_0 \in \mathbb{R}_+^3$



故 $x_0^* = (x_1, x_2, -x_3)$

二. $B_R(0)$ 球



那么 $\overset{\rho}{|x-x_0^*|}$ 与 $\overset{\rho^*}{|x-x_0|}$ 恒成比例 ⇒ 相似

$$\frac{R}{|x_0^*|} = \frac{\rho}{\rho^*} = \frac{|x_0|}{R} \Rightarrow |x_0^*| = \frac{R^2}{|x_0|} \Rightarrow x_0^* = \frac{R^2}{|x_0|^2} x_0$$

故 $G(x, x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{R}{|x_0|} \frac{1}{4\pi|x-x_0^*|}$

Thm 2.2.5

$$\begin{cases} \Delta u = f, & x \in B_R(0) \\ u|_{\partial B_R(0)} = \varphi \end{cases}$$

$\nabla \frac{1}{|x|} = -\frac{x}{|x|^3}$

故 $\nabla G(x) = \frac{x-x_0}{4\pi|x-x_0|^3} - \frac{R}{|x_0|} \frac{x-x_0^*}{4\pi|x-x_0^*|^3}$

而 $|x-x_0^*| = \frac{R}{|x_0|}|x-x_0|$ ⇒ $\nabla G(x) = \frac{x-x_0}{4\pi|x-x_0|^3} - \frac{|x_0|^2}{R^2} \frac{x-x_0^*}{4\pi|x-x_0|^3}$

$= \frac{1}{4\pi|x-x_0|^3}\left(x-x_0 - \frac{|x_0|^2}{R^2}x + \frac{|x_0|^2}{R^2}x_0^*\right) = \frac{1}{4\pi|x-x_0|^3} \frac{R^2-|x_0|^2}{R^2} x$

$\frac{\partial G}{\partial n} = \frac{x}{|x|} \nabla G = \frac{R^2-|x_0|^2}{4\pi R^2|x-x_0|^3}|x| = \frac{R^2-|x_0|^2}{4\pi R|x-x_0|^3}$ $x \in \partial B_R(0)$

所以 $u(x) = \int_\Omega G(y,x) f(y)\,dy + \int_{\partial\Omega} \varphi(y) \frac{\partial G}{\partial n}(y,x)\,dS(y)$

$= \int_{B_R(0)} G(y,x) f(y)\,dy + \frac{R^2-|x|^2}{4\pi R} \int_{\partial B_R(0)} \frac{\varphi(y)}{|y-x|^3}\,dS(y)$ ⇝ Poisson 公式

Thm 2.2.6 Harnack 不等式

u 在 $B_R(x_0)$ 内调和，且 $u \geq 0$. 则 $\frac{R(R-r)}{(R+r)^2} u(x_0) \leq u(x) \leq \frac{R(R+r)}{(R-r)^2} u(0)$

其中 $r = |x-x_0| < r$ （控制一个球内的值）.

事实上仍只需考虑 $x_0 = 0$ 即可

由 Poisson 公式：$u(x) = \frac{R^2-|x|^2}{4\pi R} \int_{|y|=R} \frac{u(y)}{|y-x|^3}\,dS(y)$

$|x|=r,\ |y|=R$ 故 $R-r\le|x-y|\le R+r$

故 $u(x)\le \frac{R^2-r^2}{4\pi R}\int_{|y|=R}\frac{u(y)}{(R-r)^3}dy = \frac{R+r}{4\pi R(R-r)^2}\int_{|y|=R}u(y)dy$

平均值性质 $\frac{R+r}{4\pi R}\frac{1}{(R-r)^2}4\pi R^2 u(0) = \frac{R(R+r)}{(R-r)^2}u(0)$

左不等式同理

Thm 2.2.7 Liouville定理 设 u 是 $\mathbb{R}^n$ 上的有上界(或有下界)的调和函数,则 u 是一个常数

设 $u(x)\le M,\ x\in\mathbb{R}^n$, 令 $V(x)=M-u(x)$, 则 $V(x)\ge 0$ 且调和

利用Harnack不等式 $\frac{R(R-r)}{(R+r)^2}V(x_0)\le V(x)\le\frac{R(R+r)}{(R-r)^2}V(x_0)\quad \forall x\in\mathbb{R}^n,\ R>|x-x_0|$

令 $R\to+\infty \Rightarrow V(x)\equiv V(x_0),\ V(x)=\text{const}$

Thm 2.3.1 弱极大值原理 $(f<0)$

$-\Delta u+c(x)u=f,\ x\in\Omega\subset\mathbb{R}^n,\ c(x)\ge 0.\ \forall x\in\Omega,\ u\in C^2(\Omega)\cap C^1(\bar\Omega),\ f<0$

则 $u(x)$ 不能在 Ω 上达到它在 $\bar\Omega$ 上的非负最大值

设 u 在 $x_0\in\Omega$ 达到在 $\bar\Omega$ 上最大值 $(\Delta u)(x_0)\le 0,\ (\nabla u)(x_0)=0.\ u(x_0)\ge 0$

于是 $-\Delta u+c(x)u\big|_{x=x_0}\ge 0$, 但 $f<0$ 矛盾

Thm 2.3.2 弱极大值原理 $(f\le 0)$

$-\Delta u+c(x)u=f,\ x\in\Omega\subset\mathbb{R}^n,\ c(x)\ge 0,\ \forall x\in\Omega\quad u\in C^2(\Omega)\cap C^1(\bar\Omega),\ f\le 0$

有 $\max\limits_{x\in\bar\Omega}u(x)\le\max\limits_{\partial\Omega}u^+(x),\ u^+=\max\{u(x),0\}$

若 u 在 $\bar\Omega$ 有正的最大值,只能在 $\partial\Omega$ 上取到

不妨设 $0\in\Omega$, 令 $d=\operatorname{diam}\Omega$, 令 $v(x)=-(d^2-|x|^2)\le 0$

$u^\varepsilon(x)=u(x)+\varepsilon v(x),\ L=-\Delta+c(x)$

则 $Lu^\varepsilon(x)=Lu(x)+\varepsilon Lv(x)=f+\varepsilon[-\Delta(-d^2+|x|^2)+c(x)(-d^2+|x|^2)]$

$=f-2n\varepsilon+\varepsilon c(x)(-d^2+|x|^2)<0$

应用"$f<0$", 故 $\max\limits_{\bar\Omega}u^\varepsilon\le\max\limits_{\partial\Omega}u^{\varepsilon+}$

又 $\max\limits_{\bar\Omega}u^\varepsilon\ge\max\limits_{\bar\Omega}u-\varepsilon d^2$

$\max_{\partial\Omega} u^{\varepsilon +} \le \max_{\partial\Omega} u^+ \Rightarrow \max_{\partial\Omega} u^+ \ge \max_{\Omega} u - \varepsilon d^2$ 令 $\varepsilon \to 0$

$\max_{\partial\Omega} u^+ \ge \max_{\Omega} u$

Thm 2.3.3 Hopf 引理

设 B_R 是 $\mathbb{R}^n$ $(n=2,3)$ 中一个以 R 为半径的球. 在 B_R 上 $c(x)\ge 0$ 有界, 若 $u \in C($

有: (1) $Lu = -\Delta u + c(x)u \le 0$, $\forall x \in B_R$

(2) $\exists x_0 \in \partial B_R$, s.t. $u(x)$ 在 x_0 达到它在 $\bar{\Omega}$ 上严格的非负最大值

即 $u(x_0) = \max_{\bar{B}_R} u$, 且 $u(x_0) > u(x)$, $x \in B_R$

则 $\frac{\partial u}{\partial \nu}(x_0) > 0$, ν 与 ∂B_R 在 x_0 的单位外法向夹角小于 $\pi/2$

令 $u^\varepsilon(x) = u(x) + \varepsilon v(x)$, $w_\varepsilon(x)$ 在 x_0 达到它在 $\bar{B}_R$ 的严格最大值

$Lu^\varepsilon(x) = Lu - \varepsilon Lv \le 0$ $\forall x \in B_R$. 则 $\frac{\partial u}{\partial n}(x_0) \ge -\varepsilon \frac{\partial v}{\partial n}(x_0) > 0$ ($\frac{\partial v}{\partial n}(x_0) < 0$)

"≥ 0" 由 u 在 $x_0 \in \partial B_R$ 取严格最大值决定, 我们要说明 $\frac{\partial u}{\partial \nu}(x_0) \ne 0$

令 $v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}$ $(\alpha > 0)$, $u^\varepsilon(x) = u(x) + \varepsilon v(x)$

$\frac{\partial v}{\partial n} = \frac{x}{|x|}\cdot \nabla u = \frac{x}{|x|}(-2\alpha x)e^{-\alpha|x|^2} = -2\alpha|x|e^{-\alpha|x|^2} \boxed{< 0}$

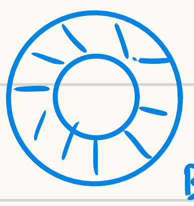
$\partial_{x_i} v = -2\alpha x_i e^{-\alpha|x|^2}$, $\partial_{x_i}^2 v = -2\alpha e^{-\alpha|x|^2} + 4\alpha x_i^2 e^{-\alpha|x|^2}$

$\Delta v = -2\alpha n e^{-\alpha|x|^2} + 4\alpha |x|^2 e^{-\alpha|x|^2}$

$Lv = -\Delta v + c(x)v = (-4\alpha^2|x|^2 + 2\alpha n + c(x))e^{-\alpha|x|^2} - c(x)e^{-\alpha R^2}$

$\le \boxed{(-4\alpha^2|x|^2 + 2\alpha n + c(x))}\, e^{-\alpha|x|^2} \overset{?}{\le} 0$

$\because$ 这里 $2\alpha n + c(x)$ 为正项, 我们需对 $|x|$ 的下限有要求



B_R^*

令 $B_R(0) \setminus \overline{B_{\frac{R}{2}}(0)} = B_R^*$

则 $Lv(x) \le (-\alpha^2 R^2 + 2\alpha n + c(x))e^{-\alpha|x|^2}$

只需 α 充分大, $Lv(x) \le 0$.

$|x| \le \frac{R}{2}$ 时, $u(x_0) + \varepsilon(e^{-\alpha R^2} - e^{-\alpha R^2}) \ge u(x) + \overset{<0}{(e^{-\alpha|x|^2} - e^{-\alpha R^2})}$

由 Thm 2.3.2, 在 B_R^* 上对 u^ε 运用弱极大值原理

u 在 $\overline{B_R^*}$ 上非负最大值 (也即 $\bar{B}_R$ 上最大值) 必在 ∂B_R 上取到.

事实上 $u^{\varepsilon}(x)$ 的最大值也在 x_0 处取到，这是显然的

Thm 2.3.4 强极值原理

假设 Ω 是 $\mathbb{R}^n$ 上的有界连通开集，$c(x)\geq 0$ 有界，若 $u\in C^2(\Omega)\cap C^1(\bar{\Omega})$ 在 Ω 上满足 $Lu\leq 0$，且 $u(x)$ 在 Ω 内达到其在 $\bar{\Omega}$ 上的非负最大值，则 u 在 Ω 上常数

设 $M=\max\limits_{\bar{\Omega}} u\geq 0$，令 $O=\{x\in\Omega \mid u(x)=M\}\subset\Omega$，即证 $O=\Omega$

即证 $O\neq\phi$，且关于 Ω 即开又闭，由于 $\exists x_0\in\Omega$，s.t. $u(x_0)=M$，故 $x_0\in\Omega$

O 非空，若 $x_n\in O$，$x_n\to\bar{x}$，由于 $u\in C^1(\bar{\Omega})$

$u(\bar{x})=\lim\limits_{n\to\infty} u(x_n)=M$，故 $\bar{x}\in O$ $\rightsquigarrow$ O 是闭集

$O=\Omega$，否则 $\Omega\setminus O$ 非空开集

$x_0\in\Omega\setminus O$，$\exists R>0$，s.t. $B_R(x_0)$ 与 ∂O 相切于 y_0

则：① u 在 y_0 上取到 $B_R(x_0)$ 的非负最大值 ② $\forall x\in B_R(x_0)$，$u(x)<M$

由Hopf引理 $\frac{\partial u}{\partial n}(y_0)>0$

而 u 在 $y_0\in\Omega$ 上取到最大值，则 $\nabla u(y_0)=0$，$\frac{\partial u}{\partial n}(y_0)=\nabla u\cdot\vec{n}=0$ 矛盾！

Thm 2.4 最大模估计 稳定性 解的唯一性

Thm 2.4.1 最大模估计（第一类边值）

$u\in C^2(\Omega)\cap C^1(\bar{\Omega})$ 是Dirichlet问题 $\begin{cases}\Delta u=f & \text{in } \Omega\\ u=g & \text{on } \partial\Omega\end{cases}$ 的解

则 $\max\limits_{\bar{\Omega}}|u(x)|\leq G+CF$，$G=\max\limits_{\partial\Omega}|g(x)|$，$F=\max\limits_{\bar{\Omega}}|f(x)|$，$C$ 是一个仅依赖于维数 n 以及直径 $d=\sup\limits_{x,y\in\Omega}|x-y|$ 的常数

令 $V(x)=u(x)-z(x)$，$\Delta V=f-\Delta z\geq 0 \Rightarrow -\Delta z\geq F$ ①

$V|_{\partial\Omega}=u|_{\partial\Omega}-z|_{\partial\Omega}=g-z|_{\partial\Omega}\leq 0 \Rightarrow z|_{\partial\Omega}\geq G$ ②

∇ 如此利用弱极值原理 $u(x)\leq z(x)$

令 $z(x)=\frac{F}{2n}(d^2-|x|^2)+G$ （不妨设 Ω 包含原点）

$\Delta V=f+\Delta z=f+F>0$，$V|_{\partial\Omega}=g-G\leq 0$

故而 $\max\limits_{\Omega} \bar{u} \le \frac{d^2}{2n}F + G$，考虑 $-u(x) \ge -z(x) \ge -G - \frac{d^2}{2n}F$

$|u(x)| \le G + \frac{d^2}{2n}F \rightsquigarrow \max\limits_{\Omega} |u(x)| \le G + CF$

Thm 2.4.2 利用最大模估计证明稳定性

假设 $u_i \in C^2(\Omega) \cap C(\bar{\Omega})$ 满足 $\begin{cases} \Delta u_i = f_i & \text{in } \Omega \\ u_i = g_i & \text{on } \partial\Omega \end{cases}$ $i=1,2$

则 $\max\limits_{\bar{\Omega}} |u_1(x) - u_2(x)| \le \max\limits_{\partial\Omega} |g_1(x) - g_2(x)| + C \max\limits_{\bar{\Omega}} |f_1(x) - f_2(x)|$

令 $u = u_1 - u_2$，$f = f_1 - f_2$，$g = g_1 - g_2$. 使用最大模估计即可

Thm 2.4.3 利用最大模估计证明解的唯一性

若 $f_1 = f_2$，$g_1 = g_2 \Rightarrow u_1 = u_2$

Thm 2.4.4 最大模估计(第三类边值)

$\begin{cases} -\Delta u + c(x) u = f(x) & c(x) \ge 0,\ x \in \Omega \\ \frac{\partial u}{\partial n} + \alpha(x) u = g(x) & \alpha(x) \ge \alpha_0 > 0,\ x \in \partial\Omega \end{cases}$

$u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. $F = \max\limits_{\bar{\Omega}} |f(x)|$，$G = \max\limits_{x \in \partial\Omega} |g(x)|$. 则 $\max\limits_{\bar{\Omega}} |u| \le C(G+F)$

$C = C(n, d)$，$d = \mathrm{diam}\,\Omega$

▷ $f(x) = 0$，$g(x) \le 0$. $-\Delta u + c(x) u = (\le) 0$ claim: $u \le 0$

否则由弱极值原理，u 在 $\partial\Omega$ 上取到正的最大值且 $\frac{\partial u}{\partial n}(x_0) > 0$

故而 $\frac{\partial u}{\partial n} + \alpha(x) u > 0$ 矛盾！

不妨设 $0 \in \Omega$，令 $w(x) = u(x) - z(x)$

则 $-\Delta w + c(x) w = f - (-\Delta z + c(x) z) \Rightarrow -\Delta z + c(x) z \ge F \quad x \in \Omega$ ①

$\frac{\partial w}{\partial n} + \alpha(x) w = \frac{\partial u}{\partial n} + \alpha(x) u - (\frac{\partial z}{\partial n} + \alpha(x) z) = g(x) - (\frac{\partial z}{\partial n} + \alpha(x) z)$

$\Rightarrow \frac{\partial z}{\partial n} + \alpha(x) z \ge G \quad x \in \partial\Omega$ ②

令 $z(x) = \frac{F}{2n}(d^2 - |x|^2) + \frac{Fd}{n\alpha_0} + \frac{G}{\alpha_0}$ $\Rightarrow$ ① 显然满足

$\frac{\partial z}{\partial n} + \alpha(x) z = \nabla z \cdot \vec{n} + \alpha(x) z = \vec{n}(-\frac{F}{n}\vec{x}) + \alpha(x) z \ge \underline{\vec{n}(-\frac{F}{n}\vec{x})} + \alpha(x)\frac{F}{2n}(d$

$0 \le (\underline{+\frac{Fd}{n}} + G \ge G$

由弱极值原理与 ∇. 可知 $u(x) \le z(x) \le \frac{Fd^2}{2n} + \frac{Fd}{\alpha_0 n} + \frac{G}{\alpha_0} \le C(F+G)$

同理有 $-u \le C(F+G) \Rightarrow \max_{\Omega} |u| \le C(F+G)$

§3. 热传导方程

Thm 3.1 Fourier 变换

定义：$\hat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\xi}dx$ ，$f\in\mathcal{S}(\mathbb{R}^n)$

$\mathcal{S}(\mathbb{R}^n)=\{f\in C^\infty(\mathbb{R}^n)$，任意阶导数衰减任意快，则不存在分部积分

① 令 $(T_{x_0}f)(x)=f(x-x_0)$，则 $\widehat{T_{x_0}f}(\xi)=e^{-2\pi i x_0\xi}\hat{f}(\xi)$

$\triangledown$ $\widehat{T_{x_0}f}(\xi)=\int_{\mathbb{R}^n} f(x-x_0)e^{-2\pi i x\xi}dx=\int_{\mathbb{R}^n} f(y)e^{-2\pi i(y+x_0)\xi}dy$

$=e^{-2\pi i x_0\xi}\hat{f}(\xi)$ $\rightsquigarrow$ 平移和Fourier变换可交换

② 令 $(S_\lambda f)(x)=f(\lambda x)$，则 $\widehat{S_\lambda f}(\xi)=\lambda^{-n}\hat{f}(\lambda^{-1}\xi)$

$\triangledown$ $\widehat{S_\lambda f}(\xi)=\int_{\mathbb{R}^n} f(\lambda x)e^{-2\pi i x\xi}dx \overset{\lambda x=y}{=\!=} \int_{\mathbb{R}^n} f(y)\cdot\lambda^{-n}e^{-2\pi i y\frac{\xi}{\lambda}}dy$

$=\lambda^{-n}\hat{f}(\lambda^{-1}\xi)$ $\rightsquigarrow$ 伸缩和Fourier变换可交换

③ 设 $\alpha=(\alpha_1\cdots\alpha_n)$ 是多重指标，$|\alpha|=\alpha_1+\cdots+|\alpha_n|$，$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$

$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. 则 $\widehat{\partial_x^\alpha f}(\xi)=(2\pi i\xi)^\alpha\hat{f}(\xi)$

$\triangledown$ $\widehat{\partial_{x_j}f}(\xi)=\int_{\mathbb{R}^n}\partial_{x_j}f(x)e^{-2\pi i x\xi}dx=-\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\xi}(-2\pi i\xi_j)dx$

$=2\pi i\xi_j\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\xi}dx=2\pi i\xi_j\hat{f}(\xi)$

$\widehat{\partial_{x_j}^{\alpha_j}f}(\xi)=(2\pi i\xi_j)^{\alpha_j}\hat{f}(\xi) \Rightarrow \widehat{\partial_x^\alpha f}(\xi)=(2\pi i\xi)^\alpha\hat{f}(\xi)$

④ $\widehat{(-2\pi i x)^\alpha f}(\xi)=\partial_\xi^\alpha\hat{f}(\xi)$

$\triangledown$ $\widehat{-2\pi i x_j f}(\xi)=\int_{\mathbb{R}^n}-2\pi i x_j f(x)e^{-2\pi i\xi x}dx=\int_{\mathbb{R}^n}\partial_{\xi_j}(e^{-2\pi i\xi x})f(x)dx$

$=\partial_{\xi_j}\int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi x}dx=\partial_{\xi_j}\hat{f}(\xi)$

⑤ 令 $(f*g)(x)=\int_{\mathbb{R}^n} f(x-y)g(y)dy=\int_{\mathbb{R}^n} f(y)g(x-y)dy$

$\forall f,g\in\mathcal{S}(\mathbb{R}^n)$ 则 $\widehat{f*g}(\xi)=\hat{f}(\xi)\hat{g}(\xi)$

⑥ 逆变换 $\check{f}(x)=\int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\xi}dx$. 则若 $f\in\mathcal{S}(\mathbb{R}^n)$，有 $\hat{f}\in\mathcal{S}(\mathbb{R}^n)$

$\check{\hat{f}}=f$

⑦ $\widehat{\Delta u}(\xi)=\widehat{\partial_{x_1}^2 u}+\cdots+\widehat{\partial_{x_n}^2 u}=[(2\pi i\xi_1)^2+\cdots+(2\pi i\xi_n)^2]\hat{u}(\xi)=-4\pi^2\xi^2\hat{u}(\xi)$

$\triangledown$ 求函数 $f(x)=\begin{cases}e^{-x} & x>0\end{cases}$ 的Fourier变换

$$\begin{cases} & \\ 0 & x\le 0\end{cases}$$

$\hat{f}(\xi)=\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx=\int_0^{+\infty}e^{-x}e^{-2\pi i x\xi}dx=\int_0^{+\infty}e^{(-2\pi i\xi+1)x}dx$

$=-\frac{1}{1+2\pi i\xi}e^{-x(1+2\pi i\xi)}\Big|_0^{+\infty}=\frac{1}{1+2\pi i\xi}$

▷ 求函数 $f(x)=e^{-|x|}$ 的 Fourier 变换.

$\hat{f}(\xi)=\frac{1}{1+2\pi i\xi}-\frac{1}{1-2\pi i\xi}=\frac{2}{1+4\pi^2|\xi|^2}$

▷ 求函数 $f(x)=e^{-x^2}$ 的 Fourier 变换

$\hat{f}(\xi)=\int_{\mathbb{R}}e^{-x^2}e^{-2\pi i\xi x}dx=F(\xi)$

$F'(\xi)=\int_{\mathbb{R}}e^{-x^2}(-2\pi i x)e^{-2\pi i\xi x}dx=\pi i\int_{\mathbb{R}}\partial_x(e^{-x^2})e^{-2\pi i\xi x}dx$

$=-\pi i\int_{\mathbb{R}}e^{-x^2}\partial_x(e^{-2\pi i\xi x})dx=-2\pi^2\xi\int_{\mathbb{R}}e^{-x^2}e^{-2\pi i\xi x}dx$

$=-2\pi^2\xi F(\xi)$

于是 $\begin{cases}F'(\xi)+2\pi^2\xi F(\xi)=0\\ F(0)=\int_{\mathbb{R}^n}e^{-x^2}dx=\sqrt{\pi}\end{cases}\Rightarrow F(\xi)=\sqrt{\pi}e^{-\pi^2|\xi|^2}$

▷ 求 $f(x)=e^{-x^2}$, $x\in\mathbb{R}^n$ 的 Fourier 级数

$\hat{f}(\xi)=\int_{\mathbb{R}^n}e^{-|x|^2}e^{-2\pi i x\xi}dx=\int_{\mathbb{R}}e^{-x_1^2}e^{-2\pi i x\xi_1}dx_1\cdots\int_{\mathbb{R}}e^{-x_n^2}e^{-2\pi i x\xi_n}dx_n$

$=\pi^{\frac{n}{2}}e^{-\pi^2|\xi|^2}$

Thm 3.2 $\begin{cases}\partial_t u-\Delta u=0 & x\in\mathbb{R}^n\\ u(x,0)=\varphi(x)\end{cases}$

▷ $\hat{u_t}(\xi)=\int u_t(x,t)e^{-2\pi i x\xi}dx=\frac{\partial}{\partial t}\hat{u}$

作 Fourier 变换

$\begin{cases}\partial_t\hat{u}(\xi)+4\pi^2|\xi|^2\hat{u}(\xi)=0\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi)\end{cases}$

$\Rightarrow\hat{u}(\xi)=e^{-4\pi^2|\xi|^2t}\hat{\varphi}(\xi)$, 两边同时做 Fourier 逆变换

$u(\xi)=(e^{-4\pi^2|\xi|^2t}\hat{\varphi}(\xi))^{\vee}=(e^{-4\pi^2|\xi|^2t})^{\vee}*\varphi$

▷ $\check{f}(x)=\int_{\mathbb{R}^n}e^{-4\pi^2|\xi|^2t}e^{2\pi i x\xi}d\xi$

令 $\xi' = 2\sqrt{t}\,\xi \Rightarrow \check{f}(x) = \int_{\mathbb{R}^n} e^{-\pi^2 \xi'^2} e^{2\pi i \xi' \frac{x}{2\sqrt{t}}} (2\sqrt{t})^{-n} d\xi'$

$= \frac{1}{(4t)^{\frac{n}{2}}} f\left(\frac{x}{2\sqrt{t}}\right) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$

$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int e^{-\frac{|x-y|^2}{4\pi}} \varphi(y)\, dy$

▽ 令 $k(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$，$k_t(x) = (t)^{-\frac{n}{2}} k\left(\frac{x}{\sqrt{t}}\right)$

对于 $\{k_t(x)\}_{t>0}$：

① $\int_{\mathbb{R}^n} k_t(x)\, dx = \int_{\mathbb{R}^n} k\left(\frac{x}{\sqrt{t}}\right) d\frac{x}{\sqrt{t}} = \int_{\mathbb{R}^n} k(x)\, dx = 1$

② $\int_{\mathbb{R}^n} |k_t(x)|\, dx = 1$

③ $\forall \eta > 0$, $\int_{|x|>\eta} |k_t(x)|\, dx = \int_{|y|>\frac{\eta}{\sqrt{t}}} k(y)\, dy \xrightarrow{t\to 0^+} 0$

▽ 若 φ 连续，有界，则 $\lim_{t\to 0^+} u(x,t) = \varphi(x)$

$u(x,t) - \varphi(x) = \int_{\mathbb{R}^n} k_t(y)\, \varphi(x-y) - \int_{\mathbb{R}^n} k_t(y)\, \varphi(x)\, dy$

$= \int_{\mathbb{R}^n} k_t(y)\, (\varphi(x-y) - \varphi(x))\, dy = \int_{\mathbb{R}^n} k(z)\, (\varphi(x-\sqrt{t}z) - \varphi(x))\, dz$

$\forall \varepsilon > 0, \exists \delta$，若 $|y| < \delta$，则 $|\varphi(x-y) - \varphi(x)| < \varepsilon$（连续性）

由于 $k(z)$ 在 $\mathbb{R}^n$ 上可积，$\forall \varepsilon > 0, \exists R$，$\int_{|z|>R} k(z)\, dz < \varepsilon$

于是 $|u(x,t) - \varphi(x)| = \underbrace{\int_{|z|>R}}_{\text{有界}+\int k \text{收敛}} + \underbrace{\int_{|z|\le R}}_{\text{连续}+k\text{有界}} < \varepsilon'$

ε' 的任意性：$\lim_{t\to 0^+} u(x,t) = \varphi(x)$

Thm3.3 从 $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\, dy$ 推得性质

(1) $\forall t > 0$，$u(x,t) \in C^\infty(\mathbb{R}^n)$

(2) $\sup_x |u(x,t)| \le \sup_x |\varphi(x)|$，若 $\varphi(x) > 0$，则 $u(x) > 0$

▽ $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} \varphi(x-y)\, dy$

$\sup_x |u(x,t)| \le \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} \sup_x \varphi(x)|\, dy = \sup_x |\varphi(x)|$

(3) 无限传播速度

(4) 沿时间不能反向演化

Thm3.4 $\begin{cases} \partial_t u - \Delta u = f(x,t) & x \in \mathbb{R}^n \end{cases}$

$u(x,t)=\varphi(x)$

$$\hat{u}(\xi,t)=e^{-4\pi^2|\xi|^2 t}\hat{\varphi}(\xi)+\int_0^t e^{-4\pi^2|\xi|^2(t-s)}\hat{f}(\xi,s)\,ds$$

$$u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\varphi(y)\,dy+\int_0^t\frac{1}{(4\pi(t-s))^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)\,dy\,ds$$

Thm 3.5 能量估计

$$\begin{cases}\partial_t u-\Delta u=f(x,t)\\ u(x,0)=\varphi(x)\\ u|_{\partial\Omega}=0\end{cases}$$

对 $\partial_t u-\Delta u=0$ 进行能量估计

$$u\partial_t u-u\Delta u=fu \Rightarrow \frac{1}{2}\partial_t(u^2)-\sum_{i=1}^{n}\partial_{x_i}(u\partial_{x_i}u)+\sum_{i=1}^{n}(\partial_{x_i}u)^2=fu$$

$\frac{1}{2}\partial_t(u^2)-\nabla\cdot(u\nabla u)+(\nabla u)^2=fu$ ⇝ 在 Ω 上积分得

$$\partial_t\int_\Omega \frac{1}{2}u^2dx-\int_{\partial\Omega}\underbrace{u\frac{\partial u}{\partial n}}_{0}dx+\int_\Omega(\nabla u)^2dx=\int_\Omega fu\,dx$$

$$\partial_t\frac{1}{2}\int_\Omega u^2dx+\int_\Omega(\nabla u)^2dx\le\frac{1}{2}\int_\Omega u^2dx+\frac{1}{2}\int_\Omega f^2dx$$

$$\Rightarrow \partial_t\left(\frac{1}{2}\int_\Omega u^2dx\right)\le\frac{1}{2}\int_\Omega u^2dx+\frac{1}{2}\int_\Omega f^2dx$$

$$\Rightarrow \partial_t(E(t)\cdot e^{-t})\le C^T\frac{1}{2}\int_\Omega f^2dx$$

$$\Rightarrow E(t)\cdot e^{-t}\le\frac{1}{2}\int_\Omega\varphi^2dx+C^T\frac{1}{2}\int_0^T dt\int_\Omega f^2dx$$

$$\Rightarrow \int_\Omega u^2dx\le C^T\left(\int_\Omega\varphi^2dx+\int_0^T\int_\Omega f^2dx\,dt\right)$$

Thm 3.6 极值原理

$$\begin{cases}\partial_t u-\Delta u=0\\ u(x,0)=\varphi(x)\\ u(x,t)=h(x,t) \quad x\in\partial\Omega\end{cases}$$

▽ $Q_T=\Omega\times(0,T]$

侧面&下底面：抛物边界 Γ

→关于 t C^1, x C^2 → 保证能取到最小值

假设 $u\in C^{1,2}(Q_T)\cap C(\bar{Q}_T)$. 满足方程 $Lu=\partial_t u-\Delta u\le 0$

则 u 在 $\bar{Q}_T$ 上的最大值在抛物边界取到, 即 $\overset{M}{\max_{\bar{Q}_T}}u\le\overset{m}{\max_{\Gamma}}u$

Step 1. $f<0$. 否则 $M>m$, 则 u 在 $(x_*,t_*)\in Q_T$ 达到 M

则 $\partial_x u(x_*,t_*)=0$，$\Delta u(x_*,t_*)\le 0$，$\partial_t u(x_*,t_*)\ge 0$，

因此 $(\partial_t u-\Delta u)(x_*,t_*)\ge 0$ 矛盾

step 2. $f\le 0$，令 $v(x,t)=u(x,t)-\varepsilon t$ $\varepsilon>0$

则 $\partial_t v-\Delta v=f-\varepsilon<0$. 由 step 1

$\max\limits_{Q_T} u-\varepsilon T\le \max\limits_{Q_T} v\le \max\limits_{\Gamma} v\le \max\limits_{\Gamma} u$

令 $\varepsilon\to 0 \Rightarrow \max\limits_{Q_T} u\le \max\limits_{\Gamma} u$

▷Cor: 设 $u\in C^{1,2}(Q_T)\cap C(\overline{Q_T})$，满足 $Lu=\partial_t u-\Delta u=f\ge 0$，则 $\min\limits_{Q_T} u\ge \min\limits_{\Gamma} u$

Cor:(比较定理) 设 $u,v\in C^{1,2}(Q_T)\cap C(\overline{Q_T})$ 满足 $Lu\le Lv$，$u|_\Gamma\le v|_\Gamma$

则在 $\overline{Q_T}$ 上有 $u(x,t)-v(x,t)$

Thm 3.7 最大模估计 (第一类边值)

设 $u\in C^{1,2}(Q_T)\cap C(\overline{Q_T})$ 是 $\begin{cases} Lu=\partial_t u-\Delta u=f & x\in[0,l),\ 0<t<T \\ u(x,0)=\varphi(x) \\ u(0,t)=g_1(t),\ u(l,t)=g_2(t) \end{cases}$

则 $\max\limits_{Q_T}|u|\le FT+B$

$F=\max\limits_{Q_T}|f|$，$B=\max\{\max\limits_{[0,l]}|\varphi(x)|,\ \max\limits_{[0,T]}|g_1(t)|,\ \max\limits_{[0,T]}|g_2(t)|\}$

令 $v(x,t)=u(x,t)-(Ft+B)$

则 $\partial_t v-\Delta v=f-F\le 0$，$v(x,0)=\varphi-B\le 0$，$v(0,t)=g_1(t)-B-Ft\le 0$

$v(l,t)=g_2(t)-B-Ft\le 0$

故由极值原理 $\max\limits_{Q_T} v=\max\limits_{\Gamma} v\le 0$ 于是 $\max\limits_{Q_T} u\le Ft+B$.

类似可证: $\max\limits_{Q_T} -u\le Ft+B \Rightarrow \max\limits_{Q_T}|u|\le Ft+B$.

▷由此可得唯一性、稳定性

Thm 3.8 最大模估计 (第三类边值)

$\begin{cases} \partial_t u-\partial_x^2 u=f \\ u(x,0)=\varphi(x) \\ u(0,t)=g_1(t),\ (u_x+hu)(l,t)=g_2(t) & h>0. \end{cases}$

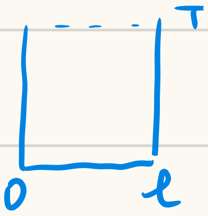
唯一性

只需证明 $\begin{cases} \partial_t u-\partial_x^2 u=0 \\ u(x,0)=0 \\ u(0,t)=0,\ (u_x+hu)(l,t)=0 \end{cases}$ (★) 只有零解

否则(★)有非零解,它有正的最大值或负的最小值.设u有正的最大值

由极值原理,u的最大值在边界取到,由于 $u(x,0)=0,\ u(0,t)=0$

故u最大值只能在 $x=l$ 上取到. 设u在 $(l,\bar{t})$ 达到最大值



则 $\partial_x u(l,\bar{t})\geq 0,\ u(l,\bar{t})>0$, $(\partial_x u+hu)(l,t)>0$ 矛盾!

类似可证u不能有负的最小值 $\Rightarrow u\equiv 0$

唯一性得证

Thm 3.9 (第二类边值)

$$\begin{cases} \partial_t u-\partial_x^2 u=0 \\ u(x,0)=0 \\ u(0,t)=0,\ u_x(l,t)=0 \end{cases}$$

唯一性

令 $\hat{u}(x,t)=u(x,t)w$

$$\partial_t u=\frac{\partial_t\hat{u}}{w},\quad \partial_x u=\frac{\partial_x\hat{u}}{w}-\frac{\partial_x w\,\hat{u}}{w^2}$$

$$\partial_x^2 u=\frac{\partial_x^2\hat{u}}{w}-\frac{\partial_x w\,\partial_x\hat{u}}{w^2}-\frac{(\partial_x^2 w\,\hat{u}+\partial_x w\,\partial_x\hat{u})w^2-2(\partial_x w)^2\hat{u}w}{w^4}$$

$$=\frac{\partial_x^2\hat{u}}{w}-\frac{2\partial_x w\,\partial_x\hat{u}}{w^2}+2\frac{(\partial_x w)^2}{w^3}\hat{u}-\frac{\partial_x^2 w\,\hat{u}}{w^2}$$

$$\Rightarrow \partial_t\hat{u}-\partial_x^2\hat{u}+\frac{2\partial_x w}{w}\partial_x\hat{u}-\left(\frac{2(\partial_x w)^2}{w^2}-\frac{\partial_x^2 w}{w}\right)\hat{u}=0$$

$$u_x(l,t)=0 \Rightarrow \left(\frac{\hat{u}_x}{w}-\frac{\partial_x w}{w^2}\hat{u}\right)(l,t)=0 \Rightarrow \left(\hat{u}_x-\frac{\partial_x w}{w}\hat{u}\right)(l,t)=0$$

$\hat{u}(0,t)=0,\ \hat{u}(x,0)=0$

取 $w(x)=l-x+1,\ w(x)\geq 1$ $\quad -\frac{\partial_x w}{w}=\frac{1}{l-x+1},\quad -\frac{w(l)}{w}=1$

则 $\hat{u}$: $\begin{cases} \partial_t\hat{u}-\partial_x^2\hat{u}-\frac{2}{l-x+1}\partial_x\hat{u}-\frac{2}{(l-x+1)^2}\hat{u}=0 \quad * \\ \hat{u}(0,x)=0 \\ \hat{u}(0,t)=0,\ (\hat{u}_x+\hat{u})(l,t)=0 \end{cases}$

令 $v(x,t)=e^{-\lambda t}\hat{u}$, $\hat{u}=e^{\lambda t}v$ $\quad \partial_t\hat{u}=\lambda e^{\lambda t}v+e^{\lambda t}\partial_t v$

于是 $\partial_t v-\partial_x^2 v-\frac{2}{l-x+1}\partial_x v+\left(\lambda-\frac{2}{(l-x+1)^2}\right)v=0$

$V(x,0)=0,\ V(0,t)=0,\ (V_x+hV)(l,t)=0$

若 $V\not\equiv 0$，则 V 有正的最大值和负的最小值

设 V 在 $(x^*,t^*)\in Q_T$ 达到正的最大值，则

$\partial_t V(x^*,t^*)\geq 0,\ \partial_x^2 V(x^*,t^*)\leq 0,\ \partial_x V(x^*,t^*)=0,\ V(x^*,t^*)>0$

这与其满足 * 矛盾！同理可证没有负的最小值

故 V 的正的最大值只能在边界取到，由边值 $V(x,0)=0,\ V(0,t)=0$

知最大值只能在 $x=l$ 处取到，则 $\partial_x V(l,\bar{t})\geq 0$，这与 $(V_x+hV)(l,t)=0$

矛盾，故 V 没有正的最大值 $\Rightarrow V\equiv 0 \Rightarrow \hat{u}\equiv 0 \Rightarrow u\equiv 0$