2024秋 微分几何 H

参考书：① 彭家贵 陈卿 ② 沈一兵 ③ Spivak Vol II Chap 1-3 Vol III

④ do Carmo ⑤ O'Neil

Chap 1 向量与微分形式

$$C^\infty(\mathbb{R}^3) \xrightarrow{\text{grad}} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\text{rot}} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\text{div}} C(\mathbb{R}^3)$$

$$\downarrow \qquad \downarrow u_i \to dx^i \qquad \downarrow u_1 \to dx^1 \wedge dx^2 \qquad \downarrow \qquad \text{交换}$$

$$\Lambda_0(\mathbb{R}) \xrightarrow{d} \Lambda_1(\mathbb{R}) \xrightarrow{d} \Lambda_2(\mathbb{R}) \xrightarrow{d} \Lambda_3(\mathbb{R})$$

Chap 2 曲线的局部理论

$r:(a,b)\to E^3$ 正则曲线 若 (i) $x(t), y(t), z(t)$ 光滑

$t\mapsto (x(t), y(t), z(t))$

(ii) $|r'(t)| = |(x'(t), y'(t), z'(t))| > 0$ (浸入)

习题 1-1

$s:(a,b)\to(0,L)$

弧长 $s(t) = \int_a^b |r'(t)|dt$. $s'(t)>0$ 可逆. 逆函数记 $t(s)$

弧长参数化 $c(s) = r(t(s)) : (0,L)\to E^3$

$|c'(s)| = 1$.

平面曲线： $r:(0,b)\to E^2$.

弯曲程度？ 密切圆半径倒数

$t_1, t_2, t_3 \to t$ 确定的圆 圆心记 $C(t_1, t_2, t_3)$

(存在 $r(t_1)\sim r(t_3)$ 不共线) ⊙

$f(t) = |r(t) - C(t_1, t_2, t_3)|^2$ $f(t_1) = f(t_2) = f(t_3)$

故$\exists \xi_1, \xi_2$ s.t $0 = \langle r'(t), r(t) - C(t_1, t_2, t_3) \rangle \Big|_{t=\xi_1, \xi_2}$

再令 $g(t) = \langle r'(t), r(t) - C(t_1, t_2, t_3) \rangle$

则$\exists \eta \in (\xi_1, \xi_2)$ s.t. $g'(\eta) = \langle r''(\eta), r(\eta) - C(t_1, t_2, t_3) \rangle + \langle r'(\eta), r'(\eta) \rangle$

令 $t_1, t_2, t_3 \to t_0$ 则 $C(t_1, t_2, t_3) \to C$

$$\begin{cases} \langle r'(t_0), r(t_0) - C \rangle = 0 \\ \langle r''(t_0), r(t_0) - C \rangle + \langle r'(t_0), r'(t_0) \rangle = 0 \end{cases}$$

注 设 r 弧长参数化 $r(s)$ 则 $\langle r'(s), r'(s) \rangle = 1$

$$\langle r''(s), r'(s) \rangle = 0$$

$$\begin{cases} \langle r'(s_0), r(s_0) - C \rangle = 0 \\ \langle r''(s_0), r(s_0) - C \rangle = -1 \end{cases}$$

若 $r''(s) \neq 0$ 则 $r(s_0) - C = a r''(s_0) \Rightarrow a = -\frac{1}{|r''(s)|^2}$ ②

故曲率 $= \frac{1}{R} = \frac{1}{|r''(s)|} = |r''(s)|$

由于推出①!

Thm. r 弧长参数化平面曲线.

(1) 若 $r''(s_0) \neq 0$. 则 $s_1 \sim s_3 \to s_0$ 时 $r(s_1)$ 不共线 且

$r(s_1)$ 可确定一个半径为 $\frac{1}{|r''(s_0)|}$ 的圆. 圆心在与 $r'(s_0)$ 垂直的线上

(2) 若 $r''(s_0) = 0$. 则圆不收敛. (无论 $r(s_1)$ 共线与否)

故定义曲率为 $k(s) = |r''(s)|$ (不带符号)

Frenet标架：$\{r(s); t(s), n(s)\}$

$\dot{t}(s)=\ddot{r}(s)=k(s)n(s)$

又$\langle n(s), n(s)\rangle=1 \Rightarrow \langle n(s), \dot{n}(s)\rangle=0 \Rightarrow \dot{n}(s)$ $t(s)$共线

$\langle n(s), t(s)\rangle=0 \Rightarrow \langle \dot{n}(s), t(s)\rangle = -\langle n(s), k(s)n(s)\rangle$

$= -k(s)$

故 $\dot{n}(s)=-k(s)t(s)$

$$\begin{pmatrix} \dot{t}(s) \\ \dot{n}(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \end{pmatrix}$$

$$k(s)=\det(\dot{r}(s), \ddot{r}(s)) = \det\begin{pmatrix} \dot{x}(s) & \dot{y}(s) \\ \ddot{x}(s) & \ddot{y}(s) \end{pmatrix}$$

方向变化改变 $k(s)$ 符号 （带符号曲率）

一般参数 $r(t)$ 则 $k(r(t)) = \det\left(\dot{r}(t(s))\frac{dt}{ds}, \ddot{r}(t(s))\left(\frac{dt}{ds}\right)^2 + \dot{r}(t)\frac{d^2t}{ds^2}\right)$

$$= \det(\dot{r}(t(s)), \ddot{r}(t(s)))\left(\frac{dt}{ds}\right)^3$$

$$= \frac{\det(\dot{r}(t), \ddot{r}(t))}{|\dot{r}(t)|^3} = \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$$

合理性：① $k(s)\equiv 0 \Leftrightarrow \ddot{r}(s)=0 \Leftrightarrow \dot{r}(s)\equiv u \Leftrightarrow$ 直线

② $k(s)=a \Rightarrow \ddot{r}(s)=a n(s) \Rightarrow \frac{d}{ds}\left(r(s)+\frac{1}{a}n(s)\right)=0$

$\dot{n}(s)=-a t(s)$ 记 $r(s)+\frac{1}{a}n(s)=v_0$

$|r(s)-v_0| = \frac{1}{a} \Rightarrow$ 为圆

k 的符号：Gauss 映射的像 $\begin{cases} + : \text{逆时针转} \\ - : \text{顺时针转} \end{cases}$

Thm(平面曲线基本定理) 设 $k:[a,b]\to\mathbb{R}$ 可连续可微，则存在弧长参数正则曲线 $r:[a,b]\to\mathbb{R}^2$，使其在 s 处的带符号曲率为 $k(s)$

且若存在 r 二阶导条件，则存在刚体变换 A s.t. $\tilde{r}=Ar$

Pf. 若存在这样的曲线 $r(s)$，则 $\begin{cases}\frac{dr}{ds}=t(s)\\ \frac{dt}{ds}=k(s)n(s)\\ \frac{dn}{ds}=-k(s)t(s)\end{cases}$

(Picard-Lindelöf: $\begin{cases}\frac{d\vec{\alpha}(t)}{dt}=\underbrace{f(t,\vec{\alpha}(t))}_{\text{Lipschitz}}\\ \vec{\alpha}(0)=\vec{\alpha}_0\end{cases}$ 解局部存在

更进一步地，若 $f(t)=A(t)\vec{\alpha}(t)$ (线性)

则整体有解)

$\begin{cases}\frac{dr}{ds}=\begin{pmatrix}t_1(s)\\ t_2(s)\end{pmatrix}\\ \frac{dt}{ds}=k(s)\begin{pmatrix}-t_2(s)\\ t_1(s)\end{pmatrix}\\ r(0)=r_0\\ t(0)=\begin{pmatrix}t_1(0)\\ t_2(0)\end{pmatrix}\text{ 且 }\langle t(0),t(0)\rangle=1\end{cases}$ Picard-Lindelöf 有解 $r(s)$ $t(s)$

检验. 弧长参数: $\frac{d}{ds}\langle t(s),t(s)\rangle=2\langle t(s),\dot{t}(s)\rangle$

$=2\left(t(s),k(s)\begin{pmatrix}-t_2(s)\\ t_1(s)\end{pmatrix}\right)=0$

$\Rightarrow\langle t(s),t(s)\rangle=1$

切存在性 ✓

刚体变换下唯一性: 若有 $r(s)$, $\tilde{r}(s)$

考察刚体运动 $A(r)=Br+C$. s.t. $Ar(0)=\tilde{r}(0)$

$At(0)=\tilde{t}(0)$

$\begin{cases}\frac{d}{ds}(Ar(s))=Bt(s)\\ \frac{d}{ds}(Bt(s))=Bk(s)\begin{pmatrix}-t_2(s)\\ t_1(s)\end{pmatrix}=k(s)\begin{pmatrix}-(Bt_2)(s)\\ (Bt_1)(s)\end{pmatrix}\end{cases}$

$\Rightarrow$ 初值方程相同 $\Rightarrow Ar=\tilde{r}$. #

空间曲线

密切平面：$r(s_1)\ r(s_2)\ r(s_3)$不共线，则为唯一平面，法向量 $\vec{n}(s_1,s_2,s_3)$

取该平面上一点 $P(s_1,s_2,s_3)$

则 $\langle r(s_i)-p(s_1,s_2,s_3),\ a(s_1,s_2,s_3)\rangle=0$

$\exists j,\ s.t.\ \langle \dot{r}(s),\ a(s_1,s_2,s_3)\rangle|_{s_1,s_2}=0$

$\exists \eta\in(s_1,s_2)\quad s.t.\ \langle \ddot{r}(s),\ a(s_1,s_2,s_3)\rangle|_{\eta}=0$

设 $\{\dot{r}(s),\ \ddot{r}(s),\ \vec{a}(s_1,s_2,s_3)\}$ 成右手系

若平面收敛，即 $\vec{a}(s_1,s_2,s_3)\to\vec{a}$，则若 $\ddot{r}(s_0)\neq 0$

密切平面为 $\dot{r}(s_0),\ \ddot{r}(s_0)$ 张成的平面

定理：r 为长为弧长的则曲线 $\ddot{r}(s_0)\neq 0$

则 (1) $s_1\sim s_3\to s_0$ 时 $r(s_1)\sim r(s_3)$ 存在

(2) $r(s_i)$ 极限的平面收敛到过 $r(s_0)$ 由 $\dot{r}(s_0),\ \ddot{r}(s_0)$ 张成平面

(3) 密切圆半径 $\dfrac{1}{|\ddot{r}(s_0)|}$，曲率 $k(s)=|\ddot{r}(s)|$

切向量 $t(s)=\dot{r}(s)$

主法向量 $n(s)=\dfrac{\ddot{r}(s)}{|\ddot{r}(s)|}$

副法向量 $b(s)=t(s)\wedge n(s)$

$\langle b(s),b(s)\rangle=1\Rightarrow\langle \dot{b}(s),b(s)\rangle=1\quad\Rightarrow \dot{b}(s)=f(s)t(s)+g(s)n(s)$

又 $\langle \dot{b}(s),\ t(s)\rangle=-\langle b(s),\dot{t}(s)\rangle=-\langle b(s),\ |\ddot{r}(s)|\,n(s)\rangle=0\quad\Rightarrow f(s)=0$

引入记号 $T(s)=-g(s)$ 挠率

$\frac{dt}{ds} = k(s)n(s)$

$\frac{db}{ds} = -T(s)n(s)$

$\frac{dn}{ds} = ?$

$\langle n, n\rangle = 1 \Rightarrow \langle n, \dot{n}\rangle = 0$

$\langle n, t\rangle = 0 \Rightarrow \langle t, \dot{n}\rangle = -\langle n, \dot{t}\rangle = -k(s)$

$\langle n, b\rangle = 0 \Rightarrow \langle \dot{n}, b\rangle = -\langle n, \dot{b}\rangle = T(s)$

$\Rightarrow \frac{dn}{ds} = -k(s)t(s) + T(s)b(s)$

$$\Rightarrow \frac{d}{ds}\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} & k & \\ -k & & T \\ & -T & \end{pmatrix}\begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

$n-b$平面：法平面

$t-b$平面：从切平面

$k(s) = |\ddot{r}(s)| = |\dot{r}(s) \wedge \ddot{r}(s)|$

$T(s) = -\langle n(s), \dot{b}(s)\rangle = -\langle \frac{\dot{r}(s)}{k(s)}, t(s) \wedge \dot{n}(s)\rangle$

$= -\langle \frac{\ddot{r}(s)}{k(s)}, t(s) \wedge \left(\frac{\ddot{r}(s)}{k(s)}\right)'\rangle$

$= \frac{(\dot{r}, \ddot{r}, \dddot{r})}{|\dot{r}(s) \wedge \ddot{r}(s)|^2} = \frac{(\dot{r}, \ddot{r}, \dddot{r})}{|\dot{r}|^2}$

一般：$k = \frac{|\dot{r} \wedge \ddot{r}|}{|\dot{r}|^3}$ $\quad T = \frac{(\dot{r}, \ddot{r}, \dddot{r})}{|\dot{r} \wedge \ddot{r}|^2}$

定理：若$k > 0$，则平面曲线 $\Leftrightarrow T = 0$

pf $\Rightarrow$ 在平面上点P，法向量a

$\langle r(s) - P, a\rangle = 0 \Rightarrow \langle t(s), a\rangle = 0 \Rightarrow k(s)\langle n(s), a\rangle = 0$

即$\langle n(s), a\rangle = 0 \Rightarrow \langle -k(s)t(s) + T(s)b(s), a\rangle = 0$

$\Rightarrow T(s)\langle b(s), a\rangle = 0$

$\Rightarrow T = 0$

$\Leftarrow$. $\tau=0 \Rightarrow \dot{b}(s)=0 \Rightarrow \frac{d}{ds}\langle r(s), b\rangle = \langle t(s), b\rangle = 0$

即 $\langle r(s), b\rangle \equiv C_0 = \langle v_0, b\rangle \Rightarrow \langle r(s)-v_0, b\rangle = 0$ 即平面. #.

为定 $T: E^3 \to E^3$ 保长变换 $\tilde{r}(s) = T \circ r(s)$

$\frac{d}{ds}\tilde{r}(s) = T \circ t(s) \Rightarrow$ 也为弧长参数

$\tilde{t}(s) = T \circ t(s) \quad \tilde{n}(s) = T \circ n(s)$

$\tilde{b}(s) = (T \circ t(s)) \wedge (T \circ n(s)) = (\det T)\, T \circ b(s)$

故若 $\det T = 1$ $\tilde{r}(s) = r(s)$ 保定向

Taylor展开: $r(s) = r(0) + s\underbrace{\dot{r}(0)}_{=t(0)} + \frac{s^2}{2}\underbrace{\ddot{r}(0)}_{=k(0)n(0)} + \frac{s^3}{6}\underbrace{\dddot{r}(0)}_{=\dot{k}(0)n(0)+k(0)(-k(0)t(0)+\tau(0)b(0))} + \varepsilon(s)$

Frenet标架下 $r(s) = r(0) + \left(s - \frac{k(0)^2}{6}s^3 + o(s^3)\right)t(0)$

$+ \left(\frac{k(0)}{2}s^2 + \frac{\dot{k}(0)}{6}s^3 + o(s^3)\right)n(0)$

$+ \left(\frac{k(0)\tau(0)}{6}s^3 + o(s^3)\right)b(0)$

圆柱螺旋线 $(a\cos t, a\sin t, bt)$ 可算 $k(t) = \frac{|a|}{a^2+b^2}$

$\tau(t) = \frac{b}{a^2+b^2}$

常曲率 常挠率

空间曲线基本定理：$k,\tau:[a,b]\to\mathbb{R}$ 连续，$k>0$

则(1)存在弧长参数曲线 $r:[a,b]\to\mathbb{R}^3$ 以 k τ 为曲率、挠率

(2)若 r_1、r_2 均满足 则刚体运动 T $s.t.\ r_1=Tor_2$

pf
$$\begin{cases}\frac{dr}{ds}=t(s)\\ \frac{dt}{ds}=k(s)n(s)\\ \frac{dn}{ds}=-k(s)t(s)+\tau(s)b(s)\\ \frac{db}{ds}=-\tau(s)n(s)\end{cases}$$

给定初值 $r(0),t(0),n(0),b(0)$
(标准正交)
则解唯一 存在

(1)存在性 check $r(s)$ 弧长参数 标架规范

$$\frac{d}{ds}\langle V_i(s),V_j(s)\rangle=\langle\sum_{k=1}^3 a_{ik}(s)V_k,V_j\rangle+\langle V_i,\sum_{k=1}^3 a_{jk}V_k\rangle$$
$$=\sum_{k=1}^3 a_{ik}\langle V_k,V_j\rangle+\sum_{k=1}^3 a_{jk}\langle V_i,V_k\rangle$$

$$\frac{d}{ds}\begin{pmatrix}\langle V_1,V_2\rangle\\ \langle V_1,V_3\rangle\\ \langle V_2,V_3\rangle\\ \langle V_1,V_1\rangle\\ \langle V_2,V_2\rangle\\ \langle V_3,V_3\rangle\end{pmatrix}=(\quad)\begin{pmatrix}\quad\end{pmatrix}$$

初值 $\langle V_i(0),V_j(0)\rangle=\delta_{ij}$
解存在唯一

$\forall\langle V_i,V_j\rangle(s)=\delta_{ij}$ 回代方程有 $0=\sum_{k=1}^3 a_{ik}\delta_{kj}+a_{jk}\delta_{ik}$
$=a_{ij}+a_{ji}=0$ $\left((-k_{ij})\right)$

即 $\langle V_i,V_j\rangle=\delta_{ij}$ 为解 $\Rightarrow$ 恒为标准正交

$s\mapsto(t(s),n(s),b(s))\in\{\pm1\}$ 初值为1 由连续性
恒为1 $\Rightarrow$ 恒为右手系

(2)唯一性：类似平面曲线证法 #

Cor 常曲率常挠率只能来自圆柱螺旋线和刚体运动

Chap3 曲面的局部理论

正则曲面(片)

正则参数化曲面: $D\subset\mathbb{R}^2\to E^3$

$(u,v)\mapsto (x(u,v),y(u,v),z(u,v))=r(u,v)$

① x,y,z 光滑

② $r_u=(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u},\frac{\partial z}{\partial u})$

$r_v=(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v},\frac{\partial z}{\partial v})$

$r_u\wedge r_v\neq 0\ (\forall u,v)$

切平面: M 上过 p 点所有光滑曲线在该点处切向量的全集

T_pM

切平面是2维向量空间

$\forall\ P=r(u_0,v_0)$

$u\mapsto r(u,v_0)$ 在此处切向量为 $r_u(u_0,v_0)$

$v\mapsto r(u_0,v)$ ------ $r_v(u_0,v_0)$ } "坐标切向量"

① $\forall a\in\mathbb{R}$

$s\mapsto r(u_0+as,v_0)$ 在 … 为 $ar_u(u_0,v_0)$

故 $ar_u(u_0,v_0)\in T_pM$ $ar_r(u_0,v_0)\in T_pM$

② $\forall a,b\in\mathbb{R}$ $s\mapsto r(u_0+as,v_0+bs)$ 在 … 为 $ar_u+br_v\in T_pM$

$\mathbb{R}PT_pM=\{ar_u(u_0,v_0)+br_v(u_0,v_0)\mid a,b\in\mathbb{R}\}$ #

法向量 $r_u\wedge r_v$ 单位法向量 $\frac{r_u\wedge r_v}{|r_u\wedge r_v|}$

局部把正则曲面可写为 $(u,v,f(u,v))$

$K(p)=\lim\limits_{A\to p}\frac{\text{area}\,g(A)}{\text{area}\,A}$

长度 $l=\int_0^t |r'(t)|dt$

光滑曲线 $t\mapsto r(t)$ 在点处可写为 $r(u(t),v(t))$

$$|r'(t)|^2 = \underbrace{\langle r_u, r_u\rangle}_{E}\left(\frac{du}{dt}\right)^2 + 2\underbrace{\langle r_u, r_v\rangle}_{F}\frac{du}{dt}\frac{dv}{dt} + \underbrace{\langle r_v, r_v\rangle}_{G}\left(\frac{dv}{dt}\right)^2$$

坐标函数 $u: M\to\mathbb{R}$ $r(u,v)\mapsto u$ $\quad v: M\to\mathbb{R}$ $r(u,v)\mapsto v$

$$du(V)=\frac{d}{dt}\Big|_{t=0} u(r(t)) \quad (r'(0)=V)$$

$du_{(r_u)}(r_u)=1$ $\quad du_{(r_v)}(r_v)=0$

$dv_{(r_v)}(r_v)$ $\quad dv_{(r_u)}(r_u)$

$I = E\,du\otimes du + F\,du\otimes dv + F\,dv\otimes du + G\,dv\otimes dv$

$I_p: T_pM\times T_pM\to$ 双线性的映射

$\forall V,W\in T_pM.\quad I(V,W)=\langle V,W\rangle$

I 自双线性性. 算 $I(r_u,r_v)$ $\quad I(r_u,r_u)$ $\quad I(r_v,r_v)$ 已知

则对 $V_1=\lambda_1 r_u+\lambda_2 r_v$

$W=\mu_1 r_u+\mu_2 r_v$

$$I(V,W)=(\lambda_1\ \lambda_2)\underbrace{\begin{pmatrix} E & F\\ F & G\end{pmatrix}}_{\text{正!}}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$$

面积. $\quad |(r_u\Delta u)\wedge(r_v\Delta v)| = |r_u\wedge r_v|\Delta u\Delta v$

$$\iint_D |r_u\wedge r_v|dudv = \iint_D \sqrt{EG-F^2}\,dudv$$

$$\tilde r_{\tilde u}\wedge\tilde r_{\tilde v} = \left|\frac{\partial(u,v)}{\partial(\tilde u,\tilde v)}\right| r_u\wedge r_v$$

$E=g_{11}\quad F=g_{12}=g_{21}\quad G=g_{22}$

$x^1=u\quad x^2=v$

$$I=\sum g_{ij}\,dx^i\otimes dx^j \xrightarrow{\text{等距变换}} \tilde{I}=\sum \tilde{g}_{ij}\,d\tilde{x}^i\otimes d\tilde{x}^j$$

$$\tilde{g}_{ij}=\langle r_{\tilde{x}^i},\, r_{\tilde{x}^j}\rangle=\langle r_{x^k}\frac{\partial x^k}{\partial \tilde{x}^i},\, r_{x^l}\frac{\partial x^l}{\partial \tilde{x}^j}\rangle$$

$$=g_{kl}\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}$$

$$d\tilde{x}^i=\frac{\partial \tilde{x}^i}{\partial x^k}dx^k \qquad d\tilde{x}^j=\frac{\partial \tilde{x}^j}{\partial x^e}dx^l$$

$$\Rightarrow \tilde{I}=\sum_{i,j,k,l} g_{kl}\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}\left(\frac{\partial \tilde{x}^i}{\partial x^k}dx^k\right)\otimes\left(\frac{\partial \tilde{x}^j}{\partial x^e}dx^l\right)$$

$$=\sum_{k,l} g_{kl}\,dx^k\otimes dx^l=I$$

① $\mathbb{R}^2$ $\quad E=G=1\quad F=0$ $\quad I=du\otimes du+dv\otimes dv$ $\quad M_1$

② 柱面 $\quad r(u,v)=(x(u),y(u),v)$ $\quad M_2$

$$E=(x')^2+(y')^2 \qquad F=0 \qquad G=1$$

$$I=(x'^2+y'^2)\,du\otimes du+dv\otimes dv$$

若 $u\mapsto(x(u),y(u))$ 弧长参数 则 $I=du\otimes du+dv\otimes dv$

$f: M_1\to M_2$

$(u,v,1)\mapsto(x(u),y(u),v)$ 等距变换

③ $\bar{D}=\{(\theta,\varphi)\mid -\frac{\pi}{2}<\theta<\frac{\pi}{2},\ 0<\varphi<2\pi\}\to E^3$

$$(\theta,\varphi)\mapsto(a\cos\theta\cos\varphi,\ a\cos\theta\sin\varphi,\ a\sin\theta)$$

$$I=a^2(d\theta\otimes d\theta+\cos^2\theta\,d\varphi\otimes d\varphi)$$

对北极的球极投影

$$\gamma(u,v)=\left(\frac{2a^2u}{a^2+u^2+v^2},\ \frac{2a^2v}{a^2+u^2+v^2},\ a\cdot\frac{u^2+v^2-a^2}{a^2+u^2+v^2}\right)$$

$$I=\frac{4}{\left(1+\frac{1}{a^2}(u^2+v^2)\right)^2}(du\otimes du+dv\otimes dv)$$

$$K(p)=\lim_{A\to p}\frac{Area(g(A))}{Area(A)}=\lim_{A\to p}\frac{\iint|n_u\wedge n_v|dudv}{\iint|r_u\wedge r_v|dudv}$$

定义Gauss曲率 $K(g)=\frac{n_u\wedge n_v}{r_u\wedge r_v}$ (涉及的因式线性)

$(U,V)=G(\bar u,\bar v)$ $\quad \bar r(\bar u,\bar v)=r\circ G$ $\quad \bar r_{\bar u}\wedge\bar r_{\bar v}=\frac{\partial(U,V)}{\partial(\bar u,\bar v)}r_u\wedge r_v$

$$\Rightarrow \bar n_{\bar u}\wedge\bar n_{\bar v}=\frac{\partial(U,V)}{\partial(\bar u,\bar v)}n_u\wedge n_v$$

故$K(p)$well-defined为变换

$r=r(u,v)$ $\quad \tilde r=T\circ r$ $\quad \tilde r(u,v)=T\circ r(u,v)$

$$\tilde r_u\wedge\tilde r_v=\det T\quad T(r_u\wedge r_v)$$

$$\tilde n=\frac{\tilde r_u\wedge\tilde r_v}{|\tilde r_u\wedge\tilde r_v|}=\det T\quad T(n)$$

$$\Rightarrow \tilde n_u\wedge\tilde n_v=\det T\quad T(n_u\wedge n_v)=\det T\quad T(K(p)\,r_u\wedge r_v)$$

$$=T(K(p))\,\tilde r_u\wedge\tilde r_v$$

故合同变换不改变Gauss曲率

$r(u,v) = (u, v, f(u,v))$

$r_u = (1, 0, f_u)$

$r_v = (0, 1, f_v)$

$r_u \wedge r_v = (-f_u, -f_v, 1)$

$$n = \frac{1}{\sqrt{1+f_u^2+f_v^2}}(-f_u, -f_v, 1)$$

$n_u = ((n_u)_1, (n_u)_2, (n_u)_3)$ $\quad n_v = ((n_v)_1, (n_v)_2, (n_v)_3)$

$$K(p) = \det\begin{pmatrix} (n_u)_1 & (n_u)_2 \\ (n_v)_1 & (n_v)_2 \end{pmatrix} = \frac{f_{uu} f_{vv} - f_{uv}^2}{\sqrt{1+f_u^2+f_v^2}}$$

不妨设 $f(0,0)=0$ $\quad$ 则 $\langle n(0,0), (0,0,1)\rangle = \dfrac{1}{\sqrt{1+f_u^2+f_v^2}}$

作仿射变换 γ $\quad \gamma = T \circ r$ $\quad$ s.t $\langle n(0,0), (0,0,1)\rangle = 0$

$T \circ r(u,v) = (u, v, h(u,v)) \Rightarrow h_u = h_v = 0$

则 $K(p) = h_{uu} h_{vv} - h_{uv}^2$

再旋转 θ (角 θ) $\quad (\bar{u}, \bar{v}) \to (\bar{u}, \bar{v}, h(\bar{u}\cos\theta - \bar{v}\sin\theta, \bar{u}\sin\theta + \bar{v}\cos\theta))$

满足 假设.

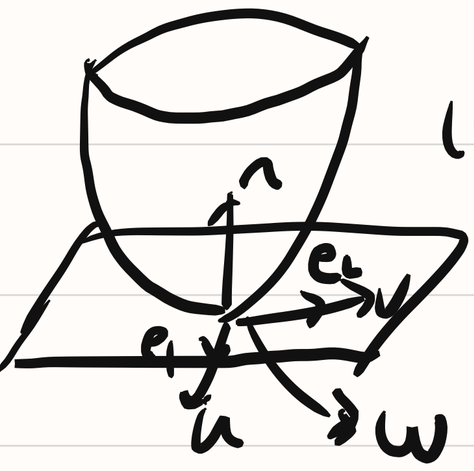
即 $\exists r(u,v) = (u, v, f(u,v))$ $\quad$ s.t $f_u(0,0) = f_v(0,0) = f_{uv}(0,0) = 0$

则 $K(0,0) = \dfrac{\partial^2 f}{\partial u^2}(0,0) \dfrac{\partial^2 f}{\partial v^2}(0,0)$

曲线 $u \mapsto (u, f(u,0))$ $\quad$ 其曲率 $K(0) = f_{uu}(0,0)$

$t \mapsto (t, f(t\cos\theta, t\sin\theta))$ $\quad$ 计算有 $K(0) = f_{uu}(0,0)\cos^2\theta + f_{vv}(0,0)\sin^2\theta$

用 $r(0)$ 和 $(0,0,1)$ 张成的平面截曲面



$(u, v, h(u,v))$

$w = e_1 \cos\theta + e_2 \sin\theta$

w-n平面截 $t \mapsto (t, h(t\cos\theta, t\sin\theta))$

$$k(t) = \frac{\det\begin{pmatrix} x' & y' \\ x'' & y'' \end{pmatrix}}{((x')^2 + (y')^2)^{\frac{3}{2}}}$$

$$= \frac{d^2}{dt^2}\Big|_{t=0} h(t\cos\theta, t\sin\theta)$$

$$= \frac{\partial^2 h}{\partial u^2}(0,0)\cos^2\theta + \frac{\partial^2 h}{\partial v^2}(0,0)\sin^2\theta$$

法曲率 $r = r(u,v)$ 正则曲面 $p = r(0,0)$ $v \in T_pM$ n为p处法向量

则M在p处沿方向v的法曲率 $k_n(v)$ 定义为 v-n平面与M所得平面曲线的带符号曲率 （参数化 $t \mapsto r(t)$ 使得 $|r'(0)| = |v|$，正向 v-n）

注：$k_n(v) = k_n(\lambda v)$ $(\lambda \neq 0)$

$r(u(s), v(s)) = r(s)$ 关于s为弧长参数.

则 $r'(0) = r_u \frac{du}{ds} + r_v \frac{dv}{ds}\Big|_{s=0}$

$$r''(0) = r_{uu}\left(\frac{du}{ds}\right)^2 + 2r_{uv}\frac{du}{ds}\frac{dv}{ds} + r_{vv}\left(\frac{dv}{ds}\right)^2 + r_u\frac{d^2u}{ds^2} + r_v\frac{d^2v}{ds^2}$$

为定 $\langle r''(0), n(0)\rangle = \langle r_{uu}, n\rangle\left(\frac{du}{ds}\right)^2 + 2\langle r_{uv}, n\rangle\frac{du}{ds}\frac{dv}{ds} + \langle r_{vv}, n\rangle\left(\frac{dv}{ds}\right)^2$

只依赖于 $\frac{du}{ds}$ $\frac{dv}{ds}$ 也即只依赖于 $r'(0)$

特殊情况：$r(s)$ 为 $r'(0)$-n平面截得 此时 $\langle r''(0), n(0)\rangle = k_n(r'(0))$

故对一般的曲线，也有 $\langle r''(0), n(0)\rangle = k_n(r'(0))$

$\Rightarrow$ 对 $w = \lambda r_u + \mu r_v \in T_pM$.

$$k_n(w) = k_n\left(\frac{w}{|w|}\right) = \langle r_{uu}, n\rangle\frac{\lambda^2}{|w|^2} + 2\langle r_{uv}, n\rangle\frac{\lambda\mu}{|w|^2} + \langle r_{vv}, n\rangle\frac{\mu^2}{|w|^2}$$

$$\text{记} \quad L=\langle r_{uu}, n\rangle \qquad M=\langle r_{uv}, n\rangle \qquad N=\langle r_{vv}, n\rangle$$

$$=-\langle r_u, n_u\rangle \qquad =-\langle r_u, n_v\rangle=-\langle r_v, n_u\rangle \qquad =-\langle r_v, n_v\rangle$$

$$\text{第二基本形式} \quad \mathrm{II}=L\,du\otimes du+M\,du\otimes dv+M\,dv\otimes du+N\,dv\otimes dv$$

$$\text{则} \quad k_n(w)=\mathrm{II}\left(\frac{w}{|w|},\frac{w}{|w|}\right)=\frac{1}{|w|^2}\mathrm{II}(w,w)=\frac{\mathrm{II}(w,w)}{\mathrm{I}(w,w)}$$

$$\underline{\mathrm{II}}=-\langle r_{u^i}, n_{u^j}\rangle du^i\otimes du^j$$

$$\underline{\overline{\mathrm{II}}}=-\langle r_{\bar u^i}, \bar n_{\bar u^j}\rangle d\bar u^i\otimes d\bar u^j=-\left\langle r_{u^k}\frac{\partial u^k}{\partial \bar u^i}, n_{u^\ell}\frac{\partial u^\ell}{\partial \bar u^j}\right\rangle d\bar u^i\otimes d\bar u^j$$

$$\left(\bar n=\frac{r_{\bar u}\wedge r_{\bar v}}{|r_{\bar u}\wedge r_{\bar v}|}\right.\qquad r_{\bar u}\wedge r_{\bar v}=\left(r_u\frac{\partial u}{\partial \bar u}+r_v\frac{\partial v}{\partial \bar u}\right)\wedge\left(r_u\frac{\partial u}{\partial \bar v}+r_v\frac{\partial v}{\partial \bar v}\right)$$

$$=\det\begin{pmatrix}\frac{\partial u}{\partial \bar u} & \frac{\partial v}{\partial \bar u}\\ \frac{\partial u}{\partial \bar v} & \frac{\partial v}{\partial \bar v}\end{pmatrix} r_u\wedge r_v$$

$$\left.\Rightarrow \bar n=\operatorname{sgn}(\det J)\,n\right)$$

$$=\pm\langle r_{u^k}, n_{u^\ell}\rangle\frac{\partial u^k}{\partial \bar u^i}\frac{\partial u^\ell}{\partial \bar u^j}\frac{\partial \bar u^i}{\partial u^k}\frac{\partial \bar u^j}{\partial u^\ell}du^k\otimes du^\ell$$

$$=\pm\mathrm{II}$$

正向变换 II 不变　　反向变换：II 差一个符号

(进而 正向变换 k_n 不变，反向变换 k_n 反号)

再进而 Gauss 曲率不依赖参数变换

故由特殊参数下的记法 $K(p)=\left(\max_{v\in T_pM,|v|=1} k_n(v)\right)\left(\min_{v\in T_pM,|v|=1} k_n(v)\right)$

按线性变换 $W: T_pM \to T_pM$　　对 $\forall \mathrm{II}(V,W) = \langle V, W(W)\rangle$

对 $V = \lambda r_u + \mu r_v$

$W = \xi r_u + \eta r_v$.

$$\mathrm{II}(V,W) = L\lambda\xi + M(\lambda\eta+\mu\xi) + N\mu\eta$$

$$= (\lambda\ \mu)\begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

$$= -\langle r_u, n_u\rangle \lambda\xi - \langle r_u, n_v\rangle \lambda\eta$$

$$-\langle r_v, n_u\rangle \mu\xi - \langle r_v, n_v\rangle \mu\eta$$

Weingarten变换

↑ (Gauss映射的differential)

故 $W: T_pM \to T_pM$

$r_u \mapsto -n_u$

$r_v \mapsto -n_v$

$$= \langle \underbrace{\lambda r_u + \mu r_v}_{V}, \underbrace{-\xi n_u - \eta n_v}_{W(W)}\rangle$$

又 $\mathrm{II}(V,W) = \mathrm{II}(W,V) \Rightarrow \langle V, W(W)\rangle = \langle W(V), W\rangle$

$\Rightarrow W$ 自伴

$\Rightarrow$ 2个实特征值 对应特征向量垂直

取 e_1, e_2 为 W 的标准正交特征向量　对应 λ_1, λ_2

$\forall\ K_n(W) = \dfrac{\mathrm{II}(W,W)}{\mathrm{I}(W,W)}$ $\overset{w = a^1e_1 + a^2e_2}{=\!=\!=}$ $\dfrac{\langle a^1e_1 + a^2e_2,\ a^1\lambda_1e_1 + a^2\lambda_2e_2\rangle}{(a^1)^2 + (a^2)^2}$

$$= \frac{\lambda_1(a^1)^2 + \lambda_2(a^2)^2}{(a^1)^2 + (a^2)^2}$$

$\Rightarrow K(p) = \lambda_1\lambda_2 = \det(W)$

$\langle W, W(W)\rangle$

$\mathrm{II}(W,W) = (\xi\ \eta)\begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$

$= (\xi\ \eta)\begin{pmatrix} E & F \\ F & G\end{pmatrix}\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$

则在基 r_u r_v 下. $W = \begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1}$

故 $K(p) = \det(W) = \dfrac{\det\begin{pmatrix} L & M \\ M & N\end{pmatrix}}{\det\begin{pmatrix} E & F \\ F & G\end{pmatrix}}$

$$= \frac{LN - M^2}{EG - F^2} \qquad (= \det(dg))$$

例1 ① 平面 $L=M=N=0 \Rightarrow II=0$

② 柱面 $r(u,v)=(x(u),y(u),v)$ $x_u^2+y_u^2=1$

$r_u=(x_u,y_u,0)$ $r_v=(0,0,1)$

$I=du\otimes du+dv\otimes dv$

$n=(y_u,-x_u,0)$ $r_{uu}=(x_{uu},y_{uu},0)$ $r_{uv}=(0,0,0)=r_{vv}$

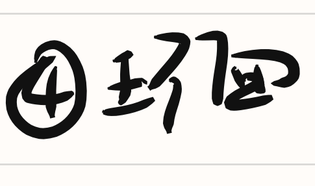
$\Rightarrow L=x_{uu}y_u-x_uy_{uu}$ $M=N=0$

$\Rightarrow II=(x_{uu}y_u-x_uy_{uu})du\otimes du$

$=-\frac{x_uy_{uu}-x_{uu}y_u}{(x_u^2+y_u^2)^{\frac{3}{2}}}=-k(u)du\otimes du$

$\because k_n(r_u)=-k(u)$ $k_n(r_v)=0$

$k_n(w)=-k(u)du(w)du(w)=-k(u)\cos^2\theta$



③ 球面 $r(\theta,\varphi)=(a\cos\theta\cos\varphi,\ a\cos\theta\sin\varphi,\ a\sin\theta)$

$I(\theta,\varphi)=a^2(d\theta\otimes d\theta+\cos^2\theta\, d\varphi\otimes d\varphi)$

$II(\theta,\varphi)=a(d\theta\otimes d\theta+\cos^2\theta\, d\varphi\otimes d\varphi)$

$\Rightarrow k_n(w)\equiv\frac{1}{a}$

④ 环面 $r(u,v)=((R+r\cos u)\cos v,\ (R+r\cos u)\sin v,\ r\sin u)$

$I(u,v)=r^2du\otimes du+(R+r\cos u)^2dv\otimes dv$

$II(u,v)=rdu\otimes du+(R+r\cos u)\cos u\, dv\otimes dv$

$K=\frac{LN-M^2}{EG-F^2}=\frac{\cos u}{r(R+r\cos u)}$

定理：正則曲面 $r(u,v)$ 在 p 點有

$$4(EG-F^2)^2K(p) = E(E_vG_v - 2F_vG_v + (G_v)^2)$$
$$+F(E_uG_v - E_vG_u - 2E_vF_v + 4F_uF_v - 2F_uG_u)$$
$$+G(E_uG_u - 2E_uF_v + (E_v)^2)$$
$$-2(EG-F^2)(E_{vv} - 2F_{uv} + G_{uu})$$

$K(p)$ 只與 E, F, G 有關

pf. $K(p)(EG-F^2)^2 = (LN-M^2)(EG-F^2)$

$$= \langle r_{uu}, r_u \wedge r_v \rangle \langle r_{vv}, r_u \wedge r_v \rangle - \langle r_{uv}, r_u \wedge r_v \rangle^2$$

$$= \det\begin{pmatrix} r_{uu} \\ r_u \\ r_v \end{pmatrix} \det(r_{vv}\ r_u\ r_v) - \det\begin{pmatrix} r_{uv} \\ r_u \\ r_v \end{pmatrix} \det(r_{uv}\ r_u\ r_v)$$

$$= \det\begin{pmatrix} \langle r_{uu}, r_{vv}\rangle & \langle r_{uu}, r_u\rangle & \langle r_{uu}, r_v\rangle \\ \langle r_u, r_{vv}\rangle & E & F \\ \langle r_v, r_{vv}\rangle & F & G \end{pmatrix} - \det\begin{pmatrix} \langle r_{uv}, r_{uv}\rangle & \langle r_{uv}, r_u\rangle & \langle r_{uv}, r_v\rangle \\ \langle r_u, r_{uv}\rangle & E & F \\ \langle r_v, r_{uv}\rangle & F & G \end{pmatrix}$$

$\langle r_{uu}, r_u\rangle = \frac{1}{2}\partial_u\langle r_u, r_u\rangle = \frac{1}{2}E_u$; $\langle r_{uu}, r_v\rangle = \frac{1}{2}E_v$; $\langle r_{vv}, r_u\rangle = \frac{1}{2}G_u$; $\langle r_{vv}, r_v\rangle = \frac{1}{2}G_v$

$\langle r_{uv}, r_u\rangle = \frac{1}{2}E_v$; $\langle r_{uv}, r_v\rangle = \frac{1}{2}G_u$

$\langle r_{uu}, r_{uv}\rangle = \frac{1}{2}E_u$; $\langle r_u, r_{uv}\rangle = \frac{1}{2}E_v$; $\langle r_v, r_{uv}\rangle = \frac{1}{2}G_u$

$\langle r_{uu}, r_{vv}\rangle = \frac{\partial}{\partial v}\langle r_{uu}, r_v\rangle - \langle r_{uuv}, r_v\rangle$

$\langle r_{uv}, r_{uv}\rangle = \frac{\partial}{\partial u}\langle r_{uv}, r_v\rangle - \langle r_{uuv}, r_v\rangle$

$\Downarrow$

$\langle r_{uu}, r_{vv}\rangle - \langle r_{uv}, r_{uv}\rangle = F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu}$

$$= \det\begin{pmatrix} F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

$= \cdots$ #

平均曲率 $H=\frac{1}{2}(k_1+k_2)=\frac{1}{2}\mathrm{tr}(W)$

$$=\frac{1}{2}\mathrm{tr}\begin{pmatrix}E&F\\F&G\end{pmatrix}^{-1}\begin{pmatrix}L&M\\M&N\end{pmatrix}$$

$$=\frac{1}{2}\frac{1}{EG-F^2}\mathrm{tr}\begin{pmatrix}G&-F\\-F&E\end{pmatrix}\begin{pmatrix}L&M\\M&N\end{pmatrix}$$

$$=\frac{1}{2}\frac{GL-2FM+EN}{EG-F^2}$$

$r:D\to E^3$ 正则曲面　　2D分段 正则闭曲线

先消映射 $\alpha:(-\varepsilon,\varepsilon)\times D\to E^3$

$\alpha(0,u,v)=r(u,v)$

且对 t，　$r_t(u,v)=\alpha(t,u,v)$ 正则小曲面　　用 α 为一个变分

$W(u_0,v_0)=\frac{d\alpha}{dt}\big|_{t=0}$　$\alpha(t,u_0,v_0)$ 变分向量场

极值变分　　$\alpha:(-\varepsilon,\varepsilon)\times D\to E^3$

$$(t,u,v)\mapsto r(u,v)+t\varphi(u,v)n(u,v)$$

$W(u,v)=\varphi(u,v)n(u,v)$

性质：r_t 的面积 $A(t)=\iint_D\sqrt{E(t)G(t)-F^2(t)}\,du\,dv$

$A'(0)=-\iint_D 2\varphi H|r_u\wedge r_v|\,du\,dv$

$E_t=\langle(r_t)_u,(r_t)_u\rangle=\langle r_u+t\varphi_u n+t\varphi n_u,\ r_u+t\varphi_u n+t\varphi n_u\rangle$

$$=E-2t\varphi L+t^2(\varphi_u^2+\varphi^2\langle n_u,n_u\rangle)$$

$F_t=F-2t\varphi M+t^2(\cdots)$　　$G_t=G-2t\varphi N+t^2(\cdots)$

$\Rightarrow E_tG_t-F_t^2=(E-2t\varphi L)(G-2t\varphi N)-(F-2t\varphi M)^2+t^2(\cdots)$

$=EG-F^2-2t\varphi(GL+EN-2MF)+t^2(\cdots)$

$=(EG-F^2)(1-4t\varphi H)+t^2(\cdots)$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0} A(t) = \iint_D \frac{\frac{d}{dt}(E_tG_t - F_t^2)}{2\sqrt{E_tG_t - F_t^2}}\Big|_{t=0} dudv$$

$$= \iint_D \frac{-4H\varphi(EG-F^2)}{2\sqrt{EG-F^2}} dudv = \iint_D -2H\varphi |r_u \times r_v| dudv$$

故若 $H \neq 0$. 可动使 A 变小

例. 二次曲面 $z = \frac{1}{2}(ax^2 + by^2)$

$$\mathrm{I} = (1+a^2x^2)dx\otimes dx + abxy\, dx\otimes dy + abxy\, dy\otimes dx + (1+b^2y^2)dy\otimes dy$$

$$\mathrm{II} = \frac{1}{\sqrt{1+a^2x^2+b^2y^2}}(a\,dx\otimes dx + b\,dy\otimes dy)$$

P处 切向量 $V = \lambda r_x + \mu r_y$ 单位长

$$K_n(V) = \mathrm{II}(V,V) = \frac{a\lambda^2 + b\mu^2}{\sqrt{1+a^2x^2+b^2y^2}}$$

高度函数 $h(u,v) = \langle r(u,v) - r(0,0),\ n(0,0)\rangle$

$$\mathrm{Hess}\, h = \begin{pmatrix} h_{uu} & h_{uv} \\ h_{vu} & h_{vv} \end{pmatrix} = \begin{pmatrix} \langle r_{uu}, n\rangle & \langle r_{uv}, n\rangle \\ \langle r_{vu}, n\rangle & \langle r_{vv}, n\rangle \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

重新参数化 s.t. r_u, r_v 为 e_1, e_2 (主方向) 则 $L = k_1,\ M = 0,\ N = k_2$

$\Rightarrow h \overset{\text{Taylor}}{=} \frac{1}{2}(k_1u^2 + k_2v^2)$

$(u, v, \frac{1}{2}(k_1u^2 + k_2v^2))$: 密切抛物面

(1) 都有同原: k_1, k_2 同号. 即 $\det\begin{pmatrix} L & M \\ M & N \end{pmatrix} > 0$ 同侧

(2) 双曲点: 异号. $\det\begin{pmatrix} L & M \\ M & N \end{pmatrix} < 0$ 异侧

(3) 抛物点: 有一个为 0 始终有两侧

(4) 平点 全为 0

正螺面 $(u,v)\mapsto(\sinh v\sin u,\ -\sinh v\cos u,\ u)$ （K和H不全为0的极小曲面）

v线 直线　　u线：螺旋线

旋转曲面 $(u,v)\mapsto(f(u)\cos v,\ f(u)\sin v,\ g(u))$

例：悬链线 $(\cosh u,u)$ 旋转 是旋转面

$r_u=(f'(u)\cos v,\ f'(u)\sin v,\ g'(u))$　　$r_v=(-f(u)\sin v,\ f(u)\cos v,\ 0)$

$E=(f'(u))^2+(g'(u))^2$　　$F=0$　　$G=f(u)^2$

$r_u\wedge r_v=(-f(u)g'(u)\cos v,\ -f(u)g'(u)\sin v,\ f(u)f'(u))$

$n=\frac{1}{\sqrt{f'^2+g'^2}}(-g'(u)\cos v,\ -g'(u)\sin v,\ f'(u))$

$r_{uu}=(f''(u)\cos v,\ f''(u)\sin v,\ g''(u))$

$r_{uv}=(-f'(u)\sin v,\ f'(u)\cos v,\ 0)$　　$r_{vv}=(-f(u)\cos v,\ -f(u)\sin v,\ 0)$

$L=\frac{f'g''-f''g'}{\sqrt{f'^2+g'^2}}$　　$M=0$　　$N=\frac{fg'}{\sqrt{f'^2+g'^2}}$

$W=(L\ N)(E\ G)^{-1}=\frac{1}{\sqrt{f'^2+g'^2}}\begin{pmatrix}\frac{f'g''-f''g'}{f'^2+g'^2} & \\ & \frac{g'}{f}\end{pmatrix}$

设$(f(u),g(u))$弧参 $\Rightarrow f'^2+g'^2=1$ $\Rightarrow$ 上述 $k_1=f'g''-f''g'$

$ff'f''+gg'g''=0$　　$k_2=\frac{g'}{f}$

$K=\frac{(f'g''-f''g')g'}{f}$

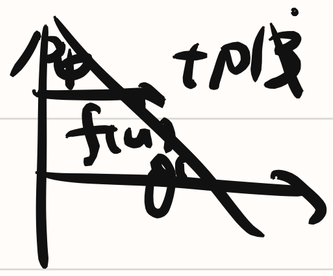
$=\frac{-f'^2f''-f''g'^2}{f}=-\frac{f''}{f}$　　$\Rightarrow f''+Kf=0$

$H=\frac{1}{2}\left(f'g''-f''g'+\frac{g'}{f}\right)=\frac{1}{2}\left(\frac{g'}{f}-\frac{f''}{g'}\right)$

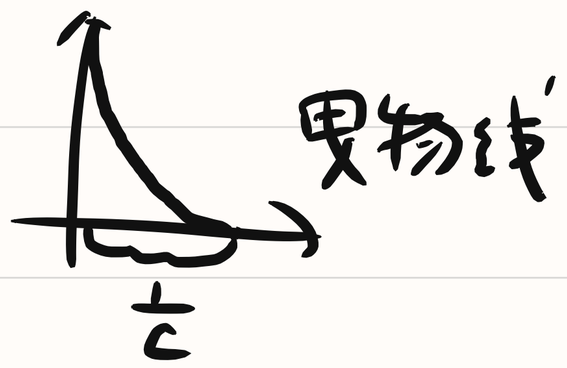
K常值 $C^2=K$　　$f=A\cos cu+B\sin cu$

$-C^2=K$　　$f=A\cosh cu+B\sinh cu$　（分正负）

$$\begin{cases} f(u)=\frac{1}{c}e^{cu} \\ g(u)=\pm\int_0^u \sqrt{1-e^{2ct}}\,dt \end{cases}$$



$$\frac{f(u)}{\cos\theta}=\frac{f}{\frac{f'}{\sqrt{f'^2+g'^2}}}=f'=\frac{1}{c}$$



曳物线'

$x-z$ 平面曲线 $C: u\mapsto(f(u),g(u))$ $\quad C'(u)=r_u$

r_u 垂直于 $x-z$ 平面 $\Rightarrow n$ 垂于 $x-z$ 平面 $(\forall u)$

$\Rightarrow n_u$ 垂于 $x-z$ 平面. 又 $n_u\in$ 切平面 $\Rightarrow n_u$ 为 $C'=r_u$ 法矢

$$-n_u=W(r_u)=k_1 r_u$$

同理 $-n_v=W(r_v)=k_2 r_v$

$H=0 \iff ff''=(g')^2=1-(f')^2 \Rightarrow (ff')'=1 \Rightarrow ff'=u+A$

$$\Rightarrow \frac{1}{2}f^2=\frac{1}{2}u^2+Au+B$$

$$f=\sqrt{u^2+2Au+B} \qquad B>A^2$$

$$f'=\frac{u+A}{\sqrt{u^2+2Au+B}} \qquad g'=\pm\sqrt{\frac{B-A^2}{u^2+2Au+B}}$$

设 $f(u)=\sqrt{u^2+a^2}$ $\Rightarrow g'(u)=\pm\frac{a}{\sqrt{u^2+a^2}}$

$$\Rightarrow (f(u),g(u))=\left(\sqrt{u^2+a^2},\ \pm a\,\operatorname{arcsinh}\frac{u}{a}\right)$$

$$=\left(a\cosh\frac{z}{a},\ z\right)$$

$\Rightarrow$ 悬链面

直纹面 $r(u,v)=a(u)+vb(u)$

$r_v=b(u)$ $r_{vv}=0 \Rightarrow N=0 \Rightarrow K=\frac{-M^2}{EG-F^2}$

例.

① 正螺旋面 $r(u,v)=(0,0,v)+u(\cos v,\sin v,0)$

② $b(u)\equiv b_0$ $r(u,v)=a(u)+vb_0$ 柱状面

③ $a(u)\equiv a_0$ $r(u,v)=a_0+vb(u)$ 锥面

$\Big\} \to K=0$

④ 切线面 $r(u,v)=a(u)+v\alpha'(u)$

$r_u=a'(u)+va''(u)$

$r_{uv}=a''(u)$ $\langle r_{uv}, n\rangle=0 \Rightarrow M=0 \Rightarrow K=0$.

回转双曲面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

$r(u,v)=(a(\cos u-v\sin u), b(\sin u+v\cos u), cv)$

$=(a\cos u, b\sin u, 0)+v(-a\sin u, b\cos u, c)$

⑤ 双曲抛物面 $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

$r(u,v)=(a(u+v), bv, u^2+2uv)$

$=(au, 0, u^2)+v(a, b, 2u)$

$K=0$ 的直纹面 (可展曲面)

性质 直纹面 $K=0 \Leftrightarrow (a', b, b')=0$

pf. $K=0 \Leftarrow M=0 \Leftarrow \langle r_{uv}, n\rangle=0$ $\Leftrightarrow r_{uv}=\lambda r_u+\mu r_v$ (b' 共面)

$=\lambda(a'+vb')+\mu b$

$\Leftrightarrow (a', b, b')=0$

Case1. $b\wedge b'=0 \Leftrightarrow b$ 为常矢量

则 $r(u,v)=a(u)+v|b(u)|\dfrac{b(u)}{|b(u)|}$ 常向量

$\Rightarrow$ 柱状面

Case2 $b\wedge b'\neq 0$ $\quad a'(u)=\lambda(u)b(u)+\mu(u)b'(u)$

令 $\hat{a}(u)=a(u)-\mu(u)b(u)$

则 $\hat{a}'(u)=(\lambda(u)-\mu'(u))b(u)$

$r(u,v)=\hat{a}(u)+(\mu(u)+v)b(u)$

若 $\hat{a}'(u)\neq 0$ 则 $r(u,v)=\hat{a}(u)+\dfrac{\mu(u)+v}{\lambda(u)-\mu'(u)}\hat{a}'(u)$

$\begin{cases}\bar{u}=u\\ \bar{v}=\dfrac{\mu(u)+v}{\lambda(u)-\mu'(u)}\end{cases}\Rightarrow r(\bar{u},\bar{v})=\hat{a}(\bar{u})+\bar{v}\hat{a}'(u)$ 柱锥面

若 $\hat{a}'(u)=0$ $\quad \hat{a}(u)=a_0\Rightarrow a(u)=\mu(u)b(u)+a_0$

$r(u)=a_0+(v+\mu(u))b(u)$ 广义柱面

#

直纹面 $b'\neq 0$ $H=0\Rightarrow$ 正则曲面/平面

Pf. 参变换 $\rightarrow$ 设 $|b|=|b'|=1$ $(\Rightarrow\langle b,b'\rangle=0)$

再令 $G(u)=a(u)+\lambda(u)b(u)$

$\langle G',b'\rangle=\langle a',b'\rangle+\lambda(u)$

则再参变换 $\Rightarrow\langle a',b'\rangle=0$

取 $\lambda(u)=-\langle a',b'\rangle$.

$H=0\Leftrightarrow 2G=2MF$

$F=\langle r_u,r_u\rangle=\langle a'+vb',b\rangle=\langle a',b\rangle$

$\Leftrightarrow L-2\langle a',b\rangle M=0$

$G=1$

$L=\left\langle a''+vb'',\dfrac{(a'+vb')\wedge b}{|r_u\wedge r_v|}\right\rangle$

$M=\dfrac{1}{|r_u\wedge r_v|}\langle b',(a'+vb')\wedge b\rangle$

$\Rightarrow \det\begin{pmatrix} a''+vb'' \\ a'+vb' \\ b \end{pmatrix} - 2\langle a', b\rangle \det\begin{pmatrix} b' \\ a'+vb' \\ b \end{pmatrix} = 0$

$\Leftrightarrow 0 = -2\langle a', b\rangle \det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix} + \det\begin{pmatrix} a'' \\ a' \\ b \end{pmatrix} + v\left(\det\begin{pmatrix} a'' \\ b' \\ b \end{pmatrix} + \det\begin{pmatrix} b'' \\ a' \\ b \end{pmatrix}\right) + v^2 \det\begin{pmatrix} b'' \\ b' \\ b \end{pmatrix}$

$-2\langle a', b\rangle \det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix} + \det\begin{pmatrix} a'' \\ a' \\ b \end{pmatrix} = 0$ ①

$\Leftrightarrow \begin{cases} \det\begin{pmatrix} a'' \\ b' \\ b \end{pmatrix} + \det\begin{pmatrix} b'' \\ a' \\ b \end{pmatrix} = 0 & ② \\ \det\begin{pmatrix} b'' \\ b' \\ b \end{pmatrix} = 0 & ③ \end{cases}$

③ $\Rightarrow b'' = \langle b'', b'\rangle b' + \langle b'', b\rangle b = \langle b'', b\rangle b = \tau \cdot -b$

$\because \langle b, b'\rangle = 0 \Rightarrow \langle b', b'\rangle + \langle b, b''\rangle = 0$

又 $b'' = K \underset{\text{主法线方向}}{n(u)} = n(u)$

$\Rightarrow$ 主法线向 $-b$

副法线向 $= b' \wedge (-b)$

$(b' \wedge -b)' = -\tau(-b) = \tau b$

$\Rightarrow \tau = 0$

$\Rightarrow b$ 为单位向

$\Rightarrow b$ 的曲率为 1

$b'' \wedge -b = (-b) \wedge (-b) = 0$

故存在 $b(u) = (\cos u, \sin u, 0)$

$\alpha(u) = (\alpha(u), \beta(u), \gamma(u))$

② $\Rightarrow \det\begin{pmatrix} a'' \\ b' \\ b \end{pmatrix} = 0 \Leftrightarrow a''$ 落在 $x-y$ 平面

$\Leftrightarrow \ddot{\gamma}(u) = 0$

$\Leftrightarrow \gamma(u) = \lambda u + C$

多次复投 $\Rightarrow$ 不妨 $\alpha(u) = (\alpha(u), \beta(u), \lambda u)$

① $\begin{pmatrix} a'' \\ a' \\ b \end{pmatrix} = \det\begin{pmatrix} \langle a'', b'\rangle b' + \langle a'', b\rangle b \\ a' \\ b \end{pmatrix} = \langle a'', b'\rangle \det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix}$

$\langle a', b'\rangle = 0 \Rightarrow \langle a'', b'\rangle + \langle a', b''\rangle = 0 \Rightarrow \langle a'', b'\rangle = \langle a', b''\rangle = \langle a', b\rangle$

$\Rightarrow -\langle a', b\rangle \underbrace{\det\begin{pmatrix} b' \\ a' \\ b \end{pmatrix}}_{\text{混合积}} = 0$

$\lambda=0 \Rightarrow$ 平面

$\lambda\neq 0$. $\langle a', b\rangle = 0 \leftrightarrow \langle (\alpha', \beta', \lambda), (\cos u, \sin u, 0)\rangle = 0$

$\alpha'\cos u + \beta'\sin u$

$\langle a', b'\rangle = 0 \Rightarrow -\alpha'\sin u + \beta'\cos u = 0$ $\Leftrightarrow \alpha', \beta' = 0$ 即 α β 常数

$\Rightarrow r(u,v) = (\alpha, \beta, \lambda u) + v(\cos u, \sin u, 0)$

$= (\alpha, \beta, 0) + (v\cos u, v\sin u, \lambda u)$ $\Rightarrow$ 此时是圆

#

E, F, G, L, M, N $D \subset \mathbb{R}^2$ 上函数 ($\begin{matrix} E & F \\ F & G \end{matrix}$) 正 是否存在曲面 r,

使其 I II 为 E, F, G, L, M, N?

特别: ① $L=M=N=0$ r 是否被为平面 (一部分)?

pf 是的

$\begin{cases} \langle n_u, r_u\rangle = 0 \\ \langle n_u, r_v\rangle = 0 \end{cases}$ $\begin{cases} \langle n_v, r_u\rangle = 0 \\ \langle n_v, r_v\rangle = 0 \end{cases}$

$n_u = \lambda r_u + \mu r_v \Rightarrow \begin{cases} \lambda E + \mu F = 0 \\ \lambda F + \mu G = 0 \end{cases} \Rightarrow \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0 \Rightarrow n_u = 0$

同理 $n_v = 0$ $\Rightarrow$ n 常向量

$\langle r, n\rangle_u = 0 = \langle r, n\rangle_v \Rightarrow \langle r, n\rangle$ 常数 故 r 平面 #

进一步地 若反 E, F, G s.t. $K=1$ 是否存在 r s.t. I 为 E, F, G !

II 为 0

② 若 $\frac{L}{E} = \frac{M}{F} = \frac{N}{G} = k$. 是否存在 r …. 若存在 是否为球面? 平面

pf 若存在 $W = \lambda r_u + \mu r_v$

$$K_n(W) = \frac{\lambda^2 L + 2\lambda\mu M + \mu^2 N}{\lambda^2 E + 2\lambda\mu F + \mu^2 G} = k$$

(任一方向法曲率相同的点称为脐点)

$k=0$ 平点 k 相同 同点

故 为全脐点曲面

此时 $W(r_u) = -n_u = kr_u$ $\Rightarrow$ $-n_{uv} = k_v r_u + k r_{uv}$

$W(r_v) = -n_v = k r_v$ $-n_{uv} = k_u r_v + k r_{uv}$

$\Rightarrow k_v r_u = k_u r_v$ $\Rightarrow$ 线性相关 $\Rightarrow k_u = k_v = 0$ $\Rightarrow$ k常值

$\Rightarrow n = -kr + C_0'$ $k=0$: 平面

$k \neq 0$. $|r - C_0| = \frac{1}{k}$ $\Rightarrow$ 球面.

但仍然可以找 E. F. G 不为0的点 #]

$K = \frac{LN - M^2}{EG - F^2} \xlongequal{\text{Gauss绝妙}} f(E, F, G)$

$LN - M^2 = (EG - F^2) f(E, F, G)$ $\Rightarrow$ 能否由I. II决定曲面?

仍不充分!

(对上面的曲面下充分).

第二优分类: Codazzi方程.

12.5 $u = u^1$ $v = u^2$

$r_1 = r_u$ $r_2 = r_v$ $r_{\alpha\beta} = \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta}$ $g_{\alpha\beta} = \langle r_\alpha, r_\beta \rangle$

$b_{\alpha\beta} = \langle r_{\alpha\beta}, n \rangle$

$(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix}$

$$\begin{cases} \frac{\partial r_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^\gamma r_\gamma + Cn \\ \frac{\partial n}{\partial u^\alpha} = D_\alpha^\beta r_\beta + A_\alpha n \end{cases}$$

$\langle n_\alpha, n \rangle = A_\alpha = 0$ $C = b_{\alpha\beta}$

$\langle n_\alpha, r_\gamma \rangle = D_\alpha^\beta g_{\beta\gamma}$ $\Rightarrow$ $-b_{\alpha\gamma} g^{\beta\gamma} = D_\alpha^\beta$

$-b_{\alpha\gamma}$ $-b_\alpha^\beta$

$$\langle r_{\alpha\beta}, r_\xi \rangle = \langle \Gamma_{\alpha\beta}^\gamma r_\gamma, r_\xi \rangle = \Gamma_{\alpha\beta}^\gamma g_{\gamma\xi}$$

$$\| $$

$$\frac{\partial}{\partial u^\beta}\langle r_\alpha, r_\xi \rangle - \langle r_\alpha, r_{\beta\xi} \rangle$$

$$\Rightarrow \frac{\partial g_{\alpha\xi}}{\partial u^\beta} = \langle r_{\alpha\beta}, r_\xi \rangle + \langle r_\alpha, r_{\beta\xi} \rangle$$

$$\frac{\partial g_{\beta\alpha}}{\partial u^\xi} = \langle r_{\beta\xi}, r_\alpha \rangle + \langle r_\beta, r_{\alpha\xi} \rangle \qquad \frac{\partial g_{\xi\beta}}{\partial u^\alpha} = \langle r_{\xi\alpha}, r_\beta \rangle + \langle r_\xi, r_{\alpha\beta} \rangle$$

①+②−③

$$\Rightarrow \langle r_\alpha, r_{\beta\xi} \rangle = \frac{1}{2}\left(\frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\alpha}}{\partial u^\xi} - \frac{\partial g_{\xi\beta}}{\partial u^\alpha}\right)$$

$$\| $$

$$\Gamma_{\xi\beta}^\gamma g_{\gamma\alpha} = \frac{1}{2}\left(g_{\alpha\xi,\beta} + g_{\beta\alpha,\xi} - g_{\xi\beta,\alpha}\right)$$

$$\Rightarrow \Gamma_{\xi\beta}^\gamma g_{\gamma\alpha}\, g^{\alpha\eta} = \cdots$$

第一类 = 类 Christoffel 符号

$$\Rightarrow \Gamma_{\beta\xi}^\eta = \frac{1}{2} g^{\alpha\eta}\left(g_{\alpha\xi,\beta} + g_{\beta\alpha,\xi} - g_{\beta\xi,\alpha}\right)$$

定义 $\Gamma_{\alpha\xi\beta} = \Gamma_{\xi\beta}^\gamma g_{\gamma\alpha} \Rightarrow \Gamma_{\alpha\xi\beta} = \frac{1}{2}\left(g_{\alpha\xi,\beta} + g_{\beta\alpha,\xi} - g_{\beta\xi,\alpha}\right)$

可解性条件

(A) $\frac{\partial}{\partial u^\gamma}\left(\frac{\partial r_\alpha}{\partial u^\beta}\right) = \frac{\partial}{\partial u^\beta}\left(\frac{\partial r_\alpha}{\partial u^\gamma}\right)$

$$LHS = \frac{\partial}{\partial u^\gamma}\left(\Gamma_{\alpha\beta}^\xi r_\xi + b_{\alpha\beta} n\right) = \frac{\partial \Gamma_{\alpha\beta}^\xi}{\partial u^\gamma} r_\xi + \Gamma_{\alpha\beta}^\xi\left(\Gamma_{\xi\gamma}^\eta r_\eta + b_{\xi\gamma} n\right) + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} n + b_{\alpha\beta}\left(-b_\gamma^\eta r_\eta\right)$$

$$= \left(\frac{\partial \Gamma_{\alpha\beta}^\xi}{\partial u^\gamma} + \Gamma_{\alpha\beta}^\eta \Gamma_{\eta\gamma}^\xi - b_{\alpha\beta} b_\gamma^\xi\right) r_\xi + \left(\Gamma_{\alpha\beta}^\xi b_{\xi\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma}\right) n$$

$$RHS = \left(\frac{\partial \Gamma_{\alpha\gamma}^\xi}{\partial u^\beta} + \Gamma_{\alpha\gamma}^\eta \Gamma_{\eta\beta}^\xi - b_{\alpha\gamma} b_\beta^\xi\right) r_\xi + \left(\Gamma_{\alpha\gamma}^\xi b_{\xi\beta} + \frac{\partial b_{\alpha\gamma}}{\partial u^\beta}\right) n$$

$$\Rightarrow \begin{cases} \dfrac{\partial \Gamma_{\alpha\beta}^\xi}{\partial u^\gamma} - \dfrac{\partial \Gamma_{\alpha\gamma}^\xi}{\partial u^\beta} + \Gamma_{\alpha\beta}^\eta \Gamma_{\eta\gamma}^\xi - \Gamma_{\alpha\gamma}^\eta \Gamma_{\eta\beta}^\xi - b_{\alpha\beta} b_\gamma^\xi + b_{\alpha\gamma} b_\beta^\xi = 0 & \text{(Gauss)} \\ \Gamma_{\alpha\beta}^\xi b_{\xi\gamma} - \Gamma_{\alpha\gamma}^\xi b_{\xi\beta} + \dfrac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \dfrac{\partial b_{\alpha\gamma}}{\partial u^\beta} = 0 & \text{(Codazzi)} \end{cases}$$

(B) ⇒ Codazzi Eq. (作业)

Gauss Eq ⇒ $\left(\frac{\partial \Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma} - \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta}\right) g_{\xi\delta} = (b_{\alpha\beta}b^{\xi}_{\gamma} - b_{\alpha\gamma}b^{\xi}_{\beta}) g_{\xi\delta}$

$= b_{\alpha\beta}b_{\gamma\delta} - b_{\alpha\gamma}b_{\beta\delta}$

$-R_{\delta\alpha\beta\gamma} \triangleq$ LHS.

$R_{1212} = \frac{1}{2}\left(2\frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{\partial^2 g_{11}}{\partial u^2 \partial u^2} - \frac{\partial^2 g_{22}}{\partial u^1 \partial u^1}\right)$

$+ \Gamma^{\eta}_{12}\Gamma^{\xi}_{12} g_{\eta\xi} - \Gamma^{\eta}_{11}\Gamma^{\xi}_{22} g_{\xi\eta}$

$R_{\delta\alpha\beta\gamma} = -R_{\delta\alpha\gamma\beta} = -R_{\alpha\delta\beta\gamma} = R_{\beta\gamma\delta\alpha}$

⇒只有关于 R_{1212} 的非平凡 $\quad -R_{1212} = b_{21}b_{21} - b_{22}b_{11} = -(LN - M^2)$

$\Rightarrow \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = K$

$R_{\delta\alpha\beta\gamma} = -g_{\xi\delta}\frac{\partial \Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}} + g_{\delta\xi}\frac{\Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}} - g_{\delta\xi}\Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma} + g_{\delta\xi}\Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta}$

$= -\frac{\partial \Gamma_{\alpha\beta\delta}}{\partial u^{\gamma}} + \Gamma^{\xi}_{\alpha\beta}\frac{\partial g_{\xi\delta}}{\partial u^{\gamma}} + \frac{\partial \Gamma_{\alpha\gamma\delta}}{\partial u^{\beta}} - \Gamma^{\xi}_{\alpha\gamma}\frac{\partial g_{\xi\delta}}{\partial u^{\beta}} - \Gamma^{\eta}_{\alpha\beta}\Gamma_{\eta\gamma\delta} + \Gamma^{\eta}_{\alpha\gamma}\Gamma_{\eta\beta\delta}$

$= -\frac{1}{2}\frac{\partial}{\partial u^{\gamma}}\left(g_{\beta\delta,\alpha} + g_{\beta\alpha,\delta} + g_{\delta\alpha,\beta}\right) + \frac{1}{2}\frac{\partial}{\partial u^{\beta}}\left(g_{\delta\gamma,\alpha} - g_{\alpha\gamma,\delta} + g_{\delta\alpha,\gamma}\right)$

$- \Gamma^{\eta}_{\alpha\beta}\left(\Gamma_{\eta\gamma\delta} - \frac{g_{\eta\delta}}{\partial u^{\gamma}}\right) + \Gamma^{\eta}_{\alpha\gamma}\left(\Gamma_{\eta\beta\delta} - \frac{\partial g_{\eta\delta}}{\partial u^{\beta}}\right)$

$\Gamma_{\eta\gamma\delta} - g_{\delta\eta,\gamma} = \frac{1}{2}\left(g_{\eta\delta,\gamma} + g_{\gamma\delta,\eta} - g_{\eta\gamma,\delta}\right) - g_{\delta\eta,\gamma}$

$= \frac{1}{2}\left(g_{\gamma\delta,\eta} - g_{\eta\delta,\gamma} - g_{\eta\gamma,\delta}\right) = -\Gamma_{\eta\delta\gamma}$

$\Rightarrow R_{\delta\alpha\beta\gamma} = \frac{1}{2}\left(-\frac{\partial^2 g_{\beta\delta}}{\partial u^{\gamma}\partial u^{\alpha}} + \frac{\partial^2 g_{\beta\alpha}}{\partial u^{\gamma}\partial u^{\delta}} - \frac{\partial^2 g_{\alpha\gamma}}{\partial u^{\beta}\partial u^{\delta}} + \frac{\partial^2 g_{\delta\gamma}}{\partial u^{\alpha}\partial u^{\beta}}\right)$

$+ \Gamma^{\eta}_{\alpha\beta}\Gamma_{\eta\gamma\delta} - \Gamma^{\eta}_{\alpha\gamma}\Gamma_{\eta\delta\beta}$ $\quad = g^{\eta\xi}\left(\Gamma_{\xi\alpha\beta}\Gamma_{\eta\delta\gamma} - \Gamma_{\xi\alpha\gamma}\Gamma_{\eta\delta\beta}\right)$

$$\Rightarrow R_{1212} = \frac{1}{2}(-E_{vv}+2F_{uv}-G_{uu}) + g^{11}(\Gamma_{121}\Gamma_{12}-\Gamma_{122}\Gamma_{11})$$
$$+g^{12}(\Gamma_{221}\Gamma_{12}-\Gamma_{222}\Gamma_{11}+\Gamma_{121}\Gamma_{12}-\Gamma_{122}\Gamma_{11})$$
$$+g^{22}(\Gamma_{221}\Gamma_{212}-\Gamma_{222}\Gamma_{211})$$

$$\Gamma_{12}\Gamma_{12}-\Gamma_{22}\Gamma_{11} = \frac{1}{4}(g_{12,1}+g_{11,2}-g_{12,1})(g_{11,2}+g_{12,1}-g_{12,1})$$
$$-\frac{1}{4}(g_{11,2}+g_{12,2}-g_{22,1})(g_{11,1})$$
$$= \frac{1}{4}E_v^2 - \frac{1}{4}(2F_v-G_u)E_u$$

$$4(EG-F^2)R_{1212} = 2(-E_{vv}+2F_{uv}-G_{uu})(EG-F^2)+G(E_v^2-2E_uF_v+G_uE_u)$$
$$+\cdots$$

有定义与 Gauss 绝妙 Thm 一致

Codazzi Eq $\quad \Gamma_{\alpha\beta}^{\xi} b_{\xi\gamma} - \Gamma_{\alpha\gamma}^{\xi} b_{\xi\beta} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} = 0 \qquad \alpha,\beta,\gamma = 1,2$

关于 γ 反称 $\Rightarrow$ 只有 $(\beta,\gamma)=(1,2)$ 非平凡

再 $\alpha=1,2$，故只有2个非平凡方程

$$\begin{cases} \frac{\partial b_{11}}{\partial u^2} - \frac{\partial b_{12}}{\partial u^1} + \Gamma_{11}^{\xi} b_{\xi 2} - \Gamma_{12}^{\xi} b_{\xi 1} = 0 \\ \frac{\partial b_{12}}{\partial u^2} - \frac{\partial b_{22}}{\partial u^1} + \Gamma_{12}^{\xi} b_{\xi 2} - \Gamma_{22}^{\xi} b_{\xi 1} = 0 \end{cases}$$

Gauss $R_{1212} = LN - M^2$

足存 $\begin{cases} \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} + \Gamma_{\alpha\beta}^{\xi} b_{\xi\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} \Gamma_{\alpha\gamma}^{\xi} b_{\xi\beta} = 0 \end{cases}$ 能够决定曲面片

若决定出 $\frac{L}{E} = \frac{M}{F} = \frac{G}{N} = k$ 则是全脐曲面

$$R_{1212} = \frac{1}{2}\left(2\frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{\partial^2 g_{11}}{\partial u^2 \partial u^2} - \frac{\partial^2 g_{22}}{\partial u^1 \partial u^1}\right) + \Gamma_{12}^{\gamma}\Gamma_{12}^{\delta} g_{\delta\gamma} - \Gamma_{11}^{\gamma}\Gamma_{22}^{\delta} g_{\delta\gamma}$$

曲面论基本定理：

定理：$D\times V\subset \mathbb{R}^2\times\mathbb{R}^3$ 开区域， $f_i: D\times V\to \mathbb{R}$ 光滑

则 方程 $\frac{\partial^2 r}{\partial u^i\partial u^j}(u^1,u^2)=f_j(u^1,u^2,F(u^1,u^2))$ $F:D\to V$

①唯一性：若有解 $F,\tilde F$ 且 $F(u_0^1,u_0^2)=\tilde F(u_0^1,u_0^2)$ 则 $F_1=F_2$ on D

②存在性：若 $\frac{\partial}{\partial u^i}(f_j(u^1,u^2,F(u^1,u^2)))=\frac{\partial}{\partial u^j}(f_i(u^1,u^2,F(u^1,u^2)))$

则存在 (u_0^1,u_0^2) 的邻域 W，W 上有光滑解 F 若线性方程 则 $W=D$

成为是方程 $\begin{cases}\frac{\partial r}{\partial u^\alpha}=r_\alpha\\ \frac{\partial r_\alpha}{\partial u^\beta}=\Gamma_{\alpha\beta}^\gamma r_\gamma+b_{\alpha\beta}n\\ \frac{\partial n}{\partial u^\alpha}=-b_\alpha^\beta r_\beta\end{cases}$ $\Rightarrow \frac{\partial}{\partial u^\alpha}\begin{pmatrix}r\\ r_1\\ r_2\\ n\end{pmatrix}=\underset{12\times 12}{(\quad)}\begin{pmatrix}r\\ r_1\\ r_2\\ n\end{pmatrix}$

线性方程

$\Rightarrow$ 唯一性.

存在性定理：$\begin{pmatrix}E&F\\ F&G\end{pmatrix}>0$ 且有 Gauss 和 Codazzi 成立 则有 r s.t.

E,F,G,L,M,N 为 r 的第一二基本形式

[pf. 由上 PDE 有解 则存在 $\begin{pmatrix}r\\ n\end{pmatrix}$ 满足方程

只须验证 (r_1,r_2,n) 确实为一对活动标架

已知初值处满足这些 则由唯一性 恒为右手系

$\begin{cases}\langle r_\alpha,r_\beta\rangle=g_{\alpha\beta}\\ \langle r_\alpha,n\rangle=0\\ \langle n,n\rangle=1\\ r_1,r_2,n\text{ 右手系}\end{cases}$

$\frac{\partial}{\partial u^\alpha}\begin{pmatrix}\langle r_\alpha,r_\beta\rangle\\ \langle r_\alpha,n\rangle\\ \langle n,n\rangle\end{pmatrix}=A\begin{pmatrix}\langle r_\alpha,r_\beta\rangle\\ \langle r_\alpha,n\rangle\\ \langle n,n\rangle\end{pmatrix}$ 解性-

又 $\begin{pmatrix}g_{\alpha\beta}\\ 0\\ 1\end{pmatrix}$ 初值存在为解

故只需 $\begin{cases}\langle r_\alpha,r_\beta\rangle=g_{\alpha\beta}\\ \langle r_\alpha,n\rangle=0\\ \langle n,n\rangle=1\end{cases}$ (*)

下证 (*) 成立 (K)

$\frac{\partial}{\partial u^\gamma}\langle r_\alpha,r_\beta\rangle=\langle\Gamma_{\alpha\xi}^\gamma r_\xi+b_{\alpha\gamma}n,r_\beta\rangle+\langle r_\alpha,\Gamma_{\beta\xi}^\gamma r_\xi+b_{\beta\gamma}n\rangle$

$=\Gamma_{\alpha\xi}^\gamma\langle r_\xi,r_\beta\rangle+b_{\alpha\gamma}\langle n,r_\beta\rangle+b_{\beta\gamma}\langle r_\alpha,n\rangle+\Gamma_{\beta\xi}^\gamma\langle r_\alpha,r_\xi\rangle$

$\frac{\partial}{\partial u^\gamma}\langle r_\alpha,n\rangle=\langle\Gamma_{\alpha\gamma}^\xi r_\xi+b_{\alpha\gamma}n,n\rangle+\langle r_\alpha,-b_\gamma^\beta r_\beta\rangle$

$=b_{\alpha\gamma}\langle n,n\rangle+\Gamma_{\alpha\gamma}^\xi\langle r_\xi,n\rangle-b_\gamma^\beta\langle r_\alpha,r_\beta\rangle$

$\frac{\partial}{\partial u^\gamma}\langle n,n\rangle = 2\langle n,-b_\gamma^\beta r_\beta\rangle = -2b_\gamma^\beta\langle r_\beta,n\rangle$ 以及第三A

证明只须验证 $\begin{pmatrix}g_{\alpha\beta}\\ b_{\alpha\beta}\end{pmatrix}$ 满足上述即可. 略 #

空间曲线:

正交活动标架 $\{r;e_1,e_2,e_3\}$ $\frac{de_i}{ds}=a_i^j e_j$

$\frac{d}{ds}\langle e_i,e_j\rangle = \langle a_i^k e_k,e_j\rangle + \langle e_i,a_j^k e_k\rangle = a_{ij}+a_{ji}=0$

选新标架 $\begin{cases}\tilde{e}_1=e_1\\ \begin{pmatrix}\tilde{e}_2\\ \tilde{e}_3\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e_2\\ e_3\end{pmatrix}\end{cases}$

$\frac{d\tilde{e}_1}{ds}=a_1^2e_2+a_1^3e_3$

$=\tilde{a}_1^2\tilde{e}_2+\tilde{a}_1^3\tilde{e}_3$

$=(\tilde{a}_1^2\ \ \tilde{a}_1^3)\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e_2\\ e_3\end{pmatrix}$

$\Rightarrow\begin{pmatrix}\tilde{a}_1^2\\ \tilde{a}_1^3\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}a_1^2\\ a_1^3\end{pmatrix}$

要求 $k^2(s)=(a_1^2)^2+(a_1^3)^2>0$ 不妨设 $a_1^2\neq 0$ 取 $\tan\theta=\frac{a_1^3}{a_1^2}$ $\begin{cases}\theta\in(-\frac{\pi}{2},\frac{\pi}{2})\ (a_1^2>0)\\ \theta\in(\frac{\pi}{2},\frac{3}{2}\pi)\ (a_1^2<0)\end{cases}$

则 $\tilde{a}_1^3=0,\ \tilde{a}_1^2>0$

$\Rightarrow\frac{d}{ds}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix}=\begin{pmatrix}0 & \tilde{a}_1^2 & 0\\ -\tilde{a}_1^2 & 0 & \tilde{a}_2^3\\ 0 & -\tilde{a}_2^3 & 0\end{pmatrix}\begin{pmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{pmatrix}$

曲面?

$$\begin{cases}dr=r_1du^1+r_2du^2\\ dr_1=\Gamma_{1\beta}^\gamma r_\gamma du^\beta+b_{1\beta}n\,du^\beta\\ dr_2=\Gamma_{2\beta}^\gamma r_\gamma du^\beta+b_{2\beta}n\,du^\beta\\ dn=-b_\alpha^\beta r_\beta du^\alpha\end{cases}\Rightarrow d\begin{pmatrix}r_1\\ r_2\\ n\end{pmatrix}=\begin{pmatrix}\Gamma_{1\beta}^1du^\beta & \Gamma_{1\beta}^2du^\beta & b_{1\beta}du^\beta\\ \Gamma_{2\beta}^1du^\beta & \Gamma_{2\beta}^2du^\beta & b_{2\beta}du^\beta\\ -b_\alpha^1du^\alpha & -b_\alpha^2du^\alpha & 0\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ n\end{pmatrix}$$

标准化 $d\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}\omega_1^1 & \omega_1^2 & \omega_1^3\\ \omega_2^1 & \omega_2^2 & \omega_2^3\\ \omega_3^1 & \omega_3^2 & \omega_3^3\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$

若$\langle e_i,e_j\rangle=\delta_{ij}$. 同上 $\omega_i^j+\omega_j^i=0$

e_1,e_2 为对偶形式为 ω^1,ω^2 若设 $e_3=n$ 想用 ω^1,ω^2 表示 ω_i^3.

$\begin{pmatrix}r_1\\r_2\end{pmatrix}=A\begin{pmatrix}e_1\\e_2\end{pmatrix}=\begin{pmatrix}a_1^1 & a_1^2\\ a_2^1 & a_2^2\end{pmatrix}\begin{pmatrix}e_1\\e_2\end{pmatrix}$

$\mathbb{R}^3\ dr=r_1du^1+r_2du^2=(du^1\ du^2)\begin{pmatrix}a_1^1 & a_1^2\\ a_2^1 & a_2^2\end{pmatrix}\begin{pmatrix}e_1\\e_2\end{pmatrix}$

$=(a_1^1du^1+a_2^1du^2)e_1+(a_1^2du^1+a_2^2du^2)e_2$

$\forall X\in T_pM\quad X=X^1r_1+X^2r_2.\quad \mathbb{R}^3\ \omega^1(X)=\langle X,e_1\rangle=X^1a_1^1+X^2a_2^1$

$(a_1^1du^1+a_2^1du^2)(X)=X^1a_1^1+X^1a_2^1$

$\Rightarrow a_1^1du^1+a_2^1du^2=\omega^1\quad\Rightarrow dr=\omega^1e_1+\omega^2e_2$

$I(V,W)=\omega^1(V)\omega^1(W)+\omega^2(W)\omega^2(W)\quad\Rightarrow I=\omega^1\otimes\omega^1+\omega^2\otimes\omega^2$

$II(V,W)=\langle -W(V),W\rangle$

$W\begin{pmatrix}e_1\\e_2\end{pmatrix}=B\begin{pmatrix}e_1\\e_2\end{pmatrix}\Rightarrow\begin{pmatrix}-n_1\\-n_2\end{pmatrix}=AB\begin{pmatrix}e_1\\e_2\end{pmatrix}\qquad B=B^T$

$de_3=(n_1\ n_2)\begin{pmatrix}du^1\\du^2\end{pmatrix}\quad\Rightarrow(du^1\ du^2)\begin{pmatrix}n_1\\n_2\end{pmatrix}=-(\omega_1^3\ \omega_2^3)\begin{pmatrix}e_1\\e_2\end{pmatrix}$

$(\omega_3^1\ \omega_3^2)\begin{pmatrix}e_1\\e_2\end{pmatrix}$

$-(du^1\ du^2)AB\begin{pmatrix}e_1\\e_2\end{pmatrix}\Rightarrow(du^1\ du^2)AB=(\omega_1^3\ \omega_2^3)$

$(\omega^1\ \omega^2)B$

$\Rightarrow\begin{pmatrix}\omega_1^3\\\omega_2^3\end{pmatrix}=B^T\begin{pmatrix}\omega^1\\\omega^2\end{pmatrix}$

故 $\mathrm{II}(v,w)=(\omega^1(v),\omega^2(v))\,B\begin{pmatrix}\omega^1(w)\\ \omega^2(w)\end{pmatrix}$

$$=(\omega^1(v),\omega^2(v))\begin{pmatrix}\omega_1^3(w)\\ \omega_2^3(w)\end{pmatrix}$$

$$\Rightarrow \mathrm{II}=\omega^1\otimes\omega_1^3+\omega^2\otimes\omega_2^3$$

由: $\omega^1\wedge\omega^2=\sqrt{EG-F^2}\,du^1\wedge du^2$

$$\omega_1^3\wedge\omega_2^3=K\omega^1\wedge\omega^2$$

$$\omega_1^3\wedge\omega^2+\omega^1\wedge\omega_2^3=2H\omega^1\wedge\omega^2$$

$$\omega_2^1\wedge\omega_3^1+\omega_2^1\wedge\omega_2^3$$

$$0=d(dv)=d(\omega^\alpha e_\alpha)=d\omega^\alpha\wedge e_\alpha-\omega^\alpha\wedge de_\alpha$$

$$=d\omega^\alpha\wedge e_\alpha-\omega^\beta\wedge\omega_\beta^j e_j$$

$$=(d\omega^1-\omega^2\wedge\omega_2^1)e_1+(d\omega^2-\omega^1\wedge\omega_1^2)e_2$$

$$-(\omega^1\wedge\omega_1^3+\omega^2\wedge\omega_2^3)e_3$$

$$\Rightarrow\begin{cases}d\omega^1=\omega^2\wedge\omega_2^1\\ d\omega^2=\omega^1\wedge\omega_1^2\\ \omega_1\wedge\omega_1^3+\omega_2\wedge\omega_2^3=0\end{cases}\quad(3)$$

$$\begin{pmatrix}\omega_1^3\\ \omega_2^3\end{pmatrix}=\begin{pmatrix}b_1^1 & b_2^1\\ b_1^2 & b_2^2\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$

$$(3)\Rightarrow 0=\omega_1\wedge(b_1^1\omega^1+b_2^1\omega^2)+\omega_2\wedge(b_1^2\omega^1+b_2^2\omega^2)$$

$$=(b_2^1-b_1^2)\,\omega_1\wedge\omega_2\quad\Leftrightarrow\quad B=B^T\ \text{对称}$$

$$\Rightarrow\begin{cases}d\omega^1=\omega^2\wedge\omega_2^1\\ d\omega^2=\omega^1\wedge\omega_1^2\end{cases}(*1$$

$$\omega_1^2=a\omega^1+b\omega^2$$

$$\Rightarrow d\omega^1=a\omega^1\wedge\omega^2\quad\Rightarrow a=d\omega^1(e_1,e_2)$$

$$d\omega^2=b\omega^1\wedge\omega^2\qquad b=d\omega^2(e_1,e_2)$$

(claim: i满足(*)的ω_i^2唯一

若ω_i^2, $\tilde{\omega}_i^2$ 均满足(*) 则 $\begin{cases} \omega^i \wedge (\omega_i^2 - \tilde{\omega}_i^2) = 0 \\ \omega^i \wedge (\omega_i^2 - \tilde{\omega}_i^2) = 0 \end{cases}$ $\Rightarrow \omega_i^2 - \tilde{\omega}_i^2 = 0$ ✓

故ω^1, ω^2, ω^3 的组合唯一决定

$$0 = d(de_\alpha) = d(\omega_\alpha^j e_j) = d\omega_\alpha^j e_j - \omega_\alpha^j \wedge de_j$$

$$= d\omega_\alpha^j e_j - \omega_\alpha^j \wedge \omega_j^k e_k$$

$$= (d\omega_\alpha^k - \omega_\alpha^j \wedge \omega_j^k) e_k$$

$$\Rightarrow \begin{cases} d\omega_\alpha^1 = \omega_\alpha^j \wedge \omega_j^1 & ① \\ d\omega_\alpha^2 = \omega_\alpha^j \wedge \omega_j^2 & ② \quad (\alpha = 1,2) \\ d\omega_\alpha^3 = \omega_\alpha^j \wedge \omega_j^3 & ③ \end{cases}$$

给的方程

$$\Rightarrow \begin{cases} d\omega_1^2 = \omega_1^3 \wedge \omega_3^2 & (a) \\ d\omega_1^3 = \omega_1^2 \wedge \omega_2^3 & (c) \\ d\omega_2^3 = \omega_2^1 \wedge \omega_1^3 \end{cases}$$

① $\alpha = 1$ 平凡　　$\alpha = 2$　　$d\omega_2^1 = \omega_2^3 \wedge \omega_3^1$

② $\alpha = 2$ 平凡　　$\alpha = 1$　　$d\omega_1^2 = \omega_1^3 \wedge \omega_3^2$ ↑

③ $\alpha = 1$　　$d\omega_1^3 = \omega_1^2 \wedge \omega_2^3$

$\alpha = 2$　　$d\omega_2^3 = \omega_2^1 \wedge \omega_1^3$

$$0 = d(de_j) = d(\omega_j^i e_i) = (d\omega_j^k - \omega_j^i \wedge \omega_i^k) e_k$$

$k=1$: $d\omega_j^1 = \omega_j^2 \wedge \omega_2^1$

$k=2$: $d\omega_j^2 = \omega_j^1 \wedge \omega_1^2$　　不产生新方程

$k=3$: 平凡

$$d\omega_1^2 = \omega_1^3 \wedge \omega_3^2 = -\omega_1^3 \wedge \omega_2^3 = -K\omega^1 \wedge \omega^2$$

$$\Rightarrow K = -\frac{d\omega_1^2}{\omega^1 \wedge \omega^2}$$　　(⟶ 高斯绝妙定理)

设 $F=0$ $\quad I=E du\otimes du+G dv\otimes dv$

取 $e_1=\frac{r_u}{\sqrt{E}}$ $\quad e_2=\frac{r_v}{\sqrt{G}}$

$$dr=(r_u\ r_v)\begin{pmatrix}du\\dv\end{pmatrix}=(\omega^1\ \omega^2)\begin{pmatrix}e_1\\e_2\end{pmatrix}\Rightarrow \omega^1=\sqrt{E}du$$

$$\omega^2=\sqrt{G}dv$$

$$d\omega^1=(\sqrt{E})_v dv\wedge du=-\frac{(\sqrt{E})_v}{\sqrt{EG}}\omega^1\wedge\omega^2$$

同理 $d\omega^2=\frac{(\sqrt{G})_u}{\sqrt{EG}}\omega^1\wedge\omega^2$

$$\Rightarrow \omega_1^2=-\frac{(\sqrt{E})_v}{\sqrt{EG}}\omega^1+\frac{(\sqrt{G})_u}{\sqrt{EG}}\omega^2=-\frac{(\sqrt{E})_v}{\sqrt{G}}du+\frac{(\sqrt{G})_u}{\sqrt{E}}dv$$

$$d\omega_1^2=\left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right)du\wedge dv$$

$$\mathrm{I\!I}\begin{pmatrix}e_1\\e_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{E}}&\\&\frac{1}{\sqrt{G}}\end{pmatrix}\mathrm{I\!I}\begin{pmatrix}r_u\\r_v\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{E}}&\\&\frac{1}{\sqrt{G}}\end{pmatrix}\begin{pmatrix}L&M\\M&N\end{pmatrix}\begin{pmatrix}E&\\&G\end{pmatrix}^{-1}\begin{pmatrix}r_u\\r_v\end{pmatrix}$$

$$B\begin{pmatrix}e_1\\e_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{E}}&\\&\frac{1}{\sqrt{G}}\end{pmatrix}\begin{pmatrix}L&M\\M&N\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{E}}&\\&\frac{1}{\sqrt{G}}\end{pmatrix}\begin{pmatrix}e_1\\e_2\end{pmatrix}$$

$$\begin{pmatrix}\omega_1^3\\\omega_2^3\end{pmatrix}=B\begin{pmatrix}\omega^1\\\omega^2\end{pmatrix}=B\begin{pmatrix}\sqrt{E}&\\&\sqrt{G}\end{pmatrix}\begin{pmatrix}du\\dv\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{E}}&\frac{M}{\sqrt{E}}\\\frac{M}{\sqrt{E}}&\frac{N}{\sqrt{G}}\end{pmatrix}\begin{pmatrix}du\\dv\end{pmatrix}$$

$$\Rightarrow \omega_1^3\wedge\omega_3^2=-\omega_1^3\wedge\omega_2^3$$

$$d\omega_1^2=-\det\left(B\begin{pmatrix}\sqrt{E}&\\&\sqrt{G}\end{pmatrix}\right)du\wedge dv$$

$$=-\frac{LN-M^2}{\sqrt{EG}}du\wedge dv$$

$$\Rightarrow\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u=-\frac{LN-M^2}{\sqrt{EG}}$$ Gauss Eq

若 $M=0$.

Codazzi $\overset{(1)}{\Rightarrow}$ $dw_1^3 = w_1^2 \wedge w_2^3 = -\frac{(\sqrt{E})_v}{\sqrt{G}} \frac{N}{\sqrt{G}} du\wedge dv$

$d(\overset{''}{\frac{L}{\sqrt{E}}} du) = -\frac{(\sqrt{E})_v N}{G} du\wedge dv$

$-(\overset{''}{\frac{L}{\sqrt{E}}})_v du\wedge dv$

$-\frac{\overset{''}{L_v}\sqrt{E} - L(\sqrt{E})_v}{E}$ $\Rightarrow L_v = \frac{1}{2}(\frac{L}{E}+\frac{N}{G})(\sqrt{E})_v\sqrt{E}\,2$

$= H E_v$

(2) $\Rightarrow$ $M_u = H G_u$

常平均曲率曲面的分类，

① $k_1 = k_2$ 全脐点 $\Rightarrow$ 球面/平面

② $k_1 \neq k_2$ e_1, e_2 主方向 $e_3 = n$

$w\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \Rightarrow \begin{pmatrix} w_1^3 \\ w_2^3 \end{pmatrix} = \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$

$dw_1^3 = k_1 dw^1 = k_1 w^2 \wedge w_2^1$

$\overset{''}{w_1^2} \wedge w_2^3 = k_2 w_1^2 \wedge w^2 \Rightarrow (k_1 - k_2) w^2 \wedge w_2^1 = 0$

$\Rightarrow w^2 \wedge w_2^1 = 0$

同理 $w^1 \wedge w_2^1 = 0$ $\Rightarrow w_1^2 = 0$ $\Rightarrow dw_1^2 = 0$

$\Rightarrow k = 0$

不妨 $k_2 = 0$ $k_1 \neq 0$ 则 $w_2^3 = 0$

$de_1 = w_1^3 e_3 = k_1 w^1 e_3$

$de_2 = w_2^3 e_3 = 0$ $\Rightarrow e_2$ 常向量

$de_3 = w_3^1 e_1 = -k_1 w^1 e_1$

$\{e_2$ 垂直的平面交出曲于一平面直线，切向为 e_1 $\Rightarrow \begin{cases} \frac{de_1}{ds} = k e_3 \Rightarrow$ 四线为圆 \\ \frac{de_3}{ds} = -k e_1 \Rightarrow$ 为圆柱面 \end{cases}$

又 $w^1(e_1) = a \Rightarrow w^1 = ds$ (弧参)

没有为了正交标架 $(r, \bar{e}_1, \bar{e}_2, \underbrace{e_3=n}_{\bar{e}_1\wedge\bar{e}_2})$ $\begin{pmatrix}\bar{e}_1\\ \bar{e}_2\end{pmatrix}=R_\theta\begin{pmatrix}e_1\\ e_2\end{pmatrix}$ R_θ正交

$$dr=(\omega^1\ \omega^2)\begin{pmatrix}e_1\\ e_2\end{pmatrix}=(\bar{\omega}^1\ \bar{\omega}^2)\begin{pmatrix}\bar{e}_1\\ \bar{e}_2\end{pmatrix}=(\bar{\omega}^1\ \bar{\omega}^2)R_\theta\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\bar{\omega}^1\\ \bar{\omega}^2\end{pmatrix}=R_\theta\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$

$$\mathrm{I}=\sum_\alpha \bar{\omega}^\alpha\otimes\bar{\omega}^\alpha=\sum_\alpha (R_\theta)^\alpha_\beta\omega^\beta\otimes(R_\theta)^\alpha_\gamma\omega^\gamma$$

$$=\left(\sum_\alpha (R_\theta)^\alpha_\beta (R_\theta)^\alpha_\gamma\right)\omega^\beta\otimes\omega^\gamma$$

$$=(R_\theta)^\alpha_\beta\,(R_\theta^T)^\gamma_\alpha\,\omega^\beta\otimes\omega^\gamma=\sum_\beta\omega^\beta\otimes\omega^\beta=\mathrm{I}$$

$$de_3=d\bar{e}_3=(\omega^1_3\ \omega^2_3)\begin{pmatrix}e_1\\ e_2\end{pmatrix}=(\bar{\omega}^1_3\ \bar{\omega}^2_3)R_\theta\begin{pmatrix}e_1\\ e_2\end{pmatrix}\Rightarrow\begin{pmatrix}\bar{\omega}^1_3\\ \bar{\omega}^2_3\end{pmatrix}=R_\theta\begin{pmatrix}\omega^1_3\\ \omega^2_3\end{pmatrix}$$

用上面的记号 $\mathrm{II}=\mathrm{II}$

$$\bar{\omega}^2_1=\langle d\bar{e}_1,\bar{e}_2\rangle=\langle d(\cos\theta e_1+\sin\theta e_2),-\sin\theta e_1+\cos\theta e_2\rangle$$

$$=\langle\cos\theta de_1+\sin\theta de_2-\sin\theta e_1 d\theta+\cos\theta e_2 d\theta,-\sin\theta e_1+\cos\theta e_2\rangle$$

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}=\cos^2\theta\omega^2_1+\sin^2\theta\omega^2_1+\sin^2\theta d\theta+\cos^2\theta d\theta=\omega^2_1+d\theta$$

$$\bar{\omega}^2_1=-\omega^2_1-d\theta$$

$$\left(\det R_\theta=-1\ \begin{pmatrix}-\sin\theta & \cos\theta\\ \cos\theta & \sin\theta\end{pmatrix}\right)$$

求曲面上两点最短线

$r=r(u^1,u^2)$ $\quad r=r(s)=r(u^1(s),u^2(s))$ 弧长参数 on $[0,l]$

并设 $e_2(s)=a^1(s)r_1(s)+a^2(s)r_2(s)$ $\quad e_1(s)=\dot r(s)$

选一族曲线 $r\big((u^1(s),u^2(s))+\lambda f(s)(a^1(s),a^2(s)\big)$ $\quad f:[0,l]\to$ 光滑 $f(0)=0=f(l)$

$\overset{\perp}{\underset{r_f^2}{}}$

$$\frac{dr_{f(s)}^2}{d\lambda}\Big|_{\lambda=0}=f(s)e_2(s)$$

若 r 极短 则 $\dfrac{dL(r_f^1)}{d\lambda}\Big|_{\lambda=0}=0$

$$\int_0^l \overset{''}{<}\frac{\partial r_f^1}{\partial s},\frac{\partial r_f^1}{\partial s}>^{-\frac12} 2<\frac{\partial}{\partial\lambda}\frac{\partial r_f^1}{\partial s},\frac{\partial r_f^1}{\partial s}>ds\Big|_{\lambda=0}$$

$$\int_0^l \overset{''}{<}\frac{\partial}{\partial\lambda}\frac{\partial r_f^1}{\partial s},\frac{\partial r_f^1}{\partial s}>ds\Big|_{\lambda=0}$$

$$\int_0^l <\frac{\partial}{\partial s}\overset{''}{\frac{\partial}{\partial s}}\frac{\partial r_f^1}{\partial\lambda},\frac{\partial r_f^1}{\partial s}>ds\Big|_{\lambda=0}$$

故 $0=\int_0^l <\frac{\partial}{\partial s}(f(s)e_2(s)),e_1(s)>ds$

$$=\int_0^l \frac{d}{ds}(<f(s)e_2(s),e_1(s)>)-<f(s)e_2(s),\frac{\partial}{\partial s}e_1(s)>ds$$

$$=-\int_0^l f(s)<e_2(s),\frac{d}{ds}e_1(s)>ds$$

$$\Rightarrow\ <e_2(s),\frac{d}{ds}e_1(s)>=0 \quad\Rightarrow \frac{de_1}{ds}\ /\!/\ n$$

$$\Leftarrow\ <de_1(\dot r(s)),e_2>=<(w_1^2e_2+w_1^3e_3)(\dot r(s)),e_2>=0$$

$$\Leftrightarrow\ w_1^2(\dot r(s))=0$$

$$\overset{''}{w_1^2}(e_1)$$

测地线：$r(s)$ 弧长参数曲线 $\quad <\underset{\text{主法向量}}{\underset{\uparrow}{\frac{de_1}{ds}}},e_2>=0$ 其中 $e_1(s)=\dot r(s)$

$e_2(s)=-e_1(s)\wedge n(s)$

$\frac{d^2r}{ds^2} - (\frac{d^2r}{ds^2}, n>n = e_2 k_g$ k_g: 测地曲率

$<\frac{d^2r}{ds^2}, n> = k_n(r(s))$ 法曲率 $k^2 = k_n^2 + k_g^2$ > 测地线 $\Rightarrow k_g = 0$

$\bar{e}_1 = \dot{r}(s)$

$\bar{e}_2 = e_3 \wedge \bar{e}_1$

$$\begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$k_g = \bar{\omega}_1^2(\bar{e}_1) = (\omega_1^2 + d\theta)(\cos\theta e_1 + \sin\theta e_2)$$

$k_g = \frac{\omega_1^2}{ds} + \frac{d\theta}{ds}$ $= \cos\theta\, \omega_1^2(e_1) + \sin\theta\, \omega_1^2(e_2) + d\theta(\bar{e}_1)$

$= \frac{d\theta}{ds} + \omega_1^2(e_1)\cos\theta + \omega_1^2(e_2)\sin\theta$

特别. 若 $\Gamma = 0$ 取 $e_1 = \frac{r_u}{\sqrt{E}}$ $e_2 = \frac{r_v}{\sqrt{G}}$

$k_g(r(s)) = \frac{d\theta}{ds} + (k_g)_u \cos\theta + (k_g)_v \sin\theta$

$\omega^1 = \sqrt{E}\, du$ $d\omega^1 = -\frac{(\sqrt{E})_v}{\sqrt{EG}}\, \omega^1 \wedge \omega^2$

$\omega^2 = \sqrt{G}\, dv$ $d\omega^2 = \frac{(\sqrt{G})_u}{\sqrt{EG}}\, \omega^1 \wedge \omega^2$

$\Rightarrow \omega_1^2 = -\frac{(\sqrt{E})_v}{\sqrt{EG}}\, \omega^1 + \frac{(\sqrt{G})_u}{\sqrt{EG}}\, \omega^2$

$\Rightarrow k_g = \frac{d\theta}{ds} - \frac{(\sqrt{E})_v}{\sqrt{EG}}\cos\theta + \frac{(\sqrt{G})_u}{\sqrt{EG}}\sin\theta$

$k_g = \frac{d\theta}{ds} - \frac{(\sqrt{E})_v}{\sqrt{G}}\frac{du}{ds} + \frac{(\sqrt{G})_u}{\sqrt{E}}\frac{dv}{ds}$

$\frac{(\sqrt{E})_v}{\sqrt{EG}} = \frac{1}{2\sqrt{G}}(\ln E)_v$

$\cos\theta = \frac{<r_u, \dot{r}(s)>}{\sqrt{E}} = \sqrt{E}\frac{du}{ds}$ (由 E, F, G 决定)

$\frac{d}{ds}\frac{dr}{ds} = r_\alpha \frac{d^2u^\alpha}{ds^2} + \frac{dr_\alpha}{ds}\frac{du^\alpha}{ds} = r_\alpha \frac{d^2u^\alpha}{ds^2} + \frac{\partial r_\alpha}{\partial u^\beta}\frac{du^\beta}{ds}\frac{du^\alpha}{ds}$

$= \left(\frac{d^2u^\gamma}{ds^2} + \Gamma^\gamma_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\right) r_\gamma + b_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} n$

故测地线 $\Leftrightarrow \frac{d^2u^\gamma}{ds^2}+\Gamma^\gamma_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}=0$

性质：$\forall p\in M$. $v\in T_pM$. $\exists$ 测地线 $r(s)=r(u^1(s),u^2(s))$ s.t. $r(0)=p$, $r'(0)=v$

$v=v^1r_1+v^2r_2$

[Pf. $\Leftrightarrow$ $\begin{cases}\frac{d^2u^\gamma}{ds^2}+\Gamma^\gamma_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}=0 \ (\Gamma^\gamma_{\alpha\beta}\text{可}C^\infty\text{开}) \\ u^1(0)=u^1_0,\ u^2(0)=u^2_0 \\ \frac{du^1}{ds}(0)=v^1,\ \frac{du^2}{ds}(0)=v^2\end{cases}$ 唯一性 故由ODE…]

等距变换

$M,\bar{M}$ 正则曲面 $\sigma: M\to\bar{M}$ 一对一. 可导 σ^{-1} 也 C^∞

称 σ 为等距变换. 若 σ 保持所有曲线的长度

$\Rightarrow \int_0^l \sqrt{I(r'(s),r'(s))}\,ds=\int_0^l \sqrt{\bar{I}(\sigma\circ r'(s),\sigma\circ r'(s))}\,ds$

$\Rightarrow I(r'(s),r'(s))=\bar{I}(\sigma\circ r'(s),\sigma\circ r'(s))$

$(\sigma_*)_p: T_pM\to T_{\sigma(p)}\bar{M}$

$v\mapsto \frac{d}{dt}\big|_{t=0}\sigma(r(s))$

$r(0)=p$, $r'(0)=v$

$\Rightarrow I(v,w)=\bar{I}(\sigma_*(v),\sigma_*(w))$

$\Rightarrow \sigma^*\bar{I}=I$

内蕴几何

$$\int_{\triangle ABC} K\,d\sigma = \angle A+\angle B+\angle C-\pi$$

测地线 γ $\quad \frac{DV(s)}{ds} = \frac{dV}{ds} - \langle \frac{dV}{ds}, n\rangle n$ $\quad$ 测地线 $\Leftrightarrow \frac{DV(s)}{ds}=0$

沿曲线 $\gamma(s)$ 的光滑向量场 $V(s)$ 为平行的 若 $\frac{DV}{ds}=0$

$$\frac{dV}{ds} \overset{(V=V^\alpha r_\alpha)}{=} \frac{dV^\alpha}{ds} r_\alpha + V^\alpha \frac{\partial r_\alpha}{\partial u^\beta}\frac{du^\beta}{ds} = \frac{dV^\alpha}{ds} r_\alpha + V^\alpha(\Gamma_{\alpha\beta}^\gamma r_\gamma + b_{\alpha\beta} n)\frac{du^\beta}{ds}$$

故 $\frac{DV}{ds} = \left(\frac{dV^\alpha}{ds} + V^\gamma \Gamma_{\gamma\beta}^\alpha \frac{du^\beta}{ds}\right) r_\alpha \quad \Leftrightarrow \frac{dV^\alpha}{ds} + V^\gamma \Gamma_{\gamma\beta}^\alpha \frac{du^\beta}{ds} = 0$

$\parallel$ 一阶线性ODE

给初值 $V(0)=(V^1(0), V^2(0))$ 则可以整体平行移动

性质 (1) $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$

(2) $\frac{D(fV)}{dt} = f'(t)V + f(t)\frac{DV}{dt}$

(3) $\frac{d\langle V,W\rangle}{dt} = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$ (Cor 平行移动保长度和角度)

$P_{\gamma,p,q} : T_pM \to T_qM$

$V \mapsto P_{\gamma,p,q}(V)$ 平行移动 (依赖于路径选取)

保内积线性映射

Fact① 两个曲面 M_1 M_2 沿着曲线 $C(t)$ 相切

设 $V_0 \in T_{C(t_0)} M_1 = T_{C(t_0)} M_2$. 则 V_0 在 M_1 上沿 $C(t)$ 的平行移动

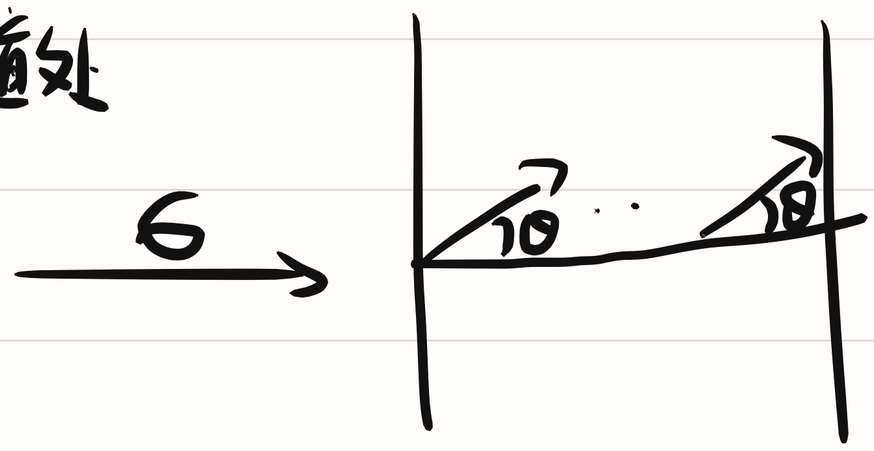
等于在 M_2 上沿 $C(t)$ 平行移动

② $G: M_1 \to M_2$ 等距. $V(t)$ 沿 $C(t)$ 在 M_1 上平行

则 $G_*(V(t))$ 沿 $G(C(t))$ 在 M_2 上平行

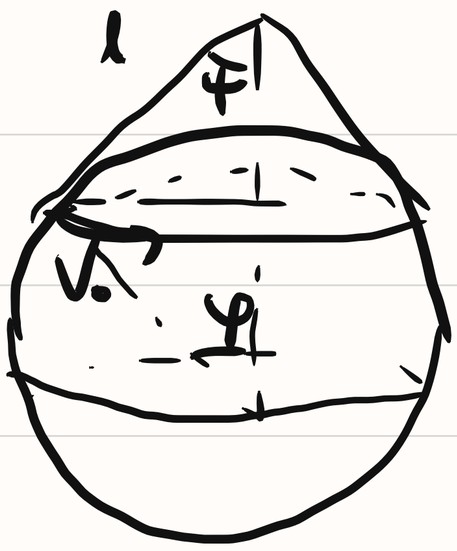


未直处



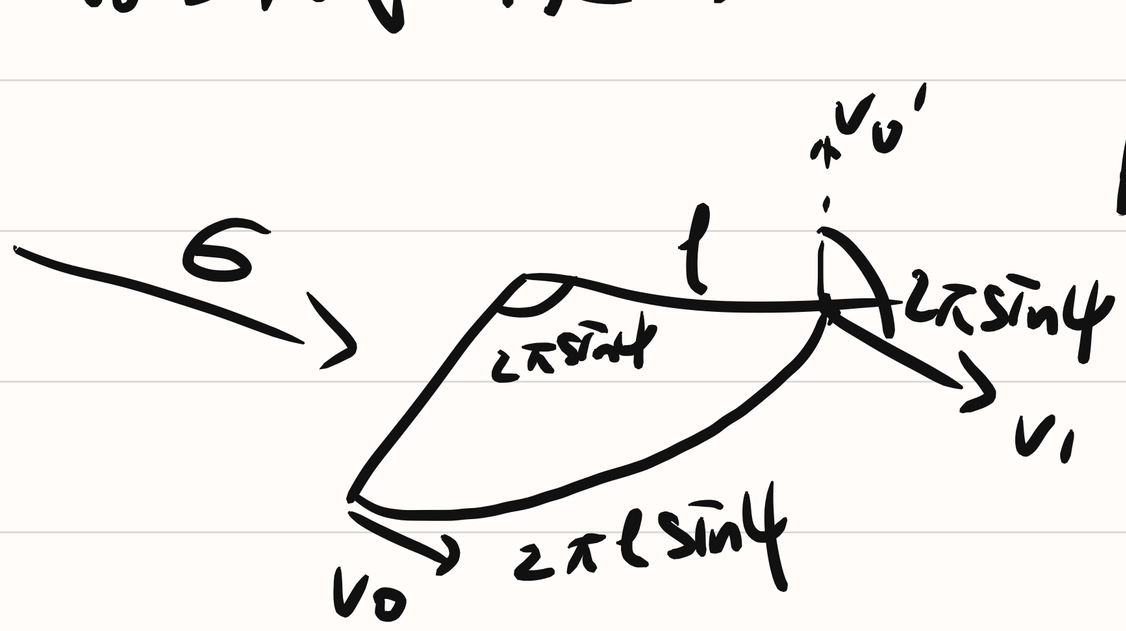
平行移动

一圈回来了变



$\psi + \varphi = \frac{\pi}{2}$

V_0 沿纬线 构造回锥面相切



$R\cos\psi = l\sin\psi$

故圈后有

两方向夹角 $2\pi\sin\psi$

测地线不一定最短 但局部最短！

指数映射 $\exp_p: T_pM \to M$

$v \mapsto \exp_p(v)$

测地线 $\gamma(0)=p \quad \dot\gamma(0)=\frac{v}{|v|} \qquad \exp_p(v) \triangleq \gamma(|v|)$

法坐标系： $e_1, e_2 \in T_pM$ 标准正交

$$r(u^1,u^2)=\exp_p(e_1u^1+e_2u^2) = r_{u^1e_1+u^2e_2}\left(\sqrt{(u^1)^2+(u^2)^2}\right)$$

$r_{u^1e_1+u^2e_2}$ 测地线 $r(0)=p \quad \dot r(0)=\frac{u^1e_1+u^2e_2}{\sqrt{(u^1)^2+(u^2)^2}}$

$\exists \varepsilon>0$ s.t. $\exp_p$ 在 $B(0,\varepsilon)$ 光滑 且 $r_1 \wedge r_2 \neq 0$

$$r_1(p)=\frac{\partial r}{\partial u_1}(p)=\frac{\partial}{\partial u^1} r(u^1,0)\Big|_{u^1=0} = \frac{\partial}{\partial u^1}\Big|_{u^1=0} r_{u^1e_1}(u^1) = e_1$$

$r_2(p)=e_2 \quad \Rightarrow r_1(p)\wedge r_2(p)\neq 0$

故可取更小的邻域 $B(0,\varepsilon')$ 使 $\exp_p: B(0,\varepsilon') \to M$ 正则参数化

Rmk $(\exp_p)_*|_0 : T_0(T_pM) \to T_pM$ （$T_pM \cong T_pM$）

$e_1 \mapsto e_1$

$e_2 \mapsto e_2$

故为 Id

① $\langle r_\alpha, r_\beta\rangle = g_{\alpha\beta}$

$g_{\alpha\beta}(p)=\delta_{\alpha\beta}$

$g_{\alpha\beta\gamma}(p)=0$ （只能在这一点处！）

但 $g_{\alpha\beta\gamma\delta}(p)=0$ 一般不成立（否则 可推出 $K(p)=0$）

在 p 处

$$g_{\alpha\beta,\gamma}=\langle r_{\alpha\gamma}, r_\beta\rangle+\langle r_\alpha, r_{\beta\gamma}\rangle = \langle \Gamma_{\alpha\gamma}^\eta r_\eta + b_{\alpha\gamma}n, r_\beta\rangle + \langle r_\alpha, \Gamma_{\beta\gamma}^\eta r_\eta + b_{\beta\gamma}n\rangle$$

$$= \Gamma_{\alpha\gamma}^\eta \delta_{\beta\eta} + \Gamma_{\beta\gamma}^\eta \delta_{\alpha\eta} = \Gamma_{\alpha\gamma}^\beta + \Gamma_{\beta\gamma}^\alpha$$

Fact: $\Gamma^{\alpha}_{\beta\gamma}(p)=0$

$(u^1,u^2)=(\rho\cos\theta,\rho\sin\theta)$ (θ固定) 是等距地线的坐标

即 $\rho\mapsto \exp_p(\rho\cos\theta,\rho\sin\theta)$ 为测地线

$$\Rightarrow \underbrace{\frac{d^2u^{\alpha}}{d\rho^2}}_{0}+\Gamma^{\alpha}_{\beta\gamma}\frac{du^{\beta}}{d\rho}\frac{du^{\gamma}}{d\rho}=0$$

$$\Rightarrow \Gamma^{\alpha}_{11}(\rho\cos\theta,\rho\sin\theta)\cos^2\theta+2\Gamma^{\alpha}_{12}(\rho\cos\theta,\rho\sin\theta)\cos\theta\sin\theta+\Gamma^{\alpha}_{22}(\rho\cos\theta,\rho\sin\theta)\sin^2\theta=0$$

$\Rightarrow \Gamma^{\alpha}_{11}(\rho,0)=0$ ($\theta=0$)　　$\Gamma^{\alpha}_{22}(0,\rho)=0$ ($\theta=\frac{\pi}{2}$)

$\Rightarrow \Gamma^{\gamma}_{\alpha\beta}(p)=0$

参数变换：$u^1=\rho\cos\theta$　　$r(u^1,u^2)=F(\rho,\theta)$

$u^2=\rho\sin\theta$

$$\begin{vmatrix}\frac{\partial u^1}{\partial\rho} & \frac{\partial u^1}{\partial\theta}\\ \frac{\partial u^2}{\partial\rho} & \frac{\partial u^2}{\partial\theta}\end{vmatrix}=\rho\neq 0$$

ρ线：测地线

θ线：测地圆

定理(Gauss)：从p出发的测地线与以p为心的测地圆正交

$$|dr|^2=d\rho\otimes d\rho+G(\rho,\theta)d\theta\otimes d\theta$$

pf. 记 $F=0$

① $\lim_{\rho\to 0}F(\rho,\theta)=0$：　　$r_{\theta}=-r_1\rho\sin\theta+r_2\rho\cos\theta$　　$\lim_{\rho\to 0}r_{\theta}=0$

② $\frac{\partial}{\partial\rho}F(\rho,\theta)=0$：　　$\frac{\partial}{\partial\rho}F(\rho,\theta)=\langle r_{\rho\rho},r_{\theta}\rangle+\langle r_{\rho},r_{\rho\theta}\rangle$

$=\frac{1}{2}\frac{\partial}{\partial\theta}\langle r_{\rho},r_{\rho}\rangle$，$\langle r_{\rho},r_{\rho}\rangle\to 0$

$=\langle\Gamma^{\alpha}_{11}r_{\alpha},r_{\theta}\rangle$　　$=0$

$u^1=\rho$，$u^2=\theta$　u^1线测地线 $\Rightarrow \frac{d^2u^{\alpha}}{d\rho^2}+\Gamma^{\alpha}_{\beta\gamma}\frac{du^{\beta}}{d\rho}\frac{du^{\gamma}}{d\rho}=0\Rightarrow\Gamma^{\alpha}_{11}(\rho,\theta)=0$

$\Leftarrow$

故 $F=0$.

定理. $\forall p\in M$ $\exists p$的邻域U s.t. $\forall q\in U$ 包含在U内的连接pq的测地线是整个曲面上连接pq的所有曲线中最短的

pf 首先令U上可建立测地极坐标系

$$\{\exp_p(w) \mid w\in B(0,\varepsilon)\subset T_pM\}$$

设$r(t)$ ($r(0)=p$ $r(t_0)=q$) 为连接p-q的曲线

(1) $C_{r(t)}$包含在U内. 则 $r(t)=r(\rho(t),\theta(t))$

$$\dot r(t)=r_\rho\frac{d\rho}{dt}+r_\theta\frac{d\theta}{dt} \qquad |\dot r(t)|=\sqrt{\left(\frac{d\rho}{dt}\right)^2+G\left(\frac{d\theta}{dt}\right)^2}$$

$$\Rightarrow \text{length}\,C=\int_0^{t_0}|\dot r(t)|dt \geq \int_0^{t_0}\left|\frac{d\rho}{dt}\right|dt \geq \left|\int_0^{t_0}\frac{d\rho}{dt}dt\right|$$

$$=|\rho(t_0)-\rho(0)| =\rho(t_0)$$

(2) 不经过p

$$t_1=\inf\{t : r(t)\notin U\}$$

$$\Rightarrow \text{length}\,C\geq \text{length}\,r|_{[0,t_2]} \geq \rho(t_2) \Rightarrow \text{length}\,C\geq\varepsilon$$

$$(\forall t_2<t_1) \qquad >\rho(t_0)$$

$$e_1=r_\rho \qquad e_2=\frac{r_\theta}{\sqrt{G}} \Rightarrow d\omega^1=0$$

$$\omega^1=d\rho \qquad \omega^2=\sqrt{G}\,d\theta \qquad d\omega^2=(\sqrt{G})_\rho\, d\rho\wedge d\theta=\frac{(\sqrt{G})_\rho}{\sqrt{G}}\omega^1\wedge\omega^2$$

$$\Rightarrow \omega_1^2=\frac{(\sqrt{G})_\rho}{\sqrt{G}}\omega^2=(\sqrt{G})_\rho\, d\theta$$

$$K=\frac{-d\omega_1^2}{\omega^1\wedge\omega^2}=\frac{-(\sqrt{G})_{\rho\rho}\,d\rho\wedge d\theta}{\sqrt{G}\,d\rho\wedge d\theta}$$

$$=-\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}$$

K为常数 $C \Leftrightarrow (\sqrt{G})_{\rho\rho} + C\sqrt{G} = 0$

性质 ① $\lim_{\rho\to 0}\sqrt{G} = 0$ ② $\lim_{\rho\to 0}(\sqrt{G})_\rho = 1$

pf $\begin{cases} u^1 = \rho\cos\theta \\ u^2 = \rho\sin\theta \end{cases}$ $G = g_{\alpha\beta}\frac{\partial u^\alpha}{\partial\theta}\frac{\partial u^\beta}{\partial\theta}$

$(\sqrt{G})_\rho = \frac{1}{2}\frac{G_\rho}{\sqrt{G}} = \frac{1}{2}\frac{g_{\alpha\beta,\rho}\rho\frac{\partial u^\alpha}{\partial\theta}\frac{\partial u^\beta}{\partial\theta} + g_{\alpha\beta}\frac{2}{\partial\theta}\left(\frac{\partial u^\alpha}{\partial\theta}\frac{\partial u^\beta}{\partial\theta}\right)}{\sqrt{G}}$

$\to 1$

(i) $C=0 \Rightarrow (\sqrt{G})_{\rho\rho}=0$ $(\sqrt{G})_\rho = f(\theta)$ ② $(\sqrt{G})_\rho = 1$ $\sqrt{G} = \rho + g(\theta)$

① $\sqrt{G} = \rho$

$I = d\rho\otimes d\rho + \rho^2 d\theta\otimes d\theta$

(ii) $C = \frac{1}{a^2} > 0 \Rightarrow \sqrt{G} = a\sin\frac{\rho}{a}$

(iii) $C = -\frac{1}{a^2} < 0 \Rightarrow \sqrt{G} = a\sinh\frac{\rho}{a}$

测地三角形内角和

定理，测地三角形 ABL 包含在 A 附近的测地极坐标邻域内.

[R] $\int_{\triangle ABC} K\, dVol = \angle A + \angle B + \angle C - \pi$

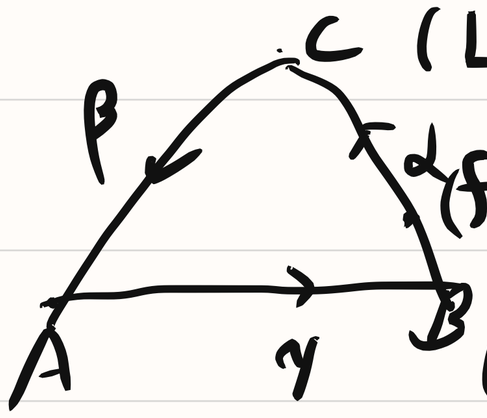
[pf

$I = d\rho\otimes d\rho + G\, d\theta\otimes d\theta$ $e_1 = r_\rho$ $e_2 = \frac{1}{\sqrt{G}}r_\theta$ $\omega^1 = d\rho$ $\omega^2 = \sqrt{G}\,d\theta$

$\Rightarrow \omega_1^2 = \frac{d\omega^1}{\omega^1\wedge\omega^2}\omega^1 + \frac{d\omega^2}{\omega^1\wedge\omega^2}\omega^2$

$= (\sqrt{G})_\rho\, d\theta$ $dVol = \sqrt{G}\,d\rho d\theta$

$K = -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}$



$$LHS=\int_0^{\angle A} d\theta \int_0^{f(\theta)} -(\sqrt{G})_{\rho\rho}\, d\rho$$

$$=\int_0^{\angle A} 1-(\sqrt{G})_\rho(f(\theta),\theta)\, d\theta$$

$$=\angle A-\int_0^{L(\alpha)} (\sqrt{G})_\rho(f(\alpha(s)),\theta(s))\cdot\frac{d\theta}{ds}ds$$

$\alpha(s)=(f(\alpha(s)),\theta(s))$

$\alpha'(s)=\underbrace{\langle\alpha'(s),r_\rho\rangle}_{\cos\varphi} r_\rho+\underbrace{\langle\alpha'(s),\frac{r_\theta}{\sqrt{G}}\rangle}_{\sin\varphi}\frac{r_\theta}{\sqrt{G}}$

$\cos\varphi=\langle r_\rho\frac{d\rho}{ds}+r_\theta\frac{d\theta}{ds}, r_\rho\rangle=\frac{d\rho}{ds}$

$\sin\varphi=\sqrt{G}\frac{d\theta}{ds}$

测地线方程 $\Rightarrow 0=\frac{d^2\rho}{ds^2}+\Gamma_{22}^1\left(\frac{d\theta}{ds}\right)^2$ $\left(\Gamma_{22}^1=-\frac{1}{2}\frac{\partial G}{\partial\rho}\right)$ (*)

$\frac{d^2\rho}{ds^2}=\frac{d\cos\varphi}{ds}=-\sin\varphi\frac{d\varphi}{ds}\Rightarrow \sqrt{G}\frac{d\theta}{ds}\frac{d\varphi}{ds}=-\frac{1}{2}\frac{\partial G}{\partial\rho}\left(\frac{d\theta}{ds}\right)^2$

$\Rightarrow\frac{d\varphi}{ds}=-(\sqrt{G})_\rho\frac{d\theta}{ds}$ (**)

左边 $=\angle A+\int_0^{L(\alpha)}\frac{d\varphi}{ds}ds=\angle A+\varphi(L(\alpha))-\varphi(0)$

$=\angle A+\angle C-(\pi-\angle B)=\angle A+\angle B+\angle C-\pi$

(*) 极坐标下 $\Gamma_{12}^2=\Gamma_{21}^2=\frac{1}{2G}G_\rho$ 事实上 $\Gamma_{\alpha\beta}^\gamma=0$.

$\Gamma_{22}^1=-\frac{1}{2}G_\rho$ $\Gamma_{22}^2=\frac{1}{2G}G_\theta$.

(直接带即可)

测地坐标下测地曲率 $k_g=\frac{d\varphi}{ds}+(k_g)_\rho\cos\varphi+(k_g)_\theta\sin\varphi$ ← ρ线为测地线

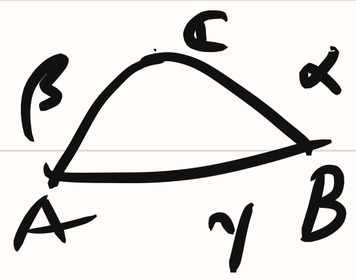
$=\frac{d\varphi}{ds}+(k_g)_\theta\sin\varphi$

$=\frac{d\varphi}{ds}+\frac{G_\rho}{2G}\sin\varphi=\frac{d\varphi}{ds}+(\sqrt{G})_\rho\frac{d\theta}{ds}$

$k_g=\frac{d\varphi}{ds}+(\sqrt{G})_\rho\frac{d\theta}{ds}$

(对为特殊情况)

定理 $\triangle ABC$ 包含在 A的测地极坐标邻域内



α 平均为 $(f(\theta),\theta)$

(GBC1) 则 $\iint_{\triangle ABC} K\,dV + \int_\alpha k_g\,ds = \angle A+\angle B+\angle C-\pi$

[证明 $\iint K\,dV = \int_0^{\angle A}\int_0^{f(\theta)} -(\sqrt{G})_\rho\,\rho\,d\rho\,d\theta$

$= \angle A - \int_0^{l(\alpha)} (\sqrt{G})_\rho(f(\theta),\theta)\frac{d\theta}{ds}ds$ ($k_g = \frac{d\varphi}{ds} + (\sqrt{G})_\rho\frac{d\theta}{ds}$)

$= \angle A + \int_0^{l(\alpha)} \frac{d\varphi}{ds}ds - \int_0^{l(\alpha)} k_g\,ds$

$= \angle A+\angle B+\angle C-\pi-\int_\alpha k_g\,ds.$]

推论 (GBC2) 多边形时 $\int_{\Omega_{A_1\cdots A_n}} K\,dV + \int_{\partial\Omega_{A_1\cdots A_n}} k_g\,ds = 2\pi - \sum(\pi - A_i)$

定理 (GBC3) C为M上分段光滑闭曲线，所围区域落在一个测地极坐标邻域内. 则 $\iint_\Omega K\,dV + \oint_C k_g\,ds + \sum(\pi-A_i) = 2\pi$

(GBC4) C为 、- - - - - 所围区域 Ω 与 R^2 上一个单连通开区域微分同胚. 则 $\iint_\Omega K\,dV + \oint_C k_g\,ds + \sum(\pi-A_i) = 2\pi$

[GBC3: 分割，将边界的 $(f(\theta),\theta)$ 平行消去

GBC4 分割，使每一块都在测地极坐标邻域中]

(要积分技)

GBC 框架下 $\iint_\Omega K\,dV = -\iint_\Omega d\omega_1^2 = -\oint_C \omega_1^2 = -\oint_C \frac{\omega_1^2}{ds}ds$ (*)

设 C 的切向量 $t(s) = \cos\alpha\, e_1 + \sin\alpha\, e_2$.

定义

则 $k_g = \frac{d\alpha}{ds} + \frac{\omega_1^2}{ds} \Rightarrow k_g ds = d\alpha + \omega_1^2$

由 GBC + (*) $\Rightarrow \iint_\Omega K\,dV = \oint_C d\alpha - k_g ds$

$\Rightarrow \iint_\Omega K\,dV + \oint_C k_g ds = \oint_C d\alpha$

$\|$

2π

定理(Hopf) C为光滑简单闭平面曲线 则 $\oint_C k(s)ds = 2\pi = \oint_C d\alpha$

(pf. 平面上 $k=0$ $k_g=k(s)$ 由上述论证即得

等号来源于定义于平面定向等价)

命题 C为M上光滑简单闭曲线. 所围区域D微分同胚于R中

单位开集. $C: r(s)$ 弧参. $V:[0,L(C)]$ 为平行向量场

设 $V(s)=\cos\beta(s)\,e_1(s)+\sin\beta(s)\,e_2(s)$ 则 $\beta(L(C))-\beta(0)=\iint_D K\,dV$

[pf $0=\frac{DV}{ds}=-\sin\beta\frac{d\beta}{ds}e_1+\cos\beta\frac{de_1}{ds}+\cos\beta(s)\frac{w_1^2e_2}{ds}+\sin\beta(s)\frac{w_2^1e_1}{ds}$

$=(-\sin\beta e_1+\cos\beta e_2)\frac{d\beta}{ds}+\frac{w_1^2}{ds}(-\sin\beta e_1+\cos\beta e_2)$

$\exists\ 0=\frac{d\beta}{ds}+\frac{w_1^2}{ds}$ $\times\ k_g=\frac{d\alpha}{ds}+\frac{w_1^2}{ds}$

$\Rightarrow -\frac{d\beta}{ds}=\frac{d\alpha}{ds}-k_g$ $d\beta=d\alpha-k_g ds$

$\beta(L(C))-\beta(0)=\oint_C d\beta=\oint_C d\alpha-k_g ds=\iint_D K dV$]

$\frac{De_1}{ds}=\frac{de_1}{ds}-\langle\frac{de_1}{ds},n\rangle n$

$=\langle\frac{de_1}{ds},e_2\rangle e_2=\frac{w_1^2}{ds}e_2$

光滑曲面 $M=\bigcup_\alpha M_\alpha$ $\begin{cases}\text{每}M_\alpha\text{同胚于圆盘 即存在 } D_\alpha\xrightarrow{U_1}M_\alpha\xrightarrow{R^3}R^3 \text{ 为正则参数化}\\ \text{在}M_\alpha\cap M_\beta\text{中处处 } U_\beta^{-1}\circ U_\alpha\text{光滑}\end{cases}$

三角剖分 0维面. 1维面:线 2维面

M划分为{T_α}. 若$T_{\lambda_1}\cap T_{\lambda_2}\neq\emptyset$ 则$T_{\lambda_1}\cap T_{\lambda_2}$为$T_{\lambda_1}$. T_{λ_2}共有的k维面 ($k=0,1$)

V-E+F: Euler-示性数

定理GBC. M为闭定向曲面 则 $\iint_M kdV=2\pi\chi(M)=2\pi(2-2g)$

[pf. 取三角剖分{T_j}. 则

$\iint_M kdV=\sum_{j=1}^{F}\iint_{T_j}kdV$ (用GBC(4))

$$= \sum_{j=1}^{F} \left(-\underbrace{\int_{\partial T_j} k_g \, ds}_{2\pi} + \angle A_j + \angle B_j + \angle C_j - 3\pi + 2\pi \right)$$

$$= \underbrace{\sum_{j=1}^{F} -\int_{\partial T_j} k_g \, ds}_{\text{(1对顶向内拐角)}} + \underbrace{\sum_{j=1}^{F} \angle A_j + \angle B_j + \angle C_j}_{2\pi V \text{(顶点数)}} - 3\pi F + 2\pi F$$

$= 2\pi V - 3\pi F + 2\pi F$

(3F=2E: 5F=三角形边中边数
=2E (每边2次))

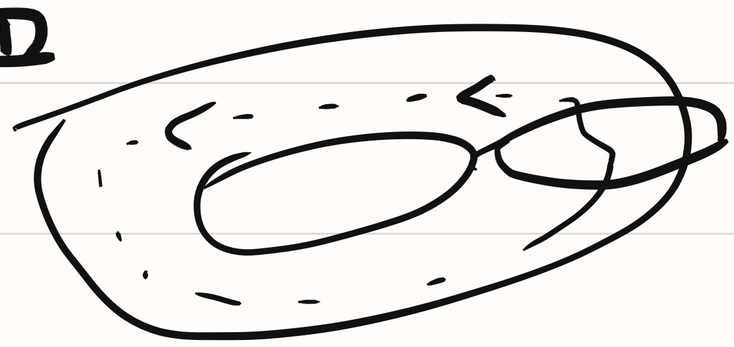
$= 2\pi \chi(M)$ ⌋

一个曲体曲面上是否存在处处非零切向量场?

若存在 我们设此向量场为 e_1 (单位化) 则记 $e_2 = n \wedge e_1$ $e_3 = n$

则 $\iint_M k dV = \iint_M k w^1 \wedge w^2 = -\iint_M dw_1^2 \overset{\text{Stokes}}{=} 0$ ⇒ 环面

反之对环面



存在处处非零向量场

从闭定向曲面 V 光滑向量场 若 $V(p)=0$ 且存在 p 的邻域内除 p 外均非零 则称 p 为孤立奇点

此时 Ω 为 p 的一个邻域 仅包含 p 这一个奇点 且 Ω 包含在此邻域内有向边界 $C = \partial\Omega$ 则 Ω 上总存在处处非零的向量场 ×

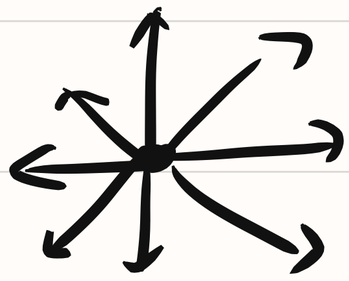
设 $\alpha:[a,b]\to C$ 为参数化 取沿 C 的有向角函数

$$\varphi=L_\alpha(X,V)$$

此时定义 V 在 p 处的指标 $ind(V,p)=\dfrac{\varphi(b)-\varphi(a)}{2\pi}$

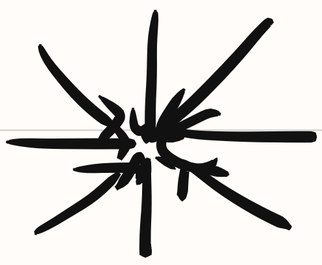
例: X 的许多指向右边

source:

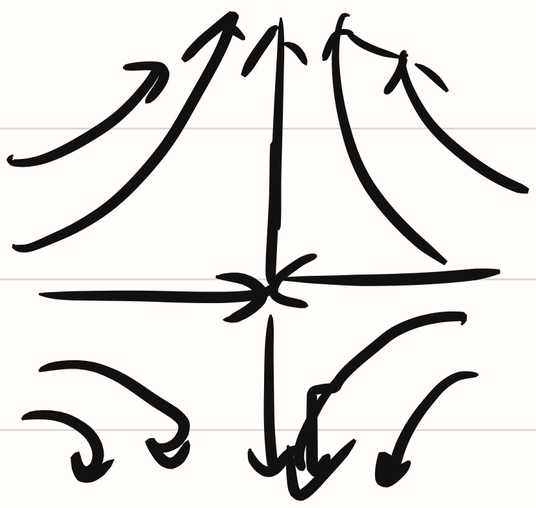


$ind=1$

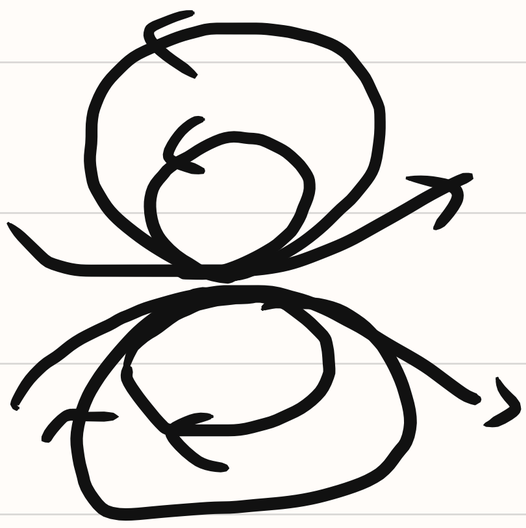
sink:



$ind=1$



$ind=-1$



$ind=2$

定理(Poincaré-Hopf) M 闭定向 V 为光滑的量场 只有孤立奇点 $p_1\cdots p_k$

则 $\chi(M)=\frac{1}{2\pi}\iint_M k\,dV=\sum_{i=1}^{k} ind(V;p_i)$

pf. p_i, D_i, X_i 令 $M=M\setminus\bigcup_{i=1}^{k}D_i$

(i) 在 M 上 V 处处非零. 取 $e_1=\frac{V}{|V|}$ $e_2=n\wedge e_1$

(ii) 在 ∂D_i 上 $e_1=\dfrac{X_i}{|X_i|}$ $e_2=n\wedge e_1$

$$\iiint_M k dV = \iiint_M K\bar{\omega}^1\wedge\bar{\omega}^2 = -\iiint_M d\bar{\omega}_1^2 = -\int_{\partial M}\bar{\omega}_1^2 = \sum_{i=1}^{k}\int_{\partial D_i}\bar{\omega}_i^2$$

设 P_i 是边 C_i 上的向量场. 记 $\bar{\varphi}_i$ 为 V 到 P_i 的方向角

则 $d\bar{\varphi}_i = -\bar{\omega}_i^2$ 沿 C_i 成立

$$-\int_{\partial D_i}\bar{\omega}_i^2 = \int_{\partial D_i} d\bar{\varphi}_i = \bar{\varphi}_i(L(C_i)) - \bar{\varphi}_i(0)$$

再 X_i 与 P_i 夹角为 φ_i 则

$$\iint_{D_i} K dV = -\int_{\partial D_i}\omega_i^2 = \int_{\partial D_i} d\varphi_i = \varphi_i(L(C_i)) - \varphi_i(0)$$

故 $\iint_M K dV = \iint_M K dV + \sum_{i=1}^{k}\iint_{D_i} K dV$

$$= -\sum_{i=1}^{k}\bar{\varphi}_i(L(C_i)) - \bar{\varphi}_i(0) + \sum_{i=1}^{k}\varphi_i(L(C_i)) - \varphi_i(0)$$

$$= \sum_{i=1}^{k}\left((-\bar{\varphi}_i + \varphi_i)(L(C_i)) - (-\bar{\varphi}_i + \varphi_i)(0)\right)$$

$$= \sum_{i=1}^{k}\angle(X_i, V)(L(C_i)) - \angle(X_i, V)(0)$$

$$= \sum_{i=1}^{k} 2\pi \operatorname{ind}(V, P_i).$$

#

Jacobi's remarkable curve Thm.

定理1 C为空间中光滑闭曲线 设其主法向量处处非零 ($\ddot{r}(s)\neq 0$)

若其主法向量的终点在单位球面上形成的曲线(主法标线)

为一条简单闭曲线. 则它将单位球面面积等分

定理2 三条空间曲线给出$\triangle ABC$ 三条曲线的主法向量处处非零且在交点

处两两不在同一大圆弧上 则主法向量在单位球面上得到$\triangle XYZ$ 有 $Area(\triangle XYZ)=\angle A+\angle B+\angle C-\pi$

(定理1为2的推论)

开始定理1 $r=r(s)$ 弧长参数 $\ddot{r}(s)\neq 0$

标架 $e_1=\dot{r}(s)$ $e_2=\frac{\ddot{r}(s)}{|\ddot{r}(s)|}$ $e_3=e_1\wedge e_2$

主法标线 $s\mapsto e_2(s)$ 设ρ为之弧长参数

$$\rho(s_0)=\int_0^{s_0}\left|\frac{de_2}{ds}\right|ds=\int_0^{s_0}\sqrt{k^2+\tau^2}\,ds \qquad \frac{d\rho}{ds}=\sqrt{k^2(s)+\tau^2(s)}$$

只须证 $\oint_{\partial D} k_g\,ds=0$ 即可 (Bonnet-Thm : $\oint_{\partial D} k_g\,ds+\iint_D K\,dV=\underbrace{\angle A+\angle B+\angle C-\pi}_{\text{总为}\pi}$)

∂D 为主法标线 D 为 $\frac{1}{2}$ Area(D) $=2\pi$

$$k_g=\left\langle \frac{d^2e_2(s(\rho))}{d\rho^2},\ -\frac{de_2(s(\rho))}{d\rho}\wedge e_1\right\rangle$$

$$\frac{de_2}{d\rho}=\frac{de_2}{ds}\frac{ds}{d\rho}=(-ke_1+\tau e_3)\frac{ds}{d\rho}\Rightarrow e_2\wedge\frac{de_2}{d\rho}=(ke_3+\tau e_1)\frac{ds}{d\rho}$$

$$\frac{d^2e_2}{d\rho^2}=\frac{d}{ds}\left(\frac{de_2}{ds}\right)\left(\frac{ds}{d\rho}\right)^2+\underbrace{\frac{de_2}{ds}}_{(-ke_1+\tau e_3)}\frac{d}{ds}\left(\frac{ds}{d\rho}\right)\frac{ds}{d\rho}$$

$\Rightarrow$ 第二项

始第次有用 $\frac{d}{ds}(-ke_1+\tau e_3)\left(\frac{ds}{d\rho}\right)^2+\cdots$

$$\frac{d}{ds}(-ke_1+\tau e_3)=-k'e_1+\tau' e_3-k^2e_2-\tau^2e_2$$

$$=\frac{-k'e_1+\tau' e_3-(k^2+\tau^2)e_2}{k^2+\tau^2}+\cdots$$

故 $Kg = \langle (-k'e_1 + \tau' e_3)(\frac{ds}{d\rho})^2, (ke_3 + \tau e_1)\frac{ds}{d\rho} \rangle$

$$= \frac{\tau' k - \tau k'}{\tau^2 + k^2} \frac{ds}{d\rho} = \frac{d}{ds}(\arctan\frac{\tau}{k})\frac{ds}{d\rho}$$

则 $\int_{\partial D} Kg\, d\rho = \int_{\partial D} d(\arctan\frac{\tau}{k}) = 0$ #

球面的刚性

Thm (Liebmann) M 为 $\mathbb{R}^3$ 中紧无边连通 Gauss 曲率 常曲面 则 M 为球面

极值原理: $\mathbb{R}^3$ 中紧无边连通曲面上连续函数 f 在一点处取最大值

若 f 光滑 则对任意曲面上过 p 的光滑曲线 $r = r(s)$, $r(0) = p$

有 $\frac{d}{ds}\big|_{s=0} f(r(s)) = 0$ $\quad \frac{d^2}{ds^2}\big|_{s=0} f(r(s)) \le 0$

引理: M 为紧曲面, 则 $\exists p_0 \in M$ s.t. $K(p_0) > 0$

Pf: $f: M \to \mathbb{R}$, $p \mapsto \langle r(p), r(p) \rangle$. 由极值原理 f 在 p_0 处最大

$\forall$ 弧长方发曲线 $r(s)$, $r(0) = p_0$

$\frac{d}{ds}\big|_{s=0} f(r(s)) = 2\langle \dot{r}(0), r(0) \rangle = 0$ $\xrightarrow{r\text{任意}}$ $r(0) = \lambda n(p_0)$

$0 \ge \frac{d^2}{ds^2}\big|_{s=0} f(r(s)) = 2\frac{d}{ds}\big|_{s=0} \langle \dot{r}(s), r(s) \rangle$

$= 2\langle \ddot{r}(0), r(0) \rangle + 2\langle \dot{r}(0), \dot{r}(0) \rangle$

$= 2(\langle \ddot{r}(0), \lambda n(p_0) \rangle + 1) \Rightarrow \lambda k_n(\dot{r}(0)) \le -1$

则 $K(p_0) = k_1(p_0) k_2(p_0) \ge \frac{1}{\lambda^2}$ #]

引理2(Hilbert) P_0为R³光滑曲面上一点. k_1,k_2主曲率函数

假设 (i) k_1在P_0局部极大 (ii)k_2在P_0局部极小 (3)$k_1(P_0)>k_2(P_0)$

则 $K(P_0)\le 0$

Pf 由(iii)知P_0非脐点. 则在P_0的邻域下 k_1,k_2光滑

曲率线网 $r(u,v)$ $I=Edu\otimes du+Gdv\otimes dv$

$e_1=\frac{r_u}{\sqrt{E}}$ $e_2=\frac{r_v}{\sqrt{G}}$ $e_3=e_1\wedge e_2$

$\omega^1=\sqrt{E}du$ $\omega^2=\sqrt{G}dv$

$\omega\begin{pmatrix}e_1'\\e_2'\end{pmatrix}=\begin{pmatrix}k_1&\\&k_2\end{pmatrix}\begin{pmatrix}e_1\\e_2\end{pmatrix}\Rightarrow\begin{pmatrix}\omega_1^3\\\omega_2^3\end{pmatrix}=\begin{pmatrix}k_1&\\&k_2\end{pmatrix}\begin{pmatrix}\omega^1\\\omega^2\end{pmatrix}=\begin{pmatrix}k_1\sqrt{E}du\\k_2\sqrt{G}dv\end{pmatrix}$

$d\omega_1^3=\omega_1^2\wedge\omega_2^3=k_2\omega_1^2\wedge\omega^2$

$dk_1\wedge\omega^1+k_1\underbrace{d\omega^1}_{\omega^2\wedge\omega_1^1}$

$\Rightarrow 0=dk_1\wedge\omega^1+(k_1-k_2)\omega_1^2\wedge\omega^2$

同理 $dk_2\wedge\omega^2+(k_1-k_2)\omega_1^2\wedge\omega^1$

$dk_1=(k_1)_u du+(k_1)_v dv\Rightarrow dk_1\wedge\omega^1=\frac{(k_1)_v}{\sqrt{G}}\omega^2\wedge\omega^1$

$=-\frac{(k_1)_v}{\sqrt{G}}\omega^1\wedge\omega^2$

$\Rightarrow 0=\left[-\frac{(k_1)_v}{\sqrt{G}}\omega^1+(k_1-k_2)\omega_1^2\right]\wedge\omega^2$

同理 $\left[-\frac{(k_2)_u}{\sqrt{E}}\omega^2+(k_1-k_2)\omega_1^2\right]\wedge\omega^1$

$\omega_1^2=a\omega^1+b\omega^2\Rightarrow -\frac{(k_1)_v}{\sqrt{G}}+(k_1-k_2)a=-\frac{(k_2)_u}{\sqrt{E}}+(k_1-k_2)b=0$

故 $(k_1-k_2)\omega_1^2=\frac{(k_1)_v}{\sqrt{G}}\omega^1+\frac{(k_2)_u}{\sqrt{E}}\omega^2$

$\Rightarrow d(k_1-k_2)\wedge\omega_1^2+(k_1-k_2)d\omega_1^2=d\left(\frac{(k_1)_v}{\sqrt{G}}\omega^1+\frac{(k_2)_u}{\sqrt{E}}\omega^2\right)$

在P_0处取值 $-(\underbrace{k_1(P_0)-k_2(P_0)}_{>0})K(P_0)\omega^1\wedge\omega^2$

$=\left(-\left(\frac{(k_1)_v}{\sqrt{G}}\right)_v\frac{1}{\sqrt{G}}+\left(\frac{(k_2)_u}{\sqrt{E}}\right)_u\frac{1}{\sqrt{E}}\right)\omega^1\wedge\omega^2$

$=\underbrace{\left(-\frac{(k_1)_{vv}}{G}+\frac{(k_2)_{uu}}{E}\right)}_{\ge 0}\omega^1\wedge\omega^2$ at P_0

$\Rightarrow K(P_0)\le 0$ #)

证明：由引理1，知M的Gauss曲率为正常数. 设k_1, k_2为M上的主曲率函数，$k_1 \geq k_2$，可知k_1, k_2为连续函数. 设k_1在P_0达极大值，由$K = k_1 k_2$为常数知k_2在P_0达极小值.

若P_0点非脐点，我们有$k_1(P_0) > k_2(P_0)$，则引理2告诉我们在P_0处，$K(P_0) \leq 0$，矛盾. 因此，P_0点为脐点，即$k_1(P_0) = k_2(P_0)$.

实际上，这意味着M上每点均为脐点：$\forall P \in M$，有

$$k_1(P_0) \geq k_1(P) \geq k_2(P) \geq k_2(P_0) = k_1(P_0) \Rightarrow k_1(P) = k_2(P).$$

回忆全脐点曲面(片)要么是平面要么是球面一部分. 由于$K > 0$，知是球面一部分. 故M为半径为$\frac{1}{\sqrt{K}}$的球面的子集. 紧性推出M为闭子集，M为光滑曲面(每点局部同胚于欧氏开球域)推出M为开子集. 即M为一个半径为$\frac{1}{\sqrt{K}}$的球面既开且闭的非空子集，故而M就是这个球面. □

定理1：设M是E^3中一个紧致连通，Gauss曲率为正的光滑曲面. 如果存在连续函数f，使得M的两个主曲率函数k_1, k_2，$k_1 \geq k_2$满足$k_2 = f(k_1)$. 则M为一个球面.

Gauss曲率$K \equiv C > 0$意味着$k_2 = \frac{C}{k_1}$. 同时，常平均曲率$H \equiv C$意味着$k_2 = 2C - k_1$!

推论：设M是E^3中一个紧致、连通、Gauss曲率为正的光滑曲面. 如果M有常平均曲率，则M为一个球面.

Gauss曲率为正可推出M同胚于球面，也即亏格$g = 0$，(高斯博内)

1950年前后，Hopf证明"$K > 0$"条件可弱化为"同胚于球面"，即E^3中一个紧致连通同胚于球面的光滑曲面若具有常平均曲率，则必为球面.

定义：E^3中的光滑曲面M称为凸曲面，如果M位于每点切平面的同一侧。也即，对$\forall P_0 \in M$，函数 $f: M \to \mathbb{R}$，

$f(P) := \langle r(P) - r(P_0), n(P_0) \rangle$ 不变号（要么$f \geq 0$ 要么$f \leq 0$），其中

$r(P)$为位置向量，$n(P)$为$\cdots$单位法向量.

Hadamard Thm：M紧连通 $K>0$ ⇒ 凸曲面

定理：设M为E^3中紧致(连通)，$K>0$的光滑曲面。 (还不行) (13)

则高斯映射 $g: M \to S^2$ 是1:1映射.

证明：我们先来证明 $g: M \to S^2$ 是满射。也即对任意 $e \in S^2$，

要证明 $\exists P_0 \in M$ s.t. $n(P_0) = g(P_0) = e$.

为此目的，考虑函数 $f: M \to \mathbb{R}$ ：$f(P) = \langle g(P), e \rangle$
$= \langle n(P), e \rangle$.
$\forall P \in M$.

观察到，若 $n(P_0)=e$，则 $f(P_0)=1$，而 $|f(P)| \leq 1$，$\forall P \in M$.

故我们考虑查函数的最大值点 P_0. 取包含P_0的曲面片，

设其有参数化 $r = r(u,v)$ 使得 $\langle r_u, r_v \rangle \equiv 0$. 由极值性质，

$F=0$

知 $df(P_0)=0$. 即

$$0 = d\langle n, e \rangle|_{P_0} = \langle dn, e \rangle|_{P_0} = \langle n_u du + n_v dv, e \rangle|_{P_0}$$

$$= \langle n_u, e \rangle du + \langle n_v, e \rangle dv \,|_{P_0}$$

$$\Rightarrow \begin{cases} 0 = \langle n_u, e \rangle|_{P_0} \\ 0 = \langle n_v, e \rangle|_{P_0} \end{cases} \quad (*)$$

回忆，基从 $\begin{cases} \langle n_u, r_u \rangle = -L \\ \langle n_u, r_v \rangle = -M \end{cases}$ 可解得 $n_u = a r_u + b r_v$，$\begin{cases} aE + bF = -L \\ bG = -M \end{cases}$

即 $n_u = -\frac{L}{E} r_u - \frac{M}{G} r_v$

类似地，有 $n_v = -\frac{M}{E} r_u - \frac{N}{G} r_v$

从而 (*) 可写成 $\begin{cases} 0 = -\frac{L}{E}\langle r_u, e \rangle - \frac{M}{G}\langle r_v, e \rangle \,|_{P_0} \\ 0 = -\frac{M}{E}\langle r_u, e \rangle - \frac{N}{G}\langle r_v, e \rangle \,|_{P_0} \end{cases}$ (**)

由假设 $K(P_0)>0$. 知

$$\det \begin{pmatrix} -\frac{L}{E} & -\frac{M}{G} \\ -\frac{M}{E} & -\frac{N}{G} \end{pmatrix}\Big|_{P_0} = \frac{LN - M^2}{EG}(P_0) > 0.$$

将方程 (4) 得 $\langle r_u, e\rangle(P_0) = \langle r_v, e\rangle(P_0) = 0$.

即 $e = \pm n(P_0)$.

再由 P_0 的极大性，知

$$\begin{cases} (i)\ 0 \geq f_{uu}(P_0) = \left\langle \left(-\frac{L}{E} r_u - \frac{M}{G} r_v\right)_u, e\right\rangle\Big|_{P_0} = \left\langle -\frac{L}{E} r_{uu} - \frac{M}{G} r_{vu}, e\right\rangle\Big|_{P_0} \\ (ii)\ 0 \geq f_{vv}(P_0) = \left\langle \left(-\frac{M}{E} r_u - \frac{N}{G} r_v\right)_v, e\right\rangle\Big|_{P_0} = \left\langle -\frac{M}{E} r_{uv} - \frac{N}{G} r_{vv}, e\right\rangle\Big|_{P_0} \end{cases}$$

若 $e = -n(P_0)$，则 $\langle r_{uu}, e\rangle|_{P_0} = -L$，$\langle r_{uv}, e\rangle|_{P_0} = -M$，

代入 (i) 得 $0 \geq \frac{L^2}{E} + \frac{M^2}{G}$

类似的，(ii) 推出 $0 \geq \frac{M^2}{E} + \frac{N^2}{G}$。也就是 P_0 点处有 $L = M = N = 0$.

这与高斯曲率 $K(P_0) > 0$ 矛盾.

因此，我们推出 $e = n(P_0)$. 满射得证。

下证高斯映射 $g: M \to S^2$ 是单的。反证法：设存在两点 $P, Q \in M$ 使得 $g(P) = g(Q) = e \in S^2$. 注意到

$K = \frac{n_u \wedge n_v}{r_u \wedge r_v}$ 是高斯映射的 Jacobi.

又由 $K > 0$，及反函数定理 知，P, Q 存在邻域 W_P, W_Q s.t.

$g: W_P \to g(W_P)$ 为微分同胚.

$g: W_Q \to g(W_Q)$

不妨设 $W_P \cap W_Q = \phi$，$g(W_P) \subset g(W_Q)$

故 Gauss 映射 g 在 $M \setminus W_P$ 上仍是满射. 故而

$$\int_{M\setminus W_P} K\,dV \geq \int_{S^2} d\sigma \quad (d\sigma \text{ 单位球面面积元})$$

$K\,dV = d\sigma$.

$= 4\pi$.

然而 $\int_M K\,dV = 4\pi$ （这是因为 $K > 0 \Rightarrow \int_M K\,dV > 0 \Rightarrow \chi = 2$）

257' (15)

这导出矛盾 $4\pi = \int_M K dV = \int_{W_p} K dV + \int_{M\setminus W_p} K dV$

$> \int_{M\setminus W_p} K dV \geq 4\pi$. □

定理(Hadamard)的证明:

反证. 若不然, $\exists P_0 \in M$ 使得 $f(p) := \langle r(p) - r(P_0), n(P_0)\rangle$ 变号. 即 $\{p\in M : f(p)>0\}$, $\{p\in M : f(p)<0\}$ 均非空.

设 Q_1, Q_2 分别为 f, $-f$ 在 M 上的最大值点. 类似于引理的证法知 $n(P_0)$ 也与 Q_1, Q_2 处切平面垂直。

也即 Q_1, Q_2, P_0 三点处法向量平行. 则少有其中两个相等。

这与 Gauss 映射为单叶值。 □

出自 Hadamard

定义: 我们称 E^3 中高斯曲率 >0 的紧致曲面为 卵形面 (ovaloid).

注记: 若将定理中假设 "$K>0$" 弱化为 "$K\geq 0$", 定理仍成立. 证明

下面我们再回到整体微分几何的另一基本工具: Stokes 公式.

这里的基本原则如下: 在 E^3 中的紧致曲面 M 上, 若能找到一个整体定义的微分一形式中, 则由 Stokes 公式 $\int_M d\phi = 0$.

若 $d\phi$ 有特别的表达式, 这将推出非常有趣的结论

设 M 为 E^3 中的一个紧致曲面. 考虑其上的形式

(r, n, dn).

其中 r 为位置向量, n 为单位法向量 (E^3中曲面 紧致 ⇒ 可定向). 此1形式是整体定义的. 下面要计算其外微分. 我们更进一步看一下这个1形式.

记 $r=(r^1,r^2,r^3)$, $n=(n^1,n^2,n^3)$. 则有

$$(r,n,dn)=\begin{vmatrix} r^1 & r^2 & r^3 \\ n^1 & n^2 & n^3 \\ dn^1 & dn^2 & dn^3 \end{vmatrix}$$

$$=(r^2n^3-r^3n^2)dn^1-(r^1n^3-r^3n^1)dn^2+(r^1n^2-r^2n^1)dn^3$$

$$=r^2dn^3\wedge dn^1+n^3dr^2\wedge dn^1-r^3dn^2\wedge dn^1-n^2dr^3\wedge dn^1$$

$$+dr^1\wedge(n^2dn^3-n^3dn^2)-dr^1\wedge(n^1dn^3-n^3dn^1)$$

$$+dr^3\wedge(n^1dn^2-n^2dn^1)+2r^1dn^2\wedge dn^3-2r^2dn^1\wedge dn^3+2r^3dn^1\wedge dn^2$$

故 $d(r,n,dn)=d(r^2n^3-r^3n^2)\wedge dn^1-d(r^1n^3-r^3n^1)\wedge dn^2+d(r^1n^2-r^2n^1)\wedge dn^3$

$$\doteq \begin{vmatrix} dr^1 & dr^2 & dr^3 \\ n^1 & n^2 & n^3 \\ dn^1 & dn^2 & dn^3 \end{vmatrix}+\begin{vmatrix} r^1 & r^2 & r^3 \\ dn^1 & dn^2 & dn^3 \\ dn^1 & dn^2 & dn^3 \end{vmatrix}$$

$$=(dr,n,dn)+(r,dn,dn)$$

限制在一个曲面上. 反对称架 $e_1,e_2,e_3=n$

$$(dr,n,dn)=(w_1^1e_1+w_1^2e_2,\ e_3,\ w_3^1e_1+w_3^2e_2)$$

$$=\langle w_1^1e_1+w_1^2e_2,\ e_3\wedge(w_3^1e_1+w_3^2e_2)\rangle$$

$$=\langle w_1^1e_1+w_1^2e_2,\ w_3^1e_2-w_3^2e_1\rangle$$

$$=w_1^1\wedge w_2^3+w_1^3\wedge w_2^1=2H\,w^1\wedge w^2$$

$$(r,dn,dn)=\langle r,(w_3^1e_1+w_3^2e_2)\wedge(w_3^1e_1+w_3^2e_2)\rangle$$

$$=\langle r,\ w_3^1\wedge w_3^2n-w_3^2\wedge w_3^1n\rangle$$

$$=2\langle r,n\rangle w_1^3\wedge w_2^3=2K\langle r,n\rangle w^1\wedge w^2$$

$$\Rightarrow 0=\iint_M d\Phi=\iint_M(2H+2K\langle r,n\rangle)w^1\wedge w^2$$

$$\Rightarrow \iint_M H\,dV=-\iint_M K\langle r,n\rangle dV \qquad \text{(Minkowski 公式①)}$$

$$=\iint_M \varphi K\,dV \qquad \varphi: M\to\mathbb{R}$$

$$p\mapsto -\langle r(p),n(p)\rangle$$

若选 $\Phi=(r,n,dr)$ 可类似可推 $\iint_M dV=\iint_M \varphi H\,dV$ (②)

选中=(r, dr, $\underline{a}$) $\Rightarrow$ $\iint_M \langle n, a\rangle dV = 0$

第1指量

$\phi = (dr, n, a) \ni$ $\iint_M H\langle n, a\rangle dV = 0$

$\phi = (n, dn, a) \ni$ $\iint_M k\langle n, a\rangle dV = 0$

由a的任意性知 $\iint_M n dV = \iint_M H n dV = \iint_M K n dV = 0$

①

再证Liebmann:

M紧、则可取单位法向使得 $\varphi > 0$ 且又由于 $\exists P$ s.t. $K(P) > 0 \ni K \equiv C > 0$

且 $H^2 = \left(\frac{k_1+k_2}{2}\right)^2 \geq K = k_1 k_2$ "=" $\Leftrightarrow k_1 = k_2$

$\Rightarrow \iint_M k\varphi dV = \iint_M H dV \geq \iint_M \sqrt{K} dV = \sqrt{K} \iint_M dV = \sqrt{K} \iint_M H\varphi dV \geq \iint_M k\varphi dV$

$\Rightarrow$ 取等 $\ni k_1 = k_2 \ni$ 闭无脐曲面 $\ni$ 球面

②再证推论: M紧连通 $K > 0$ H等值 $\ni$ 球面

Hadamard $\ni$ 为

$\iint_M k\varphi dV = \iint_M H dV = H^2 \iint_M \varphi dV \geq \iint k\varphi dV$

取等 $\ni$ 球面

φ刚面的支撑函数:

$g: M \to S^2$ --映射

$S^2 \xrightarrow{g^{-1}} M \xrightarrow{r} E^3$

故φ可视为S^2上函数

$\xi \mapsto r(g^{-1}(\xi))$

反S^2上标架 $[r; e_1, e_2, e_3]$

$\xi = e_3$

对应 $\theta^1, \theta^2, \theta_1^3, \theta_2^3, \theta_1^2$

$d\varphi = -\langle dr, n\rangle - \langle r, dn\rangle$

$\varphi_1'\theta^1 + \varphi_2\theta^2$

$\because dn = d\xi = \theta^1 e_1 + \theta^2 e_2$

$d(r\circ g^{-1})(v_p) = dr\left((g^{-1})_*(v_p)\right) \perp n \Rightarrow \langle dr, n\rangle = 0$

故 $\varphi_1\theta^1 + \varphi_2\theta^2 = -\langle r, e_1\rangle\theta^1 - \langle r, e_2\rangle\theta^2$

$\Rightarrow r = \langle r, e_1\rangle e_1 + \langle r, e_2\rangle e_2 + \langle r, e_3\rangle e_3$

$= -\varphi_1 e_1 - \varphi_2 e_2 - \varphi e_3$

若 $e_3 = n$ 与 $\bar{n} = \bar{e}_3$ 仅依赖 e_1, e_2

若 $e_3 = -n$ 与 $\bar{n} = \bar{e}_3$ 则差一个负号时 φ^*

定理(Cohn-Vossen): E^3 中的两个等距的闭曲面是合同的

pf. $M, \bar{M}$ 闭曲面 $f: M \to \bar{M}$ 等距

$\{r; e_1, e_2, e_3\}$ $\{\bar{r}; \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ 其中 $f_* e_1 = \bar{e}_1$ $f_* e_2 = \bar{e}_2$ (初值差一个合同变换)

只需证 $\omega^1 = f^*\bar{\omega}^1$ $\omega^2 = f^*\bar{\omega}^2$ $\omega_1^3 = f^*\bar{\omega}_1^3$ $\omega_2^3 = f^*\bar{\omega}_2^3$ $(\det f_* > 0)$ $(\det f_* < 0)$

(即证运动方程的系数相同 又由为线性方程 $\Rightarrow$ 整体关合同变换)

(ω_1^2 只由 ω^1, ω^2 决定)

$f^*\bar{\omega}^\alpha(e_\beta) = \bar{\omega}^\alpha(f_* e_\beta) = \bar{\omega}^\alpha(\bar{e}_\beta) = \delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2) \Rightarrow f^*\bar{\omega}^\alpha = \omega^\alpha$ ✓

$$\begin{pmatrix}\omega_1^2\\ \omega_1^3\end{pmatrix} = \begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \qquad \begin{pmatrix}\bar{\omega}_1^2\\ \bar{\omega}_1^3\end{pmatrix} = \begin{pmatrix}\bar{h}_{11} & \bar{h}_{12}\\ \bar{h}_{21} & \bar{h}_{22}\end{pmatrix}\begin{pmatrix}\bar{\omega}^1\\ \bar{\omega}^2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}f^*\bar{\omega}_1^2\\ f^*\bar{\omega}_1^3\end{pmatrix} = \begin{pmatrix}\bar{h}_{11}\circ f & \bar{h}_{12}\circ f\\ \bar{h}_{21}\circ f & \bar{h}_{22}\circ f\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}$$

$\Rightarrow$ 只需证 $\begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix} = \operatorname{sgn}(\det f)\begin{pmatrix}\bar{h}_{11}\circ f & \bar{h}_{12}\circ f\\ \bar{h}_{21}\circ f & \bar{h}_{22}\circ f\end{pmatrix}$

引理: $\begin{pmatrix}\lambda & \mu\\ \mu & \nu\end{pmatrix}$ $\begin{pmatrix}\bar\lambda & \bar\mu\\ \bar\mu & \bar\nu\end{pmatrix}$ 正定且行列式相同 则 $\det\begin{pmatrix}\lambda-\bar\lambda & \mu-\bar\mu\\ \mu-\bar\mu & \nu-\bar\nu\end{pmatrix}\le 0$

取等当且仅当两矩阵相同

B

Lpf: 设 ABA^T 对角

$$A\begin{pmatrix}\lambda & \mu\\ \mu & \nu\end{pmatrix}A^T=\begin{pmatrix}\lambda^* & \mu^*\\ \mu^* & \nu^*\end{pmatrix}$$

$$A\begin{pmatrix}\bar\lambda & \bar\mu\\ \bar\mu & \bar\nu\end{pmatrix}A^T=\begin{pmatrix}\bar\lambda^* & \bar\mu^*\\ \bar\mu^* & \bar\nu^*\end{pmatrix}$$

$\Rightarrow \mu^*=\bar\mu^*$

$\exists\ \lambda^*\nu^*=\bar\lambda^*\bar\nu^*$

$\lambda^*,\nu^*,\bar\lambda^*,\bar\nu^*>0$

$\det(ABA^T)=\det A^2\det B$

$(\lambda^*-\bar\lambda^*)(\nu^*-\bar\nu^*)=-\frac{\mu^*}{\lambda^*}(\bar\lambda^*-\lambda^*)^2\le 0$ $\Rightarrow \det B\le 0$

取等 $\Leftrightarrow \bar\lambda^*=\lambda^*\Rightarrow \mu^*=\bar\mu^*$

$\nu^*=\bar\nu^*$

$\Leftrightarrow B=0$ #]

考虑 M 上的1形式 $\phi=(r,n,f^*\bar\omega_3^1e_1+f^*\bar\omega_3^2e_2)$

整体定义? 只须证 $f^*\bar\omega_3^1e_1+f^*\bar\omega_3^2e_2$ 不依赖于标架

设 M 上另有标架 $\{r;e_1',e_2',e_3=n\}$

$$\begin{pmatrix}e_1'\\ e_2'\end{pmatrix}=\underbrace{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}}_{R_\theta}\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

$$\begin{pmatrix}\bar e_1'\\ \bar e_2'\end{pmatrix}=\begin{pmatrix}f^*e_1'\\ f^*e_2'\end{pmatrix}=R_\theta\begin{pmatrix}\bar e_1\\ \bar e_2\end{pmatrix}$$

$$\begin{pmatrix}\bar\omega_3^{1\prime}\\ \bar\omega_3^{2\prime}\end{pmatrix}=\begin{pmatrix}\langle d\bar e_3,\bar e_1'\rangle\\ \langle d\bar e_3,\bar e_2'\rangle\end{pmatrix}=R_\theta\begin{pmatrix}\langle d\bar e_3,\bar e_1\rangle\\ \langle d\bar e_3,\bar e_2\rangle\end{pmatrix}=R_\theta\begin{pmatrix}\bar\omega_3^1\\ \bar\omega_3^2\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}f^*\bar\omega_3^{1\prime}\\ f^*\bar\omega_3^{2\prime}\end{pmatrix}=R_\theta\begin{pmatrix}f^*\bar\omega_3^1\\ f^*\bar\omega_3^2\end{pmatrix}$$

$$\Rightarrow(f^*\bar\omega_3^{1\prime}\ \ f^*\bar\omega_3^{2\prime})\begin{pmatrix}e_1'\\ e_2'\end{pmatrix}$$

$$=(f^*\bar\omega_3^1\ \ f^*\bar\omega_3^2)\underbrace{R_\theta^TR_\theta}_{Id}\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$ ✓

Stokes $\Rightarrow \int_M d\phi=0$

$$0=(dr,n,f^*\bar\omega_3^1e_1+f^*\bar\omega_3^2e_2)=\langle\omega^1e_1+\omega^2e_2,\ n\wedge(f^*\bar\omega_3^1e_1+f^*\bar\omega_3^2e_2)\rangle$$

$$=\omega^2\wedge f^*\bar\omega_3^1-\omega^1\wedge f^*\bar\omega_3^2$$

$$=\omega^1\wedge f^*\bar\omega_2^3+f^*\bar\omega_1^3\wedge\omega^2$$

$$\alpha\begin{pmatrix} f^*\bar{\omega}_1^3 \\ f^*\bar{\omega}_2^3 \end{pmatrix} = \begin{pmatrix} \bar{h}_{11}\circ f & \bar{h}_{12}\circ f \\ \bar{h}_{21}\circ f & \bar{h}_{22}\circ f \end{pmatrix}\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$$

$$\Rightarrow \quad ① = (\bar{h}_{22}\circ f + \bar{h}_{11}\circ f)\,\omega^1\wedge\omega^2 = 2(\bar{H}\circ f)\,\omega^1\wedge\omega^2$$

$$② = (r, dn, f^*\bar{\omega}_3^1 e_1 + f^*\bar{\omega}_3^2 e_2)$$

$$= \langle r, (\omega_3^1 e_1 + \omega_3^2 e_2)\wedge(f^*\bar{\omega}_3^1 e_1 + f^*\bar{\omega}_3^2 e_2)\rangle$$

$$= \langle r, \omega_3^1 f^*\bar{\omega}_3^2 n - \omega_3^2 f^*\bar{\omega}_3^1 n\rangle = -\varphi(\omega_1^3\wedge f^*\bar{\omega}_2^3 + f^*\bar{\omega}_1^3\wedge\omega_2^3)$$

$$= -\varphi\left[\det\begin{pmatrix} h_{11} & h_{12} \\ \bar{h}_{21}\circ f & \bar{h}_{22}\circ f \end{pmatrix} + \det\begin{pmatrix} \bar{h}_{11}\circ f & \bar{h}_{12}\circ f \\ h_{21} & h_{22} \end{pmatrix}\right]\omega^1\wedge\omega^2$$

$$= -\varphi(2k - B)\,\omega^1\wedge\omega^2$$

$$\left[\det\begin{pmatrix} h_{11}-\bar{h}_{11}\circ f & h_{12}-\bar{h}_{12}\circ f \\ h_{21}-\bar{h}_{21}\circ f & h_{22}-\bar{h}_{22}\circ f \end{pmatrix} \triangleq B\right.$$

$$\left. = \underbrace{\det\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} + \det\begin{pmatrix} \bar{h}_{11}\circ f & \bar{h}_{12}\circ f \\ \bar{h}_{21}\circ f & \bar{h}_{22}\circ f \end{pmatrix}}_{2k} - \det\begin{pmatrix} h_{11} & h_{12} \\ \bar{h}_{21}\circ f & \bar{h}_{22}\circ f \end{pmatrix} - \det\begin{pmatrix} \bar{h}_{11}\circ f & \bar{h}_{12}\circ f \\ h_{21} & h_{22} \end{pmatrix}\right]$$

$$③ = (r, n, d(f^*\bar{\omega}_3^1 e_1 + f^*\bar{\omega}_3^2 e_2))$$

$$= (r, n, \underbrace{de_1}_{\omega_1^2 e_2 + \omega_1^3 e_3}\wedge f^*\bar{\omega}_3^1 + e_1\underbrace{d(f^*\bar{\omega}_3^1)}_{f^*(d\bar{\omega}_3^1)} + \underbrace{de_2}_{\omega_2^1 e_1 + \omega_2^3 e_3}\wedge f^*\bar{\omega}_3^2 + e_2\underbrace{d(f^*\bar{\omega}_3^2)}_{f^*(d\bar{\omega}_3^2)})$$

$$= (r, n, (f^*\bar{\omega}_3^2\wedge\underbrace{f^*\bar{\omega}_2^1}_{\omega_2^1} + \omega_2^1\wedge f^*\bar{\omega}_3^2)e_1 + (\omega_1^2\wedge f^*\bar{\omega}_3^1 + f^*\bar{\omega}_3^1\wedge\underbrace{f^*\bar{\omega}_1^2}_{\omega_1^2})e_2 + \cdots e_3)$$

$=0$

$\det(f_x) > 0$ 时. 反原点 $\exists n$ s.t. $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{21} \end{pmatrix}$ 改 $\Rightarrow \begin{pmatrix} \bar{h}_{11}\circ f & \bar{h}_{12}\circ f \\ \bar{h}_{21}\circ f & \bar{h}_{22}\circ f \end{pmatrix}$ 改.

$$\Rightarrow \int_M d\phi = 0 = 2\int_M \bar{H}\circ f\,dV - 2\int_M k\varphi dV + \int_M \varphi B\,dV.$$

$$\Rightarrow 2\int_M H - \bar{H}\circ f\,dV = \int_M \varphi\underbrace{B}_{\geq 0(\text{引理})}dV \leq 0$$

同理 $2\int_M \underbrace{\bar{H} - H\circ f^{-1}}_{(\bar{H}\circ f^{-1} - H)\circ f^{-1}} dV \leq 0$ (反 $\varphi > 0$)

$$\Rightarrow \int_M H - \bar{H}\circ f\,dV = 0$$

$$\Rightarrow B = 0 \quad \checkmark$$

$\det(f_{ik})<0$ $\varphi>0$ $H>0$ $\bar{M}$上 $\bar{\varphi}>0$ $\bar{H}>0$

(Bijie) $\phi=(r,n,f^*\bar{\omega}_3^1 e_1+f^*\bar{\omega}_3^2 e_2)=2(\bar{H}\circ f)\omega^1\wedge\omega^2-\varphi(C-2|c|)\omega^1\wedge\omega^2$

$\det\begin{pmatrix} h_{11}+\bar{h}_{11}\circ f & h_{12}+\bar{h}_{12}\circ f \\ h_{21}+\bar{h}_{21}\circ f & h_{22}+\bar{h}_{22}\circ f \end{pmatrix}\geq C\leq 0$ (因为 $\det\begin{pmatrix} -\bar{h}_{11}\circ f & -\bar{h}_{12}\circ f \\ -\bar{h}_{21}\circ f & -\bar{h}_{22}\circ f \end{pmatrix}$ 正定)

$\Rightarrow 0=\int_M d\phi=2\int_M \bar{H}\circ f\,dV+2\int_M k\varphi\,dV-\int_M \varphi C\,dV$

$\Rightarrow 2\int_M(\bar{H}\circ f+H)\,dV=\int_M \varphi C\,dV\leq 0$

同理 $\int_{\bar{M}} H\circ f^{-1}+\bar{H}\,d\bar{V}=\int_{\bar{M}}\bar{\varphi}\det(\bar{h}_{ij}+h_{ij}\circ f^{-1})\,d\bar{V}\geq 0$

$\int_M H+\bar{H}\circ f\,dV$ $\Rightarrow \int_M H+\bar{H}\circ f\,dV=0$

$\Rightarrow C=0\to h_{ij}=-\bar{h}_{ij}\circ f$ #J

Minkowski问题：给定 $K_0: S^2\to\mathbb{R}_{>0}$ 光滑，是否$\exists$闭曲面 M s.t. $\kappa\circ g^{-1}=K_0$？

唯一性

定理：$M,\bar{M}$ 为两个闭曲面，若$\exists$光滑 $f: S^2\to\mathbb{R}_{>0}$ s.t. $\kappa\circ g^{-1}=\bar{\kappa}\circ\bar{g}^{-1}=f$

则 $M,\bar{M}$ 相差一个平移

pf 在 M 的曲面片上选标架 $(r,e_1,e_2,e_3=n)$

$\forall p\in M$ $p\ g(p)$ $\bar{g}^{-1}(g(p))$ 处单位法向量判断

(e_1,e_2) 平移 S^2 平移 $(\bar{e}_1,\bar{e}_2)$ 为 $\bar{M}$ 上 $\bar{g}^{-1}(g(p))$ 处标架

M 上 1 形式 ω $\bar{M}$ 上 1 形式 $\bar{\omega}$ S^2 上 1 形式 θ

运动方程 初值差一个平移 + 方程相同 $\longrightarrow$ 整体差平移

只须证 $\omega^1=(\bar{g}^{-1}\circ g)^*\bar{\omega}^1$ $\omega^2=(\bar{g}^{-1}\circ g)^*\bar{\omega}^2$

$\omega_1^3=(\bar{g}^{-1}\circ g)^*\bar{\omega}_1^3$ $\omega_2^3=(\bar{g}^{-1}\circ g)^*\bar{\omega}_2^3$

$\Leftrightarrow$ $\begin{cases} (g^{-1})^* \omega^1 = (\bar{g}^{-1})^* \bar{\omega}^1 \\ (g^{-1})^* \omega^2 = (\bar{g}^{-1})^* \bar{\omega}^2 \end{cases}$ (I) $\begin{cases} (g^{-1})^* \omega_1^3 = (\bar{g}^{-1})^* \bar{\omega}_1^3 \\ (g^{-1})^* \omega_2^3 = (\bar{g}^{-1})^* \bar{\omega}_2^3 \end{cases}$ (II)

$\forall S^2$ 上光滑曲线 $c(t)$. $\frac{d}{dt} e_3(c(t)) = \frac{d}{dt} e_3(g^{-1}(c(t)))$

$\langle \frac{d}{dt} e_3(c(t)), e_\alpha(c(t)) \rangle = \langle \frac{d}{dt} e_3(g^{-1}(c(t))), e_\alpha(g^{-1}(c(t))) \rangle$

$(\frac{d}{dt} c(t) = v)$ $\Rightarrow \omega_3^\alpha((g^{-1})_* v) = \theta_3^\alpha(v)$ $\forall v \in T_p S^2$ $\alpha = 1, 2$

$\Rightarrow (g^{-1})^* \omega_3^\alpha = \theta_3^\alpha$ 同理 $(\bar{g}^{-1})^* \bar{\omega}_3^\alpha = \theta_3^\alpha$ $\Rightarrow$ (IV)

从上 $\begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}$ $K > 0 \Rightarrow \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ 正 $= \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$

(也可作复数 $\lambda_{\alpha\beta} > 0$)

$\Rightarrow \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} \Rightarrow \begin{pmatrix} (g^{-1})^* \omega^1 \\ (g^{-1})^* \omega^2 \end{pmatrix} = \begin{pmatrix} \lambda_{11} \circ g^{-1} & \lambda_{12} \circ g^{-1} \\ \lambda_{21} \circ g^{-1} & \lambda_{22} \circ g^{-1} \end{pmatrix} \begin{pmatrix} (g^{-1})^* \omega_1^3 \\ (g^{-1})^* \omega_2^3 \end{pmatrix}$

只须证 $\underbrace{\lambda_{\alpha\beta} \circ g^{-1}}_{\text{正定}} = \underbrace{\bar{\lambda}_{\alpha\beta} \circ \bar{g}^{-1}}_{\text{正定}}$

同所 $\det(\lambda_{\alpha\beta} \circ g^{-1} - \bar{\lambda}_{\alpha\beta} \circ \bar{g}^{-1}) \overset{C}{=} 0$

再考虑 $\Psi = (r \circ g^{-1}, \bar{r} \circ \bar{g}^{-1}, d(\bar{r} \circ \bar{g}^{-1}))$ on S^2 的形式

$d\Psi = (d(r \circ g^{-1}), \bar{r} \circ \bar{g}^{-1}, d(\bar{r} \circ \bar{g}^{-1})) + (r \circ g^{-1}, d(\bar{r} \circ \bar{g}^{-1}), d(\bar{r} \circ \bar{g}^{-1})) = A + B$

$d(r \circ g^{-1})(v) = \frac{d}{dt}|_{t=0} (r \circ g^{-1})(c(t)) = \frac{d}{dt}|_{t=0} r(g^{-1}(c(t))) = dr((g^{-1})_* v)$

$v \in T_p S^2$

$= (g^{-1})^* (dr)(v)$

$\Rightarrow d(r \circ g^{-1}) = (g^{-1})^* dr = (g^{-1})^* (\omega^1 e_1 + \omega^2 e_2)$

$= (g^{-1})^* \omega^1 e_1 + (g^{-1})^* \omega^2 e_2$

$A = ((g^{-1})^* \omega^1 e_1 + (g^{-1})^* \omega^2 e_2, \bar{r} \circ \bar{g}^{-1}, (\bar{g}^{-1})^* \bar{\omega}^1 e_1 + (\bar{g}^{-1})^* \bar{\omega}^2 e_2)$

$= (g^{-1})^* \omega^1 \wedge (\bar{g}^{-1})^* \bar{\omega}^2 (e_1, \bar{r} \circ \bar{g}^{-1}, e_2) + (g^{-1})^* \omega^2 \wedge (\bar{g}^{-1})^* \bar{\omega}^1 e_1 (e_2, \bar{r} \circ \bar{g}^{-1}, e_1)$

$$\chi(e_1, \bar{r}\circ\bar{g}^{-1}, e_2) = -\langle \bar{r}\circ\bar{g}^{-1}, e_1\wedge e_2\rangle = -\langle \bar{r}\circ\bar{g}^{-1}, n\rangle = \bar{\varphi}\circ\bar{g}^{-1}$$

$$(e_2, \bar{r}\circ\bar{g}^{-1}, e_1) = -\bar{\varphi}\circ\bar{g}^{-1}$$

$$\Rightarrow A = \bar{\varphi}\circ\bar{g}^{-1}\left((g^{-1})^*\omega^1\wedge(\bar{g}^{-1})^*\bar{\omega}^2 - (g^{-1})^*\omega^2\wedge(\bar{g}^{-1})^*\bar{\omega}^1\right)$$

$$= \bar{\varphi}\circ\bar{g}^{-1}\Big(\underbrace{\begin{pmatrix}\lambda_1\circ g^{-1} & \lambda_2\circ g^{-1}\\ \bar{\lambda}_2\circ\bar{g}^{-1} & \bar{\lambda}_2\circ\bar{g}^{-1}\end{pmatrix} - \begin{pmatrix}\bar{\lambda}_1\circ\bar{g}^{-1} & \bar{\lambda}_2\circ\bar{g}^{-1}\\ \lambda_2\circ g^{-1} & \lambda_2\circ g^{-1}\end{pmatrix}}_{D}\Big)\theta_1^3\wedge\theta_2^3$$

$$C = -D + \begin{pmatrix}\lambda_1\circ g^{-1} & \lambda_2\circ g^{-1}\\ \lambda_2\circ g^{-1} & \lambda_2\circ g^{-1}\end{pmatrix} + \begin{pmatrix}\bar{\lambda}_1\circ\bar{g}^{-1} & \bar{\lambda}_2\circ\bar{g}^{-1}\\ \bar{\lambda}_2\circ\bar{g}^{-1} & \bar{\lambda}_2\circ\bar{g}^{-1}\end{pmatrix} = D + \frac{1}{\kappa\circ g^{-1}} + \frac{1}{\bar{\kappa}\circ\bar{g}^{-1}}$$

$$= D + \frac{2}{\kappa}$$

$$B = (r\circ g^{-1}, (\bar{g}^{-1})^*\bar{\omega}^1 e_1 + (\bar{g}^{-1})^*\bar{\omega}^2 e_2, (\bar{g}^{-1})^*\bar{\omega}^1 e_1 + (\bar{g}^{-1})^*\bar{\omega}^2 e_2)$$

$$= -2\varphi\circ g^{-1}\left((\bar{g}^{-1})^*\bar{\omega}^1\wedge(\bar{g}^{-1})^*\bar{\omega}^2\right) = -2\varphi\circ g^{-1}\frac{1}{\bar{\kappa}\circ\bar{g}^{-1}}$$

$$= -\varphi\circ g^{-1}\frac{2}{\kappa}$$

$$\Rightarrow d\psi = \bar{\varphi}\circ\bar{g}^{-1}\left(-C + \frac{2}{\kappa}\right)\theta_1^3\wedge\theta_2^3 - \varphi\circ g^{-1}\frac{2}{\kappa}\theta_1^3\wedge\theta_2^3$$

$$= \left[\frac{2}{\kappa}\left(\bar{\varphi}\circ\bar{g}^{-1} - \varphi\circ g^{-1}\right) - \bar{\varphi}\circ\bar{g}^{-1}C\right]\theta_1^3\wedge\theta_2^3$$

$$\Rightarrow \int_{S^2}\frac{2}{\kappa}\left(\bar{\varphi}\circ\bar{g}^{-1} - \varphi\circ g^{-1}\right)dV = \int_{S^2}\underbrace{\bar{\varphi}\circ\bar{g}^{-1}}_{\text{反}>0}\underbrace{C}_{\leq 0}\,dV \leq 0$$

对称性 $\int_{S^2}\frac{2}{\kappa}\left(\varphi\circ g^{-1} - \bar{\varphi}\circ\bar{g}^{-1}\right)dV \leq 0 \Rightarrow C = 0$. 得证. #

Hilbert定理. E^3中不存在常负Gauss曲率的完备曲面

完备 M为E^3中光滑曲面. 定义 $d(P,Q) = \inf_\gamma l(\gamma)$

γ取遍连接P,Q的分段光滑曲线

引理. M光滑连通 则P,Q可用分段光滑曲线连接

Pf. $\forall P$ $A_P = \{Q\in M \mid P,Q$ ... $\}\neq\Phi$ $(P\in A_P)$

① A_P开: $\forall P_0\in A_P$ 取包含P_0的光滑曲面片 $M_{P_0}\Rightarrow M_{P_0}\subset A_P$

② A_P闭. $\forall P_0\notin A_P$. 则仍然 $M_{P_0}\subset A_P^c$ #]

引理2 ① $d(P,Q)=d(Q,P)$ ② $d(P,Q)+d(Q,R)\geq d(P,R)$

③ $d(P,Q)\geq 0$ 且 $d(P,Q)=0 \Leftrightarrow P=Q$

pf 只须证③ $\forall P\neq Q$

$\exists$ 充分小的 s.t. 测地球 $B(P,\varepsilon)$ s.t. $Q\notin B(P,\varepsilon)$

$\forall$ 连接 P,Q 的分段光滑曲线 $\gamma:[0,1]\to M$

$t_0=\inf\{t\in(0,1] \mid \gamma(t)\notin B(P,\varepsilon)\} \Rightarrow l(\gamma)\geq l(\gamma_{|[0,t_0]})=\varepsilon>0$

$\Rightarrow d(P,Q)\geq\varepsilon>0$ #]

命题 若 M 完备 则 $\forall P\in M$ $\exp_P$ 定义在整个 T_PM 上

pf 倒推 $\forall v\in T_PM$ 单位长 s.t. $\gamma_v(t)$ 只定义在 $(0,s_0)$

设 $P=P_0,P_1,\cdots$ 为 Cauchy 列 $\forall P_n\to Q$

取 $B(Q,r)$ 为闭测地球且 $\overline{B(Q,r)}$ 紧

$\Rightarrow \exists \varepsilon_0>0$ s.t. $\forall P\in\overline{B(Q,r)}$ 开测地球 $B(P,\varepsilon_0)$

又 $P_n\to Q$ $\exists n_0$ s.t. $d(P_{n_0},Q)<\frac{\varepsilon_0}{2}$ $\Rightarrow P_{n_0}$ 处得指标 $\leq$

$\Rightarrow \gamma_v$ 总在 $[0,s_0+\frac{\varepsilon_0}{2}]$ 内]

Rmk 测地完备 $\leftrightarrow \forall P\in M$ $\exp_P$ 总在 T_PM 上

$\updownarrow$ Hopf-Rinow

反量完备

$\updownarrow$

$\exists P\in M,\cdots$

下面证 Hilbert Thm

首先若 $K\equiv -c^2$ 则 rescale $r\mapsto cr$

有 $K=\frac{LN-M^2}{EG-F^2}=\frac{c^2}{c^4}K_0=-1$.

故不妨设 $K\equiv -1$

定义：$r:(a,b)\times(c,d)\to M$ 为一个 Chebyshev 网 若 $|r_u|=|r_v|=1$

进一步若 r_u, r_v 的为渐近线 则称为渐近 Chebyshev 网

引理3： M 为 $K\equiv -1$ 的曲面 $\forall p\in M$ $\exists \varepsilon>0$ 和曲面片 $r:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\to M$

s.t. $r(0,0)=p$ 且 r 为渐近 Chebyshev 网

pf. $K\equiv -1$ $\Rightarrow$ 无脐点 $\Rightarrow$ p 可邻域在曲率线网

(假设 $k_1>0>k_2$)

即 $r:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\to M$

$(0,0)\mapsto p$

$\langle r_u, r_v\rangle = 0$ r_u, r_v 为主方向

$e_1=\frac{r_u}{|r_u|}$ $e_2=\frac{r_v}{|r_v|}$ $e_3=e_1\wedge e_2$ 渐进方向 $\cos\varphi e_1+\sin\varphi e_2$

$k_2=-\cot\varphi$ $k_1=\tan\varphi$

$k_1\cos^2\varphi+k_2\sin^2\varphi=0$

$\Rightarrow k_1^2\cos^2\varphi-\sin^2\varphi=0$

源栈 ξ,η s.t. $\begin{cases} r_\xi=\frac{\cos\varphi}{\sqrt{E}}r_u \\ r_\eta=\frac{\sin\varphi}{\sqrt{G}}r_v \end{cases}$ $\Leftrightarrow$ 解 $\begin{cases} d\xi=\frac{\sqrt{E}}{\cos\varphi}du \\ d\eta=\frac{\sqrt{G}}{\sin\varphi}dv \end{cases}$

check 可积性条件 $\begin{cases} d^1\xi=\left(\frac{\sqrt{E}}{\cos\varphi}\right)_v dv\wedge du \\ d^2\eta=\left(\frac{\sqrt{G}}{\sin\varphi}\right)_u du\wedge dv \end{cases}$

$\omega^1=\sqrt{E}du$
$\omega^2=\sqrt{G}dv$
$\Rightarrow \omega_1^2=-\frac{(\sqrt{E})_v}{\sqrt{G}}du+\frac{(\sqrt{G})_u}{\sqrt{E}}dv$

$\omega_1^3=k_1\omega^1$ $\omega_2^3=k_2\omega^2$

$d\omega_1^3=(k_1\sqrt{E})_v dv\wedge du$
$''\omega_1^2\wedge\omega_2^3=-k_2(\sqrt{E})_v du\wedge dv$

$\Rightarrow (k_1)_v\sqrt{E}+(k_1-k_2)(\sqrt{E})_v=0$

同理 $(k_2)_u\sqrt{G}-(k_1-k_2)(\sqrt{G})_u=0$

代入 $k_1=\tan\varphi$ $k_2=-\cot\varphi$

$\Rightarrow 0=\frac{\varphi_v}{\cos^2\varphi}\sqrt{E}+\frac{(\sqrt{E})_v}{\cos^2\varphi\tan\varphi}$

$=\frac{1}{\sin\varphi}\left(\frac{\sqrt{E}}{\cos\varphi}\right)_v$

同理 $-\frac{1}{\cos\varphi}\left(\frac{\sqrt{G}}{\sin\varphi}\right)_u$ $\Rightarrow$ 可积性条件 ✓

$\Rightarrow r = r(u,v) = \bar{r}(\xi(u,v), \eta(u,v))$ 此时 $\bar{r}_\xi = \frac{\cos\varphi}{|e|} r_u$

$\bar{r}_\eta = \frac{\sin\varphi}{|G|} r_v$

再 $\begin{cases} \xi = s+t \\ \eta = s-t \end{cases} \Rightarrow$ s,t 为所求参数 #]

对 $r(\xi,\eta)$ 如上 $\omega^1, \omega^2, \omega^3, \omega^3_2, \omega^1_1$

$\omega^1 = \cos\varphi d\xi$

$\omega^2 = \sin\varphi d\eta$

由 $d\omega^2 = -\frac{(\cos\varphi)_\eta}{\cos\varphi\sin\varphi}\omega^1 + \frac{(\sin\varphi)_\xi}{\cos\varphi\sin\varphi}\omega^2$

$= \varphi_\eta d\xi + \varphi_\xi d\eta$

$\omega^3_1 = k_1\omega^1 = \sin\varphi d\xi$

$\omega^3_2 = -\cos\varphi d\eta$

Codazzi 恒等 · 可验证成立

Gauss $d\omega^1_2 = \omega^3_1 \wedge \omega^2_3 = \sin\varphi\cos\varphi d\xi\wedge d\eta$

$(\varphi_{\xi\xi} - \varphi_{\eta\eta}) d\xi\wedge d\eta \Rightarrow \varphi_{\xi\xi} - \varphi_{\eta\eta} = \frac{1}{2}\sin 2\varphi \quad (*)$

想要 φ 满足 $(*)$ 则可求一个 $K\equiv -1$ 的曲面

$(*) \overset{\alpha = 2\varphi}{\Longleftrightarrow} \alpha_{\xi\xi} - \alpha_{\eta\eta} = \sin\alpha. \quad \alpha\in(0,\pi)$

再用原面的 (s,t)

$\alpha_{st} = (\alpha)_t = (\alpha_\xi + \alpha_\eta)_t = \alpha_{\xi\xi} - \alpha_{\eta\eta}$

$\overset{(*)}{\Rightarrow} \alpha_{st} = \sin\alpha \quad \alpha\in(0,\pi)$ Sine-Gordan Eq.

E^3 中 $K\equiv -1$ 的完备曲面上有渐近 Chebyshev 网

$f: \mathbb{R}^2 \to M$

$(s,t) \mapsto f(s,t)$

$s,t: s \mapsto (s,t_0)$ 渐近

$t \mapsto (s_0,t)$ 渐近

为渐近线

If 的渐近线可无限延伸: $\forall p\in M$, 渐近方向 $v\in T_pM$

$\Rightarrow \exists \varepsilon>0$ 和 $c:(-\varepsilon,\varepsilon)\to M$ 渐近线

$c(0)=p \quad c'(0)=v$

则 c 可延伸到原点 (练习)

在角点 $c(s)$ 处取一个与 $c(s)$ 无关的渐近方向 $X(s)$

构造 $f: \mathbb{R}^2 \to M$

$(s,t) \mapsto \gamma_s(t)$

γ_s：弧长参数渐近线

$s.t\ \gamma_s(0)=c(s)$ $\dot{\gamma}_s(0)=X(s)$

断言 t_0 $s \mapsto f(s,t_0)$ 渐近线！（t_0 一时兴起）

（证明：只说 $\exists (s_0-\varepsilon, s_0+\varepsilon)$ 上为渐近线

紧合 $\{f(s,t) \in M \mid 0 \le t \le t_0\}$ 等 $\exists$ 有限个曲面片 $r_0, \dots, r_n$ 覆盖之 且 $\Gamma_i \cap \Gamma_{i+1} = \phi$

$r_0: (-\varepsilon_0, \varepsilon_0)^2 \to M$ （渐近Chebyshev网）

$r_0(s,0)=f(s,0)=c(s) \Rightarrow r_0(s,t)=f(s,t)$ $(\forall\, t \in (-\varepsilon_0, \varepsilon_0))$

$\Rightarrow s \mapsto f(s,t)$ 为渐近线 $(\forall t \in (-\varepsilon_0, \varepsilon_0))$

一步步把性质迭代 令 $\varepsilon = \min_{i=0,\dots,n} \varepsilon_i$ 即可 ）

最后引理：存在 $\alpha: \mathbb{R}^2 \to (0,\pi)$ $s.t.\ d_{st}=\sin\alpha$

若存在 $\exists \alpha_{t_0} > 0 \Rightarrow t \mapsto \alpha_s(t)$ 单增

特别 a_s 不恒为 0 存在 $\alpha_s(0,0) > 0$

$\Rightarrow \exists \delta$ $s.t.\ \alpha_s(s,0)$ 在 $[0,\delta]$ 正

$\Rightarrow \alpha_s$ 在 $[0,s_0] \times [0,+\infty)$ 上正

$\Rightarrow \exists 0 < s_1 < s_2 < s_3$ 在 $[s_1, s_2]$ 上 $\alpha(s,0) \in [\alpha(s_1,0), \alpha(s_2,0)]$

记 $\delta = \min\{\alpha(s_1,0)-\alpha(0,0), \alpha(s_3,0)-\alpha(s_2,0)\} > 0$

$\Rightarrow \alpha(s,0) \in [\delta, \pi-\delta]$ $(\forall s \in [s_1, s_2])$

又 $\alpha(s_1,t)-\alpha(0,t) = \int_0^{s_1} \alpha_s'(s,t)\,ds \ge \int_0^{s_1} \alpha_s(s,0)\,ds = \alpha(s_1,0)-\alpha(0,0)$

$\Rightarrow \alpha(s_1,t) \ge \alpha(0,t)+\delta \ge \delta$ 同理 $\alpha(s_2,t) \le \pi-\delta$

$\Rightarrow \alpha(s,t) \in [\delta, \pi-\delta]$ $\forall t \ge 0,\ s \in [s_1, s_2]$

$\Rightarrow [s_1, s_2] \times [0,+\infty)$ 上 $\sin\alpha \ge \sin\delta$

又 $\int_0^T\int_{S_1}^{S_2} d_{st}\,ds\,dt = \int_0^T\int_{S_1}^{S_2} \sin\omega\,ds\,dt \geq \sin\delta(S_1-S_2)\cdot T$

$=\alpha(S_2,T)-\alpha(S_2,0)-\alpha(S_1,T)+\alpha(S_1,0)$

左边有界. 再令 $T\to+\infty$. 矛盾!　　故 Hilbert Thm 得证

Rmk. $x^2+y^2-z^2=1$　　$(\sqrt{u^2+1}\cos v, \sqrt{u^2+1}\sin v, u)$

$\Rightarrow K=-\frac{1}{(2u^2+1)^2}$　$\Rightarrow$ 为负曲率完备曲面

Willmore 配速

M光滑 "足够"无边可定向曲面　　$f: M\to E^3$ 嵌入

G-B Thm $\Rightarrow \frac{1}{2\pi}\int_{f(M)} K\,dV = \chi(M)$

再定义 $\tau(f)=\frac{1}{2\pi}\int_{f(M)} H^2\,dV$

$\tau(M)=\inf_{f: M\to E^3 \text{光滑嵌入}} \tau(f)$ 且 $\partial=0$

thm (Willmore) M光滑"足够"定向. 则 $\tau(M)\geq\chi(M)$ 且 $\tau(f)=\tau(M)=2\Leftrightarrow M$ 为球面

[pf. $H^2=(\frac{k_1+k_2}{2})^2\geq k_1k_2=K$ $\Rightarrow \tau(f)\geq\chi(M)$

反之 $\Leftarrow$ 空间子流形曲率 $\Leftarrow$ 球面.]

$g\geq 1$? 环面 $f_{a,b}(T)=((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u)$ $(0<b<a)$

令 $\frac{b}{a}=c<1$　　可算得 $\tau(f_{a,b})=\frac{\pi}{2c\sqrt{1-c^2}}$

$c=\frac{1}{\sqrt{2}}$ 时　$\tau(f_{a,b})$ 最小. 为 π.

$g=1$ 时 $\tau(M)=\pi$?　　Willmore conjecture