中国科大电动力学期中考试试题卷(2023)

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成绩：

提示：

- 闵氏空间$\mathbb{M}_4$中洛伦兹推动变换$x^\mu \rightsquigarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$的非零矩阵元是：

$$\Lambda^0{}_0 = \gamma, \quad \Lambda^0{}_j = -\gamma\beta_j, \quad \Lambda^i{}_0 = -\gamma\beta^i, \quad \Lambda^i{}_j = \delta^i_j + \frac{\gamma^2}{\gamma+1}\beta^i\beta_j \tag{1}$$

式中$\boldsymbol{\beta} = \beta^i \boldsymbol{e}_i$为无量纲的牵连速度，$\gamma = 1/\sqrt{1-\beta^2}$.

- $\mathbb{M}_4$中的洛伦兹变换$x^\mu \rightsquigarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$也包括空间反射变换：

$$\Lambda^0{}_0 = 1, \quad \Lambda^0{}_j = \Lambda^i{}_0 = 0, \quad \Lambda^i{}_j = -\delta^i_j \tag{2}$$

- 球谐函数$Y_{lm}(\theta, \phi)$定义为：

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}, \quad l \in \mathbb{N}, \ m = -l, \ -l+1, \ \cdots, l-1, \ l. \tag{3}$$

几个低阶的球谐函数的显示形式是：

$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm 1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{4}$$

- Γ-函数的积分表达式：

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t, \quad \mathrm{Re}\{s\} > 0 \tag{5}$$

倘若s为正整数，则$\Gamma(s) = (s-1)!$.

简答(40分):

1. 假设$T_{\mu\nu}(x)$是$\mathbb{M}_4$中的二阶张量场. 据此可以构造出如下两个新的二阶4-张量场:

$$S_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) - \frac{1}{4}\eta_{\mu\nu}T^{\alpha}_{\ \alpha}, \quad A_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \tag{6}$$

式中$\eta_{\mu\nu}$是$\mathbb{M}_4$的度规矩阵, $T^{\alpha}_{\ \alpha} \equiv \eta^{\alpha\beta}T_{\beta\alpha}$是4-张量场$T_{\mu\nu}(x)$的迹.

- 请问$T_{\mu\nu}$、$S_{\mu\nu}$和$A_{\mu\nu}$各自有多少独立分量(3分)?
- 请分别明确写出$A_{\mu\nu}$和$T^{\alpha}_{\ \alpha}$在洛伦兹变换下的变换法则(5分).
- 请计算$S^{\mu\nu}A_{\mu\nu}$并明确说明计算结果适合于哪些惯性参考系(2分).

解:

- $T_{\mu\nu}$、$S_{\mu\nu}$和$A_{\mu\nu}$各自具有的独立分量的数目为16、9和6.
- 线性变换矩阵$\Lambda^{\mu}_{\ \nu}$服从赝正交条件

$$\eta_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma} = \eta_{\rho\sigma} \tag{7}$$

的线性变换$x^{\mu} \rightsquigarrow x'^{\mu} = \Lambda^{\mu}_{\ \nu}x^{\nu}$就是洛伦兹变换. $T^{\alpha}_{\ \alpha}(x)$是一个4-标量, 它在洛伦兹推动变换下不变:

$$T^{\alpha}_{\ \alpha}(x) \quad \rightsquigarrow \quad T'^{\alpha}_{\ \alpha}(x') = T^{\alpha}_{\ \alpha}(x) \tag{8}$$

而$A_{\mu\nu}(x)$在洛伦兹推动变换下变换法则是:

$$A_{\mu\nu}(x) \quad \rightsquigarrow \quad A'_{\mu\nu}(x') = A_{\rho\sigma}(x)\widetilde{\Lambda}^{\rho}_{\ \mu}\widetilde{\Lambda}^{\sigma}_{\ \nu} \tag{9}$$

式中$\widetilde{\Lambda}^{\mu}_{\ \nu} = \eta^{\mu\rho}\Lambda^{\sigma}_{\ \rho}\eta_{\sigma\nu}$是逆洛伦兹变换矩阵, $\Lambda^{\mu}_{\ \rho}\widetilde{\Lambda}^{\rho}_{\ \nu} = \delta^{\mu}_{\ \nu}$.

2. 某电场的电场强度分布在球坐标系中表达为$\boldsymbol{E} = \frac{C}{r}\cos\theta\boldsymbol{e}_r$, 式中$C$为一有量纲参数. 请问该电场是否是静电场(10分)?

解:

因为

$$\boldsymbol{\nabla} \times \boldsymbol{E} = C\boldsymbol{\nabla}\left(\frac{\cos\theta}{r}\right) \times \boldsymbol{e}_r = -\frac{C\sin\theta}{r^2}\boldsymbol{e}_\theta \times \boldsymbol{e}_r = \frac{C\sin\theta}{r^2}\boldsymbol{e}_\phi \neq 0 \tag{10}$$

所以此题考虑的电场不是静电场.

3. 设4-矢量场A_μ是电磁场的规范势, 电磁场张量可表为$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. 请问两个4-标量$F^{\mu\nu}F_{\mu\nu}$和$A^\mu A_\mu$在规范变换$A_\mu \rightsquigarrow A'_\mu = A_\mu + \partial_\mu\alpha(x)$下的变换法则(10分).

解:

4-标量$F^{\mu\nu}F_{\mu\nu}$是规范不变量, 但另一4-标量$A^\mu A_\mu$在规范变换下的变换法则是:

$$A'^{\mu}A'_{\mu} = (A^\mu + \partial^\mu\alpha)(A_\mu + \partial_\mu\alpha) = A^\mu A_\mu + 2A^\mu\partial_\mu\alpha + \partial^\mu\alpha\partial_\mu\alpha \tag{11}$$

4. Proca矢量场$A_\mu(x)$的拉氏密度可表为:

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\sigma m^2 A^\mu A_\mu \tag{12}$$

式中$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. 倘若以此拉氏密度出发构建体系的哈密顿表述, 体系中存在约束吗? 若存在, 请写出约束方程(10分).

解:

以A_μ为矢量场的广义坐标, 则场的动力学方程是如下拉氏方程:

$$0=\frac{\partial \mathscr{L}}{\partial A_\mu}-\frac{\partial}{\partial x^\nu}\left[\frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu)}\right] \tag{13}$$
$$=\sigma m^2 A^\mu+\frac{1}{2}\frac{\partial}{\partial x^\nu}\left[F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\nu A_\mu)}\right] \tag{14}$$
$$=\sigma m^2 A^\mu+\partial_\nu F^{\nu\mu} \tag{15}$$

也就是$\partial_\mu F^{\mu\nu}=\sigma m^2 A^\nu$. 接着以$A_\mu$为矢量场朴素的正则坐标建立体系的哈密顿表述. 与$A_\mu$其共轭的正则动量是:

$$\pi^\mu=\frac{\partial \mathscr{L}}{\partial(\partial_0 A_\mu)}=-\frac{1}{2}F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_0 A_\mu)}=-F^{\alpha\beta}\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_0 A_\mu)}=-F^{0\mu} \tag{16}$$

基本的非零等时泊松括号为:

$$\{A_\mu(t,\boldsymbol{x}),\ \pi^\nu(t,\boldsymbol{y})\}=\delta_\mu^\nu\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{17}$$

显然, 体系中存在约束. 初级约束为:

$$\chi_1\approx\pi^0=0 \tag{18}$$

它与基本泊松括号不相容. 为了写出体系中存在的其它约束, 我们写出体系的哈密顿密度

$$\mathscr{H}=\pi^\mu\partial_0 A_\mu-\mathscr{L}=\pi^i\partial_0 A_i+\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}\sigma m^2 A^\mu A_\mu \tag{19}$$
$$=\pi^i\pi_i+\pi^i\partial_i A_0+\frac{1}{2}F^{0i}F_{0i}+\frac{1}{4}F^{jk}F_{jk}+\frac{1}{2}\sigma m^2 A_0^2-\frac{1}{2}\sigma m^2 A^i A_i \tag{20}$$
$$=\frac{1}{2}\pi^i\pi_i+\frac{1}{4}F^{jk}F_{jk}-\frac{1}{2}\sigma m^2 A^i A_i+\partial_i(\pi^i A_0)-A_0\partial_i\pi^i+\frac{1}{2}\sigma m^2 A_0^2 \tag{21}$$

式中$\pi^i=-F^{0i}$而$\pi_i=F_{0i}$. 体系的哈密顿量是:

$$H=\int \mathrm{d}^3x\mathscr{H}=\int \mathrm{d}^3x\left(\frac{1}{2}\pi^i\pi_i+\frac{1}{4}F^{jk}F_{jk}-\frac{1}{2}\sigma m^2 A^i A_i-A_0\partial_i\pi^i+\frac{1}{2}\sigma m^2 A_0^2\right) \tag{22}$$

要求初级约束不随时间变化,

$$0\approx\frac{\mathrm{d}\chi_1}{\mathrm{d}t}=\{\chi_1,\ H\}=\{\pi^0(t,\boldsymbol{x}),\ H\} \tag{23}$$
$$=\int \mathrm{d}^3y\{\pi^0(t,\boldsymbol{x}),\ A_0(t,\boldsymbol{y})\}\Big[-\partial_i\pi^i(t,\boldsymbol{y})+\sigma m^2 A_0(t,\boldsymbol{y})\Big] \tag{24}$$
$$=\partial_i\pi^i-\sigma m^2 A_0 \tag{25}$$

我们看到体系中存在着如下次级约束:

$$\chi_2\approx\partial_i\pi^i-\sigma m^2 A_0=0 \tag{26}$$

重复上述步骤,

$$0\approx\frac{\mathrm{d}\chi_2}{\mathrm{d}t}=\{\chi_2,\ H\}=\{\partial_i\pi^i(t,\ \boldsymbol{x})-\sigma m^2 A_0(t,\boldsymbol{x}),\ H\} \tag{27}$$
$$=\{\partial_i\pi^i(t,\ \boldsymbol{x}),\ H\} \tag{28}$$
$$=\int \mathrm{d}^3y\left[\frac{1}{2}F^{jk}(t,\boldsymbol{y})\{\partial_i\pi^i(t,\boldsymbol{x}),\ F_{jk}(t,\boldsymbol{y})\}-\sigma m^2 A^j(t,\boldsymbol{y})\{\partial_i\pi^i(t,\boldsymbol{x}),\ A_j(t,\boldsymbol{y})\}\right] \tag{29}$$
$$=\int \mathrm{d}^3y\left[F^{ij}(t,\boldsymbol{y})\frac{\partial^2\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}{\partial x^i\partial y^j}+\sigma m^2 A^i(t,\boldsymbol{y})\frac{\partial\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}{\partial x^i}\right] \tag{30}$$
$$=\sigma m^2\partial_i A^i(t,\boldsymbol{x}) \tag{31}$$

体系中存在着新的次级约束:

$$\chi_3 \approx \boldsymbol{\nabla}\cdot\boldsymbol{A}(t,\boldsymbol{x}) = \partial_i A^i(t,\boldsymbol{x}) = 0 \tag{32}$$

注意到：

$$\begin{aligned}
0 \approx \frac{\mathrm{d}\chi_3}{\mathrm{d}t} &= \{\chi_3,\ H\} = \{\partial_i A^i(t,\boldsymbol{x}),\ H\} &(33)\\
&= \int \mathrm{d}^3y\left[\pi^j(t,\boldsymbol{y})\{\partial_i A^i(t,\boldsymbol{x}),\ \pi_j(t,\boldsymbol{y})\} - A_0(t,\boldsymbol{y})\{\partial_i A^i(t,\boldsymbol{x}),\ \partial_j\pi^j(t,\boldsymbol{y})\}\right] &(34)\\
&= \int \mathrm{d}^3y\left[\pi^i(t,\boldsymbol{y})\frac{\partial\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}{\partial x^i} - \eta^{ij}A_0(t,\boldsymbol{y})\frac{\partial^2\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}{\partial x^i\partial y^j}\right] &(35)\\
&= \partial_i\pi^i(t,\boldsymbol{x}) + \boldsymbol{\nabla}^2 A_0(t,\boldsymbol{x}) &(36)\\
&= \partial_i F^{i0} + \boldsymbol{\nabla}^2 A_0 = \partial_i(\partial^i A^0 - \partial^0 A^i) + \boldsymbol{\nabla}^2 A_0 &(37)\\
&= \boldsymbol{\nabla}^2 A^0 - \partial^0(\partial_i A^i) - \boldsymbol{\nabla}^2 A^0 = \partial_0(\boldsymbol{\nabla}\cdot\boldsymbol{A}) = \partial_0\chi_3 = 0 &(38)
\end{aligned}$$

故体系中不再有新的次级约束. 综合起来，自由Proca场论的哈密顿表述中共包含着如下三个约束：

$$\chi_1 = \pi^0 \approx 0,\qquad \chi_2 = \boldsymbol{\nabla}\cdot\boldsymbol{\pi} - \sigma m^2 A_0 \approx 0,\qquad \chi_3 = \boldsymbol{\nabla}\cdot\boldsymbol{A} \approx 0. \tag{39}$$

计算(60分)：

5. 倘若反对称的4-张量$F^{\mu\nu}$和

$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{40}$$

分别是电磁场张量与对偶电磁场张量.

- 请证明$F^{\mu\nu}F_{\mu\nu} \propto G^{\mu\nu}G_{\mu\nu}$并求出比例系数的精确表达式(5分).
- 请证明4-标量$F^{\mu\nu}F_{\mu\nu}$是空间反射变换下的不变量、但$F^{\mu\nu}G_{\mu\nu}$则不然(10分).
- 明确写出电场强度$\boldsymbol{E}$和磁感应强度$\boldsymbol{B}$在空间反射变换下的变换法则(5分).

解：

- 使用恒等式$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\rho\sigma} = -2(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)$知：

$$G^{\mu\nu}G_{\mu\nu} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta}F_{\rho\sigma}F^{\alpha\beta} = -\frac{1}{2}(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha)F_{\rho\sigma}F^{\alpha\beta} = -F_{\alpha\beta}F^{\alpha\beta} = -F^{\mu\nu}F_{\mu\nu} \tag{41}$$

- 倘若以$x^\mu \rightsquigarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$表示空间反射变换(属于一种离散的洛伦兹变换)，$\det\Lambda = -1$，则

$$F^{\mu\nu} \rightsquigarrow F'^{\mu\nu} = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma F^{\rho\sigma},\quad F_{\mu\nu} \rightsquigarrow F'_{\mu\nu} = F_{\rho\sigma}\widetilde{\Lambda}^\rho{}_\mu\widetilde{\Lambda}^\sigma{}_\nu \tag{42}$$

而$\epsilon^{\mu\nu\rho\sigma}$作为洛伦兹变换下的不变张量，$\epsilon^{\mu\nu\rho\sigma} \rightsquigarrow (\epsilon')^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}$，其在空间反射变换下的变换法则是：

$$\epsilon^{\mu\nu\rho\sigma} \quad\rightsquigarrow\quad (\epsilon')^{\mu\nu\rho\sigma} = (\det\Lambda)\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\lambda\Lambda^\sigma{}_\tau\epsilon^{\alpha\beta\lambda\tau} = -\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\lambda\Lambda^\sigma{}_\tau\epsilon^{\alpha\beta\lambda\tau} \tag{43}$$

注意到

$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{44}$$

我们看到$G^{\mu\nu}$在空间反射变换下的变换性质是：

$$G^{\mu\nu} \rightsquigarrow G'^{\mu\nu} = \frac{1}{2}(\epsilon')^{\mu\nu\rho\sigma}F'_{\rho\sigma} \tag{45}$$
$$= -\frac{1}{2}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\lambda}\Lambda^{\sigma}{}_{\tau}\epsilon^{\alpha\beta\lambda\tau}F_{\xi\zeta}\widetilde{\Lambda}^{\xi}{}_{\rho}\widetilde{\Lambda}^{\zeta}{}_{\sigma} \tag{46}$$
$$= -\frac{1}{2}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\,\epsilon^{\alpha\beta\lambda\tau}F_{\xi\zeta}\,(\widetilde{\Lambda}^{\xi}{}_{\rho}\Lambda^{\rho}{}_{\lambda})\,(\widetilde{\Lambda}^{\zeta}{}_{\sigma}\Lambda^{\sigma}{}_{\tau}) \tag{47}$$
$$= -\frac{1}{2}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\,\epsilon^{\alpha\beta\lambda\tau}F_{\xi\zeta}\,\delta^{\xi}_{\lambda}\delta^{\zeta}_{\tau} = -\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\left[\frac{1}{2}\epsilon^{\alpha\beta\lambda\tau}F_{\lambda\tau}\right] = -\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}G^{\alpha\beta} \tag{48}$$

类似地，

$$G_{\mu\nu} \rightsquigarrow G'_{\mu\nu} = -G_{\alpha\beta}\widetilde{\Lambda}^{\alpha}{}_{\mu}\widetilde{\Lambda}^{\beta}{}_{\nu} \tag{49}$$

所以，$F^{\mu\nu}F_{\mu\nu}$在空间反射变换下保持不变，而$F^{\mu\nu}G_{\mu\nu}$在空间反射变换下将会改变符号(赝标量)：

$$F^{\mu\nu}F_{\mu\nu} \rightsquigarrow F'^{\mu\nu}F'_{\mu\nu} = F^{\mu\nu}F_{\mu\nu}, \qquad F^{\mu\nu}G_{\mu\nu} \rightsquigarrow F'^{\mu\nu}G'_{\mu\nu} = -F^{\mu\nu}G_{\mu\nu}. \tag{50}$$

- 注意到电磁场张量$F^{\mu\nu}$的物理内涵是：

$$E^i = cF^{0i}, \quad B_i = \frac{1}{2}\epsilon_{ijk}F^{jk} \tag{51}$$

式中ϵ_{ijk}是空间转动变换(包括空间反射)下的不变张量，$\epsilon_{ijk} \rightsquigarrow \epsilon'_{ijk} = \epsilon_{ijk}$，电场强度$\boldsymbol{E}$在空间反射变换下的变换法则是：

$$E^i \rightsquigarrow E'^i = cF'^{0i} = c\,\Lambda^{0}{}_{\mu}\Lambda^{i}{}_{\nu}F^{\mu\nu} = c\,\Lambda^{0}{}_{0}\Lambda^{i}{}_{j}F^{0j} = -c\,\delta^{i}_{j}F^{0j} = -cF^{0i} = -E^i \tag{52}$$

而磁感应强度$\boldsymbol{B}$在空间反射变换下的变换法则是：

$$B_i \rightsquigarrow B'_i = \frac{1}{2}\epsilon'_{ijk}F'^{jk} = \frac{1}{2}\epsilon_{ijk}\Lambda^{j}{}_{\mu}\Lambda^{k}{}_{\nu}F^{\mu\nu} = \frac{1}{2}\epsilon_{ijk}\Lambda^{j}{}_{m}\Lambda^{k}{}_{n}F^{mn} \tag{53}$$
$$= (-1)^2\frac{1}{2}\epsilon_{ijk}\delta^{j}{}_{m}\delta^{k}{}_{n}F^{mn} \tag{54}$$
$$= \frac{1}{2}\epsilon_{ijk}F^{jk} = B_i \tag{55}$$

所以，电场强度是$\mathbb{E}_3$中的极矢量，而磁感应强度是轴矢量.

6. 电磁场的能量动量张量$\Theta^{\mu\nu}$定义为：

$$\Theta^{\mu\nu} = \frac{1}{\mu_0}\eta^{\nu\sigma}F^{\mu\rho}F_{\rho\sigma} + \frac{1}{4\mu_0}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \tag{56}$$

- 请证明$\Theta^{\mu\nu}$具有对称、无迹和规范不变的性质(10分).
- 从麦克斯韦方程组可知$\partial_\mu\Theta^{\mu\nu} = -J_\mu F^{\mu\nu}$，式中$J_\mu$为4-电流密度矢量. 若定义电磁场的角动量张量

$$M^{\mu\rho\sigma} = \Theta^{\mu\rho}x^\sigma - \Theta^{\mu\sigma}x^\rho \tag{57}$$

请建立$M^{\mu\rho\sigma}$满足的守恒定律(10分).

解：

- 注意到电磁场张量$F_{\mu\nu}$是反对称的4-张量，$F_{\mu\nu} = -F_{\nu\mu}$，我们有：

$$\Theta^{\nu\mu} = \frac{1}{\mu_0}\eta^{\mu\sigma}F^{\nu\rho}F_{\rho\sigma} + \frac{1}{4\mu_0}\eta^{\nu\mu}F^{\rho\sigma}F_{\rho\sigma} \tag{58}$$
$$= \frac{1}{\mu_0}F^{\mu}{}_{\rho}F^{\rho\nu} + \frac{1}{4\mu_0}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \tag{59}$$
$$= \frac{1}{\mu_0}\eta^{\nu\sigma}F^{\mu\rho}F_{\rho\sigma} + \frac{1}{4\mu_0}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \tag{60}$$
$$= \Theta^{\mu\nu} \tag{61}$$
$$\Theta^{\mu}{}_{\mu} = \eta_{\mu\nu}\Theta^{\nu\mu} = \frac{1}{4}\delta^{\sigma}{}_{\nu}F^{\nu\rho}F_{\rho\sigma} + \frac{1}{4\mu_0}\delta^{\nu}{}_{\nu}F^{\rho\sigma}F_{\rho\sigma} = \frac{1}{4}(F^{\sigma\rho} + F^{\rho\sigma})F_{\rho\sigma} = 0 \tag{62}$$

因为$F_{\mu\nu}$是规范变换下的不变量，$\Theta^{\mu\nu}$在规范变换下也是不变的.

- 电磁场角动量张量满足的守恒定律是:

$$\partial_\mu M^{\mu\rho\sigma} = \partial_\mu(\Theta^{\mu\rho}x^\sigma - \Theta^{\mu\sigma}x^\rho) \tag{63}$$
$$= (\partial_\mu\Theta^{\mu\rho})x^\sigma + \Theta^{\mu\rho}(\partial_\mu x^\sigma) - (\partial_\mu\Theta^{\mu\sigma})x^\rho - \Theta^{\mu\sigma}(\partial_\mu x^\rho) \tag{64}$$
$$= -J_\mu F^{\mu\rho}x^\sigma + \Theta^{\mu\rho}\delta^\sigma_\mu + J_\mu F^{\mu\sigma}x^\rho - \Theta^{\mu\sigma}\delta^\rho_\mu \tag{65}$$
$$= J_\mu(x^\rho F^{\mu\sigma} - x^\sigma F^{\mu\rho}) \tag{66}$$

7. 当不考虑原子核的运动以及原子核的电荷分布时，氢原子的电荷分布在球坐标系中可表为:

$$\rho(r,\theta) = -\frac{er^2\sin^2\theta}{64\pi}e^{-r}, \quad 0 \le r < \infty,\ \ 0 \le \theta \le \pi. \tag{67}$$

e是电子电荷量的绝对值. 请计算氢原子所有的非零电多极矩(精确到电四极矩，10分)，计算其在远处的静电势分布(10分).

解:

精确到电四极矩，体系的电多极矩可表为:

$$q_{lm} = \int \mathrm{d}^3x\rho(\boldsymbol{x})r^lY_{lm}(\theta,\phi), \quad (l=0,1,2) \tag{68}$$

因为题设$\rho(\boldsymbol{x})$在球坐标系中的表达式与方位角ϕ无关，故非零的电多极矩只有q_{l0}:

$$q_{l0} = \int_0^\infty \mathrm{d}rr^{l+2}\int \mathrm{d}\Omega\rho(r,\theta)Y_{l0}(\theta,\phi) \tag{69}$$

计及“提示”中枚举的球谐函数表达式以及

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \tag{70}$$

体系的电荷体密度可表为:

$$\rho(r,\theta) = -\frac{e}{64\pi}r^2\exp(-r)\ \sin^2\theta = -\frac{e}{96\pi}\sqrt{\frac{4\pi}{5}}\ r^2\exp(-r)\left[\sqrt{5}Y_{00}(\theta,\phi) - Y_{20}(\theta,\phi)\right] \tag{71}$$

所以,

$$q_{l0} = -\frac{e}{96\pi}\sqrt{\frac{4\pi}{5}}\ \int_0^\infty \mathrm{d}rr^{l+4}\exp(-r)\int \mathrm{d}\Omega\left[\sqrt{5}Y_{00}(\theta,\phi) - Y_{20}(\theta,\phi)\right]Y_{l0}(\theta,\phi) \tag{72}$$
$$= -\frac{e}{96\pi}\sqrt{\frac{4\pi}{5}}\ \Gamma(l+5)\left[\sqrt{5}\delta_{l0} - \delta_{l2}\right] \tag{73}$$
$$= -\frac{e}{96\pi}\sqrt{\frac{4\pi}{5}}\ (\sqrt{5}\ 4!\delta_{l0} - 6!\delta_{l2}) \tag{74}$$

换言之，此氢原子非零的电多极矩为:

$$q_{00} = -\frac{e}{\sqrt{4\pi}}, \quad q_{20} = 6e\sqrt{\frac{5}{4\pi}} \tag{75}$$

它们激发的静电势分布是:

$$\varphi(r,\theta) = \frac{1}{\epsilon_0}\sum_{lm}\frac{1}{2l+1}\frac{q_{lm}}{r^{l+1}}Y_{lm}(\theta,\phi) = \frac{1}{\epsilon_0}\left[\frac{q_{00}}{r}Y_{00}(\theta,\phi) + \frac{q_{20}}{5r^3}Y_{20}(\theta,\phi)\right] \tag{76}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{e}{r}\left[1 - \frac{3}{r^2}(3\cos^2\theta - 1)\right] \tag{77}$$

若不使用球多极展开，则计算过程略繁. 计算结果应是: 体系的总电荷量为$Q = -e$，电偶极矩为零($\boldsymbol{p} = 0$)，电四极矩张量非零的笛卡尔分量是$D_{11} = D_{22} = -3e$, $D_{33} = 6e$. 静电势的表达式同上.