- 电磁场拉氏密度的相互作用项似乎还可以有如下几个候选 4-标量：

$$J^{\mu}(x)J^{\nu}(x)\left\{\begin{array}{l}F_{\mu\nu}(x)\\ \mathscr{F}_{\mu\nu}(x)\end{array}\right. ,\quad J^{\mu}(x)A^{\nu}(x)\left\{\begin{array}{l}F_{\mu\nu}(x)\\ \mathscr{F}_{\mu\nu}(x)\end{array}\right.$$

然而前二者恒为零，后二者不具有规范变换下的不变性. 因此，它们都无资格出现在电磁场的拉氏密度表达式中.

综合起来，倘若我们不考虑 4-电流密度 $J^{\mu}(x)$ 本身的动力学，则电磁场的动力学应决定于如下拉氏函数密度：

$$\boxed{\mathscr{L}(x)=-\frac{1}{4\mu_0}F_{\mu\nu}(x)F^{\mu\nu}(x)+J^{\mu}(x)A_{\mu}(x)}$$

拉氏密度中，两项的系数原则上都是任意的，对其做出具体规定相当于选择了某种特殊的单位制[15]. 按此拉氏密度，我们有：

$$\frac{\partial\mathscr{L}}{\partial A_{\nu}(x)}=J^{\nu}(x),\qquad \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}A_{\nu}(x))}=-\frac{1}{\mu_0}F^{\mu\nu}(x)$$

[15] 我们采取国际单位制，μ_0 是真空的磁导率，并按习惯取 $\lambda=1$.

代入到电磁场的拉氏方程

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu(x))} \right) - \frac{\partial \mathscr{L}}{\partial A_\nu(x)} = 0$$

可以将其明确地表为：

$$\boxed{\partial_\mu F^{\mu\nu}(x) = -\mu_0 J^\nu(x)}$$

这就是电磁场的动力学方程. 此外, 根据电磁场张量的定义式

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

不难看出 $F_{\mu\nu}(x)$ 还须服从约束条件：

$$\boxed{\partial_\rho F_{\mu\nu}(x) + \partial_\mu F_{\nu\rho}(x) + \partial_\nu F_{\rho\mu}(x) = 0}$$

假若你愿意, 你可以把此约束条件称为电磁场张量必须满足的毕安琪 (Bianchi) 恒等式. 拉氏方程与毕安琪恒等式联合提供了描写电磁场性质的理论基础, 它们就是协变形式的 Maxwell 方程组.

计算 Levi-Civita 全反对称张量 $\epsilon^{\mu\nu\rho\sigma}$ 与电磁场毕安琪恒等式

$$\boxed{\partial_\rho F_{\mu\nu}(x)+\partial_\mu F_{\nu\rho}(x)+\partial_\nu F_{\rho\mu}(x)=0}$$

的缩并，我们有：

$$\begin{aligned}
0 &= \epsilon^{\mu\nu\rho\sigma}\left[\partial_\rho F_{\mu\nu}(x)+\partial_\mu F_{\nu\rho}(x)+\partial_\nu F_{\rho\mu}(x)\right] \\
&= -\partial_\rho\left[\epsilon^{\sigma\rho\mu\nu}F_{\mu\nu}(x)\right]-\partial_\mu\left[\epsilon^{\sigma\mu\nu\rho}F_{\nu\rho}(x)\right]-\partial_\nu\left[\epsilon^{\sigma\nu\rho\mu}F_{\rho\mu}(x)\right] \\
&= 3\partial_\rho\left[\epsilon^{\rho\sigma\mu\nu}F_{\mu\nu}(x)\right]
\end{aligned}$$

借助于对偶电磁场张量

$$\mathscr{F}^{\mu\nu}(x)=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}(x)$$

我们最终把电磁场的毕安琪恒等式简洁地表达为：

$$\boxed{\partial_\mu\mathscr{F}^{\mu\nu}(x)=0}$$

电磁场张量 $F_{\mu\nu}(x)$ 的物理内涵:

电磁场张量

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

是闵氏空间 $\mathbb{M}_4$ 中的 2 阶反对称张量, $F_{\mu\nu}(x) = -F_{\nu\mu}(x)$, 其独立分量的数目为

$$\left.\frac{\mathrm{d}(\mathrm{d}-1)}{2}\right|_{\mathrm{d}\to 4} = 6$$

我们规定这些分量的物理内涵为:

$$\boxed{F^{0i} = \frac{1}{c}E^{i}, \quad F^{ij} = \epsilon^{ijk}B_k}$$

或者等价地,

$$\boxed{E^{i} = c\,F^{0i}, \quad B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}}$$

此处 $\boldsymbol{E} = E^{i}\boldsymbol{e}_i$ 与 $\boldsymbol{B} = B^{i}\boldsymbol{e}_i$ 是欧氏空间 $\mathbb{E}_3$ 中电磁场的电场强度与磁感应强度.

现在讨论电磁场基本方程组

$$\boxed{\frac{1}{\mu_0}\partial_\mu F^{\mu\nu}(x) = -J^\nu(x), \quad \partial_\mu \mathscr{F}^{\mu\nu}(x) = 0}$$

的物理内涵:

- 拉氏方程 $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$ 在 $\nu = 0$ 时表达为:

$$-\mu_0 c\rho = -\mu_0 J^0 = \partial_\mu F^{\mu 0} = \partial_i F^{i0} = -\frac{1}{c}\partial_i E^i = -\frac{1}{c}\nabla\cdot\boldsymbol{E}$$

回忆初等电磁学或者物理光学等前置课程中学过的光速与电磁常数的关系,

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

上式可改写为:

$$\nabla\cdot\boldsymbol{E} = \rho/\epsilon_0$$

这正是电高斯定律.

- 拉氏方程 $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$ 在 $\nu = i$ 时表达为:

$$\begin{aligned}
-\mu_0 j^i &= -\mu_0 J^i = \partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_k F^{ki} \\
&= \frac{1}{c^2}\frac{\partial E^i}{\partial t} + \epsilon^{kij}\partial_k B_j \\
&= \mu_0\epsilon_0 \frac{\partial E^i}{\partial t} - (\nabla \times \boldsymbol{B})^i
\end{aligned}$$

亦即:

$$\nabla \times \boldsymbol{B} - \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{j}$$

这是安培麦克斯韦方程,其中包含着位移电流修正项.

- 欲了解毕安琪恒等式的物理内涵,须事先了解对偶电磁场张量 $\mathscr{F}^{\mu\nu}(x)$ 的物理内涵 (Optional):

$$\boxed{\mathscr{F}^{0i} = B^i, \quad \mathscr{F}^{ij} = -\frac{1}{c}\epsilon^{ijk}E_k}$$

或者等价地,

$$\boxed{B^i = \mathscr{F}^{0i}, \quad E_i = -\frac{c}{2}\epsilon_{ijk}\mathscr{F}^{jk}}$$

- 毕安琪恒等式 $\partial_\mu \mathscr{F}^{\mu\nu}(x)=0$ 在 $\nu=0$ 情形下表达为：

$$0=\partial_\mu \mathscr{F}^{\mu 0}=\partial_i \mathscr{F}^{i0}=-\partial_i B^i$$

亦即：

$$\nabla\cdot\boldsymbol{B}=0$$

这是磁高斯定律. 从归纳法的视角看，这是电磁场能获得规范场属性的实验基础. 它也在经典场论意义下排除了点磁荷存在的可能性[16].

- 毕安琪恒等式 $\partial_\mu \mathscr{F}^{\mu\nu}(x)=0$ 在 $\nu=i$ 情形下表达为：

$$\begin{aligned}0 &= \partial_\mu \mathscr{F}^{\mu i}=\partial_0 \mathscr{F}^{0i}+\partial_k \mathscr{F}^{ki}=\frac{1}{c}\frac{\partial B^i}{\partial t}-\frac{1}{c}\epsilon^{kij}\partial_k E_j\\ &= \frac{1}{c}\left(\frac{\partial \boldsymbol{B}}{\partial t}+\nabla\times\boldsymbol{E}\right)^i\end{aligned}$$

亦即：

$$\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=0$$

这是法拉第电磁感应定律.

[16]但磁高斯定律原则上仍允许磁单极的存在.

两个重要的 4-标量

使用电磁场张量与其对偶张量，可以构造出如下两个彼此独立的洛伦兹标量：

$$F_{\mu\nu}(x)F^{\mu\nu}(x), \qquad F_{\mu\nu}(x)\mathscr{F}^{\mu\nu}(x)$$

它们的物理内涵分别是：

$$\begin{aligned}
F_{\mu\nu}F^{\mu\nu} &= F_{0\nu}F^{0\nu} + F_{i\nu}F^{i\nu} = F_{0j}F^{0j} + F_{i0}F^{i0} + F_{ij}F^{ij} \\
&= -\frac{2}{c^2}E_iE^i + \epsilon_{ijk}\epsilon^{ijl}B^kB_l \\
&= 2\left(\boldsymbol{B}^2 - \frac{\boldsymbol{E}^2}{c^2}\right)
\end{aligned}$$

$$\begin{aligned}
F_{\mu\nu}\mathscr{F}^{\mu\nu} &= F_{0\nu}\mathscr{F}^{0\nu} + F_{i\nu}\mathscr{F}^{i\nu} = F_{0j}\mathscr{F}^{0j} + F_{i0}\mathscr{F}^{i0} + F_{ij}\mathscr{F}^{ij} \\
&= -\frac{2}{c}E_iB^i - \frac{1}{c}\epsilon_{ijk}\epsilon^{ijl}B^kE_l \\
&= -\frac{4}{c}\boldsymbol{E}\cdot\boldsymbol{B}
\end{aligned}$$

4-标量

$$\boxed{\frac{1}{2}F_{\mu\nu}F^{\mu\nu}=\boldsymbol{B}^2-\frac{\boldsymbol{E}^2}{c^2},\qquad -\frac{c}{4}F_{\mu\nu}\mathscr{F}^{\mu\nu}=\boldsymbol{E}\cdot\boldsymbol{B}}$$

的存在意味着：

1. 倘若电磁场在某一惯性系中表现为纯粹的电场，则一定找不到另一惯性系使得电磁场在其中表现为纯粹的磁场. 反之亦然.
2. 倘若电磁场在某一惯性系中表现为纯粹的电场或者纯粹的磁场，则在任何一个别的惯性系中此电磁场的电场强度矢量必然正交于其磁感应强度.
3. 利用毕安琪恒等式不难看出：

$$\begin{aligned}F_{\mu\nu}\mathscr{F}^{\mu\nu}&=(\partial_\mu A_\nu-\partial_\nu A_\mu)\,\mathscr{F}^{\mu\nu}=2\,(\partial_\mu A_\nu)\,\mathscr{F}^{\mu\nu}\\&=2\partial_\mu\left(A_\nu\mathscr{F}^{\mu\nu}\right)-2A_\nu\left(\partial_\mu\mathscr{F}^{\mu\nu}\right)\\&=2\partial_\mu\left(A_\nu\mathscr{F}^{\mu\nu}\right)\end{aligned}$$

这是 $\mathbb{M}_4$ 中的全散度. 对于拓扑性质平庸的 $\mathbb{M}_4$ 而言，即使 $F_{\mu\nu}\mathscr{F}^{\mu\nu}$ 项出现在拉氏密度中，也不会产生实质性的贡献.

顺便指出，虽然

$$\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}$$

也是一个 4-标量，但它实际上就是 $F_{\mu\nu}F^{\mu\nu}$. 无须重复考虑.

事实上，

$$\begin{aligned}
\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu} &= \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\alpha\beta}F^{\rho\sigma}F_{\alpha\beta} \\
&= -\frac{1}{2}\left(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\beta}_{\rho}\delta^{\alpha}_{\sigma}\right)F^{\rho\sigma}F_{\alpha\beta} \\
&= -\frac{1}{2}\left(F^{\alpha\beta} - F^{\beta\alpha}\right)F_{\alpha\beta} \\
&= -F_{\mu\nu}F^{\mu\nu}
\end{aligned}$$

直接考虑其物理内涵，我们有:

$$\begin{aligned}
\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu} &= 2\mathscr{F}_{i0}\mathscr{F}^{i0} + \mathscr{F}_{ij}\mathscr{F}^{ij} \\
&= -2B_iB^i + \frac{1}{c^2}\epsilon^{ijk}\epsilon_{ijl}E^lE_k \\
&= 2\left(\frac{\boldsymbol{E}^2}{c^2} - \boldsymbol{B}^2\right)
\end{aligned}$$

拓展：

规范不变的 4-标量

$$F_{\mu\nu}\mathscr{F}^{\mu\nu}$$

倘若出现在拓扑非平庸时空中某个场论体系的拉氏密度中，就称其为 Chern-Simions (CS) 项.

- 因为

$$\begin{aligned} F_{\mu\nu}\mathscr{F}^{\mu\nu} &= 2\partial_\mu\left(A_\nu\mathscr{F}^{\mu\nu}\right) = \partial_\mu\left(\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right) \\ &= 2\partial_\mu\left(\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma\right) \\ &= \partial_\mu K^\mu \end{aligned}$$

  4-矢量

$$K^\mu = 2\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma$$

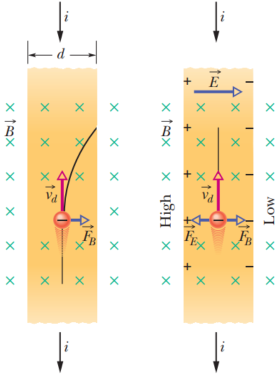
  称为（阿贝尔）Chern-Simions 流密度.

- CS 项意味着时间反演对称性的破坏与宇称不再守恒. 因此，在 QCD 的非微扰研究中常计及非阿贝尔 CS 项的贡献以企探索 $\mathcal{P}$ 与 $\mathcal{T}$ 这两个分立对称性的破坏程度.

- 磁场的存在会破坏时间反演对称性. 因此, CS 项常出现在 $(1+2)$ 维时空的电磁相互作用拉氏密度

$$\mathscr{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} + \alpha\,\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} - J^\mu A_\mu$$

中, 从理论上解释与理解 $(1+2)$ 维时空中发现的分数统计和分数量子霍尔效应[17].



[17] 例如: Daniel Arovas, Lecture notes on quantum hall effect, 2020, in progress.

Quantum Hall Effect

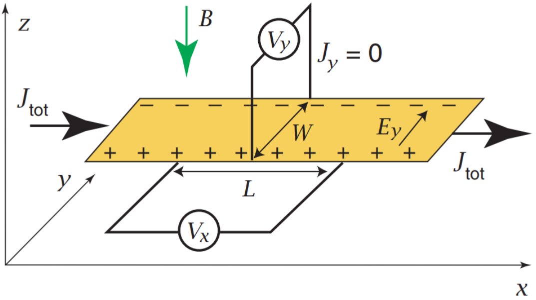


Fig. 1 Hall and diagonal resistivities R_{xy} and R_{xx} are independent of sample properties and given by $R_{xy} = V_y/J_{tot} \longrightarrow \nu^{-1}(2\pi\hbar/e^2)$, $R_{xx} = -WV_x/LJ_{tot} \longrightarrow 0$, in quantum Hall (QH) states. Here, ν is the filling factor.

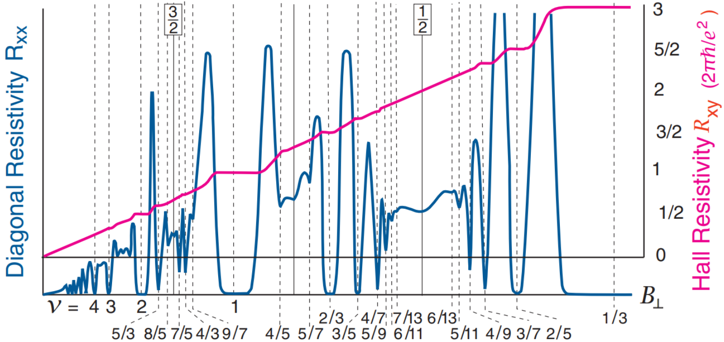
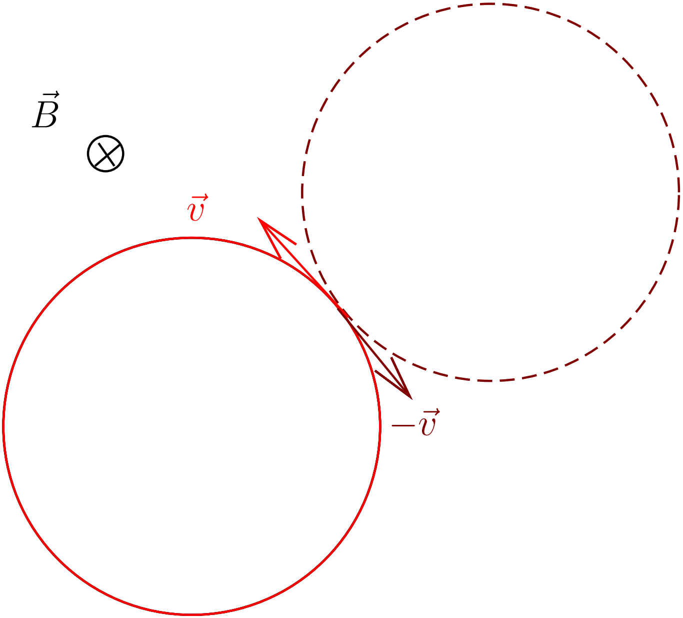


Fig. 2 QH states are detected by plateaux developed in the Hall resistivity R_{xy} or dips in the diagonal resistivity R_{xx}.

磁场的存在破坏时间反演对称性



$\mathbb{M}_3$ 与 $\mathbb{M}_4$ 中 Chern-Simions 项的比较:

如前所述, 对于 $(1+3)$ 维时空 $\mathbb{M}_4$ 中的 $U(1)$ 规范场而言, 其拉氏密度中候选的 Chern-Simions 项为:

$$\mathscr{L}_{\rm CS}^{(4)} = \alpha\, F_{\mu\nu}\mathscr{F}^{\mu\nu} = \frac{\alpha}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$\mathscr{L}_{\rm CS}^{(4)}$ 显然地是一个规范不变的 4-标量. 因为

$$\frac{\partial\mathscr{L}_{\rm CS}^{(4)}}{\partial A^\nu} = 0,$$

$$\frac{\partial\mathscr{L}_{\rm CS}^{(4)}}{\partial(\partial_\mu A_\nu)} = \alpha\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = 2\alpha\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta} = 4\mathscr{F}^{\mu\nu}$$

$\mathscr{L}_{\rm CS}^{(4)}$ 对规范势 $A_\nu(x)$ 拉氏方程的贡献是:

$$\frac{\partial\mathscr{L}_{\rm CS}^{(4)}}{\partial A^\nu} - \partial_\mu\frac{\partial\mathscr{L}_{\rm CS}^{(4)}}{\partial(\partial_\mu A_\nu)} = -4\partial_\mu\mathscr{F}^{\mu\nu}$$

考虑到毕安琪恒等式, $\partial_\mu\mathscr{F}^{\mu\nu} = 0$, 这个贡献实际上为零.

- 对于 $\mathbb{M}_4$ 中的 $U(1)$ 规范场而言，拉氏密度中计及 CS 项与否都不影响规范场的运动方程.

$(1+2)$ 时空 $\mathbb{M}_3$ 中的情形与此不同. $\mathbb{M}_3$ 中的全反对称不变张量是 $\epsilon^{\mu\nu\rho}$[18]. 因此，$\mathbb{M}_3$ 中规范场的毕安琪恒等式为：

$$\epsilon^{\mu\nu\rho}\partial_\rho F_{\mu\nu} = 0$$

$U(1)$ 规范场拉氏密度中候选的 Chern-Simions 项为：

$$\boxed{\mathscr{L}_{\text{CS}}^{(3)} = \alpha\epsilon^{\mu\nu\rho}A_\rho F_{\mu\nu}}$$

容易看出，$\mathscr{L}_{\text{CS}}^{(3)}$ 也是局域规范变换 $\delta A_\mu(x) = \partial_\mu\theta(x)$ 下的不变量：

$$\delta\mathscr{L}_{\text{CS}}^{(3)} = \alpha\epsilon^{\mu\nu\rho}\Big(\delta A_\rho\Big)F_{\mu\nu} = \alpha\epsilon^{\mu\nu\rho}\Big(\partial_\rho\theta(x)\Big)F_{\mu\nu} = \partial_\rho\Big(\alpha\theta(x)\epsilon^{\mu\nu\rho}F_{\mu\nu}\Big)$$

最后一步使用了毕安琪恒等式.

---

[18] 我们约定 $\epsilon^{012} = 1$.

$\mathscr{L}_{\mathrm{CS}}^{(3)}$ 对"广义坐标"$A_\mu(x)$ 以及"广义速度"$\partial_\nu A_\mu(x)$ 的偏导数为：

$$\frac{\partial \mathscr{L}_{\mathrm{CS}}^{(3)}}{\partial A^\nu} = \alpha\epsilon^{\nu\rho\sigma}F_{\rho\sigma},$$

$$\frac{\partial \mathscr{L}_{\mathrm{CS}}^{(3)}}{\partial(\partial_\mu A_\nu)} = \alpha\epsilon^{\beta\rho\sigma}A_\sigma\frac{\partial F_{\beta\rho}}{\partial(\partial_\mu A_\nu)} = 2\alpha\epsilon^{\mu\nu\sigma}A_\sigma$$

因此，$\mathscr{L}_{\mathrm{CS}}^{(3)}$ 对规范场拉氏方程的贡献为：

$$\frac{\partial \mathscr{L}_{\mathrm{CS}}^{(3)}}{\partial A^\nu} - \partial_\mu\frac{\partial \mathscr{L}_{\mathrm{CS}}^{(3)}}{\partial(\partial_\mu A_\nu)} = \alpha\epsilon^{\nu\rho\sigma}F_{\rho\sigma} - 2\alpha\epsilon^{\mu\nu\sigma}\partial_\mu A_\sigma = 4\alpha\epsilon^{\nu\rho\sigma}\partial_\rho A_\sigma$$

这个贡献不为零.

- 对于 $\mathbb{M}_3$ 中的 $U(1)$ 规范场而言，拉氏密度中计及 CS 项的贡献会严重影响规范场的运动方程.

电磁场强度的洛伦兹推动变换

电场强度 $\boldsymbol{E}$ 与磁感应强度 $\boldsymbol{B}$ 联合构成电磁场张量 $F_{\mu\nu}(x)$，它们在洛伦兹变换 $x^\mu \rightsquigarrow x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$ 下的变换法则自然是：

$$\boxed{F_{\mu\nu}(x) \quad \rightsquigarrow \quad F'_{\mu\nu}(x') = F_{\rho\sigma}(x)\, \tilde{\Lambda}^\rho{}_\mu\, \tilde{\Lambda}^\sigma{}_\nu}$$

对于无量纲牵连速度为 $\boldsymbol{\beta}$ 的洛伦兹推动变换，逆洛伦兹变换矩阵 $\boldsymbol{\Lambda}$ 非零的矩阵元为：

$$\tilde{\Lambda}^0{}_0 = \gamma, \quad \tilde{\Lambda}^0{}_i = \gamma\beta_i, \quad \tilde{\Lambda}^i{}_0 = \gamma\beta^i, \quad \tilde{\Lambda}^i{}_j = \delta^i_j + \frac{\gamma^2}{\gamma+1}\beta^i\beta_j.$$

式中 $\gamma = 1/\sqrt{1-\beta^2}$ 是相对论收缩因子.

回想电磁场张量的物理内涵，

$$\boxed{E_i = -cF_{0i}, \quad B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}}$$

我们看到：

$$E'_i = -cF'_{0i} = -cF_{\rho\sigma}\, \tilde{\Lambda}^\rho{}_0\, \tilde{\Lambda}^\sigma{}_i = -cF_{0\sigma}\, \tilde{\Lambda}^0{}_0\, \tilde{\Lambda}^\sigma{}_i - cF_{k\sigma}\, \tilde{\Lambda}^k{}_0\, \tilde{\Lambda}^\sigma{}_i$$

亦即：

$$\begin{aligned}
E_i' &= -cF_{0j}\,\tilde{\Lambda}^0{}_0\,\tilde{\Lambda}^j{}_i - cF_{k0}\,\tilde{\Lambda}^k{}_0\,\tilde{\Lambda}^0{}_i - cF_{kj}\,\tilde{\Lambda}^k{}_0\,\tilde{\Lambda}^j{}_i \\
&= -\gamma cF_{0i} + \frac{\gamma^2}{\gamma+1}\beta_i\beta^j cF_{0j} - \gamma c\beta^j F_{ji} \\
&= \gamma E_i - \frac{\gamma^2}{\gamma+1}\beta_i\beta^j E_j + \gamma c\beta^j\epsilon_{ijk}B^k
\end{aligned}$$

换言之，

$$\boxed{\boldsymbol{E}' = \gamma\boldsymbol{E} - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{E})\boldsymbol{\beta} + \gamma c\,\boldsymbol{\beta}\times\boldsymbol{B}}$$

同理有：

$$\begin{aligned}
B'^i &= \frac{1}{2}\epsilon^{ijk}F'_{jk} = \frac{1}{2}\epsilon^{ijk}F_{\mu\nu}\,\tilde{\Lambda}^\mu{}_j\,\tilde{\Lambda}^\nu{}_k \\
&= \frac{1}{2}\epsilon^{ijk}F_{0l}\,\tilde{\Lambda}^0{}_j\,\tilde{\Lambda}^l{}_k + \frac{1}{2}\epsilon^{ijk}F_{m0}\,\tilde{\Lambda}^m{}_j\,\tilde{\Lambda}^0{}_k + \frac{1}{2}\epsilon^{ijk}F_{ml}\,\tilde{\Lambda}^m{}_j\,\tilde{\Lambda}^l{}_k \\
&= \epsilon^{ijk}F_{0l}\,\tilde{\Lambda}^0{}_j\,\tilde{\Lambda}^l{}_k + \frac{1}{2}\epsilon^{ijk}F_{ml}\,\tilde{\Lambda}^m{}_j\,\tilde{\Lambda}^l{}_k
\end{aligned}$$

亦即：

$$
\begin{aligned}
B'^{i} &= \gamma\epsilon^{ijk}\beta_j F_{0k} + \frac{1}{2}\epsilon^{ijk}F_{jk} + \frac{\gamma^2}{\gamma+1}\epsilon^{ijk}F_{jl}\beta^l\beta_k \\
&= -\frac{\gamma}{c}\epsilon^{ijk}\beta_j E_k + B^{i} + \frac{\gamma^2}{\gamma+1}\epsilon^{ijk}\epsilon_{jlm}B^m\beta^l\beta_k \\
&= B^{i} - \frac{\gamma}{c}(\boldsymbol{\beta}\times\boldsymbol{E})^{i} + \frac{\gamma^2}{\gamma+1}\left(\delta^k_l\delta^i_m - \delta^k_m\delta^i_l\right)B^m\beta^l\beta_k \\
&= B^{i} - \frac{\gamma}{c}(\boldsymbol{\beta}\times\boldsymbol{E})^{i} + \frac{\gamma^2\beta^2}{\gamma+1}B^{i} - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B})\beta^{i} \\
&= \gamma B^{i} - \frac{\gamma}{c}(\boldsymbol{\beta}\times\boldsymbol{E})^{i} - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B})\beta^{i}
\end{aligned}
$$

换言之，

$$
\boxed{\boldsymbol{B}' = \gamma\boldsymbol{B} - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B})\boldsymbol{\beta} - \frac{\gamma}{c}\boldsymbol{\beta}\times\boldsymbol{E}}
$$

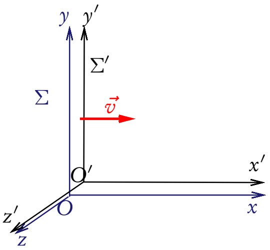
所以，电场与磁场是同一个电磁场的两个侧面. 在给定参考系中，电磁场的电场与磁场表现出不同性质. 但是当参考系变换时，它们可以互相转化.

电磁场强度在两个惯性系之间的反变换关系式是：

$$\boxed{\boldsymbol{E} = \gamma \boldsymbol{E}' - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{E}')\boldsymbol{\beta} - \gamma c\,\boldsymbol{\beta}\times\boldsymbol{B}'}$$

$$\boxed{\boldsymbol{B} = \gamma \boldsymbol{B}' - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{B}')\boldsymbol{\beta} + \frac{\gamma}{c}\boldsymbol{\beta}\times\boldsymbol{E}'}$$

例：求出以速度 $\boldsymbol{v} = c\boldsymbol{\beta}$ 相对于惯性系 Σ 作匀速直线运动的点电荷 Q 所激发的电磁场场强分布.



解:

选择 Σ' 系为点电荷自身系. 由于在 Σ' 系中点电荷始终处在静止状态, 故 Σ' 系中只存在静电场:

$$\boldsymbol{E}' = \frac{Q\boldsymbol{r}'}{4\pi\epsilon_0 r'^3}, \quad \boldsymbol{B}' = 0$$

按照洛伦兹推动变换, Q 在实验室系 Σ 中激发的电磁场场强为:

$$\boxed{\boldsymbol{E} = \gamma\boldsymbol{E}' - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\boldsymbol{E}'\right), \quad \boldsymbol{B} = \frac{\gamma}{c}\boldsymbol{\beta}\times\boldsymbol{E}'}$$

求上述第一式与牵连速度 $\boldsymbol{\beta}$ 的矢量积, 可知 $\boldsymbol{\beta}\times\boldsymbol{E} = \gamma\,\boldsymbol{\beta}\times\boldsymbol{E}'$. 所以实验室参考系中电磁场强是彼此正交的:

$$\boxed{\boldsymbol{B} = \frac{1}{c}\boldsymbol{\beta}\times\boldsymbol{E}}$$

此外, 一旦求出了实验室系中的电场强度分布, 上式可以使我们立刻获得磁感应强度的分布.

以下专心计算 Σ 系中的电场强度分布. 场点位置矢量在二惯性系 Σ 和 Σ' 之间的洛伦兹推动变换是

$$\boldsymbol{r}' = \boldsymbol{r} + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\boldsymbol{r}\right) - \gamma c\boldsymbol{\beta}t$$

假设 Σ 系中的观察者是在 $t=0$ 时刻进行测量, 则有:

$$\boldsymbol{r}' = \boldsymbol{r} + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\boldsymbol{r}\right)$$

且

$$\begin{aligned}
r'^2 &= \boldsymbol{r}'\cdot\boldsymbol{r}' \\
&= r^2 + \frac{2\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\boldsymbol{r}\right)^2 + \frac{\gamma^4\beta^2}{(\gamma+1)^2}\left(\boldsymbol{\beta}\cdot\boldsymbol{r}\right)^2 \\
&= r^2 + \gamma^2\left(\boldsymbol{\beta}\cdot\boldsymbol{r}\right)^2
\end{aligned}$$

$$\rightsquigarrow \quad \boldsymbol{E}' = \frac{Q}{4\pi\epsilon_0\left[r^2+\gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{r})^2\right]^{3/2}}\left[\boldsymbol{r} + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{r})\right]$$

于是:

$$\begin{aligned} \boldsymbol{E} &= \gamma \boldsymbol{E}' - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E}') \\ &= \frac{Q}{4\pi\epsilon_0\left[r^2+\gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{r})^2\right]^{3/2}}\left[\gamma\boldsymbol{r} + \frac{\gamma^3}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{r}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{r})\right. \\ &\qquad \left. - \frac{\gamma^4\beta^2}{(\gamma+1)^2}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{r})\right] \\ &= \frac{\gamma Q\, \boldsymbol{r}}{4\pi\epsilon_0\left[r^2+\gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{r})^2\right]^{3/2}} \end{aligned}$$

运动点电荷 Q 激发的电磁场在实验室系 Σ 中的磁感应强度为:

$$\boldsymbol{B} = \frac{1}{c}\boldsymbol{\beta}\times\boldsymbol{E} = \frac{\gamma\mu_0 Q(\boldsymbol{v}\times\boldsymbol{r})}{4\pi\left[r^2+\gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{r})^2\right]^{3/2}}$$

若点电荷作极低速运动, $v \ll c$, 可以略去 β^2 级项且取 $\gamma \approx 1$,

$$\rightsquigarrow \quad \boldsymbol{E} \approx \frac{Q\boldsymbol{r}}{4\pi\epsilon_0 r^3}, \qquad \boldsymbol{B} \approx \frac{\mu_0 Q\,(\boldsymbol{v}\times\boldsymbol{r})}{4\pi r^3}$$