场量的连续变换

为了讨论电磁相互作用服从的守恒定律，我们需要研究电磁规范势 $A_\mu(x)$ 的变换性质.

- $A_\mu(x)$ 的变换可以仅仅是场空间内部的变化，例如局域规范变换

$$A_\mu(x) \quad \rightsquigarrow \quad A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x)$$

- $A_\mu(x)$ 的变换也可以是时空点位置坐标的改变在场空间诱导的变换. 例如洛伦兹变换

$$x^\mu \quad \rightsquigarrow \quad x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \qquad \boxed{\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}}$$

可以诱导出电磁势的如下变换:

$$A_\mu(x) \quad \rightsquigarrow \quad A'_\mu(x') = A_\nu(x)\tilde{\Lambda}^\nu{}_\mu \qquad \boxed{\tilde{\Lambda}^\nu{}_\mu = \eta_{\mu\rho}\Lambda^\rho{}_\sigma \eta^{\sigma\nu}}$$

- 倘若场量 $A_\mu(x)$ 以及时空点位置坐标的改变存在无穷小形式，这类改变就称为连续变换.
- 电磁势在场空间的无穷小变换记为：

$$\delta_0 A_\mu(x) = A'_\mu(x) - A_\mu(x)$$

- 我们把时空点位置坐标的无穷小变换记为

$$x^\mu \quad \rightsquigarrow \quad x'^\mu = x^\mu + \delta x^\mu$$

它在场空间诱导的电磁势无穷小变换是：

$$\delta A_\mu(x) = A'_\mu(x') - A_\mu(x)$$

显然，

$$\begin{aligned}\delta A_\mu(x) \quad &= A'_\mu(x+\delta x) - A_\mu(x) \approx \left[A'_\mu(x) + \delta x^\nu \partial_\nu A'_\mu(x)\right] - A_\mu(x) \\ &\approx \delta_0 A_\mu(x) + \delta x^\nu \partial_\nu A_\mu(x) + \mathscr{O}\left((\delta x)^2\right)\end{aligned}$$

所以，精确到无穷小变换变换参数的一次幂，我们有：

$$\boxed{\delta A_\mu(x) = \delta_0 A_\mu(x) + \delta x^\nu \partial_\nu A_\mu(x)}$$

- 无穷小的局域规范变换：

$$\delta x^\mu = 0, \quad \delta A_\mu(x) = \delta_0 A_\mu(x) = \partial_\mu \theta(x)$$

- 通常认为电磁势等场量在时空点位置坐标的平移变换下保持不变

$$x^\mu \ \rightsquigarrow \ x'^\mu = x^\mu + b^\mu, \qquad A_\mu(x) \ \rightsquigarrow \ A'_\mu(x') = A_\mu(x)$$

所以，无穷小的时空平移变换表达为：

$$\delta x^\mu = \epsilon^\mu, \qquad \delta A_\mu(x) = 0, \qquad \delta_0 A_\mu(x) = -\epsilon^\nu \partial_\nu A_\mu(x)$$

- 洛伦兹变换矩阵 $\Lambda^{\mu}{}_{\nu}$ 服从条件 $\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$. 设 $\Lambda^{\mu}{}_{\nu}$ 的无穷小形式为：

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \eta^{\mu\rho}\omega_{\rho\nu}$$

$\omega_{\mu\nu}$ 是无穷小的洛伦兹变换实参数[19]，我们看到：

$$\begin{aligned}\eta_{\rho\sigma} &= \eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\mu\nu}\left(\delta^{\mu}_{\rho} + \omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}_{\sigma} + \omega^{\nu}{}_{\sigma}\right)\\ &= \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + \mathscr{O}(\omega^2) \qquad \rightsquigarrow \quad \boxed{\omega_{\rho\sigma} = -\omega_{\sigma\rho}}\end{aligned}$$

因此，逆洛伦兹矩阵的无穷小形式是：

$$\tilde{\Lambda}^{\mu}{}_{\nu} = \eta_{\nu\rho}\Lambda^{\rho}{}_{\sigma}\eta^{\sigma\mu} = \delta^{\mu}_{\nu} - \eta^{\mu\rho}\omega_{\rho\nu}$$

电磁势 $A_\mu(x)$ 相关的无穷小洛伦兹变换最终表达为：

$$\begin{aligned}&\delta x^{\mu} = \eta^{\mu\rho}\omega_{\rho\nu}x^{\nu}, \qquad \delta A_{\mu}(x) = \omega_{\mu\nu}A^{\nu}(x),\\ &\delta_0 A_{\mu}(x) = \omega_{\mu\nu}A^{\nu}(x) + \omega_{\rho\sigma}x^{\rho}\partial^{\sigma}A_{\mu}(x)\end{aligned}$$

[19]我们约定：$\omega^{\mu}{}_{\nu} = \eta^{\mu\rho}\omega_{\rho\nu}$.

解释：

电磁势作为一个协变的 4-矢量，它遵循的洛伦兹变换为：

$$A_\mu(x) \quad \rightsquigarrow \quad A'_\mu(x') = A_\nu(x)\, \tilde{\Lambda}^\nu{}_\mu$$

对于无穷小洛伦兹变换，

$$A'_\mu(x') = A_\nu(x)\left(\delta^\nu_\mu - \eta^{\nu\rho}\omega_{\rho\mu}\right) = A_\mu(x) + \omega_{\mu\rho}A^\rho(x)$$

所以，

$$\begin{aligned}
\delta A_\mu(x) &= \omega_{\mu\nu}A^\nu(x) \\
&= \frac{1}{2}\left(\omega_{\mu\nu} - \omega_{\nu\mu}\right)A^\nu(x) = \frac{1}{2}\omega_{\alpha\beta}\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right)A^\nu(x) \\
&= \frac{1}{2}\omega_{\alpha\beta}\left(\Sigma^{\alpha\beta}\right)_{\mu\nu}A^\nu(x)
\end{aligned}$$

式中

$$\boxed{\left(\Sigma^{\alpha\beta}\right)_{\mu\nu} = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu}$$

是洛伦兹群 $O(1,3)$ 在其矢量表示中生成元的矩阵元.

时空点位置坐标 x^μ 的无穷小洛伦兹变换为：

$$\delta x^\mu = \eta^{\mu\rho}\omega_{\rho\sigma}x^\sigma$$

因此，电磁势的无穷小洛伦兹变换可以表达为如下纯粹的场空间中 $A_\mu(x)$ 的改变：

$$\begin{aligned}
\delta_0 A_\mu(x) &= \delta A_\mu(x) - \delta x^\nu \partial_\nu A_\mu(x) \\
&= \omega_{\mu\nu}A^\nu(x) + \omega_{\rho\sigma}x^\rho\partial^\sigma A_\mu(x) \\
&= \frac{1}{2}\omega_{\rho\sigma}\left(\Sigma^{\rho\sigma}\right)_{\mu\nu}A^\nu(x) + \frac{1}{2}\omega_{\rho\sigma}\left(x^\rho\partial^\sigma - x^\sigma\partial^\rho\right)A_\mu(x)
\end{aligned}$$

亦即，

$$\boxed{\delta_0 A_\mu(x) = \frac{1}{2}\omega_{\rho\sigma}\left[\eta_{\mu\nu}\left(x^\rho\partial^\sigma - x^\sigma\partial^\rho\right) + \left(\Sigma^{\rho\sigma}\right)_{\mu\nu}\right]A^\nu(x)}$$

下面考虑电磁相互作用的诺特定理.

诺特定理

诺特定理断言：相应于体系作用量泛函在任意一种连续变换下的不变性，体系存在着一个守恒定律. 现在以电磁相互作用为例讨论诺特定理. 电磁场的作用量泛函是：

$$S[A_\mu] = \int \mathrm{d}^4x\, \mathscr{L}(x) = \int \mathrm{d}^4x\, \mathscr{L}(A_\mu(x),\, \partial_\nu A_\mu(x))$$

式中，

$$\boxed{\mathscr{L}(x) = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu + \mathscr{L}_{\mathrm{m}}(x)}$$

电磁势的规范变换 $\delta_0 A_\mu(x) = \partial_\mu \theta(x)$ 是纯粹的场空间中场量的改变，

$$\delta_0 \mathscr{L} = J^\mu \delta_0 A_\mu = J^\mu \partial_\mu \theta(x) = \partial_\mu \left[J^\mu \theta(x)\right] - \theta(x)\partial_\mu J^\mu$$

所以，作用量泛函在局域规范变换下的不变性意味着存在电荷守恒定律：

$$0 = \delta_0 S[A_\mu] = -\int \mathrm{d}^4x\, \theta(x)\, \partial_\mu J^\mu \quad \rightsquigarrow \quad \partial_\mu J^\mu = 0$$

现在考虑作用量泛函在时空点位置坐标的连续变换

$$x^\mu \quad \rightsquigarrow \quad x'^\mu = x^\mu + \delta x^\mu$$

下的不变性所导致的守恒定律. 注意到:

$$\boxed{\delta S = \int \mathrm{d}^4 x \, (\delta \mathscr{L}) + \int (\delta \mathrm{d}^4 x) \, \mathscr{L}}$$

- 因为场空间电磁势的无穷小变换 $\delta_0 A_\mu(x)$ 引起的拉氏密度改变为:

$$\begin{aligned}
\delta_0 \mathscr{L} &= \frac{\partial \mathscr{L}}{\partial A_\mu(x)} \delta_0 A_\mu(x) + \frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu(x))} \delta_0(\partial_\nu A_\mu(x)) \\
&= \left[\frac{\partial \mathscr{L}}{\partial A_\mu(x)} - \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu(x))} \right] \delta_0 A_\mu(x) \\
&\quad + \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu(x))} \delta_0 A_\mu(x) \right) \\
&= \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu(x))} \delta_0 A_\mu(x) \right)
\end{aligned}$$

- 直接因为 x^μ 的无穷小变换引起的拉氏密度改变为:

$$\delta_1\mathscr{L} = \frac{\partial\mathscr{L}(x)}{\partial x^\mu}\,\delta x^\mu = \delta x^\mu\,\partial_\mu\mathscr{L}$$

所以,

$$\delta\mathscr{L} = \delta_0\mathscr{L} + \delta_1\mathscr{L} = \partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu A_\mu(x))}\delta_0 A_\mu(x)\right) + \delta x^\mu\,\partial_\mu\mathscr{L}$$

接下来再考虑 $\delta(\mathrm{d}^4x)$. 因为

$$x'^\mu = x^\mu + \delta x^\mu \qquad \rightsquigarrow\quad \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu(\delta x^\mu)$$

其行列式为:

$$\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \prod_{a=0}^{3}\left[1 + \partial_a(\delta x^a)\right] = 1 + \partial_\mu(\delta x^\mu)$$

由此知:

$$\delta\left(\mathrm{d}^4x\right) = \mathrm{d}^4x' - \mathrm{d}^4x = \left[\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) - 1\right]\mathrm{d}^4x = \mathrm{d}^4x\,\partial_\mu(\delta x^\mu)$$

所以，时空坐标的无穷小变化

$$x^{\mu} \quad \rightsquigarrow x'^{\mu} = x^{\mu} + \delta x^{\mu}$$

可能引起的电磁相互作用体系作用量泛函的改变为：

$$\begin{aligned}
\delta S &= \int \left[\delta(\mathrm{d}^4 x)\mathscr{L} + \mathrm{d}^4 x\, \delta\mathscr{L}\right] \\
&= \int \mathrm{d}^4 x \left[\mathscr{L}\partial_{\mu}(\delta x^{\mu}) + \delta x^{\mu}\partial_{\mu}\mathscr{L} + \partial_{\nu}\left(\frac{\partial\mathscr{L}}{\partial(\partial_{\nu}A_{\mu}(x))}\delta_0 A_{\mu}(x)\right)\right] \\
&= \int \mathrm{d}^4 x\, \partial_{\mu}\left[\mathscr{L}\delta x^{\mu} + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}A_{\nu}(x))}\delta_0 A_{\nu}(x)\right] \\
&= \int \mathrm{d}^4 x\, \partial_{\mu}\mathscr{J}^{\mu}
\end{aligned}$$

式中，

$$\mathscr{J}^{\mu} = \mathscr{L}\delta x^{\mu} + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}A_{\nu}(x))}\delta_0 A_{\nu}(x)$$

- 倘若时空坐标变换 δx^μ 是电磁相互作用体系的一种对称性:

$$\delta S = 0 \qquad \rightsquigarrow \quad \partial_\mu \mathscr{J}^\mu = 0$$

如此,

$$\boxed{\mathscr{J}^\mu = \mathscr{L}\delta x^\mu + \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu(x))}\delta_0 A_\nu(x)}$$

就称为与此对称性相联系的守恒流 4-矢量.

- 回忆矢量场 $A_\mu(x)$ 的能量动量张量 $T^{\mu\nu}$ 的定义:

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - \eta^{\mu\nu}\mathscr{L}$$

可以把守恒流 4-矢量改写为:

$$\begin{aligned}\mathscr{J}^\mu &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)}\delta_0 A_\nu + \delta x_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - T^{\mu\nu}\right) \\ &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu(x))}\delta A_\nu(x) - T^{\mu\nu}(x)\,\delta x_\nu\end{aligned}$$

时空平移对称性

在无穷小的时空平移变换下，

$$\delta x_\nu = \epsilon_\nu, \quad \delta A_\nu(x) = 0$$

倘若电磁相互作用体系具有时空平移对称性，$\delta S = 0$，相应的守恒流 4-矢量为：

$$\partial_\mu \mathscr{J}^\mu(x) = 0 \qquad \looparrowleft \quad \mathscr{J}^\mu(x) = -T^{\mu\nu}(x)\,\epsilon_\nu$$

所以，

- 与时空平移对称性相联系的守恒定律本质上是电磁相互作用体系的能量动量守恒定律：

$$\partial_\mu T^{\mu\nu}(x) = 0 \qquad \rightsquigarrow \quad \frac{\partial T^{0\nu}}{\partial t} = -c\,\partial_i T^{i\nu}$$

- 定义 4-矢量 $\mathscr{M}^\mu$：

$$\mathscr{M}^\mu \equiv -\int \mathrm{d}^3x\, T^{0\mu}(x)$$

它实际上就是体系能量动量守恒定律中出现的守恒量：

$$\begin{aligned}\frac{\mathrm{d}\mathscr{M}^{\mu}}{\mathrm{d}t} &= c\int \mathrm{d}^3x\,\partial_i T^{i\mu}(x)\\ &= c\oint_{\infty}\mathrm{d}s_i T^{i\mu}(x)\\ &= 0\end{aligned}$$

- 电磁相互作用体系的能量与动量分别定义为：

$$E = -\int \mathrm{d}^3x\; T^{00}(x)$$

$$\boldsymbol{P}^{i} = -\frac{1}{c}\int \mathrm{d}^3x\; T^{0i}(x)$$

换言之，

$$\mathscr{M}^{\mu} = (E,\ c\boldsymbol{P})$$

电磁场的能量动量 4-张量

- 自由电磁场的拉氏密度为

$$\mathscr{L}_{\text{em}} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

所以，电磁场对于电磁相互作用体系能量动量张量的贡献为：

$$T^{\mu\nu}_{\text{em}} = -\frac{1}{\mu_0} F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4\mu_0} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

$\mathscr{Q}$: 能否把 $T^{\mu\nu}_{\text{em}}$ 诠释为电磁场的能量动量 4-张量？

1. 很明显，$T^{\mu\nu}_{\text{em}}$ 不是规范变换下的不变量.
2. 倘若电磁相互作用的理论可以作为一个可靠的物理理论，那么 $T^{\mu\nu}_{\text{em}}$ 不具有规范变换下的不变性就是一条不可接受的缺点. 我们的结论是：$T^{\mu\nu}_{\text{em}}$ 没有获得能被解读为电磁场能量动量 4-张量的资格.

- 因为

$$\partial^\nu A_\rho = \eta^{\nu\sigma}\partial_\sigma A_\rho = \eta^{\nu\sigma}\left(F_{\sigma\rho} + \partial_\rho A_\sigma\right) = -\eta^{\nu\sigma}F_{\rho\sigma} + \eta^{\nu\sigma}\partial_\rho A_\sigma$$

我们可以把 $T_{\text{em}}^{\mu\nu}$ 改写为：

$$\begin{aligned} T_{\text{em}}^{\mu\nu} &= \frac{1}{\mu_0}\left(\eta^{\nu\sigma}F^{\mu\rho}F_{\rho\sigma} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) - \frac{1}{\mu_0}\eta^{\nu\sigma}F^{\mu\rho}\partial_\rho A_\sigma \\ &= \Theta^{\mu\nu} + T_{\text{D}}^{\mu\nu} \end{aligned}$$

式中，

$$\boxed{\Theta^{\mu\nu} = \frac{1}{\mu_0}\left(\eta^{\nu\sigma}F^{\mu\rho}F_{\rho\sigma} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)}$$

而

$$T_{\text{D}}^{\mu\nu} = -\frac{1}{\mu_0}\eta^{\nu\sigma}F^{\mu\rho}\partial_\rho A_\sigma$$

使用自由电磁场的 Maxwell 方程 $\partial_\rho F^{\mu\rho} = 0$，我们可以把 $T_{\text{D}}^{\mu\nu}$ 表达为一个全导数：

$$\boxed{T_{\mathrm{D}}^{\mu\nu} = -\frac{1}{\mu_0}\partial_\rho\left(\eta^{\nu\sigma}F^{\mu\rho}A_\sigma\right)}$$

1. $T_{\mathrm{D}}^{\mu\nu}$ 自动满足守恒定律：

$$\partial_\mu T_{\mathrm{D}}^{\mu\nu} = -\frac{1}{\mu_0}\partial_\mu\partial_\rho\left(\eta^{\nu\sigma}F^{\mu\rho}A_\sigma\right) = 0 \quad \rightsquigarrow \quad \partial_\mu\Theta^{\mu\nu} = \partial_\mu T_{\mathrm{em}}^{\mu\nu}$$

2. $T_{\mathrm{D}}^{0\mu}$ 在 $\mathbb{E}_3$ 上的体积分为零：

$$\begin{aligned}\int \mathrm{d}^3x\, T_{\mathrm{D}}^{0\mu} &= -\frac{1}{\mu_0}\int \mathrm{d}^3x\,\partial_i\left(\eta^{\mu\sigma}F^{0i}A_\sigma\right)\\ &= -\frac{1}{\mu_0}\oint_\infty \mathrm{d}s_i\,\left(\eta^{\mu\sigma}F^{0i}A_\sigma\right)\\ &= 0\end{aligned}$$

所以，

$$\mathscr{M}_{\mathrm{em}}^{\mu} \equiv -\int \mathrm{d}^3x\, T_{\mathrm{em}}^{0\mu}(x) = -\int \mathrm{d}^3x\,\Theta^{0\mu}(x)$$

综合以上因素，通常把电磁场本身的能量动量张量定义为：

$$\Theta^{\mu\nu} = \frac{1}{\mu_0}\left(\eta^{\nu\sigma}F^{\mu\rho}F_{\rho\sigma} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

$\Theta^{\mu\nu}$ 具有如下特点：

- 它是规范不变的二阶 4-张量.
- 它是对称张量，$\Theta^{\mu\nu} = \Theta^{\nu\mu}$.
- 它是无迹张量，

$$\Theta^{\mu}{}_{\mu} = \eta_{\mu\nu}\Theta^{\mu\nu} = 0$$

- 它服从守恒定律：

$$\partial_\mu\left(\Theta^{\mu\nu} + T_{\mathrm{m}}^{\mu\nu}\right) = 0$$

分量 $\Theta^{0\mu}$ 在 $\mathbb{E}_3$ 上的体积分诠释为电磁场本身的能量与动量：

$$(E,\ cP) = \mathscr{M}^{\mu} = -\int \mathrm{d}^3x\, \Theta^{0\mu}(x)$$

现在讨论 $\Theta^{\mu\nu}$ 诸分量的物理内涵:

$$\begin{aligned}
\Theta^{00} &= \frac{1}{\mu_0}\left(\eta^{0\sigma}F^{0\rho}F_{\rho\sigma}+\frac{1}{4}\eta^{00}F_{\alpha\beta}F^{\alpha\beta}\right)\\
&= \frac{1}{\mu_0}F^{0i}F_{0i}-\frac{1}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta}\\
&= -\frac{1}{\mu_0c^2}\boldsymbol{E}^2-\frac{1}{2\mu_0}\left(\boldsymbol{B}^2-\frac{\boldsymbol{E}^2}{c^2}\right)\\
&= -\frac{1}{2}\left(\epsilon_0\boldsymbol{E}^2+\frac{\boldsymbol{B}^2}{\mu_0}\right)
\end{aligned}$$

显然,

$$\boxed{u=-\Theta^{00}=\frac{1}{2}\left(\epsilon_0\boldsymbol{E}^2+\frac{\boldsymbol{B}^2}{\mu_0}\right)}$$

正是我们期望的电磁场在 $\mathbb{E}_3$ 中的能量体密度.

$$
\begin{aligned}
\Theta^{0i} &= \frac{1}{\mu_0}\eta^{i\sigma}F^{0\rho}F_{\rho\sigma} = \frac{1}{\mu_0}\eta^{ik}F^{0j}F_{jk} \\
&= -\frac{1}{\mu_0 c}\epsilon^{ijk}E_j B_k \\
&= -\frac{1}{\mu_0 c}(\boldsymbol{E}\times\boldsymbol{B})^i
\end{aligned}
$$

换言之,

$$
\boxed{\boldsymbol{S} = -c\,\Theta^{0i}\boldsymbol{e}_i = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}}
$$

正是通常描写电磁能量传输性质的能流密度矢量.

1. 电磁场在 $\mathbb{E}_3$ 中的动量密度矢量定义为:

$$
\boxed{\boldsymbol{g} = -\frac{1}{c}\,\Theta^{0i}\boldsymbol{e}_i = \epsilon_0\boldsymbol{E}\times\boldsymbol{B}}
$$

显然, $\boldsymbol{S} = c^2\boldsymbol{g}$.

$$
\begin{aligned}
\Theta^{ij} &= \frac{1}{\mu_0}\left(\eta^{j\sigma}F^{i\rho}F_{\rho\sigma} + \frac{1}{4}\eta^{ij}F_{\alpha\beta}F^{\alpha\beta}\right) \\
&= \frac{1}{\mu_0}\eta_{\rho\sigma}F^{i\rho}F^{\sigma j} + \frac{1}{4\mu_0}\eta^{ij}F_{\alpha\beta}F^{\alpha\beta} \\
&= \frac{1}{\mu_0}F^{0i}F^{0j} - \frac{1}{\mu_0}\eta_{kl}F^{ik}F^{jl} + \frac{1}{4\mu_0}\eta^{ij}F_{\alpha\beta}F^{\alpha\beta} \\
&= \frac{1}{\mu_0 c^2}E^iE^j - \frac{1}{\mu_0}\eta^{jl}\epsilon^{imk}\epsilon_{lnk}B_mB^n + \frac{1}{2\mu_0}\eta^{ij}\left(\boldsymbol{B}^2 - \frac{\boldsymbol{E}^2}{c^2}\right) \\
&= \epsilon_0E^iE^j - \frac{1}{\mu_0}\left(\eta^{ij}\boldsymbol{B}^2 - B^iB^j\right) + \frac{1}{2\mu_0}\eta^{ij}\left(\boldsymbol{B}^2 - \frac{\boldsymbol{E}^2}{c^2}\right) \\
&= \epsilon_0E^iE^j + \frac{1}{\mu_0}B^iB^j - \frac{1}{2}\eta^{ij}\left(\epsilon_0\boldsymbol{E}^2 + \frac{\boldsymbol{B}^2}{\mu_0}\right)
\end{aligned}
$$

通常把

$$
\boxed{\mathscr{T}^{ij} = -\Theta^{ij} = -\epsilon_0E^iE^j - \frac{1}{\mu_0}B^iB^j + \frac{1}{2}\eta^{ij}\left(\epsilon_0\boldsymbol{E}^2 + \frac{\boldsymbol{B}^2}{\mu_0}\right)}
$$

称为电磁场在 $\mathbb{E}_3$ 中的动量流密度张量.

能量动量守恒定律

1. 倘若所考虑的体系是纯粹的电磁场

$$\mathscr{L}_{\rm em} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

没有电荷电流体系与其发生相互作用，则体系的能量动量守恒定律为：

$$\partial_\mu \Theta^{\mu\nu}(x) = 0$$

2. 能够对电磁相互作用体系提供完整描写的拉氏密度是：

$$\mathscr{L}_{\rm Full} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu + \mathscr{L}_{\rm m}$$

电磁场充当电荷电流分布 $(J^\mu, \mathscr{L}_{\rm m})$ 发生电磁相互作用的媒介. 体系的能量动量守恒定律为：

$$\partial_\mu \left[\Theta^{\mu\nu}(x) + T_{\rm m}^{\mu\nu}(x)\right] = 0$$

式中

$$T_{\mathrm{m}}^{\mu\nu}=\frac{\partial\mathscr{L}_{\mathrm{m}}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho-\eta^{\mu\nu}\mathscr{L}_{\mathrm{m}}$$

是电荷电流分布本身的能量动量 4-张量.

我们暂时不关心电荷电流分布的动力学[20], 仅仅把 J^μ 看作是电磁场的源:

$$\mathscr{L}=-\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}+J^\mu A_\mu$$

在此情形下, 如何表述体系的能量动量守恒定律呢?

- 虽然不完整的拉氏密度 $\mathscr{L}$ 舍弃了对作为电磁场源的电荷电流分布的动力学描写, 但它仍提供了对于电磁场本身动力学的完整描写. 根据 $\mathscr{L}$ 可以推导出 Maxwell 方程组

$$\partial_\mu F^{\mu\nu}=-\mu_0 J^\nu$$

并定义电磁场规范不变的能量动量 4-张量 $\Theta^{\mu\nu}(x)$.

[20] 即暂时不讨论 $\mathscr{L}_{\mathrm{m}}$ 与 $T_{\mathrm{m}}^{\mu\nu}$.

电磁场的能量动量 4 张量 $\Theta^{\mu\nu}(x)$

$$\boxed{\Theta^{\mu\nu}=\frac{1}{\mu_0}\left(\eta^{\nu\sigma}F^{\mu\rho}F_{\rho\sigma}+\frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)}$$

具有对称、无迹和规范变换不变性等性质，但在电磁场源存在的情形下 $\partial_\mu\Theta^{\mu\nu}(x)\neq 0$. 这是因为电磁场与其源之间不可避免的存在着能量与动量的交换：

$$\begin{aligned}
\partial_\mu\Theta^{\mu\nu} &= \frac{1}{\mu_0}\partial_\mu\left(\eta^{\nu\sigma}F^{\mu\rho}F_{\rho\sigma}+\frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)\\
&= -\eta^{\nu\sigma}J^\rho F_{\rho\sigma}+\frac{1}{\mu_0}\eta^{\nu\sigma}F^{\mu\rho}\partial_\mu F_{\rho\sigma}+\frac{1}{2\mu_0}\eta^{\mu\nu}F^{\alpha\beta}\partial_\mu F_{\alpha\beta}\\
&= -\eta^{\nu\sigma}J^\rho F_{\rho\sigma}+\frac{1}{2\mu_0}\eta^{\mu\nu}F^{\alpha\beta}\left(\partial_\mu F_{\alpha\beta}+\partial_\alpha F_{\beta\mu}+\partial_\beta F_{\mu\alpha}\right)\\
&= -\eta^{\nu\sigma}J^\rho F_{\rho\sigma}
\end{aligned}$$

亦即：

$$\partial_\mu\Theta^{\mu\nu}=-J_\mu F^{\mu\nu}$$

或者:

$$\boxed{\partial_\mu \Theta^{\mu\nu} + J_\mu F^{\mu\nu} = 0}$$

这就是计入了电磁场源(但未考虑其动力学)时电磁相互作用体系的能量动量守恒定律. 回忆

$$J^\mu = (c\rho, j), \quad \Theta^{00} = -u, \quad \Theta^{0i} = -\frac{S^i}{c}, \quad F^{0i} = \frac{E^i}{c}, \quad F^{ij} = \epsilon^{ijk} B_k$$

$\nu = 0$ 时上式的物理内涵为:

$$\begin{aligned} 0 &= \partial_\mu \Theta^{\mu 0} + J_\mu F^{\mu 0} = \frac{1}{c}\frac{\partial \Theta^{00}}{\partial t} + \partial_i \Theta^{i0} + j_i F^{i0} \\ &= -\frac{1}{c}\frac{\partial u}{\partial t} - \frac{1}{c}\partial_i S^i - j_i \frac{E^i}{c} = -\frac{1}{c}\left(\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} + \boldsymbol{j} \cdot \boldsymbol{E}\right) \end{aligned}$$

亦即,

$$\boxed{-\nabla \cdot \boldsymbol{S} = \frac{\partial u}{\partial t} + \boldsymbol{j} \cdot \boldsymbol{E}}$$

求其在 $\mathbb{E}_3$ 上某个区域 V 上的的体积分,

$$\boxed{-\oint_{\partial V} \mathrm{d}\boldsymbol{s}\cdot\boldsymbol{S} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V u\,\mathrm{d}^3x + \int_V \boldsymbol{j}\cdot\boldsymbol{E}\,\mathrm{d}^3x}$$

由于上式右端第二项恰为洛伦兹力做功的功率,上式正是我们期望的电磁场源存在情形下的能量守恒定律.

倘若 $\nu = i$, 我们有:

$$\Theta^{0i} = -cg^i, \quad \Theta^{ij} = -\mathscr{T}^{ij}$$

因此,

$$\begin{aligned}
0 &= \partial_\mu \Theta^{\mu i} + J_\mu F^{\mu i} = \frac{1}{c}\frac{\partial \Theta^{0i}}{\partial t} + \partial_k \Theta^{ki} - c\rho F^{0i} + j_k F^{ki} \\
&= -\frac{\partial g^i}{\partial t} - \partial_k \mathscr{T}^{ki} - \rho E^i + \epsilon^{kil} j_k B_l \\
&= -\left(\frac{\partial \boldsymbol{g}}{\partial t} + \nabla\cdot\mathscr{T} + \rho\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B}\right)^i
\end{aligned}$$

亦即：

$$\boxed{-\nabla\cdot\mathscr{T}=\frac{\partial \boldsymbol{g}}{\partial t}+\rho\boldsymbol{E}+\boldsymbol{j}\times\boldsymbol{B}}$$

求其在 $\mathbb{E}_3$ 上某个区域 V 上的的体积分，

$$\boxed{-\oint_{\partial V}\mathrm{d}\boldsymbol{s}\cdot\mathscr{T}=\frac{\mathrm{d}}{\mathrm{d}t}\int_V\boldsymbol{g}\,\mathrm{d}^3x+\int_V(\rho\boldsymbol{E}+\boldsymbol{j}\times\boldsymbol{B})\,\mathrm{d}^3x}$$

这恰可解读为电磁场源存在情形下的动量守恒定律.

- $\mathscr{T}$ 可解读为单位时间内垂直通过界面 ∂V 上单位面元从区域 V 流出(到外部环境中)的动量.
- 单位时间内通过界面 ∂V 流出区域 V 的总电磁动量为：

$$\boldsymbol{F}=\oint_{\partial V}\mathrm{d}\boldsymbol{s}\cdot\mathscr{T}$$

按照牛顿第二定律，这就是区域 V 中的电磁场通过界面 ∂V 施加给外部环境的合力.

- 洛伦兹力情形下的牛顿第二定律表为：

$$\frac{d\wp}{dt} = \rho \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}$$

式中 $\wp$ 是电荷电流分布机械动量体密度，

$$\boldsymbol{P} = \int_V \wp \, d^3x = \sum_i \boldsymbol{p}_i$$

因此，积分形式的动量守恒定律可重新表为：

$$\boxed{-\oint_{\partial V} d\boldsymbol{s} \cdot \mathscr{T} = \frac{d}{dt}\left(\int_V \boldsymbol{g} \, d^3x + \sum_i \boldsymbol{p}_i\right)}$$

所以，即使区域 V 内外没有电磁动量交换，$\mathscr{T}^{ij} = 0$，作为电磁场源的电荷电流分布本身的总机械动量也是不守恒的[21]：

$$\frac{d\boldsymbol{P}}{dt} = \frac{d}{dt}\sum_i \boldsymbol{p}_i \neq 0$$

[21] 换言之，牛顿第三定律在电磁相互作用过程中不成立.

空间转动对称性

为简单起见,本小节的研究对象仅限于自由电磁场:

$$\mathscr{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

在无穷小的洛伦兹变换下,

$$\delta x_\mu = \omega_{\mu\sigma} x^\sigma, \qquad \delta A_\mu(x) = \frac{1}{2}\omega_{\rho\sigma}\left(\Sigma^{\rho\sigma}\right)_{\mu\nu} A^\nu(x)$$

自由电磁场天然地具有洛伦兹变换下的不变性, $\delta S = 0$. 相应的守恒流 4-矢量为:

$$\begin{aligned}
\mathscr{J}^\mu(x) &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu(x))}\,\delta A_\nu(x) - T^{\mu\nu}(x)\,\delta x_\nu \\
&= \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu(x))}\left[\frac{1}{2}\omega_{\rho\sigma}\left(\Sigma^{\rho\sigma}\right)_{\nu\alpha} A^\alpha(x)\right] - \omega_{\nu\alpha}\, T^{\mu\nu}(x)\, x^\alpha \\
&= -\frac{1}{\mu_0} F^{\mu\nu}\left[\frac{1}{2}\omega_{\rho\sigma}\left(\Sigma^{\rho\sigma}\right)_{\nu\alpha} A^\alpha(x)\right] - \omega_{\rho\sigma}\, T^{\mu\rho}(x)\, x^\sigma
\end{aligned}$$

亦即：

$$\mathscr{J}^{\mu}=-\frac{1}{2}\omega_{\rho\sigma}\widetilde{\mathscr{M}}^{\mu\rho\sigma}$$

式中，

$$\widetilde{\mathscr{M}}^{\mu\rho\sigma}=\frac{1}{\mu_0}F^{\mu\nu}\left(\Sigma^{\rho\sigma}\right)_{\nu\alpha}A^{\alpha}(x)+\left[T^{\mu\rho}(x)\,x^{\sigma}-T^{\mu\sigma}(x)\,x^{\rho}\right]$$

因为 $\mathscr{J}^{\mu}$ 是自由电磁场与洛伦兹不变性相联系的守恒流 4-矢量，$\partial_{\mu}\mathscr{J}^{\mu}=0$，$\widetilde{\mathscr{M}}^{\mu\rho\sigma}$ 自然也就是电磁场与洛伦兹不变性相联系的守恒 3 阶 4-张量：

$$\partial_{\mu}\widetilde{\mathscr{M}}^{\mu\rho\sigma}=0$$

1. 在洛伦兹变换中，参数 ω_{ij} 描写无穷小空间转动. 自由电磁场与空间转动对称性相联系的守恒定律即为：

$$\boxed{\partial_{\mu}\widetilde{\mathscr{M}}^{\mu ij}=0}$$

现在的问题是：应该在物理上如何诠释上式中出现的 9 个守恒 4-矢量 $\widetilde{\mathscr{M}}^{\mu ij}$？

暂且回避守恒 4-矢量 $\widetilde{\mathscr{M}}^{\mu ij}$ 的诠释问题，我们先设法克服守恒 4-张量 $\widetilde{\mathscr{M}}^{\mu\rho\sigma}$ 表达式的缺点 $\rightsquigarrow$ $\widetilde{\mathscr{M}}^{\mu\rho\sigma}$ 中出现了 $T^{\mu\rho}(x)$. 由于 $T^{\mu\rho}(x)$ 不能诠释为电磁场的能量动量 4-张量，它的出现阻碍了我们合理理解 $\widetilde{\mathscr{M}}^{\mu\rho\sigma}$ 的物理内涵.

$\mathscr{Q}$: 那么，能否使用

$$\mathscr{M}^{\mu\rho\sigma} = \frac{1}{\mu_0}F^{\mu\nu}\left(\Sigma^{\rho\sigma}\right)_{\nu\alpha}A^{\alpha}(x) + \left[\Theta^{\mu\rho}(x)\, x^{\sigma} - \Theta^{\mu\sigma}(x)\, x^{\rho}\right]$$

替代 $\widetilde{\mathscr{M}}^{\mu\rho\sigma}$ 作为自由电磁场联系于洛伦兹不变性的守恒 4-张量?

如前述，$T^{\mu\nu}(x)$ 与自由电磁场能量动量张量 $\Theta^{\mu\nu}(x)$ 之间的关系是:

$$T^{\mu\nu} = \Theta^{\mu\nu} + T_{\rm D}^{\mu\nu}$$

$T_{\rm D}^{\mu\nu}$ 自动满足守恒定律 $\partial_{\mu}T_{\rm D}^{\mu\nu} = 0$ 且具有表达式

$$T_{\rm D}^{\mu\nu} = -\frac{1}{\mu_0}\partial_{\rho}\left(F^{\mu\rho}A^{\nu}\right) = -\frac{1}{\mu_0}F^{\mu\rho}\partial_{\rho}A^{\nu}$$

我们看到：

$$\widetilde{\mathscr{M}}^{\mu\rho\sigma} - \mathscr{M}^{\mu\rho\sigma} = T_{\mathrm{D}}^{\mu\rho}(x)\, x^{\sigma} - T_{\mathrm{D}}^{\mu\sigma}(x)\, x^{\rho}$$

进而，

$$\begin{aligned}
\partial_{\mu}\mathscr{M}^{\mu\rho\sigma} &= \partial_{\mu}\widetilde{\mathscr{M}}^{\mu\rho\sigma} - \partial_{\mu}\left[T_{\mathrm{D}}^{\mu\rho}(x)\, x^{\sigma} - T_{\mathrm{D}}^{\mu\sigma}(x)\, x^{\rho}\right] \\
&= -T_{\mathrm{D}}^{\mu\rho}(x)\, \partial_{\mu}x^{\sigma} + T_{\mathrm{D}}^{\mu\sigma}(x)\, \partial_{\mu}x^{\rho} \\
&= -T_{\mathrm{D}}^{\mu\rho}(x)\, \delta_{\mu}^{\sigma} + T_{\mathrm{D}}^{\mu\sigma}(x)\, \delta_{\mu}^{\rho} \\
&= T_{\mathrm{D}}^{\rho\sigma}(x) - T_{\mathrm{D}}^{\sigma\rho}(x)
\end{aligned}$$

回忆

$$T_{\mathrm{D}}^{\rho\sigma} = \frac{1}{\mu_0}\partial_{\mu}\left(F^{\mu\rho}A^{\sigma}\right)$$

前面的表达式实际上是如下连续性方程：

$$\partial_{\mu}\left[\mathscr{M}^{\mu\rho\sigma} - \frac{1}{\mu_0}\left(F^{\mu\rho}A^{\sigma} - F^{\mu\sigma}A^{\rho}\right)\right] = 0$$

所以，能替代 $\widetilde{\mathscr{M}}^{\mu\rho\sigma}$ 作为自由电磁场联系于洛伦兹不变性的守恒 4-张量并不是 $\mathscr{M}^{\mu\rho\sigma}$，而是：[22]

$$M^{\mu\rho\sigma} \equiv \mathscr{M}^{\mu\rho\sigma} - \frac{1}{\mu_0}\left(F^{\mu\rho}A^{\sigma} - F^{\mu\sigma}A^{\rho}\right)$$

进一步注意到：

$$F^{\mu\rho}A^{\sigma} - F^{\mu\sigma}A^{\rho} = F^{\mu\nu}\left(\delta^{\rho}_{\nu}\delta^{\sigma}_{\alpha} - \delta^{\sigma}_{\nu}\delta^{\rho}_{\alpha}\right)A^{\alpha} = F^{\mu\nu}\left(\Sigma^{\rho\sigma}\right)_{\nu\alpha}A^{\alpha}$$

我们有：

$$\boxed{M^{\mu\rho\sigma} = \Theta^{\mu\rho}x^{\sigma} - \Theta^{\mu\sigma}x^{\rho}}$$

- 3 阶 4-张量 $M^{\mu\rho\sigma}$ 表达式中不出现裸露的规范势因子，因此它明显地具有规范变换下的不变性.
- $M^{\mu\rho\sigma}$ 具有后二指标交换的反对称性：

$$M^{\mu\rho\sigma} = -M^{\mu\sigma\rho}$$

[22]K. Bhattacharya 等人的新著 Introduction to advanced electrodynamics 上的 (9.101) 式看起来完全错了.

- $M^{\mu\rho\sigma}$ 就是自由电磁场联系于洛伦兹不变性的守恒 4-张量:

$$\partial_\mu M^{\mu\rho\sigma} = 0$$

其中分量 $M^{\mu ij}$ 形成了自由电磁场联系于空间转动对称性的守恒角动量 4-矢量.

求连续性方程 $\partial_\mu M^{\mu\rho\sigma} = 0$ 在 $\mathbb{E}_3$ 某个区域 V 中的体积分,

$$\begin{aligned} 0 &= \int_V \partial_\mu M^{\mu\rho\sigma} \, \mathrm{d}^3 x \\ &= \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_V M^{0\rho\sigma} \, \mathrm{d}^3 x + \int_V \partial_i M^{i\rho\sigma} \, \mathrm{d}^3 x \\ &= \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_V M^{0\rho\sigma} \, \mathrm{d}^3 x + \oint_{\partial V} \mathrm{d}s_i \, M^{i\rho\sigma} = \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_V M^{0\rho\sigma} \, \mathrm{d}^3 x \end{aligned}$$

最后一步使用了场量 (法分量) 在边界面 ∂V 上为零的假设.

所以，自由电磁场联系于洛伦兹不变性的守恒定律改写为：

$$\frac{\mathrm{d}}{\mathrm{d}t} M^{\rho\sigma} = 0 \qquad \hookleftarrow \quad M^{\rho\sigma} = \int_V M^{0\rho\sigma}\, \mathrm{d}^3 x$$

守恒量 $M^{\rho\sigma}$ 的被积函数可表为：

$$M^{0\rho\sigma} = \Theta^{0\rho}\, x^{\sigma} - \Theta^{0\sigma}\, x^{\rho}$$

回忆

$$\Theta^{00} = -u, \qquad u = \frac{\epsilon_0}{2} E^2 + \frac{B^2}{2\mu_0}$$

$$\Theta^{0i} = -c g^i = -\frac{S^i}{c}, \qquad \boldsymbol{g} = \frac{\boldsymbol{S}}{c^2} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B}$$

我们有：

$$M^{00i} = \Theta^{00}\, x^i - \Theta^{0i}\, x^0 = -u\, x^i + c^2\, g^i\, t = -\left(u\boldsymbol{r} - c^2 \boldsymbol{g} t\right)^i$$

$$M^{0jk} = \Theta^{0j}\, x^k - \Theta^{0k}\, x^j = -c\, g^j\, x^k + c\, g^k\, x^j = c\, \epsilon^{jkl} (\boldsymbol{r} \times \boldsymbol{g})_l$$

我们看到：

- M^{0jk} 的物理本质是 $\boldsymbol{r}\times\boldsymbol{g}$，其在区域 V 中的体积分

$$\boxed{\boldsymbol{M}\equiv\int_V \boldsymbol{r}\times\boldsymbol{g}\,\mathrm{d}^3x}$$

  正是自由电磁场联系于空间转动对称性的守恒量. 我们把它诠释为区域 V 中电磁场的角动量.

- M^{00i} 的物理内涵是 $u\boldsymbol{r}-c^2t\boldsymbol{g}$. 其在区域 V 中的体积分是自由电磁场联系于洛伦兹推动不变性的守恒量：[23]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left(u\boldsymbol{r}-c^2t\boldsymbol{g}\right)\,\mathrm{d}^3x=0$$

自由电磁场联系于洛伦兹推动不变性的守恒定律可改写为：

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left(u\boldsymbol{r}\right)\,\mathrm{d}^3x-c^2\int_V\boldsymbol{g}\,\mathrm{d}^3x-c^2t\,\frac{\mathrm{d}}{\mathrm{d}t}\int_V\boldsymbol{g}\,\mathrm{d}^3x$$

[23]进一步的讨论请参阅：H. Muller-Kirsten, *Electrodynamics, an introduction including quantum effects*, World Scientific, 2004, Page 434.

假设区域 V 中没有电荷电流分布，V 内外也无电磁能量、动量的交换：

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V u\,\mathrm{d}^3x = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{g}\,\mathrm{d}^3x = 0$$

在此情形下，

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (u\boldsymbol{r})\,\mathrm{d}^3x = c^2\int_V \boldsymbol{g}\,\mathrm{d}^3x$$

倘若定义区域 V 中自由电磁场的"能量中心"：

$$\boxed{\boldsymbol{r}_c \equiv \frac{\int_V (u\boldsymbol{r})\,\mathrm{d}^3x}{\int_V u\,\mathrm{d}^3x}}$$

此能量中心的运动速度为：

$$\boldsymbol{v}_c = \frac{\mathrm{d}\boldsymbol{r}_c}{\mathrm{d}t} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\int_V (u\boldsymbol{r})\,\mathrm{d}^3x}{\int_V u\,\mathrm{d}^3x} = c^2\,\frac{\int_V \boldsymbol{g}\,\mathrm{d}^3x}{\int_V u\,\mathrm{d}^3x} = \frac{\boldsymbol{G}\,c^2}{U}$$

式中 U 与 $\boldsymbol{G}$ 分别是区域 V 中电磁场的总能量和总动量：

$$U=\int_V u\,\mathrm{d}^3x, \qquad \boldsymbol{G}=\int_V \boldsymbol{g}\,\mathrm{d}^3x$$

所以，电磁场作用量泛函具有洛伦兹推动变换下的不变性意味着其能量中心的运动类似于一个能量为 U 动量为 $\boldsymbol{G}$ 的相对论性质点的运动.

接下来讨论电磁场角动量的物理内涵. 我们对 $F^{\mu\nu}$ 所赋予的物理内涵意味着

$$\epsilon_{ijk}B^k=F_{ij}=\partial_i A_j-\partial_j A_i=\epsilon_{ijk}(\nabla\times\boldsymbol{A})^k \quad \rightsquigarrow \ \boldsymbol{B}=\nabla\times\boldsymbol{A}$$

进而，

$$\begin{aligned}\boldsymbol{g}\ &=\epsilon_0\,\boldsymbol{E}\times\boldsymbol{B}=\epsilon_0\boldsymbol{E}\times(\nabla\times\boldsymbol{A})=\epsilon_0\boldsymbol{e}_i\epsilon_{ijk}\epsilon^{mnk}E_j\partial^m A^n\\ &=\epsilon_0\boldsymbol{e}_iE_j(\partial^iA^j-\partial^jA^i)=\epsilon_0E_j\nabla A^j-\epsilon_0(\boldsymbol{E}\cdot\nabla)\boldsymbol{A}\end{aligned}$$

以及:

$$\begin{aligned}&\frac{1}{\epsilon_0}\boldsymbol{r}\times\boldsymbol{g}=E_j(\boldsymbol{r}\times\nabla)A^j-E^j(\boldsymbol{r}\times\partial_j\boldsymbol{A})\\ &=E_j(\boldsymbol{r}\times\nabla)A^j-\partial_j[E^j(\boldsymbol{r}\times\boldsymbol{A})]+(\nabla\cdot\boldsymbol{E})(\boldsymbol{r}\times\boldsymbol{A})+E^j[(\partial_j\boldsymbol{r})\times\boldsymbol{A}]\\ &=E_j(\boldsymbol{r}\times\nabla)A^j-\partial_j[E^j(\boldsymbol{r}\times\boldsymbol{A})]+\boldsymbol{E}\times\boldsymbol{A}\end{aligned}$$

求其在 $\mathbb{E}_3$ 某个区域 V 中的体积分，

$$\begin{aligned}
\boldsymbol{M} &= \int_V \boldsymbol{r}\times\boldsymbol{g}\,\mathrm{d}^3x \\
&= \epsilon_0 \int_V E_j(\boldsymbol{r}\times\nabla)A^j\,\mathrm{d}^3x + \epsilon_0\int_V \boldsymbol{E}\times\boldsymbol{A}\,\mathrm{d}^3x \\
&\qquad - \epsilon_0\int_V \partial_j[E^j(\boldsymbol{r}\times\boldsymbol{A})]\,\mathrm{d}^3x \\
&= \epsilon_0 \int_V E_j(\boldsymbol{r}\times\nabla)A^j\,\mathrm{d}^3x + \epsilon_0\int_V \boldsymbol{E}\times\boldsymbol{A}\,\mathrm{d}^3x - \epsilon_0\oint_{\partial V}\mathrm{d}s_j\,E^j(\boldsymbol{r}\times\boldsymbol{A})
\end{aligned}$$

假设电磁场仅分布于区域 V 之内、无场强法分量通过边界面 ∂V 溢出：

$$\oint_{\partial V}\mathrm{d}s_j\,E^j(\boldsymbol{r}\times\boldsymbol{A}) = 0$$

如此，

$$\boxed{\boldsymbol{M} = \epsilon_0 \int_V E_j(\boldsymbol{r}\times\nabla)A^j\,\mathrm{d}^3x + \epsilon_0\int_V \boldsymbol{E}\times\boldsymbol{A}\,\mathrm{d}^3x}$$

换言之，

$$\boldsymbol{M} = \boldsymbol{l} + \boldsymbol{s}$$

其中，

1. $\boldsymbol{l}$ 称为电磁场的轨道角动量：

$$\boldsymbol{l} = \epsilon_0 \int_V \mathrm{d}^3x\, E_j(\boldsymbol{r} \times \nabla) A^j$$

   之所以如此命名是因为其被积函数中含有复合算符 $(\boldsymbol{r} \times \nabla)$，而量子力学体系的轨道角动量算符是 $-i\hbar(\boldsymbol{r} \times \nabla)$.

2. $\boldsymbol{s}$ 称为电磁场的自旋角动量：

$$\boxed{\boldsymbol{s} = \epsilon_0 \int_V \mathrm{d}^3x\, \boldsymbol{E} \times \boldsymbol{A}}$$

   自旋角动量的特点是其被积函数不显含场点的位置矢径 $\boldsymbol{r}$.

为了对量子物理中光子的自旋有一点感觉，现在考虑真空中一列具有确定频率的时谐电磁波且具有圆极化. 假设该电磁波的频率为 ω，传播方向为直角坐标系的 X^3 轴正向[24]，则此时谐电磁波可用矢量势

$$\boldsymbol{A} = A(r)\big[\boldsymbol{e}_1 \cos(\omega t) + \boldsymbol{e}_2 \sin(\omega t)\big], \qquad k = \omega / c$$

描写. 把磁感应强度的定义式 $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ 与法拉第定律结合起来，

$$0 = \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right)$$

因此，在物理上可以有：

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} = \omega A(r)\big[\boldsymbol{e}_1 \sin(\omega t) - \boldsymbol{e}_2 \cos(\omega t)\big] \qquad \rightsquigarrow \ [A(r)]^2 = \frac{\boldsymbol{E}^2}{\omega^2}$$

进一步地，

$$\boldsymbol{E} \times \boldsymbol{A} = \omega [A(r)]^2 \boldsymbol{e}_3 = \frac{\boldsymbol{E}^2}{\omega} \boldsymbol{e}_3$$

[24] 直角坐标系三个独立方向的基矢分别记为 $\boldsymbol{e}_i$ $(i = 1, 2, 3)$.

所以，

$$s = \epsilon_0 \int_V \mathrm{d}^3x\, \boldsymbol{E} \times \boldsymbol{A} = \frac{\boldsymbol{e}_3}{\omega}\, \epsilon_0 \int_V \mathrm{d}^3x\, \boldsymbol{E}^2$$

回忆电磁场总能量的表达式：

$$U = \frac{\epsilon_0}{2} \int_V \mathrm{d}^3x\, \boldsymbol{E}^2 + \frac{1}{2\mu_0} \int_V \mathrm{d}^3x\, \boldsymbol{B}^2$$

且对于真空中的平面电磁波而言[25]，

$$\frac{1}{\mu_0}\boldsymbol{B}^2 = \epsilon_0 \boldsymbol{E}^2 \quad \rightsquigarrow \quad U = \epsilon_0 \int_V \mathrm{d}^3x\, \boldsymbol{E}^2$$

我们又可以把圆极化的平面电磁波自旋角动量表达为：

$$s = \frac{U}{\omega}\boldsymbol{e}_3$$

[25]证明请参见本课程后续课件，或者郭硕鸿先生的著作《电动力学》第三版第116页之(1.30)式.

1. 倘若频率为 ω 的一列平面电磁波恰好对应于量子物理中的一个光子，它的能量须表为：

$$U = \hbar\omega$$

式中的 $\hbar$ 称为 Planck 常数，它具有角动量的量纲、其实验值是：

$$\hbar \approx 1.054 \times 10^{-34}\ \text{Joul} \cdot \text{s}$$

于是，光子的自旋角动量写为：

$$\boldsymbol{s} = \hbar \boldsymbol{e}_3$$

鉴于 $\boldsymbol{s}$ 的量值只是 $\hbar$ 的一倍，所以称光子的自旋为 1.[26]

[26] $s_3 = \pm 1$，对应于平面电磁波有两个独立的极化方向.