

2024 组合学期中考试

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满分 140 分，第一和第十题 10 分，剩下题都 15 分。

Problem 1. Prove that:

$$\sum_{k=0}^m \binom{m}{k} \binom{n+k}{m} = \sum_{k=0}^m \binom{m}{k} \binom{n}{k} 2^k$$

Problem 2. We extend the definition of Ramsey's number. For two graphs G_1 and G_2 , let $R(G_1, G_2)$ denote the smallest integer v such that every two-edge-coloring of K_v contains either a red copy of G_1 or a blue copy of G_2 as a subgraph. Show that $R(C_4, C_4) = 6$.

Problem 3. There are two partitions of $[mn]$, A_1, A_2, \dots, A_m and B_1, B_2, \dots, B_m , such that $|A_1| = |A_2| = \dots = |A_m| = |B_1| = |B_2| = \dots = |B_m| = n$. Show that one can rearrange $B_{1 \sim m}$ to make that $A_i \cap B_i \neq \emptyset$ for every $i \in [m]$.

Problem 4. Count the number of permutations x_1, x_2, \dots, x_{2n} of $[2n]$ such that $x_i + x_{i+1} \neq 2n + 1$ for each $i \in [2n - 1]$ and $x_1 + x_{2n} \neq 2n + 1$.

Problem 5. Let \mathcal{F} be a family of subsets of $[n]$ that forms an antichain, where for every $A \in \mathcal{F}$, the size $|A| \leq k$. Show that $|\mathcal{F}| \leq \binom{n}{k}$.

Problem 6. One can step by one or two stairs at a time. Let A_n be the number of ways to step by n stairs. Determine the closed-form expression for the generating function $\sum_{n \geq 1} A_n x^n$.

Problem 7. Find a closed formula of:

$$\sum_{k=0}^{\lfloor n/7 \rfloor} \binom{n}{7k}$$

Problem 8. Let S be a set of n points on a plane such that any three points are not collinear. Moreover, for every $v \in S$, there are at least k points in S which are equidistant from v . Show that $k < \frac{1}{2} + \sqrt{2n}$.

Problem 9. Show that there is a constant C only depending on k , such that $\text{ex}(n, K_{k,k}) \leq Cn^{2-1/k}$.

Problem 10. Let $K_n - \{e\}$ be the complete graph deleting an edge. Show that the number of spanning trees of $K_n - \{e\}$ is $(n - 2)n^{n-3}$.