2019秋复变试题答案

11.24 renyixin

题一:

(1)

$$z^3 = -3\bar{z}$$

令

$$z = re^{i\phi}$$

代入得

$$r^3 e^{3i\phi} = -3re^{-i\phi} \Rightarrow r^2 e^{4i\phi} = -3 \Rightarrow r = \sqrt{3}, 4\phi = \pi + 2k\pi \ldots (k \in \mathbb{Z})$$

即

$$r = \sqrt{3}, \phi = \frac{\pi + 2k\pi}{4} \ldots (k \in Z)$$
$$\Rightarrow z = \sqrt{3} e^{i(\frac{\pi}{4} + \frac{k}{2}\pi)} \ldots (k \in \mathbb{Z})$$

(2)

$$\sin z = 3$$
$$\Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 3$$
$$\Rightarrow e^{2iz} - 6ie^{iz} - 1 = 0$$
$$\Rightarrow e^{iz} = (3 \pm 2\sqrt{2})i$$
$$\Rightarrow iz = Ln((3 \pm 2\sqrt{2})i)$$
$$\Rightarrow z = -i(ln(3 \pm 2\sqrt{2}) + \frac{\pi}{2} i + 2k\pi i) = \frac{\pi}{2} + 2k\pi - iln(3 \pm 2\sqrt{2}) \ldots (k \in \mathbb{Z})$$

题二:

$$u(x, y) = e^{\alpha y} \cos(3x) + 3x$$

代入C.R.方程

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -3e^{\alpha y} \sin 3x + 3 \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \alpha e^{\alpha y} \cos 3x \end{cases}$$
$$\Rightarrow \begin{cases} v = -\frac{3}{\alpha} e^{\alpha y} \sin 3x + 3y + \phi(x) \\ v = -\frac{\alpha}{3} e^{\alpha y} \sin 3x + \psi(y) \end{cases}$$
$$\Rightarrow \alpha = 3, \phi(x) = 0, \psi(y) = 3y$$
$$\Rightarrow v = -e^{3y} \sin 3x + 3y$$
$$\Rightarrow f(z) = u + vi = e^{-3iz} + 3z$$

题三:

(1)

$$f(z) = z^5 e^{2z} = z^5 \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{n+5} = \sum_{n=5}^{\infty} \frac{2^{n-5}}{(n-5)!} z^n \ldots (|z| < \infty)$$

(2)

$$g(z) = \frac{1}{(z-2)(z-4)^2} = \frac{1}{4} \frac{1}{z-2} \frac{1}{(1 - \frac{z-2}{2})^2} = \frac{1}{4} \frac{1}{z-2} \sum_{n=0}^{\infty} \frac{(n+1)}{2^n} (z-2)^n = \sum_{n=-1}^{\infty} \frac{n+2}{2^{n+3}} (z-2)^n$$

题四:

(1)

$$\int_0^{\pi i}(2019z^2-\cos z)dz=\frac{2019}{3}z^3-\sin z|_0^{\pi i}=\frac{2019}{3}(\pi i)^3-sin(\pi i)=-i\frac{2019}{3}\pi^3-ish\pi=(-\frac{2019}{3}\pi^3-sh\pi)i=-(673\pi^3+sh\pi)i$$

(2)

$$\int_{|z|=6}\frac{e^{2z}}{(z-1)(z-3)}dz=2\pi i(Res[\frac{e^{2z}}{(z-1)(z-3)},1]+Res[\frac{e^{2z}}{(z-1)(z-3)},3])=2\pi i(\frac{e^6-e^2}{2})=\pi(e^6-e^2)i$$

(3)

$$\int_{|z|=\frac{5}{2}}\frac{z^2-8z+5}{z^3(z+2)(z-3)^2}dz$$
$$=2\pi i(Res[\frac{z^2-8z+5}{z^3(z+2)(z-3)^2},0]+Res[\frac{z^2-8z+5}{z^3(z+2)(z-3)^2},-2])$$
$$=2\pi i(\frac{1}{2!}\frac{d^2}{dz^2}\frac{z^2-8z+5}{(z+2)(z-3)^2}|_{z=0})+2\pi i(\frac{z^2-8z+5}{z^3(z-3)^2}|_{z=-2})=-\frac{4}{27}\pi i$$

附注:

$$\frac{z^2-8z+5}{(z+2)(z-3)^2}=\frac{1}{z+2}-\frac{2}{(z-3)^2}$$
$$\frac{d^2}{dz^2}\frac{z^2-8z+5}{(z+2)(z-3)^2}=\frac{2}{(z+2)^2}-\frac{12}{(z-3)^4}$$

(4)

$$\int_{|z|=3}\frac{\cos\frac{1}{z-2}}{4-z}dz$$

法一:

令

$$w=\frac{1}{z}$$

则

$$\int_{|z|=3}\frac{\cos\frac{1}{z-2}}{4-z}dz=\int_{|w|=1/3,\text{顺时针}}\frac{\cos\frac{1}{1/w-2}}{4-1/w}-\frac{1}{w^2}dw=\int_{|w|=1/3}\frac{\cos\frac{1}{1/w-2}}{4-1/w}\frac{1}{w^2}dw$$
$$=\frac{1}{4}\int_{|w|=1/3}\frac{\cos\frac{w}{1-2w}}{w(w-\frac{1}{4})}dw=\frac{1}{4}2\pi i(Res[\frac{\cos\frac{w}{1-2w}}{w(w-\frac{1}{4})},0]+Res[\frac{\cos\frac{w}{1-2w}}{w(w-\frac{1}{4})},\frac{1}{4}])=\frac{\pi i}{2}(-4+4\cos\frac{1}{2})=(2\cos\frac{1}{2}-2)\pi i$$

法二:

$$\int_{|z|=3}\frac{\cos\frac{1}{z-2}}{4-z}dz=2\pi iRes(\frac{\cos\frac{1}{z-2}}{4-z},2)$$

$$\frac{\cos\frac{1}{z-2}}{4-z}=\cos\frac{1}{z-2}\frac{1}{2}\frac{1}{1-\frac{z-2}{4}}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}(z-2)^{-2n}\sum_{m=0}^{\infty}(\frac{z-2}{2})^m$$

$$Res(\frac{\cos\frac{1}{z-2}}{4-z},2)=a_{-1}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n)!}\frac{1}{2^{2n-1}}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n)!}\frac{1}{2^{2n}}=\cos\frac{1}{2}-1$$

(5)

$$\int_{|z|=3} \frac{z+5}{1-\cos(z-2)}dz = 2\pi i Res[\frac{z+5}{1-\cos(z-2)},2]$$

$$\frac{z+5}{1-\cos(z-2)} = \frac{z-2+7}{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-2)^{2n}}{(2n)!}}$$

$$a_{-1} = \frac{1}{\frac{1}{2!}} = 2$$

则

$$\int_{|z|=3} \frac{z+5}{1-\cos(z-2)}dz = 2\pi i Res[\frac{z+5}{1-\cos(z-2)},2] = 4\pi i$$

(6)

$$\int_{|z|=2} \frac{|dz|}{|z-i|^4}dz$$

$$= \int_0^{2\pi} \frac{2d\phi}{|2e^{i\phi}-i|^4}d\phi = \int_0^{2\pi} \frac{2d\phi}{(5-4\sin\phi)}d\phi$$

$$= 2\int_{|z|=1} \frac{1}{(5-4\frac{z-\frac{1}{z}}{2i})^2}\frac{dz}{iz} = \frac{i}{2}\int_{|z|=1} \frac{z}{(z^2+\frac{5}{2i}z-1)^2}dz$$

$$= \frac{i}{2}2\pi i Res[\frac{z}{(z^2+\frac{5}{2i}z-1)^2},\frac{i}{2}]$$

$$= -\pi\frac{d}{dz}\frac{z}{(z-2i)^2}|_{z=\frac{i}{2}} = \frac{20}{27}\pi$$

题五:

(1)

$$\int_0^{2\pi} \frac{\cos 2\theta}{3-2\cos 2\theta}d\theta$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{3-2\cos 2\theta}d\theta = Re(\int_0^{2\pi} \frac{e^{i2\theta}}{3-2\cos 2\theta}d\theta)$$

$$\int_0^{2\pi} \frac{e^{i2\theta}}{3-2\cos 2\theta}d\theta = \int_{|z|=1} \frac{z^2}{3-z-\frac{1}{z}}\frac{dz}{iz}$$

$$= i\int_{|z|=1} \frac{z^2}{z^2-3z+1}dz = i2\pi i Res[\frac{z^2}{z^2-3z+1},\frac{3-\sqrt{5}}{2}]$$

$$= -2\pi\frac{z^2}{z-\frac{3+\sqrt{5}}{2}}|_{z=\frac{3-\sqrt{5}}{2}} = \frac{7\sqrt{5}-15}{5}\pi$$

(2)

$$\int_0^{\infty} \frac{x^3\sin 4x}{(x^2+4)^2}dx = Im(\int_0^{\infty} \frac{x^3e^{i4x}}{(x^2+4)^2}dx)$$

$$\int_0^{\infty} \frac{x^3e^{i4x}}{(x^2+4)^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^3e^{i4x}}{(x^2+4)^2}dx = \frac{1}{2}\int_{C_R+C_1} \frac{z^3e^{i4z}}{(z^2+4)^2}dz = \frac{1}{2}2\pi i\frac{d}{dz}\frac{z^3e^{i4z}}{(z+2i)^2}|_{z=2i}$$

$$= \pi i(\frac{3z^2e^{i4z}+z^3 4ie^{i4z}}{(z+2i)^2} - 2z^3e^{i4z}\frac{1}{(z+2i)^3})|_{z=2i} = \pi i(\frac{20e^{-8}}{-16} - \frac{1}{4}e^{-8}) = -\frac{3}{2}e^{-8}\pi i$$

$$\int_0^{\infty} \frac{x^3\sin 4x}{(x^2+4)^2}dx = Im(\int_0^{\infty} \frac{x^3e^{i4x}}{(x^2+4)^2}dx) = -\frac{3}{2}e^{-8}\pi$$

题六:

$$f(z) = z^9 - 8z^3 - 2z^2 - z - 2 = 0$$

(i)考虑$|z| < 5$内

令

$$f_1(z)=z^9,\phi_1(z)=-8z^3-2z^2-z-2$$

在$|z|=5$上

$$|f_1(z)|=5^9>8*5^3+2*5^2+5+2\geq|\phi_1(z)|$$

则由歇儒定理，$f(z)$在$|z|<5$内的零点与$f_1(z)$相同，为9个

(ii)考虑$|z|<1$内

令

$$f_2(z)=-8z^3,\phi_2(z)=z^9-2z^2-z-2$$

在$|z|=1$上

$$|f_2(z)|=8>1+2+1+2\geq|\phi_2(z)|$$

则由歇儒定理，$f(z)$在$|z|<5$内的零点与$f_2(z)$相同，为3个

综上：在$1<|z|<5$内有6个零点

题七：

$$p^2Y-py(0)-y'(0)+Y=\mathcal{L}(e^t\cos 2t)=\frac{p-1}{(p-1)^2+4}$$
$$\Rightarrow Y=\frac{1}{p^2+1}[\frac{p-1}{(p-1)^2+4}+4p]=\frac{41}{10}\frac{p}{p^2+1}-\frac{3}{10}\frac{1}{p^2+1}-\frac{1}{10}\frac{p-1}{(p-1)^2+4}+\frac{1}{5}\frac{2}{(p-1)^2+4}$$
$$\Rightarrow y=\frac{41}{10}\cos t-\frac{3}{10}\sin t-\frac{1}{10}e^t\cos 2t+\frac{1}{5}e^t\sin 2t\ldots(t>0)$$

题八：

$$\frac{1}{2\pi i}\int_{|\xi|=1}(\frac{f(\xi)}{\xi-a}-\frac{ag(\xi)}{\xi(\xi-a)})d\xi$$

令

$$\phi(\xi)=g(\frac{1}{\xi})$$

由已知可得

$$\phi(\xi)\text{在}|z|\leq 1\text{上解析}$$

则

$$\frac{1}{2\pi i}\int_{|\xi|=1}(\frac{f(\xi)}{\xi-a}-\frac{ag(\xi)}{\xi(\xi-a)})d\xi$$
$$=\frac{1}{2\pi i}\int_{|\xi|=1}\frac{f(\xi)}{\xi-a}d\xi-\frac{1}{2\pi i}\int_{|\xi|=1,\text{顺时针}}\frac{ag(\frac{1}{\xi})}{\frac{1}{\xi}(\frac{1}{\xi}-a)}-\frac{1}{\xi^2}d\xi$$
$$=\frac{1}{2\pi i}\int_{|\xi|=1}\frac{f(\xi)}{\xi-a}d\xi-\frac{1}{2\pi i}\int_{|\xi|=1}\frac{ag(\frac{1}{\xi})}{\frac{1}{\xi}(\frac{1}{\xi}-a)}\frac{1}{\xi^2}d\xi$$
$$=\frac{1}{2\pi i}\int_{|\xi|=1}\frac{f(\xi)}{\xi-a}d\xi+\frac{1}{2\pi i}\int_{|\xi|=1}\frac{\phi(\xi)}{\xi-\frac{1}{a}}d\xi$$

(i)$|a|<1$

$$\frac{1}{2\pi i}\int_{|\xi|=1}\frac{f(\xi)}{\xi-a}d\xi+\frac{1}{2\pi i}\int_{|\xi|=1}\frac{\phi(\xi)}{\xi-\frac{1}{a}}d\xi=f(a)+0=f(a)$$

(ii)$|a|>1$

$$\frac{1}{2\pi i}\int_{|\xi|=1}\frac{f(\xi)}{\xi - a}d\xi + \frac{1}{2\pi i}\int_{|\xi|=1}\frac{\phi(\xi)}{\xi - \frac{1}{a}}d\xi = 0 + \phi(\frac{1}{a}) = g(a)$$