第三章　随机变量的数字特征

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1.

$E\xi_1 = \frac{1}{4} \times (0 + 10 + 20 + 30) = 15$

$E\xi_2 = 0 \times \frac{1}{8} + 10 \times \frac{1}{8} + 20 \times \frac{3}{8} + 30 \times \frac{3}{8} = 20$

$E\xi_3 = 0 \times \frac{1}{2} + 30 \times \frac{1}{2} = 15$

3.

(A):$\frac{1}{10} \times (1 + 1 + 2 + 2 + 1 + 2 + 2 + 3 + 3 + 1) = 1.8$

(B):$\frac{1}{10} \times (1 + 1 + 1 + 1 + 2 + 2 + 2 + 3 + 3 + 1) = 1.7$

(C):$\frac{1}{10} \times (1 + 1 + 2 + 4 + 1 + 2 + 1 + 3 + 4 + 1) = 2.0$

$\therefore$ 使用(B)组砝码时所需的平均砝码数最少

5.

(1)$\int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{+\infty} \frac{x}{2a} \exp\left\{-\frac{|x-b|}{a}\right\} dx \xlongequal{t=x-b} \int_{-\infty}^{+\infty} \frac{t+b}{2a} \exp\left\{-\frac{|t|}{a}\right\} dt$

$= \int_{-\infty}^{+\infty} \frac{b}{2a} \exp\left\{-\frac{|t|}{a}\right\} dt = \int_{0}^{+\infty} \frac{b}{a} \exp\left\{-\frac{t}{a}\right\} dt = b$

(3)$\int_{-\infty}^{+\infty} xf(x)dx = \int_{-\pi/2}^{\pi/2} \frac{2x}{\pi} \cos^2 x \, dx = 0$ (奇函数)

6.

定义$I_i = \begin{cases} 1 \ , & \text{第i个人选中自己的帽子} \\ 0 \ , & \text{第i个人未选中自己的帽子} \end{cases}$ ，$i = 1, 2, \ldots, N$

则$I_i, i = 1, 2, \ldots, N$同分布(不独立)，$EI_i = \frac{1}{N}$

选中自己帽子的人数$X = \sum_{i=1}^{N} I_i$

$\therefore EX = \sum_{i=1}^{N} EI_i = \sum_{i=1}^{N} \dfrac{1}{N} = 1$

7.

定义$X_i = \begin{cases} 1 \ , & 第i个盒子里一黑一白 \\ 0 \ , & 其他 \end{cases} \ , \ i = 1, 2, \ldots, 200$

则$X_i, i = 1, 2, \ldots, 200$同分布(不独立)，$EX_i = \dfrac{C_{300}^1 C_{100}^1}{C_{400}^2}$

$X = \sum_{i=1}^{200} X_i$

$\therefore EX = \sum_{i=1}^{200} EX_i = \sum_{i=1}^{200} \dfrac{C_{300}^1 C_{100}^1}{C_{400}^2} = \dfrac{10000}{133}$

8.

$F(m) = 1 - e^{-\lambda m} = \dfrac{1}{2}$

$\therefore m = \dfrac{\ln 2}{\lambda}$

$E|\xi - m| = \displaystyle\int_0^m (m - x)\lambda e^{-\lambda x} dx + \int_m^{+\infty} (x - m)\lambda e^{-\lambda x} dx$

$= \left(m - \dfrac{1}{\lambda} + \dfrac{1}{\lambda} e^{-\lambda m}\right) + \dfrac{1}{\lambda} e^{-\lambda m} = \dfrac{\ln 2}{\lambda}$

9.

$h(a) = E|\xi - a| = \displaystyle\int_{-\infty}^a (a - x) f(x) dx + \int_a^{+\infty} (x - a) f(x) dx$，对$a$求导

令$h'(a) = \displaystyle\int_{-\infty}^a f(x) dx - \int_a^{+\infty} f(x) dx = 0$ 可得

$\displaystyle\int_{-\infty}^a f(x) dx = \int_a^{+\infty} f(x) dx = \dfrac{1}{2}$

$\therefore a$等于ξ的中位数m时，$h(a)$最小.

(验证过程参见陈希孺文集:《概率论与数理统计》习题提示与解答347面第19题)

19.

(1)

$E(X) = 1.63$, $Var(X) = 0.9931$

$E(Y) = 3.14$, $Var(Y) = 2.0604$

(2)

$E(X+Y) = 4.77$, $E(X-Y) = -1.51$, $E(XY) = 5.82$

$Var(X+Y) = 4.4571$, $Var(X-Y) = 1.6499$, $Var(XY) = 21.7876$

(3)

$Cov(X,Y) = 0.7018$, $Corr(X,Y) = 0.4906$

(4)

$E(X^2|Y=1) = 2$

20.

(1)

$$P(\xi+\eta=m) = \sum_{i=0}^{m} P(\xi+\eta=m|\eta=i)P(\eta=i) = \sum_{i=0}^{m} P(\xi=m-i)P(\eta=i)$$

$$= \sum_{i=0}^{m} e^{-\lambda}\frac{\lambda^{m-i}}{(m-i)!}e^{-\mu}\frac{\mu^i}{i!} = \frac{e^{-(\lambda+\mu)}}{m!}\sum_{i=0}^{m} C_m^i \lambda^{m-i}\mu^i = e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^m}{m!}$$

$$P(\xi=k|\xi+\eta=m) = \frac{P(\xi=k,\eta=m-k)}{P(\xi+\eta=m)} = \frac{e^{-\lambda}\dfrac{\lambda^k}{k!}e^{-\mu}\dfrac{\mu^{m-k}}{(m-k)!}}{e^{-(\lambda+\mu)}\dfrac{(\lambda+\mu)^m}{m!}} = C_m^k\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{m-k}$$

$$\therefore \xi\big|\xi+\eta=m \sim B\left(m, \frac{\lambda}{\lambda+\mu}\right)$$

$$\therefore E(\xi|\xi+\eta=m) = \frac{m\lambda}{\lambda+\mu}$$

(2)

$$P(\xi+\eta=m) = \sum_{i=0}^{m} P(\xi+\eta=m|\eta=i)P(\eta=i) = \sum_{i=0}^{m} P(\xi=m-i)P(\eta=i)$$

$$= \sum_{i=0}^{m} C_n^i p^i(1-p)^{n-i}C_n^{m-i}p^{m-i}(1-p)^{n-m+i} = p^m(1-p)^{2n-m}\sum_{i=0}^{m} C_n^i C_n^{m-i} = p^m(1-p)^{2n-m}C_{2n}^m$$

$$P(\xi=k|\xi+\eta=m) = \frac{P(\xi=k,\eta=m-k)}{P(\xi+\eta=m)} = \frac{C_n^k p^k(1-p)^{n-k}C_n^{m-k}p^{m-k}(1-p)^{n-m+k}}{p^m(1-p)^{2n-m}C_{2n}^m} = \frac{C_n^k C_n^{m-k}}{C_{2n}^m}$$

$$\therefore E(\xi|\xi+\eta=m)=\frac{1}{C_{2n}^m}\sum_{k=0}^{m}kC_n^kC_n^{m-k}=\frac{1}{C_{2n}^m}\sum_{k=1}^{m}nC_{n-1}^{k-1}C_n^{m-k}=\frac{nC_{2n-1}^{m-1}}{C_{2n}^m}=\frac{m}{2}$$

(3)

$$P(\xi+\eta=m)=\sum_{i=0}^{m}P(\xi+\eta=m|\eta=i)P(\eta=i)=\sum_{i=0}^{m}P(\xi=m-i)P(\eta=i)$$

$$=\sum_{i=0}^{m}(1-p)^{m-i}p(1-p)^ip=(m+1)p^2(1-p)^m$$

$$P(\xi=k|\xi+\eta=m)=\frac{P(\xi=k,\eta=m-k)}{P(\xi+\eta=m)}=\frac{(1-p)^kp(1-p)^{m-k}p}{(m+1)p^2(1-p)^m}=\frac{1}{m+1}$$

$$\therefore E(\xi|\xi+\eta=m)=\sum_{k=0}^{m}\frac{k}{m+1}=\frac{m}{2}$$

注: 这里的几何分布与陈希孺文集:《概率论与数理统计》第48面中一致,
即$P(\xi=k)=(1-p)^kp$, $k=0,1,2,\cdots$

22.

$Cov(\xi,\eta)=E\xi\eta-E\xi E\eta=0\Longrightarrow\xi,\eta$不相关

$P(\xi=-1)=\frac{3}{8}$, $P(\eta=-1)=\frac{3}{8}$

$P(\xi=-1,\eta=-1)=\frac{1}{8}\neq P(\xi=-1)P(\eta=-1)$

$\therefore\xi,\eta$不独立

24.

(1)

$$f_X(x)=\int_{-x}^{x}I_{[0,1]}(x)dy=2xI_{[0,1]}(x)$$

$$f_Y(y)=\int_{|y|}^{1}I_{[-1,1]}(y)dx=(1-|y|)\,I_{[-1,1]}(y)$$

(2)

$$EX=\int_0^1 2x^2dx=\frac{2}{3}$$

$$EY=\int_{-1}^{1}y(1-|y|)dy=0$$

$$EX^2=\int_0^1 2x^3dx=\frac{1}{2}$$

$$EY^2 = \int_{-1}^{1} y^2(1-|y|)dy = 2\int_{0}^{1}(y^2-y^3)dy = \frac{1}{6}$$

$$\therefore Var(X) = EX^2 - (EX)^2 = \frac{1}{18}$$

$$\therefore Var(Y) = EY^2 - (EY)^2 = \frac{1}{6}$$

(3)

$$EXY = \int_0^1 \int_{-x}^{x} xy\, dydx = 0$$

$$Cov(X,Y) = EXY - EX \cdot EY = 0$$

25.

令$Z = X - Y$，则$Z \sim N(0,1)$

$$\therefore Var(|Z|) = EZ^2 - (E|Z|)^2 = 1 - (E|Z|)^2$$

$$又\because E|Z| = \int_{-\infty}^{+\infty} \frac{|z|}{\sqrt{2\pi}} e^{-z^2/2} dz = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} z e^{-z^2/2} dz = \sqrt{\frac{2}{\pi}}$$

$$\therefore Var(|Z|) = 1 - \frac{2}{\pi}$$

26.

ξ	1	2	3	4	5	6
P	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

η	1	2	3	4	5	6
P	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{1}{4}$	$\frac{11}{36}$

$E\xi = \frac{7}{2}$，$Var(\xi) = \frac{35}{12}$

$E\eta = \frac{161}{36}$，$Var(\eta) = \frac{2555}{1296}$

$\xi\eta$	1	2	3	4	5	6	8	9	10	12	15	16	18	20	24	25	30	36
P	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{6}$

$$\therefore E\xi\eta = \frac{154}{9}$$

$$\therefore Cov(\xi,\eta) = E\xi\eta - E\xi E\eta = \frac{35}{24}$$

28.

$$Var(Z)=\pi^2Var(X)+(1-\pi)^2Var(Y)+2\pi(1-\pi)Cov(X,Y)=\left(3\pi^2-3\pi+1\right)\sigma^2$$

$$=\left[3(\pi-\frac{1}{2})^2+\frac{1}{4}\right]\sigma^2$$

$\because 0\le\pi\le 1$

$\therefore Var(Z)\le\sigma^2$

当$\pi=\dfrac{1}{2}$时，$Var(Z)$取最小值$\dfrac{1}{4}\sigma^2$

30.

(1)

$$f_\xi(x)=\frac{1}{2}\int_{-\infty}^{+\infty}\varphi_1(x,y)dy+\frac{1}{2}\int_{-\infty}^{+\infty}\varphi_2(x,y)dy$$

$$=\frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}+\frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}$$

$$=\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}$$

同理$f_\eta(y)=\dfrac{1}{\sqrt{2\pi}\sigma_2}\exp\left\{-\dfrac{(x-\mu_2)^2}{2\sigma_2^2}\right\}$

$\therefore \xi\sim N(\mu_1,\sigma_1^2)\quad,\quad \eta\sim N(\mu_2,\sigma_2^2)$

(2)

$$E\xi\eta=\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xy\varphi_1(x,y)dxdy+\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xy\varphi_2(x,y)dxdy$$

$$=\frac{1}{2}EX_1Y_1+\frac{1}{2}EX_2Y_2=\frac{1}{2}\left(\rho\sigma_1\sigma_2+\mu_1\mu_2\right)+\frac{1}{2}\left(-\rho\sigma_1\sigma_2+\mu_1\mu_2\right)=\mu_1\mu_2$$

$\therefore Cov(\xi,\eta)=E\xi\eta-E\xi E\eta=0$

$\therefore Corr(\xi,\eta)=0$

(3) 不独立

32.

$\because f(x,y)\ne f_X(x)f_Y(y)$ $\left(\text{或者说}f(x,y)\text{不能表示成}f_1(x)f_2(y)\text{的形式}\right)$

$\therefore X,Y$不独立

令$Z=X^2$，$W=Y^2$，则$0\le z,w\le 1$时

$$F(z,w)=P(Z\le z,W\le w)=P(X^2\le z,Y^2\le w)$$

$$= \int_{-\sqrt{w}}^{\sqrt{w}} \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{4}(1 + xy)\, dxdy = \sqrt{zw}$$

$\therefore f(z, w) = \dfrac{1}{4\sqrt{zw}} I_{[0,1]}(z) I_{[0,1]}(w)$可分离为分别只关于$z$和$w$的函数的乘积

$\therefore X^2$，Y^2独立

37.

$$P(|X + Y| \geq 6) \leq \frac{Var(X + Y)}{36}$$

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y) = 1 + 4 - 2 \times 0.5 \times 1 \times 2 = 3$$

$$\therefore P(|X + Y| \geq 6) \leq \frac{1}{12}$$

41.

(1)

设$X_i = \begin{cases} 1 \quad , \quad 第i个部件正常工作 \\ 0 \quad , \quad 否则 \end{cases}$，令$S_{100} = X_1 + X_2 + \cdots + X_{100}$

则$S_{100} \sim B(100, 0.9)$，整个系统正常工作的概率

$$P(S_{100} \geq 85) = P\left(\frac{S_{100} - ES_{100}}{\sqrt{Var(S_{100})}} \geq \frac{85 - ES_{100}}{\sqrt{Var(S_{100})}}\right) = P\left(\frac{S_{100} - ES_{100}}{\sqrt{Var(S_{100})}} \geq \frac{85 - 90}{3}\right) = 1 - \Phi\left(-\frac{5}{3}\right)$$

$$= \Phi\left(\frac{5}{3}\right) = 0.952$$

(2)

设$X_i = \begin{cases} 1 \quad , \quad 第i个部件正常工作 \\ 0 \quad , \quad 否则 \end{cases}$，令$S_n = X_1 + X_2 + \cdots + X_n$

则$S_n \sim B(n, 0.9)$，整个系统正常工作的概率

$$P(S_n \geq 0.8n) = P\left(\frac{S_n - ES_n}{\sqrt{Var(S_n)}} \geq \frac{0.8n - ES_n}{\sqrt{Var(S_n)}}\right) = P\left(\frac{S_n - ES_n}{\sqrt{Var(S_n)}} \geq \frac{0.8n - 0.9n}{0.3\sqrt{n}}\right) = 1 - \Phi\left(-\frac{\sqrt{n}}{3}\right)$$

$$= \Phi\left(\frac{\sqrt{n}}{3}\right) \geq 0.95$$

$$\therefore \frac{\sqrt{n}}{3} \geq 1.64$$

$$\therefore n \geq 24.20$$

$\therefore n$至少为25

44.

(1)

做法与前类似，概率0.18

(2)

至多443次加法运算

47.

需要842kw的电力

48.

至少调查9604个人